The London School of Economics and Political Science

Essays on Information and Frictions in Financial Markets

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

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Statement of conjoint work

I confirm that Chapter 2 was jointly co-authored with Dr Georgy Chabakauri and I contributed 1/2 of this work.

Abstract

The first chapter studies the dynamics of information acquisition and uncertainty in a noisy rational expectations model. Investors choose to acquire most information at times when uncertainty and risk premia are high; this choice feeds back and endogenously reduces subsequent uncertainty. Within the model, uncertainty can be measured directly from risk-neutral variance—analogous to the VIX index—so this translates into the concrete prediction that risk-neutral variance mean-reverts rapidly following spikes in volatility, as is observed empirically. The cyclicality of information acquisition depends on the skewness of the underlying asset: if the market is negatively skewed, market-level information acquisition is countercyclical. Conversely, information acquisition and risk premia are high following good news for positively skewed assets such as individual stocks, which gives rise to momentum in the stock market.

In the second chapter, my co-author and I consider an economy populated by investors with heterogeneous preferences and beliefs who receive non-pledgeable labor incomes. We study the effects of collateral constraints that require investors to maintain sufficient pledgeable capital to cover their liabilities. We show that these constraints inflate stock prices, give rise to clusters of stock return volatilities, and produce spikes and crashes in price-dividend ratios and volatilities. Furthermore, the mere possibility of a crisis significantly decreases interest rates and increases Sharpe ratios. The stock price has a large collateral premium over non-pledgeable incomes. Asset prices are in closed form, and investors survive in the long run.

The third chapter studies information acquisition with a long-lived risky asset that generates dividends in each period. The investors can either be informed or uninformed, and the informed investors actively acquire information on the time-varying dividend growth rate. Informed investors take short positions in the variance swap to realize their informational advantage; the uninformed investor takes a long position to hedge his risks. Serial correlation of returns is decreasing in information acquisition of informed investors. Low uncertainty induces investors to acquire less information and decreases the cross-sectional dispersion of beliefs in expected returns.

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Chapter 1

Dynamic Information Acquisition and Asset Prices

This paper studies the dynamics of information acquisition and uncertainty in a noisy rational expectations model. Investors choose to acquire most information at times when uncertainty and risk premia are high; this choice feeds back and endogenously reduces subsequent uncertainty. Within the model, uncertainty can be measured directly from risk-neutral variance—analogous to the VIX index—so this translates into the concrete prediction that risk-neutral variance mean-reverts rapidly following spikes in volatility, as is observed empirically. The cyclicality of information acquisition depends on the skewness of the underlying asset: if the market is negatively skewed, market-level information acquisition is countercyclical. Conversely, information acquisition and risk premia are high following good news for positively skewed assets such as individual stocks, which gives rise to momentum in the stock market.

Keywords: Dynamic information acquisition, Uncertainty, Investor attention, Risk-neutral variance

1.1. Introduction

Investors acquire information on asset fundamentals when they trade in financial markets. Information acquisition determines portfolio choices and thus affects asset prices and the uncertainty that investors face. Conversely, uncertainty about asset payoffs changes investors' incentives to acquire information. This interplay between information acquisition and uncertainty is central to understanding how asset prices dynamically evolve. The literature on information acquisition and asset prices typically treats information acquisition as a one-off decision. This paper investigates the endogenous dynamics of information acquisition and uncertainty in a noisy rational expectations model.

Within this model, uncertainty can be measured directly from the risk-neutral variance of the asset payoff, analogous to the volatility index VIX. In periods of high uncertainty, investors acquire more information. This choice feeds back and endogenously reduces subsequent uncertainty. As a result, the risk-neutral variance mean reverts rapidly following spikes in volatility, in line with empirical evidence. The cyclicality of information acquisition depends on the skewness of the asset payoff: market-level information acquisition is countercyclical because aggregate stock market displays negative skewness; in contrast, firm-level information acquisition is procyclical because individual stocks are positively skewed. Following good news for individual stocks, information acquisition and risk premia are high, which gives rise to momentum in the stock market.

I start by analyzing an economy with a single risky asset. Most findings and predictions extend to a multiple-asset setup that I later consider. The economy is populated by a continuum of ex-ante identical investors. Exogenous noisy supply of the asset prevents the price from fully revealing the asset's final payoff. Knowledge about the payoff is gradually acquired over multiple time periods and comes to investors in the form of a stream of private signals. The sources of private information are different across investors, as in Hellwig (1980). I use investor attention to represent the precision of the private signal. Information is costly to investors and the cost is increasing and convex in attention.

In this paper, attention represents both the effort in gathering information and the amount of information acquired by investors in a given period.[1] This study focuses on the variation in investors' attention levels over time rather than the static allocation of attention across assets. Investors acquire less information in aggregate when the market is devoid of profitable investment opportunities.

In this economy, investor attention is endogenously determined by the level of uncertainty. I show that uncertainty is represented by the risk-neutral variance of the asset's terminal payoff, which quantifies the value (in utility terms) of a marginal piece of information. It can therefore be directly measured, in principle, from option prices, on similar lines to the construction of the volatility index, VIX.

Investors also learn from past prices. This generates rich dynamics as the public information set, which consists of the entire history of asset prices, grows over time. I show that four state variables summarize the public information and that these state variables have a natural economic interpretation (discussed further in Section 3.2). The four state variables jointly define a system of PDEs, which characterize equilibrium and can be solved numerically.

I allow the payoff of the risky asset to have an arbitrary distribution. Non-normality is more plausible empirically. It gives rise to endogenously fluctuating uncertainty and to comovement between asset prices and information acquisition. If the payoff is normally distributed, uncertainty no longer varies across states because the risk-neutral variance becomes a deterministic function of time.

A simple illustration of the dynamic interactions between information acquisition and asset prices is as follows. Suppose that the economy enters a period of high uncertainty about asset payoffs. If the level of information acquisition does not change, then the volatility of asset returns increases, and remains uniformly high during the high-uncertainty period. Agents, however, respond to the high uncertainty by acquiring more information. This causes asset returns to be even more volatile in the early stages of the high-uncertainty period, as agents learn more information. Returns become less volatile in the later stages, as learning gradually reduces payoff uncertainty. Thus, dynamic information acquisition causes return volatility to vary more over time and to be less persistent.

[1]Attention in this paper represents information acquisition. I use attention and information acquisition interchangeably throughout the paper.

The model generates a rich set of implications supported by empirical observations. First, uncertainty is mean-reverting. High uncertainty induces investors to devote more attention to the asset, in turn creating downward pressure on uncertainty itself. As a result, the model explains why peaks in VIX are usually followed by a rapid decline.[2]

Second, the expected return of an asset is increasing in investor attention. When attention is high today, investors gather information at a rapid pace, leading to a rapid reduction in uncertainty and the amount of risk that this asset entails. Then, the asset becomes less risky tomorrow and enjoys a lower risk premium tomorrow and beyond. This quick reduction in tomorrow's risk premium compared to today's implies a high expected return for the asset.

A growing list of empirical evidence supports this prediction. Da, Engelberg and Gao (2011) find that an increase in Google Search Volume Index predicts higher stock prices in the next two weeks. Lou (2014) documents that increased advertising spending is associated with more attention and a rise in abnormal stock returns. Lee and So (2017) show that abnormal analyst coverage predicts improvements in firms' fundamental performance. Because attention is determined by uncertainty, the expected return of the asset is also positively associated with uncertainty. Martin (2017) find that the risk-neutral variance predicts the return of the market at horizons from one month to one year.

Third, the cyclicality of information acquisition depends on the skewness of the underlying asset. When the distribution of the payoff is right-skewed, investors are more excited about potential upside gains. Therefore they devote more attention to the asset when the price is high. Conversely, when the distribution is left-skewed, investors are more worried about potential losses in the downside and acquire more information when the price is low. Aggregate stock market displays negative skewness and individual stocks display positive skewness (e.g., Bakshi, Kapadia and Madan (2003); Albuquerque(2012)). Therefore, attention is procyclical for firm-specific information and countercyclical for market-wide information. Hong, Lim and Stein (2000) find that negative firm-specific news travels more slowly compared

[2]Martin (2017) find that the leading models including Campbell and Cochrane (1999), Bansal and Yaron (2004), Bollerslev, Tauchen, and Zhou (2009), and Wachter (2013) cannot explain the relatively low autocorrelation of VIX.

to positive news. In contrast, Garcia (2013) documents that investors react more strongly to business cycle news at times of recession.

I extend the model to multiple assets to explain momentum in the stock market. Investors trade several individual stocks. Each stock's payoff consists of a common market component and a stock-specific (idiosyncratic) component. Information acquisition on this idiosyncratic component determines the dynamics of the stock's excess return. Stocks that performed well relative to the market in the past are likely to attract more attention and thus continue to generate higher expected excess returns in the future.

Related Literature

This paper relates to an extensive literature on information acquisition in financial markets, initiated by Grossman and Stiglitz (1980) and Verrecchia (1982) and developed by Holden and Subrahmanyam (2002), Mendelson and Tunca (2004), Veldkamp(2006), Huang and Liu (2007), and Andrei and Hasler (2014), among others. The model is in the spirit of Verrecchia (1982). It allows for a group of ex-ante identical investors to learn from both prices and diverse private information.

Kacperczyk, Van Nieuwerburgh, and Veldkamp (2016) study how mutual fund managers allocate attention across different assets and used the state of the business cycle to predict information choices. Banerjee and Breon-Drish(2017) consider a strategic trader who optimizes the time to acquire costly information about an asset's payoff in a Kyle(1985) framework. They show that equilibrium with smooth trading and a pure acquisition strategy cannot exist when the market maker cannot observe acquisition. In contrast to the above papers, where the incentive to acquire information is affected by exogenous business cycle variations or public news, investor attention in this model is determined by the endogenously generated risk-neutral variances.

The explanation of momentum also differs from existing behavioral and rational models. Hong and Stein (1999) explain underreaction and overreaction in asset markets through two groups of bounded rational investors and the assumption that information diffuses gradually across the population. Andrei and Cujean (2017) build a rational-expectations model where investors use word-of-mouth communication to acquire information and show that price exhibits momentum when information flows at an increasing rate. In both cases, momentum arises because signals first observed by a small group of investors are subsequently released to a larger group. This paper's explanation relies on no such channel and is based on the interaction between attention and risk premium.

This study also contributes to the analysis of rational expectations equilibrium with general payoff distributions. Breon-Drish (2015) relax the normality assumption and proved the existence of and characterized the equilibrium for a class of models that nests the standard Grossman and Stiglitz (1980) and Hellwig (1980) setups. Chabakauri, Yuan and Zachariadis (2017) analyze asset prices in both complete and incomplete markets for realistic multi-asset economies with non-normal payoff distributions. Previous studies employed static setups but this paper works with a dynamic one.

The paper is organized as follows. Section 2 introduces the model setup. Section 3 solves investors' portfolio and attention choice problems and characterizes the equilibrium. Section 4 illustrates model predictions and empirical implications. Section 5 extends the model to a multi-asset setup. Section 6 concludes.

1.2. Model

The economy features a single risky asset and a risk-free asset. It is populated by a continuum of ex-ante identical investors who actively acquire private information on the asset's final payoff. To facilitate the exposition, I start with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \ldots, T$, and later take a continuous-time limit when I characterize the equilibrium in Section 3.3 .

The main differences with a standard rational expectations model (e.g. Hellwig(1980)) are that information acquisition is endogenous and that the payoff of the risky asset is not necessarily normally distributed.

Assets

The risk-free asset is the numéraire in this economy, and its price is normalized to 1 for all dates. The interest rate is equal to 0 for all periods. The risky asset realizes a liquidating payoff y at the final date T and pays no dividend between 0 and $T-\Delta t$. The distribution of the terminal payoff $G(y)$ is not restricted to normal distribution.

I assume that the moment generating function of this distribution $M_y(\theta)$ exists for any θ. This technical assumption guarantees the existence of investors' asset demands in equilibrium.

$$M_y(\theta) = \mathbb{E}[e^{\theta y}] = \int_{-\infty}^{\infty} e^{\theta y} dG(y) < \infty, \quad \theta \in R. \tag{1.1}$$

Assets are traded at dates $t = 0, \Delta t, 2\Delta t, \ldots, T - \Delta t$. Let p_t represent the price of the risky asset at date t. For the period between date t and $t + \Delta t$, the return on the risky asset is $p_{t+\Delta t} - p_t$. Noisy supply of the risky asset prevents the price from fully revealing the final payoff. I assume that this supply z_t follows a random walk and its increment $z_{t+\Delta t} - z_t$ is normally distributed with a mean of zero and variance of $\sigma_z^2 \Delta t$:

$$z_{t+\Delta t} - z_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_z^2 \Delta t). \tag{1.2}$$

Information Acquisition

Investors start with no information about the payoff of the risky asset and gradually acquire a stream of private signals about this payoff. Investor attention determines the precision of these signals.

Suppose investor $i \in [0, 1]$ devotes attention a_{it} to payoff y for the period between date t and date $t + \Delta t$. New private information for this investor in this period is represented by a signal $s_{t+\Delta t}$, which communicates y perturbed by a normal noise with precision $a_{it}\Delta t$. Investors acquire different pieces of information and their sources of private information are independent, as in Hellwig (1980) and Verrecchia (1982). In this setup, the noises in signals are independent both across time and across investors:

$$s_{t+\Delta t} = y + \epsilon_{t+\Delta t}, \qquad \epsilon_{t+\Delta t} \sim \mathcal{N}(0, (a_{it}\Delta t)^{-1}). \tag{1.3}$$

New information comes at a cost that is increasing and convex in attention. This cost takes the form of $C(a_{it})\Delta t$, where $C(a_{it})$ is a function with continuous first and second order derivatives. Marginal cost of attention $C'(a_{it})$ is increasing in a_{it}.

Investors also learn from past prices. The price history up to the current date p_0, $p_{\Delta t}$, ..., p_t represents all available public information. Investor i's information set at date t consists of her private signals $s_{\Delta t}, s_{2\Delta t}, \ldots s_{it}$ and the public information set. Investors are rational and use all available private and public information to update their beliefs about the asset payoff and future investment opportunities.

Preference and Investor Optimization

Investor i is endowed with initial wealth W_{i0} at date 0. At date t, she allocates her wealth W_{it} to θ_{it} units of risky asset and $W_{it} - \theta_{it} p_t - C(a_{it})\Delta t$ units of the risk-free asset. She also decides at this date how much attention a_{it} to devote to payoff y for the period between t and $t + \Delta t$. It is worth noting that the corresponding private signal $s_{i,t+\Delta t}$ arrives at date $t+\Delta t$ and could only be incorporated in investor i's portfolio choice from that date onwards.

All investors have constant absolute risk aversion (CARA) preference with the risk aversion parameter A. They make portfolio and attention choices (θ_{it}, a_{it}) to maximize expected utility over the terminal wealth:

$$\max_{\theta_{it}, a_{it}} \mathbb{E}_t^i \left[U(W_{iT}) \right], \qquad U(W_{iT}) = -e^{-AW_{iT}}, \tag{1.4}$$

subject to the self-financing budget constraint, given by:

$$W_{i,t+\Delta t} = W_{it} + \theta_{it}(p_{t+\Delta t} - p_t) - C(a_{it})\Delta t. \tag{1.5}$$

Price and Market Clearing

Equilibrium price p_t is determined by market clearing:

$$\int_{i=0}^{1} \theta_{it} di = z_t. \tag{1.6}$$

where the left-hand side represents the aggregate demand of the risky asset and the right-hand side, its supply.

Definition of Equilibrium

The definition of equilibrium is standard. Investors make the optimal portfolio and attention choices and the market clears.

Definition 1. *The equilibrium is a set of risky asset prices p_t and portfolio and attention policies (θ_{it}, a_{it}) that solve the optimization problem (1.4) for each investor and satisfies market clearing condition (1.6).*

1.3. Equilibrium

In this model, the asset payoff y and the noisy supply of the risky asset $z_0, z_{\Delta t}, \ldots, z_{T-\Delta t}$ are exogenously given. Price p_t, attention a_{it} and uncertainty (measured by the risk-neutral variance of the asset payoff) are endogenously determined.

I characterize the equilibrium in a three-step process. First, I solve investors' portfolio and attention choices. Second, I use the market clearing condition to express the asset demand in terms of the exogenous variables noisy supply and asset payoff. The time series of asset demands is informationally equivalent to the time series of past asset prices. I further define state variables that summarize the information content of these time series. Third, I derive a system of equations for the price and the risk-neutral variance.

1.3.1 Portfolio and Attention Choice

Let us first consider a suggestive three-period example. Suppose that the payoff is realized at the final date $T = 2$ and assets are traded at dates $t = 0, 1$ with a time interval of $\Delta t = 1$. I solve investors' optimization problems through backward induction, starting from date 1 and then moving on to date 0.

Date 1 is the last trading opportunity before the realization of payoff. Any information arriving after date 1 is worthless to investors because they could no longer change their portfolios. As a result, investors will not devote any attention to the payoff for the period between date 1 and date 2.

At date 1, from investor i's perspective, y follows distribution $G(y|p_0, p_1, s_{i1})$. She chooses θ_{i1} to maximize expected utility:

$$\max_{\theta_{i1}} \int -\exp\left(-A\theta_{i1}(y - p_1)\right) \; dG(y|p_0, p_1, s_{i1}). \tag{1.7}$$

Since investors are infinitely small and their sources of private information are independent, the asset price does not depend on one particular investor's signal. This implies that conditional on payoff y, past prices p_0, p_1 are independent from signal s_{i1}. To put it another way, the private signal is a source of information independent of the price history. Applying Bayes theorem and noting that s_{i1} is normally distributed with mean y and precision a_{i0}, the optimization problem (1.7) is equivalent to:

$$\max_{\theta_{i1}} \int -\exp\left(-A\theta_{i1}(y - p_1)\right) \cdot \frac{1}{\sqrt{2\pi(a_{i0})^{-1}}} \exp\left(-\frac{a_{i0}}{2}(s_{i1} - y)^2\right) \; dG(y|p_0, p_1). \tag{1.8}$$

The coefficient of y in the above expression is $-A\theta_{i1} + a_{i0}s_{i1}$, a linear combination of asset demand and private signal. Investors' utility maximization problems are similar despite the differences in the private signals. Taking the first-order condition, I find that $-A\theta_{i1}+a_{i0}s_{i1}$ is identical across investors. As a result, the asset demand θ_{i1} is additively separable in the signal, as in Breon-Drish (2015). It also contains a common component θ_1 that only depends on the price history. Lemma 1 reports this asset demand. Proofs of all lemmas and propositions are given in the appendix.

Lemma 1. *Asset demand θ_{i1} is the sum of the attention weighted private signal and a component θ_1 that is common across investors and only depends on the prices p_0 and p_1:*

$$\theta_{i1} = \theta_1(p_0, p_1) + A^{-1}a_{i0}s_{i1}. \tag{1.9}$$

At date 0, investors do not possess any private signals and have identical asset demands $\theta_{i0} = \theta_0(p_0)$ that only depend on price p_0. They also decide on this date how much information to acquire for each asset for the period between date 0 and date 1. A higher level of attention improves portfolio choices at date 1 and increases investors' expected utility. Substituting in θ_{i0} and θ_{i1} and integrating over signal s_{i1}, I find that:

$$\begin{aligned}
\mathbb{E}_0^i[U(W_{i2})] =& \mathbb{E}_0^i\left[-\exp\left(-AW_{i0} - A\theta_{i0}(p_1-p_0) - A\theta_{i1}(y-p_1) + A\cdot C(a_{i0})\right)\right] \\
=& \mathbb{E}_0^i\Bigg[\int -\exp\Bigg(- AW_{i0} - A\theta_{i0}(p_1-p_0) - A\left(\theta_{i1} - A^{-1}a_{i0}s_{i1}\right)(y-p_1) \\
& - \frac{a_{i0}}{2}(y^2-p_1^2) + A\cdot C(a_{i0})\Bigg)\cdot\frac{1}{\sqrt{2\pi(a_{i0})^{-1}}}\exp\left(-\frac{a_{i0}}{2}(s_{i1}-p_1)^2\right)ds_{i1}\Bigg] \\
=& \mathbb{E}_0^i\Bigg[-\exp\Bigg(- AW_{i0} - A\theta_{i0}(p_1-p_0) - A\theta_1(y-p_1) \\
& - \underbrace{\frac{a_{i0}}{2}(y^2-p_1^2)}_{\text{Value of Information}} + \underbrace{A\cdot C(a_{i0})}_{\text{Cost of Information}}\Bigg)\Bigg].
\end{aligned} \tag{1.10}$$

In the above expression, $A\cdot C(a_{i0})$ represents the cost of information and $a_{i0}/2\cdot(y^2-p_1^2)$ measures the expected gain in utility from a precision a_{i0} signal. Let $\mathbb{E}^*(X)$ represent the risk-neutral expectation of a random variable X, defined by $\mathbb{E}\left[U'(W_{i2})/\,\mathbb{E}[U'(W_{i2})]\cdot X\right]$, where $U'(W_{i2})$ is investor i's marginal utility.

Differentiate date 0 expected utility $\mathbb{E}_0^i[U(W_{i2})]$ with respect to attention level

a_{i0}:

$$\begin{aligned} C'(a_{i0}) &= \frac{1}{2}A^{-1}\frac{\mathbb{E}_0^i\left[U'(W_{i2})(y^2-p_1^2)\right]}{\mathbb{E}_0^i\left[U'(W_{i2})\right]} \\ &= \frac{1}{2}A^{-1}\,\mathbb{E}_0^*\left[y^2-p_1^2\right] \\ &= \frac{1}{2}A^{-1}\,\mathbb{E}_0^*\left[\mathrm{Var}_1^*(y)\right]. \end{aligned} \tag{1.11}$$

Marginal cost $C'(a_{i0})$ is an increasing function of attention a_{i0}. It is proportional to the expectation of $y^2 - p_1^2$ under the risk-neutral measure. Since the interest rate is zero, p_1 is equal to $\mathbb{E}_1^*[y]$, and thus $\mathbb{E}_1[y^2 - p_1^2]$ is equal to the risk-neutral variance of the payoff $\mathrm{Var}_1^*(y)$. This risk-neutral variance measures the marginal value of information for signal s_{i1} and represents the uncertainty in payoff y from the investors' perspective. The precision of this signal needs to be decided one period ahead, at date 0. As a result, attention a_{i0} is determined by the risk-neutral expectation of next-period risk-neutral variance $\mathbb{E}_0^*\left[\mathrm{Var}_1^*(y)\right]$.

Investors are ex-ante identical and have the same prior belief and cost of information. In equilibrium, they choose the same level of attention a_{i0}. The risk-neutral expectations and variances of the payoff computed from different investors' marginal utilities are identical.

The risk-neutral variance $\mathrm{Var}_1^*(y)$ can be directly measured from option prices on similar lines to the construction of the volatility index VIX[3], thereby relating the unobservable information acquisition to an empirically observable measure. Allowing options to be traded will not change the equilibrium allocation. If there are no exogenous supply, investors' demand for options will also be identical at a level of zero.

Equations (1.9) and (1.11) report investors' portfolio and attention choices for this particular example where $T = 2$ and $\Delta t = 1$. Proposition 1 generalizes these findings to arbitrary T and Δt.

Proposition 1. *Let v_t represent the risk-neutral variance of the payoff at date t:*

$$v_t \equiv \mathrm{Var}_t^*[y] \tag{1.12}$$

Attention a_{it} is identical across investors and independent of private signals. Let a_t

[3]The risk-neutral variance in this paper more closely resembles volatility index SVIX introduced by Martin(2017). SVIX differs from VIX if the setting is not conditionally log-normal.

represent this identical level of attention. It is determined by the price history and satisfies:

$$C'(a_t) = \frac{1}{2}A^{-1}\,\mathbb{E}_t^*\left[v_{t+\Delta t}\right]. \tag{1.13}$$

Asset demand θ_{it} is the sum of attention weighted private signals and a component θ_t that is common across investors and only depends on the price history up to the current date p_0, $p_{\Delta t}$, ..., p_t:

$$\theta_{it} = \theta_t(p_0, p_{\Delta t}, \ldots, p_t) + A^{-1}\sum_{u=0}^{t-\Delta t} a_u s_{i,u+\Delta t}\Delta t, \tag{1.14}$$

Additive separability of signals in asset demand functions is a feature of CARA utility assumption and the normal noise signal structure given in equation (1.3). Substituting the asset demands into the utility function and simplify, I find that investors face the same attention optimization problem since they have the same risk-aversion and cost of information. Therefore, attention and the risk-neutral variance in equilibrium are identical across investors and independent of the realization of private signals.

Equation (1.13) establishes a link between investor attention and the risk-neutral variance of the payoff. The right-hand side of this equation represents the marginal value of an additional piece of information to investors. The left-hand side is the marginal cost of information, which is increasing in attention because of the convexity of the cost function $C(a_t)$. When the risk-neutral variance $v_{t+\Delta t}$ is expected to be high, investors are willing to devote more attention to the asset payoff. Intuitively, people acquire more information when they are uncertain about the state of the world.

This risk-neutral variance v_t resembles the volatility index VIX in that they both measure investors' perceptions of uncertainty in the asset. However, here investors are interested in the variation of the final payoff, as opposed to that of a one-period return defined by the ratio of next period price $p_{t+\Delta t}$ to the current price p_t. Falling asset price is usually accompanied by an upward spike in the VIX index because of the leverage effect. This is not necessarily true for the risk-neutral variance of the terminal payoff. Its correlation with the asset price changes with the skewness and support of the payoff distribution and will be further analyzed in Section 4.3.

1.3.2 Market Clearing and State Variables

From market clearing, the aggregate demand for the asset is equal to its supply. Integrate θ_{it} in (1.14) over investors. By the law of large numbers, noises in private signals are canceled out and the average of s_{iu} is exactly equal to y

$$\int_0^1 \left(\theta_t + A^{-1} \sum_{u=0}^{t-\Delta t} a_u s_{i,u+\Delta t} \Delta t \right) di = \theta_t + A^{-1} \left(\sum_{u=0}^{t-\Delta t} a_u \Delta t \right) y = z_t. \tag{1.15}$$

Investors are infinitely small and the impact of their private signals on the prices is canceled out by the law of large numbers. It is sufficient to characterize the equilibrium using the information publicly available to all investors, which is the history of prices up to the current date p_0, $p_{\Delta t}$, ..., p_t.

The demand curve is downward sloping in this model: the common component of asset demand θ_t, implicitly defined in (1.14), is strictly decreasing in the current price p_t. The mapping from $(p_0, p_{\Delta t}, \ldots, p_t)$ to $(\theta_0, \theta_{\Delta t}, \ldots, \theta_t)$ is one-to-one.[4] In other words, the time series θ_0, $\theta_{\Delta t}$,..., θ_t is informationally equivalent to the price history.

At date $t + \Delta t$, the sequence θ_0, $\theta_{\Delta t}$,..., $\theta_{t+\Delta t}$ represents all publicly available information. Let us consider $A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$

$$A(a_t \Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t) = y + A(a_t \Delta t)^{-1}(z_{t+\Delta t} - z_t). \tag{1.16}$$

It is a public signal that communicates payoff perturbed by a normal noise with precision $A^{-2}\sigma_z^{-2}a_t^2\Delta t$. This signal represents new public information that arrives at date $t + \Delta t$. The noise in this signal is proportional to the increment in asset supply $z_{t+\Delta t} - z_t$ and is independent of all previous public signals because the supply follows a random walk.

State Variables

One difficulty involved in solving this model is that the dimension of state space grows as time increases. θ_t itself is not Markovian. Both $p_0, p_{\Delta t}, \ldots, p_t$ and θ_0, $\theta_{\Delta t}$,..., θ_t have dimensions equal to the number of trading dates. Fortunately, the

[4] I prove this result by mathematical induction. At the starting date, $p_0 \to \theta_0$ is injective because $\theta_0(p_0)$ is strictly decreasing. Suppose that for date $t - \Delta t$, each set of asset prices $(p_0, p_{\Delta t}, \ldots, p_{t-\Delta t})$ correspond to only one set of demands $(\theta_0, \theta_{\Delta t}, \ldots, \theta_{t-\Delta t})$. Fixing p_0, $p_{\Delta t}$, ..., $p_{t-\Delta t}$, each p_t corresponds to only one θ_t because $\theta_t(\ldots, p_t)$ is strictly decreasing in p_t. This completes the mathematical induction.

information content of these sequences can be summarized into 4 state variables, including the common component of asset demand θ_t and other 3 state variables that I define below.

Definition 2. Expected payoff m_t, public information precision χ_t and private information precision τ_t are defined by:

$$m_t \equiv \mathbb{E}\left[y|p_0, p_{\Delta t}, \ldots, p_t\right]. \tag{1.17}$$

$$\chi_t \equiv \sum_{u=0}^{t-\Delta t} A^{-2}\sigma_z^{-2}a_u^2\Delta t. \tag{1.18}$$

$$\tau_t \equiv \sum_{u=0}^{t-\Delta t} a_u\Delta t. \tag{1.19}$$

Investors use the public signals from (1.16) to update their beliefs about the final payoff. Public information is represented by two state variables, expected payoff m_t and public information precision χ_t. Expected payoff is the expected value of y using only public signals, ignoring all private signals. Public information precision is defined by the aggregate precision of all public signals from date 0 to date t. It represents the aggregate amount of information that is publicly available to investors.

Analogously, I use τ_t to denote the aggregate precision of private signals. It measures the amount of information privately acquired by investors from date 0 to date t. By substituting the definition of τ_t (1.19) into the market clearing equation (1.15), I show that the common component of asset demand θ_t can be expressed as a linear combination of the terminal payoff and the noisy supply:

$$\theta_t = z_t - A^{-1}\tau_t y. \tag{1.20}$$

The price of and the risk-neutral variance of the terminal payoff can be expressed as functions of these 4 state variables. Price is decreasing in asset demand θ_t and increasing in m_t, which represents investors' expectation of the terminal payoff from public information. The risk-neutral variance v_t is decreasing in information precisions χ_t and τ_t, because investors feel less uncertain about the asset payoff if they are more informed.

State Variable Dynamics

Information about the final payoff is gradually incorporated into private and public signals. The amount of public and private information increases and investors update their beliefs about the expected payoff. Lemma 2 summarizes how these state variables evolve from date t to $t + \Delta t$.

Lemma 2. *The dynamics of state variables θ_t, m_t, χ_t and τ_t are given by:*

$$\theta_{t+\Delta t} = \theta_t \ - A^{-1} a_t y \Delta t + z_{t+\Delta t} - z_t, \tag{1.21}$$

$$m_{t+\Delta t} = m_t + A^{-2} \sigma_z^{-2} a_t^2 h_t (y - m_t) \Delta t - A^{-1} \sigma_z^{-2} a_t h_t (z_{t+\Delta t} - z_t) + o(\Delta t), \tag{1.22}$$

$$\chi_{t+\Delta t} = \chi_t \ + A^{-2} \sigma_z^2 a_t^2 \Delta t, \tag{1.23}$$

$$\tau_{t+\Delta t} = \tau_t \ + a_t \Delta t. \tag{1.24}$$

where

$$h_t = \mathrm{Var}\left[y | p_0, p_{\Delta t}, \ldots, p_t \right] \tag{1.25}$$

is a deterministic function of m_t and χ_t and represents the variance of y in the objective physical measure after observing public information $p_0, p_{\Delta t}, \ldots, p_t$.

The aggregate amount of private information is increasing at a speed of attention, as expressed in the equation (1.24). The amount of public information is growing at a speed proportional to the square of attention, which is also the rate at which the expected payoff m_t drifts towards the direction of its true value y. A higher level of attention indicates that information is both acquired and disseminated at a quicker pace.

1.3.3 Equations for Price and Risk-neutral Variance

In the following section, I derive a system of recursive equations for price p_t and the risk-neutral variance v_t. Because the interest rate is zero, the price is a martingale under the risk-neutral measure:

$$p_t = \mathbb{E}_t^*(p_{t+\Delta t}). \tag{1.26}$$

Using the law of total variance, I decompose v_t as the sum of expected next-period variance and variance of next-period expectation:

$$\begin{aligned} v_t = \mathrm{Var}_t^*(y) &= \mathbb{E}_t^*[\mathrm{Var}_{t+\Delta t}^*(y)] + \mathrm{Var}_t^*[\mathbb{E}_{t+\Delta t}^*(y)] \\ &= \mathbb{E}_t^*[v_{t+\Delta t}] + \mathrm{Var}_t^*(p_{t+\Delta t}). \end{aligned} \tag{1.27}$$

The stochastic discount factor from date t to date T is required to change the probability measure from the risk-neutral one to the objective one. Lemma 3 specifies this stochastic discount factor $\xi_{t,T}$.

Lemma 3. *Let $\xi^i_{t,T}$ denote the ratio of investor i's marginal utility $U'(W_{iT})$ and its date t conditional expectation $\mathbb{E}^i_t[U'(W_{iT})]$:*

$$\xi^i_{t,T} = \frac{U'(W_{iT})}{\mathbb{E}^i_t[U'(W_{iT})]} \tag{1.28}$$

Let $\xi_{t,T}$ represent the average of $\xi^i_{t,T}$ across investors. It is a valid SDF and is given by:

$$\begin{aligned}
\xi_{t,T} &= \mathbb{E}\left[\xi^i_{t,T}|p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right] \qquad (1.29)\\
&= \exp\left(-\sum_{u=t}^{T-\Delta t}\left[A\theta_u(p_{u+\Delta t}-p_u)+\frac{1}{2}\tau_u(p^2_{u+\Delta t}-p^2_u)-A\cdot C(a_u)\Delta t\right]-f_t\right), \qquad (1.30)
\end{aligned}$$

where f_t is a normalizing variable defined by:

$$f_t = \ln \mathbb{E}_t\left[\exp\left(-\sum_{u=t}^{T-\Delta t}\left[A\theta_u(p_{u+\Delta t}-p_u)+\frac{1}{2}\tau_u(p^2_{u+\Delta t}-p^2_u)-A\cdot C(a_u)\Delta t\right]\right)\right]. \tag{1.31}$$

such that $\mathbb{E}_t[\xi_{t,T}] = 1$.

f_t is an auxiliary variable that helps form a system of equations involving the price and the risk-neutral variance. Substituting (1.29) into (1.27) and (1.26), use law of iterated expectations, and simplify, I arrive at the following recurrence equations for p_t and v_t:

$$\begin{aligned}
p_t &= \mathbb{E}_t\left[\exp\left(-A\theta_t(p_{t+\Delta t}-p_t)-\frac{1}{2}\tau_t(p^2_{t+\Delta t}-p^2_t)+A\cdot C(a_t)\Delta t - f_t + f_{t+\Delta t}\right)p_{t+\Delta t}\right],\\
v_t &= \mathbb{E}_t\left[\exp\left(-A\theta_t(p_{t+\Delta t}-p_t)-\frac{1}{2}\tau_t(p^2_{t+\Delta t}-p^2_t)+A\cdot C(a_t)\Delta t - f_t + f_{t+\Delta t}\right)v_{t+\Delta t}\right]\\
&\quad + \mathrm{Var}^*_t(p_{t+\Delta t}). \qquad (1.32)
\end{aligned}$$

Definition (1.31) can also be rewritten recursively:

$$\exp(f_t) = \mathbb{E}_t\left[\exp\left(-A\theta_t(p_{t+\Delta t}-p_t)-\frac{1}{2}\tau_t(p^2_{t+\Delta t}-p^2_t)+A\cdot C(a_t)\Delta t + f_{t+\Delta t}\right)\right]. \tag{1.33}$$

Equilibrium in the Continuous-time Limit

A continuous-time limit approach confers several advantages over approaching (1.32) and (1.33) directly in the discrete-time setup. It is challenging to express $\mathbb{E}^*_t[v_{t+\Delta t}]$ in (1.13) as a function of date t state variables. This issue is sidestepped by taking the limit $\Delta t \to 0$, in which case $\mathbb{E}^*_t[v_{t+\Delta t}] \to v_t$.

In addition, these integral equations simplify to partial differential equations in continuous time. Let μ_{pt} and σ_{pt} denote the instantaneous drift and volatility of price p_t in the continuous-time limit:

$$\mu_{pt} = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t[p_{t+\Delta t} - p_t]}{\Delta t}, \qquad \sigma_{pt}^2 = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t[(p_{t+\Delta t} - p_t)^2]}{\Delta t}. \tag{1.34}$$

Applying the Taylor series expansion to $p(t, \theta_t, m_t, \chi_t, \tau_t)$ and substituting in the state variable dynamics from Lemma 1, drift μ_{pt} and volatility σ_{pt} are given by:

$$\begin{aligned}
\mu_{pt} =& \frac{\partial p}{\partial t} - \frac{\partial p}{\partial \theta} A^{-1} a_t m_t + \frac{\partial p}{\partial \chi} A^{-2} \sigma_z^{-2} a_t^2 + \frac{\partial p}{\partial \tau} a_t + \frac{1}{2} \frac{\partial^2 p}{\partial \theta^2} \sigma_z^2, \\
&+ \frac{1}{2} \frac{\partial^2 p}{\partial m^2} \left(A^{-1} \sigma_z^{-1} a_t h_t \right)^2 - \frac{\partial^2 p}{\partial \theta \partial m} A^{-1} a_t h_t \qquad (1.35) \\
\sigma_{pt} =& \frac{\partial p}{\partial m} A^{-1} \sigma_z^{-1} a_t h_t - \frac{\partial p}{\partial \theta} \sigma_z. \qquad (1.36)
\end{aligned}$$

μ_{vt}, σ_{vt}, μ_{ft} and σ_{ft}, the drifts and volatilities of variables v_t and f_t, are similarly defined by replacing p with v or f in (1.34) and (1.36).

Proposition 2 describes this system of partial differential equations. The terminal conditions are reported in the appendix.

Proposition 2. *Price, risk-neutral variance, and the normalizing variable in stochastic discount factor, as functions of time and state variables θ_t, m_t, χ_t and τ_t satisfy:*

$$\mu_{pt} + \sigma_{pt} \sigma_{ft} - (A\theta_t + \tau_t p_t) \sigma_{pt}^2 = 0, \tag{1.37}$$

$$\mu_{vt} + \sigma_{vt} \sigma_{ft} - (A\theta_t + \tau_t p_t) \sigma_{vt} \sigma_{pt} + \sigma_{pt}^2 = 0, \tag{1.38}$$

$$\mu_{ft} + \frac{1}{2} (\sigma_{ft})^2 - \frac{1}{2} (A\theta_t + \tau_t p_t)^2 \sigma_{pt}^2 + A \cdot C(a_t) = 0. \tag{1.39}$$

where attention is implicitly determined by:

$$C'(a_t) = \frac{1}{2} A^{-1} v_t. \tag{1.40}$$

This system of equations is solved numerically by the finite-difference method on a five-dimensional grid of time t and four state variables θ_t, m_t, χ_t, and τ_t. A noteworthy by-product of this analysis is the drift in price dynamics μ_{pt}, which also represents the instantaneous expected return $\mathbb{E}_t[dp_t]/dt$. The solution to price, risk-neutral variance, attention, and expected return facilitates the analysis of this equilibrium.

1.4. Asset Pricing Implications

Different realizations of the exogenous noisy supply of the asset leads to different endogenous dynamics of price, uncertainty and information acquisition. The price is decreasing in the noisy supply of the asset. When the payoff distribution is not normal, uncertainty measured by the risk-neutral variance also varies with the noisy supply. If uncertainty is high (compared to other realizations of exogenous shocks), investors acquire more information.

In this section, I investigate how uncertainty, expected return, and past prices are endogenously related through investor attention. Section 4.1 studies the interplay between information acquisition and uncertainty. In Section 4.2, I explore the link between information acquisition and the future expected return. Section 4.3 relates cyclicality of attention to skewness in the payoff distribution. Findings and predictions in this section are also valid if multiple assets are traded.

1.4.1 Endogenous Uncertainty Dynamics

Suppose that the economy enters a period of high uncertainty about the asset payoff. High uncertainty induces investors to acquire more information. This causes asset returns to be even more volatile in the early stages of the high-uncertainty period, as agents learn more information. Returns become less volatile in the later stages, as learning gradually reduces payoff uncertainty.

Proposition 1 establishes the link between uncertainty and information acquisition. The risk-neutral variance of the final payoff v_t measures uncertainty and represents the marginal value of an additional piece of information to investors. With a high level of uncertainty, information becomes valuable, and investors accumulate it at a quicker pace through a higher level of attention a_t. The instantaneous volatility of the asset return σ_{pt} also rises as a result.

$$\frac{\partial \sigma_{pt}}{\partial a_t} = \frac{\partial p}{\partial m} A^{-1} \sigma_z^{-1} > 0 \tag{1.41}$$

When investors acquire more precise private signals, they also make the asset demand and thus the asset price more informative with regard to the final payoff, which increases the amount of public information available to investors. The dynamics of private and public information precisions in the continuous-time limit are as follows:

$$\tau_t' = a_t, \qquad \chi_t' = A^{-2}\sigma_z^2 a_t^2. \tag{1.42}$$

Private information accumulates at the speed of attention, and public information accumulates at a speed proportional to the square of attention. Information reduces investors' perception of uncertainty in the asset. A higher level of attention indicates that this reduction of uncertainty happens more quickly. Let μ_{vt}^* denote the instantaneous drift of v_t in the risk-neutral measure:

$$\mu_{vt}^* = \underbrace{\lim_{\Delta t \to 0} \frac{\mathbb{E}_t^*[v_{t+\Delta t} - v_t]}{\Delta t}}_{\text{Expected reduction of uncertainty}}. \tag{1.43}$$

Proposition 3 reports the expression of this expected change in risk-neutral variance.

Proposition 3. *The risk-neutral drift of risk-neutral variance v_t is given by:*

$$\mu_{vt}^* = -\left(\frac{\partial p}{\partial \theta}\right)^2 \sigma_z^2 + 2\frac{\partial p}{\partial \theta}\frac{\partial p}{\partial m} A^{-1} a_t h_t - \left(\frac{\partial p}{\partial m}\right)^2 A^{-2}\sigma_z^2 a_t^2 h_t^2, \tag{1.44}$$

It is decreasing in investor attention:

$$\frac{\partial \mu_{vt}^*}{\partial a_t} < 0. \tag{1.45}$$

The last term in equation (1.44) represents the contribution of public information to the reduction of uncertainty. It is decreasing in attention and contains $A^{-2}\sigma_z^2 a_t^2$, the speed at which public information disseminates. This effect is prominent at high levels of uncertainty because it is proportional to attention squared. The arrival of public information updates investors' belief and makes the distribution of the final payoff more concentrated around its mean, which contributes to the decline of the risk-neutral variance v_t.

The second term in (1.44) is also a decreasing function of attention. A higher expected payoff m_t shifts the risk-neutral distribution of the final payoff to the right. As a result, the current price p_t is increasing in m_t. Similarly, the price is decreasing in the common component of asset demand θ_t:

$$\frac{\partial p}{\partial m} > 0, \qquad \frac{\partial p}{\partial \theta} < 0. \tag{1.46}$$

The drift of uncertainty in the objective physical measure μ_{vt} differs from μ_{vt}^* by the variance risk premium. Depending on the shape and skewness of the risk-neutral

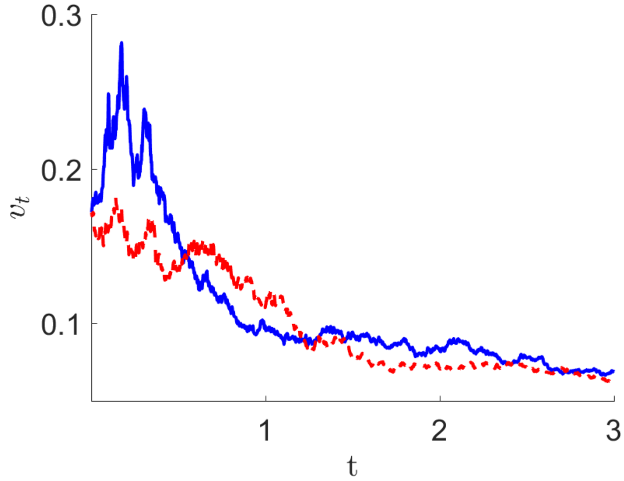


Figure 1.1: Sample Paths of the Risk-Neutral Variance

The figure demonstrates 2 sample paths of the risk-neutral variance v_t simulated from the model. The parameters are set as follows: $T = 3$, $y \sim \text{Lognormal}(0, \sigma_y^2)$ where $\sigma_y = 0.702$, $z_0 = 5$, $\sigma_z = 0.15$, $A = 0.2$, and $C(a_{it}) = 5(a_{it})^2$.

distribution, it either reinforces or diminishes the positive correlation between attention and the reduction in uncertainty. However, for a realistic choice of parameters this channel is unlikely to overtake the direct effect of information dissemination analyzed above.

When uncertainty is high, investor attention increases and reduction in uncertainty happens more quickly. An upward spike of the risk-neutral variance is usually followed by a rapid reduction. As uncertainty decreases, the return of the asset also becomes less volatile. This is consistent with the empirical observation that peaks in VIX are usually followed by a rapid decline.[5] Figure 1 illustrates 2 sample paths of the risk-neutral variance. The risk-neutral variance in the solid line initially shoots up but soon falls below the dashed line because of intensive information acquisition. The model is not stationary and the risk-neutral variance always have a tendency to decline.

The dynamics of uncertainty and risk-neutral variance also have a profound

[5]Harvey and Whaley (1992) demonstrate that changes in implied volatilities are negatively predicted by the lagged changes and that the explanatory power is higher in the sample that includes the crash. Mencia and Sentana (2013) suggest that during the 2008-2009 crisis volatility exhibits more mean reversion than that in the past.

impact on the dynamics of information acquisition. Each piece of information diminishes the value of the following piece. High attention levels make subsequent information acquisition less profitable and are therefore unlikely to be sustained.

1.4.2 Expected Return and Attention

The expected return of an asset is positively associated with investor attention. When attention is high today, investors acquire information at a rapid pace, leading to a rapid reduction in uncertainty. Then, the asset becomes less risky tomorrow and enjoys a lower risk premium tomorrow and beyond. This quick reduction in tomorrow's risk premium compared to today's corresponds to a high expected return for the asset.

The asset's expected payoff is m_t. The expectation of payoff in the risk-neutral measure is the price p_t. Their difference $m_t - p_t$ is this asset's risk premium. Investors' acquisition and dissemination of information make the asset less risky and shrink its risk premium towards zero.

Let us move back to the discrete-time setup for a moment. The risk premium at date $t + \Delta t$ is $m_{t+\Delta t} - p_{t+\Delta t}$. Because the expected payoff is a martingale $m_t = \mathbb{E}_t[m_{t+\Delta t}]$, the expected return from date t to $t+\Delta t$ is equal to the expected reduction of risk premium in the same period:

$$\mathbb{E}_t[p_{t+\Delta t} - p_t] = \mathbb{E}_t\Bigg[\underbrace{(m_t - p_t)}_{\text{date } t \text{ risk premium}} - \underbrace{(m_{t+\Delta t} - p_{t+\Delta t})}_{\text{date } t+\Delta t \text{ risk premium}}\Bigg]. \tag{1.47}$$

When the asset is trading at a price p_t below the expected payoff m_t, its risk premium is positive and shrinks towards zero from above. High attention implies a quick reduction in risk premium and hence a high expected return. Proposition 4 reports the expected return in the continuous-time limit and relates it to attention and risk premium.

Proposition 4. *The instantaneous expected return of the asset μ_{pt} is given by:*

$$\mu_{pt} = \Bigg[\underbrace{(m_t - p_t)}_{\text{risk premium}} A^{-1}\sigma_z^{-1}a_t + \sigma_z\,\mathbb{E}_t^*\left[\int_t^T \frac{\partial^2 p_u}{\partial\theta_u^2}A\sigma_z^2 du\right] + A^{-1}\sigma_z^{-1}h_t a_t\,\mathbb{E}_t^*\left[\int_t^T \frac{\chi_u}{\chi_t}\frac{\partial^2 p_u}{\partial m_u^2}a_u h_u\right]\Bigg]$$
$$\cdot \underbrace{\left(\frac{\partial p_t}{\partial m_t}A^{-1}\sigma_z^{-1}h_t a_t + (-\frac{\partial p_t}{\partial\theta_t}\sigma_z)\right)}_{>0}. \tag{1.48}$$

If the risk premium $m_t - p_t$ is greater than the second-order derivatives terms in the above expression, the expected return μ_{pt} is increasing in attention a_t.

The expected return is positively associated with investor attention as long as the risk premium is not exceptionally low, or the curvature of asset demand is not exceptionally negative. The second-order derivatives $\partial^2 p_u/\partial\theta_u^2$ and $\partial^2 p_u/\partial m_u^2$ appear because of the non-normality in the payoff distribution.

This prediction provides an alternative explanation to a list of well-documented empirical regularities. Da, Engelberg and Gao (2011) use Google Search Volume Index to measure investor attention directly. They find that an increase in search frequency predicts higher stock prices in the following two weeks. Lou (2014) documents that advertising can attract investor attention and impact stock returns in the short run: an increase in advertising spending is accompanied by a contemporaneous rise in retail buying and higher abnormal stock returns. Lee and So (2017) show that analyst coverage predicts stock return: firms with abnormally high analyst coverage subsequently outperform firms with abnormally low coverage by approximately 80 basis points per month.

The above analysis suggests that investor attention predicts the asset return. Because attention is determined by uncertainty, the model also implies that the risk-neutral variance v_t positively forecasts the expected return at the same date. v_t is also useful at predicting $\mu_{p,t+u}$, the expected return at a future date $t+u$, which changes with attention and uncertainty at that date. Since information is acquired and diffused only gradually through the investing public, date $t+u$ risk-neutral variance v_{t+u} is positively correlated to the date t measure. This translates into a correlation between v_t and $\mu_{p,t+u}$, illustrated in the right panel of Figure 2. This prediction is also supported by empirical evidence. Martin (2017) find that the risk-neutral variance predicts the return of the market at horizons from one month to one year.

However, this correlation becomes weaker as the time interval u increases, because the autocorrelation of the risk-neutral variance decreases over time. High uncertainty implies increased attention and information acquisition, which in turn drives down uncertainty. Consequently, the correlation between v_t and v_{t+u} decreases as the time interval u increases from 1 month to 24 months, as is shown in the left panel of Figure 2.

1.4.3 Cyclicality of Information Acquisition

The cyclicality of investor attention depends on the skewness of the payoff distribution. When the distribution is positively skewed and bounded from below, investors are more excited about potential upside gains. Therefore they acquire more information when the price is high. Conversely, when the distribution is negatively skewed, investors are more worried about potential losses in the downside. They devote more attention to the asset when the price is low.

First, let us consider an individual stock where the payoff distribution $G(y)$ has a lower bound and a fat right tail. At date t, investors observe the entire price history up to this date and use this information to update their beliefs about the risk-neutral distribution of payoff. Figure 3 demonstrates how the risk-neutral distribution depends on price in this example.

A low price implies that this probability distribution is concentrated near the lower bound. Conversely, a high price indicates that the payoff is more likely to be in a region with high variation and, thus, more uncertain from the investors' perspective. This establishes a positive correlation between price and the risk-neutral variance. Therefore, attention is procyclical, suggesting that more information is acquired and disseminated in good times than in bad times.

Now, consider a second example where the asset is a bond that pays 0 if it defaults and 1 if not. The risk-neutral distribution of payoff is completely characterized by the bond price p_t, which represents the risk-neutral probability that it does not default. The risk-neutral variance is:

$$v_t = \mathbb{E}_t^*[(y - p_t)^2] = p_t(1 - p_t). \tag{1.49}$$

If the risk-neutral probability of default does not exceed $1/2$, p_t is greater than $1/2$ and in this region, the risk-neutral variance is decreasing in price. When the bond is trading at a price near its face value 1, investors do not acquire much information because the value of an additional signal is close to zero. Any negative news that leads to a decline in price increases the value of information and attracts more attention. Thus information acquisition is countercyclical when $p_t \in (1/2, 1)$.

For general payoff distributions, the cyclicality of information acquisition is determined by $\partial v/\partial m$, the sensitivity of the risk-neutral variance to the expected payoff. Attention is determined by uncertainty and the price is increasing in the expected payoff m_t. Therefore, attention is procyclical if $\partial v/\partial m$ is positive and countercyclical if it is negative. Proposition 5 discusses how uncertainty moves in line with the price at date $T-\Delta t$, the last trading date before the asset realizes its payoff.

Proposition 5. *At date $T-\Delta t$, uncertainty is increasing in price if the risk-neutral distribution is positively skewed. The sensitivity of the risk-neutral variance v_t to the expected payoff m_t is given by:*

$$\frac{\partial v}{\partial m} = \frac{1}{h_{T-\Delta t}} \mathbb{E}^*_{T-\Delta t}\left[(y - p_{T-\Delta t})^3\right]. \tag{1.50}$$

$\partial v/\partial m$ has the same sign as the risk-neutral skewness of the payoff. If the distribution is right-skewed, uncertainty is increasing in the expected payoff and so is the price. The converse is true if the distribution is left-skewed. Unfortunately, $\partial v/\partial m$ does not admit a simple expression at dates other than $T-\Delta t$. It is not only influenced by the risk-neutral skewness but also relies on the interaction between distribution and other state variables.

The model predicts that information acquisition is procyclical for firm-specific information and countercyclical for market-wide information, consistent with empirical evidence. Idiosyncratic components of stock returns and payoffs tend to be right-skewed[6], implying that stock-specific information is more valuable in good times. Hong, Lim and Stein (2000) find that negative firm-specific information diffuses more slowly compared to positive news. The market return, on the contrary, is left-skewed. Garcia (2013) finds that in times of hardship investors react strongly to business cycle news, while in good times the predictability of media content on Dow Jones Industrial Average is much weaker.

[6]Kothari and Warner(1997) document that abnormal returns estimated using four models (market model, market-adjusted model, capital asset pricing model, and Fama French three-factor model) are all positively skewed.

1.5. Extension

I consider a simple multi-asset extension of the model that explains momentum in the stock market. In this economy, a risk-free asset, several individual stocks, and the market are traded. Each individual stock's final payoff is the sum of a stock-specific idiosyncratic component and a market component identical for all stocks. The focus of analysis in this section is information acquisition with respect to this idiosyncratic component of the payoff. It determines the return of a stock in excess of the market and explains why it is positively autocorrelated.

1.5.1 Setup

Investors trade 1 risk-free asset, n individual stocks, and 1 asset representing the market. Stock $j \in \{1, 2, \ldots, n\}$ has final payoff $y_0 + y_j$, which consists of a market component y_0 and a stock-specific idiosyncratic component y_j that is independent of the market and across stocks. The market payoff itself is also traded and there is a market asset that pays y_0 at the final date.

Payoffs y_0 and y_j are unobservable to investors at the start and they are distributed with cumulative distribution functions $G_0(y_0)$ and $G_i(y_j)$. The distribution of stock-specific component y_j, $G_j(y_j)$, is bounded from below and positively skewed. A number of factors contribute to this asymmetry in the stock payoff. Limited liability for equity holders indicates that the investments in stocks have bounded downside risk but some potential for a large upside gain. Besides, the firm may possess a real option to expand the business when it is doing well which further boosts the upside potential.

Let p_{0t} denote the date t price of the market asset and p_{jt} denote the price of a claim that pays y_j at the final date. The price of individual stock j which pays $y_0 + y_j$ is $p_{0t} + p_{jt}$. p_{jt} represents the price of the idiosyncratic payoff component, and it is equal to the difference in price between stock j and the market asset. For the period between date t and $t + \Delta t$, the return of the market asset is $p_{0,t+\Delta t} - p_{0t}$ and the return of stock j in excess of the market is:

$$[(p_{0,t+\Delta t} + p_{j,t+\Delta t}) - (p_{0t} + p_{jt})] - [p_{0,t+\Delta t} - p_{0t}] = p_{j,t+\Delta t} - p_{jt}. \tag{1.51}$$

Let z_{jt} represent the supply of stock j and z_{0t} represent the aggregate supply of all risky assets, including the market asset and all n individual stocks. Since the payoff of any risky asset contains a market payoff component, z_{0t} also represents the supply of this component y_0 in the economy. Similar to Kacperczyk, Van Nieuwerburgh and Veldkamp (2016), increments in the supplies of payoff components $z_{0,t+\Delta t} - z_{0t}$ and $z_{j,t+\Delta t} - z_{jt}$ are assumed to be independent. As in the baseline model, they both follow random walks and have variances $\sigma_{z0}^2 \Delta t$ and $\sigma_{zj}^2 \Delta t$ respectively:

$$\begin{aligned} z_{0,t+\Delta t} - z_{0t} &\overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_{z0}^2 \Delta t), \\ z_{j,t+\Delta t} - z_{jt} &\overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_{zj}^2 \Delta t), \qquad j = 1, 2, \ldots, n. \end{aligned} \tag{1.52}$$

The cost of information acquisition is additive across different payoffs. Let a_{it}^j represent investor i's attention towards payoff y_j. The aggregate cost of information for investor j from date t to date $t + \Delta t$ is the sum of that for each payoff $\sum_{j=0}^{n} C(a_{it}^j)\Delta t$. Contrary to standard rational inattention models (e.g., Sims (2003)), which impose a fixed capacity upper bound for the aggregate attention on all assets, I assume that information acquisition is independent across payoffs and different assets do not compete for investor attention. Increased attention on one payoff does not raise the cost of information for other payoffs. Investors acquire less information in aggregate when the market is devoid of profitable investment opportunities.

The above assumptions about asset supply and information acquisition simplify the analysis of equilibrium. The portfolio choice for different payoffs can be solved separately because payoffs $y_0, y_1, \ldots, y_n$ are independent and investors have CARA preference. Since the cost of information is additive across assets, attention choice can also be solved separately. The characterization of equilibrium is therefore identical to the single asset baseline model, with θ_{jt}, m_{jt}, χ_{jt} and τ_{jt} replacing θ_t, m_t, χ_t and τ_t as state variables.

1.5.2 Analysis of Equilibrium

I consider a numerical example with parameters provided in Table 1.[7] The final payoff is realized in $T = 3$ years. For simplicity, I assume the idiosyncratic payoff components for different stocks have the same distribution $y_j \sim \text{Lognormal}(0, \sigma_y^2)$,

[7]The analysis here focuses on idiosyncratic payoffs of individual stocks and does not extend to the market. In section 5.2, j to refers one of $1, 2, \ldots, n$ and does not include 0.

Table 1.1: Parameter Values

The table lists values assigned to the length of investment T; the distribution of idiosyncratic payoff $G_j(y_j)$ and parameter σ_y; the starting supply of individual stock z_{j0}; the volatility of stock supply σ_{zj}; the absolute risk aversion parameter A; and the cost of information $C(a_{it}^j)$.

T	$G_j(y_j)$	σ_y	z_{j0}	σ_{zj}	A	$C(a_{it}^j)$
3	Lognormal$(0, \sigma_y^2)$	0.702	5	0.15	0.2	$5(a_{it}^j)^2$

where σ_y is set at 0.702 to match the skewness of 36-month buy-and-hold CAPM abnormal return at a level of 2.90. [8]

The starting supply of individual stock is $z_{j0} = 5$, and the volatility of stock supply is set at $\sigma_{zj} = 0.15$, such that daily trading volume is approximately 0.2 percent of the outstanding shares. The absolute risk aversion is $A = 0.2$, which corresponds to relative risk aversion around 10 at the initial date for an investor with wealth equal to 10 times its investment in one stock. Last, the cost of information is assumed to be a quadratic function of attention $C(a_{it}^j) = 5(a_{it}^j)^2$.

Attention and Past Excess Return

For the period between date 0 and date t, the return of stock j in excess of the market is $p_{jt} - p_{j0}$. In equilibrium this return is determined by state variables θ_{jt}, m_{jt}, χ_{jt} and τ_{jt}. In particular, it is increasing in the expected payoff m_{jt}, which summarizes public information about this stock's idiosyncratic payoff. Attention to this stock-specific payoff tends to be procyclical because its distribution is bounded from below and skewed to the right.

In this numerical example, the risk-neutral variance of idiosyncratic payoff v_{jt} is indeed increasing in its expected value. Figure 4 illustrates the relationship between v_{jt} and m_{jt} at date $t = 1$, one year after the starting date. The other 3 state variables θ_{jt}, χ_{jt} and τ_{jt} are fixed at the median of their distributions.

Excess return and risk-neutral variance are both increasing in the expected payoff. This contributes to a positive correlation between a stock's past performance

[8] Kothari and Warner (1997) find that 36-month buy-and-hold abnormal return with respect to CAPM has a skewness of 2.90. Other moments including kurtosis and quartiles are also similar to those implied by a log-normal distribution.

in excess of the market and investors' current attention to its idiosyncratic payoff. High excess return in the past indicates that this payoff is likely to be in a range with high variation, and thus it is more valuable for investors to focus on this piece of information. Consider a group of stocks with similar payoff distributions. This model predicts that stocks that performed well relative to the cross-sectional average are likely to attract more attention. The left panel of Figure 5 shows a scatter plot for excess return in the first year $p_{j1} - p_{j0}$ and attention at the end of the first year for $20,000$ simulated time-series.

Expected Excess Return and Attention

In the baseline model, Section 4.2 establishes that the expected return is increasing in attention if the asset enjoys a positive risk premium. Similar results hold true for the multi-asset case. High attention to the stock's idiosyncratic payoff implies a quick reduction in the risk-premium concerning this payoff, which in turn contributes to high excess return for the stock. The right panel of Figure 5 demonstrates the end of the first year attention a_t and instantaneous expected excess return μ_{pt}^j for simulated data.

Serial Correlation of Excess Return

Investor attention to a stock is positively correlated with its past excess return. The expected excess return of this stock in the future increases with attention. Combining these two results, I find that excess return of a stock exhibits positive autocorrelation because of endogenous information acquisition.

The average return of a sufficiently large group of individual stocks approximates that of the market. Therefore, a stock's excess return is high if and only if it performed well relative to others. Past winners in this group tend to attract more attention and thus continue to generate higher excess returns. This time-series result explains the cross-sectional momentum.

This model also predicts that this momentum effect weakens over time. The autocorrelation of excess return decreases with the horizon. Past excess return $p_{jt} - p_{j0}$ affects its future expectation through the risk-neutral variance v_{jt}. As time interval u increases, the connection between v_{jt} and $v_{j,t+u}$ weakens and the serial correlation of excess return decreases. Let β_{t+u}^j denote the regression coefficient of instantaneous expected return μ_{pt}^j on $p_{jt} - p_{j0}$. Figure 6 reports the estimated β_{t+u}^j

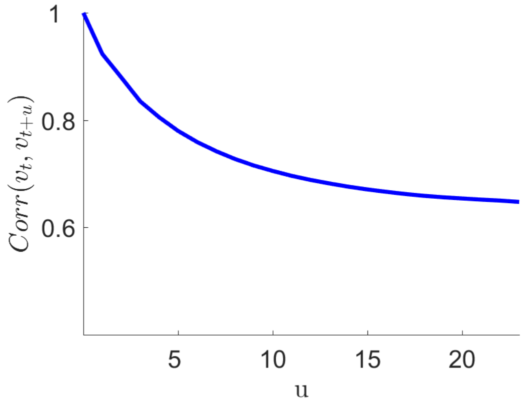
for u from 1 to 24 month using simulated time-series. $\widehat{\beta^j_{t+u}}$ exhibits a decreasing pattern.

1.6. Concluding Remarks

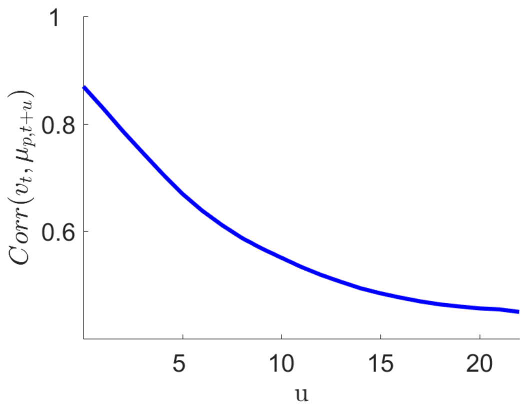
In this paper, I developed a noisy rational expectations model with endogenous information acquisition and used it to analyze the joint dynamics of attention, price, and uncertainty. The starting point of this analysis is equation (1.13), which shows that investor attention is determined by the uncertainty measure risk-neutral variance. It is empirically measurable from option prices and resembles the volatility index VIX. Conversely, attention determines investment choices and thus affects the dynamics of asset prices and of uncertainty. This interaction between attention and uncertainty creates rich asset pricing dynamics.

This model generates predictions that are qualitatively different from those in static and normal distribution models. First, high uncertainty attracts more attention, which in turn reduces both uncertainty and attention. Episodes of high uncertainty and attention are therefore unlikely to be sustained. Second, information acquisition drives down both uncertainty and risk premium. The expected return, which is identical to the expected reduction of risk premium, increases with investor attention. Third, the correlation between price and the risk-neutral variance depends on the skewness and support of the payoff distribution. Information acquisition tends to be procyclical for right-skewed payoffs and countercyclical for left-skewed ones. These predictions are consistent with empirical observations.

In the extension, I applied the above results to a multi-asset setup and illustrated that past winners tend to continue to perform well relative to the market. The idiosyncratic component of stock payoff is right-skewed because of limited liability and real option to expand. Stocks that performed well relative to the market have high uncertainty and attract more attention and, hence, are expected to continue to generate high excess returns. Because the dynamics of uncertainty contain a mean-reverting component, the serial correlation of excess return weakens as the horizon increases.



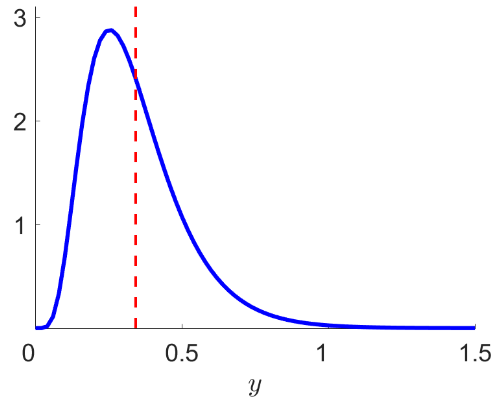
(a) Autocorrelation of v_t



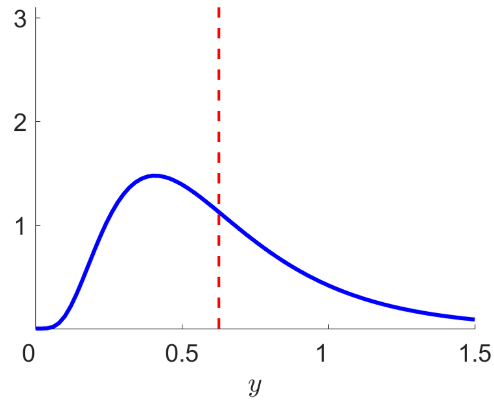
(b) v_t predicts $\mu_{p,t+u}$

Figure 1.2: Uncertainty and Expected Return of the Asset

The figure shows the autocorrelation between the risk-neutral variance in panel (a) and the correlation between present risk-neutral variance v_t and future expected return $\mu_{p,t+u}$ in panel (b). The current date is $t = 1$. Other parameters are set as follows: $T = 3$, $y \sim \text{Lognormal}(0, \sigma_y^2)$ where $\sigma_y = 0.702$, $z_0 = 5$, $\sigma_z = 0.15$, $A = 0.2$, and $C(a_{it}) = 5(a_{it})^2$.



(a) price p_t is low



(b) price p_t is high

Figure 1.3: Risk-Neutral Distributions of Stock Payoff

The figure demonstrates how asset price changes the shape of the payoff's risk-neutral distribution. The solid line represents the probability density, and the dashed line correspond to the asset price, which is identical to the mean of this distribution. The prior distribution of the payoff is $y \sim \text{Lognormal}(0, \sigma_y^2)$ where $\sigma_y = 0.702$.

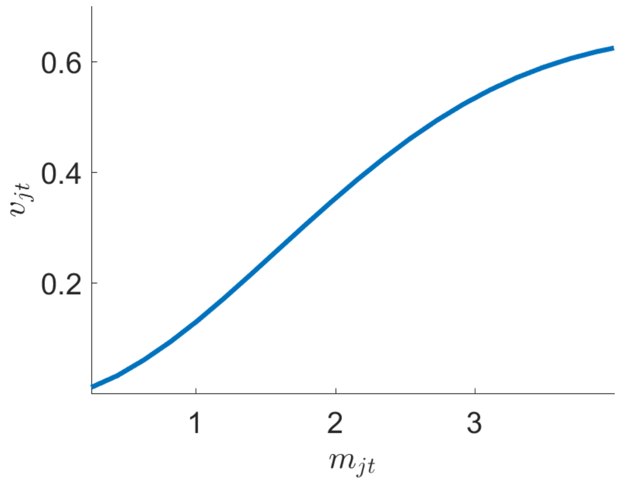
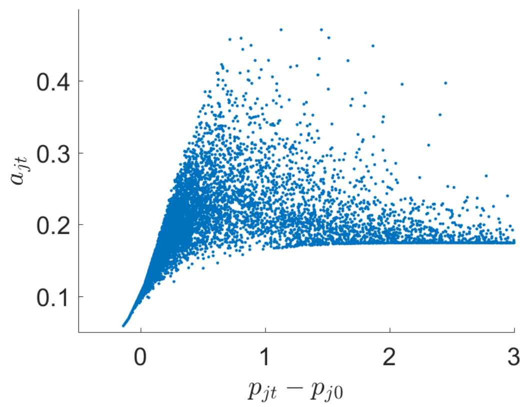
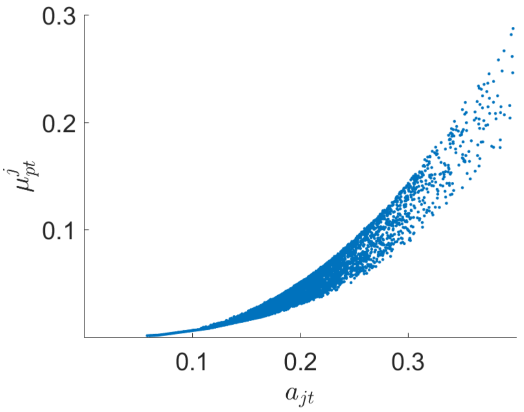


Figure 1.4: Risk-Neutral Variance and Expected Payoff

The figure plots the risk-neutral variance v_{jt} as a function of the expected payoff m_{jt} for the idiosyncratic payoff component of a stock. Other state variables are fixed at the middle of their distribution. The parameters are set in Table 1.



(a) attention and Past Return



(b) Expected Return and attention

Figure 1.5: attention and Excess Return

The figure plots past excess return $p_{jt} - p_{j0}$, attention a_{jt} and the instantaneous expected excess return μ^j_{pt} at year $t = 1$ for 20,000 simulated time-series. Panel (a) shows that attention is increasing in past excess return, and Panel (b) illustrates that expected excess return is increasing in attention. The parameters are set in Table 1.

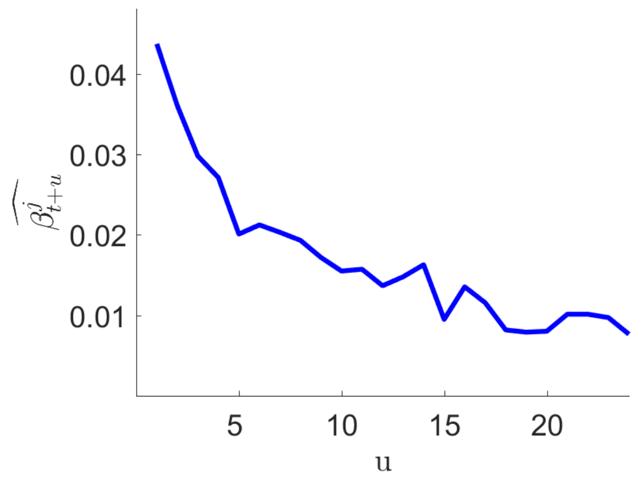


Figure 1.6: Serial Correlation of Excess Return

The figure shows the estimated value of the regression coefficient β^j_{t+u} for the equation $\mu^j_{pt} = \alpha^j_{t+u} + \beta^j_{t+u}(p_{jt} - p_{j0}) + e^j_{t+u}$ from simulated data. The parameters are set in Table 1.

Bibliography

Rui Albuquerque. Skewness in stock returns: reconciling the evidence on firm versus aggregate returns. *The Review of Financial Studies*, 25(5):1630–1673, 2012.

Rui Albuquerque and Jianjun Miao. Advance information and asset prices. *Journal of Economic Theory*, 149:236–275, 2014.

Daniel Andrei and Julien Cujean. Information percolation, momentum and reversal. *Journal of Financial Economics*, 123(3):617–645, 2017.

Daniel Andrei and Michael Hasler. Investor attention and stock market volatility. *The Review of Financial Studies*, 28(1):33–72, 2014.

Andrew Ang, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang. The cross-section of volatility and expected returns. *The Journal of Finance*, 61(1):259–299, 2006.

Efstathios Avdis. Information tradeoffs in dynamic financial markets. *Journal of Financial Economics*, 122(3):568–584, 2016.

Gurdip Bakshi, Nikunj Kapadia, and Dilip Madan. Stock return characteristics, skew laws, and the differential pricing of individual equity options. *The Review of Financial Studies*, 16(1):101–143, 2003.

Turan G. Bali and Nusret Cakici. Idiosyncratic volatility and the cross section of expected returns. *Journal of Financial and Quantitative Analysis*, 43(1):29–58, 2008.

Snehal Banerjee and Bradyn Breon-Drish. Dynamic information acquisition and strategic trading. *Working Paper*, 2017.

Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. X-capm: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24, 2015.

Bradyn Breon-Drish. On existence and uniqueness of equilibrium in a class of noisy rational expectations models. *The Review of Economic Studies*, 82(3):868–921, 2015.

Georgy Chabakauri, Kathy Yuan, and Konstantinos Zachariadis. Multi-asset noisy rational expectations equilibrium with contingent claims. *Working Paper*, 2016.

Zhi Da, Joseph Engelberg, and Pengjie Gao. In search of attention. *The Journal of Finance*, 66(5):1461–1499, 2011.

Itamar Drechsler. Uncertainty, time-varying fear, and asset prices. *The Journal of Finance*, 68(5):1843–1889, 2013.

Diego Garcia. Sentiment during recessions. *The Journal of Finance*, 68(3):1267–1300, 2013.

Sanford J. Grossman and Joseph E. Stiglitz. On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408, 1980.

Campbell R Harvey and Robert E Whaley. Market volatility prediction and the efficiency of the s&p 100 index option market. *Journal of Financial Economics*, 31(1):43–73, 1992.

Martin F Hellwig. On the aggregation of information in competitive markets. *Journal of Economic Theory*, 22(3):477–498, 1980.

Craig W. Holden and Avanidhar Subrahmanyam. News events, information acquisition, and serial correlation. *The Journal of Business*, 75(1):1–32, 2002.

Harrison Hong and Jeremy C Stein. A unified theory of underreaction, momentum trading, and overreaction in asset markets. *The Journal of Finance*, 54(6):2143–2184, 1999.

Harrison Hong, Terence Lim, and Jeremy C Stein. Bad news travels slowly: Size, analyst coverage, and the profitability of momentum strategies. *The Journal of Finance*, 55(1):265–295, 2000.

Lixin Huang and Hong Liu. Rational inattention and portfolio selection. *The Journal of Finance*, 62(4):1999–2040, 2007.

Narasimhan Jegadeesh and Sheridan Titman. Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of finance*, 48(1):65–91, 1993.

Marcin Kacperczyk, Stijn Van Nieuwerburgh, and Laura Veldkamp. A rational theory of mutual funds' attention allocation. *Econometrica*, 84(2):571–626, 2016.

S.P. Kothari and Jerold B. Warner. Measuring long-horizon security price performance. *Journal of Financial Economics*, 43(3):301–339, 1997.

Charles M.C. Lee and Eric C. So. Uncovering expected returns: Information in analyst coverage proxies. *Journal of Financial Economics*, 124(2):331–348, 2017.

Dong Lou. Attracting investor attention through advertising. *The Review of Financial Studies*, 27(6):1797–1829, 2014.

Ian Martin. What is the expected return on the market? *The Quarterly Journal of Economics*, 132(1):367–433, 2017.

Ian Martin and Christian Wagner. What is the expected return on a stock? *The Journal of Finance*, 2018.

Javier Mencia and Enrique Sentana. Valuation of vix derivatives. *Journal of Financial Economics*, 108(2):367–391, 2013.

Haim Mendelson and Tunay I. Tunca. Strategic trading, liquidity, and information acquisition. *The Review of Financial Studies*, 17(2):295–337, 2004.

Tobias J. Moskowitz, Yao Hua Ooi, and Lasse Heje Pedersen. Time series momentum. *Journal of Financial Economics*, 104(2):228–250, 2012.

Christopher A Sims. Implications of rational inattention. *Journal of Monetary Economics*, 50(3):665–690, 2003.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information acquisition and under-diversification. *The Review of Economic Studies*, 77(2):779–805, 2010.

Dimitri Vayanos and Paul Woolley. An institutional theory of momentum and reversal. *The Review of Financial Studies*, 26(5):1087–1145, 2013.

Laura Veldkamp. Media frenzies in markets for financial information. *American Economic Review*, 96(3):577–601, 2006.

Robert E Verrecchia. Information acquisition in a noisy rational expectations economy. *Econometrica*, pages 1415–1430, 1982.

Jiang Wang. A model of intertemporal asset prices under asymmetric information. *The Review of Economic Studies*, 60(2):249–282, 1993.

X. Frank Zhang. Information uncertainty and stock returns. *The Journal of Finance*, 61(1):105–137, 2006.

Appendix

Proof of Lemma 1.

Optimization (1.8) is equivalent to:

$$\max_{\theta_{i1}} \int -\exp\left(-A\left(\theta_{i1} - A^{-1}a_{i0}s_{i1}\right)(y-p_1) - \frac{a_{i0}}{2}(y^2-p_1^2) + A\cdot C(a_{i0})\right)$$
$$\cdot \frac{1}{\sqrt{2\pi(a_{i0})^{-1}}}\exp\left(-\frac{a_{i0}}{2}(s_{i1}-p_1)^2\right)\ dG(y|p_0,p_1). \tag{A1}$$

$A\cdot C(a_{i0})$ and $\exp\left(-a_{i0}/2\cdot(s_{i1}-p_1)^2\right)$ only contain known information at date 1 and could be taken out of the expression. Choosing optimal θ_{i1} is equivalent to choosing optimal $\theta_{i1} - A^{-1}a_{i0}s_{i1}$:

$$\max_{\theta_{i1}-A^{-1}a_{i0}s_{i1}} \int -\exp\left(-A\left(\theta_{i1} - A^{-1}a_{i0}s_{i1}\right)(y-p_1) - \frac{a_{i0}}{2}(y^2-p_1^2)\right)\ dG(y|p_0,p_1). \tag{A2}$$

This optimization problem is identical for different investors and concave in $\theta_{i1} - A^{-1}a_{i0}s_{i1}$. Take the first-order condition and simplify:

$$\int -\exp\left(-A\left(\theta_{i1} - A^{-1}a_{i0}s_{i1}\right)(y-p_1) - \frac{a_{i0}}{2}(y^2-p_1^2)\right)\cdot(y-p_1)\ dG(y|p_0,p_1) = 0. \tag{A3}$$

Because the moment generating function always exists, $G(y)$ has exponentially bounded tails. For price p_1 that belongs to the support of $G(y)$, the solution to (A3) exists and is unique. As a result, $\theta_{i1} - A^{-1}a_{i0}s_{i1}$ must be the same across investors. Define

$$\theta_1 = \theta_{i1} - A^{-1}a_{i0}s_{i1}. \tag{A4}$$

θ_1 represents a component of asset demand that is identical across investors. θ_1 is a function of prices p_0 and p_1 because (A2) relies on $dG(y|p_0,p_1)$. Identity (A4) is equivalent to (1.9).

Proof of Proposition 1.

I prove this proposition by backward induction. Suppose asset demand equation (1.14) holds for dates $t = t_0+\Delta t, \ldots, T-\Delta t$ and attention equation (1.13) holds for dates $t = t_0, \ldots, T-\Delta t$.

First I prove that (1.14) is valid for date $t = t_0$.

Let p^{t_0} denote future prices $p_{t_0+\Delta t}$, $\ldots$, $p_{T-\Delta t}$, and s^{t_0} denote investor i's future signals $s_{i,t_0+\Delta t}$, $\ldots$, $s_{i,T-\Delta t}$. The joint distribution of future prices and signals conditional on all public and private information possessed by investor i at date t_0 is represented by $G(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}, s_{i,\Delta t}, \ldots, s_{i,t_0})$. I use $\pi(s_{i,\Delta t}, \ldots, s_{i,t_0})$ to denote the probability density function of the signals.

At date t_0, investor i's utility is:

$$
\begin{aligned}
&\mathbb{E}^i_{t_0}[-\exp(-AW_{iT})] \\
&= \int -\exp(-AW_{iT})\, dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}, s_{i,\Delta t}, \ldots, s_{i,t_0}) \\
&= \int -\exp(-AW_{iT}) \frac{\pi(s_{i,\Delta t}, \ldots, s_{i,t_0}|y)}{\pi(s_{i,\Delta t}, \ldots, s_{i,t_0}|p_0, \ldots, p_{t_0})}\, dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0}) \\
&= \int -\exp\left(-AW_{i,t_0} - A\sum_{t=t_0}^{T-\Delta t} \theta_{it}(p_{i,t+\Delta t} - p_{it}) + A\sum_{t=t_0}^{T-\Delta t} C(a_{it})\Delta t\right) \\
&\quad \cdot \prod_{t=0}^{t_0-\Delta t} \frac{1}{\sqrt{2\pi(a_{it}\Delta t)^{-1}}} \exp(-\frac{a_{it}\Delta t}{2}(s_{i,t+\Delta t} - y)^2) \frac{dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0})}{\pi(s_{i,\Delta t}, \ldots, s_{i,t_0}|p_0, \ldots, p_{t_0})}.
\end{aligned}
\tag{A5}
$$

In the second equality, I applied the conditional Bayes theorem on the probability density function and used the fact that signals $s_{i,\Delta t}$,...,s_{i,t_0} as a group is conditionally independent with prices p_0, ..., p_{t_0}. In the third inequality, I further used the fact that the signals themselves are conditionally independent.

Substitute in the expressions of θ_{it} for $t = t_0 + \Delta t, \ldots, T - \Delta t$ and simplify,

$$
\begin{aligned}
&\mathbb{E}^i_{t_0}[-\exp(-AW_{iT})] \\
&= \int -\exp\Bigg[-AW_{i,t_0} - \left(A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu}s_{i,u+\Delta t}\Delta t\right)(p_{t_0+\Delta t} - p_{t_0}) + A\sum_{t=t_0}^{T-\Delta t} C(a_{it})\Delta t \\
&\quad - \sum_{t=t_0+\Delta t}^{T-\Delta t}\left(A\theta_{it} - \sum_{u=0}^{t_0-\Delta t} a_{iu}s_{i,u+\Delta t}\Delta t\right)(p_{t+\Delta t} - p_t) - \sum_{t=0}^{t_0-\Delta t} \frac{a_{it}\Delta t}{2}(y^2 - p_{t_0}^2)\Bigg] \\
&\quad \cdot \prod_{t=0}^{t_0-\Delta t} \frac{1}{\sqrt{2\pi(a_{it}\Delta t)^{-1}}} \exp(-\frac{a_{it}\Delta t}{2}(s_{i,t+\Delta t} - p_{t_0})^2) \frac{dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0})}{\pi(s_{i,\Delta t}, \ldots, s_{i,t_0}|p_0, \ldots, p_{t_0})} \\
&= \int -\exp\Bigg[-AW_{i,t_0} - \left(A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu}s_{i,u+\Delta t}\Delta t\right)(p_{t_0+\Delta t} - p_{t_0}) + A\sum_{t=t_0}^{T-\Delta t} C(a_{it})\Delta t \\
&\quad - \sum_{t=t_0+\Delta t}^{T-\Delta t}\left(A\theta_t + \sum_{u=t_0}^{t-\Delta t} a_{iu}s_{i,u+\Delta t}\Delta t\right)(p_{t+\Delta t} - p_t) - \sum_{t=0}^{t_0-\Delta t} \frac{a_{it}\Delta t}{2}(y^2 - p_{t_0}^2)\Bigg] \\
&\quad \cdot \prod_{t=0}^{t_0-\Delta t} \frac{1}{\sqrt{2\pi(a_{it}\Delta t)^{-1}}} \exp(-\frac{a_{it}\Delta t}{2}(s_{i,t+\Delta t} - p_{t_0})^2) \frac{dG(y, p^{t_0}, s^{t_0}|p_0, \ldots, p_{t_0})}{\pi(s_{i,\Delta t}, \ldots, s_{i,t_0}|p_0, \ldots, p_{t_0})}.
\end{aligned}
\tag{A6}
$$

Investors' utility maximization problems are similar despite the differences in private signals received. In the above integral, $s_{i,\Delta t}$,...,s_{i,t_0} only appears as coefficients of $p_{t_0+\Delta t} - p_{t_0}$. Taking the first order condition, $A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu} s_{i,u+\Delta t}\Delta t$ must be identical across investors. Define

$$\theta_{t_0} = \frac{1}{A}\left(A\theta_{i,t_0} - \sum_{u=0}^{t_0-\Delta t} a_{iu} s_{i,u+\Delta t}\Delta t\right). \tag{A7}$$

θ_{t_0} represents the common component of asset demand and only relies on the price history p_0,...,p_{t_0}. This completes the backward induction for the asset demand equation.

Next, I prove that (1.13) is correct for date $t = t_0 - \Delta t$. Attention at this date is decided without the knowledge of date t_0 information. Substitute in (A6) and differentiate date $t_0 - \Delta t$ expected utility $\mathbb{E}^i_{t_0-\Delta t}\left[\mathbb{E}^i_{t_0}[U(W_{iT})]\right]$ with respect to $a_{i,t_0-\Delta t}$:

$$\mathbb{E}\left[U'(W_{iT})\left(A \cdot C'(a_{i,u_{t_0-\Delta t}})\Delta t - \frac{\Delta t}{2}(y^2 - p^2_{t_0})\right)\middle| p_0, \dots, p_{t_0-\Delta t}\right] = 0. \tag{A8}$$

Apply the law of iterated expectations and use the fact that $p_{t_0} = E^*_{t_0}[y]$,

$$C'(a_{i,u_{t_0-\Delta t}}) = \frac{1}{A}\ \mathbb{E}^*_{t_0-\Delta t}\left[\mathbb{E}^*_{t_0}(y^2 - p^2_{t_0})\right] = \frac{1}{A}\ \mathbb{E}^*_{t_0-\Delta t}[v_t]. \tag{A9}$$

This completes the backward induction for the attention equation.

Proof of Lemma 2.

I arrive at (1.21), (1.23), and (1.24) by taking first difference of θ_t, χ_t and τ_t from equations (1.18)-(1.20).

Let $G_t(y)$ denote the posterior distribution of y after observing all public information up to date t. m_t and h_t respectively represent the mean and variance of this distribution:

$$\begin{aligned} m_t &= \mathbb{E}[y|p_0, p_1, \cdots, p_t] = \int y\, dG_t(y), \\ h_t &= \mathrm{Var}[y|p_0, p_1, \cdots, p_t] = \int (y - m_t)^2\, dG_t(y). \end{aligned} \tag{A10}$$

The signal $A(a_t\Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$ communicates payoff y perturbed by a normal noise with precision $A^{-2}\sigma_z^{-2}a_t^2\Delta t$. It represents new public information arrived at date $t+\Delta t$ and is independent of all previous public signals. Applying Bayes formula,

$$\begin{aligned}
m_{t+\Delta t} &= \mathbb{E}[y|p_0, p_1, \cdots, p_{t+\Delta t}] \\
&= \frac{\int \exp\left(-\frac{1}{2}A^{-2}\sigma_z^{-2}a_t^2\Delta t\,(y - A(a_t\Delta t)^{-1}(\theta_{t+\Delta t}-\theta_t))^2\right) y\, dG_t(y)}{\int \exp\left(-\frac{1}{2}A^{-2}\sigma_z^{-2}a_t^2\Delta t\,(y - A(a_t\Delta t)^{-1}(\theta_{t+\Delta t}-\theta_t))^2\right) dG_t(y)} \\
&= \left(1 - A^{-2}\sigma_z^{-2}a_t^2 h_t\Delta t\right) m_t + A^{-2}\sigma_z^{-2}a_t^2 h_t \Delta t \cdot A(a_t\Delta t)^{-1}(\theta_{t+\Delta t}-\theta_t) + o(\Delta t).
\end{aligned} \tag{A11}$$

Substituting in the expression of $A(a_t\Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$ from (1.16), I obtain (1.22).

Proof of Lemma 3.

First I prove that $\xi_{t,T}$ defined in (1.29) is a valid stochastic discount factor.

At date t, investor i makes portfolio choice θ_{it} to maximize her expected utility $\mathbb{E}_t[U(W_{iT})]$, where

$$W_{iT} = W_{it} + \sum_{u=t}^{T-\Delta t} \theta_{iu}(p_{u+\Delta t} - p_t) + A \sum_{u=t}^{T-\Delta t} C(a_{iu})\Delta t. \tag{A12}$$

The first-order condition with respect to θ_{iu} suggests that

$$\mathbb{E}_u^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = 0. \tag{A13}$$

Apply the law of iterated expectations:

$$\mathbb{E}_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = \mathbb{E}_t^i\left[\mathbb{E}_u^i[U'(W_{iT})(p_{u+\Delta t} - p_u)]\right] = 0, \tag{A14}$$

$$\mathbb{E}_t^i[U'(W_{iT})(y - p_t)] = \sum_{u=t}^{T-\Delta t} \mathbb{E}_t^i[U'(W_{iT})(p_{u+\Delta t} - p_u)] = 0, \tag{A15}$$

Substitute in the definition of $\xi_{t,T}^i$,

$$\mathbb{E}_t^i[\xi_{t,T}^i(y - p_t)] = 0. \tag{A16}$$

$\mathbb{E}_t^i$ represents the conditional expectation using investor i's private information set at time t. $\mathbb{E}_t$ represents the conditional expectation using the public information set (the price history). The private information set contains the public one. Apply law of iterated expectations:

$$\mathbb{E}_t[\xi_{t,T}^i(y - p_t)] = \mathbb{E}_t[\mathbb{E}_t^i[\xi_{t,T}^i(y - p_t)]] = 0. \tag{A17}$$

Now consider $\xi_{t,T} = \mathbb{E}\left[\xi^i_{t,T}|p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right]$.

$$
\begin{aligned}
\mathbb{E}_t[\xi_{t,T}(y-p_t)] &= \mathbb{E}_t\left[\mathbb{E}\left[\xi^i_{t,T}|p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right](y-p_t)\right] \\
&= \mathbb{E}_t\left[\mathbb{E}\left[\xi^i_{t,T}(y-p_t)|p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right]\right] \\
&= \mathbb{E}_t[\xi^i_{t,T}(y-p_t)] = 0. \qquad \text{(A18)}
\end{aligned}
$$

In the second equality, I used the fact that both p_t and y are measurable with respect to $p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y$. In the third equality I applied the law of iterated expectations. As a result, $\xi_{t,T}$ prices the risky asset correctly. Furthermore, $\mathbb{E}_t[\xi_{t,T}] = 1$. This completes the proof that $\xi_{t,T}$ is a valid SDF.

Next, I prove that $\xi_{t,T}$ has the expression given in (1.30). From the definition of $\xi^i_{t,T}$,

$$
\xi^i_{t,T} = \frac{U'(W_{iT})}{\mathbb{E}^i_t[U'(W_{iT})]} = \frac{-AU(W_{iT})}{\mathbb{E}^i_t[-AU(W_{iT})]} = \frac{U(W_{iT})}{\mathbb{E}^i_t[U(W_{iT})]}. \qquad \text{(A19)}
$$

$U(W_{iT})$ is given by:

$$
U(W_{iT}) = -\exp\left[-\sum_{u=t}^{T-\Delta t}\left(A\theta_t + \sum_{u_0=0}^{u-\Delta t} a_{i,u_0} s_{i,u_0+\Delta t}(p_{u+\Delta t} - p_u)\right) + A\sum_{u=t}^{T-\Delta t} C(a_{iu})\Delta t\right]. \qquad \text{(A20)}
$$

The derivation of $\mathbb{E}^i_t[U(W_{iT})]$ is similar to (A6):

$$
\begin{aligned}
&\mathbb{E}^i_t[U(W_{iT})] \\
&= \int -\exp\left[-\sum_{u=t}^{T-\Delta t}\left(A\theta_t + \sum_{u_0=t}^{u-\Delta t} a_{i,u_0} s_{i,u_0+\Delta t}(p_{u+\Delta t} - p_u)\right) + A\sum_{u=t}^{T-\Delta t} C(a_{iu})\Delta t - \frac{\tau_t}{2}(y^2 - p_t^2)\right] \\
&\quad \cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu}\Delta t)^{-1}}}\exp(-\frac{a_{iu}\Delta t}{2}(s_{i,u+\Delta t} - p_t)^2) \cdot \frac{dG(y, p^t, s^t|p_0, \ldots, p_t)}{\pi(s_{i,\Delta t}, \ldots, s_{it}|p_0, \ldots, p_t)} \\
&= \int -\exp\left(-\sum_{u=t}^{T-\Delta t}\left[A\theta_u(p_{u+\Delta t} - p_u) + \frac{1}{2}\tau_u(p^2_{u+\Delta t} - p^2_u) - A\cdot C(a_u)\Delta t\right]\right) \\
&\quad \cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu}\Delta t)^{-1}}}\exp(-\frac{a_{iu}\Delta t}{2}(s_{i,u+\Delta t} - p_t)^2) \cdot \frac{dG(y, p^t, s^t|p_0, \ldots, p_t)}{\pi(s_{i,\Delta t}, \ldots, s_{it}|p_0, \ldots, p_t)} \\
&= -\exp(f_t)\cdot \prod_{u=0}^{t-\Delta t} \frac{1}{\sqrt{2\pi(a_{iu}\Delta t)^{-1}}}\exp(-\frac{a_{iu}\Delta t}{2}(s_{i,u+\Delta t} - p_t)^2) \cdot \frac{1}{\pi(s_{i,\Delta t}, \ldots, s_{it}|p_0, \ldots, p_t)}.
\end{aligned}
\qquad \text{(A21)}
$$

In the second equality, I applied the conditional Bayes theorem and then integrated over the signals $s_{i,t+\Delta t}, \ldots, s_{i,T-\Delta t}$, similar to (A5) and (A6). In the third equality, I used the definition of f_t (1.31). Therefore,

$$\xi_{t,T} = \mathbb{E}\left[\frac{U(W_{iT})}{\mathbb{E}_t^i[U(W_{iT})]}\middle| p_0, p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right] \tag{A22}$$

$$= \mathbb{E}\left[\exp\left[-\sum_{u=t}^{T-\Delta t}\left(A\theta_t + \sum_{u_0=0}^{u-\Delta t} a_{i,u_0} s_{i,u_0+\Delta t}(p_{u+\Delta t} - p_u)\right) + A\sum_{u=t}^{T-\Delta t} C(a_{iu})\Delta t - f_t\right]\right.$$

$$\cdot \pi(s_{i,\Delta t}, \ldots, s_{it}|p_0, \ldots, p_t) \cdot \prod_{u=0}^{t-\Delta t}\left(\frac{1}{\sqrt{2\pi(a_{iu}\Delta t)^{-1}}}\exp(-\frac{a_{iu}\Delta t}{2}(s_{i,u+\Delta t} - p_t)^2)\right)^{-1} \Bigg| p_0,$$

$$\left. p_{\Delta t}, \ldots, p_{T-\Delta t}, y\right]. \tag{A23}$$

Applying conditional Bayes theorem and integrating over the signals once again, I arrive at (1.30).

Proof of Proposition 2.

First, I prove the expression of μ_{pt} and σ_{pt} given in (1.36). Apply Taylor series expansion to $p(t+\Delta t, \theta_{t+\Delta t}, m_{t+\Delta t}, \chi_{t+\Delta t}, \tau_{t+\Delta t})$ around $(t, \theta_t, m_t, \chi_t, \tau_t)$,

$$\begin{aligned}
&p(t+\Delta t, \theta_{t+\Delta t}, m_{t+\Delta t}, \chi_{t+\Delta t}, \tau_{t+\Delta t}) \\
=&p(t, \theta_t, m_t, \chi_t, \tau_t) + \frac{\partial p}{\partial t}\Delta t + \frac{\partial p}{\partial \theta}(\theta_{t+\Delta t} - \theta_t) + \frac{\partial p}{\partial m}(m_{t+\Delta t} - m_t) + \frac{\partial p}{\partial \chi}(\chi_{t+\Delta t} - \chi_t) \\
&+ \frac{\partial p}{\partial \tau}(\tau_{t+\Delta t} - \tau_t) + \frac{1}{2}\frac{\partial^2 p}{\partial \theta^2}(\theta_{t+\Delta t} - \theta_t) + \frac{1}{2}\frac{\partial^2 m}{\partial \theta^2}(m_{t+\Delta t} - m_t) \\
&+ \frac{\partial^2 p}{\partial \theta \partial m}(\theta_{t+\Delta t} - \theta_t)(m_{t+\Delta t} - m_t) + o(\Delta t).
\end{aligned} \tag{A24}$$

Substituting in (1.21)-(1.24) and use the fact the expectation of payoff $\mathbb{E}_t[y]$ is m_t, the conditional expectations of state variable increments are given by:

$$\begin{aligned}
\mathbb{E}_t[\theta_{t+\Delta t} - \theta_t] &= \mathbb{E}_t[-A^{-1}a_t y\Delta t + z_{t+\Delta t} - z_t] = -A^{-1}a_t m_t \Delta t, \\
\mathbb{E}_t[m_{t+\Delta t} - m_t] &= \mathbb{E}_t[A^{-2}\sigma_z^{-2}a_t^2 h_t(y - m_t)\Delta t - A^{-1}\sigma_z^{-1}a_t h_t(z_{t+\Delta t} - z_t) + o(\Delta t)] \\
&= A^{-2}\sigma_z^{-2}a_t^2 h_t \Delta t\ \mathbb{E}_t[y - m_t] + o(\Delta t) = o(\Delta t), \\
\mathbb{E}_t[\chi_{t+\Delta t} - \chi_t] &= A^{-2}\sigma_z^2 a_t^2 \Delta t, \\
\mathbb{E}_t[\tau_{t+\Delta t} - \tau_t] &= a_t \Delta t.
\end{aligned} \tag{A25}$$

The conditional variance of $\theta_{t+\Delta t} - \theta_t$ is:

$$\begin{aligned}
\mathrm{Var}_t[\theta_{t+\Delta t} - \theta_t] &= \mathrm{Var}_t[-A^{-1}a_t y\Delta t + z_{t+\Delta t} - z_t] \\
&= \mathrm{Var}_t[-A^{-1}a_t y\Delta t] + \mathrm{Var}_t[z_{t+\Delta t} - z_t] \\
&= A^{-2}a_t^2\ \mathrm{Var}_t[y] \cdot (\Delta t)^2 + \sigma_z^2\Delta t = \sigma_z^2\Delta t + o(\Delta t).
\end{aligned} \tag{A26}$$

Similarly,

$$\begin{aligned}
&\mathrm{Var}_t[m_{t+\Delta t} - m_t] = A^{-2}\sigma^{-4}a_t^2 h_t^2 \,\mathrm{Var}_t[z_{t+\Delta t} - z_t] = A^{-2}\sigma^{-2}a_t^2 h_t^2 \Delta t, \\
&\mathrm{Cov}_t[\theta_{t+\Delta t} - \theta_t, m_{t+\Delta t} - m_t] = A^{-1}\sigma^{-2}a_t h_t \,\mathrm{Var}_t[z_{t+\Delta t} - z_t] = A^{-1}a_t h_t \Delta t. \quad \text{(A27)}
\end{aligned}$$

Substituting (A24) and (A25)-(A27) into (1.34) and simplify, I arrive at the expression of drift μ_{pt} and volatility σ_{pt}.

Next, I derive the system of partial differential equations (1.37)-(1.39). Let $\xi_{t,t+\Delta t}$ denote the stochastic discount factor from date t to $t + \Delta t$:

$$\xi_{t,t+\Delta t} = \frac{\xi_{t,T}}{\xi_{t+\Delta t,T}} = \exp\left(-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A\cdot C(a_t)\Delta t - f_t + f_{t+\Delta t}\right). \quad \text{(A28)}$$

Because the interest rate is zero, $\mathbb{E}_t[\xi_{t,t+\Delta t}]$ is equal to 1. From (1.26),

$$\begin{aligned}
p_t = \mathbb{E}_t[\xi_{t,t+\Delta t}p_{t+\Delta t}] &= \mathbb{E}_t[\xi_{t,t+\Delta t}]\ \mathbb{E}_t[p_{t+\Delta t}] + \mathrm{Cov}_t[\xi_{t,t+\Delta t},\ p_{t+\Delta t}] \\
&= p_t + \mu_{pt}\Delta t + \mathrm{Cov}_t\left[-A\theta_t p_{t+\Delta t} - \frac{1}{2}\tau_t p_{t+\Delta t}^2 + f_{t+\Delta t},\ p_{t+\Delta t}\right] + o(\Delta t) \\
&= p_t + \left(\mu_{pt} + \sigma_{pt}\sigma_{ft} - (A\theta_t + \tau_t p_t)\sigma_{pt}^2\right)\Delta t + o(\Delta t). \quad \text{(A29)}
\end{aligned}$$

The first equation (1.37) in the system of PDEs is obtained by taking the limit $\Delta t \to 0$. Similarly, the second equation (1.38) comes from

$$\begin{aligned}
v_t &= \mathbb{E}_t[\xi_{t,t+\Delta t}v_{t+\Delta t}] + \mathrm{Var}_t^*[p_{t+\Delta t}] \\
&= \mathbb{E}_t[\xi_{t,t+\Delta t}v_{t+\Delta t}] + \mathrm{Var}_t[p_{t+\Delta t}] + o(\Delta t) \\
&= v_t + \left(\mu_{vt} + \sigma_{vt}\sigma_{ft} - (A\theta_t + \tau_t p_t)\sigma_{pt}^2 + \sigma_{pt}^2\right)\Delta t + o(\Delta t). \quad \text{(A30)}
\end{aligned}$$

Apply Taylor expansion to $\xi_{t,t+\Delta t} = \exp(\ln \xi_{t,t+\Delta t})$ around $\ln \xi_{t,t+\Delta t} = 0$:

$$\begin{aligned}
\xi_{t,t+\Delta t} &= 1 + \ln \xi_{t,t+\Delta t} + \frac{1}{2}(\ln \xi_{t,t+\Delta t})^2 + o(\Delta t) \\
\mathbb{E}_t[\xi_{t,t+\Delta t}] &= 1 + \mathbb{E}_t\left[-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A\cdot C(a_t)\Delta t - f_t + f_{t+\Delta t}\right] \\
&\quad + \frac{1}{2}\mathbb{E}_t\left[\left(-A\theta_t(p_{t+\Delta t} - p_t) - \frac{1}{2}\tau_t(p_{t+\Delta t}^2 - p_t^2) + A\cdot C(a_t)\Delta t - f_t + f_{t+\Delta t}\right)^2\right] + o(\Delta t) \\
&= 1 + \left[-(A\theta_t + \tau_t p_t)\mu_{pt} + A\cdot C(a_t) + \mu_{ft} + \frac{1}{2}\left(-(A\theta_t + \tau_t p_t)\sigma_{pt} + \sigma_{ft}\right)^2\right]\Delta t + o(\Delta t).
\end{aligned} \quad \text{(A31)}$$

Taking the limit $\Delta t \to 0$ and substitute in (1.37) to eliminate μ_{pt}, I arrive at (1.39). The equation that relates attention to risk-neutral variance (1.13) converges to equation (1.40) in the continuous-time limit.

The terminal conditions for this system of equations come from the continuous-time limit of $p_{T-\Delta t}$, $v_{T-\Delta t}$ and $f_{T-\Delta t}$:

$$
\begin{aligned}
p_{T-\Delta t} &= \mathbb{E}_{T-\Delta t}\left[\exp\left(-A\theta_{T-\Delta t}(y-p_{T-\Delta t}) - \frac{1}{2}\tau_{T-\Delta t}(y^2 - p^2_{T-\Delta t}))\Delta t - f_{T-\Delta t}\right)\cdot y\right],\\
v_{T-\Delta t} &= \mathbb{E}_{T-\Delta t}\left[\exp\left(-A\theta_{T-\Delta t}(y-p_{T-\Delta t}) - \frac{1}{2}\tau_{T-\Delta t}(y^2 - p^2_{T-\Delta t}))\Delta t - f_{T-\Delta t}\right)\cdot (y-p_{T-\Delta t})^2\right.\\
f_{T-\Delta t} &= \log\left(\mathbb{E}_{T-\Delta t}\left[\exp\left(-A\theta_{T-\Delta t}(y-p_{T-\Delta t}) - \frac{1}{2}\tau_{T-\Delta t}(y^2 - p^2_{T-\Delta t})\right)\right]\right). \qquad \text{(A32)}
\end{aligned}
$$

Proof of Proposition 3.

Risk-neutral variance at date t and $t+\Delta t$ is related by equation (1.27):

$$
\begin{aligned}
\mathbb{E}^*_t[v_{t+\Delta t} - v_t] = -\mathrm{Var}^*_t[p_{t+\Delta t}] &= -\left(\mathbb{E}_t[\xi_{t,t+\Delta t}\, p^2_{t+\Delta t}] - p^2_t\right)\\
&= -(\mathbb{E}_t[p^2_{t+\Delta t}] - p^2_t) - \mathrm{Cov}_t[\xi_{t,t+\Delta t}, p^2_{t+\Delta t}]\\
&= -\sigma^2_{pt}\Delta t + o(\Delta t), \qquad \text{(A33)}
\end{aligned}
$$

As a result, the instantaneous drift of v_t in the risk-neutral measure is given by:

$$
\begin{aligned}
\mu^*_{vt} = \lim_{\Delta t\to 0}\frac{\mathbb{E}^*_t[v_{t+\Delta t} - v_t]}{\Delta t} &= -\sigma^2_{pt}\\
&= -\left(\frac{\partial p}{\partial m}A^{-1}\sigma_z^{-1}a_t h_t - \frac{\partial p}{\partial \theta}\sigma_z\right)^2. \qquad \text{(A34)}
\end{aligned}
$$

Since price is increasing in m_t and decreasing in θ_t, μ^*_{vt} is decreasing in attention:

$$
\frac{\partial \mu^*_{vt}}{\partial a_t} = 2\frac{\partial p}{\partial \theta}\frac{\partial p}{\partial m}A^{-1}h_t - 2\left(\frac{\partial p}{\partial m}\right)^2 A^{-2}\sigma_z^2 a_t h_t^2 < 0. \qquad \text{(A35)}
$$

I introduce Lemma A1 and Lemma A2 to prove Proposition 4.

Lemma A1. *$\partial f_t/\partial \theta_t$ is given by:*

$$
\frac{\partial f_t}{\partial \theta_t} = (A\theta_t + \tau_t p_t)\frac{\partial p_t}{\partial \theta_t} + \mathbb{E}^*_t\left[\int_t^T \frac{\partial^2 p_u}{\partial \theta_u^2}A\sigma_z^2 du\right], \qquad \text{(A36)}
$$

Proof Lemma A1.

I start from the definition of f_t in discrete time (1.31):

$$
f_t = \ln \mathbb{E}_t\left[\exp\left(-\sum_{u=t}^{T-\Delta t}\left[A\theta_u(p_{u+\Delta t} - p_u) + \frac{1}{2}\tau_u(p^2_{u+\Delta t} - p^2_u) - A\cdot C(a_u)\Delta t\right]\right)\right]. \qquad \text{(A37)}
$$

Differentiate f_t in (1.31) with respect to p_t,

$$
\frac{\partial f_t}{\partial p_t} = (A\theta_t + \tau_t p_t).
$$

An increase in θ_t implies the same change in θ_u for $u = t+\Delta t, \ldots, T-\Delta t$. From the dynamics of state variables (1.21),

$$\frac{\partial \theta_u}{\partial \theta_t} = 1, \quad u = t+\Delta t, \ldots, T-\Delta t. \tag{A38}$$

Now differentiate f_t in (1.31) with respect to θ_u

$$\begin{aligned}
\frac{\partial f_t}{\partial \theta_u} =& \frac{1}{f_t}\mathbb{E}_t\Bigg[\exp\Bigg(-\sum_{u=t}^{T-\Delta t}\Big[A\theta_u(p_{u+\Delta t}-p_u)+\frac{1}{2}\tau_u(p_{u+\Delta t}^2-p_u^2)-A\cdot C(a_u)\Delta t\Big]\Bigg)\\
&\Bigg(-A(p_{u+\Delta t}-p_t)+(A\theta_u-A\theta_{u-\Delta t}+a_{u-\Delta t}\Delta t p_u)\frac{\partial p_u}{\partial \theta_u}\Bigg)\Bigg].\\
=&\mathbb{E}_t^*\left[-A(p_{u+\Delta t}-p_t)+(A\theta_u-A\theta_{u-\Delta t}+a_{u-\Delta t}\Delta t p_u)\frac{\partial p_u}{\partial \theta_u}\right]\\
=&0+\mathrm{Cov}_t^*\left[(A\theta_u-A\theta_{u-\Delta t}+a_{u-\Delta t}\Delta t p_u), \frac{\partial p_u}{\partial \theta_u}\right]\\
=&\mathrm{Cov}_t^*\left[(A\theta_u-A\theta_{u-\Delta t}+a_{u-\Delta t}\Delta t p_u), \frac{\partial^2 p_u}{\partial \theta_u^2}(\theta_u-\theta_{u-\Delta t})\right]+o(\Delta t)\\
=&\frac{\partial^2 p_u}{\partial \theta_u^2}A\sigma_z^2\Delta t+o(\Delta t).
\end{aligned} \tag{A39}$$

The third equality in the above expression follows from the fact that price is a martingale under the risk-neutral measure and that $\mathbb{E}_t^*[A\theta_u - A\theta_{u-\Delta t} + a_{u-\Delta t}\Delta t p_u] = 0$. From (A38) and (A39), I obtain (A36).

Lemma A2. *$\partial f_t/\partial m_t$ is given by:*

$$\frac{\partial f_t}{\partial m_t} = (p_t - m_t)h_t^{-1} + (A\theta_t + \tau_t p_t)\frac{\partial p_t}{\partial m_t} - \mathbb{E}_t^*\left[\int_t^T \frac{\chi_u}{\chi_t}\frac{\partial^2 p_u}{\partial m_u^2}a_u h_u\right]. \tag{A40}$$

Proof Lemma A2.

From equation (1.16), $A(a_t\Delta t)^{-1}(\theta_{t+\Delta t} - \theta_t)$ is a public signal of y with precision $A^{-2}\sigma_z^{-2}a_t^2\Delta t$. χ_t represents the aggregate precision of all public signals. Let η_t denote the average of public signals using precisions $A^{-2}\sigma_z^{-2}a_t^2\Delta t$ as weights:

$$\chi_t = \sum_{u=0}^{t-\Delta t} A^{-2}\sigma_z^{-2}a_u^2\Delta t, \tag{A41}$$

$$\eta_t = \sum_{u=0}^{t-\Delta t}\frac{A^{-2}\sigma_z^{-2}a_u^2\Delta t}{\chi_{T-\Delta t}}\cdot A(a_t\Delta t)^{-1}(\theta_{t+\Delta t}-\theta_t). \tag{A42}$$

Because public signals in different periods are independent, they are informationally equivalent to a signal η_t with precision χ_t. Applying Bayes formula, the expected payoff m_t could be expressed as:

$$m_t = \frac{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \cdot y \, dG(y) \right]}{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \, dG(y) \right]}. \tag{A43}$$

Differentiate m_t with respect to η_t:

$$\begin{aligned} \frac{dm_t}{d\eta_t} &= \frac{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \cdot y \cdot \chi_t (y-\eta_t) \, dG(y) \right]}{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \, dG(y) \right]} \\ &\quad - m_t \cdot \frac{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \cdot \chi_t (y-\eta_t) \, dG(y) \right]}{\int \left[\exp\left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) \, dG(y) \right]} \\ &= \mathbb{E}_t[y \cdot \chi_t(y-\eta_t)] - m_t \, \mathbb{E}_t[\chi_t(y-\eta_t)] \\ &= \chi_t \left[\mathbb{E}_t[y^2] - m_t \, \mathbb{E}_t[y] \right] = \chi_t h_t. \end{aligned} \tag{A44}$$

Similarly, f_t could be expressed as:

$$\begin{aligned} f_t = \ln \left(\int \exp \left(- \sum_{u=t}^{T-\Delta t} \left[A\theta_u (p_{u+\Delta t} - p_u) + \frac{1}{2}\tau_u (p_{u+\Delta t}^2 - p_u^2) - A \cdot C(a_u)\Delta t \right] - \frac{\chi_t}{2}(y-\eta_t)^2 \right) dG \right. \\ - \ln \left(\int \exp \left(-\frac{\chi_t}{2}(y-\eta_t)^2 \right) dG(y) \right) \end{aligned} \tag{A45}$$

Differentiate f_t with respect to η_t:

$$\frac{df_t}{d\eta_t} = \chi_t \, \mathbb{E}_t^*[y-\eta_t] - \chi_t \, \mathbb{E}_t[y-\eta_t] = \chi_t (p_t - m_t). \tag{A46}$$

(A44) and (A46) give rise to the first term in (A40). The second term is similar to Lemma 1.

An increase in m_t implies less than one-to-one change in m_u for $u = t + \Delta t, \ldots, T - \Delta t$. From the dynamics of state variables (1.22),

$$\frac{dm_u}{dm_t} = \frac{\chi_u}{\chi_t} + o(\Delta t), \quad u = t + \Delta t, \ldots, T - \Delta t. \tag{A47}$$

Similar to (A39),

$$\frac{\partial f_t}{\partial m_u} = -\frac{\partial^2 p_u}{\partial m_u^2} a_u h_u + o(\Delta t). \tag{A48}$$

The above expression gives the third term in (A40).

Proof of Proposition 4.

Using (1.37) and (1.36), the instantaneous expected return μ_{pt} is given by:

$$\begin{aligned}
\mu_{pt} &= \sigma_{pt}(-\sigma_{ft} + (A\theta_t + \tau_t p_t)\sigma_{pt}) \\
&= \left(\frac{\partial p_t}{\partial m_t} A^{-1}\sigma_z^{-1} h_t a_t - \frac{\partial p_t}{\partial \theta_t}\sigma_z\right)(-\sigma_{ft} + (A\theta_t + \tau_t p_t)\sigma_{pt}) \\
&= \left(\frac{\partial p_t}{\partial m_t} A^{-1}\sigma_z^{-1} h_t a_t - \frac{\partial p_t}{\partial \theta_t}\sigma_z\right)(\frac{\partial f_t}{\partial \theta_t}\sigma_z - \frac{\partial f_t}{\partial m_t} A^{-1}\sigma_z^{-1} h_t a_t + (A\theta_t + \tau_t p_t)\sigma_{pt}).
\end{aligned} \tag{A49}$$

Combine (A49) with the results from Lemma A1 and Lemma A2, I arrive at (1.48).

Proof of Proposition 5.

Define $\eta_{T-\Delta t}$ similar to η_t in (A42). Public signals from date 0 to date $T - \Delta t$ are informationally equivalent to a signal $\eta_{T-\Delta t}$ with precision $\chi_{T-\Delta t}$. Similar to (A44),

$$\frac{dm_{T-\Delta t}}{d\eta_{T-\Delta t}} = \chi_{T-\Delta t}\left[\mathbb{E}_{T-\Delta t}[y^2] - m_{T-\Delta t}\,\mathbb{E}_{T-\Delta t}[y]\right] = \chi_{T-\Delta t} h_{T-\Delta t}. \tag{A50}$$

Applying the Bayes formula, the risk-neutral variance $v_{T-\Delta t}$ can be expressed as follows:

$$v_{T-\Delta t} = \frac{\int \left[\xi_{T-\Delta t,T}\cdot \exp\left(-\frac{\chi_{T-\Delta t}}{2}(y - \eta_{T-\Delta t})^2\right)\cdot (y - p^2_{T-\Delta t})\, dG(y)\right]}{\int \left[\xi_{T-\Delta t,T}\cdot \exp\left(-\frac{\chi_{T-\Delta t}}{2}(y - \eta_{T-\Delta t})^2\right)\, dG(y)\right]}. \tag{A51}$$

And the partial derivative of $v_{T-\Delta t}$ to $\eta_{T-\Delta t}$ is:

$$\frac{\partial v_{T-\Delta t}}{\partial \eta_{T-\Delta t}} = \chi_{T-\Delta t}\,\mathbb{E}^*_{T-\Delta t}[(y - p^3_{T-\Delta t})].$$

Combine (A50) and (A52), I arrive at (1.50).

Chapter 2

Collateral Requirements and Asset Prices

We consider an economy populated by investors with heterogeneous preferences and beliefs who receive non-pledgeable labor incomes. We study the effects of collateral constraints that require investors to maintain sufficient pledgeable capital to cover their liabilities. We show that these constraints inflate stock prices, give rise to clusters of stock return volatilities, and produce spikes and crashes in price-dividend ratios and volatilities. Furthermore, mere possibility of a crisis significantly decreases interest rates and increases Sharpe ratios. The stock price has large collateral premium over non-pledgeable incomes. Asset prices are in closed form, and investors survive in the long run.

Journal of Economic Literature Classification Numbers: D52, G12.
Keywords: collateral, non-pledgeable labor income, heterogeneous preferences, disagreement, asset prices, stationary equilibrium.

2.1. Introduction

Financial markets play a key role in facilitating risk sharing and efficient allocation of assets among investors. However, trading in financial assets often entails moral hazard due to investors' incentives to default on their risky positions. The moral hazard can be alleviated by collateralized trades whereby risky positions are backed by financial capital that can be seized in the event of default. The latter arrangement restores the functionality of financial markets at a cost of restricting risk sharing among investors. In this paper, we develop a parsimonious model which sheds light on the economic effects of such restrictions and show that they give rise to rich dynamics of asset prices. In particular, we show how collateralization inflates asset prices, generates repeated booms and busts in the stock market, and leads to spikes, crashes, and clustering of stock return volatilities, as well as cycles of high and low leverage. Our analysis is facilitated by closed-form solutions and the stationarity of equilibrium processes.

We consider a pure exchange economy with one consumption good produced by a tree with i.i.d. shocks, similar to Lucas (1978). The economy is populated by two representative investors with heterogeneous constant relative risk aversion (CRRA) preferences over consumption and heterogeneous beliefs about the output growth rate. Each investor receives a fraction of the tree's output as labor income and invests total wealth in financial assets such as bonds and stocks. The investors have limited liability and can re-enter the financial market following defaults on risky positions in financial assets. In the event of default the financial assets can be seized by counterparties but labor income cannot be expropriated. The arising moral hazard problem is resolved by requiring risky positions to be backed by collateral in such a way that each investor's total financial wealth stays positive at all times, and hence, investors can always pay back to counterparties. We also allow the aggregate consumption to experience rare large negative shocks, which help us explore how mere anxiety about the possibility of a production crisis affects the economy by tightening collateral requirements. Our closed form solutions allow us to prove some of the results for general model parameters rather than for particular calibrations.

First, we show that collateral requirements increase the prices of all tradable assets with positive cash flows relative to a frictionless economy. Moreover, these increases in prices are larger when investors are closer to their default boundaries. In particular, the stock price-dividend ratio is a U-shaped function of one of the investor's share of the aggregate consumption. Consequently, it spikes upwards in response to small economic shocks near default boundaries giving rise to repeated periods of high and low stock prices.

The intuition for the latter results is as follows. Absent any frictions, the investors' consumption shares gradually approach zero or one, and hence the economic impact of one of the investors vanishes in the long-run (e.g., Blume and Easley, 2006; Yan, 2008; Chabakauri, 2015). The collateral requirements restrict financial losses and protect investors from losing their consumption shares. The result is that the consumption shares are bounded away from zero and one. Moreover, the constraints never bind simultaneously for both investors, and at each moment one of the investors is unconstrained. The unconstrained investor's marginal utility of consumption is proportional to the prices of Arrow-Debreu securities. This marginal utility is expected to be higher in the economy with constraints because the unconstrained investor's consumption is expected to be lower than in the unconstrained economy due to the upper bound on the consumption share, discussed above. Consequently, the prices of Arrow-Debreu securities, and hence, the prices of all assets with positive cash flows, are higher in the constrained economy.

The dynamics of the price-dividend ratio determines the effect of constraints on volatilities. We show that collateral requirements dampen volatilities in bad times, when aggregate consumption is low, and amplify them in good times, when aggregate consumption is high. The latter effect makes collateral requirements a useful tool for curbing excessive volatility in bad times. The explanation is that the U-shaped price-dividend ratio is procyclical in good and countercyclical in bad times. As a result, the price-dividend ratio and the dividend move in the same direction in good times and in opposite directions in bad times. Because the stock price is the product of the price-dividend ratio and the dividend, the stock return volatility increases in good times and decreases in bad times. The volatility experiences spikes and crashes due to the sensitivity of price-dividend ratios to small shocks when investors are close to hitting their constraints. Moreover, the periods of high and low volatility are persistent because of the persistence of periods when constraints are likely to bind, as discussed below, which gives rise to the clustering of volatilities.

We also derive the distributions of investors' consumption shares in analytic form and show that they are stationary and non-degenerate (i.e., their support is a closed interval rather than a single point). The analysis of these distributions yields three economic insights. First, there is non-trivial time-variation of asset prices in the long run. Second, periods of binding collateral requirements are persistent. That is, the economy stays close to default boundaries for some time because hitting a constraint makes likely hitting it again in the near future due to slow accumulation of wealth over time. Third, we show that all investors, including those with incorrect beliefs, survive in the long run and can have large economic impact in equilibrium because the constraints prevent investors from losing their consumption shares, similar to the related literature (e.g., Blume and Easley, 2006; Cao, 2018). We note that the non-degeneracy of consumption share distributions and the persistence of the periods of binding constraints are more difficult to demonstrate than survival, and, to our best knowledge, these results are new to the literature.

Next, we show that mere possibility of a large (albeit unpredictable) drop in the aggregate output next period decreases interest rates and increases Sharpe ratios in the current period when the irrational optimist is close to hitting the collateral constraint. The latter effect only occurs when production crises and collateral requirements are jointly present in the economy. Hence, the collateral requirements amplify the spillover of the production crisis to the financial market. The amplification effect arises because investors "fly to quality" by buying riskless bonds when there is a possibility of hitting the collateral constraint next period. We note that lower interest rates and higher Sharpe ratios can be generated by alternative mechanisms and constraints, discussed in the literature review below. However, the amplification mechanism, to our best knowledge, has not been studied before. We also show that investor heterogeneity and the stationarity of equilibrium give rise to cycles of high and low leverage. In particular, the leverage is high when investors are far away from their default boundaries, and drops to zero when investors hit their constraints.

Finally, we measure the collateral liquidity premium of the stock versus labor income. This premium arises because dividends and labor incomes are collinear but incomes are non-pledgeable. First, we derive shadow prices of claims to labor incomes such that exchanging marginal units of these claims for the consumption good at shadow prices does not affect investors' welfare. Then, we construct portfolios of stocks that replicate labor incomes. We define the collateral liquidity premium as the percentage difference in the value of the replicating portfolio and the shadow price. The premium from the view of a particular investor widens close to that investor's default boundary and ranges from 0% to 35% in our calibration, which demonstrates the economic importance of collateralization. We also show that the non-tradability of labor income does not contribute to this premium. This is because in economies with pledgeable labor income investors circumvent non-tradability by taking short positions in the stock, and hence, the liquidity premium is such economies is zero.

The paper develops new methodology for studying the effects of collateral requirements. This new methodology allows us to obtain closed-form equilibrium processes and prove their properties which previously could only be studied numerically. For example, we prove that collateral requirements always increase price-dividend ratios and generate spikes in asset prices, and lead to non-degeneracy and stationarity of consumption share distributions. Hence, collateralization emerges as a tractable way of inducing the stationary of equilibrium. Finally, the paper introduces a tractable discrete-time framework that makes exposition less technical and permits taking continuous-time limits. The tractability and stationarity make our model a convenient benchmark for asset pricing research that can be extended in various directions.

Related Literature. Closest to us are papers that study economies where investors have limited liability and face solvency constraints. Deaton (1990) considers a partial equilibrium model in which investors trade in a riskless asset with an exogenous interest rate and face a non-negativity constraint on their financial wealth. Detemple and Serrat (2003) also study the non-negative wealth constraint in a model where investors have heterogeneous beliefs and identical risk aversions. They show that this constraint introduces a singularity component into interest rates when the constraint binds while stock risk premia have the same structure as in unconstrained economies. They do not compute price-dividend ratios, volatilities, and consumption share distributions as we do in this paper, which are more difficult to characterize than interest rates and risk premia. Moreover, in our paper the constraint has an effect on interest rates and risk premia in the internal area of the state-space when there are rare production crises in the economy.

Chien and Lustig (2010) study a similar constraint in an economy with a continuum of ex ante identical investors that receive non-pledgeable labor incomes affected by idiosyncratic shocks. Lustig and Van Nieuwerburgh (2005) study the role of housing collateral when labor income is non-pledgeable. The main difference of our paper from the latter two papers is that our investors are ex ante heterogeneous and are not affected by idiosyncratic shocks to labor income. The economic effects of heterogeneity in preferences and beliefs are different from the effects of ex-post heterogeneity in realized idiosyncratic labor income shocks in the above literature. For example, Krueger and Lustig (2010) show the irrelevance of market incompleteness induced by these income shocks for the risk premia. Kubler and Schmedders (2013) show the existence of stationary equilibria in economies with collateral constraints.

Cao (2018) proves that investors with incorrect beliefs have strictly positive shares of consumption in the long run (i.e., survive in the long-run) in economies with general collateral constraints and stationary endowment processes bounded away from zero. Similar results are also shown numerically in an example with non-stationary endowments. Blume et al (2015) explore potential benefits from imposing trading restrictions, such as natural borrowing constraints, in economies with bounded endowments and investors with heterogeneous beliefs. In contrast to these works, our results do not rely on bounded endowments. Moreover, in addition to showing the survival of investors, we derive consumption share distributions in closed form, and establish their bimodality, stationarity, and non-degeneracy (i.e., their support is a closed interval rather than a single point), and derive new equilibrium effects. Rampini and Viswanathan (2018) study household insurance in an economy with collateral constraints with limited enforcement and deep-pocket risk-neutral lenders, who provide state-contingent claims to households at zero risk premium. In contrast to their model, all investors in our economy are risk averse, and risk-premia are non-zero and time-varying.

Geanakoplos (2003, 2009), Fostel and Geanakoplos (2008, 2014), and Geanakoplos and Zame (2014) develop the theory of collateral constraints in two- and three-period multinomial general equilibrium economies. Similar to the baseline analysis in Geanakoplos (2003, 2009), our constraint also prevents investors from defaulting in the worst-case scenario. Simsek (2013) studies a two-period economy with a continuum of states and shows that collateral constraints have asymmetric disciplining effects, depending on investor's beliefs, and also shows how defaultable debt endogenously emerges in equilibrium. Biais, Hombert, and Weill (2018) study a two-period economy with multiple trees and imperfect collateral pledgeability. In contrast to this literature, we focus on the non-pledgeability of labor income rather than imperfect pledgeability of assets. Hence, our model generates a different set of predictions. Our constraint is also more tractable and allows us to study multiperiod economies where investors have different preferences and beliefs and the output follows a geometric Brownian motion with jumps.

Kehoe and Levine (1993), Kocherlakota (1996), Tsyrennikov (2012), and Osambela (2015) study economies where investors are weakly better off not defaulting and are permanently excluded from securities markets if they default. Alvarez and Jermann (2000) show that such constraints can be implemented by imposing certain "not too tight" solvency portfolio constraints. Alvarez and Jermann (2001) find that such constraints help explain equity premia in the U.S. economy. They solve a simple example in closed form and develop a numerical method for the general case. In contrast to this literature, our investors have limited liability and can re-enter the market after a default.

Our constraint restricts borrowing and short-selling in equilibrium. Consequently, the paper is related to the literature on economic effects of borrowing, margin, short-sale and position limit constraints (e.g., Harrison and Kreps, 1978; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Gromb and Vayanos, 2002, 2010; Pavlova and Rigobon, 2008; Brunnermeier and Pedersen, 2009; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014; Brumm et al, 2015; Buss et al, 2016), portfolio insurance (e.g., Basak, 1995) and VaR constraints (e.g., Basak and Shapiro, 2001). Our economic results are different from the results in this literature. First, the latter constraints can increase or decrease stock prices depending on whether the investors' risk aversions are greater or less than one (e.g., Chabakauri, 2015), whereas our collateral requirements always increase stock prices irrespective of risk aversions and beliefs. Second, these constraints typically dampen stock return volatility whereas our collateral requirements amplify them in some states of the economy. We also uncover new effects such as spikes and crashes of volatilities and stock prices, and clusters of volatility.

We also note that our collateral requirements are conceptually different from the margin and borrowing constraints in some of the related models discussed above (e.g., Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015; Rytchkov, 2014). The latter models focus on partial non-pledgeability of stocks and do not incorporate labor incomes. The economic effects of constrains in these models are driven by reduced risk-sharing. In contrast to these works, we explore the non-pledgeability of labor incomes in a setting with fully pledgeable financial assets serving as collateral. The effects of constraints in our model are driven by increased marginal utilities of investors and collateral premia, which inflate asset prices.

Our methodology is also different from the approaches in the related literature. The equilibrium in models with constraints is often constructed using fictitious complete market economies (Cvitanić and Karatzas, 1992; Cuoco, 1997; Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Pavlova and Rigobon, 2008; Gârleanu and Pedersen, 2011; Chabakauri, 2013, 2015). Moreover, when investors have non-logarithmic utilities the equilibrium is characterized in terms of non-linear differential equations that can only be solved numerically (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014). In contrast to these works, we do not employ fictitious markets and derive the equilibrium using the envelope theorem. To our best knowledge this paper is the first to derive analytical prices, distributions of consumption shares, and conditions for the constraints to be binding.

The paper is also related to macro-finance, financial intermediation, and banking literatures that study economies with frictions (Kiyotaki and Moore, 1997; Brunnermeier and Sannikov, 2014; Klimenko, Pfeil, Rochet, and De Nicolo, 2016; Kondor and Vayanos, 2018) and to the literature on frictionless economies with heterogeneous investors (e.g., Chan and Kogan, 2002; Basak, 2005; Yan, 2008; Bhamra and Uppal, 2014; Atmaz and Basak, 2018; Borovička, 2018; Massari, 2018, among others).

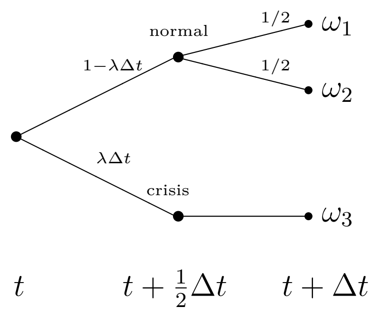


Figure 2.1

States of the Economy

After time t the economy moves to a normal state with probability $1-\lambda\Delta t$ and to a crisis state with probability $\lambda\Delta t$. Conditional on being in a normal state the economy moves to either ω_1 or ω_2 with equal probabilities.

2.2. Economic setup

We consider a pure-exchange infinite-horizon economy with one consumption good produced by an exogenous Lucas (1978) tree. The economy is populated by two representative heterogeneous investors A and B that hold shares in the tree and receive labor income each period. To facilitate the exposition, we start with a discrete-time economy with dates $t = 0, \Delta t, 2\Delta t, \dots$, and later take a continuous-time limit.

At each point of time $t = 0, \Delta t, 2\Delta t, \dots$ the economy is in one of the three states: ω_1, ω_2, and ω_3. With probability $1-\lambda\Delta t$ the economy is either in state ω_1 or state ω_2, which we call *normal states*, and with probability $\lambda\Delta t$ in state ω_3, which we call the *crisis state*. Parameter $\lambda > 0$ is the *crisis intensity*. States ω_1 and ω_2 have probabilities 1/2 conditional on the economy being in a normal state. Figure 2.1 depicts the structure of uncertainty.

2.2.1 Output, financial markets, and investor heterogeneity

At date t the tree produces $D_t\Delta t$ units of aggregate output, where D_t follows a process

$$\Delta D_t = D_t[\mu_D\Delta t + \sigma_D\Delta w_t + J_D\Delta j_t], \tag{2.1}$$

where $\mu_D \geq 0$, $\sigma_D > 0$, and $J_D \leq 0$ are output growth mean, volatility, and drop during a crisis, respectively, and $\Delta D_t = D_{t+\Delta t} - D_t$ is the change in output. Processes w_t and j_t are discrete-time analogues of a Brownian motion and Poisson processes, respectively.[1] These processes follow dynamics $w_{t+\Delta t} = w_t + \Delta w_t$ and $j_{t+\Delta t} = j_t + \Delta j_t$, where increments Δw_t and Δj_t are i.i.d. random variables given by:

$$\Delta w_t = \begin{cases} +\sqrt{\Delta t}, & \text{in state } \omega_1, \\ -\sqrt{\Delta t}, & \text{in state } \omega_2, \\ 0, & \text{in state } \omega_3, \end{cases} \qquad \Delta j_t = \begin{cases} 0, & \text{in state } \omega_1, \\ 0, & \text{in state } \omega_2, \\ 1, & \text{in state } \omega_3. \end{cases} \tag{2.2}$$

It can be easily verified that $\mathbb{E}_t[\Delta w_t|\text{normal}] = 0$ and $\text{var}_t[\Delta w_t|\text{normal}] = \Delta t$, similar to a Brownian motion, where $\mathbb{E}_t[\cdot]$ and $\text{var}_t[\cdot]$ are expectation and variance conditional on time-t information. Parameters μ_D, σ_D, and J_D are such that $D_t > 0$ for all t.

The economy is populated by two representative price-taking investors A and B. Each investor stands for a continuum of identical investors of unit mass. Fractions l_A and l_B of the aggregate output $D_t\Delta t$ are paid to investors A and B as their labor incomes, respectively. Labor incomes are non-tradable. Fractions l_A and l_B can be also interpreted as non-tradable shares in the aggregate output such as holdings of illiquid assets. The remaining fraction $1 - l_A - l_B$ is paid as a dividend to the shareholders.

The investors can trade three securities at each date t: 1) a riskless bond in zero net supply, which pays one unit of consumption at date $t + \Delta t$; 2) one stock in net supply of one unit, which is a claim to the stream of dividends $(1-l_A-l_B)D_t\Delta t$; 3) a one-period insurance contract in zero net supply, which pays one unit of consumption in the crisis state ω_3 and zero otherwise. Absent any frictions the market is complete. Market completeness and the absence of idiosyncratic shocks to labor income are required for tractability, and allow us to solve the model in closed form. Bond, stock, and insurance prices B_t, S_t, and P_t, respectively, are determined in equilibrium.

[1] Chabakauri (2014) shows that process (2.1) converges to a continuous-time Lévy process as $\Delta t \to 0$.

2.2.2 Investor heterogeneity and optimization problems

The investors have heterogeneous CRRA preferences over consumption, given by

$$u_i(c) = \begin{cases} \dfrac{c^{1-\gamma_i}}{1-\gamma_i}, & \text{if } \gamma_i \neq 1, \\ \ln(c), & \text{if } \gamma_i = 1, \end{cases} \tag{2.3}$$

where $i = A, B$. The investors agree on time-t asset prices and the aggregate output but disagree on the probabilities of states. Investor A is rational and has correct probabilities

$$\pi_A(\omega_1) = \frac{1-\lambda\Delta t}{2}, \quad \pi_A(\omega_2) = \frac{1-\lambda\Delta t}{2}, \quad \pi_A(\omega_3) = \lambda\Delta t, \tag{2.4}$$

where λ is such that probabilities (2.4) are positive. Investor B has biased probabilities

$$\pi_B(\omega_1) = \frac{1-\lambda_B\Delta t}{2}(1+\delta\sqrt{\Delta t}), \quad \pi_B(\omega_2) = \frac{1-\lambda_B\Delta t}{2}(1-\delta\sqrt{\Delta t}), \quad \pi_B(\omega_3) = \lambda_B\Delta t, \tag{2.5}$$

where crisis intensity λ_B and disagreement parameter δ are such that probabilities (2.5) are positive. It is immediate to verify that $\pi_B(\omega_1) + \pi_B(\omega_2) + \pi_B(\omega_3) = 1$, and hence, $\pi_B(\omega)$ is a probability measure. Throughout the paper, by $\mathbb{E}_t^i[\cdot]$ and $\mathrm{var}_t^i[\cdot]$ we denote conditional expectations and variances under the probability measure of investor i.

It can be easily verified that time-t conditional expected output growth rate in normal times under the beliefs of investor B is given by:

$$\mathbb{E}_t^B\left[\frac{\Delta D_t}{D_t}\middle|\text{normal}\right] = (\mu_D + \delta\sigma_D)\Delta t, \tag{2.6}$$

Therefore, parameter δ measures the extent of the investor disagreement about the expected output growth during normal times. For tractability, we assume that investor B does not update probabilities over time. We also assume that investor B is weakly less risk averse and more optimistic than investor A: $\gamma_A \geq \gamma_B$, $\lambda \geq \lambda_B$ and $\delta \geq 0$. The assumption that the less risk averse investor is also more optimistic is imposed to simplify the exposition and does not affect the qualitative results in the paper.[2] We allow for the heterogeneity in both risk aversions and beliefs for generality. Main qualitative results do not change if we keep only one source of heterogeneity.

At date 0 the investors have certain endowments of financial assets. The total time-t disposable wealth of investor i is given by $W_{it} + l_i D_t \Delta t$, where W_{it} is the financial wealth, defined as the time-t value of all positions in financial assets acquired at the previous date, and $l_i D_t \Delta t$ is the labor income. At date t, investor i allocates wealth to $c_{it}\Delta t$ units of consumption, b_{it} units of bond, and a portfolio of risky assets $n_{it} = (n_{i,St}, n_{i,Pt})$, where $n_{i,St}$ and $n_{i,Pt}$ are units of stock and insurance, respectively. The bond and the risky assets are pledgeable, i.e., can be used as collateral, but the labor income is not.

In a frictionless economy, the financial wealth W_{it} can become negative when investors take risky positions backed by their future labor income. However, we assume that labor income is not pledgeable, and the investors can default when their financial wealth becomes negative. The investors also have limited liability and can re-enter the market after default, which gives rise to a moral hazard problem, similar to the related literature (e.g., Chien and Lustig, 2010; Geanakoplos, 2009). This problem is addressed here by requiring the investors to keep their next-period financial wealth $W_{i,t+\Delta t}$ positive at all times, so that their pledgeable capital is sufficient to cover all liabilities such as debt and short positions.

Investor $i = A, B$ maximizes expected discounted utility with time discount ρ

$$\max_{c_{it}, b_{it}, n_{it}} \mathbb{E}_t^i \left[\sum_{\tau=t}^{\infty} e^{-\rho\tau} u_i(c_{i\tau}) \Delta t \right], \tag{2.7}$$

subject to the self-financing budget constraints, given by

$$\begin{aligned} W_{it} + l_i D_t \Delta t &= c_{it}\Delta t + b_{it} B_t + n_{it}(S_t, P_t)^\top, & (2.8) \\ W_{i,t+\Delta t} &= b_{it} + n_{it}\Big(S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t}\Delta t,\ \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}}\Big)^\top, & (2.9) \end{aligned}$$

and the collateral constraint:

$$W_{i,t+\Delta t} \geq 0, \tag{2.10}$$

[2] Assuming that the less risk averse investor is more optimistic makes our main state variable $s_t = c^*_{At}/D_t$ (introduced in Section 2.3 below) countercyclical, which facilitates the analysis of the results. If this assumption is relaxed, the qualitative results remain the same, but additional analysis is required to determine whether the state variable s is counter- or pro-cyclical.

where $W_{i,t+\Delta t}$ is the financial wealth at date $t + \Delta t$ given by equation (2.9). Constraint (2.10) requires investors to cross-collateralize their positions in such a way that losses on one position are always offset by gains on the other positions. As a result, default is prevented even in the worst-case scenario, similar to Geanakoplos (2003, 2009).

Remark 1 (Partially pledgeable labor income). Our model can be easily extended to economies where fraction $k_i \in [0, 1]$ of investor i's labor income can be pledged. The requirement to keep next-period pledgeable wealth is then given by:

$$W_{i,t+\Delta t} + \underbrace{\frac{k_i l_i}{1 - l_A - l_B}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big)}_{\text{measure of pledgeable labor income}} \geq 0. \tag{2.11}$$

The second term in constraint (2.11) measures the value of the pledgeable income. Let $k_i l_i D_t \Delta t$ be the pledgeable income of investor i. This income is proportional to stock dividends $(1 - l_A - l_B)D_t\Delta t$, and hence, can be replicated by a portfolio of $\widehat{n}_i = k_i l_i/(1-l_A-l_B)$ units of stock with cum-dividend value $\widehat{n}_i(S_t+(1-l_A-l_B)D_t\Delta t)$. The investors can circumvent the non-tradability of pledgeable income by shorting stocks against this income. Hence, the claims to pledgeable income are, effectively, tradable and have the same value as the replicating portfolio. The requirement to have positive pledgeable wealth then becomes $W_{i,t+\Delta t} + \widehat{n}_i\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big) \geq 0$, which is equivalent to constraint (2.11). Lemma A.1 in the Appendix shows that models with $k_i \neq 0$ reduce to models with $k_i = 0$ by a change of variable. Hence, the economic implications of our baseline model with constraint (2.10) and the model with a more general constraint (2.11) are the same.

2.2.3 Equilibrium

Definition. *An equilibrium is a set of asset prices $\{B_t, S_t, P_t\}$ and of consumption and portfolio policies $\{c^*_{it}, b^*_{it}, n^*_{it}\}_{i\in\{A,B\}}$ that solve optimization problem (2.7) for each investor, given processes $\{B_t, S_t, P_t\}$, and consumption and securities markets clear:*

$$c^*_{At} + c^*_{Bt} = D_t, \quad b^*_{At} + b^*_{Bt} = 0, \quad n^*_{A,St} + n^*_{B,St} = 1, \quad n^*_{A,Pt} + n^*_{B,Pt} = 0. \tag{2.12}$$

In addition to asset prices, we derive price-dividend and wealth-output ratios $\Psi = S/\Big((1 - l_A - l_B)D\Big)$ and $\Phi_i = W^*_i/D$, respectively. We also derive annualized

Δt-period riskless interest rates r_t, stock mean-returns μ_t and volatilities σ_t in normal times, and the percentage change of the stock price in the crisis state, denoted by J_t.

We derive the equilibrium in terms of state variable υ_t given by the log-ratio of marginal utilities of investors evaluated at their shares of the aggregate consumption c^*_{it}/D_t:

$$\upsilon_t = \ln\left(\frac{(c^*_{At}/D_t)^{-\gamma_A}}{(c^*_{Bt}/D_t)^{-\gamma_B}}\right). \tag{2.13}$$

Substituting consumption shares of investors A and B, denoted by $s_t = c^*_{At}/D_t$ and $1 - s_t = c^*_{Bt}/D_t$, into equation (2.13), we express υ_t as a function of s_t:

$$\upsilon_t = \gamma_B \ln(1 - s_t) - \gamma_A \ln(s_t). \tag{2.14}$$

Variable υ_t is a decreasing function s_t, and hence, s_t is an alternative state variable.

We assume that the exogenous model parameters are such that

$$\mathbb{E}^i\left[e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_i}\right] < 1, \quad i = A, B. \tag{2.15}$$

Condition (2.15) is necessary and sufficient for the existence of equilibrium in homogeneous-agent economies populated only by investor A or investor B.

2.3. Characterization of equilibrium

First, we derive the investors' state price densities (SPD) ξ_{At} and ξ_{Bt} defined as processes such that asset prices can be expressed as follows (e.g., Duffie (2001, p.23)):

$$B_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big], \tag{2.16}$$

$$S_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\Big(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t}\Delta t\Big)\Big], \tag{2.17}$$

$$P_t = \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}\mathbf{1}_{\{\omega_{t+\Delta t}=\omega_3\}}\Big], \tag{2.18}$$

where $i = A, B$. The state price density ξ_{it} exists for each investor i due to the absence of arbitrage opportunities in our economy.[3] The investors can eliminate arbitrage because strategies with zero investment and non-negative payoffs are feasible given constraints (2.8)–(2.10). The SPDs ξ_{At} and ξ_{Bt} differ due to heterogeneity in beliefs and are linked by the *change of measure equation*[4]

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = \frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{\pi_A(\omega_{t+\Delta t})}{\pi_B(\omega_{t+\Delta t})}. \tag{2.19}$$

We find the SPDs from the first order conditions in terms of investors' marginal utilities of consumption and Lagrange multipliers for collateral requirements (2.10). First, we rewrite the budget equations (2.8)–(2.9) in a static form that expresses the current wealth in terms of current consumption and the expected discounted future wealth (e.g., Cox and Huang, 1989). Then, we solve investor optimizations by dynamic programming and the method of Lagrange multipliers. Lemma 1 below reports the results.

Lemma 1 (Dynamic programming and the first order condition).

1) *Let $V_i(W_{it}, v_t; l_i)$ denote the value function of investor i, where v_t is the state variable. Then, the value function solves the following equation of dynamic programminng:*

$$V_i(W_{it}, v_t; l_i) = \max_{c_{it}} \left\{ u_i(c_{it})\Delta t + e^{-\rho \Delta t} \mathbb{E}_t^i [V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)] \right\}, \tag{2.20}$$

subject to the static budget and collateral constraints:

$$W_{it} + l_i D_t \Delta t = c_{it}\Delta t + \mathbb{E}_t^i \left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t} \right], \tag{2.21}$$

$$W_{i,t+\Delta t} \geq 0. \tag{2.22}$$

2) *Value function $V_i(W_{it}, v_t; l_i)$ is a concave function of wealth W_{it}.*

3) *The SPDs ξ_{it} and optimal consumptions c_{it}^* satisfy the first order conditions*

$$\frac{\xi_{i,t+\Delta t}}{\xi_{it}} = e^{-\rho \Delta t} \frac{(c_{i,t+\Delta t}^*)^{-\gamma_i} + \ell_{i,t+\Delta t}}{(c_{it}^*)^{-\gamma_i}}, \tag{2.23}$$

[3]The proof of existence of the SPD in arbitrage-free economies can be found in Duffie (2001, p.4).

[4]Three equations (2.16)–(2.18) can be rewritten as equations for three unknowns $\pi_i(\omega_k)\xi_{i,t+\Delta t}(\omega_k)/\xi_{it}$, where $k = 1, 2, 3$ and i is set to either A or B. The solution of these equations is unique when the matrix of asset payoffs is invertible, and hence, $\pi_B(\omega_{t+\Delta t})\xi_{B,t+\Delta t}/\xi_{Bt} = \pi_A(\omega_{t+\Delta t})\xi_{A,t+\Delta t}/\xi_{At}$ for all states.

where $\ell_{i,t+\Delta t} \geq 0$ is the Lagrange multiplier for collateral requirement (2.22) *satisfying the complementary slackness condition $\ell_{i,t+\Delta t} W^*_{i,t+\Delta t} = 0$.*

We use Lemma 1 to derive the dynamics of state variable v_t. First, suppose constraints do not bind. In this case, Lagrange multipliers $\ell_{i,t+\Delta t}$ vanish and the first order conditions (2.23) are the same as in the unconstrained economy. The dynamics of the state variable v_t in the unconstrained region of the state-space is then the same as in the unconstrained economy, and is found in closed form below. Next, let $\overline{v}$ and $\underline{v}$ be the values of the state variable v_t when constraints (2.10) of investors A and B bind, respectively. We show that state variable v_t stays within boundaries $\underline{v} \leq v_t \leq \overline{v}$. Intuitively, binding collateral constraints restrict the investors' losses of wealth and consumption, which traps the state variable in the interval $[\underline{v}, \overline{v}]$. The boundaries $\overline{v}$ and $\underline{v}$ are found from the condition that the constraints bind: $W_{i,t+\Delta t} = 0$. Dividing these constraints by $D_{t+\Delta t}$, we obtain equations

$$\Phi_{A}(\overline{v}) = 0, \quad \Phi_{B}(\underline{v}) = 0, \tag{2.24}$$

where $\Phi_i(v_t)$ are wealth-output ratios given by equations (A18) and (A19) in the Appendix. Proposition 1 below reports the dynamics of v_t.

Proposition 1 (Closed-form dynamics of state variable v_t).

Given the boundaries $\overline{v}$ and $\underline{v}$, the equilibrium dynamics of state variable v_t is given by:

$$v_{t+\Delta t} = \max\Big\{\underline{v}; \min\{\overline{v}; v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t\}\Big\}, \tag{2.25}$$

where drift μ_v, volatility σ_v, and jump J_v are given in closed form by:

$$\mu_v = \frac{1}{2\Delta t}\left((\gamma_A - \gamma_B)\ln[(1+\mu_D \Delta t)^2 - \sigma_D^2 \Delta t] + \ln\left(\frac{1-\lambda_B \Delta t}{1-\lambda \Delta t}\right)^2 + \ln(1-\delta^2 \Delta t)\right), \tag{2.26}$$

$$\sigma_v = \frac{1}{2\sqrt{\Delta t}}\left((\gamma_A - \gamma_B)\ln\left(\frac{1+\mu_D \Delta t + \sigma_D\sqrt{\Delta t}}{1+\mu_D \Delta t - \sigma_D\sqrt{\Delta t}}\right) + \ln\left(\frac{1+\delta\sqrt{\Delta t}}{1-\delta\sqrt{\Delta t}}\right)\right), \tag{2.27}$$

$$J_v = (\gamma_A - \gamma_B)\ln(1+\mu_D \Delta t + J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right) - \mu_v \Delta t. \tag{2.28}$$

Boundaries $\overline{v}$ and $\underline{v}$ are reflecting when Δt is sufficiently small; that is, v_t does not stay at the boundaries forever: $\mathrm{Prob}(\overline{v} > v_{t+\Delta t} | v_t = \overline{v}) > 0$ *and* $\mathrm{Prob}(v_{t+\Delta t} > \underline{v} | v_t = \underline{v}) > 0$.

Equation (2.25) reveals the exact structure of the state variable and sheds light on the equilibrium effects of the collateral requirement. The equation demonstrates that the constraint does not alter the dynamics of the state variable when the constraint does not bind, and all its effects are due to imposing bounds on process v_t. This property of state variable v_t plays important role in establishing the clustering of volatilities and other results in Section 4 below, and it is difficult to see using numerical methods instead of a closed-form dynamics.

Proposition B.1 in technical appendix B proves the existence of time-independent bounds $\overline{v}$ and $\underline{v}$ satisfying equations (2.24). The intuition for the existence of these bounds is as follows. Suppose, for example, that bound $\overline{v}$ does not exist. Then, equation (2.14) for the consumption share $s(v_t)$ of investor A implies that $s(v_t) \approx 0$ when v_t is sufficiently large, and hence, investor A's consumption net of labor income $(s(v_t) - l_A)D_t$ can be negative for a long period. As a result, investor A's wealth is negative for a sufficiently large v_t because this wealth equals the present value of net consumption. However, negative wealth contradicts the constraint $W_{A,t+\Delta t} \geq 0$, and hence $\overline{v}$ exists.

The closed-form dynamics (2.25) helps us build a theory of collateral constraints. In particular, we use this dynamics to prove the existence of equilibrium and stationarity of equilibrium processes, derive asset prices, and to study the effects of collateralization on asset prices. Proposition 2 below reports the SPD and the stock price.

Proposition 2 (State price density and the effects on asset prices).

1) *The state price density under the beliefs of investor A is given by:*

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} \exp\Big(\max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}\Big), \tag{2.29}$$

where investor A's time-t consumption share $s(v_t)$ solves equation (2.14).

2) *The price-dividend ratio $\Psi(v_t)$ is uniformly bounded, and the stock price S_t is given by*

$$S_t = (1 - l_A - l_B)D_t\mathbb{E}_t^A\Big[\sum_{\tau=t+\Delta t}^{+\infty}\frac{\xi_{A\tau}}{\xi_{At}}\frac{D_\tau}{D_t}\Big], \tag{2.30}$$

the prices of the bond and the insurance contract are given by $B_t = \mathbb{E}_t^A[\xi_{A,t+\Delta t}]/\xi_{At}$ and $P_t = \mathbb{E}_t^A[\xi_{A,t+\Delta t}1_{\{\omega_{t+\Delta t}=\omega_3\}}]/\xi_{At}$, respectively.

3) *The prices of bond, stock, and insurance contract are higher in the economy with collateral constraints than in the frictionless economy, conditional on two economies having the same current output D_t and the state variable v_t.*

Equation (2.29) captures the effect of collateralization on the SPD in our economy. It shows that the change in the SPD, $\xi_{t+\Delta t}/\xi_t$, can be decomposed into two terms. The first term, $e^{-\rho\Delta t}(s(v_{t+\Delta t})D_{t+\Delta t})^{-\gamma_A}/(s(v_t)D_t)^{-\gamma_A}$, given by the ratio of marginal utilities of investor A at times $t+\Delta t$ and t, is the change in SPD in the frictionless economy. The second term captures the effect of the friction on the SPD, and is only activated when the constraint of investor A is binding. An equivalent representation of SPD can be obtained in terms of the marginal utilities of investor B.

Proposition 2 demonstrates that imposing collateral requirements inflates asset prices. This is because the SPD in the constrained economy exceeds its counterpart in the frictionless economy due to the positive Lagrange multiplier $\ell_{i,t+\Delta t}$ in the first order condition (2.23). This result is in contrast to the effects of borrowing, margin, and restricted participation constraints in the related literature (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014), which increase or decrease the stock prices depending on the investors' elasticities of intertemporal substitution. Moreover, this literature evaluates the effects of frictions numerically, whereas we provide rigorous proofs aided by the closed-form dynamics of the state variable (2.25) and the SPD (2.29). We discuss the intuition and further economic differences between our constraint and the constraints in the literature in Section 4.1.

Proposition B.2 in the technical Appendix B provides the verification theorem for the optimality of investors' optimal strategies, and is not reported here for brevity. In particular, this proposition shows that in the economy where the state price density is given by equation (2.29) the dynamic programming problem (2.20)–(2.22) has a unique solution V_{it} and this solution is the indirect utility function of investor i.

2.3.1 Closed-form solution in a continuous-time limit

Next, we take continuous-time limit $\Delta t \to 0$ and derive the equilibrium in closed form. Taking the limit allows rewriting equations (A30) and (A31) for the price-dividend and wealth-consumption ratios, Ψ_t and Φ_{it}, as differential-difference equations. For tractability, we derive ratios Ψ_t and Φ_{it} in terms of a transformed ratio $\widehat{\Psi}(v;\theta)$, which satisfies a simpler equation reported in Lemma 2 below.

Lemma 2 (Differential-difference equation). *In the limit $\Delta t \to 0$, the price-dividend ratio Ψ and wealth-output ratios Φ_i are given by:*

$$\Psi(v) = \widehat{\Psi}(v; -\gamma_A) s(v)^{\gamma_A}, \tag{2.31}$$

$$\Phi_i(v) = \Big((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v; 1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v; -\gamma_A) \Big) s(v)^{\gamma_A}, \tag{2.32}$$

where $s(v)$ solves equation (2.14) *and $\widehat{\Psi}(v;\theta)$ satisfies a differential-difference equation*

$$\begin{aligned} \frac{\widehat{\sigma}_v^2}{2}\widehat{\Psi}''(v;\theta) &+ \Big(\widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v\Big)\widehat{\Psi}'(v;\theta) \\ &- \left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\widehat{\Psi}(v;\theta) \\ &+ \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v}; v + \widehat{J}_v\};\theta\Big) + s(v)^\theta = 0, \end{aligned} \tag{2.33}$$

subject to the reflecting boundary conditions

$$\widehat{\Psi}'(\underline{v};\theta) = 0, \quad \widehat{\Psi}'(\overline{v};\theta) - \widehat{\Psi}(\overline{v};\theta) = 0, \tag{2.34}$$

where $\widehat{\mu}_v$, $\widehat{\sigma}_v \geq 0$, and $\widehat{J}_v \leq 0$ are constants given by:

$$\widehat{\mu}_v = (\gamma_A - \gamma_B)\left(\mu_D - \frac{\sigma_D^2}{2}\right) + \lambda - \lambda_B - \frac{\delta^2}{2}, \tag{2.35}$$

$$\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta, \tag{2.36}$$

$$\widehat{J}_v = (\gamma_A - \gamma_B)\ln(1+J_D) + \ln\left(\frac{\lambda_B}{\lambda}\right). \tag{2.37}$$

The boundaries $\overline{v}$ and $\underline{v}$ solve the following equations:

$$\frac{\widehat{\Psi}(\overline{v}; 1-\gamma_A)}{\widehat{\Psi}(\overline{v}; -\gamma_A)} = l_A, \quad \frac{\widehat{\Psi}(\underline{v}; 1-\gamma_A)}{\widehat{\Psi}(\underline{v}; -\gamma_A)} = 1 - l_B. \tag{2.38}$$

We observe that equation (2.33) is linear, in contrast to economies with constraints directly imposed on trading strategies of investors (e.g., Gârleanu and Pedersen, 2012; Chabakauri, 2013, 2015; Rytchkov, 2014). This equation is a differential-difference equation with a "delayed" argument in the fourth term on the left-hand side of the equation because $\widehat{J}_v \leq 0$. This term is further complicated by the fact that the delayed argument is restricted to stay above the lower boundary $\underline{v}$, which gives rise to the dependence of the fourth term on a peculiar argument $\max\{\underline{v}; v + \widehat{J}_v\}$. This term captures investors' decisions in anticipation of hitting their collateral constraint.

Before deriving the equilibrium in the general case, in Corollary 1 below, we provide analytical price-dividend ratios when there is no crisis and investors have log preferences.

Corollary 1 (Analytical asset prices in a special case). *Suppose, investors A and B have logarithmic preferences and there is no production crisis, that is,* $\lambda = \lambda_B = 0$. *Then, price-dividend ratio* $\Psi(v)$ *is given by:*

$$\Psi(v) = \frac{1}{\rho} + \frac{C_1 e^{\varphi_+ v} + C_2 e^{\varphi_- v}}{1 + e^v}, \tag{2.39}$$

where $\varphi_\pm = 0.5(1 \pm \sqrt{1 + 8\rho/\delta^2})$, *and constants* C_1 *and* C_2 *are given by equations* (A44) *and* (A45) *in the Appendix, respectively.*

In Section 4 below, we argue that the analytical price-dividend ratio (2.39) captures some important properties of price-dividend ratio that hold in the general case with arbitrary risk aversions and crises. Hence, this special case can be used as a tractable benchmark in asset pricing research. Nevertheless, we undertake a comprehensive investigation of equilibrium in the general case.

Proposition B.3 in Appendix B presents the closed-form price-dividend ratio for general CRRA risk aversions and beliefs. These solutions are in terms of exogenous model parameters and do not require solving equation (2.33). Although the closed-form solution in Proposition B.3 is complex, it provides a constructive proof for the existence of price-dividend ratios and helps avoid numerical methods based on value function iterations (e.g., Krusell and Smith, 1998), widely used in the literature with market frictions, for which the convergence results, in general, are not available. We double-checked the solution reported in Proposition B.3 by solving problem (2.33)–(2.34) using the method of finite differences.

We call the interval $v \in [\underline{v}, \underline{v} - \widehat{J}_v]$ in the state-space a period of *anxious economy*, similar to Fostel and Geanakoplos (2008).[5] When the economy falls into this state,

[5]However, in contrast to Fostel and Geanakoplos (2008), the disagreement about the consump-

even a small possibility of a crisis renders the collateral requirement binding and leads to deleveraging in the economy. To explore the economic effects of the anxious economy, we provide closed-form expressions for the interest rates r_t and risk premia in normal times $\mu_t - r_t$, which can be easily obtained using previously derived equations for asset prices and the state price density. Proposition 3 below reports the results.

Proposition 3 (Interest rates and risk premia in the limit). *For a sufficiently small interval Δt, the interest rate r_t and the risk premium $\mu_t - r_t$ in normal times are given by:*

$$r_t = \begin{cases} \tilde{r}_t \;-\; \lambda(1+J_D)^{-\gamma_A}\left(\dfrac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t}\right)^{-\gamma_A} + O(\Delta t), \text{ for } \underline{v} < v_t < \overline{v}, \\[2ex] \dfrac{(1-s_t)\Gamma_t\Big(\mathbf{1}_{\{v=\underline{v}\}} - \mathbf{1}_{\{v=\overline{v}\}}\Big) - \gamma_B}{2\gamma_B\sqrt{\Delta t}}\widehat{\sigma}_v + O(1), \text{ for } v = \underline{v} \text{ or } v = \overline{v}, \end{cases} \tag{2.40}$$

$$\begin{aligned} \mu_t \;-\; r_t = {} & \left(\gamma_A\sigma_D - \frac{(1-s_t)\Gamma_t\widehat{\sigma}_v}{\gamma_B} + \frac{(1-s_t)\Gamma_t\widehat{\sigma}_v(\mathbf{1}_{\{v=\underline{v}\}} + \mathbf{1}_{\{v=\overline{v}\}}) - \gamma_B\widehat{\sigma}_v\mathbf{1}_{\{v=\overline{v}\}}}{2\gamma_B}\right)\sigma_t \\ & - \lambda(1+J_D)^{-\gamma_A}J_t\left(\frac{s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)}{s_t}\right)^{-\gamma_A} + O(\sqrt{\Delta t}), \end{aligned} \tag{2.41}$$

where $\tilde{r}_t$ is the interest rate in the unconstrained economy without crisis risk, given by:

$$\begin{aligned} \tilde{r}_t = \lambda \;&+\; \rho + \gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2 + \left(\frac{\gamma_A\sigma_D\widehat{\sigma}_v - \widehat{\mu}_v}{\gamma_B}\right)(1-s_t)\Gamma_t \\ &-\; \widehat{\sigma}_v^2\left(\frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3\right), \end{aligned} \tag{2.42}$$

drift $\widehat{\mu}_v$, volatility $\widehat{\sigma}_v$, and $\widehat{J}_v$ of the state variable v are given by equations (2.35)–(2.37), *volatility σ_t and jump size J_t are given by equations* (B27)–(B28), *respectively, and $\Gamma_t \equiv \gamma_A\gamma_B/\Big(\gamma_A(1-s_t) + \gamma_B s_t\Big)$ is the risk aversion of a representative investor.*

tion growth dynamics in our economy does not increase during these periods.

The effects of collateral requirements on interest rates and risk premia arise due to the investors' concern that a potential crisis may render the constraint binding next period when the economy is close to boundary $\underline{v}$. The last term in the first equation in (2.40) for the interest rate quantifies the impact of collateral requirements on precautionary savings due to a downward jump in the aggregate consumption, which we further discuss in Section 4.

Equations (2.40) and (2.41) also feature terms with indicator functions $\mathbf{1}_{\{v=\underline{v}\}}$ and $\mathbf{1}_{\{v=\overline{v}\}}$, which are non-zero only at the boundaries $\underline{v}$ and $\overline{v}$. For the interest rate r_t these terms have the order of magnitude proportional to $1/\sqrt{\Delta t}$, and hence, the interest rate has singularities at the boundaries $\underline{v}$ and $\overline{v}$ when $\Delta t \to 0$. Similar singularities arise in a continuous-time model of Detemple and Serrat (2003). Our discrete-time analysis sheds new light on these singularities by uncovering their order of magnitude $1/\sqrt{\Delta t}$. Consequently, the per-period rate $r_t \Delta t$ is finite and has an order of magnitude $O(\sqrt{\Delta t})$.

The intuition for the singularity is that near the boundaries $\underline{v}$ and $\overline{v}$ even a small shock Δw_t may lead to a default. Consequently, when the collateral requirement of an investor binds at time t, the investor allocates a larger fraction of labor income to bond than in the interior region $\underline{v} < v_t < \overline{v}$ and requires a higher risk premium. Therefore, the interest rate decreases and Sharpe ratio increases at the boundaries.

The interest rates and Sharpe ratios in a similar model have been studied in Detemple and Serrat (2003). However, our model is more general in that it incorporates jumps and heterogeneity of preferences. Moreover, their paper does not study price-dividend ratios and stock return volatilities, which are significantly more difficult to obtain, as can be seen from the price-dividend ratios reported in Lemma 2 above and Proposition 3.B in the technical Appendix B. We also note that the interest rate (2.40) and the risk-premium (2.41) are significantly different from those in the related literature on borrowing and margin constraints (e.g., Chabakauri, 2013, 2015; Rytchkov, 2014). The main difference is that these quantities in the related literature feature additional Lagrange multipliers for the constraints that bind in an interval of a state space and do not have singularities.

2.3.2 Stationary distribution of consumption share

Absent any frictions, state variable v follows an arithmetic Brownian motion with a jump. This process is non-stationary and induces non-stationarity of the unconstrained equilibrium where one of the investor's share of consumption gradually converges to zero. Hence, with the exception of some knife-edge parameter combinations, only one of the investors has a significant impact on asset prices in the frictionless economy in the long run (e.g., Blume and Easley, 2001; Yan, 2008; Chabakauri, 2015).

It is intuitive that imposing collateral requirements (2.10) helps both investors survive and have an impact on equilibrium in the long-run because these constraints protect investors against losing their shares of aggregate consumption beyond certain limits. This intuition for the survival of of investors in economies with market imperfections has also been discussed in the previous literature (e.g., Blume and Easley, 2001; Cao, 2018, among others). However, this intuition does not reveal the shape of the distribution of consumption share s, whether this distribution is well-defined or degenerate (e.g., fully concentrated at boundaries $\underline{s}$ or $\overline{s}$), and which parameters determine the relative dominance of investors in the economy. Our main contribution in this section is that armed with the closed-form dynamics of the state variable v_t in (2.25), we derive the probability density function (PDF) of consumption share s in closed form, show that this PDF is stationary and non-degenerate, and find parameters that determine its shape. The latter result is important because it implies non-trivial time-variation of asset prices in the long run. For simplicity, we assume that there is no crisis risk so that $\lambda = \lambda_B = 0$. Proposition 4 reports the results.

Proposition 4 (Stationary distribution of consumption share). *Suppose, $\lambda = \lambda_B = 0$. Then, the PDF $f(s, \tau; s_t; \tau)$ of consumption share s at time τ conditional on observing share s_t at time t is given in closed form by expression* (A65) *in the Appendix. Furthermore, the stationary PDF of consumption share s is given by:*

$$f(s) = \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\frac{\gamma_A}{s} + \frac{\gamma_B}{1-s}\right)\frac{\Big((1-s)^{\gamma_B}/s^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}{\Big((1-\underline{s})^{\gamma_B}/\underline{s}^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2} - \Big((1-\overline{s})^{\gamma_B}/\overline{s}^{\gamma_A}\Big)^{2\widehat{\mu}_v/\widehat{\sigma}_v^2}}\mathbf{1}_{\{\underline{s}\le s\le\overline{s}\}}, \tag{2.43}$$

where $\widehat{\mu}_v = (\gamma_A - \gamma_B)(\mu_D - \sigma_D^2/2) - \delta^2/2$, $\widehat{\sigma}_v = (\gamma_A - \gamma_B)\sigma_D + \delta$, $\mathbf{1}_{\{\underline{s} \le s \le \overline{s}\}}$ *is an indicator function and* $\underline{s}$ *and* $\overline{s}$ *are the bounds on the consumption share* s*, which solve equation* (2.24) *for* $\overline{v}$ *and* $\underline{v}$*, respectively.*

Proposition 4 confirms that both investors survive in the long run, and that consumption share s has well-defined stationary distribution. The beliefs enter PDF (2.43) via the ratio of the drift and variance of process v_t, given by $\widehat{\mu}_v/\widehat{\sigma}_v^2$. This ratio determines the relative dominance of investors in the economy. In particular, for bounds $\underline{s}$ and $\overline{s}$ that are symmetric around 0.5 the PDF is concentrated around $\underline{s}$ if $\widehat{\mu}_v > 0$ and around $\overline{s}$ if $\widehat{\mu}_v < 0$.

Figure 2.2 plots the stationary PDF (2.43) and transition densities $f(s, t; s_0, 0)$, for parameters described in the legend and explained in Section 4 below. The stationary PDF has a larger mass around $s = 0.1$ because the labor share $l_B = 0.14$ of investor B exceeds the labor share $l_A = 0.123$ of investor A in this example in order to get boundary values $\underline{s} = 0.1$ and $\overline{s} = 0.9$ symmetric around 0.5. From Figure 2.2 we observe that both rational and irrational investors can occasionally have large consumption shares.

Another notable feature of PDF (2.43) is that it is bimodal, with a large mass of the distribution concentrated around boundaries $\underline{s}$ and $\overline{s}$. The economic implication of this bimodality is that the periods of binding constraints are likely to be persistent. The closed-form dynamics (2.25) for the state variable v helps explain the bimodality of the PDF. From this dynamics, we observe that after hitting a boundary the process v_t remains in its vicinity for some time. Hence, because variable v follows an arithmetic Brownian motion in the interval $(\underline{v}, \overline{v})$, the probability of hitting the same boundary again is high.

Proposition 4 implies that the PDF of consumption share s is always stationary when investors have positive labor incomes $l_B > 0$ and $l_A > 0$ because in this case $1 > \overline{s} > \underline{s} > 0$, and hence, the PDF (2.43) is well-defined. The PDF of s is also stationary when $l_A = 0$, $l_B > 0$, and $\widehat{\mu}_v < 0$, or $l_A > 0$, $l_B = 0$, and $\widehat{\mu}_v > 0$. In the latter cases, $\underline{s} = 0$ or $\overline{s} = 1$, respectively. Then, we observe that the PDF (2.43) is well-defined when $\underline{s} = 0$ and $\widehat{\mu}_v < 0$, and when $\overline{s} = 1$ and $\widehat{\mu}_v > 0$, and hence is stationary. The PDF of s is non-stationary when $l_A = 0$ and $l_B = 0$, and is derived in closed form in Chabakauri (2015). In the latter case only investor A survives if

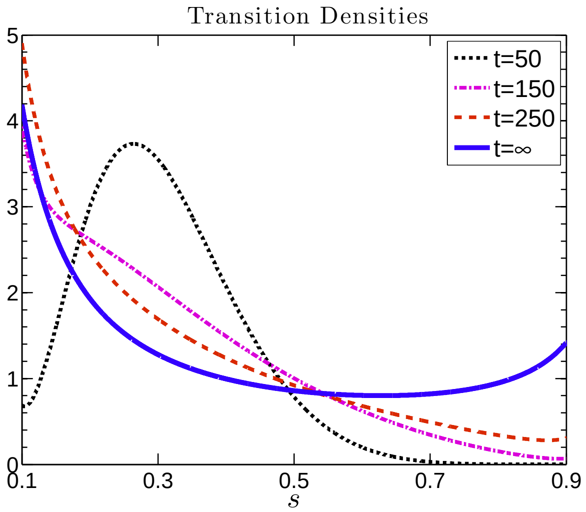


Figure 2.2

Convergence to stationary distribution of consumption share $s_t = c^*_{A,t}/D_t$

The Figure shows transition densities $f(s,t;s_0,0)$ for the starting point $s_0 = 0.2$ and the stationary distribution $f(s)$ (i.e., density for $t = \infty$). We set $\gamma_A = 2$, $\gamma_B = 1.5$, $\mu_D = 0.018$, $\sigma_D = 0.032$, $\lambda = \lambda_B = 0$, $\rho = 0.02$, $\delta = 0.1125$, $\underline{s} = 0.1$, $\overline{s} = 0.9$, $l_A = 0.123$, and $l_B = 0.14$.

$\widehat{\mu}_v < 0$, and only investor B survives when $\widehat{\mu}_v > 0$.

2.4. Analysis of Equilibrium

In this section, we demonstrate the economic implications of our model. In Section 4.1, we show that capital requirements amplify the effect of rare crises on generating lower interest rates and higher Sharpe ratios, lead to spikes and crashes of stock prices and stock return volatilities, amplify volatility in good times and decrease it in bad times, and generate volatility clusters. Section 4.2 measures the economic significance of collateralization by quantifying the collateral premium of the stock.

We study the equilibrium for calibrated parameters. We set the parameters of the aggregate consumption process to $\mu_D = 0.018$, $\sigma_D = 0.032$, $J_D = -0.2$, and the crisis intensities of investors A and B to $\lambda = 0.017$ and $\lambda_B = 0.01$, respectively.[6] The risk aversions are $\gamma_A = 2$ and $\gamma_B = 1.5$, and the time discount is $\rho = 0.02$. The disagreement parameter is $\delta = 0.1125$, which corresponds to the mean growth rate (2.6) under investor B's beliefs equal to $1.2\mu_D$. The shares of labor income

[6]Drift μ_D and volatility σ_D are within the ranges considered in the literature (e.g., Basak and Cuoco, 1998; Chan and Kogan, 2002; Rytchkov, 2014), intensity $\lambda = 0.017$ is from Barro (2009).

$l_A = 0.123$ and $l_B = 0.14$ are chosen to generate symmetric bounds on investor A's consumption share: $\underline{s} = 0.1$ and $\overline{s} = 0.9$.[7]

We plot the equilibrium distributions and processes as functions of consumption share $s_t = c^*_{At}/D_t$ because s lies in the interval $[0, 1]$ and is more intuitive than variable v. We observe that consumption share s is countercyclical in the sense that $\mathrm{corr}_t(ds_t, dD_t) < 0$. Intuitively, the aggregate wealth and consumption shift to (away from) investor A following negative (positive) shocks to output because this investor is more risk averse and pessimistic than investor B. We call a process procyclical (countercyclical) if that process is a decreasing (increasing) function of s. We interpret periods of low (high) s_t as good (bad) times in the economy, because during these periods the output D_t is high (low).

2.4.1 Equilibrium processes

Figure 2.3 depicts investor B's leverage/market ratio L_t/S_t and stock holdings n_{Bt} in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) demonstrates the cyclicality of leverage. The leverage is lowest when either investor A or investor B bind on their constraints. Intuitively, when $s = \overline{s}$, investor B's financial wealth is zero, and hence, B lacks collateral and cannot borrow. When $s = \underline{s}$, investor A's financial wealth is zero and the labor income $l_A D_t \Delta t$ is infinitesimally small in the continuous-time limit. The liquidity dries up because investor A cannot supply credit. The leverage cycles are present only in the constrained economy. They do not occur in the unconstrained economy where the state variable s is non-stationary and gradually converges to 0 or 1.

Panel (b) presents the number of stocks held by investor B. Consider first the unconstrained economy where the labor income is pledgeable. From panel (b) we observe that in this economy investor B shorts stocks despite being more optimistic than investor A when consumption share s is close to 1. The intuition is that in bad times, following a sequence of negative shocks to output, investor B shorts

[7] To avoid finding bounds $\underline{s}$ and $\overline{s}$ numerically, we set them exogenously to $\underline{s} = 0.1$ and $\overline{s} = 0.9$ and then recover the shares of labor incomes $l_A = 0.123$ and $l_B = 0.14$ that imply these bounds in equilibrium. First, we find $\underline{v}$ and $\overline{v}$ from equation (2.14) for v, and then find l_A and l_B from equations (2.38).

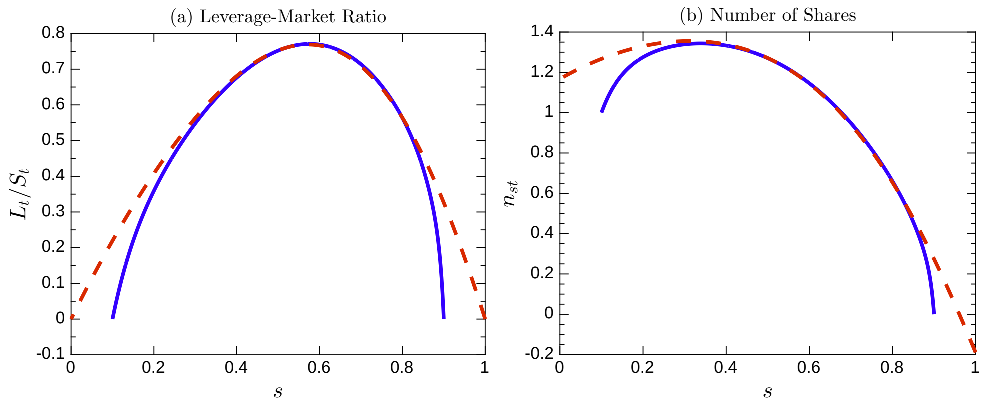


Figure 2.3

Leverage and stock holdings of optimistic and less risk averse investor B

Panels (a) and (b) depict optimistic and less risk averse investor B's leverage/market price ratio L_t/S_t and the number of shares n_{st}, respectively, as functions of consumption share $s_t = c^*_{At}/D_t$. The solid and dashed lines correspond to constrained and unconstrained economies, respectively.

stocks to finance consumption and backs short positions by the pledgeable labor income. The stream of labor income $l_B D_t \Delta t$ is equivalent to dividends from holding $\widehat{n}_B = l_B/(1 - l_A - l_B)$ units of non-tradable shares in the Lucas tree. Short-selling allows the investor to circumvent the non-tradability of labor income and freely adjust the effective share $\hat{n}_B + n_{B,st}$ in the Lucas tree. Overcoming the non-tradability of labor incomes makes this economy similar to the non-stationary unconstrained economy where investors can freely trade shares in the Lucas tree. The financial wealth can then become negative. The collateral requirement imposes non-negative wealth constraint, which precludes investor B from shorting. The trading strategy of investor A equals $1 - n^*_{Bt}$ in equilibrium and can be analyzed similarly. Investor A also has an additional motive to short stocks due to being more pessimistic than investor B.

Figure 2.4 depicts the interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ, and excess stock return volatility $(\sigma_t - \sigma_D)/\sigma_D$ in the constrained (solid line) and unconstrained (dashed line) economies. Panel (a) shows the interest rates r_t.[8] The interest rate declines sharply when the economy enters into an anxious state close to the boundary $\overline{s}$ where even a small possibility of a crisis next period makes

[8]We exclude the singularities in the dynamics of r_t and focus on the dynamics in the unconstrained region because the economy spends an infinitesimal amount of time at the boundaries.

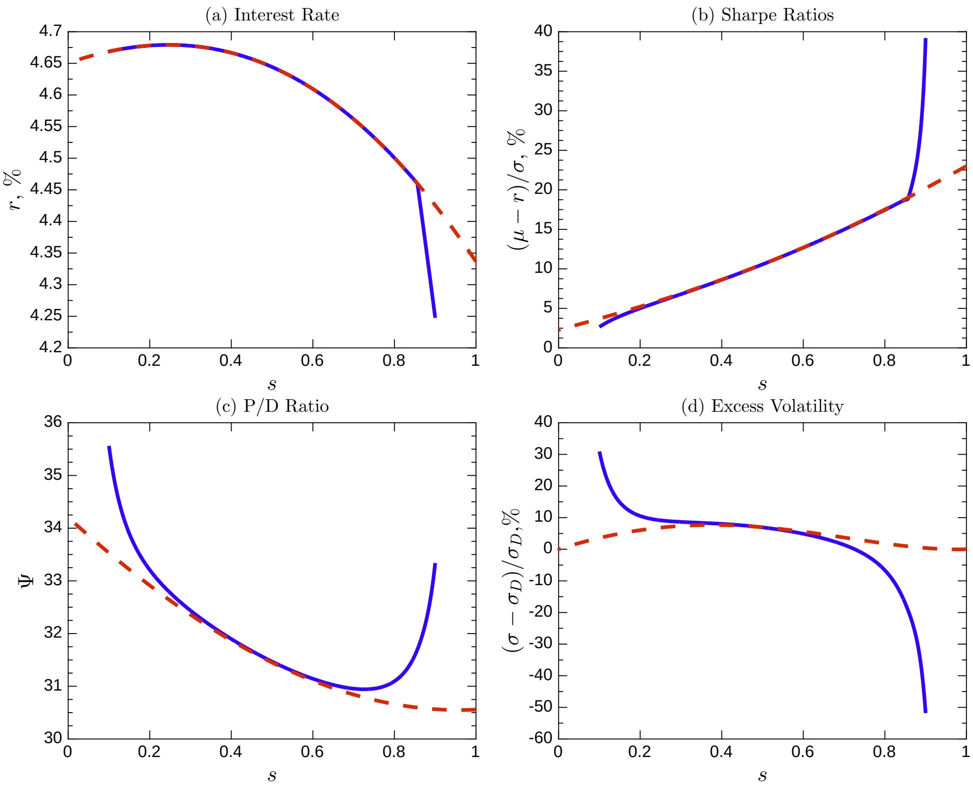


Figure 2.4

Equilibrium processes

Panels (a)–(d) show interest rate r_t, Sharpe ratio $(\mu_t - r_t)/\sigma_t$, price-dividend ratio Ψ_t, and excess volatility $(\sigma_t - \sigma_D)/\sigma_D$ as functions of $s_t = c^*_{At}/D_t$ for the constrained (solid lines) and unconstrained (dashed lines) economies.

the constraint of investor B binding. The intuition is as follows. In the unconstrained economy, a crisis around state $\overline{s}$ generates wealth transfer to the pessimistic and more risk averse investor A and increases her consumption share s above $\overline{s}$. In the constrained economy, consumption share s is capped by $\overline{s}$. Consequently, following a crisis, investor A's marginal utility $(c^*_A)^{-\gamma_A}$ is higher in the constrained than in the unconstrained economy. As a result, investor A is more willing to smooth consumption in the constrained economy, and hence, the interest rate declines due to the precautionary savings motive. In particular, the investor buys more bonds, which drives interest rates down. Panel (b) of Figure 2.4 shows that the Sharpe ratio increases to compensate investor A for buying risky assets from investor B.

Our results on interest rates and Sharpe ratios indicate that the rare crises and collateral requirements reinforce the effects of each other. In particular, the decreases in interest rates and increases in Sharpe ratios during anxious times arise only when both the crises and the constraints (2.10) are simultaneously present. Removing the constraint but keeping the crisis risk increases the interest rate and decreases the Sharpe ratio. Equation (2.40) for the interest rate and equation (2.41) for the risk premium show that removing the crisis risk (i.e., setting $\lambda = \lambda_B$=0) but keeping the constraint leads to r_t and $\mu_t - r_t$ which are the same as in the frictionless economy when $\underline{v} < v_t < \overline{v}$, consistent with findings in Detemple and Serrat (2003). Absent any crises, the constraints affect r_t and $\mu_t - r_t$ only at the boundaries of the state-space, as shown in Section 3.1.

From panel (c), we observe that the collateral requirements give rise to higher price-dividend ratio Ψ than in the unconstrained economy, $\Psi_t^{constr} - \Psi_t^{unc} > 0$, as proven in Proposition 2. The increases in ratio Ψ are larger around the boundaries $\underline{s}$ and $\overline{s}$, which makes ratio Ψ a U-shaped function of s sensitive to small shocks close to boundaries. The U-shape is a robust phenomenon that does not require rare crises or investors that differ both in risk aversions and beliefs. When both investors have identical risk aversions $\gamma_A = \gamma_B = 1$ but different beliefs and there is no crisis risk (i.e., $\lambda_A = \lambda_B = 0$), the U-shape is an analytical result that follows from the closed-form expression (2.39) for ratio Ψ. This ratio remains U-shaped when investors have different risk aversions but identical beliefs.

The intuition for the U-shape is as follows. Suppose, consumption share s is close to the boundary $\overline{s}$, where investor B's constraint is likely to bind but investor A is unconstrained. Because investor A's constraint is loose the state price density ξ_{At} is proportional to investor A's marginal utility $(c^*_{At})^{-\gamma_A}$. In the constrained economy the consumption share of investor A is capped by $\overline{s} < 1$ whereas in the unconstrained economy it can increase above $\overline{s}$. Therefore, the marginal utility of investor A and, hence, the state price density are expected to be higher in the constrained than in the unconstrained economy. Consequently, stocks are more valuable in the constrained economy around the boundary $\overline{s}$. The intuition around $\underline{s}$ can be analyzed in a similar way. An additional economic force contributing to higher stock price is that the stock can be used as collateral that helps relax the constraint, which gives rise to a premium. This force is explored in Section 4.2.

The results on panel (d) demonstrate that the constraint makes volatility more procyclical, reduces volatility in bad times (around $\overline{s}$) and increases it in good times (around $\underline{s}$). This is because U-shaped price-dividend ratio in the constrained economy is more procyclical in good times (i.e., around $\underline{s}$) and more countercyclical in bad times (i.e., around $\overline{s}$) than in the unconstrained economy. Stock price $S_t = \Psi_t D_t$ is more volatile in good times (around $\underline{s}$) because both Ψ and D_t change in the same direction, and is less volatile in bad times (around $\overline{s}$) because Ψ and D_t change in opposite directions and partially offset the effects of each other. Lower volatility in bad times is in line with the previous literature on the effects of portfolio constraints on asset prices (e.g., Chabakauri, 2013, 2015; Brunnermeier and Sannikov, 2014, among others). The empirical literature finds that the volatility tends to be higher in bad times (e.g., Schwert, 1989). However, high volatility can be explained by high uncertainty about the economic growth and learning effects in bad times (e.g., Veronesi, 1999), which we do not study in this paper to focus on the effects of collateral constraints which are not confounded by other effects.

Boundary conditions (2.34) allow us to explore volatility σ_t near the boundaries $\underline{s}$ and $\overline{s}$ using closed form expressions in Corollary 2 below.

Corollary 2 (Stock return volatility at the boundaries). *Stock return volatility in normal times σ_t satisfies the following boundary conditions:*

$$\sigma(\underline{s}) = \sigma_D + \frac{\gamma_B \underline{s} \widehat{\sigma}_v}{\gamma_A(1-\underline{s}) + \gamma_B \underline{s}} > \sigma_D, \quad \sigma(\overline{s}) = \sigma_D - \frac{\gamma_A(1-\overline{s})\widehat{\sigma}_v}{\gamma_A(1-\overline{s}) + \gamma_B \overline{s}} < \sigma_D. \quad (2.44)$$

By continuity, inequalities (2.44) also hold in a vicinity of the boundaries. Panel (d) shows that volatility σ_t is very steep at the boundaries: it spikes close to $\underline{s}$ and crashes close to $\overline{s}$, consistent with Corollary 2. It also evolves in three regimes of low, medium, and high volatility, which resembles volatility clustering documented in the empirical literature (e.g., Bollerslev, 1987). The distribution of consumption share s on Figure 2.2 implies that the economy persists in these clusters for some time.

Figure 2.5 plots the simulated dynamics of P/D ratio and stock return volatility over a period of 50 years. Consistent with our discussion above, the dynamics of P/D ratio on panel (a) exhibits intervals of booms and busts around the times when the collateral requirements become binding. These intervals resemble periods of inflating and deflating bubbles in the economy. The volatility σ on panel (b) evolves

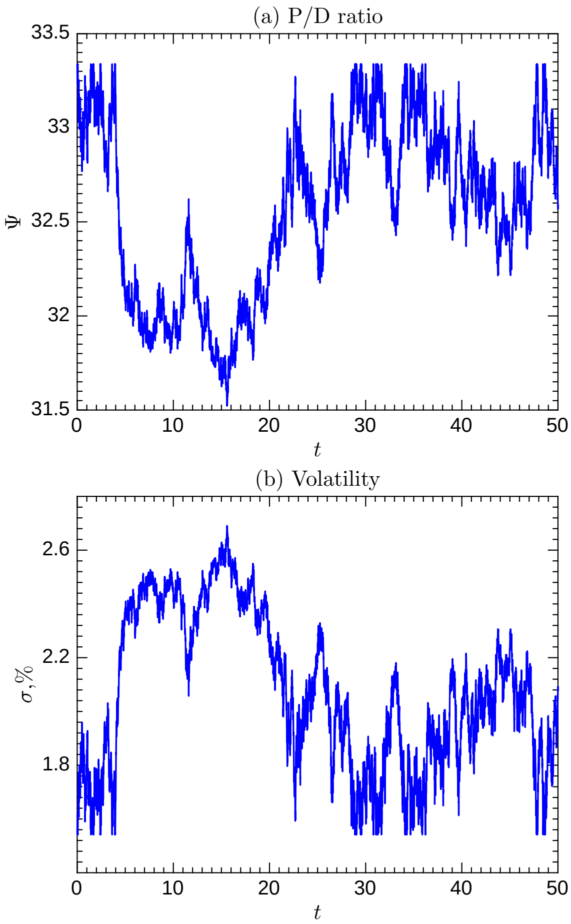


Figure 2.5

Simulated P/D ratio Ψ and stock return volatility σ over time

Panels (a) and (b) show the spikes and crashes of simulated P/D ratio and volatility σ, and clustering of volatility σ over the period of 50 years.

in clusters of high and low volatility, as explained above.

The economic effects of collateral requirements are different from the effects of margin and borrowing constraints in the related literature discussed in the introduction. In particular, those constraints increase or decrease price-dividend ratios and make them pro- or counter-cyclical depending on investors' risk aversions (e.g., Chabakauri, 2015). They also shrink volatility towards the output volatility σ_D by reducing the risk-sharing. The main mechanism in our paper is different, and is driven by the increased marginal utilities due to endogenously arising bounds on the consumption share s. As a result, in our model price-dividend ratios increase irrespective of the beliefs and risk aversions of investors, and the volatility deviates further away from the output volatility (Figure 2.4). Other new effects relative to this literature include the cyclicality of leverage, mutual amplification of effects of rare crises and collateral constraints, and spikes in price-dividend ratios and volatilities.

2.4.2 Collateral liquidity premium

In this section, we measure the liquidity premium of stocks over labor income arising because stocks can be used as collateral. We consider a marginal representative investor i that does not affect asset prices and characterize this investor's shadow indifference price $\widehat{S}_{it}$ of labor income. We define $\widehat{S}_{it}$ as the price such that exchanging marginal Δl_i units of labor income for $\widehat{S}_{it}\Delta l_i$ units of wealth leaves the investor's utility unchanged. Consider the investor's value function $V_i(W_{it}, v_t; l_i)$ satisfying the dynamic programming equation (2.20) subject to constraints (2.21) and (2.22). Price $\widehat{S}_{it}$ is the solution of equation $V_i(W^*_{it}, v_t; l_i) = V_i(W^*_{it} + \widehat{S}_{it}\Delta l_i, v_t; l_i - \Delta l_i)$ when $\Delta l_i \to 0$. In the limit, we find:

$$\widehat{S}_{it} = \frac{\partial V_i(W^*_{it}, v_t; l_i)/\partial l_i}{\partial V_i(W^*_{it}, v_t; l_i)/\partial W_{it}}. \tag{2.45}$$

The definition of shadow indifference price $\widehat{S}_{it}$ comes from the literature on the valuation of derivative securities in incomplete markets (e.g., Davis, 1997).

The labor incomes $l_i D_t \Delta t$ are proportional to dividends $(1-l_A-l_B)D_t\Delta t$. Therefore, if claims on labor incomes were tradable and pledgeable, shadow price $\widehat{S}_{it}$ would have been equal to $S_t/(1-l_A-l_B)$. However, labor incomes are non-tradable and non-pledgeable. Hence, from the view of investor i, the stock enjoys liquidity premium, which we define as

$$\Lambda_{it} = \frac{S_t/(1-l_A-l_B) - \widehat{S}_{it}}{S_t/(1-l_A-l_B)}. \tag{2.46}$$

We find derivatives in equation (2.45) using the envelope theorem. Then, we derive prices $\widehat{S}_{it}$ and show that premia (2.46) are positive and large. Proposition 5 reports our results.

Proposition 5 (Shadow prices and the liquidity premium). *In the limit $\Delta t \to 0$, investor i's shadow price of a unit of labor income is given by:*

$$\widehat{S}_{it} = \widehat{\Psi}_i(v; -\gamma_A)s(v)^{\gamma_A}D_t, \quad i = A, B, \tag{2.47}$$

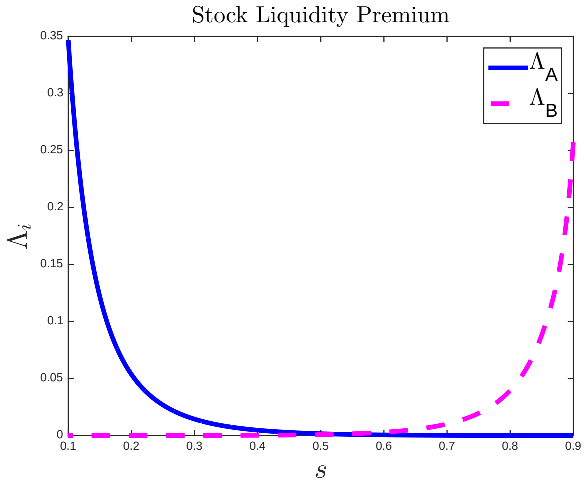


Figure 2.6

Collateral liquidity premia from the view of investors A and B

The Figure shows the collateral liquidity premia (2.46) of stocks over non-pledgeable labor incomes from the view of investors A and B.

where $\widehat{\Psi}_i(v;\theta)$ *satisfies differential-difference equation* (2.33) *subject to the following boundary conditions for investors* A *and* B

$$\widehat{\Psi}'_A(\underline{v};\theta) = 0, \qquad \widehat{\Psi}'_A(\overline{v};\theta) = 0, \tag{2.48}$$

$$\widehat{\Psi}'_B(\underline{v};\theta) = \widehat{\Psi}_B(\underline{v};\theta), \qquad \widehat{\Psi}'_B(\overline{v};\theta) = \widehat{\Psi}_B(\overline{v};\theta). \tag{2.49}$$

The investors' liquidity premia for stocks Λ_A *and* Λ_B *are positive, and hence,*

$$S_t/(1 - l_A - l_B) > \widehat{S}_{At}, \quad S_t/(1 - l_A - l_B) > \widehat{S}_{Bt}. \tag{2.50}$$

The premium $\Lambda_{it} > 0$ arises because the stock can be used as a collateral whereas the labor income cannot. We note that this premium is zero in the unconstrained economy, and hence the non-tradability of labor income and the possibility of shorting stocks do not contribute to the premium. This is because, as discussed in Section 4.1, in an unconstrained economy with fully pledgeable labor income the investors can circumvent the non-tradability of labor income by shorting stocks. We further remark that the shadow prices and liquidity premia can be found in closed form, similar to stock prices in Section 3, but we do not present them for brevity.

Figure 2.6 plots the liquidity premia (2.46) for the same calibrated parameters as in Section 4.1. We observe that investors A and B have different valuations of their labor incomes due to differences in preferences and beliefs. Their premia Λ_i are close to zero when the investors are far away from the boundaries where their respective collateral requirements become binding. The premia increase up to 35% close to the boundaries where the stock is more valuable for the purposes of relaxing the constraints. Large premia Λ_{it} imply the economic significance of stock pledgeability.

2.5. Conclusion

We develop a parsimonious and tractable theory of asset pricing under collateral requirements. We show that requiring investors to collateralize their trades has significant effects on asset prices and their moments. The constraints decrease interest rates and increase Sharpe ratios when optimistic investors are close to default boundaries. They also increase price-dividend ratios, amplify volatilities in good states and dampen them in bad states. Hence, collateral requirements emerge as viable instruments for stabilizing markets in bad times. The tractability of our model allows us to obtain asset prices and the distributions of consumption shares in closed form.

Bibliography

Alvarez, F., and U. J. Jermann, 2000, "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica* 68, 775-797.

Alvarez, F., and U. J. Jermann, 2001, "Quantitative Asset Pricing Implications of Endogenous Solvency Constraints," *Review of Financial Studies* 14, 1117-1151.

Atmaz, A., and S. Basak, 2018, "Belief Dispersion in the Stock Market," *Journal of Finance* 73, 1225-1279.

Back, K., E., 2010, *Asset Pricing and Portfolio Choice Theory*, Oxford University Press.

Barro, R. J., 2009, "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review* 99, 243-264.

Basak, S., 1995, "A General Equilibrium Model of Portfolio Insurance," *Review of Financial Studies* 8, 1059-1090.

Basak, S., 2005, "Asset Pricing with Heterogeneous Beliefs," *Journal of Banking and Finance* 29, 2849-2881.

Basak, S., and B. Croitoru, 2000, "Equilibrium Mispricing in a Capital Market with Portfolio Constraints," *Review of Financial Studies* 13, 715-748.

Basak, S., and B. Croitoru, 2006, "On the Role of Arbitrageurs in Rational Markets," *Journal of Financial Economics* 81, 143-173.

Basak, S., and D. Cuoco, 1998, "An Equilibrium Model with Restricted Stock Market Participation," *Review of Financial Studies* 11, 309-341.

Basak, S., and A. Shapiro, 2001, "Value-at-Risk-Based Management: Optimal Policies and Asset Prices," *Review of Financial Studies* 14, 371-405.

Bhamra, H. S., and R. Uppal, 2014, "Asset Prices with Heterogeneity in Preferences and Beliefs," *Review of Financial Studies* 27, 519-580.

Biais, B., Hombert, J., and P.-O. Weill, 2018, "Incentive Constrained Risk Sharing, Segmentation, and Asset Pricing," Working Paper.

Blume, L. E., T. Cogley, D. Easley, T. J. Sargent, and V. Tsyrennikov, 2018, "A Case for Incomplete Markets," *Journal of Economic Tehory* 178, 191-221.

Blume, L. E., and D. Easley, 2006, "If You're so Smart, Why Aren't You so Rich? Belief selection in Complete and Incomplete Markets," *Econometrica* 74, 929-966.

Bollerslev, T., 1987, "A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return," *Review of Economics and Statistics* 69, 542-547.

Borovička, J., 2018, "Survival and Long-run Dynamics with Heterogeneous Beliefs under Recursive Preferences," *Journal of Political Economy*, forthcoming.

Brumm, J., M. Grill, F. Kubler, and K. Schmedders, 2015, "Margin Regulation and Volatility," *Journal of Monetary Economics* 75, 54-68.

Brunnermeier, M. K., and L. H. Pedersen, 2009, "Market Liquidity and Funding Liquidity," *Review of Financial Studies* 22, 2201-2238.

Brunnermeier, M. K., and Y. Sannikov, 2014, "A Macroeconomic Model with a Financial Sector," *American Economic Review* 104, 379-421.

Buss, A., Dumas, B., Uppal, R., Vilkov, G., 2016, "The Intended and Unintended Consequences of Financial-Market Regulations: a General Equilibrium Analysis," *Journal of Monetary Econoics* 81, 25-43.

Cao, D., 2018, "Speculation and Financial Wealth Distribution under Belief Heterogeneity," *Economic Journal 128*, 2258-2281.

Chabakauri, G., 2013, "Dynamic Equilibrium with two Stocks, Heterogeneous Investors, and Portfolio Constraints," *Review of Financial Studies* 26, 3104-3141.

Chabakauri, G., 2014, "Dynamic Equilibrium with Rare Events and Heterogeneous Epstein-Zin Investors," Working Paper, London School of Economics.

Chabakauri, G., 2015, "Asset Pricing with Heterogeneous Preferences, Beliefs, and Portfolio Constraints," *Journal of Monetary Economics* 75, 21-34.

Chan, Y. L., and L. Kogan, 2002, "Catching Up with the Joneses: Heterogeneous Preferences and the Dynamics of Asset Prices," *Journal of Political Economy* 110, 1255-1285.

Chien, Y., and H. Lustig, 2010, "The Market Price of Aggregate Risk and the Wealth Distribution," *Review of Financial Studies* 23, 1596–1650.

Cox, J., and C.-F. Huang, 1989, "Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process," *Journal of Economic Theory* 49, 33-83.

Cuoco, D., 1997, "Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income," *Journal of Economic Theory* 72, 33-73.

Cvitanić, J., and I. Karatzas, 1992, "Convex Duality in Constrained Portfolio Optimization," *Annals of Applied Probability* 2, 767-818.

Davis, M. H. A., 1997, "Option pricing in incomplete markets," in M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities, pages 216-227.* Cambridge University Press, Cambridge.

Deaton, A., 1991, "Saving and Liquidity Constraints," *Econometrica* 59, 1221-1248.

Detemple, J. B., and S. Murthy, 1997, "Equilibrium Asset Prices and No-Arbitrage with Portfolio Constraints," *Review of Financial Studies* 10, 1133-1174.

Detemple, J., and A. Serrat, 2003, "Dynamic Equilibrium with Liquidity Constraints," *Review of Financial Studies* 16, 597-629.

Duffie, D., 2001, *Dynamic Asset Pricing Theory*, Princeton, NJ: Princeton University Press.

Fostel, A., and J. Geanakoplos, 2008, "Leverage Cycles and Anxious Economy," *American Economic Review* 98, 1211-1244.

Fostel, A., and J. Geanakoplos, 2014, "Endogenous Collateral Constraints and Leverage Cycles," *Annual Review of Economics* 6, 771-799.

Gârleanu, N., and L. H. Pedersen, 2011, "Margin Based Asset Pricing and the Deviations from the Law of One Price," *Review of Financial Studies* 24, 1980-2022.

Geanakoplos, J., 2003, "Liquidity, Default, and Crashes: Endogenous Contracts in General Equilibrium," *A*dvances in Economics and Econometrics: Theory and Applications 2, 170-205. Econometric Society Monographs. New York: Cambridge University Press.

Geanakoplos, J., 2009, "The Leverage Cycle," *NBER Macroeconomic Annual* 24, 1-65.

Geanakoplos, J., and W. R. Zame, 2014, "Collateral Equilibrium, I: a Basic Famework," *Economic Theory* 56, 443-492.

Gradshteyn, I.S., and I.M. Ryzhik, 2007, *Table of Integrals, Series, and Products* (6th ed.), New York: Academic Press.

Gromb, D., and D. Vayanos, 2002, "Equilibrium and Welfare in Markets with Financially Constrained Arbitrageurs," *Journal of Financial Economics* 66, 361-407.

Gromb, D., and D. Vayanos, 2010, "Limits of Arbitrage: The State of the Theory," *Annual Review of Financial Economics* 2, 251-275.

Harrison, J. M., and D. M. Kreps, 1978, "Speculative Investor Behavior in a Stock Market with Heterogeneous Expectations," *Quarterly Journal of Economics* 92, 323-336.

Kehoe, T. J., and D. K. Levine, 1993, "Debt-Constrained Asset Markets," *Review of Economic Studies* 60, 865-888.

Kiyotaki, N., and J. Moore, 1997, "Credit Cycles," *Journal of Political Economy* 105, 211-248.

Klimenko, N., S. Pfeil, J.-C. Rochet, and G. De Nicolo, 2016, "Aggregate Bank Capital and Credit Dynamics," Working Paper.

Kocherlakota, N. R., 1996, "Implications of Efficient Risk Sharing Without Commitment," *Review of Economic Studies* 63, 595-609.

Kondor, P., and D. Vayanos, 2018, "Liquidity Risk and the Dynamics of Arbitrage Capital," *Journal of Finance*, forthcoming.

Krueger, D., and H. Lustig, 2010, "When is market incompleteness irrelevant for the price of aggregate risk (and when is it not)?" *Journal of Economic Theory 145*, 1-41.

Krusell, P., and A. A. Smith, 1998, "Income and Wealth Heterogeneity in the Macroeconomy," *Journal of Political Economy* 106, 867-896.

Kubler, F., and K. Schmedders, 2003, "Stationary Equilibria in Asset-Pricing Models with Incomplete Markets and Collateral," *Econometrica* 71, 1767-1793.

Lucas, R. E., 1978, "Asset Prices in an Exchange Economy," *Econometrica* 46, 1429-1445.

Lustig, H. N., and S. Van Nieuwerburgh, 2005, "Housing Collateral, Consumption Insurance, and Risk Premia: An Empirical Perspective," *Journal of Finance* 60, 1167-1219.

Massari, F., 2018, "Market Selection in Large Economies: A Matter of Luck," *Theoretical Economics*, forthcoming.

Osambela, E., 2015, "Differences of Opinion, Endogenous Liquidity, and Asset Prices," *Review of Financial Studies* 28, 1914-1959.

Pavlova, A., and R. Rigobon, 2008, "The Role of Portfolio Constraints in the International Propagation of Shocks," *Review of Economic Studies* 75, 1215-1256.

Rampini, A. A., and S. Viswanathan, 2018, "Financing Insurance," Working Paper.

Rytchkov, O., 2014, "Asset Pricing with Dynamic Margin Constraints," *Journal of Finance* 69, 405-452.

Schwert, W., 1989, "Why Does Stock Market Volatility Change Over Time?" *Journal of Finance* 44, 1115-1153.

Simsek, A., 2013, "Belief Disagreements and Collateral Constraints," *Econometrica* 81, 1-53.

Stokey, N. L., and R. E. Lucas, 1989, *Recursive Methods in Economic Dynamics*, Harvard University Press.

Tsyrennikov, V., 2012, "Heterogeneous Beliefs, Wealth Distribution, and Asset Markets with Risk of Default," *American Economic Review* 102, 156-160.

Veestraeten, D., 2004, "The Conditional Probability Density Function for a Reflected Brownian Motion," *Computational Economics* 24, 185-207.

Veronesi, P., 1999, "Stock Market Overreaction to bad News in Good Times: A Rational Expectations Equilibrium Model," *Review of Financial Studies* 12, 975-1007.

Yan, H., 2008, "Natural Selection in Financial Markets: Does it Work?" *Management Science* 54, 1935-1950.

Appendix A: Proofs

Lemma A.1 (Change of variable). *Let* $\widehat{n}_i = k_i l_i/(1 - l_A - l_B)$. *Maximization of expected discounted utility* (2.7) *subject to budget constraints* (2.8) *and* (2.9), *and constraint* (2.11) *is equivalent to maximizing* (2.7) *with respect to* c_{it}, b_{it} *and* $\widetilde{n}_{it}$ *subject to the following set of constraints:*

$$\begin{aligned} \widetilde{W}_{it} + l_i D_t \Delta t &= c_{it}\Delta t + b_{it} B_t + \widetilde{n}_{it}(S_t, P_t)^\top, && \text{(A1)} \\ \widetilde{W}_{i,t+\Delta t} &= b_{it} + \widetilde{n}_{it}\Big(S_{t+\Delta t} + (1 - l_A - l_B) D_{t+\Delta t}\Delta t,\, \mathbf{1}_{\{\omega_{t+\Delta t} = \omega_3\}}\Big)^\top, && \text{(A2)} \\ \widetilde{W}_{i,t+\Delta t} &\geq 0, && \text{(A3)} \end{aligned}$$

where $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ *and* $\widetilde{W}_{i,t+\Delta t} = W_{i,t+\Delta t} + \widehat{n}_i(S_{t+\Delta t} + (1 - l_A - l_B)D_{t+\Delta t})$.

Proof of Lemma A.1. Substituting $n_{it} = \widetilde{n}_{it} - (\widehat{n}_i, 0)$ into (2.8) and (2.9), we obtain constraints (A1) and (A2). Rewriting constraint (2.11) in terms of variable $\widetilde{W}_{i,t+\Delta t}$, we obtain (A3). Finally, we note that $\widetilde{W}_{it} = W_{it} + \widehat{n}_i S_t$ is worth $\widetilde{W}_{i,t+\Delta t}$ next period. Hence, (A1) and (A2) can be seen as self-financing budget constraints. ■

Proof of Lemma 1.

1) We start by demonstrating the equivalence of the dynamic (2.8)–(2.9) and static budget constraints (2.21). Multiplying equation (2.9) by $\xi_{i,t+\Delta t}/\xi_{it}$, taking expectation operator $\mathbb{E}_t^i[\cdot]$ on both sides, and using equations (2.16)–(2.18) for asset prices, we obtain:

$$\mathbb{E}_t^i\left[\frac{\xi_{i,t+\Delta t}}{\xi_{it}} W_{i,t+\Delta t}\right] = b_{it} B_t + n_{it}(S_t, P_t)^\top. \tag{A4}$$

From the budget constraint equation (2.8), we observe that the right-hand side of (A4) equals $W_{it} + l_i D_t \Delta t$, and hence, we obtain the static budget constraint (2.21). Conversely, if there exists $W_{i,t+\Delta t}$ satisfying constraints (2.21) and (2.22) there exist trading strategies b_{it} and n_{it} that replicate $W_{i,t+\Delta t}$ because the underlying market is effectively complete (i.e., the payoff matrix is invertible). Then, rewriting the optimization problem (2.7) in a recursive form, we obtain the dynamic programming equation (2.20) for the value function.

2) Consider wealth levels W_{it} and $\widehat{W}_{it}$. Let $\{c_{it}^*, b_{it}^*, n_{it}^*\}$ and $\{\widehat{c}_{it}^*, \widehat{b}_{it}^*, \widehat{n}_{it}^*\}$ be optimal consumptions and portfolios that correspond to W_{it} and $\widehat{W}_{it}$, respectively, and satisfy

constraints (2.8)–(2.10). For any $\alpha \in [0,1]$, policies $\{\alpha\widehat{c}^*_{it} + (1-\alpha)c^*_{it}, \alpha\widehat{b}^*_{it} + (1-\alpha)b^*_{it}, \alpha\widehat{n}^*_{it} + (1-\alpha)n^*_{it}\}$ are admissible for wealth $\alpha W_{it} + (1-\alpha)\widehat{W}_{it}$. By concavity of CRRA utilities:

$$\begin{aligned}
V_i(\alpha W_{it} + (1-\alpha)\widehat{W}_{it}, v_t; l_i) &\geq \sum_{\tau=t}^{\infty} u_i(\alpha\widehat{c}^*_{it} + (1-\alpha)c^*_{it}) \\
&\geq \sum_{\tau=t}^{\infty} (\alpha u_i(\widehat{c}^*_{it}) + (1-\alpha)u_i(c^*_{it})) \\
&= \alpha V_i(W_{it}, v_t; l_i) + (1-\alpha)V_i(\widehat{W}_{it}, v_t; l_i).
\end{aligned} \tag{A5}$$

Therefore, $V_i(W_{it}, v_t; l_i)$ is a concave function of wealth.

3) Consider the following Lagrangian:

$$\begin{aligned}
\mathscr{L} &= u_i(c_{it})\Delta t + e^{-\rho\Delta t}\mathbb{E}^i_t\Big[V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)\Big] \\
&+ \eta_{it}\left(W_{it} + l_i D_t\Delta t - c_{it}\Delta t - \mathbb{E}^i_t\Big[\frac{\xi_{i,t+\Delta t}}{\xi_{it}}W_{i,t+\Delta t}\Big]\right) + \mathbb{E}^i_t\Big[e^{-\rho\Delta t}\ell_{i,t+\Delta t}W_{i,t+\Delta t}\Big],
\end{aligned} \tag{A6}$$

where multiplier $\ell_{i,t+\Delta t}$ satisfies the complementary slackness condition $\ell_{i,t+\Delta t}W_{i,t+\Delta t} = 0$. Differentiating the Lagrangian (A6) with respect to c_{it} and $W_{i,t+\Delta t}$, we obtain:

$$u'_i(c^*_{it}) = \eta_{it}, \tag{A7}$$

$$e^{-\rho\Delta t}\left(\frac{\partial V_i(W_{t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} + \ell_{i,t+\Delta t}\right) = \eta_{it}\frac{\xi_{i,t+\Delta t}}{\xi_{it}}. \tag{A8}$$

By the envelope theorem (e.g, Back (2010, p.162)):

$$\frac{\partial V_i(W_{i,t+\Delta t}, v_{t+\Delta t}; l_i)}{\partial W} = u'_i(c^*_{i,t+\Delta t}). \tag{A9}$$

Substituting the partial derivative of the value function (A9) and the marginal utility (A7) into equation (A8), and then dividing both sides of the equation by $u'_i(c^*_{it})$, we obtain the expression for the SPD (2.23). ■

Proof of Proposition 1.

Step 1. Consider the case when constraints do not bind, and hence, $\ell_{i,t+\Delta t} = 0$. Then, using equation (2.13) for the state variable v_t and the first order conditions (2.23), we obtain:

$$v_{t+\Delta t} - v_t = \ln\left(\frac{(c^*_{A,t+\Delta t}/c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) = \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right).$$

From the above equation and the change of measure equation (2.19), which relates SPDs $\xi_{A,t+\Delta t}$ and $\xi_{B,t+\Delta t}$, we obtain the dynamics of v_t when constraints do not bind:

$$v_{t+\Delta t} - v_t = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right). \tag{A10}$$

Let $\overline{v}$ and $\underline{v}$ be the boundaries satisfying Equations (2.24), at which the constraints of investors A and B bind, respectively. Let investor A's constraint be binding so that $v_{t+\Delta t} = \overline{v}$, and hence, $\ell_{A,t+\Delta t} \geq 0$. Using Equation (2.13) for v_t, first order conditions (2.23), and $\ell_{A,t+\Delta t} \geq 0$, we obtain:

$$\begin{aligned}
\overline{v} - v_t &\leq \ln\left(\frac{((c^*_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t})/(c^*_{At})^{-\gamma_A}}{(c^*_{B,t+\Delta t}/c^*_{Bt})^{-\gamma_B}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\
&= \ln\left(\frac{\xi_{A,t+\Delta t}/\xi_{At}}{\xi_{B,t+\Delta t}/\xi_{Bt}}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) = \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right).
\end{aligned} \tag{A11}$$

Similarly, for $v_{t+\Delta t} = \underline{v}$ we obtain that $\underline{v} - v_t \geq \ln\Big(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\Big(D_{t+\Delta t}/D_t\Big)^{\gamma_A-\gamma_B}\Big)$. The latter two inequalities imply that when the constraint binds $v_{t+\Delta t}$ is given by:

$$v_{t+\Delta t} = \max\left\{\underline{v};\min\left\{\overline{v}; v_t + \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right)\right\}\right\}. \tag{A12}$$

We observe that (A12) is also satisfied in the unconstrained case where $\underline{v} < v_{t+\Delta t} < \overline{v}$. It remains to prove that v_t does not escape $[\underline{v}, \overline{v}]$ interval. Consider a marginal investor of type A. We guess that v_t follows dynamics (A12) and verify that the consumption choice of investor A indeed implies this dynamics. The analysis for investor B is similar.

We have shown above that v_t satisfies inequality (A11) when investor A is constrained. Now, we show the opposite: investor A is constrained when v_t satisfies (A11). Hence, $v_{t+\Delta t}$ cannot exceed $\overline{v}$. Consider v_t such that $v_t + \ln\big(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\,(D_{t+\Delta t}/$ $\overline{v}$ for some $\omega_{t+\Delta t}$ and $v_t \in (\underline{v}, \overline{v})$. Because $\underline{v} < v_t < \overline{v}$, investor A consumes $c^*_{At} = s(v_t)D_t$, as shown above. We show that the constraint of investor A binds and $c^*_{A,t+\Delta t} = s(\overline{v})D_{t+\Delta t}$. This consumption level confirms that $v_{t+\Delta t} = \overline{v}$ is indeed an equilibrium outcome.

Consider the constraint of investor A at date t in the state $\omega_{t+\Delta t}$ where $v_{t+\Delta t} = \overline{v}$:

$$W_{A,t+\Delta t} \geq 0 \equiv \Phi_A(\overline{v}) D_{t+\Delta t}, \tag{A13}$$

where the last equality holds by the definition of $\overline{v}$. Using the concavity of the value function, proven in Lemma 1, and condition (A9) from the envelope theorem, we obtain:

$$u'_A(c^*_{A,t+\Delta t}) = \frac{\partial V_A(W_{A,t+\Delta t}, \overline{v}; l_A)}{\partial W} \leq \frac{\partial V_A(\Phi_A(\overline{v}) D_{t+\Delta t}, \overline{v}; l_A)}{\partial W} = u'_A(s(\overline{v}) D_{t+\Delta t}). \tag{A14}$$

Because $u'_i(c)$ is a decreasing function, we find that $c^*_{A,t+\Delta t}/D_{t+\Delta t} \geq s(\overline{v})$.

Investor B is unconstrained when $v_{t+\Delta t} = \overline{v}$, and hence, has SPD

$$\frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} = e^{-\rho\Delta t}\left(\frac{c^*_{B,t+\Delta t}}{c^*_{Bt}}\right)^{-\gamma_B} = e^{-\rho\Delta t}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}. \tag{A15}$$

From the change of measure equation (2.19) and the FOC (2.23), the SPD of investor A is

$$\begin{aligned} \frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= \frac{\xi_{B,t+\Delta t}}{\xi_{Bt}} \frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} \\ &= e^{-\rho\Delta t} \frac{(c^*_{A,t+\Delta t})^{-\gamma_A} + \ell_{A,t+\Delta t}}{(c^*_{At})^{-\gamma_A}}. \end{aligned} \tag{A16}$$

From (A16) and (A15), we find the Lagrange multiplier:

$$\begin{aligned} \frac{l_{A,t+\Delta t}}{(c^*_{A,t+\Delta t})^{-\gamma_A}} &= \left(\frac{c^*_{A,t+\Delta t}}{c^*_{At}}\right)^{\gamma_A}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &\geq \left(\frac{s(\overline{v})D_{t+\Delta t}}{s(v_t)D_t}\right)^{\gamma_A}\left(\frac{(1-s(\overline{v}))D_{t+\Delta t}}{(1-s(v_t))D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})} - 1 \\ &= \left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) e^{v_t-\overline{v}} - 1 > 0. \end{aligned}$$

The first inequality follows from the fact that $c^*_{A,t+\Delta t} \geq s(\overline{v})D_{t+\Delta t}$ we proved above. The second equality holds by the definition of state variable (2.13). The second inequality comes from the assumption that $v_t + \ln\left(\pi_B(\omega_{t+\Delta t})/\pi_A(\omega_{t+\Delta t})\left(D_{t+\Delta t}/D_t\right)^{\gamma_A-\gamma_B}\right) >$

$\overline{v}$. Hence, the Lagrange multiplier $l_{A,t+\Delta t}$ is strictly positive. From the complementary slackness condition, the constraint (A13) must be binding. Therefore, inequality (A14) becomes an equality, and hence, $c^*_{A,t+\Delta t} = s(\overline{v})D_{t+\Delta t}$.

Step 2. We now look for coefficients μ_v, σ_v and J_v such that:

$$\begin{aligned} \mu_v \Delta t \;+\; \sigma_v \Delta w_t + J_v \Delta j_t &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\right) \\ &= \ln\left(\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+\sigma_D\Delta w_t + J_v\Delta j_t). \end{aligned} \tag{A17}$$

We write identity (A17) in each of the states $\omega_{t+\Delta t} \in \{\omega_1, \omega_2, \omega_3\}$ and obtain the following system of three linear equations with three unknowns μ_v, σ_v and J_v:

$$\mu_v\Delta t \;+\; \sigma_v\sqrt{\Delta t} \;=\; \ln\left(\frac{(1-\lambda_B\Delta t)(1+\delta\Delta t)}{1-\lambda\Delta t}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+\sigma_D\sqrt{\Delta t}),$$

$$\mu_v\Delta t \;-\; \sigma_v\sqrt{\Delta t} \;=\; \ln\left(\frac{(1-\lambda_B\Delta t)(1-\delta\Delta t)}{1-\lambda\Delta t}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t-\sigma_D\sqrt{\Delta t}),$$

$$\mu_v\Delta t \;+\; J_v \;=\; \ln\left(\frac{\lambda_B}{\lambda}\right) + (\gamma_A-\gamma_B)\ln(1+\mu_D\Delta t+J_D).$$

Solving the above system, we obtain μ_v, σ_v and J_v reported in Proposition 1.

Step 3. Finally, we show that the boundaries are reflecting for a sufficiently small Δt. Suppose, two conditions are satisfied: $\mu_v\Delta t - \sigma_v\sqrt{\Delta t} < 0$ and $\mu_v\Delta t + \sigma_v\sqrt{\Delta t} > 0$. Then, the boundaries are reflecting: 1) if $v_t = \overline{v}$, then $v_{t+\Delta t} = \overline{v} + \mu_v\Delta t - \sigma_v\sqrt{\Delta t} < \overline{v}$ with positive probability; 2) if $v_t = \underline{v}$, then $v_{t+\Delta t} = \underline{v} + \mu_v\Delta t + \sigma_v\sqrt{\Delta t} > \underline{v}$ with positive probability. It can be easily verified that as $\Delta t \to 0$, $\mu_v \to \widehat{\mu}_v$ and $\sigma_v \to \widehat{\sigma}_v$, where $\widehat{\mu}_v$ and $\widehat{\sigma}_v$ are constants given by equations (2.35) and (2.36), respectively. Because $\sigma_v > 0$ and $\sqrt{\Delta t}$-terms dominate Δt-terms for small Δt, we find that $\mu_v\Delta t - \sigma_v\sqrt{\Delta t} < 0$ and $\mu_v\Delta t + \sigma_v\sqrt{\Delta t} > 0$ for all sufficiently small Δt. Hence, the boundaries are reflecting. ■

Lemma A.2 (Wealth-output ratios). *The investors' wealth-output ratios Φ_i are* *uniformly bounded and given by:*

$$\Phi_A(v_t) = \mathbb{E}_t^A\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\left(\frac{s(v_\tau)}{s(v_t)}\right)^{-\gamma_A}(s(v_\tau)-l_A)\Delta t\right], \quad \text{(A18)}$$

$$\Phi_B(v_t) = \mathbb{E}_t^B\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_B}\left(\frac{1-s(v_\tau)}{1-s(v_t)}\right)^{-\gamma_B}(l_B-s(v_\tau))\Delta t\right]. \quad \text{(A19)}$$

Proof of Lemma A.2. Substituting FOC (2.23) into the budget constraint (2.21) and using the complementary slackness condition $\ell_{i,t+\Delta t}W^*_{i,t+\Delta t} = 0$, we obtain:

$$W^*_{At} = \mathbb{E}_t^A\left[e^{-\rho\Delta t}\left(\frac{c^*_{A,t+\Delta t}}{c^*_{At}}\right)^{-\gamma_A} W^*_{A,t+\Delta t}\right] + (c^*_{At} - l_A D_t)\Delta t. \quad \text{(A20)}$$

Substituting $W^*_{At} = \Phi_{At}D_t$ and $c^*_{At} = s(v_t)D_t$ into equation (A20) and iterating, we obtain equation (A18). Let $\overline{s} = s(\underline{v})$ $\underline{s} = s(\overline{v})$, where $s(v)$ is given by equation (2.14). Then, $\overline{s} \geq s \geq \underline{s} > 0$. Using the bounds on s_t, we obtain the following uniform bound on Φ_A:

$$\Phi_A(v_t) \leq Const \times \mathbb{E}_t^A\left[\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right].$$

The series on the right-hand side of the latter inequality is convergent due to condition (2.15) on model parameters. Equation (A19) is obtained along the same lines. ■

Proof of Proposition 2. 1) First, we derive the SPD ξ_{At} under the correct beliefs of investor A. When investor A's constraint does not bind, substituting $c^*_{At} = s(v_t)D_t$ into the first order condition (2.23) we find that

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}. \quad \text{(A21)}$$

Equation (A21) is consistent with SPD (2.29) because when the constraint does not bind $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t < \overline{v}$, and hence the exponential term in (2.29) vanishes.

When the constraint of investor A binds, the constraint of investor B is loose: the constraints cannot bind simultaneously because stock market would not clear otherwise. Therefore, the ratio $\xi_{B,t+\Delta t}/\xi_{Bt}$ is given by FOC (2.23) for investor B with $\ell_B = 0$. Using equation (2.19), we rewrite the latter SPD under the correct beliefs of investor A:

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t}\left(\frac{1-s(v_{t+\Delta t})}{1-s(v_t)}\right)^{-\gamma_B}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_B}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}. \tag{A22}$$

Next, from equation (2.14) for consumption share s we find that $(1-s_t)^{-\gamma_B} = e^{-v_t}s_t^{-\gamma_A}$. Substituting the latter equality into equation (A22), and also using equation (A17) for the increment $v_{t+\Delta t} - v_t$, we obtain:

$$\begin{aligned}\frac{\xi_{A,t+\Delta t}}{\xi_{At}} &= e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} e^{v_t - v_{t+\Delta t}}\frac{\pi_B(\omega_{t+\Delta t})}{\pi_A(\omega_{t+\Delta t})}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{\gamma_A-\gamma_B}\\ &= e^{-\rho\Delta t}\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A}\exp\{v_t - v_{t+\Delta t} + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\}.\end{aligned} \tag{A23}$$

The fact that the constraint of investor A is binding means that $v_{t+\Delta t} = \overline{v}$ and $v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v \geq \overline{v}$ (because otherwise $v_{t+\Delta t} < \overline{v}$, and hence, the constraint does not bind). Therefore, the exponential term $\exp(v_t - v_{t+\Delta t})$ in equation (A23) can be replaced with $\exp(\max\{0, v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\})$. When the constraint of investor A does not bind the latter term vanishes and we obtain equation (A21). Therefore, both equations (A21) and (A23) are summarized by equation (2.29) for $\xi_{A,t+\Delta t}/\xi_{At}$.

2) Lemma A.2 derives the wealth-output ratios $\Phi_i(v_t)$ and shows that they are uniformly bounded. From the market clearing condition $S_t = W_{At} + W_{Bt}$. Dividing by D_t, we obtain that $\Psi(v_t) = \Phi_A(v_t) + \Phi_B(v_t)$. Hence, $\Psi(v_t)$ is uniformly bounded. The fact that stock price S_t is given by (2.30) can be verified by substituting S_t into the recursive equation (2.17).

3) In the unconstrained economy, the state variable v_t^{unc} follows dynamics:

$$v_t^{unc} = \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t. \tag{A24}$$

Define processes $U_{t+\Delta t} = U_t + \Delta U_t$ and $V_{t+\Delta t} = V_t + \Delta V_t$, where increments are given by:

$$\Delta U_t = \max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}, \quad \Delta V_t = \max\{0; \underline{v} - v_t - \mu_v\Delta t - \sigma_v\Delta w_t - J_v\Delta j_t\}. \tag{A25}$$

The process for the state variable in the constrained economy can be rewritten as

$$v_{t+\Delta t} = v_t + \mu_v \Delta t + \sigma_v \Delta w_t + J_v \Delta j_t + \Delta V_t - \Delta U_t. \tag{A26}$$

If the state variables have the same value at time 0, i.e., $v_0 = v_0^{unc}$, we obtain:

$$v_t = v_t^{unc} + V_t - U_t \tag{A27}$$

Next, we prove that the SPD is higher in the constrained economy.

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho \Delta t} \left(\frac{s(v_{t+\Delta t})}{s(v_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} \exp(\Delta U_t), \tag{A28}$$

$$\frac{\xi^{unc}_{A,t+\Delta t}}{\xi^{unc}_{At}} = e^{-\rho \Delta t} \left(\frac{s(v^{unc}_{t+\Delta t})}{s(v^{unc}_t)} \frac{D_{t+\Delta t}}{D_t} \right)^{-\gamma_A} . \tag{A29}$$

Iterating the above equations, we obtain:

$$\frac{\xi_{At}}{\xi_{A0}} = e^{-\rho t} \left(\frac{s(v_t)}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A} \exp(U_t),$$

$$\frac{\xi^{unc}_{At}}{\xi^{unc}_{A0}} = e^{-\rho t} \left(\frac{s(v^{unc}_t)}{s(v_0)} \frac{D_t}{D_0} \right)^{-\gamma_A} .$$

By the definition of $s(v)$ in equation (2.14), we have $e^v = (1 - s(v))^{\gamma_B} \cdot s(v)^{-\gamma_A}$. Hence,

$$\begin{aligned}
\frac{\xi_{At}/\xi_{A0}}{\xi^{unc}_{At}/\xi^{unc}_{A0}} &= \left(\frac{s(v_t)}{s(v^{unc}_t)} \right)^{-\gamma_A} \exp(U_t) = \left(\frac{s(v^{unc}_t + V_t - U_t)}{s(v^{unc}_t)} \right)^{-\gamma_A} e^{v^{unc}_t} e^{-(v^{unc}_t - U_t)} \\
&\geq s(v^{unc}_t - U_t)^{-\gamma_A} e^{-(v^{unc}_t - U_t)} \cdot s(v^{unc}_t)^{\gamma_A} e^{v^{unc}_t} \\
&= (1 - s(v^{unc}_t - U_t))^{-\gamma_B} \cdot (1 - s(v^{unc}))^{\gamma_B} \geq 1.
\end{aligned}$$

Therefore, we conclude that $\xi_{At}/\xi_{A0} \geq \xi^{unc}_{At}/\xi^{unc}_{A0}$. The latter inequality and the equations for asset prices (2.16)–(2.18) imply that prices are higher in the constrained economy. ■

Proof of Lemma 2. The price-dividend ratio Ψ and wealth-aggregate consumption ratios Φ_i are functions of the state variable v, and satisfy equations:

$$\Psi(v_t) = \mathbb{E}^A_t \left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{D_{t+\Delta t}}{D_t} \Big(\Psi(v_{t+\Delta t}) + \Delta t \Big) \right], \tag{A30}$$

$$\Phi_i(v_t) = \mathbb{E}^A_t \left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}} \frac{D_{t+\Delta t}}{D_t} \Phi_i(v_{t+\Delta t}) \right] + \Big(\mathbf{1}_{\{i=A\}} s(v_t) + \mathbf{1}_{\{i=B\}} (1 - s(v_t)) - l_i \Big) \Delta t \tag{A31}$$

These equations are obtained by substituting $S_t = (1-l_A-l_B)D_t\Psi(v_t)$ into equation (2.17) for the stock price, and $\Psi_i = D_t W_{it}$ into static budget constraints (2.21). Define the following function in discrete time:

$$\widehat{\Psi}(v_t;\theta) = \mathbb{E}_t^A\left[e^{-\rho\Delta t+\Delta U_t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] + s(v_t)^{\theta}\Delta t, \tag{A32}$$

where ΔU_t is given by equation

$$\Delta U_t = \max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}. \tag{A33}$$

Comparing equation (A32) with equations (A30) and (A31) for Ψ and Φ_i and using the linearity of equation (A32), it easy to observe that $\Psi(v_t)$ and $\Phi_i(v_t)$ are given by the following equations in terms of $\widehat{\Psi}(v_t;\theta)$:

$$\begin{aligned}
\Psi(v_t) &= \widehat{\Psi}(v_t, -\gamma_A)s(v_t)^{\gamma_A} - \Delta t,\\
\Phi(v_t) &= \Big((\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}})\widehat{\Psi}(v;1-\gamma_A) + (\mathbf{1}_{\{i=B\}} - l_i)\widehat{\Psi}(v;-\gamma_A)\Big)\, s(v)^{\gamma_A}.
\end{aligned}$$

Taking limit $\Delta t \to 0$, we obtain equations (2.31) and (2.32) for $\Psi(v_t)$ and $\Phi_i(v_t)$.

First, we derive the equation for $\widehat{\Psi}(v_t;\theta)$ when v_t belongs to the interior $(\underline{v}, \overline{v})$. For a sufficiently small Δt we have $\Delta U_t = 0$, where ΔU_t is given by (A33). Then, we rewrite the expectation of $(D_{t+\Delta t})/D_t)^{1-\gamma_A}\widehat{\Psi}(v_t;\theta)$ as follows:

$$\begin{aligned}
\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] &= (1-\lambda\Delta t)\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{normal}\right]\\
&\quad +\lambda\Delta t\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\Big|\text{crisis}\right].
\end{aligned} \tag{A34}$$

Noting that in the crisis $D_{t+\Delta t}/D_t = 1 + \mu_v\Delta t + J_D$ and $v_{t+\Delta t} = \max\{\underline{v}; v_t + \mu_v\Delta t + J_v\}$ and in the normal state $D_{t+\Delta t}/D_t = 1 + \mu_D\Delta t + \sigma_D\Delta w_t$ and $v_{t+\Delta t} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t$, using Taylor expansions for $(D_{t+\Delta t}/D_t)^{1-\gamma_A}$ and $\widehat{\Psi}(v_{t+\Delta t};\theta)$, we find:

$$\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{crisis}\right] = (1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v};v_t+J_v\};\theta\Big). \tag{A35}$$

$$\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)|\text{normal}\right] = \left[1+\left((1-\gamma_A)\mu_D+\frac{(1-\gamma_A)\gamma_A\sigma_D^2}{2}\right)\Delta t\right]\widehat{\Psi}(v_t;\theta)$$

$$+\Big(\mu_v+(1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v_t;\theta)\Delta t + \frac{\sigma_v^2}{2}\widehat{\Psi}''(v_t;\theta)\Delta t+o(\Delta t). \tag{A36}$$

Substituting (A35)-(A36) into (A32), we obtain:

$$\begin{aligned}\widehat{\Psi}(v_t;\theta) &= \left[1-\left(\lambda+\rho-(1-\gamma_A)\mu_D+\frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)\Delta t\right]\widehat{\Psi}(v_t;\theta)\\ &+ \Big(\mu_v+(1-\gamma_A)\sigma_D\sigma_v\Big)\widehat{\Psi}'(v;\theta)\Delta t+\frac{\sigma_v^2}{2}\widehat{\Psi}''(v;\theta)\Delta t\\ &+ \lambda(1+J_D)^{1-\gamma_A}\widehat{\Psi}\Big(\max\{\underline{v};v_t+J_v\};\theta\Big)\Delta t+s(v)^{\theta}\Delta t+o(\Delta t).\end{aligned} \tag{A37}$$

Canceling similar terms, diving by Δt, taking limit $\Delta t\to 0$, and noting that μ_v, σ_v and J_v converge to $\widehat{\mu}_v$, $\widehat{\sigma}_v$ and $\widehat{J}_v$ given by (2.35)-(2.37), we obtain equation (2.33) for $\widehat{\Psi}(v_t;\theta)$.

Next, we derive the boundary conditions for $\widehat{\Psi}(v_t;\theta)$. From equation (2.25), the state variable dynamics at lower bound is $v_{t+\Delta t}=\underline{v}+\max\{0,\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t\}$. We use Δv_t to denote the difference of $v_{t+\Delta t}$ and v_t. In this case,

$$\Delta v_t=\max\{0,\mu_v\Delta t+\sigma_v\Delta w_t+J_v\Delta j_t\}. \tag{A38}$$

For sufficiently small Δt increment Δv_t is positive only in state ω_1 and is zero otherwise. In state ω_1, $\Delta v_t=\mu_v\Delta t+\sigma_v\sqrt{\Delta t}$. Therefore, the order of $\mathbb{E}_t^A[\Delta v_t]$ is $\sqrt{\Delta t}$, but second order terms involving Δv_t have lower order:

$$\begin{aligned}\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A[\Delta v_t]}{\sqrt{\Delta t}} &= \frac{\widehat{\sigma}_v}{2},\\ \lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A\left[(\Delta v_t)^2\right]}{\sqrt{\Delta t}}=\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A[\Delta v_t\Delta t]}{\sqrt{\Delta t}} &= \lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A[\Delta v_t\Delta w_t]}{\sqrt{\Delta t}}=\lim_{\Delta t\to 0}\frac{\mathbb{E}_t^A[\Delta v_t\Delta j_t]}{\sqrt{\Delta t}}=0.\end{aligned} \tag{A39}$$

Taylor expansion of $\widehat{\Psi}(v_{t+\Delta t};\theta)$ at $v_t = \underline{v}$ is given by

$$\widehat{\Psi}(v_{t+\Delta t};\theta) = \widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2 + o(\sqrt{\Delta t}). \tag{A40}$$

In subsequent calculations we keep terms with order of $\sqrt{\Delta t}$. Using the above results, we obtain the following expansion:

$$\begin{aligned}
&\mathbb{E}_t^A\left[\left(\frac{D_{t+\Delta t}}{D_t}\right)^{1-\gamma_A}\widehat{\Psi}(v_{t+\Delta t};\theta)\right] \\
&= \mathbb{E}_t^A\left[(1+\mu_D\Delta t + \sigma_D\Delta w_t + J_v\Delta j_t)^{1-\gamma_A}\left(\widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\Delta v_t + \frac{1}{2}\widehat{\Psi}''(\underline{v};\theta)\Delta v_t^2\right)\right] \\
&= \widehat{\Psi}(\underline{v};\theta) + \widehat{\Psi}'(\underline{v};\theta)\mathbb{E}_t^A[\Delta v_t] + o(\sqrt{\Delta t}).
\end{aligned} \tag{A41}$$

Substituting (A41) into (A32), taking into account that $\Delta U_t = 0$ at $v_t = \underline{v}$, and canceling $\widehat{\Psi}(\underline{v};\theta)$ on both sides, we obtain the first boundary condition $\widehat{\Psi}'(\underline{v};\theta) = 0$.

At the upper bound $v_t = \overline{v}$ investor A is constrained, and hence, ΔU_t in (A33) is positive. From (2.25) the state variable at the upper bound is

$$v_{t+\Delta t} = \min\{\overline{v}, v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t\} = v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \Delta U_t. \tag{A42}$$

The order of $\mathbb{E}_t^A[\Delta U_t]$ is $\sqrt{\Delta t}$, but second order terms involving ΔU_t have order $o(\sqrt{\Delta t})$.

Proceeding in the same way as (A39)-(A41), we arrive at

$$\widehat{\Psi}(\overline{v};\theta) = \widehat{\Psi}(\overline{v};\theta) + \left[\widehat{\Psi}(\overline{v};\theta) - \widehat{\Psi}'(\overline{v};\theta)\right]\mathbb{E}_t^A[\Delta U_t] + o(\sqrt{\Delta t}).$$

Canceling similar terms, taking limit $\Delta t \to 0$, we obtain condition $\widehat{\Psi}(\overline{v};\theta) - \widehat{\Psi}'(\overline{v};\theta) = 0$.

Finally, we derive the equations for $\overline{v}$ and $\underline{v}$. Taking limit $\Delta t \to 0$ in equations (2.24), we find that these equations become: $\Phi_A(\overline{v}) = 0$, $\Phi_B(\underline{v}) = 0$. Substituting $\Phi_i(v)$ and $\Psi(v)$ in terms of $\widehat{\Psi}(v;\theta)$ from equations (2.32) into the latter equations for the boundaries, after some algebra, we obtain equations (2.38). ■

Proof of Corollary 1. Consider the case $\lambda = \lambda_B = 0$ and $\gamma_A = \gamma_B = 1$. Then, $s(v)$ solving equation (2.25) is given by $s(v) = 1/(1+e^v)$, $\Psi(v) = \widehat{\Psi}(v)s(v)$, where

$\widehat{\Psi}(v)$ solves a special case of equation (2.33) given by:

$$\frac{\delta^2}{2}\widehat{\Psi}''(v) - \frac{\delta^2}{2}\widehat{\Psi}'(v) - \rho\widehat{\Psi}(v) + 1 + e^v = 0, \tag{A43}$$

subject to boundary conditions (2.34). It can be easily verified that $\widehat{\Psi}(v) = C_1 e^{\varphi_- v} + C_2 e^{\varphi_+ v} + (1+e^v)/\rho$ satisfies (A43). Substituting $\widehat{\Psi}(v)$ into boundary conditions (2.34), we obtain the following system for coefficients C_1 and C_2:

$$C_1\varphi_- e^{\varphi_- \underline{v}} + C_2\varphi_+ e^{\varphi_+ \underline{v}} + e^{\underline{v}}/\rho = 0; \quad C_1(\varphi_- - 1)e^{\varphi_- \overline{v}} + C_2(\varphi_+ - 1)e^{\varphi_+ \overline{v}} - 1/\rho = 0.$$

Solving these equations, we obtain:

$$C_1 = \frac{1}{\rho}\frac{(\varphi_+ - 1)e^{\underline{v}+\varphi_+\overline{v}} + \varphi_+ e^{\varphi_+\underline{v}}}{\varphi_+(\varphi_- - 1)e^{\varphi_-\overline{v}+\varphi_+\underline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}, \tag{A44}$$

$$C_2 = -\frac{1}{\rho}\frac{(\varphi_- - 1)e^{\underline{v}+\varphi_-\overline{v}} + \varphi_- e^{\varphi_-\underline{v}}}{\varphi_+(\varphi_- - 1)e^{\varphi_+\underline{v}+\varphi_-\overline{v}} - \varphi_-(\varphi_+ - 1)e^{\varphi_+\overline{v}+\varphi_-\underline{v}}}. \blacksquare \tag{A45}$$

Proof of Proposition 3. From equation (2.16) for the bond price and the fact that $1 = B_t(1 + r_t\Delta t)$ we find that the riskless interest rate r_t is given by:

$$\begin{aligned} r_t &= \frac{1 - \mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t} \\ &= \frac{1 - (1-\lambda\Delta t)\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}] - \lambda\Delta t\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]}{\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}]\Delta t}, \end{aligned} \tag{A46}$$

where $\xi_{A,t+\Delta t}/\xi_{At}$ is given by equation (2.29). We separately calculate $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ and $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{crisis}]$, and then take the limit $\Delta t \to 0$.

We start with the derivation of $\mathbb{E}_t[\xi_{A,t+\Delta t}/\xi_{At}|\text{normal}]$ when $\underline{v} < v_t < \overline{v}$, and hence, by continuity, for a sufficiently small Δt the economy is unconstrained next period, so that $\underline{v} < v_{t+\Delta t} < \overline{v}$. In the unconstrained region $\Delta v_t = \widehat{\mu}_v\Delta t + \widehat{\sigma}_v\Delta w_t$ and the SPD is given by (A21). From the expression for the SPD, using expansions (A55) and (A57), we obtain:

$$\begin{aligned} \mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Big|\text{normal}\right] &= \mathbb{E}_t\left[\Big(1 + a_t\Delta v_t + b_t(\Delta v_t)^2\Big)\Big(1 - r_A\Delta t - \kappa_A\Delta w_t\Big)\Big|\text{normal}\right] + o(\Delta t) \\ &= \mathbb{E}_t\Big[1 + a_t\Delta v_t + b_t(\Delta v_t)^2 - r_A\Delta t - \kappa_A\Delta w_t - \kappa_A a_t\Delta v_t\Delta w_t\Big|\text{normal}\Big] + o(\Delta t) \\ &= 1 + a_t\widehat{\mu}_v\Delta t + b_t\widehat{\sigma}_v^2\Delta t - r_A\Delta t - \kappa_A a_t\widehat{\sigma}_v\Delta t + o(\Delta t). \end{aligned} \tag{A47}$$

Conditioning on the crisis state, we have:

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{crisis}\right] &= (1-\rho\Delta t)(1+\mu_D\Delta t+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\mu_v\Delta t+J_v\})}{s(v_t)}\right)^{-\gamma_A} \\
&= (1+J_D)^{-\gamma_A}\left(\frac{s(\max\{\underline{v}, v_t+\widehat{J}_v\})}{s(v_t)}\right)^{-\gamma_A} + o(\Delta t).
\end{aligned} \tag{A48}
$$

Substituting a_t and b_t from (A56) into equation (A47), and then substituting (A47) and (A48) into equation (A46), after simple algebra, we obtain r_t in (2.40) for the case $\underline{v} < v_t < \overline{v}$.

Now, we derive r_t at the boundaries $\underline{v}$ and $\overline{v}$. The SPD is given by (2.29). Using expansions (A55) and (A57), we obtain the following expansion:

$$
\begin{aligned}
\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{normal}\right] &= \mathbb{E}_t\Big[\Big((1+a_t\Delta v_t+b_t(\Delta v_t)^2\Big)\Big(1-r_A\Delta t-\kappa_A\Delta w_t\Big) \\
&\qquad \times(1+\Delta U_t+0.5(\Delta U_t)^2)|\text{normal}\Big]+o(\Delta t) \\
&= \mathbb{E}_t\Big[1+a_t\Delta v_t+b_t(\Delta v_t)^2-r_A\Delta t-\kappa_A\Delta w_t-\kappa_A a_t\Delta v_t\Delta w_t \\
&+ \Delta U_t-\kappa_A\Delta w_t\Delta U_t+a_t\Delta U_t\Delta v_t+0.5(\Delta U_t)^2\Big|\text{normal}\Big]+O(\Delta t),
\end{aligned} \tag{A49}
$$

where ΔU_t is given by equation (A33). Using equation (2.25) for the process v_t and equation (A33) for ΔU_t, for a fixed v_t and sufficiently small Δt, we find that Δv_t and ΔU_t at the boundaries are given by:

$$
\Delta v_t = \begin{cases} \min(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \\ \max(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t = \underline{v}, \end{cases} \tag{A50}
$$

$$
\Delta U_t = \begin{cases} 0, & \text{if } v_t < \overline{v}, \\ \max(0, \mu_v\Delta t+\sigma_v\Delta w_t), & \text{if } v_t = \overline{v}, \end{cases} \tag{A51}
$$

We note that for a sufficiently small Δt the sign of $\mu_v\Delta t+\sigma_v\Delta w_t$ is solely determined by the second term $\sigma_v\Delta w_t$ because it has the order of magnitude $\sqrt{\Delta t}$. Volatility σ_v is positive because under our assumptions investor A is more risk averse and more pessimistic. Using the latter observation, substituting equations (A50) and (A51) into equation (A49) and computing the expectation, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\middle|\text{normal}\right]=1+\begin{cases}\left(\frac{a_t(\mu_v-\kappa_A\sigma_v)}{2}+\frac{b_t\sigma_v^2}{2}+\frac{\mu_v+\kappa_A\sigma_v+\sigma_v^2}{2}-r_A\right)\Delta t\\ \qquad +\frac{\sigma_v(1-a_t)}{2}\sqrt{\Delta t}+O(\Delta t), \quad \text{if} \quad v_t=\underline{v},\\ \left(\frac{a_t\mu_v-a_t\kappa_A\sigma_v+b_t\sigma_v^2}{2}\right)\Delta t+\frac{a_t\sigma_v}{2}\sqrt{\Delta t}+O(\Delta t), \quad \text{if} \quad v_t=\overline{v}.\end{cases} \tag{A52}$$

Substituting (A52) and (A48) into equation (A46) for the interest rate r_t, we obtain r_t in (2.40) for the case when v_t is at the boundary.

To obtain the risk premium, we first decompose stock returns as follows:

$$\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}=\mu_t\Delta t+\sigma_t\Delta w_t+J_t\Delta j_t. \tag{A53}$$

Multiplying both sides of (A53) by $\xi_{A,t+\Delta t}/\xi_{At}$ and taking expectations, we obtain:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}\right]=\mu_t\Delta t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right]+\sigma_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right.$$

On the other hand, from the equation for stock price (2.17) we find that:

$$\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\frac{\Delta S_t+(1-l_A-l_B)D_{t+\Delta t}\Delta t}{S_t}\right]=1-\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\right].$$

Combining the last two equations and the equation (A46) for the interest rate, we obtain:

$$\mu_t-r_t=-\left(\sigma_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta w_t\right]+J_t\mathbb{E}_t\left[\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\Delta j_t\right]\right)\frac{1+r_t\Delta t}{\Delta t}. \tag{A54}$$

Then, proceeding in the same way as with the calculation of interest rates and using similar expansions, we obtain equation (2.41) for the risk premium. ■

Lemma A.3 (Useful expansions).
1) *For small increment* $\Delta v_t=v_{t+\Delta t}-v_t$ *the ratio* $\Big(s(v_{t+\Delta t})/s(v_t)\Big)^{-\gamma_A}$ *has expansion:*

$$\left(\frac{s(v_{t+\Delta t})}{s(v_t)}\right)^{-\gamma_A}=1+a_t\Delta v_t+b_t(\Delta v_t)^2+o(\Delta t), \tag{A55}$$

where coefficients a_t and b_t are given by:

$$a_t = \frac{(1-s_t)\Gamma_t}{\gamma_B}, \quad b_t = \frac{1}{2\gamma_B^2}(1-s_t)^2\Gamma_t^2 + \frac{1}{2\gamma_A^2\gamma_B^2}s_t(1-s_t)\Gamma_t^3, \tag{A56}$$

$\Gamma_t = \gamma_A\gamma_B/(\gamma_A(1-s)+\gamma_B s)$ *is the risk aversion of the representative investor and* s_t *is consumption share of investor A that solves equation* (2.14).

2) *For the case $J_D = 0$, the SPD in a one-investor economy can be expanded as follows:*

$$e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right)^{-\gamma_A} = 1 - r_A\Delta t - \kappa_A\Delta w_t + o(\Delta t), \tag{A57}$$

where r_A and κ_A are the riskless rate and the Sharpe ratio in an economy populated only by investor A, given by:

$$r_A = \rho + \gamma_A\mu_D - \frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2, \quad \kappa_A = \gamma_A\sigma_D. \tag{A58}$$

Proof of Lemma A.3. 1) We expand the ratio on the left-hand side of (A55) using Taylor's formula, and observe that $a_t = (s(v_t)^{-\gamma_A})'/s(v_t)^{-\gamma_A}$ and $b_t = 0.5(s(v_t)^{-\gamma_A})''/s(v_t)^{-\gamma_A}$. Differentiating, we obtain the following expressions for a_t and b_t:

$$a_t = -\gamma_A\frac{s'(v_t)}{s(v_t)}, \quad b_t = \frac{\gamma_A(1+\gamma_A)}{2}\left(\frac{s'(v_t)}{s(v_t)}\right)^2 - \frac{\gamma_A}{2}\frac{s''(v)}{s(v)}. \tag{A59}$$

To find derivatives $s'(v)$ and $s''(v)$, we differentiate equation (2.14) twice with respect to v, and obtain two equations for the derivatives:

$$1 = -\left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s'(v_t), \tag{A60}$$

$$0 = \left(\frac{\gamma_A}{s_t^2} - \frac{\gamma_B}{(1-s_t)^2}\right)(s'(v_t))^2 - \left(\frac{\gamma_A}{s_t} + \frac{\gamma_B}{1-s_t}\right)s''(v_t). \tag{A61}$$

Finding $s'(v)$ and $s''(v)$ from the system (A60)–(A61) and substituting them into expressions (A59) for coefficients a_t and b_t, after some algebra, we obtain expressions (A56).

2) Substituting $D_{t+\Delta t}/D_t$ from (2.1) into equation (A57), after some algebra, we

obtain:

$$
\begin{aligned}
e^{-\rho\Delta t}\left(\frac{D_{t+\Delta t}}{D_t}\right) &= e^{-\rho\Delta t}\left(1+\mu_D\Delta t+\sigma_D\Delta w_t\right)^{-\gamma_A} \\
&= (1-\rho\Delta t)\left(1-\left(\gamma_A\mu_D-\frac{\gamma_A(1+\gamma_A)}{2}\sigma_D^2\right)\Delta t-\gamma_A\sigma_D\right)+o(\Delta t) \\
&= 1-r_A\Delta t-\kappa_A\Delta w_t+o(\Delta t). \quad \blacksquare
\end{aligned} \tag{A62}
$$

Proof of Proposition 4. Consider a reflected arithmetic Brownian motion with boundaries $\underline{v}$ and $\overline{v}$ and dynamics $dv_t = \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t$ when $\underline{v} < v_t < \overline{v}$, where w_t is a Brownian motion. The transition density for this process is given by (see Veestraeten, 2004):

$$
\begin{aligned}
f_v(v,\tau;v_t,t) = \frac{1}{\sqrt{2\pi\widehat{\sigma}_v^2(\tau-t)}}\sum_{n=-\infty}^{+\infty}&\left[\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}n(\overline{v}-\underline{v})-\frac{\left(v-v_t-\widehat{\mu}_v(\tau-t)+2n(\overline{v}-\underline{v})\right)^2}{2\widehat{\sigma}_v^2(\tau-t)}\right)\right. \\
&\left.+\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(v_t-\underline{v}+n(\overline{v}-\underline{v})\right)-\frac{\left(v-v_t-\widehat{\mu}_v(\tau-t)+2(v_t-\underline{v}+n[\overline{v}-\underline{v}])\right)^2}{2\widehat{\sigma}_v^2(\tau-t)}\right)\right] \\
+\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\sum_{n=0}^{+\infty}&\left[\exp\left(-\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(\overline{v}-v+n[\overline{v}-\underline{v}]\right)\right)\mathcal{N}\left(\frac{v_t+\widehat{\mu}_v(\tau-t)-v-2(\overline{v}-v+n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}}\right)\right. \\
&\left.-\exp\left(\frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\left(v-\underline{v}+n[\overline{v}-\underline{v}]\right)\right)\left(1-\mathcal{N}\left(\frac{v_t+\widehat{\mu}_v(\tau-t)-v+2(v-\underline{v}+n[\overline{v}-\underline{v}])}{\widehat{\sigma}_v\sqrt{\tau-t}}\right)\right)\right],
\end{aligned} \tag{A63}
$$

where $\mathcal{N}(\cdot)$ is the cumulative distribution of a standard normal distribution. By $F_v(v,\tau;v_t,t) = \text{Prob}\{v_\tau \le v|v_t\}$ we denote the corresponding cumulative distribution function of v conditional on observing v_t at time t. We observe that $s_t = s(v_t)$ is a decreasing function of v_t implicitly defined by equation (2.14). From the latter equation we also find that $s^{-1}(x) = \gamma_B \ln(1-s) - \gamma_A \ln(s)$. The cumulative distribution function of consumption share s_τ at time τ conditional on observing s_t at time

t can then be found as follows:

$$\begin{aligned} F(x,\tau;s_t,t) &= \text{Prob}\{s_\tau \le x|s_t\} \equiv \text{Prob}\{s(v_\tau) \le x|s_t\} \\ &= 1-\text{Prob}\{v_\tau \le s^{-1}(x)|v_t\} \\ &= 1-\text{Prob}\{v_\tau \le \gamma_B \ln(1-x)-\gamma_A \ln(x)|v_t\} \\ &= 1-F_v(\gamma_B \ln(1-x)-\gamma_A \ln(x),\tau;v_t,t). \end{aligned} \tag{A64}$$

Substituting $v_t = \gamma_B \ln(1-s_t)-\gamma_A \ln(s_t)$ into (A64), differentiating CDF $F(x,\tau;s_t,t)$ with respect to x and setting $x = s$, we find that the transition PDF for s is given by:

$$f(s,\tau;s_t,t) = \left(\frac{\gamma_A}{s}+\frac{\gamma_B}{1-s}\right) f_v\Big(\gamma_B \ln(1-s)-\gamma_A \ln(s),\tau;\gamma_B \ln(1-s_t)-\gamma_A \ln(s_t),t\Big), \tag{A65}$$

where transition density $f_v(v,\tau;v_t,t)$ is given by equation (A63).

The stationary distribution of variable v, calculated in Veestraeten (2004), is given by:

$$f_v(v) = \frac{2\widehat{\mu}_v}{\widehat{\sigma}_v^2}\frac{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)v\Big)}{\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\overline{v}\Big)-\exp\Big((2\widehat{\mu}_v/\widehat{\sigma}_v^2)\underline{v}\Big)}. \tag{A66}$$

Proceeding in the same way as for the derivation of transition PDF (A65), we obtain stationary PDF (2.43) for consumption share s. ■

Proof of Corollary 2. The proof easily follows by substituting boundary conditions (2.34) into the equation (B27) for volatility σ_t at the boundary values $\underline{v}$ and $\overline{v}$. ■

Proof of Proposition 5. Consider Lagrangian (A6) for the dynamic optimization of investor i. Differentiating this Lagrangian with respect to l_i and c_{it}, we obtain:

$$\frac{\partial V_i(W_{it}^*,v_t;l_i)}{\partial l_i} = \eta_{it}D_t\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i\left[\frac{\partial V_i(W_{i,t+\Delta t}^*,v_{t+\Delta t};l_i)}{\partial l_i}\right], \tag{A67}$$

$$u'(c_{it}^*) = \eta_{it}. \tag{A68}$$

By the envelope theorem (e.g., Back (2010, p.162)):

$$\frac{\partial V_i(W_{it},v_t;l_i)}{\partial W} = u_i'(c_{it}^*), \tag{A69}$$

$$\frac{\partial V_i(W_{i,t+\Delta t},v_{t+\Delta t};l_i)}{\partial W} = u_i'(c_{i,t+\Delta t}^*). \tag{A70}$$

Substituting (2.45), (A68), (A69), and (A70) into equation (A67), and simplifying, we find:

$$\widehat{S}_{it} = D_t\Delta t + \mathbb{E}_t^i\left[e^{-\rho\Delta t}\frac{u_i'(c^*_{i,t+\Delta t})}{u_i'(c^*_{it})}\widehat{S}_{i,t+\Delta t}\right]. \tag{A71}$$

From equation (2.29), we recall that the SPD of investor A is given by

$$\frac{\xi_{A,t+\Delta t}}{\xi_{At}} = e^{-\rho\Delta t+\Delta U_t}\frac{(c^*_{A,t+\Delta t})^{-\gamma_A}}{(c^*_{At})^{-\gamma_A}}\frac{D_{t+\Delta t}}{D_t}, \tag{A72}$$

where $\Delta U_t = \max\{0; v_t + \mu_v\Delta t + \sigma_v\Delta w_t + J_v\Delta j_t - \overline{v}\}$. Rewriting equation (A71) for investor A in terms of SPD (A72), we obtain:

$$\widehat{S}_{At} = D_t\Delta t + \mathbb{E}_t^A\left[e^{-\Delta U_t}\frac{\xi_{A,t+\Delta t}}{\xi_{At}}\widehat{S}_{A,t+\Delta t}\right]. \tag{A73}$$

Following the same steps as in the proof of Lemma 2, we find that $\widehat{S}_{At} = \widehat{\Psi}_i(v_t; -\gamma_A)s(v_t)^{\gamma_A}D_t$, where $\widehat{\Psi}_i(v;\theta)$ satisfies differential-difference equation (2.33) with boundary conditions (2.48).

Iterating equation (2.17) for stock and equation (A73) for shadow prices, we obtain:

$$S_t + (1 - l_A - l_B)D_t\Delta t = \frac{1}{\xi_t}\mathbb{E}_t^A\left[\sum_{\tau=t}^{\infty}\xi_\tau(1 - l_A - l_B)D_\tau\Delta t\right], \tag{A74}$$

$$\widehat{S}_{At} = \frac{1}{\xi_t}\mathbb{E}_t^A\left[\sum_{\tau=t}^{\infty}e^{-(U_\tau - U_t)}\xi_\tau D_\tau\Delta t\right]. \tag{A75}$$

Inequality $(S_t + (1 - l_A - l_B)D_t\Delta t)/(1 - l_A - l_B) > \widehat{S}_{At}$ follows from the fact that $U_t = \sum_{\tau=0}^{t}\Delta U_\tau$ is a non-decreasing processes. In the continuous-time limit, we obtain that $S_t/(1 - l_A - l_B) > \widehat{S}_{At}$. Hence, the liquidity premium Λ_{At} is positive. The derivation of the shadow price of investor B is analogous and available upon request. ■

Appendix B: Technical results.

Proposition B.1 (Existence of boundaries $\underline{v}$ and $\overline{v}$). *There exist constant boundaries $\underline{v}$ and $\overline{v}$ for the state variable v_t process* (2.25) *that solve equations* (2.24).

Proof of Proposition B.1. We here show the existence of $\overline{v}$ that solves $\Phi_A(\overline{v}) = 0$, where $\Phi_A(v)$ is given by equation (A18) in Appendix A. The proof for $\underline{v}$ is analogous.

We note that $\Phi_A(v_t) \geq 0$ because of the constraint $W_{At} \geq 0$. Suppose, $\overline{v}$ does not exist, and hence $\Phi_A(v_t) > 0$ for all v_t. From equation (2.14) for consumption share s we observe that $s(v_t) \to 0$ when $v_t \to +\infty$. For arbitrary $\varepsilon \in (0, l_A)$ choose v_t sufficiently large, so that $s(v_t) - l_A < -\varepsilon$. Let $T(v_t)$ be the stopping time, defined as

$$T(v_t) = \inf\{\tau : s(v_\tau) - l_A \geq -\varepsilon\}. \tag{B1}$$

From equation (A18) for $\Phi_A(v_t)$ we obtain the following inequality:

$$\begin{aligned}
\Phi_A(v_t)s(v_t)^{-\gamma_A} \quad &\leq \quad -\varepsilon \mathbb{E}_t^A\left[\sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A}\Delta t\right] \\
&+ \quad \mathbb{E}_t^A\left[\sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A} s(v_\tau)^{-\gamma_A}(s(v_\tau)-\varepsilon)1_{\{s(v_\tau)\geq\varepsilon\}}\Delta t\right] \\
&\leq \quad -\varepsilon(l_A-\varepsilon)^{-\gamma_A}\mathbb{E}_t^A\left[\sum_{\tau=t}^{T(v_t)} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right] \\
&\quad + \max(1;\varepsilon^{1-\gamma_A})\mathbb{E}_t^A\left[\sum_{\tau=T(v_t)+\Delta t}^{+\infty} e^{-\rho(\tau-t)}\left(\frac{D_\tau}{D_t}\right)^{1-\gamma_A}\Delta t\right].
\end{aligned} \tag{B2}$$

Next, we show that $T(v_t) \to +\infty$ as $v_t \to +\infty$. Let $\widehat{v}$ be such that $s(\widehat{v}) = l_A - \varepsilon$. Then, because $s(v_t)$ is a decreasing function, $v_t \geq \widehat{v}$ and the stopping time (B1) can be rewritten as $T(v_t) = \inf\{\tau : v_\tau \leq \widehat{v}\}$. We note that $T(v_t) \geq \widehat{T}$, where $\widehat{T}$ is the minimal time required to get from v_t to $\widehat{v}$, which is the time when $\Delta w_t = -\sqrt{\Delta t}$ and $\Delta j_t = 1$ along the path. Time $\widehat{T}$ is found from the condition $v_t + (\widehat{T}/\Delta t)(\mu_v\Delta t - \sigma_v\sqrt{\Delta t} + J_v) = \widehat{v}$, where $J_v < 0$. We observe that $\widehat{T} \to +\infty$ as $v_t \to +\infty$, and hence $T(v_t) \to +\infty$. We also note that $\mathbb{E}_t[\sum_{\tau=t}^{\infty} e^{-r(\tau-t)}D_\tau^{1-\gamma_A}\Delta t] < +\infty$ by condition (2.15). Therefore, for a sufficiently large v_t we obtain from inequality (B2) that

$\Phi_A(v_t) < 0$, which contradicts initial assumption that $\Phi_A(v_t) > 0$ for all v_t. Hence, there exists $\overline{v}$ such that $\Phi_A(\overline{v}) = 0$. ■

Lemma B.1 (Unconstrained optimization). *Consider an infinitesimal unconstrained investor with risk aversion γ_i and labor income $l_i D_t$, $i = A, B$, who lives in the economy where the state price density is given by (2.29). The investor's value function is given by*

$$V_i^{unc}(W_t, v_t) = \frac{(W_t + l_i/(1 - l_A - l_B)S_t)^{1-\gamma_i}}{1-\gamma_i} h_i(v_t)^{\gamma_i}, \tag{B3}$$

where $h(v_t)$ is a uniformly bounded wealth-consumption ratio, given by:

$$h_i(v_t) = \mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \Big(\frac{\xi_{i\tau}}{\xi_{it}} \Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i} \Delta t \Big]. \tag{B4}$$

The investor's optimal consumption is given by $c_{i\tau}^* = \ell(\xi_{i\tau} e^{\rho(\tau-t)})^{-1/\gamma_i}$, where ℓ is a constant. Moreover, for all feasible consumptions c_t the following inequalities are satisfied:

$$\sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_\tau)\Delta t \le \sum_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(c_\tau^*)\Delta t \quad = \quad V_i^{unc}(W_t, v_t), \tag{B5}$$

$$\lim_{T\to\infty} \sup e^{-\rho T} \mathbb{E}_t \Big[V_i^{unc}(W_T, v_T) \Big] \quad \le \quad 0. \tag{B6}$$

Proof of Lemma B.1. We solve the problem using the martingale method. The static budget constraint is given by:

$$\mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} c_\tau^* \Big] = W_t + \frac{l_i S_t}{1 - l_A - l_B}, \tag{B7}$$

where the last term is the value of the labor income. Because the dividends and labor incomes are collinear, the value of the labor income is given by:

$$\mathbb{E}_t^i \Big[\sum_{\tau=t}^{+\infty} \frac{\xi_{i\tau}}{\xi_{it}} (l_i D_\tau) \Big] = \frac{l_i S_t}{1 - l_A - l_B}.$$

The first order condition gives the optimal consumption $c_\tau^* = \ell(\xi_{i\tau}/\xi_{it} e^{\rho(\tau-t)})^{-1/\gamma_i}$, where ℓ is the Lagrange multiplier that can be found by substituting c_τ^* into (B7). Finding the multiplier ℓ and substituting c_τ^* into the objective function, we obtain the value function (B3), where $h(v_t)$ is given by (B4).

Next, we show that $h(v_t)$ is uniformly bounded. First, we consider the case $\gamma_i \ge 1$. Using equation (B4) and Hölder's inequality, we obtain:

$$h_i(v_t) = \mathbb{E}_t^i\Big[\sum_{\tau=t}^{\infty}\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Big] \le \Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^{\infty}\frac{\xi_{i\tau}D_\tau}{\xi_{it}D_t}\Big]\Big)^{1-1/\gamma_i}\Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^{\infty} e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]\Big)^{1/\gamma_i}.$$

We note that both multipliers on the right-hand side of the latter inequality are bounded. The first multiplier equals the price-dividend ratio and is bounded by Proposition 2. The second multiplier is bounded due to condition (2.15) on the model parameters. Consider now the case $\gamma_i \le 1$. From the FOCs (2.23) and the fact that $\underline{s} \le s \le \overline{s}$, we obtain:

$$\frac{\xi_{iT}}{\xi_{it}} \ge e^{-\rho(T-t)}\Big(\frac{c_T^*}{c_t^*}\Big)^{-\gamma_i} \ge e^{-\rho(T-t)}\Big(\frac{D_T}{D_t}\Big)^{-\gamma_i}\Big(\frac{\overline{s}}{\underline{s}}\Big)^{-\gamma_i}.$$

From the latter inequality it follows that

$$\mathbb{E}_t^i\Big[\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Big] \le \Big(\frac{\overline{s}}{\underline{s}}\Big)^{1-\gamma_i}\mathbb{E}_t^i\Big[e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]. \tag{B8}$$

The inequality (B8) and condition (2.15) imply that the infinite series in (B4) converges and function $h_i(v)$ is uniformly bounded. We also observe that $h_i(v) \ge \Delta t > 0$.

Now, we prove inequality (B5). We consider feasible consumption streams satisfying condition $W_t + l_i/(1-l_A-l_B)S_t \ge 0$ for all t, which means that investor's aggregate wealth is non-negative at all times so that investor does not go bankrupt. From the investor's budget constraint and the latter inequality for all feasible consumptions we obtain:

$$W_t + \frac{l_iS_t}{1-l_A-l_B} \ge \mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}\frac{\xi_{i\tau}}{\xi_{it}}c_\tau\Delta t\Big] + \mathbb{E}_t^i\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T + \frac{l_iS_T}{1-l_A-l_B}\Big)\Big] \ge \mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}\frac{\xi_{i\tau}}{\xi_{it}}c_\tau\Delta t\Big]. \tag{B9}$$

Consider the weighting function w_t given by

$$w_\tau = \frac{\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}}{\widehat{h}_{iT}(v_t)}, \quad \text{where } \widehat{h}_{iT}(v_t) = \mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}\Big(\frac{\xi_{i\tau}}{\xi_{it}}\Big)^{1-1/\gamma_i} e^{-\rho(\tau-t)/\gamma_i}\Delta t\Big]. \tag{B10}$$

We note that $\mathbb{E}_t^i[\sum_{\tau=t}^{T} w_\tau\Delta t] = 1$. Using Jensen's inequality and inequality (B9), we obtain:

$$\begin{aligned}
\mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}\frac{e^{-\rho(\tau-t)}c_\tau^{1-\gamma_i}}{1-\gamma_i}\Delta t\Big] &= \mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}\frac{\Big((\xi_{i\tau}/\xi_{it})^{1/\gamma_i}e^{\rho(\tau-t)/\gamma_i}c_\tau\Big)^{1-\gamma_i}w_\tau\Delta t}{1-\gamma_i}\Big]\widehat{h}_{iT}(v_t)\\
&\le \frac{\Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}(\xi_{i\tau}/\xi_{it})^{1/\gamma_i}e^{\rho(\tau-t)/\gamma_i}c_\tau w_\tau\Delta t\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t)\\
&= \frac{\Big(\mathbb{E}_t^i\Big[\sum_{\tau=t}^{T}(\xi_{i\tau}/\xi_{it})c_\tau\Delta t\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t)^{\gamma_i}\\
&\le \frac{\Big(W_t+\frac{l_iS_t}{1-l_A-l_B}\Big)^{1-\gamma_i}}{1-\gamma_i}\widehat{h}_{iT}(v_t)^{\gamma_i}.
\end{aligned} \tag{B11}$$

Taking limit $T\to\infty$ in (B11), and noting that $\widehat{h}_{iT}(v_t)\to h_i(v_t)$, we obtain (B5).

Finally, we prove inequality (B6). Because $c_\tau\ge 0$, from inequality (B9), we obtain:

$$\mathbb{E}_t^i\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T+\frac{l_iS_T}{1-l_A-l_B}\Big)\Big]\le W_t+\frac{l_iS_t}{1-l_A-l_B}. \tag{B12}$$

Using Jensen's inequality following the same steps as in inequality (B11), we obtain:

$$\begin{aligned}
\frac{\mathbb{E}_t^i\Big[\Big(W_T+\frac{l_iS_T}{1-l_A-l_B}\Big)^{1-\gamma_i}\Big]}{1-\gamma_i} &\le \frac{\Big(\mathbb{E}_t^i\Big[\frac{\xi_{iT}}{\xi_{it}}\Big(W_T+\frac{l_iS_T}{1-l_A-l_B}\Big)\Big]\Big)^{1-\gamma_i}}{1-\gamma_i}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}}\Big]\Big)^{\gamma_i}\\
&\le \frac{\Big(W_t+\frac{l_iS_t}{1-l_A-l_B}\Big)^{1-\gamma_i}}{1-\gamma_i}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}}\Big]\Big)^{\gamma_i}.
\end{aligned}$$

The above inequality and the boundedness of $h_i(v_t)$ then imply the following inequality:

$$e^{-\rho(\tau-t)}\mathbb{E}_t^i[V_{iT}^{unc}]\le Const\times V_{it}^{unc}\Big(\mathbb{E}_t^i\Big[\Big(\frac{\xi_{iT}}{\xi_{it}}\Big)^{-\frac{1-\gamma_i}{\gamma_i}}e^{-\rho(\tau-t)/\gamma_i}\Big]\Big)^{\gamma_i}. \tag{B13}$$

Inequality (B13) also holds for $\gamma_i=1$ if CRRA preferences are replaced with logarithmic preferences. Suppose, $\gamma_i>1$. Then, inequality (B6) is satisfied because $V_i^{unc}<0$. Suppose, $\gamma_i\le 1$. Then, using inequalities (B8), (B13), and condition (2.15), we obtain:

$$e^{-\rho(\tau-t)}\mathbb{E}_t^i[V_{iT}^{unc}]\le Const\times\Big(\mathbb{E}_t^i\Big[e^{-\rho(\tau-t)}\Big(\frac{D_\tau}{D_t}\Big)^{1-\gamma_i}\Big]\Big)^{\gamma_i}\to 0,\ \text{as}\ \ T\to\infty.\ \blacksquare$$

Lemma B.2. *Let $\mathcal{P}(V)$ be a point-wise monotone operator such that for all point-wise bounded functions V_1 and V_2 such that $V_1 \leq V_2 \Rightarrow \mathcal{P}(V_1) \leq \mathcal{P}(V_2)$. Suppose further there exist point-wise bounded functions $\underline{V}$ and $\overline{V}$ such that $\underline{V} \leq \overline{V}$, $\mathcal{P}(\underline{V}) \geq \underline{V}$, and $\mathcal{P}(\overline{V}) \leq \overline{V}$. Then, there exists a point-wise bounded function V^* such that:* 1) $\underline{V} \leq V^* \leq \overline{V}$; 2) $V^* \leq \mathcal{P}(V^*)$; 3) $\mathcal{P}^n(\underline{V}) \to V^*$ *point-wise as $n \to \infty$.*

Proof of Lemma B.2. From the monotonicity of the operator $\mathcal{P}(V)$ and the definitions of $\underline{V}$ and $\overline{V}$, we obtain:

$$\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}(\overline{V}) \leq \overline{V}. \tag{B14}$$

Applying operator $\mathcal{P}$ to inequalities (B14), and then using the definitions of $\underline{V}$ and $\overline{V}$, we obtain: $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \overline{V}$. Proceeding in the same way n times we obtain $\underline{V} \leq \mathcal{P}(\underline{V}) \leq \mathcal{P}^2(\underline{V}) \leq \cdots \leq \mathcal{P}^n(\underline{V}) \leq \overline{V}$. Consequently, $\mathcal{P}^n(\underline{V})$ is point-wise increasing and bounded, and hence converges to some function V^* such that $\underline{V} \leq V^* \leq \overline{V}$ and $\mathcal{P}^n(\underline{V}) \leq V^*$. Applying operator to both sides of the latter inequality, we find that $\mathcal{P}^{n+1}(\underline{V}) \leq \mathcal{P}(V^*)$. Taking limit, we find that $V^* \leq \mathcal{P}(V^*)$. ■

Proposition B.2 (Verification of optimality). *Consider an infinitesimal investor i who lives in an economy where the state price density is given by equation* (2.29). *Suppose, this investor maximizes expected discounted utility* (2.7) *subject to a self-financing budget constraint and the collateral constraint* (2.10). *Then, there exists unique bounded value function V_i^* satisfying the dynamic programming equation* (2.20) *and the transversality condition, such that for all feasible consumptions*

$$V_{it}^* \geq \mathbb{E}_t^i\Big[\sum_{\tau=t}^{+\infty} u(c_{i\tau})\Delta t\Big], \tag{B15}$$

and, moreover,

$$V_{it}^* = \mathbb{E}_t^i\Big[\sum_{\tau=t}^{+\infty} u(c_{i\tau}^*)\Delta t\Big], \tag{B16}$$

for the optimal consumptions given by FOCs (2.23).

Proof of Proposition B.2. Consider the following operator:

$$\mathcal{P}_i(V) = \max_{c_t} \Big\{ u_i(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t^i[V_{i,t+\Delta t})] \Big\}, \quad i = A, B \tag{B17}$$

where maximization is subject to budget constraint (2.21) and collateral constraint

(2.22). Consider the following functions:

$$\underline{V}_{it} = \begin{cases} 0, & \gamma_i < 1, \\ \mathbb{E}_t^i \Big[\sum\limits_{\tau=t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_\tau)\Delta t \Big], & \gamma_i \geq 1, \end{cases} \qquad \overline{V}_{it} = \begin{cases} V_{it}^{unc}, & \gamma_i \leq 1, \\ 0, & \gamma_i > 1, \end{cases} \tag{B18}$$

where V_t^{unc} is given by (B3).

We observe that for $\gamma_i \geq 1$ function $\underline{V}_i$ is bounded due to condition (2.15) imposed on model parameters. Because $c_t = l_i D_t$ is feasible, we obtain that

$$\mathcal{P}(\underline{V}_i) \geq u_i(l_i D_t) + e^{-\rho\Delta t}\mathbb{E}_t^i \Big[\sum_{\tau=t+\Delta t}^{+\infty} e^{-\rho(\tau-t)} u_i(l_i D_t)\Delta t \Big] = \underline{V}_i.$$

For $\gamma_i < 1$ it is easy to see that $\mathcal{P}(\underline{V}_i) \geq \underline{V}_i$ because $u_i(c) > 0$. Next, we prove that $\mathcal{P}_i(\overline{V}_i) \leq \overline{V}_i$. The latter inequality is straightforward for $\gamma_i > 1$ because $\mathcal{P}_i(0) \leq 0$. Suppose now, $\gamma_i \leq 1$. Consider operator $\widetilde{\mathcal{P}}_i(V_i)$ given by equation (B17), where the maximization is subject to the budget constraint (2.21), but without the collateral constraint (2.22). Hence, $\mathcal{P}_i(V_i) \leq \widetilde{\mathcal{P}}_i(V_i)$. By Lemma B.1, $\overline{V}_i^{unc}$ is the solution of the unconstrained optimization, and hence $\overline{V}_i = \widetilde{\mathcal{P}}_i(\overline{V}_i)$. Therefore, $\mathcal{P}_i(\overline{V}_i) \leq \widetilde{\mathcal{P}}_i(\overline{V}_i) = \overline{V}_i$.

We drop subscript and superscript i for convenience. Consider the sequence $V_{n+1} = \mathcal{P}(V_n)$, with $V_0 = \underline{V}$, where $\underline{V}$ is given in (B18). Then, by Lemma B.2, $V_n \to V^*$ point-wise as $n \to \infty$. Next, we show that V^* is the value function and $\mathcal{P}(V^*) = V^*$. By the definition operator $\mathcal{P}(V)$ in (B17), for all feasible consumption streams

$$\begin{aligned} V_{n+1} &\geq u(c_t)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t \left[V_n(W_{t+\Delta t}; v_{t+\Delta t}) \right] \\ &\geq \mathbb{E}_t \Big[\sum_{\tau=t}^{n\Delta t} e^{-\rho(\tau-t)} u(c_\tau)\Delta t \Big] + e^{-\rho n\Delta t}\mathbb{E}_t[\underline{V}]. \end{aligned} \tag{B19}$$

Taking point-wise limit $n \to \infty$ in (B19) and taking into account that $\mathbb{E}_t[\underline{V}]$ is point-wise bounded, we obtain inequality (B15).

By Lemma B.2, $V^* \leq \mathcal{P}(V^*)$ and $V^* \leq \overline{V}$, where $\overline{V}$ is given in (B18), and hence

$$\begin{aligned} V^*(W_t, v_t) &\leq u(c_t^*)\Delta t + e^{-\rho\Delta t}\mathbb{E}_t \left[V^*(W_{t+\Delta t}; v_{t+\Delta t}) \right] \\ &\leq \mathbb{E}_t \Big[\sum_{\tau=t}^{T} u(c_\tau^*)\Delta t \Big] + e^{-\rho T}\mathbb{E}_t[V^*(W_T, v_T)] \\ &\leq \mathbb{E}_t \Big[\sum_{\tau=t}^{T} u(c_\tau^*)\Delta t \Big] + e^{-\rho T}\mathbb{E}_t[\overline{V}(W_T, v_T)], \end{aligned} \tag{B20}$$

where c^* is the optimal consumption that solves optmization in equation (B17).

We note that $\overline{V} = 0$ for $\gamma > 1$ and $\limsup e^{-\rho T}\mathbb{E}_t[\overline{V}(W_T, v_T)] \leq 0$ as $T \to \infty$ for $\gamma \leq 1$, by Lemma B.1. Taking limit $T \to \infty$ in (B20) we find that $V^* \leq \mathbb{E}_t\Big[\sum_{\tau=t}^{+\infty} u(c_\tau^*)\Delta t\Big]$, which along with inequality (B15) yields (B16). Equation (B16) along with inequality (B20) also imply that $V^* = \mathcal{P}(V^*)$. Moreover, V^* is point-wise bounded because $\underline{V} \leq V^* \leq \overline{V}$. Then, given the existence of the value function, the optimal consumptions are given by (2.23). Finally, we show that V^* satisfies the transversality condition. We note that $e^{-\rho(T-t)}\mathbb{E}_t[\underline{V}_T] \leq e^{-\rho(T-t)}\mathbb{E}_t[V_T^*] \leq e^{-\rho(T-t)}\mathbb{E}_t[\overline{V}_T]$. Taking limit $T \to 0$ we find that the upper and lower bound in the latter equation converge to 0, and hence the transversality condition is satisfied for V^*. ■

Proposition B.3 (Closed-form solutions).

1) *In the limit* $\Delta t \to 0$ *the price-dividend ratio* Ψ *and wealth-consumption ratios* Φ_i *are given by equations* (2.31) *and* (2.32), *where function* $\widehat{\Psi}(v;\theta)$ *is given by:*

$$\widehat{\Psi}(v;\theta) = \int_{\underline{v}}^{v} s(y)^\theta \widehat{\psi}(v-y)dy + \frac{\int_{\underline{v}}^{\overline{v}} s(y)^\theta \left[\widehat{\psi}'(\overline{v}-y) - \widehat{\psi}(\overline{v}-y)\right] dy}{1 + H\left(\widehat{\psi}(\overline{v}-\underline{v}) - \int_0^{\overline{v}-\underline{v}} \widehat{\psi}(y)dy\right)}\left(1 - H\int_0^{v-\underline{v}} \widehat{\psi}(y)dy\right), \tag{B21}$$

where $s(y)$ *solves equation*[9] (2.14), and $\widehat{\psi}(x)$, H *and some auxiliary variables are given by:*

$$\widehat{\psi}(x) = \frac{2}{\widehat{\sigma}_v^2}\sum_{n=0}^{\infty}\left[\left(\frac{2\lambda(1+J_D)^{1-\gamma_A}}{\widehat{\sigma}_v^2}\right)^n \frac{\exp\left((\zeta_+ + \zeta_-)(x + n\widehat{J}_v)/2\right)}{(\zeta_+ - \zeta_-)^{2n+1} n!}\right. \tag{B22}$$

$$\left. \times\, Q_n\left(\frac{(\zeta_+ - \zeta_-)(x + n\widehat{J}_v)}{2}\right)\mathbf{1}_{\{x+n\widehat{J}_v \geq 0\}}\right], \tag{B23}$$

$$Q_n(x) = \exp(-x)\sum_{m=0}^{n}(2x)^{n-m}\frac{(n+m)!}{m!(n-m)!} - \exp(x)\sum_{m=0}^{n}(-2x)^{n-m}\frac{(n+m)!}{m!(n-m)!}, \tag{B24}$$

$$H = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2 - \lambda(1+J_D)^{1-\gamma_A}, \tag{B25}$$

[9] Although $s(y)$ is not in closed form, we observe from equation (2.14) that its inverse is given by $s^{-1}(x) = \gamma_B \ln(x) - \gamma_A \ln(1-x)$. The change of variable $x = s(y)$ eliminates implicit functions, similar to Chabakauri (2015). We keep all integrals in terms of $s(y)$ because $s(y)$ is intuitive and easily computable from (2.14).

$$\zeta_{\pm} = -\frac{\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D \mp \sqrt{(\widehat{\mu}_v + (1-\gamma_A)\widehat{\sigma}_v\sigma_D)^2 + 2\widehat{\sigma}_v^2\left(\lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2\right)}}{\widehat{\sigma}_v^2}. \tag{B26}$$

2) *Stock return volatility in normal times and the jump size J_t are given by:*

$$\sigma_t = \sigma_D + \left(\frac{\widehat{\Psi}'(v_t; -\gamma_A)}{\widehat{\Psi}(v_t; -\gamma_A)} - \frac{\gamma_A(1-s(v_t))}{\gamma_A(1-s(v_t)) + \gamma_B s(v_t)}\right)\widehat{\sigma}_v, \tag{B27}$$

$$J_t = \frac{(1+J_D)\widehat{\Psi}\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}; -\gamma_A\Big)s\Big(\max\{\underline{v}; v_t + \widehat{J}_v\}\Big)^{\gamma_A}}{\widehat{\Psi}(v_t; -\gamma_A)s(v_t)^{\gamma_A}} - 1. \tag{B28}$$

Numbers of shares $n^*_{i,st}$ and leverage $L_{it} = -b_{it}B_{it}$ to market price S_t ratio are given by:

$$n^*_{i,st} = \frac{\Phi_i(v_t)\sigma_D + \Phi_i'(v_t)\widehat{\sigma}_v}{\Psi(v_t)\sigma_t}, \quad \frac{L_{it}}{S_t} = n_{i,st} - \frac{\Phi_i(v_t)}{\Psi(v_t)(1-l_A-l_B)}. \tag{B29}$$

Proof of Proposition B.3. 1) First, we solve the differential-difference equation in Lemma 2. We denote $g(x) = \widehat{\Psi}(x + \underline{v}; \theta)$ and apply the following changes of variables:

$$x = v - \underline{v}, \quad \tilde{\sigma} = \widehat{\sigma}_v, \quad \tilde{\mu} = \widehat{\mu}_v + (1-\gamma_A)\sigma_D\widehat{\sigma}_v, \quad \tilde{J} = -\widehat{J}_v, \quad \tilde{\lambda} = \lambda(1+J_D)^{1-\gamma_A},$$

$$\tilde{\rho} = \lambda + \rho - (1-\gamma_A)\mu_D + \frac{(1-\gamma_A)\gamma_A}{2}\sigma_D^2. \tag{B30}$$

Equations (2.33) and (2.34) with new variables now become:

$$\frac{\tilde{\sigma}^2}{2}g''(x) + \tilde{\mu}g'(x) - \tilde{\rho}g(x) + \tilde{\lambda}g(\max\{x - \tilde{J}, 0\}) + s(x+\underline{v})^{\theta} = 0, \tag{B31}$$

$$g'(0) = 0, \quad g(\overline{v} - \underline{v}) - g'(\overline{v} - \underline{v}) = 0. \tag{B32}$$

Let $\mathcal{L}[g(x)] = \int_0^\infty e^{-zx}g(x)dx$ be the Laplace transform of $g(x)$, and similarly for other functions. The Laplace transforms of $g'(x)$, $g''(x)$ and $g(\max\{x - \tilde{J}, 0\})$ are given by:

$$
\begin{aligned}
\mathcal{L}\left[g'(x)\right] &= z\mathcal{L}\left[g(x)\right] - g(0), \\
\mathcal{L}\left[g''(x)\right] &= z^2\mathcal{L}\left[g(x)\right] - zg(0) - g'(0), \\
\mathcal{L}\left[g(\max\{x-\tilde{J},0\})\right] &= \int_0^\infty e^{-zx} g(\max\{x-\tilde{J},0\})dx \\
&= \int_0^{\tilde{J}} e^{-zx} g(0)dx + \int_{\tilde{J}}^\infty e^{-zx} g(x-\tilde{J})dx \\
&= \frac{1}{z}(1-e^{-\tilde{J}z})g(0) + e^{-\tilde{J}z}\mathcal{L}\left[g(x)\right].
\end{aligned}
\tag{B33}
$$

Applying the transform to equation (B31), we arrive at the following equation:

$$
\begin{aligned}
\frac{\tilde{\sigma}^2}{2}\left(z^2\mathcal{L}\left[g(x)\right] - zg(0) - g'(0)\right) &+ \tilde{\mu}\left(z\mathcal{L}\left[g(x)\right] - g(0)\right) - \tilde{\rho}\mathcal{L}\left[g(x)\right] \\
&+ \tilde{\lambda}\left(e^{-\tilde{J}z}\mathcal{L}\left[g(x)\right] + \frac{1}{z}(1-e^{-\tilde{J}z})g(0)\right) + \mathcal{L}\left[s(x+\underline{v})^\theta\right] = 0.
\end{aligned}
\tag{B34}
$$

Applying boundary condition $g'(0) = 0$ and solving for $\mathcal{L}\left[g(x)\right]$, we obtain:

$$
\mathcal{L}\left[g(x)\right] = \frac{\mathcal{L}\left[s(x+\underline{v})^\theta\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} + g(0)\left(\frac{1}{z} - \frac{\tilde{\rho}-\tilde{\lambda}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right). \tag{B35}
$$

Now define a new function $\widehat{\psi}(x)$ through inverse Laplace transform

$$
\widehat{\psi}(x) = \mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right]. \tag{B36}
$$

Next, we apply inverse transform to each term in (B35). Noting that $\mathcal{L}^{-1}[1/z] = 1$ and using the theorem which states that Laplace transform of a convolution is the product of Laplace transforms, we derive the following inverse transforms:

$$
\begin{aligned}
\mathcal{L}^{-1}\left[\frac{\mathcal{L}\left[s(x+\underline{v})^\theta\right]}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}}\right] &= \int_0^x s(y+\underline{v})^\theta \cdot \widehat{\psi}(x-y)dy, \\
\mathcal{L}^{-1}\left[\frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \cdot \frac{1}{z}\right] &= \int_0^x \mathbf{1}_{\{y\geq 0\}} \cdot \widehat{\psi}(x-y)dy = \int_0^x \widehat{\psi}(y)dy.
\end{aligned}
\tag{B37}
$$

The linearity of the Laplace transform gives the following equation:

$$
g(x) = \mathcal{L}^{-1}\left[\mathcal{L}\left[g(x)\right]\right] = \int_0^x s(y+\underline{v})^\theta \cdot \widehat{\psi}(x-y)dy + g(0)\left[1 - \left(\tilde{\rho}-\tilde{\lambda}\right)\int_0^x \widehat{\psi}(y)dy\right]. \tag{B38}
$$

We calculate $g(0)$ below, and then after changing the variable back from x to $v = x + \underline{v}$, substituting in expressions for $\tilde{\rho}$ and $\tilde{\lambda}$ from (B30), we obtain (B21).

Next, we solve for $\widehat{\psi}(x)$ in closed form. We expand $\mathcal{L}\left[\widehat{\psi}(x)\right]$ as series, and sum up the inverse transforms of each term in the summation to get $\widehat{\psi}(x)$.

$$\begin{aligned}
\mathcal{L}\left[\widehat{\psi}(x)\right] &= \frac{1}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2 - \tilde{\lambda}e^{-\tilde{J}z}} \\
&= (\tilde{\rho} - \tilde{\mu}z - \tfrac{\tilde{\sigma}^2}{2}z^2)^{-1} \cdot (1 - \frac{\tilde{\lambda}e^{-\tilde{J}z}}{\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2})^{-1} \qquad \text{(B39)} \\
&= \sum_{n=0}^{\infty} \frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}.
\end{aligned}$$

The above series converges for z such that $|\tilde{\rho} - \tilde{\mu}z - (\tilde{\sigma}^2/2)z^2| > |\tilde{\lambda}\exp(-\tilde{J}z)|$. This holds if the real part of z is sufficiently large, e.g., $\Re(z) > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho} + \tilde{\lambda}}$. The inverse Laplace transform can then be calculated along the line $(\overline{z} - i\infty, \overline{z} + i\infty)$ in the complex domain where $\overline{z} > 4|\tilde{\mu}|/\tilde{\sigma}^2 + (2/\tilde{\sigma})\sqrt{\tilde{\rho} + \tilde{\lambda}}$, and hence, the inequality for $\Re(z)$ is satisfied.

Let $\zeta_- < \zeta_+$ be roots of $\tilde{\rho} - \tilde{\mu}z - \tilde{\sigma}^2 z^2/2 = 0$, given by (B26). We use the following inversion formula for $1/[(z - \zeta_+)(z - \zeta_-)]^{n+1}$ from Gradshteyn and Ryzhik (2007, p. 1117):

$$\mathcal{L}^{-1}\left[\frac{1}{[(z - \zeta_+)(z - \zeta_-)]^{n+1}}\right] = \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{x^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}x} I_{n+\frac{1}{2}}\left(\frac{\zeta_+ - \zeta_-}{2}x\right). \qquad \text{(B40)}$$

Function $e^{-n\tilde{J}z}$ in the complex domain corresponds to a shift from x to $x - n\tilde{J}$. Therefore,

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{\tilde{\lambda}^n e^{-n\tilde{J}z}}{(\tilde{\rho} - \tilde{\mu}z - \frac{\tilde{\sigma}^2}{2}z^2)^{n+1}}\right] &= \tilde{\lambda}^n \left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1} \mathbf{1}_{x \geq n\tilde{J}} \qquad \text{(B41)} \\
&\times \frac{\sqrt{\pi}}{\Gamma(n+1)} \frac{(x - n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x - n\tilde{J})} I_{n+\frac{1}{2}}\left(\frac{(\zeta_+ - \zeta_-)(x - n\tilde{J})}{2}\right).
\end{aligned}$$

Consequently, the explicit expression for $\widehat{\psi}(x)$ is given by:

$$\widehat{\psi}(x) = \sum_{n=0}^{\infty} \tilde{\lambda}^n \left(-\frac{\tilde{\sigma}^2}{2}\right)^{-n-1} \frac{\mathbf{1}_{\{x \geq n\tilde{J}\}}\sqrt{\pi}}{\Gamma(n+1)} \frac{(x - n\tilde{J})^{n+\frac{1}{2}}}{(\zeta_+ - \zeta_-)^{n+\frac{1}{2}}} e^{\frac{\zeta_+ + \zeta_-}{2}(x - n\tilde{J})} I_{n+\frac{1}{2}}\left(\frac{(\zeta_+ - \zeta_-)(x - n\tilde{J})}{2}\right), \qquad \text{(B42)}$$

where function $I_{n+\frac{1}{2}}(\cdot)$ is a modified Bessel function of the first kind, $\zeta_- < \zeta_+$ are given by (B26) and $\tilde{\rho}$, $\tilde{\mu}$, $\tilde{\sigma}$, $\tilde{\lambda}$, and $\tilde{J}$ are defined in (B30). Bessel function $I_{n+\frac{1}{2}}(\cdot)$ is given by (see equation 8.467 in Gradshteyn and Ryzhik (2007)):

$$I_{n+\frac{1}{2}}(z) = \frac{1}{\sqrt{2\pi z}}\left[e^z \sum_{m=0}^{n} \frac{(-1)^m (n+m)!}{m!(n-m)!(2z)^m} + (-1)^{n+1} e^{-z} \sum_{m=0}^{n} \frac{(n+m)!}{m!(n-m)!(2z)^m}\right]. \tag{B43}$$

Substituting (B43) into (B42), after minor algebra, we obtain expression (B23) for $\widehat{\psi}(x)$. The infinite series (B42) has only finite number of non-zero terms because for a fixed x indicators $1_{\{x \geq n\tilde{J}\}}$ vanish for sufficiently large n, and hence, (B42) is well-defined.

To find $g(0)$ in equation (B38), we first evaluate $\widehat{\psi}(0)$. From the above formula (B42), because $\mathbf{1}_{\{0 \geq n\tilde{J}\}} = 0$ for all $n > 0$, we obtain

$$\widehat{\psi}(0) = -\frac{2}{\tilde{\sigma}^2} \cdot \frac{e^{\zeta_+ \cdot 0} - e^{\zeta_- \cdot 0}}{\zeta_+ - \zeta_-} = 0. \tag{B44}$$

Differentiating (B38) and using $\widehat{\psi}(0) = 0$, we find:

$$g'(x) = \int_0^x s(y + \underline{v})^\theta \cdot \widehat{\psi}'(x - y)dy - g(0) \cdot \left(\tilde{\rho} - \tilde{\lambda}\right) \widehat{\psi}(x), \tag{B45}$$

We solve for $g(0)$ from the boundary condition $g(\overline{v} - \underline{v}) - g'(\overline{v} - \underline{v}) = 0$ and obtain:

$$g(0) = \frac{\displaystyle\int_0^{\overline{v}-\underline{v}} s(y + \underline{v})^\theta \cdot \left[\widehat{\psi}'(\overline{v} - \underline{v} - y) - \widehat{\psi}(\overline{v} - \underline{v} - y)\right] dy}{1 - \left(\tilde{\rho} - \tilde{\lambda}\right) \displaystyle\int_0^{\overline{v}-\underline{v}} \widehat{\psi}(y)dy + \left(\tilde{\rho} - \tilde{\lambda}\right) \widehat{\psi}(\overline{v} - \underline{v})}. \tag{B46}$$

Substituting (B46) into (B38), we derive equation (B21) for $\widehat{\Psi}(v;\theta)$.

2) Next we solve for stock volatility and jump size. In the unconstrained region $\underline{v} < v_t < \overline{v}$, stock price S_t, dividend D_t and state variable v_t follow processes:

$$\begin{aligned} dS_t &= S_t[\mu_t d_t + \sigma_t dw_t + J_t dj_t], \\ dD_t &= D_t[\mu_D d_t + \sigma_D dw_t + J_D dj_t], \\ dv_t &= \widehat{\mu}_v dt + \widehat{\sigma}_v dw_t + \left(\max\{\underline{v}; v_t + \widehat{J}_v\} - v_t\right) dj_t. \end{aligned} \tag{B47}$$

Applying Ito's lemma to $S_t = (1 - l_A - l_B)\widehat{\Psi}(v_t; -\gamma_A)s(v_t)^{\gamma_A} D_t$, and matching dw_t and dj_t terms, after some algebra, we obtain σ_t and J_t in Proposition B.3.

Equation equation (2.9) for $W_{i,t+\Delta t}$, implies the following expressions for $n^*_{i,st}$ and b^*_{it}:

$$
\begin{aligned}
n^*_{i,st} &= \sqrt{\frac{\mathrm{var}_t[W_{i,t+\Delta t} - W_{it}|\mathrm{normal}]}{\mathrm{var}_t[\Delta S_t + (1 - l_{\scriptscriptstyle A} - l_{\scriptscriptstyle B}) D_t \Delta t|\mathrm{normal}]}}, \\
b^*_{it} &= \mathbb{E}_t[W_{i,t+\Delta t|}\mathrm{normal}] - n_{it}\mathbb{E}_t[S_{t+\Delta t} + (1 - l_{\scriptscriptstyle A} - l_{\scriptscriptstyle B}) D_{t+\Delta t}\Delta t|\mathrm{normal}].
\end{aligned}
$$

Taking limit $\Delta t \to 0$ in the above expressions and using expansions similar to those in the proof of Lemma 2, we obtain the number of stocks and the leverage per the market value of stocks in equation (B29).

Chapter 3

Information Acquisition with Long Lived Assets

This paper studies information acquisition with a long-lived risky asset that generates dividends in each period. The investors can either be informed or uninformed, and the informed investors actively acquire information on the time-varying dividend growth rate. Informed investors take short positions in the variance swap to realize their informational advantage; the uninformed investor takes a long position to hedge his risks. Serial correlation of returns is decreasing in information acquisition of informed investors. Low uncertainty induces investors to acquire less information and decreases the cross-sectional dispersion of beliefs in expected returns.

Keywords: Information acquisition, Dividend growth, Variance swap, Rational expectations equilibrium

3.1. Introduction

Investors in financial markets acquire private signals on asset fundamentals and employ this information in their trading strategies. This fundamental value changes over time and the value of old signals decays as time progresses. Therefore information acquisition on this time-varying fundamental affects how asset returns are correlated in time-series and cross-sectional dimensions. In this paper, I develop a rational expectations model where investors endogenously acquire information and trade a long-lived dividend-paying asset as well as a volatility derivative.

The economy is populated by a continuum of informed investors and a representative uninformed investor. They trade a long-lived risky asset that generates dividends in each period, and the growth rate of dividends consists of a time-varying fundamental component and a noise component. The informed investors could acquire private signals about the fundamental at a cost that is quadratic in signal precision. Exogenous noisy supply of the asset prevents the price from fully revealing the fundamental. A variance swap that pays the difference between the realized variance of the excess return and a fixed strike is also traded in this economy.

The uninformed investor learns from the realized dividend growth and changes in prices. The informed investors have access to private signals in addition to the information that is available to the uninformed investor. An informed investor's expected excess return is increasing in the difference between his estimate of the dividend growth and that of the uninformed. Information acquisition leads to more accurate estimates of the excess return and reduces asset volatility from the informed investors' perspective.

The informed investors possess an information advantage over the uninformed investor and are willing to take short positions in the variance swap to profit from this advantage. The uninformed investor takes a long position because the variance swap insures his wealth against extreme movements of asset prices. In equilibrium, both investors take up positions such that their marginal utility weighted risk-neutral variances of returns are equal to the price of variance, measured by the strike of the variance swap.

The informed investors choose a level of signal precision that equates the marginal cost of information to the marginal value of information. The cost of information is quadratic and this limits the amount of information the investor is willing to acquire in a single period. The value of information is increasing in uncertainty about the fundamental dividend growth rate and noise trader demands. When uncertainty is high, investors face better investment opportunities and acquire more precise private signals as a result.

An investor's unexpected return and future expected returns are both increasing in the difference between the true value of the fundamental and this investor's estimate. This estimation error component contributes to positive serial correlation in asset returns. Information acquisition reduces the magnitude of this component and therefore decreases the autocorrelation of returns for the informed investors.

Highest cross-sectional dispersion of beliefs in asset returns is realized when information acquisition is at an intermediate level. When private signals are imprecise, all informed investors' estimates of both the return and the dividend growth are all close to those of the uninformed investor. When investors acquire highly precise private signals, all investors' estimates of the dividend growth are close to its true value. In both cases dispersion of beliefs is small. Lower uncertainty about the fundamental decreases the value of information, induces investors to acquire less information, and reduces the cross-sectional dispersion of beliefs.

This paper is related to an extensive literature on information acquisition in financial markets, initiated by Grossman and Stiglitz (1980) and Verrecchia (1982). The model builds upon the dynamic asymmetric information asset pricing framework developed by Wang (1993). That paper works with the assumption that informed investors are endowed with perfect information about the fundamental component of dividend growth. Wang (1993) solves the rational expectations equilibrium in closed form, and demonstrates that information asymmetry among investors increases the risk premium, price volatility, and negative autocorrelation in returns.

This paper instead allows investors to endogenously choose their signal precision. In equilibrium, no investor chooses to acquire perfectly informative signals because the fundamental is time-varying and the value of old signals decays as time progresses. Investors acquire different pieces of information and their sources of private information are assumed to be independent, as in Hellwig (1980). This assumption introduces cross-sectional dispersion of beliefs on expected returns among informed investors. In addition, the equilibrium is symmetric within the group of informed investors if the cost of information acquisition is the same across investors. This assumption facilitates the analytical solution of the model. If instead investors are competing for the same stream of information, no equilibrium where informed investors acquire the same precision of information can exist.

I further analyze the impact of information acquisition on the serial correlation of returns and investors' trading strategies. Both Wang (1993) and Brennan and Cao (1996) find that uninformed investors behave as rational trend-followers, while more informed investors follow a contrarian strategy. I establish a link between investors' trading strategies and the amount of information they privately acquired. Because information acquisition is affected by uncertainty, this analysis opens up a channel between uncertainty and differences in trading strategies.

This paper introduces a volatility derivative market which facilitates risk-sharing between informed and uninformed investors. The informed investors possess an information advantage on asset returns and sell insurance-like variance swaps to the uninformed investor. Brennan and Cao (1996) consider a model with one risky asset realizing a terminal payoff and a quadratic derivative is written on it. They found that the derivative does not reveal any additional information and effectively completes the market. Chabakarui, Yuan and Zachariadis (2016) further extend this result to economies without payoff normality. These papers assume that the fundamental value stays the same over time, whereas in this model the fundamental that investors acquire private signals on is time-varying. The market is not effectively complete because dividend strips are not traded and a noise component of dividend growth cannot be learned.

This paper is organized as follows. Section 2 introduces the model setup. Section 3 characterizes the equilibrium, solves the signal precision choice for the informed investor, and solves the portfolio demands for both informed and uninformed investors. Section 4 analyzes the impact of information acquisition on return autocorrelation and cross-sectional dispersion of beliefs. Section 5 concludes.

3.2. Model

I consider an infinite-horizon discrete time economy that features a single long-lived risky asset. The asset generates dividend d_t at date t and its ex-dividend price is denoted by p_t. The dividend growth consists of two components, one fundamental component f_t and one "noise" component $\epsilon^d_{t+1} \sim \mathcal{N}(0, \tau_d^{-1})$, where τ_d is a constant precision parameter.

$$d_{t+1} - d_t = f_t + \epsilon^d_{t+1}. \tag{3.1}$$

The fundamental component of dividend growth is persistent. It follows an AR(1) process and reverts to its long-run mean $\overline{f}$. The speed of mean reversion is represented by parameter λ.

$$f_{t+1} - f_t = -\lambda(f_t - \overline{f}) + \epsilon^f_{t+1}, \quad \epsilon^f_{t+1} \sim \mathcal{N}(0, \tau_f^{-1}). \tag{3.2}$$

Investors and Information Acquisition

The investors in the economy can be either informed or uninformed about the fundamental component of dividend growth f_t. There is a continuum of informed investors indexed by $i \in [0, 1]$ and a representative uninformed investor with a mass of unity. Both types of investors learn from the time series of prices and dividends.

The informed investors have access to the public information and also have an opportunity to acquire private signals. The signal at date t for investor i is represented by s_{it}. The signal is acquired before trading starts and the investor could employ this signal in his portfolio choice at date t. The precision of the private signal a_{it} is endogenously determined by the investor.

$$s_{it} = f_t + \epsilon^s_{it}, \quad \epsilon^s_{it} \sim \mathcal{N}(0, a_{it}^{-1}). \tag{3.3}$$

New information comes at a cost that is quadratic in the signal precision. This cost takes a quadratic form $ka_{it}^2/2$. The marginal cost of information ka_{it} is proportional to the signal precision.

For simplicity, I assume that investors acquire different pieces of information and their sources of private information are independent. The noises in private signals ϵ^s_{it} are independent both over time and across investors.

Assets

All investors have access to a risk-free storage technology with a constant rate of return r. They can borrow and lend at this interest rate with no additional cost. The excess return of the risky asset is $(p_{t+1} - p_t + d_{t+1}) - rp_t$, which we denote as r^e_{t+1}.

Demand from the noise traders prevents the price from fully revealing the fundamental. Let z_t denote the difference between the total supply of the risky asset and its noisy demand. The residual supply z_t has a long-run stationary level $\overline{z}$:

$$z_{t+1} - z_t = -\lambda(z_t - \overline{z}) + \epsilon^z_{t+1}, \quad \epsilon^z_{t+1} \sim \mathcal{N}(0, \sigma^2_z). \tag{3.4}$$

There is a volatility derivative market in this economy, and the underlying asset is the risky asset. At date t, investors could trade a one-period variance swap exchanges the realized variance of the excess return $(r^e_{t+1})^2$ for some fixed "strike" v_t at date $t+1$. v_t is typically set to make the value of the payoff $(r^e_{t+1})^2 - v_t$ zero at the initiation of the trade. The net supply of this one-period variance swap is zero.

Preferences and Investor Optimization

At date t, the representative uninformed investor allocates his wealth W_{ut} to c_{ut} units of consumption, θ_{ut} units of the risky asset, and ψ_{ut} units of one-period variance swap. He invests the remaining $W_{ut} - c_{ut} - \theta_{ut}p_t$ in the risk-free storage technology. Let W_{it} represent the wealth of informed investor i before trading starts but after the acquisition of information at date t. Similarly, his consumption and portfolio choices are denoted by c_{it} and (θ_{it}, ψ_{it}).

All investors have constant absolute risk aversion (CARA) preference with time preference parameter ρ and risk aversion parameter A. They maximize their expected utilities:

$$\mathbb{E}^u_t \left[\sum_{s=t}^{\infty} - \exp\left(-\rho(s-t) - Ac_{us}\right) \right], \tag{3.5}$$

$$\mathbb{E}^i_t \left[\sum_{s=t}^{\infty} - \exp\left(-\rho(s-t) - Ac_{is}\right) \right]. \tag{3.6}$$

subject to their self-financing budget constraints:

$$W_{u,t+1} = (1+r)(W_{ut} - c_{ut}) + \theta_{ut} r^e_{t+1} + \psi_{ut} \left[(r^e_{t+1})^2 - v_t \right], \tag{3.7}$$

$$W_{i,t+1} = (1+r)(W_{it} - c_{it}) + \theta_{it} r^e_{t+1} + \psi_{it} \left[(r^e_{t+1})^2 - v_t \right] - \frac{k}{2} a^2_{i,t+1}. \tag{3.8}$$

Definition of Equilibrium

The definition of equilibrium is standard. Investors make the optimal portfolio and signal precision choices and the market clears.

Definition 1. *The equilibrium is a set of risky asset prices p_t, variance swap strikes v_t, portfolio policies (θ_{ut}, ψ_{ut}) and (θ_{it}, ψ_{it}), and signal precision policies a_{it} that solve the optimization problem (3.5) and (3.6) for uninformed and informed investors and clear the asset markets:*

$$\theta_{ut} + \int_0^1 \theta_{it} di = z_t, \tag{3.9}$$

$$\psi_{ut} + \int_0^1 \psi_{it} di = 0. \tag{3.10}$$

3.3. Characterization of Equilibrium

I characterize a stationary equilibrium in a three-step process similar to that of Wang (1993). I first conjecture an equilibrium price function and derive investors' belief processes. Next, I solve the signal precision choices for the informed investors and the portfolio choices for both informed and uninformed investors. Market clearing is then imposed to pin down the coefficients in the conjectured price function.

3.3.1 Asset Price and Evolution of Beliefs

The uninformed investor learn from the entire history of prices and dividends. Let $\widehat{f_{ut}} = \mathbb{E}_t^u[f_t]$ denote the expectation of the fundamental dividend growth rate by the uninformed investor and $\tau_{ut} = (\mathrm{Var}_t^u[f_t])^{-1}$ denote the precision of this belief. In contrast to Wang(1993), the informed investors do not directly observe the fundamental f_t. They have access to private signals in addition to the information that is available to the uninformed investor. I use $\tau_{it} = (\mathrm{Var}_t^i[f_t])^{-1}$ to represent the belief precision of investor i after information acquisition at date t, and $\widehat{f_{it}} = \mathbb{E}_t^i[f_t]$ to represent the conditional expectation after incorporating the private signal s_{it}.

The uninformed investor's expectation of the present value of future dividends is given by:

$$\mathbb{E}_t^u\left[\sum_{s=t+1}^{\infty} e^{-r(s-t)} d_s\right] = \frac{1}{r} d_t + \frac{1+r}{r(r+\lambda)}\widehat{f_u} + \frac{(1+r)\lambda}{r^2(r+\lambda)}\overline{f} \tag{3.11}$$

The equilibrium price differs in two aspects from the above expression. Informed investors also incorporate private signals in their trading strategies. Investor i's asset demand is increasing in $\widehat{f_{it}} - \widehat{f_{ut}}$, the difference between his estimate of the dividend growth rate and that of the uninformed. Because noises in informed investors' private signals are independent, the average difference in beliefs between informed and uninformed investors is equal to $f_t - \widehat{f_{ut}}$. As a result, the asset price is increasing in the average difference in beliefs. The investors also demand a risk premium for the risk involved in uncertain cash flows. This risk premium is larger when the supply of the risky asset z_t is higher.

Because investors have CARA preferences and all state variables are jointly normally distributed, we look for the equilibrium price in a form that is linear in the average difference in beliefs $f_t - \widehat{f_{ut}}$ and the asset supply z_t:

$$p_t = \frac{1}{r}d_t + \frac{1+r}{r(r+\lambda)}\widehat{f_{ut}} + \frac{(1+r)\lambda}{r^2(r+\lambda)}\overline{f} - b_0 + b_f(f_t - \widehat{f_{ut}}) - b_z(z_t - \overline{z}). \tag{3.12}$$

Coefficients b_f and b_z are both positive and respectively measure the sensitivity of price to $f_t - \widehat{f_{ut}}$ and z_t. b_0 represents the average discount of the asset price when the average difference is zero and the asset supply is equal to its long-run stationary level. Because $\widehat{f_{ut}}$ belongs to the information set of both informed and uninformed investors, price p_t is informationally equivalent to its sufficient statistic $b_f f_t - b_z z_t$. This sufficient statistic also belongs to both investors' information set. This implies that the uninformed investor's expectation of this sufficient statistic $\mathbb{E}_t^u[b_f f_t - b_z z_t]$ is equal to its true value $b_f f_t - b_z z_t$. As a result, the uninformed investor's expectation of the asset supply $\widehat{z_{ut}} = \mathbb{E}_t^u[z_t]$ is given by:

$$\widehat{z_{ut}} = z_t + \frac{b_f}{b_z}(\widehat{f_{ut}} - f_t). \tag{3.13}$$

The uninformed investor updates his belief from both changes in prices and the realized dividend growth. This belief is therefore affected by the shocks to asset supply and the noise component of dividend growth. Lemma 1 reports how their expectation and belief precision evolve. Proofs of all lemmas and propositions are given in the appendix.

Lemma 1. *The law of motion of the uninformed investor's belief precision τ_{ut}, expectation of the dividend growth rate $\widehat{f_{ut}}$, and expectation of asset supply $\widehat{z_{ut}}$ are* *given by:*

$$\tau_{u,t+1} = \left[(1-\lambda)^2(\tau_{ut}+\tau_d)^{-1} + \frac{b_z^2\tau_f^{-1}\tau_z^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\right]^{-1}, \tag{3.14}$$

$$\begin{aligned}\widehat{f_{u,t+1}} &= \lambda\overline{f} + (1-\lambda)\frac{\tau_{ut}}{\tau_{ut}+\tau_d}\widehat{f_{ut}} + (1-\lambda)\frac{\tau_d}{\tau_{ut}+\tau_d}f_t \\ &\quad + (1-\lambda)\frac{\tau_d}{\tau_{ut}+\tau_d}\epsilon^d_{t+1} + \frac{b_f\tau_f^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon^f_{t+1} - b_z\epsilon^z_{t+1}\right),\end{aligned} \tag{3.15}$$

$$\begin{aligned}\widehat{z_{u,t+1}} &= \lambda\overline{z} + (1-\lambda)\frac{\tau_{ut}}{\tau_{ut}+\tau_d}\widehat{z_{ut}} + (1-\lambda)\frac{\tau_d}{\tau_{ut}+\tau_d}z_t \\ &\quad + \frac{b_f}{b_z}(1-\lambda)\frac{\tau_d}{\tau_{ut}+\tau_d}\epsilon^d_{t+1} - \frac{b_z\tau_z^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon^f_{t+1} - b_z\epsilon^z_{t+1}\right).\end{aligned} \tag{3.16}$$

The uninformed investor's belief precision is constant over time in a stationary equilibrium and I use τ_u to represent this constant level. The dynamics of the informed investors' expectations and belief precision are similar to that of the uninformed investor and are provided in Lemma 2.

Lemma 2. *The law of motion of the informed investors i's belief precision τ_{it} and expectation $\widehat{f_{it}}$ are given by:*

$$\tau_{i,t+1} = \left[(1-\lambda)^2(\tau_{it}+\tau_d)^{-1} + \frac{b_z^2\tau_f^{-1}\tau_z^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\right]^{-1} + a_{i,t+1}, \tag{3.17}$$

$$\begin{aligned}\widehat{f_{i,t+1}} &= \left(1-\frac{a_{i,t+1}}{\tau_{i,t+1}}\right)\Bigg[\lambda\overline{f} + (1-\lambda)\frac{\tau_{it}}{\tau_{it}+\tau_d}\widehat{f_{it}} + (1-\lambda)\frac{\tau_d}{\tau_{it}+\tau_d}f_t \\ &\quad +(1-\lambda)\frac{\tau_d}{\tau_u+\tau_d}\epsilon^d_{t+1} + \frac{b_f\tau_f^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon^f_{t+1} - b_z\epsilon^z_{t+1}\right)\Bigg] \\ &\quad + \frac{a_{i,t+1}}{\tau_{i,t+1}}\left(f_{t+1} + \epsilon^s_{i,t+1}\right).\end{aligned} \tag{3.18}$$

3.3.2 Portfolio and Signal Precision Choices

I start this section with analysis on investor's investment opportunities, with focus on the excess return of the asset $r^e_{t+1} = p_{t+1} + d_{t+1} - (1+r)p_t$. Because the asset price depends on the uninformed investor's estimate of the fundamental $\widehat{f_{ut}}$, the return also relies on the belief dynamics of the uninformed investors. Let $b_{fu} = (1+r)/r(r+\lambda) - b_f$ denote the coefficient of $\widehat{f_{ut}}$ in the equilibrium price function (3.12). From equations (3.12) and (3.15)-(3.18) I arrive at the excess return of the asset and investors' conditional expectations. Lemma 3 reports these results.

Lemma 3. *The excess return of the risky asset r^e_{t+1} is given by:*

$$\begin{aligned} r^e_{t+1} = rb_0 &+ \left[r + \lambda + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d} \right] b_{fu}(f_t - \widehat{f_{ut}}) + (r+\lambda) b_z (z_t - \overline{z}) \\ &+ \left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u+\tau_d} b_{fu} \right) \epsilon^d_{t+1} + \left(1 + \frac{b_f \tau_f^{-1}}{b_f^2 \tau_f^{-1} + b_z^2 \tau_z^{-1}} b_{fu} \right) \left(b_f \epsilon^f_{t+1} - b_z \epsilon^z_{t+1} \right). \end{aligned} \tag{3.19}$$

The expected excess return from the uninformed investor's perspective is:

$$\mathbb{E}^u_t[r^e_{t+1}] = rb_0 + (r+\lambda) b_z (\widehat{z_{ut}} - \overline{z}). \tag{3.20}$$

For informed investor i, the expected excess return is:

$$\mathbb{E}^i_t[r^e_{t+1}] = rb_0 + \left[\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u+\tau_d} b_{fu} \right] (\widehat{f_{it}} - \widehat{f_{ut}}) + (r+\lambda) b_z (\widehat{z_{ut}} - \overline{z}). \tag{3.21}$$

Similar to the asset price, the excess return r^e_{t+1} is increasing in the asset supply z_t and the average difference in beliefs between informed and uninformed investors $f - \widehat{f_{ut}}$. Both investors' expected returns depend on their estimation of the asset supply. From informed investor i's perspective, the expected return is linear in the difference between his estimate of the fundamental $\widehat{f_{it}}$ and that of the uninformed investor $\widehat{f_{ut}}$. The private signals acquired by the informed investors provide them a more precise estimate of the time series of asset returns.

Let $\sigma^2_{ut} = \mathrm{Var}^u_t[r^e_{t+1}]$ denote the conditional variance of the excess return for the uninformed investor and $\sigma^2_{it} = \mathrm{Var}^i_t[r^e_{t+1}]$ denote that for the informed investor. The informed investors face less uncertainty compared to the uninformed investor, and this difference is increasing in the amount of information they privately acquired. Different beliefs about uncertainty provide the informed investors an incentive to take short positions in the variance swap and the uninformed investor an incentive to take a long position.

Consider the situation where I weight each outcome of the excess return by one investor's marginal utility. The resulting measure is this investor's risk-neutral distribution of r^e_{t+1}, and I refer to the variance of this distribution as the risk-neutral variance. In the absence of the variance swap market, the risk-neutral variances of the excess return differ across investors. The introduction of the variance swap market equalizes this difference.

When the uninformed investor takes zero position in the variance swap market, his risk-neutral variance is higher than the strike of variance swap v_t. Therefore it is profitable for him to take a long position to hedge extreme risks. Long positions in the variance swap increase the utility when price movements are large and therefore decreases the risk-neutral variance. The uninformed investor will continue to take long positions until his risk-neutral variance is equal to the strike of the variance swap. Similarly, the informed investors start with a risk-neutral variance lower than the strike of variance swap and will continue to take short positions until their risk-neutral variances reach v_t. Thus v_t represents the variance of the excess return for all investors in their risk-neutral measures.

Now I investigate the optimization problem of the uninformed investor. The only state variable that plays a role in this investor's next period expected return is his estimate of the asset supply $\widehat{z_{ut}}$. Furthermore, this state variable is Markovian from the uninformed investor's perspective. From equation (3.16), the conditional expectation of $\widehat{z_{u,t+1}}$ only relies on $\widehat{z_{ut}}$:

$$\mathbb{E}_t^u[\widehat{z_{u,t+1}}] = \lambda \overline{z} + (1-\lambda)\widehat{z_{ut}}. \tag{3.22}$$

Therefore all information about future investment opportunities at date t is summarized by $\widehat{z_{ut}}$. As a result, the value function of the uninformed investor only depends on $\widehat{z_{ut}}$ and not $\widehat{f_{ut}}$.

With the above analysis in mind, I proceed to solve the uninformed investor's optimization problem. Proposition 1 reports the portfolio choice of the uninformed investor.

Proposition 1. *Let* $\sigma_{uz}^2 = \mathrm{Var}_t^u[\widehat{z_{u,t+1}}]$ *denote the conditional variance of* $\widehat{z_{u,t+1}}$ *and* $\rho_{urz} = \mathrm{Corr}_t^u[r_{t+1}^e, \widehat{z_{u,t+1}}]$ *denote the conditional correlation of* $\widehat{z_{u,t+1}}$ *and* r_{t+1}^e*. The value function of the uninformed investor has the following form:*

$$V(W_{ut}, \widehat{z_{ut}}) = -\exp\left(-rAW_{u,t} - \frac{\alpha_{uz}}{2}(\widehat{z_{ut}} - \overline{z})^2 - \beta_{uz}(\widehat{z_{ut}} - \overline{z}) - \gamma_{uz}\right). \tag{3.23}$$

The uninformed investor's demand for the risky asset is:

$$\theta_{ut} = (1 + \alpha_{uz}^* \rho_{urz}^2 \sigma_{uz}^2)\frac{\mathbb{E}_t^u[r_{t+1}^e]}{rA\sigma_{ut}^2} - [\alpha_{uz}^*(1-\lambda)(\widehat{z_{ut}} - \overline{z}) + \beta_{uz}^*]\frac{\rho_{urz}\sigma_{uz}}{rA\sigma_{ut}}. \tag{3.24}$$

where α_{uz}^* *and* β_{uz}^* *are functions of* α_{uz}*,* β_{uz}*,* ρ_{iuz} *and* σ_{uz}

His position in the variance swap is given by:

$$\psi_{ut} = \frac{1}{2rAv_t} - (1 + \alpha^*_{uz}\rho^2_{urz}\sigma^2_{uz})\frac{1}{2rA\sigma^2_{ut}}. \tag{3.25}$$

The myopic demand for the risky asset is determined by $\mathbb{E}^u_t[r^e_{t+1}]$, which is linear in the uninformed investor's estimate of the supply $\widehat{z_{ut}}$. $\widehat{z_{ut}}$ also naturally appears in the investor's hedging demand. Therefore the uninformed investor's demand for the risky asset is linear in $\widehat{z_{ut}}$.

If the variance swap is not traded, the risk-neutral variance of excess return for the uninformed investor is equal to $\sigma^2_{ut}/(1 + \alpha^*_{uz}\rho^2_{rz}\sigma^2_z)$. $\widehat{z_{u,t+1}}$ measures the next period expected return of this asset and is correlated with the current period unexpected return $r^e_{t+1} - \mathbb{E}^u_t[r^e_{t+1}]$. The investor benefits from better future investment opportunities when price movements are large. This increase in utility in these states reduces the risk-neutral variance to a level below that in the objective physical measure σ^2_{ut}. The investor takes positions in the variance swap if $\sigma^2_{ut}/(1 + \alpha^*_{uz}\rho^2_{rz}\sigma^2_z)$ is different from the strike v_t and the demand is given by equation (3.25).

Next I look at the portfolio choice of the informed investors. Different from that of the uninformed investor, the expected excess return of the informed investors consists of two components: difference in belief $\widehat{f_{it}} - \widehat{f_{ut}}$ and the uninformed investor's estimate of asset supply $\widehat{z_{ut}}$. A high precision of belief provides an accurate estimate of the current period excess return. It also translates into precise predictions about future investment opportunities. The dynamics of both $\widehat{f_{it}} - \widehat{f_{ut}}$ and $\widehat{z_{ut}}$ rely on investor i's estimate of f_t:

$$\mathbb{E}^i_t[\widehat{f_{i,t+1}} - \widehat{f_{u,t+1}}] = (1-\lambda)\frac{\tau_u}{\tau_u + \tau_d}(\widehat{f_{it}} - \widehat{f_{ut}}), \tag{3.26}$$

$$\mathbb{E}^i_t[\widehat{z_{u,t+1}}] = \overline{z} + (1-\lambda)(\widehat{z_{ut}} - \overline{z}) + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}\frac{b_f}{b_z}(\widehat{f_{it}} - \widehat{f_{ut}}). \tag{3.27}$$

All information about future investment opportunities at date t is summarized by $\widehat{f_{it}} - \widehat{f_{ut}}$ and $\widehat{z_{ut}}$, and both state variables enter the informed investors value function. In this model, W_{it} represents the date t wealth before trading but after the acquisition of information. I similarly define the value function as the investor's expected utility after the incorporation of the private signal into his belief. The value function and the portfolio choice of the informed investors are provided in Proposition 2.

Proposition 2. *Let* $\sigma_{iz}^2 = \mathrm{Var}_t^i[\widehat{z_{u,t+1}}]$ *and* $\sigma_{if}^2 = \mathrm{Var}_t^i[\widehat{f_{it}} - \widehat{f_{ut}}]$ *denote informed investor i's conditional variances. Let* $\rho_{irz} = \mathrm{Corr}_t^i[r_{t+1}^e, \widehat{z_{u,t+1}}]$ *and* $\rho_{irf} = \mathrm{Corr}_t^i[r_{t+1}^e, \widehat{f_{i,t+1}} - \widehat{f_{u,t+1}}]$ *denote the conditional correlations. The value function of the informed investor i has the following form:*

$$V(W_{ut}, \widehat{z_{ut}}, \widehat{f_{it}} - \widehat{f_{ut}}, \tau_{it}) = -\exp\Big(-rAW_{it} - \frac{\alpha_z}{2}(\widehat{z_{ut}} - \overline{z})^2 - \frac{\alpha_f \tau_{it}}{2}(\widehat{f_{it}} - \widehat{f_{ut}})^2 \\ -\alpha_{zf}(\widehat{z_{ut}} - \overline{z})(\widehat{f_{it}} - \widehat{f_{ut}}) - \beta_z(\widehat{z_{ut}} - \overline{z}) - \beta_f(\widehat{f_{it}} - \widehat{f_{ut}}) - \gamma(\tau_{it})\Big). \tag{3.28}$$

The uninformed investor's demand for the risky asset is:

$$\begin{aligned}\theta_{it} = &\left(1 + \alpha_z^* \rho_{irz}^2 \sigma_{iz}^2 + \alpha_f^* \tau_{i,t+1} \rho_{irf}^2 \sigma_{if}^2 + 2\alpha_{zf}^* \rho_{irz}\rho_{irf}\sigma_{iz}\sigma_{if}\right) \frac{\mathbb{E}_t^u[r_{t+1}^e]}{rA\sigma_{ut}^2}, \\ &- \left[\alpha_z^*\, \mathbb{E}_t^i[\widehat{z_{u,t+1}} - \overline{z}] + \alpha_{zf}^*\, \mathbb{E}_t^i[\widehat{f_{i,t+1}} - \widehat{f_{u,t+1}}] + \beta_z^*\right] \frac{\rho_{irz}\sigma_{iz}}{rA\sigma_{it}} \\ &- \left[\alpha_{zf}^*\, \mathbb{E}_t^i[\widehat{z_{u,t+1}} - \overline{z}] + \alpha_f^* \tau_{i,t+1}\, \mathbb{E}_t^i[\widehat{f_{i,t+1}} - \widehat{f_{u,t+1}}] + \beta_f^*\right] \frac{\rho_{irf}\sigma_{if}}{rA\sigma_{it}}.\end{aligned} \tag{3.29}$$

His position in the variance swap is given by:

$$\psi_{it} = \frac{1}{2rAv_t} - \left(1 + \alpha_z^* \rho_{irz}^2 \sigma_{iz}^2 + \alpha_f^* \tau_{i,t+1} \rho_{irf}^2 \sigma_{if}^2 + 2\alpha_{zf}^* \rho_{irz}\rho_{irf}\sigma_{iz}\sigma_{if}\right) \frac{1}{2rA\sigma_{it}^2}. \tag{3.30}$$

Informed investor i's demand for the risky asset is linear in the two state variables $\widehat{z_{ut}}$ and $\widehat{f_{it}} - \widehat{f_{ut}}$. The investor's position on the variance swap ψ_{it} depends on both the strike v_t and his risk-neutral variance in the absence of the variance market.

Last I come to the informed investors' information acquisition choices. Because of normal distribution and CARA preference, the signal precision choices do not depend on the level of wealth or previous estimates of the asset supply $\widehat{z_{u,t-1}}$ or the fundamental $\widehat{f_{i,t-1}} - \widehat{f_{u,t-1}}$. The value function $V(W_{ut}, \widehat{z_{ut}}, \widehat{f_{it}} - \widehat{f_{ut}}, \tau_{it})$ given in Proposition 2 represents investor i's expected utility after the acquisition of information. At the beginning of date t, investor i chooses the precision of his private signal a_{it} to maximize this objective. Investor i's belief precision τ_{it} follows the law of motion given in Lemma 2 and maps one-to-one to a_{it}:

$$\tau_{it} = \left[(1-\lambda)^2(\tau_{i,t-1} + \tau_d)^{-1} + \frac{b_z^2 \tau_f^{-1} \tau_z^{-1}}{b_f^2 \tau_f^{-1} + b_z^2 \tau_z^{-1}}\right]^{-1} + a_{it} \tag{3.31}$$

Therefore the problem could be reformulated a choice of the optimal τ_{it}. This belief precision choice determines the distribution of the state variable $\widehat{f_{it}} - \widehat{f_{ut}}$.

Substituting the expression of $\widehat{f_{it}} - \widehat{f_{ut}}$ into the value function (3.28), integrating over the realization of the private signal s_{it}, and differentiating with respect to τ_{it}:

$$\frac{\partial}{\partial \tau_{it}} V(W_{ut}, \widehat{z_{ut}}, \widehat{f_{it}} - \widehat{f_{ut}}, \tau_{it}) = V(W_{ut}, \widehat{z_{ut}}, \widehat{f_{it}} - \widehat{f_{ut}}, \tau_{it}) \left[\frac{\alpha_f}{2} + \frac{\beta_f}{2\tau_{it}} + \gamma'(\tau_{it}) \right]. \tag{3.32}$$

The above expression gives the marginal value of information in terms of utility. The cost of information in terms of wealth is $ka_{it}^2/2$ and the marginal cost of information is ka_{it}. In equilibrium, the investor selects a signal precision level that equalizes the value of information with the cost of information:

$$a_{it} = \frac{1}{k} \left[\frac{\alpha_f}{2} + \frac{\beta_f}{2\tau_{it}} + \gamma'(\tau_{it}) \right]. \tag{3.33}$$

There is a limit to how much information the investor is willing to acquire. The cost of information is quadratic in this setup, and it is prohibitively expensive to acquire a large amount of information in a short time. In addition, the marginal value of information is decreasing in the investor's belief precision τ_{it}. Even if the investor pushes this precision to infinity, his utility is still finite and bounded from above. Two factors contribute to this result. First, the noise component in dividend growth ϵ_{t+1}^d is assumed to be unlearnable. An investor endowed with complete information about the fundamental component of dividend growth would still face investment risks. Second, the value of a signal on the fundamental at a given date decreases as time progresses. The fundamental f_t also evolves over time and signals on past values could not by itself contribute to an accurate estimate of the current value.

The analysis focuses on a stationary equilibrium where belief and signal precision are all constant over time. It is still helpful to intuitively think about what will happen if the informed investors' belief precision start at a level different from that in the stationary equilibrium. If the informed investors are endowed with more information, the marginal value of information will be smaller and investors will acquire less information compared to the stationary level. Conversely, investors will acquire more information if they start with less precise beliefs. As time progresses, investors' belief and signal precision will converge to the stationary level.

3.3.3 Market Clearing

First I analyze equilibrium in the variance swap market. The equilibrium is symmetric within the group of informed investors: belief precision τ_{it}, signal precision a_{it}, and all conditional correlations and variances ρ_{irz}, ρ_{irf}, σ^2_{iz} and σ^2_{if} are constant across investors. The reason is that the cost of information is the same and that realizations of belief and private signals do not affect precision in a normal distribution setup. As a result, the risk-neutral variance in the absence of the variance swap is the same for all informed investors. Their positions in the variance swap are also the same from equation (3.30).

The informed and uninformed investors have equal mass and take opposite positions in the variance swap. Substituting (3.30) and (3.25) into (3.10):

$$\psi_{ut} = \frac{1}{2}\left[(1+\alpha^*_{uz}\rho^2_{urz}\sigma^2_{uz})\frac{1}{2\sigma^2_{ut}} - \left(1+\alpha^*_z\rho^2_{irz}\sigma^2_{iz} + \alpha^*_f\tau_{i,t+1}\rho^2_{irf}\sigma^2_{if} + 2\alpha^*_{zf}\rho_{irz}\rho_{irf}\sigma_{iz}\sigma_{if}\right)\frac{1}{2\sigma^2_{it}}\right], \tag{3.34}$$

$$\psi_{it} = -\psi_{ut}, \tag{3.35}$$

$$v_t = \left[(1+\alpha^*_{uz}\rho^2_{urz}\sigma^2_{uz})\frac{1}{2\sigma^2_{ut}} + \left(1+\alpha^*_z\rho^2_{irz}\sigma^2_{iz} + \alpha^*_f\tau_{i,t+1}\rho^2_{irf}\sigma^2_{if} + 2\alpha^*_{zf}\rho_{irz}\rho_{irf}\sigma_{iz}\sigma_{if}\right)\frac{1}{2\sigma^2_{it}}\right]^{-1} \tag{3.36}$$

The strike of the variance swap v_t is equal to the harmonic mean of two groups of investors' risk-neutral variances without variance swaps. The uninformed investor takes a long position in the variance swap market and the informed investors take short positions. Long positions decrease investors' risk-neutral variances and the converse happens for short positions. In equilibrium, the risk-neutral variances for both informed and uninformed are equal to v_t.

Next, I impose market clearing for the risky asset itself. Because the equilibrium is symmetric within the group of informed investors, their asset demands (3.29) can be simplified to the following expression:

$$\theta_{it} = h_0 + h_f(\widehat{f_{it}} - \widehat{f_{ut}}) + h_z(\widehat{z_{ut}} - \overline{z}). \tag{3.37}$$

where h_0, h_f, and h_z are constants and do not depend on the index of the investor.

Similarly for the uninformed investor:

$$\theta_{it} = h_{u0} + h_{uz}(\widehat{z_{ut}} - \overline{z}). \tag{3.38}$$

The average of difference in beliefs $\widehat{f_{it}} - \widehat{f_{ut}}$ is equal to $f - \widehat{f_{ut}}$. From (3.13), $f - \widehat{f_{ut}} = b_z(z_t - \widehat{z_{ut}})/b_f$. Substituting (3.37) and (3.38) into (3.9):

$$
\begin{aligned}
z_t &= \int_0^1 \theta_{it} di + \theta_{ut} \\
&= h_0 + h_{u0} + h_f(f_t - \widehat{f_{ut}}) + (h_z + h_{uz})(\widehat{z_{ut}} - \overline{z}) \\
&= h_0 + h_{u0} + h_f \cdot \frac{b_z}{b_f}(z_t - \widehat{z_{ut}}) + (h_z + h_{uz})(\widehat{z_{ut}} - \overline{z}) \qquad (3.39)
\end{aligned}
$$

Comparing both sides of the equation:

$$h_z + h_{uz} = 1, \qquad (3.40)$$

$$\frac{b_z}{b_f} \cdot h_f = 1, \qquad (3.41)$$

$$h_0 + h_{u0} = \overline{z}. \qquad (3.42)$$

The above system of equations allows me to numerically pin down the coefficients in the conjectured equilibrium price function (3.12). I start from an initial guess of b_f and b_z and proceed to solve the uninformed investor's value function and portfolio demands. The uninformed investor's belief precision is solved by setting $\tau_{u,t+1}$ equal to τ_{ut} in equation (3.14).

The informed investors' belief precision and signal precision satisfy two equations: the law of motion of beliefs (3.17) and the information acquisition decision (3.33). In the former equation, the stationary level of belief precision is increasing in the amount of information acquired. In the latter equation, higher belief precision reduces the value of information and thus the signal precision. This system of equations for τ_{it} and a_{it} thus have a unique solution given b_f and b_z. The asset demand equations (3.40) and (3.41) then determine b_f and b_z and then (3.42) determines the average discount of the asset price b_0.

3.4. Analysis of Equilibrium

I investigate how information acquisition influences return autocorrelation and cross-sectional dispersion of beliefs in this section. I first analyze how uncertainty affects information acquisition. In section 4.1, I study the serial correlation of returns for both informed and uninformed investors. In section 4.2, I study the cross-sectional dispersion of beliefs among the informed investors. All results come from comparative statics analysis and do not involve the motion towards a new stationary equilibrium. A fully dynamic model where information acquisition endogenously reacts to exogenous news shocks would be a good direction for future research.

The fundamental that investors acquire private signals on is time-varying in this model. In a setup where the fundamental value stays the same over time, information acquisition is determined by uncertainty about the fundamental which is measured by its risk-neutral variance. No such simple relationship exists in this setup, because not all components of asset returns can be learned and information acquired today also plays a role in future portfolio decisions.

However, uncertainty about the fundamental dividend growth and supply shocks is still the primary driving force behind investors' information acquisition. The value of information is increasing in the innovation variance of the fundamental τ_f^{-1}. When the noise trader demands are more volatile, investors face better investment opportunities and they could expect to realize higher profits from the same piece of information.

Investors acquire more information in response to higher uncertainty and this mitigates the initial impact of uncertainty on asset volatility. Compared to the case where private signals with fixed precision are received, the informed investors in this high uncertainty situation have a more precise belief and thus trade more aggressively in the asset market. As a result, asset prices become more informative about the fundamental and future expected return. This is reflected in the variance swap market with a lower level of risk-neutral variance or strike v_t.

3.4.1 Serial Correlation of Returns

The uninformed investor's unexpected return in the current period consists of 3 components: the difference between the true value of fundamental and his estimate $f_t - \widehat{f_{ut}}$, the noise component of dividend growth ϵ_{t+1}^d, and innovations in dividend growth and asset supply $b_f \epsilon_{t+1}^f - b_z \epsilon_{t+1}^z$:

$$
\begin{aligned}
r_{t+1}^e - \mathbb{E}_t^u[r_{t+1}^e] = &\left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d} b_{fu}\right)(f_t - \widehat{f_{ut}}) \\
&+ \left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d} b_{fu}\right)\epsilon_{t+1}^d + \left(1 + \frac{b_f \tau_f^{-1}}{b_f^2 \tau_f^{-1} + b_z^2 \tau_z^{-1}} b_{fu}\right)\left(b_f \epsilon_{t+1}^f - b_z \epsilon_{t+1}^z\right)
\end{aligned}
\tag{3.43}
$$

The next period expected return $\mathbb{E}_{t+1}^{u}[r_{t+2}^{e}]$ also consists of its date t expectation $\mathbb{E}_{t}^{u}[r_{t+2}^{e}]$ and these three components:

$$\mathbb{E}_{t+1}^{u}[r_{t+2}^{e}] = \mathbb{E}_{t}^{u}[r_{t+2}^{e}] + (r+\lambda)(1-\lambda)\frac{\tau_d}{\tau_u+\tau_d}(f_t - \widehat{f_{ut}}) \\ + (r+\lambda)(1-\lambda)\frac{\tau_u}{\tau_u+\tau_d} b_f \epsilon_{t+1}^{d} - (r+\lambda)\frac{b_z^2\tau_z^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon_{t+1}^{f} - b_z\epsilon_{t+1}^{z}\right). \tag{3.44}$$

Both $f_t - \widehat{f_{ut}}$ and ϵ_{t+1}^{d} components contribute to positive autocorrelation in returns. When price and dividend news indicate that dividend growth is likely to be above his current expectation, the investor revises his beliefs upwards and this leads to persistence. The $b_f\epsilon_{t+1}^{f} - b_z\epsilon_{t+1}^{z}$ term on the contrary contributes to negative autocorrelation in returns. Supply shocks that lead to a positive return in the current period are expected to reverse itself in future periods.

Positive serial correlation from the estimation error in the fundamental $f_t - \widehat{f_{ut}}$ decays more quickly compared to the negative serial correlation from supply shocks. From equation (3.4), the asset supply z_t reverts to its mean at a speed of $1-\lambda$. The estimation error $f_t - \widehat{f_{ut}}$, however, reverts at a faster speed $(1-\lambda)\tau_u/(\tau_u+\tau_d)$:

$$f_{t+1} - \widehat{f_{u,t+1}} = (1-\lambda)\frac{\tau_u}{\tau_u+\tau_d}(f_t - \widehat{f_{ut}}) \\ + \epsilon_{t+1}^{f} - (1-\lambda)\frac{\tau_d}{\tau_{ut}+\tau_d}\epsilon_{t+1}^{d} - \frac{b_f\tau_f^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon_{t+1}^{f} - b_z\epsilon_{t+1}^{z}\right) \tag{3.45}$$

As the horizon increases, positive autocorrelation from the estimation error becomes weaker in comparison with negative autocorrelation from supply shocks. Therefore returns are more likely to exhibit reversal at long horizons.

The informed investor i's unexpected return is given by:

$$r_{t+1}^{e} - \mathbb{E}_{t}^{i}[r_{t+1}^{e}] = \left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u+\tau_d}b_{fu}\right)(f_t - \widehat{f_{it}}) \\ + \left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u+\tau_d}b_{fu}\right)\epsilon_{t+1}^{d} + \left(1 + \frac{b_f\tau_f^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}b_{fu}\right)\left(b_f\epsilon_{t+1}^{f} - b_z\epsilon_{t+1}^{z}\right) \tag{3.46}$$

This investor's next period expected return also has the form:

$$\mathbb{E}_{t+1}^{i}[r_{t+2}^{e}] = \mathbb{E}_{t}^{i}[r_{t+2}^{e}] + e_i(f_t - \widehat{f_{it}}) + e_s\epsilon_{t+1}^{s} + e_d\epsilon_{t+1}^{d} + e_f\epsilon_{t+1}^{f} + e_z\epsilon_{t+1}^{z}.$$

where e_i, e_s, e_d, e_f, and e_s are constant coefficients and e_i is positive.

Both the unexpected return and next period expected return are increasing in estimation error of the informed investor $f_t - \widehat{f_{it}}$. The variance of this component τ_{it}^{-1} is smaller than that of the uninformed investor τ_{ut}^{-1}. Therefore this component contributes less to positive serial correlation for the informed investors compared to the uninformed. Information acquisition increases the discrepancy of return autocorrelation between the beliefs of informed and uninformed investors.

Within the group of informed investors, the autocorrelation of returns is decreasing in information acquisition. The variance of $f_t - \widehat{f_{it}}$ is decreasing in the amount of information acquired. In addition, the incentive to learn from prices and dividends is smaller when investors have accurate private signals. In the limit where informed investors have complete knowledge about the fundamental, the estimation error $f_t - \widehat{f_{it}}$ becomes zero, and this component ceases to exist.

Information acquisition also has an impact on informed investors' trading behavior. Consider an informed investor who has infinitesimal mass and no impact on the equilibrium asset prices. When this investor acquires imprecise private signals, his trading strategies is similar to that of the uninformed investor. An unexpected price increase corresponds to an upward revision of belief and leads to increased demand for risky assets. On the contrary, if this investor's private signals are precise, high return today correspond to lower expected returns in the future and therefore decreases his asset demand. Investors with more information adopt a contrarian strategy while those with less information behave like trend followers.

3.4.2 Cross-sectional Dispersion of Beliefs

Cross-sectional dispersion measures the extent to which individual investor's expected return diverge from the market average. The expected return of investor i given by equation (3.21) is linear in $\widehat{f_{it}} - \widehat{f_{ut}}$, the difference in beliefs between herself and the uninformed investor. The variance of the expected return across different investors is equal to:

$$\mathrm{Var}(\mathbb{E}_t^i[r_{t+1}^e]) = \left(\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d} b_{fu}\right)^2 \cdot \mathrm{Var}(\widehat{f_{it}}) \tag{3.47}$$

where the cross-section variance of belief $\widehat{f_{it}}$ is

$$\mathrm{Var}(\widehat{f_{it}}) = \frac{\tau_{it} - \tau_u}{\tau_{it}^2} \tag{3.48}$$

$\text{Var}(\widehat{f_{it}})$ is an inverted-U shaped function of the informed investors' precision τ_{it}. Highest dispersion is realized when information acquisition is at an intermediate level. When there is no learning from private signals, the informed investors have the same information set as the uninformed investor and all investors' estimates of the dividend growth rate are equal to $\widehat{f_{ut}}$. When the private signals are infinitely precise, all informed investors' estimates are equal to f_t. In both cases there is no dispersion of beliefs.

Low uncertainty about the fundamental almost always decreases the cross-sectional dispersion of beliefs. All investors form their belief using the information that is publicly available. In this situation, the time series of prices and dividends already provide relatively accurate information about the dividend growth rate. Lower uncertainty also decreases the value of information and induces investors to acquire less accurate private signals. The difference of precision between informed and uninformed investors $\tau_{it} - \tau_{ut}$ is small relative to τ_{ut} in this case, and all informed investors' estimates will be close to that of the uninformed.

3.5. Concluding Remarks

In this paper, I develop a rational expectations model with endogenous information acquisition where investors trade a long-lived dividend-paying asset and a volatility derivative. The analysis starts with the conjectured price function (3.12) from which I derive the evolution of beliefs for both informed and uninformed investors. The expected excess return of informed investors is linear in the difference between his estimate of the dividend growth and that of the uninformed. Information acquisition increases the precision of belief about asset returns and reduces volatility.

The uninformed investor possesses less information and feels that investment in the asset is riskier. He takes a long position in the variance swap to insure against extreme price movements. This long position drives up the price of the variance, measured by the strike of the variance swap. The informed investors believe that the variance of asset returns is smaller and find it profitable to take short positions in the variance swap.

Uncertainty about the fundamental increases the value of information and induces more information acquisition. The serial correlation of returns for more informed investors is smaller and they adopt a contrarian trading strategy. Low uncertainty about the fundamental, on the other hand, induces investors to acquire less information and reduces the cross-sectional dispersion of beliefs in asset returns.

This paper opens up interesting directions for future research. A non-normal setup would allow the level of expectation to affect the conditional variances and correlations in investors' beliefs, which create cross-sectional dispersion in the value of information and amount of information investors acquire. Alternatively, a fully dynamic model where information acquisition endogenously reacts to exogenous news shocks would be a good setup to study the impact of FOMC and other macroeconomic announcements on VIX and the risk premium.

Bibliography

Rui Albuquerque. Skewness in stock returns: reconciling the evidence on firm versus aggregate returns. *The Review of Financial Studies*, 25(5):1630–1673, 2012.

Rui Albuquerque and Jianjun Miao. Advance information and asset prices. *Journal of Economic Theory*, 149:236–275, 2014.

Daniel Andrei and Julien Cujean. Information percolation, momentum and reversal. *Journal of Financial Economics*, 123(3):617–645, 2017.

Daniel Andrei and Michael Hasler. Investor attention and stock market volatility. *The Review of Financial Studies*, 28(1):33–72, 2014.

Andrew Ang, Robert J. Hodrick, Yuhang Xing, and Xiaoyan Zhang. The cross-section of volatility and expected returns. *The Journal of Finance*, 61(1):259–299, 2006.

Efstathios Avdis. Information tradeoffs in dynamic financial markets. *Journal of Financial Economics*, 122(3):568–584, 2016.

Gurdip Bakshi, Nikunj Kapadia, and Dilip Madan. Stock return characteristics, skew laws, and the differential pricing of individual equity options. *The Review of Financial Studies*, 16(1):101–143, 2003.

Turan G. Bali and Nusret Cakici. Idiosyncratic volatility and the cross section of expected returns. *Journal of Financial and Quantitative Analysis*, 43(1):29–58, 2008.

Snehal Banerjee and Bradyn Breon-Drish. Dynamic information acquisition and strategic trading. *Working Paper*, 2017.

Nicholas Barberis, Robin Greenwood, Lawrence Jin, and Andrei Shleifer. X-capm: An extrapolative capital asset pricing model. *Journal of Financial Economics*, 115(1):1–24, 2015.

Bradyn Breon-Drish. On existence and uniqueness of equilibrium in a class of noisy rational expectations models. *The Review of Economic Studies*, 82(3):868–921, 2015.

Georgy Chabakauri, Kathy Yuan, and Konstantinos Zachariadis. Multi-asset noisy rational expectations equilibrium with contingent claims. *Working Paper*, 2016.

Zhi Da, Joseph Engelberg, and Pengjie Gao. In search of attention. *The Journal of Finance*, 66(5):1461–1499, 2011.

Itamar Drechsler. Uncertainty, time-varying fear, and asset prices. *The Journal of Finance*, 68(5):1843–1889, 2013.

Diego Garcia. Sentiment during recessions. *The Journal of Finance*, 68(3):1267–1300, 2013.

Sanford J. Grossman and Joseph E. Stiglitz. On the impossibility of informationally efficient markets. *The American economic review*, 70(3):393–408, 1980.

Campbell R Harvey and Robert E Whaley. Market volatility prediction and the efficiency of the s&p 100 index option market. *Journal of Financial Economics*, 31(1):43–73, 1992.

Martin F Hellwig. On the aggregation of information in competitive markets. *Journal of Economic Theory*, 22(3):477–498, 1980.

Craig W. Holden and Avanidhar Subrahmanyam. News events, information acquisition, and serial correlation. *The Journal of Business*, 75(1):1–32, 2002.

Harrison Hong and Jeremy C Stein. A unified theory of underreaction, momentum trading, and overreaction in asset markets. *The Journal of Finance*, 54(6):2143–2184, 1999.

Harrison Hong, Terence Lim, and Jeremy C Stein. Bad news travels slowly: Size, analyst coverage, and the profitability of momentum strategies. *The Journal of Finance*, 55(1):265–295, 2000.

Lixin Huang and Hong Liu. Rational inattention and portfolio selection. *The Journal of Finance*, 62(4):1999–2040, 2007.

Narasimhan Jegadeesh and Sheridan Titman. Returns to buying winners and selling losers: Implications for stock market efficiency. *The Journal of finance*, 48(1):65–91, 1993.

Marcin Kacperczyk, Stijn Van Nieuwerburgh, and Laura Veldkamp. A rational theory of mutual funds' attention allocation. *Econometrica*, 84(2):571–626, 2016.

S.P. Kothari and Jerold B. Warner. Measuring long-horizon security price performance. *Journal of Financial Economics*, 43(3):301–339, 1997.

Charles M.C. Lee and Eric C. So. Uncovering expected returns: Information in analyst coverage proxies. *Journal of Financial Economics*, 124(2):331–348, 2017.

Dong Lou. Attracting investor attention through advertising. *The Review of Financial Studies*, 27(6):1797–1829, 2014.

Ian Martin. What is the expected return on the market? *The Quarterly Journal of Economics*, 132(1):367–433, 2017.

Ian Martin and Christian Wagner. What is the expected return on a stock? *The Journal of Finance*, 2018.

Javier Mencia and Enrique Sentana. Valuation of vix derivatives. *Journal of Financial Economics*, 108(2):367–391, 2013.

Haim Mendelson and Tunay I. Tunca. Strategic trading, liquidity, and information acquisition. *The Review of Financial Studies*, 17(2):295–337, 2004.

Tobias J. Moskowitz, Yao Hua Ooi, and Lasse Heje Pedersen. Time series momentum. *Journal of Financial Economics*, 104(2):228–250, 2012.

Christopher A Sims. Implications of rational inattention. *Journal of Monetary Economics*, 50(3):665–690, 2003.

Stijn Van Nieuwerburgh and Laura Veldkamp. Information acquisition and under-diversification. *The Review of Economic Studies*, 77(2):779–805, 2010.

Dimitri Vayanos and Paul Woolley. An institutional theory of momentum and reversal. *The Review of Financial Studies*, 26(5):1087–1145, 2013.

Laura Veldkamp. Media frenzies in markets for financial information. *American Economic Review*, 96(3):577–601, 2006.

Robert E Verrecchia. Information acquisition in a noisy rational expectations economy. *Econometrica*, pages 1415–1430, 1982.

Jiang Wang. A model of intertemporal asset prices under asymmetric information. *The Review of Economic Studies*, 60(2):249–282, 1993.

X. Frank Zhang. Information uncertainty and stock returns. *The Journal of Finance*, 61(1):105–137, 2006.

Appendix

Proof of Lemma 1.

The dynamics of fundamental f_t could be rewritten as:

$$f_{t+1} = \lambda \overline{f} + (1-\lambda) f_t + \epsilon^f_{t+1}. \tag{A1}$$

The expectations of f_t and ϵ^f_{t+1} conditional on the date $t+1$ information set of the uninformed investor are given by:

$$\mathbb{E}^u_{t+1}[f_t] = \frac{\tau_{ut}}{\tau_{ut}+\tau_d}\widehat{f_{ut}} + \frac{\tau_d}{\tau_{ut}+\tau_d}\left(f_t + \epsilon^d_{t+1}\right), \tag{A2}$$

$$\mathbb{E}^u_{t+1}[\epsilon^f_{t+1}] = \frac{b_f \tau_f^{-1}}{b_f^2 \tau_f^{-1} + b_z^2 \tau_z^{-1}}\left(b_f \epsilon^f_{t+1} - b_z \epsilon^z_{t+1}\right). \tag{A3}$$

Substituting (A2) and (A3) into (A1), I obtain (3.15).

From equation (3.13),

$$\widehat{z_{u,t+1}} = z_t + \frac{b_f}{b_z}(\widehat{f_{u,t+1}} - f_{t+1}). \tag{A4}$$

Substituting (A1) and (3.15) into (A4), I obtain (3.16).

Proof of Lemma 2.

If we limit the information set to investor i's date t information and date $t+1$ price and dividend, the conditional expectation of f_{t+1} is given by:

$$\begin{aligned}\mathbb{E}^i[f_{t+1}|\widehat{f_{it}}, p_{t+1}, d_{t+1}] = {} & \lambda\overline{f} + (1-\lambda)\frac{\tau_{it}}{\tau_{it}+\tau_d}\widehat{f_{it}} + (1-\lambda)\frac{\tau_d}{\tau_{it}+\tau_d} f_t \\ & + (1-\lambda)\frac{\tau_d}{\tau_u+\tau_d}\epsilon^d_{t+1} + \frac{b_f\tau_f^{-1}}{b_f^2\tau_f^{-1}+b_z^2\tau_z^{-1}}\left(b_f\epsilon^f_{t+1} - b_z\epsilon^z_{t+1}\right).\end{aligned} \tag{A5}$$

The noise $\epsilon^s_{i,t+1}$ in signal $s_{i,t+1}$ is independent from $\widehat{f_{it}}$, p_{t+1}, and d_{t+1}. Investor i's estimate of f_{t+1} given his full information set at date $t+1$ is the weighted average of $\mathbb{E}^i[f_{t+1}|\widehat{f_{it}}, p_{t+1}, d_{t+1}]$ and signal $s_{i,t+1}$:

$$\mathbb{E}^i_{t+1}[f_{t+1}] = \frac{\tau_{i,t+1} - a_{i,t+1}}{\tau_{i,t+1}}\mathbb{E}^i[f_{t+1}|\widehat{f_{it}}, p_{t+1}, d_{t+1}] + \frac{a_{i,t+1}}{\tau_{i,t+1}} s_{i,t+1}. \tag{A6}$$

Substituting (A5) into (A6), I obtain (3.18).

Proof of Lemma 3.

Substituting (3.12) and (3.15) into $r^e_{t+1} = p_{t+1} + d_{t+1} - (1+r)p_t$, I obtain (3.19).

$$\mathbb{E}^u_t[r^e_{t+1}] = rb_0 + \left[r + \lambda + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}\right] b_{fu}(\mathbb{E}^u_t[f_t] - \widehat{f_{ut}}) + (r+\lambda)b_z(\mathbb{E}^u_t[z_t] - \overline{z}) \tag{A7}$$

From definition $\mathbb{E}^u_t[f_t] - \widehat{f_{ut}} = 0$, $\mathbb{E}^u_t[z_t] = \widehat{z_{ut}}$. The above equation is equivalent to (3.20).

For informed investor i:

$$\begin{aligned}
\mathbb{E}^i_t[r^e_{t+1}] &= rb_0 + \left[r + \lambda + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}\right] b_{fu}(\mathbb{E}^i_t[f_t] - \widehat{f_{ut}}) + (r+\lambda)b_z(\mathbb{E}^i_t[z_t] - \overline{z}) \\
&= rb_0 + \left[r + \lambda + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}\right] b_{fu}(\widehat{f_{it}} - \widehat{f_{ut}}) + (r+\lambda)b_z(\widehat{z_{it}} - \widehat{z_{ut}}) + (r+\lambda)b_z(\widehat{z_{ut}} - \\
&= rb_0 + \left[r + \lambda + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}\right] b_{fu}(\widehat{f_{it}} - \widehat{f_{ut}}) + (r+\lambda)b_z\frac{b_f}{b_z}(\widehat{f_{it}} - \widehat{f_{ut}}) + (r+\lambda)b_z(\widehat{z_u} \\
&= rb_0 + \left[\frac{1+r}{r} + (1-\lambda)\frac{\tau_d}{\tau_u + \tau_d}b_{fu}\right](\widehat{f_{it}} - \widehat{f_{ut}}) + (r+\lambda)b_z(\widehat{z_{ut}} - \overline{z}).
\end{aligned} \tag{A8}$$

This is equation (3.21). The third equality comes from the fact that the sufficient statistic $b_f f_t - b_z z_t$ belongs to both informed and uninformed investors' information set.

$$\mathbb{E}^i_t[b_f f_t - b_z z_t] = b_f f_t - b_z z_t = \mathbb{E}^u_t[b_f f_t - b_z z_t] \tag{A9}$$

Proof of Proposition 1.

The value function at date $t+1$ is given by:

$$\begin{aligned}
V(W_{u,t+1}, \widehat{z_{u,t+1}}) = -\exp\Big(&-rAW_{ut} + rAc_{ut} - rA\theta_{ut}r^e_{t+1} - rA\psi_{ut}\left((r^e_{t+1})^2 - v_t\right) \\
&- \frac{\alpha_{uz}}{2}(\widehat{z_{u,t+1}} - \overline{z})^2 - \beta_{uz}(\widehat{z_{u,t+1}} - \overline{z}) - \gamma_{uz}\Big).
\end{aligned} \tag{A10}$$

I decompose $\widehat{z_{u,t+1}} - \overline{z}$ into 3 components:

$$\widehat{z_{u,t+1}} - \overline{z} = (1-\lambda)(\widehat{z_{ut}} - \overline{z}) + \frac{\rho_{urz}\sigma_{uz}}{\sigma_{ut}}(r^e_{t+1} - \mathbb{E}^u_t(r^e_{t+1})) + \widehat{\epsilon_{u,t+1}}. \tag{A11}$$

The first component is the conditional expectation of $\widehat{z_{u,t+1}}$. The second component is linear in the unexpected component of excess return. The third component $\widehat{\epsilon_{u,t+1}}$ is orthogonal to the second component and its mean is equal to zero.

Substituting (A11) into (A10) and integrating over $\widehat{\epsilon_{u,t+1}}$, I arrive at the conditional expectation of value function given r^e_{t+1}. Solving this optimization problem, I obtain (3.24) and (3.25).

The proof of Proposition 2 is similar to that of Proposition 1.