The London School of Economics and Political Science

# Essays in Financial Markets

Chutiorn Tontivanichanon

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# Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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I confirm that Chapter 3 is a revised version of the paper I submitted for the Master of Research (MRes) degree in Economics at the LSE, awarded in July 2015.

I declare that my thesis consists of approximately 38,000 words.

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# Abstract

This thesis contains three chapters discussing different aspects of financial markets.

The first chapter studies the impact of learning information about future non-fundamental shocks on stock price dynamics and provides a new insight into how rational speculators can cause inefficiency and volatility to stock markets during periods of information technology advancement. I construct an infinite-period competitive market model and analyze how an increase in non-fundamental signal precision affects trading strategy of rational investors, price formation, and efficiency. Contradicting to traditional rational speculative theory, I found that higher non-fundamental signal precision can increase stock return volatility, increase price sensitivity to both current and future non-fundamental shocks, and decrease price informativeness. Moreover, even though investors have better information about future stock prices, they can predict stock returns less accurately. This is because the investors expect that their future counterparts will also trade more aggressively on new non-fundamental information that arrives in the future, causing future stock prices to be endogenously more volatile to new non-fundamental shocks which are unpredictable to the investors at the present.

The second chapter develops a game-theoretic dynamic model to study strategic dealer choice of buy-side investors in over-the-counter (OTC) secondary asset markets and provides a new theory of why periphery dealers, despite locating at inferior positions in OTC dealer network, can survive and co-exist with core dealers. My theory is based on a premise that buy-side investors form a non-binding long-term relationship with core dealers to obtain costly liquidity in bad periods. The main finding is that periphery dealers can help investors with infrequent liquidity needs, those who cannot form the relationship with core dealers directly due to commitment problem, successfully obtain costly liquidity in bad periods. By connecting with several investors and forming relationship with a core dealer on their behalf, periphery dealer will have enough power to pressure the core dealer to commit to the relationship. Therefore, investors with infrequent liquidity needs will trade with periphery dealers to obtain the benefit of long-term relationship, granting market power to periphery dealers to co-exist with core dealers.

The third chapter develops a game-theoretic model to study strategic formation of financial network. In the model, a finite number of risk-averse agents who invest in risky projects can issue and trade forward contracts (i.e. assets) to obtain fractions of investment return of other agents. All trades are bilateral, each involving two parties with trading relationship privately bargaining

on asset price and quantity. The main objective is to examine how structural properties of trading network determine trading decision of the agents and equilibrium asset allocation. To this end, I use the concept of line graph transformation to identify network of asset flows and map positions of trading links onto the equilibrium outcome. The main insight is that equilibrium asset allocation corresponds to a generalized Bonacich centrality of the network of asset flows.

# Chapter 1

# A Hidden Curse of Non-fundamental Information in Stock Markets

## 1.1 Introduction

In the past years, rapid rise of information technology has shifted competition and profitability nature of financial markets towards information processing capability. Nowadays, all market participants can store and analyze financial data more efficiently than ever before. However, there is widespread debate on how this may cause inefficiency due to shifting nature of information processing towards non-fundamental information, the information about temporary price movement unrelated to the fundamentals. A recent empirical literature has found that stock price informativeness has been declining over time (Farboodi et al (2017)).[1] We aim to shed light on this issue by studying the impact of learning non-fundamental information toward price efficiency and implications of information technology growth in stock markets.

This paper focuses on future noise information – the information about temporary *future* price deviation from the fundamentals. Based on traditional rational speculative theory (Friedman (1953)), learning more information about future non-fundamental shocks reduces asset volatility while having ambiguous effect on price informativeness. Intuitively, when speculators are more informed about future prices, they will buy more assets when asset price is low (and vice versa). Therefore, asset price becomes less sensitive to current non-fundamental shock, resulting in lower asset volatility and possible improvement on price efficiency. However, this argument assumes that speculative return is exogenous which might be misleading for stock markets, because future stock price dynamics depend on trading decision of future investors who can also learn more non-fundamental information. We contribute to the literature by revisiting

[1] Farboodi et al (2017) found that price informativeness of all publicly traded firms in the US has declined. When considering subsamples of S&P500 firms VS other public firms, price informativeness of S&P500 firms has been increasing while that of other listed firms has been decreasing. The analysis suggested that the increase in price informativeness of S&P500 firms can be explained by a change in size composition of S&P500 firms.

this issue in an infinite-period dynamic setting, in which asset return depends on trading strategies of future investors, and examine the impact of a permanent increase in non-fundamental signal precision on price formation, volatility, and efficiency.

We develop an infinite-period competitive market model based on Farboodi and Veldkamp (2018). There is a long-lived divisible risky asset (i.e. stock) which is a lifetime claim of stochastic dividends that follow AR(1) process. In every period, a new generation of short-lived investors is born and enters the market. Each cohort contains a unit continuum of rational risk-averse investors who face an idiosyncratic non-tradable endowment shock and live for one period. Prior to trading, they receive noisy private signals about dividend shock (i.e fundamental shock) and next-period aggregate endowment shock (i.e future non-fundamental shock). At the end of each period, the dividend is publicly observable and paid out to all investors. We follow the notion of competitive rational expectation equilibrium and characterize the most informative stationary equilibrium.

Our main finding is that, contradicting to classical rational speculative theory, an increase in non-fundamental signal precision can increase asset return volatility and reduce price informativeness. Also, asset price becomes more responsive to both current and future non-fundamental shocks. The intuition is as follows. Consistent with the rational speculative theory, better non-fundamental information partly reduces investment risk which stimulates trade and moves the price to the right direction. However, there is a hidden adverse dynamic effect from an endogenous change in trading strategies of future investors. When the quality of non-fundamental information improves, future investors will also trade more assets based on future non-fundamental information, causing future prices to be endogenously more volatile to non-fundamental shock of further periods. Unfortunately, the investors cannot learn the future non-fundamental information that their future counterparts can learn. As a result, future asset price becomes endogenously harder to predict for the investors even though they have more financial information. This higher future non-fundamental information risk feeds back into trading decision of the investors, resulting in higher asset volatility, higher price sensitivity to non-fundamental shocks, and lower price informativeness.

In addition, we found that price efficiency when investors do not have non-fundamental information can be optimal (i.e. maximum price informativeness with minimum price sensitivity to both non-fundamental shocks) if two conditions are met: 1) the investors have accurate private fundamental signals and 2) discount factor approaches one. From the model, price efficiency can increase when non-fundamental information quality improves due to two reasons. First, the benefit of more financial information starts dominating the adverse dynamic feedback effect. Second, when the investors trade more stocks based on non-fundamental information, past stock prices are more informative about current non-fundamental shock, helping the investors extract fundamental information from current stock price more accurately. Consequently, price efficiency when the investors have good non-fundamental signals can improve from the level which the investors have no non-fundamental signals. However, if the investors already have perfect fundamental signal, they will use the information from past prices to extract only non-fundamental information from the current price. Also, when discount factor is high, the future noise information risk will cause large adverse impact toward the investors, resulting in no efficiency improvement from learning non-fundamental information.

To study the implications of non-fundamental information production on information technology growth, we introduce endogenous information acquisition to the model. Prior to trading, investors strategically choose precisions of their fundamental and non-fundamental private signals subject to an information technology constraint, a constraint which requires the sum of acquired signal precisions to not exceed a value which we called information-processing capacity. We examine how an increase in the processing capacity which permits the investors to obtain higher quality of financial information will affect equilibrium information choice of the investors and stock price informativeness.

Provided that 1) non-fundamental information is relatively hard to process and that 2) discount factor is sufficiently high, we found that advancement in the information-processing capacity can deteriorate price informativeness when the processing capacity is high. When the processing capacity is low, financial information is scarce and stock price is uninformative. Thus, the investors will acquire only fundamental information. However, once the processing capacity is high, the fundamental information, both from stock price and from private signals of the investors, is abundant. To outperform the market, the investors will start processing the more costly non-fundamental information and free-riding the fundamental information from stock prices. They may also process more fundamental information at this stage. However, it will reduce their investment risks marginally, as they already have good quality of the fundamental information. During this stage, an increase in the processing capacity causes price informativeness to (non-monotonically) decrease and converge to a lower level.

At first sight, one may expect that there should be no non-fundamental information production when the fundamental information is abundant since stock prices will be fully revealing. Interestingly, the model predicts that price inefficiency from the non-fundamental information production is bound to happen, because stock prices are never fully revealing. Due to the infinite-period payoff structure of stocks, stock investors cannot learn perfectly future information, both fundamental and non-fundamental information, that their future counterparts can learn. Thus, regardless of how developed the information technology is, the stock price will not be fully revealing, driving the investors to start processing the costly non-fundamental information to make more speculative profits in due course.

## Related literature

Our paper belongs to the early literatures that study how rational speculators can destabilize asset prices. Hart and Kreps (1986) constructed a model consisting of rational speculators who can store commodity across periods. They established that the speculators, who intend to buy and store the commodity to resell at a higher price in the future, can cause price destabilization from their de-storage strategies. Stein (1987) pointed out that an entry of rational speculators who have noisy private fundamental information can bring noise to the market price, causing negative informational externality to those who are already in the market. De Long et al (1989) constructed a model which contains speculators and positive feedback traders – those who buy securities when the asset price increases and sell securities when the asset price declines. They established that rational speculators can destroy price informativeness, because their speculative trades can trigger positive feedback traders to react to the price move and destabilize the price further away from the fundamentals. A more recent theoretical study by Madrigal (1996) proved that an entry of non-fundamental speculators – those who have superior information about past non-fundamental shocks which allows them to infer fundamental information from the market price – can cause price destabilization. This is because their information free-riding behaviour reduces trading profits of informed investors who will in turn respond by modifying their trading strategies to convey less information. This paper contributes to the literature by showing that future speculative investors can deteriorate price efficiency by creating future non-fundamental information risk toward present speculative investors when the signal quality of future noise shocks increases.

This paper also belongs to a growing literature which studies the impact of learning non-fundamental information. Several literatures focus on information about short-term deviation of *past* or *current* asset prices from the fundamentals (e.g. Ganguli and Yang (2009), Manzano and Vives (2010), Guo and Yang (2014), Marmora and Rytchkov (2015), and Yang and Zhu (2016), Farboodi and Veldkamp (2018)). Such noise information can improve price efficiency as it helps risk-averse investors extract fundamental information from asset prices more accurately. A main drawback which can cause inefficiency is that investors may excessively free-ride information from asset prices rather than using their own private information. Another line of literature focused on other types of information unrelated to the fundamentals such as beliefs of other traders (Banerjee et al (2018)), trading motives of other investors (Banerjee and Green (2015)), variance of non-fundamental factor (Hong and Rady (2002)), or the number of informed traders (Gao et al (2013)). We contribute to this literature by studying a different type of non-fundamental information – the information about future non-fundamental shock – and shedding light on how the dynamic feedback effect from learning non-fundamental information can cause price inefficiency.

The most closely related literature to this paper is Farboodi and Veldkamp (2018) which studied evolution of information choice between fundamental and non-fundamental information (about current noise) when there is long-run information-processing technology growth. In their model, there is future fundamental information risk which prevents asset prices from being fully informative. This is similar to the future non-fundamental information risk which causes inefficiency in our paper. Farboodi and Veldkamp (2018) also established that investors will initially acquire fundamental information when the technology is poor and will start acquiring non-fundamental information to extract fundamental information from asset price later on. However, price efficiency improves because the information about current noise helps the investors extract fundamental information more accurately from the asset price. This is different from our finding, particularly on implications towards price efficiency, which states that the investors start processing costly non-fundamental information (about future noise) to beat the market when trading on fundamental information is no longer profitable.

## Structure of the paper

The rest of the paper is organized as follows. Section 2 outlines the model setting. Section 3 analyses the model without non-fundamental information production. Section 4 examines the impact of learning information about future non-fundamental shock. Section 5 analyzes how learning information about current noise shock can deteriorate price informativeness in our framework. Section 6 extends the model to include endogenous information acquisition and analyzes the impact of information technology growth. Section 7 concludes. Throughout the paper, *NF information* refers to information about *future non-fundamental shock* while *F information* refers to information about *fundamental shock* unless stated otherwise. The equilibrium characterization and all omitted proofs are in appendix.

## 1.2 The model

Consider a discrete-time infinite-period model based on Farboodi and Veldkamp (2018) consisting of a risky asset (i.e stock) which is a claim on lifetime stochastic dividend payments $\{d_t\}_{\forall t}$ and unlimited cash. The dividend payment follows $AR(1)$ process of $d_t = \alpha d_{t-1} + \theta_t + \bar{\theta}$ in which $\theta_t \sim N(0, \sigma_\theta^2)$ is an i.i.d. random dividend shock and $\bar{\theta} \in \mathcal{R}^+$ is the mean. Denote $\tau_\theta = \frac{1}{\sigma_\theta^2}$. At the end of period $t$, dividend $d_t$ is publicly observable and paid out to all investors.

In every period, a new generation of risk-averse investors is born and enters the market. Each cohort contains a unit continuum of investors who live for one period. Each investor $i$ has the following mean-variance utility function:

$$U_i(\pi_{it}|I) = E(\pi_{it}|I) - \frac{\rho}{2} Var(\pi_{it}|I)$$

where $\rho$ is the degree of risk aversion and $\pi_{it} = (\delta p_{t+1} + d_t - p_t)(x_{it} - u_{it})$ is the return from buying $x_{it}$ units of the risky asset at price $p_t$ and selling them in the next period at $p_{t+1}$. The non-tradable endowment shock $u_{it} = u_t + \sigma_{ui}\mu_{uit}$ consists of i.i.d. market-wide shock $u_t \sim N(0, \sigma_u^2)$ and i.i.d. white noise idiosyncratic shock $\mu_{uit}$. Denote $\tau_u = \frac{1}{\sigma_u^2}$. This endowment shock is privately observable and represents investor sentiments or unexpected idiosyncratic liquidity needs.[2] We refer to dividend shock $\theta_t$ as the fundamental shock and aggregate endowment shock $u_t$ as the non-fundamental shock. Note that an idiosyncratic endowment shock $u_{it}$ which each investor $i$ can observe is also informative about the non-fundamental shock $u_t$ with precision of $\tau_{ui} = \frac{1}{\sigma_{ui}^2}$. For simplicity, we assume that $\tau_{ui} = 0$, implying that the investors cannot learn about non-fundamental factor $u_t$ from observing their own endowment shocks.[3]

The trading timeline is as follows. At the beginning of period $t$, a cohort of investors enters the market and observes public information $d_{t-1}$, their private signals $s_{it}$, and their endowment shocks $u_{it}$.[4] Afterwards, all investors submit their demand schedules $x_{it}(p_t, s_{it}, u_{it}, d_{t-1})$ contingent on market price $p_t$ to an auctioneer who will set a price to clear the market. Then, all

[2]Note that this non-tradable endowment shock creates stochastic intercept in the demand function which serves as a non-fundamental shock that prevents the stock price from being fully revealing. In fact, the stochastic aggregate endowment shock in market price ($u_t$) is identical to aggregate random demand of noise traders in the standard competitive noise-trading model. See Manzano and Vives (2010) and Farboodi and Veldkamp (2018) for more details.

[3]This assumption will not affect our main findings. We will discuss the role of information about current noise factor $u_t$ in Section 1.5.

[4]For tractability purpose, we assume that the investors cannot observe past prices. This allows us to focus on the main role of NF information in the model while avoiding complicated posterior updating about $u_t$ from past prices. We discuss how relaxing this assumption would affect our results in Section 1.4.3.

trades take place and the market continues to the next period in which 1) all investors close their positions and realize their investment returns and 2) a new generation of investors enters the market.

### The equilibrium

Let $I_{it} = \{s_{it}, u_{it}, d_{t-1}\}$ be the information set of investor $i$ at the time of submitting demand schedule to the auctioneer. The equilibrium definition is as follows.

**Definition 1.1 (Competitive rational expectation equilibrium)** *An equilibrium consists of a sequence of market price function $\{p_t(\cdot)\}_{\forall t}$ and demand function of all investors $X(p)$ such that*

1. *each investor chooses demand schedule optimally.*

$$x_{it}(p_t) \in \arg\max_x U_i(\pi_{it}(x)|I_{it})$$

2. *market-clearing condition of the asset market is satisfied.*

$$\int_i x_{it} di = 0$$

This is the standard competitive rational expectation equilibrium which imposes optimality conditions in a competitive market framework. We also restrict our focus to the stationary most-informative linear equilibrium as described below.

**Definition 1.2 (Equilibrium selection)** *The equilibrium is such that*

1. *market price function is linear and stationary.*

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

2. *provided that multiple linear equilibria exist, choose the most informative linear equilibrium.*

### Measures of volatility and efficiency

First, our measure of stock volatility is unconditional asset return volatility as defined below. Intuitively, this volatility metric evaluates the asset riskiness from a viewpoint of uninformed investors.

**Definition 1.3 (Volatility)** *Asset volatility in period $t$ is $var(\delta p_{t+1} + d_t - p_t)$.*

It is worth noting that, even though this metric provides a good estimation of asset riskiness in general, it might not reflect an actual risk of stock returns faced by individual investors. When the investors have more financial information, they actually face lower return risk provided that the stock volatility does not change. Therefore, we will also consider the variance of stock return conditional on all information available to investors as defined below.

**Definition 1.4 (Perceived asset risk)** *Perceived asset risk of investor $i$ in period $t$ is $var(\delta p_{t+1} + d_t - p_t | I_{it}, p_t)$.*

To measure price informativeness, we follow the notion of revelatory price efficiency (Bond et al. (2012), Bai et al. (2016)) which estimates how much stock prices reveal the information about fundamental components that have impact to real economic activity. In our model, the assumption of exogenous random process of dividend shocks (i.e. no feedback channel between stock price dynamics and firm's decision making) does not allow us to clearly identify the key factor that has real economic impact. To proceed, we will assume that the information about dividend shocks which should convey new information about the firm's fundamental in practice can have real economic impact. This is intuitive, as real decision makers (e.g firm managers, business partners of the firms, policymakers) depend on secondary market prices to learn new information about the firm fundamentals to take appropriate actions (Bond et al. (2012)). We define the level of price informativeness as follows.

**Definition 1.5 (Informativeness)** *Price informativeness at period $t$ is $var(\theta_t) - var(\theta_t | p_t)$*

From the definition, price informativeness estimates how much fundamental shock volatility from an outsider viewpoint is reduced if the outsider observes the market price. An alternative measure of price informativeness is price impact of fundamental shock ($\frac{\partial p_t}{\partial \theta_t}$). However, this measure does not capture possible changes of non-fundamental components in stock prices. As such, it may not be suitable in our study.

To measure market liquidity, the standard metric is price impact of non-fundamental shock ($\frac{\partial p_t}{\partial u_t}$). In the standard noise-trader model, this is the price impact of a trade from noise traders. Intuitively, lower price impact of non-fundamental shock implies that market price can absorb trade orders from noise traders well, reflecting a liquid (deep) market. In our model, price impact of non-fundamental shock will indirectly measure how well an asset price can absorb non-fundamental-driven portion of asset trading of investors. However, the price impact $\frac{\partial p_t}{\partial u_t}$ might not contain enough information for measuring market liquidity in our context, as stock prices, say in period $t$, might depend on both current fundamental shock $u_t$ and future non-fundamental shock $u_{t+1}$. Therefore, we measure market liquidity based on the following condition.

**Definition 1.6 (Liquidity comparison)** *Market liquidity is lower(higher) if both $\frac{\partial p_t}{\partial u_t}$ and $\frac{\partial p_t}{\partial u_{t+1}}$ are higher(lower).*

## 1.3 Baseline equilibrium without NF information.

In this section, we will analyze the model without non-fundamental (NF) information production to understand fundamental properties of stock prices in our dynamic setting. We assume that the investors obtain a noisy private signal about fundamental factor $\theta_t$. Define $s_{\theta,it} = \theta_t + \sqrt{\frac{1}{\hat{\tau}_\theta}}\epsilon_{\theta,it}$ in which $\epsilon_{\theta,it}$ is an i.i.d. random variable drawn from a normal distribution $N(0,1)$ with signal precision $\hat{\tau}_\theta$. We obtain the following remark.

**Remark 1.1** *If private signal profile is $s_{it} = \{s_{\theta,it}\}$, then*

1. *an equilibrium in which the investors trade the asset exists if and only if the degree of risk aversion is sufficiently low,*

2. *price informativeness increases if one of the following is true:*

   a) *the degree of risk aversion decreases*

   b) $\hat{\tau}_\theta$ *increases if the degree of risk aversion is sufficiently low.*

3. *market price is never fully revealing for any positive value of risk-aversion degree.*

Firstly, we found that the existence condition of equilibrium with trade depends on the degree of risk aversion. When the investors are more risk-averse, they trade the asset less aggressively, causing the asset prices to be more sensitive to non-fundamental shock and harder to predict. This high volatility feeds back into trading decision of investors in previous periods, causing asset price volatility to amplify further. Thus, if the investors are highly risk-averse, asset return volatility can be indefinite which causes the market to shut down.[5]

Interpretation of the second remark is straightforward. Price informativeness is higher if the investors are more risk-tolerant or if the investors are more informed about the asset fundamental, because they are willing to buy more assets to speculate against price deviation from the fundamentals. What is more interesting is the last remark which states that the asset price will never be fully revealing even when the investors are perfectly informed about the fundamental shock, consistent with the finding in Farboodi and Veldkamp (2018).

Why is the price not fully revealing when the investors have perfect information about dividend shock $\theta_t$? This seems counterintuitive as the investors with perfect fundamental signal should continue trading until the asset price returns to its fundamental value, the expected discounted sum of all future dividends $E(\sum_{i=t}^{\infty} \delta^{i-t} d_i | d_{t-1}, \theta_t)$. However, this is not true because the investors still face positive investment risk. Indeed, when trading assets, they do not know the information about future dividend shock $\theta_{t+1}$ which will arrive in the future. To see this, consider the mathematical expression of the asset prices below. In period $t$, the investors cannot perfectly forecast $p_{t+1}$, because they are uninformed about $\theta_{t+1}$. As such, they will be reluctant to trade more assets and move the price towards the fundamental value, even though there is profitable opportunity to do so.

$$p_t = f(d_{t-1}, \theta_t, \dots)$$

$$p_{t+1} = f(d_t, \underbrace{\theta_{t+1}}_{\text{unknown}}, \dots)$$

At this point, one may claim that, if the investors in every period can also learn about one-period ahead dividend shock (i.e. $s_{it} = \{\theta_t, \theta_{t+1}\}$), then market price can be fully revealing. Unfortunately, this is invalid because, from the viewpoint of investors, their future counterparts will also learn information about fundamental shocks of further periods that they cannot learn

[5] Note also that non-stationary equilibrium with trade cannot exist too since there is no finite real-valued coefficients in asset price function that are consistent with expected future price.

today. This is illustrated mathematically below. In this situation, the investors in period $t$ now face future price risk from being uninformed about two-period ahead dividend shock $\theta_{t+2}$.

$$p_t = f(d_{t-1}, \theta_t, \theta_{t+1}, \dots)$$

$$p_{t+1} = f(d_t, \theta_{t+1}, \underbrace{\theta_{t+2}}_{\text{unknown}}, \dots)$$

Let us look into this issue by considering the case when the investors have more signals about future dividend shocks. We obtain the following remark.

**Remark 1.2** *If private signal is $s_{it} = \{\theta_k\}_{k=t}^{t+n}$, then*

1. *asset price is not fully revealing for any finite value of $n$,*

2. *if discount factor is strictly less than one, price sensitivity to noise factor strictly decreases when $n$ increases,*

3. *if discount factor is one, price sensitivity to noise factor does not change when $n$ increases.*

This remark highlights an important distinction between stocks and other asset classes with fixed maturity date. For the latter, asset prices may be fully informative if the investors have perfect information about all fundamental shocks of the whole remaining asset lifetime. However, because common stocks are expected to pay dividend perpetually, future price risk will persist. Unless the investors have perfect information about all future dividends; that is, when the investors in every period $t$ have perfect information about all future dividend shocks $\{\theta_i\}_{i=t}^{\infty}$ or when there are no dividend shocks (i.e $\sigma_\theta^2 = 0$), the asset price will not be fully informative. Moreover, when the discount factor is high, a reduction in price sensitivity to noise factor can be minimal.

One important implication from this finding is that any inefficiency from learning non-fundamental information can be long-lasting, because the investors will always have incentive to learn non-fundamental information. This is particularly true when the investors have good fundamental information, because non-fundamental-information-based trading can be more profitable to the investors who always look for new information that other investors do not have. We will shed more light on this issue in the next two sections.

## 1.4 Analysis of learning NF information.

In this section, we will study how learning NF information, the information about next-period aggregate endowment shock $u_{t+1}$, can cause inefficiency to asset prices. We will first consider the impact of learning NF information using a standard two-period model. Then, we analyze the impact of an increase in NF information quality in our infinite-period model.

### 1.4.1 Static effect of learning NF information

Consider a classical two-period trading model with similar setting to our infinite-period dynamic model. There is a risky divisible asset which pays a stochastic return $\theta + u_2$ in the second period where $\theta \sim N(0, \sigma_\theta^2)$ is an i.i.d. stochastic fundamental factor and $u_2 \sim N(0, \sigma_u^2)$ is an i.i.d. random non-fundamental factor. There is a unit continuum of investors with the following mean-variance utility function:

$$U_i(\pi_i|I) = E(\pi_i|I) - \frac{\rho}{2} Var(\pi_i|I)$$

where $\rho$ is the degree of risk aversion and $\pi_i = (\theta + u_2 - p_1)(x_i - u_i)$ is the return from buying $x_i$ units of risky assets at price $p_1$ in period 1. As noted before, the investors also face non-tradable endowment shock $u_i = u_1 + \sqrt{\frac{1}{\tau_{ui}}}\mu_{ui}$ with market-wide shock $u_1 \sim N(0, \sigma_u^2)$ and i.i.d white noise idiosyncratic shock $\mu_{ui1}$ in period 1. Assume that $\tau_{ui} = 0$. Let private signal profile of investor $i$ be $s_i = \{s_{\theta i}, s_{ui}\}$ in which

$$s_{\theta i} = \theta + \sqrt{\frac{1}{\hat{\tau}_\theta}}\epsilon_{\theta i} \quad \text{and} \quad s_{ui} = u_2 + \sqrt{\frac{1}{\hat{\tau}_u}}\epsilon_{ui}$$

This setup allows the investors to observe both private fundamental signal $s_{\theta i}$ and non-fundamental signal $s_{ui}$. The trading protocol and equilibrium solution concept are identical to our main infinite-period model. Note that this static setting is a one-period snapshot of our infinite-period model with an additional assumption that future asset prices remain unchanged regardless. We obtain the following remark.

**Remark 1.3 (Static effect of learning NF information)** *In the static setting, if $\hat{\tau}_\theta$ is sufficiently high, then an increase in $\hat{\tau}_u$ causes*

1. *perceived asset risk $var(\theta + u_2|I_i, p_1)$ to decrease*
2. *price impact of future noise factor ($\frac{\partial p_1}{\partial u_2}$) to increase*
3. *price impact of current noise factor ($\frac{\partial p_1}{\partial u_1}$) to decrease if and only if $\hat{\tau}_u > \frac{\rho(\tau_u - \rho)}{\tau_u}$.*
4. *stock volatility to decrease if $\hat{\tau}_u > \frac{\rho(\tau_u - \rho)}{\tau_u}$*
5. *price informativeness to decrease if $\tau_u - \rho$ is positive and sufficiently high*
6. *price informativeness to increase if $\tau_u < \rho$ and $\hat{\tau}_u$ is small.*

*Moreover,*

1. $\lim_{\hat{\tau}_u \to \infty} \frac{\partial p_1}{\partial u_1} = 0$,
2. $\lim_{\hat{\tau}_u \to \infty} var(\theta|p) < \lim_{\hat{\tau}_u \to 0} var(\theta|p)$ *if and only if $\rho > \tau_u$, and*
3. $\lim_{\hat{\tau}_u \to \infty} var(\theta + u_2 - p_1) = 0$.

Interpretation of this result is straightforward. When the investors have more financial information to estimate their investment returns, their investment risks are lower, increasing their willingness to trade more assets. Due to more aggressive trade, price sensitivity to current noise factor ($\frac{\partial p_1}{\partial u_1}$) decreases while price sensitivity to future noise factor ($\frac{\partial p_1}{\partial u_2}$) increases. An exception is when the noise factor volatility is relatively low, because the investors will free-ride the non-fundamental information from asset price excessively when their own private signals are relatively low (i.e. when $\hat{\tau}_u < \frac{\rho(\tau_u - \rho)}{\tau_u}$). This free-riding behavior decelerates aggregation speed of non-fundamental information in the price and causes the price to be more sensitive to current non-fundamental shock temporarily. As shown in Figure 1.1, price impact of non-fundamental shock $u_1$ increases temporarily before declining towards zero when non-fundamental signal precision increases. Stock volatility declines, but price informativeness deteriorates. Eventually, when the investors have perfect non-fundamental signal, asset price in period 1 will no longer fluctuate with current non-fundamental shock $u_1$ and stock volatility converges to zero. That is, the asset price becomes fully informative about asset return $\theta + u_2$.

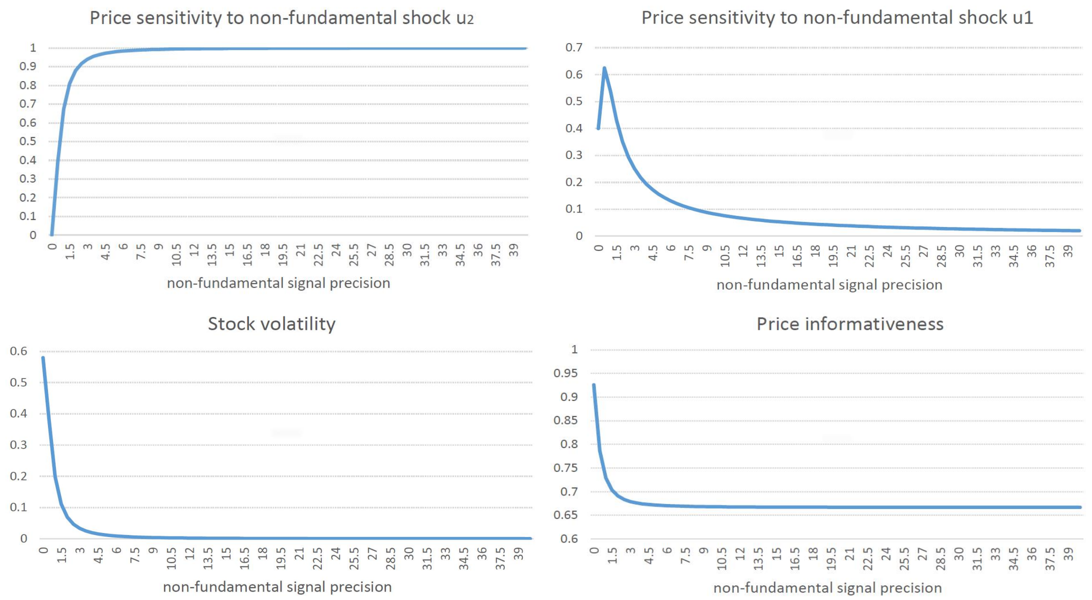


Figure 1.1: Simulation results of an increase in non-fundamental signal precision $\hat{\tau}_u$ with $\tau_u = 3$, $\rho = 0.8$, and $\tau_\theta = 1$.

Interestingly, price informativeness could improve when non-fundamental signal precision increases, because asset price is less volatile to current non-fundamental shock. From the remark, this scenario is possible when degree of risk aversion is so large that the asset price when there is no non-fundamental information would be relatively volatile to current non-fundamental shock $u_1$. As shown in Figure 1.2, an increase in non-fundamental signal precision causes price informativeness to increase and converge to a higher level.

In a nutshell, from efficiency point of view, learning NF information can improve price efficiency and reduce asset volatility. However, this conclusion might be misleading for stock markets since we have not yet considered dynamics effect from endogenous change of future price dynamics.

### 1.4.2 Dynamic feedback effect and price inefficiency.

Our next step is to analyze the impact of learning NF information on price formation in our dynamic setting. Let $s_{\theta,it} = \theta_t + \sqrt{\frac{1}{\hat{\tau}_\theta}}\epsilon_{\theta,it}$ in which $\epsilon_{\theta,it}$ is an i.i.d. random variable drawn from a normal distribution $N(0,1)$ and $\hat{\tau}_\theta$ is precision of the fundamental signal. Let $s_{u,it} = u_{t+1} + \sqrt{\frac{1}{\hat{\tau}_u}}\epsilon_{u,it}$ in which $\epsilon_{u,it}$ is an i.i.d. random variable drawn from a normal

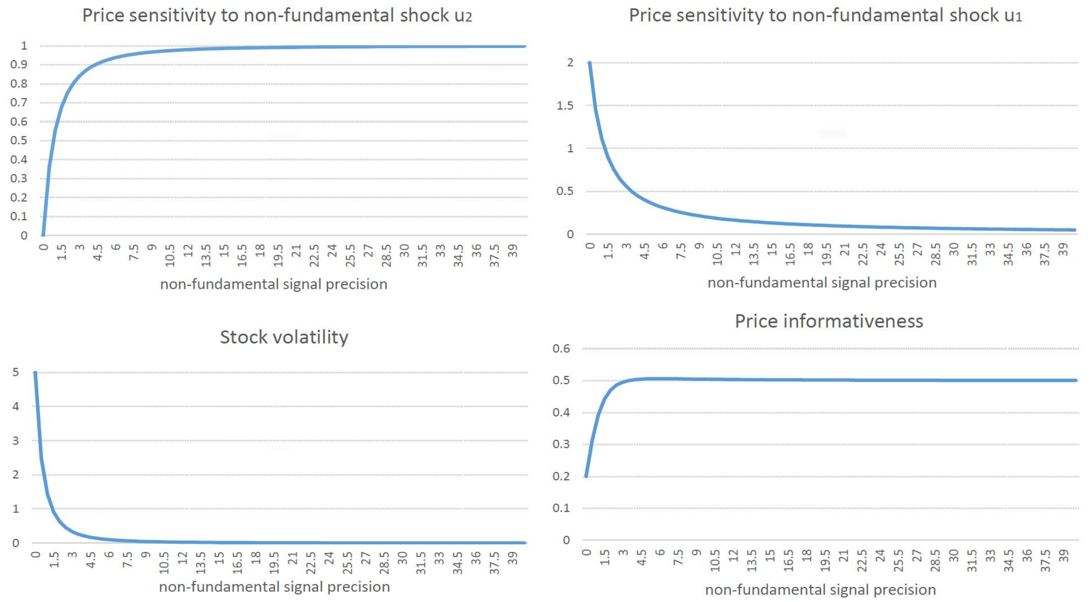


Figure 1.2: Simulation results of an increase in non-fundamental signal precision $\hat{\tau}_u$ with $\tau_u = 1$, $\rho = 2$, and $\tau_\theta = 1$.

distribution $N(0, 1)$ and $\hat{\tau}_u$ is precision of the non-fundamental signal. Throughout Subsection 1.4.2, we make the following assumption unless stated otherwise.

**Assumption 1.1** *Private signal profile is* $s_{it} = \{s_{\theta,it}, s_{u,it}\}$

Now, consider an exogenous increase in non-fundamental signal precision $\hat{\tau}_u$. Undeniably, better NF information must partly improve price efficiency based on our static analysis. In the static model, risk-averse investors who learn more NF information will trade more aggressively which will reduce price sensitivity to current non-fundamental shock. Additionally, since the investors trade more assets on their NF information, current asset price becomes more informative about future non-fundamental shock $u_{t+1}$, providing more information about future price to all investors. This in turn should reduce the price impact of noise factor $u_t$ even more. Surprisingly, we found that this conjecture is invalid as established in the following proposition.

**Proposition 1.1 (Inefficiency from learning NF information I)** *If 1) fundamental signal precision and discount factor are sufficiently high and 2) non-fundamental signal precision is sufficiently low, an increase in non-fundamental signal precision leads to*

1. *lower market liquidity,*
2. *higher stock volatility,*
3. *lower price informativeness, and*
4. *higher perceived asset risk as measured by* $var(\delta p_{t+1} + d_t | I_{it}, p_t)$.

This proposition establishes contradicting predictions from classical rational speculative theory. From the proposition, when the investors have better quality of NF information, the stock volatility increases, price informativeness declines, and price sensitivity to non-fundamental factors increases (i.e. lower market liquidity). More surprisingly, the investors view that the asset is riskier (i.e. higher $var(\delta p_{t+1} + d_t | I_{it}, p_t)$) despite having more information about future price. This is because of higher future price risk endogenously created by future investors. When receiving better NF information, the future investors will also trade more assets on their NF information, causing the future prices to be more volatile to non-fundamental shocks of further periods. However, the investors cannot learn the NF information that the future investors can learn. This is illustrated mathematically below.

$$p_t = f(d_{t-1}, \theta_t, u_{t+1}, u_t)$$
$$p_{t+1} = f(d_t, \underbrace{\theta_{t+1}, u_{t+2}}_{\text{unknown}}, u_{t+1})$$

When the investors learn NF information, asset price in period $t+1$ now depends on both fundamental factor $\theta_{t+1}$ and two-period ahead non-fundamental factor $u_{t+2}$, both of which are unknown to investors in period $t$. When the non-fundamental signal precision increases, asset price $p_{t+1}$ becomes more sensitive to noise factor $u_{t+2}$ which adversely affects the investors in period $t$. This higher stock return volatility feeds back into trading decision of investors, resulting in lower price efficiency.

The next question is whether learning NF information can improve price efficiency at some point; that is, whether the negative dynamic feedback effect will always dominate the positive effect of learning more information. Indeed, we found that when the noise signal precision is high, price efficiency might improve when the noise signal precision increases as implied in the following proposition.

**Proposition 1.2 (Inefficiency from learning NF information II)** *Provided that fundamental signal precision and discount factor are sufficiently high,*

1. $\lim_{\hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t} < \frac{\partial p_t}{\partial u_t}$ *for any* $\hat{\tau}_u > 0$*,*
2. $\lim_{\hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t} \leq \frac{\partial p_t}{\partial u_t}$ *at* $\hat{\tau}_u = 0$*.*

From the proposition, the price will be least sensitive to the current noise shock when the investors have perfect non-fundamental signals. Interestingly, the sensitivity can even go below the value which the investors do not have non-fundamental information. The is because the marginal benefit of learning more information exceeds the negative dynamic feedback effect. This finding suggests that it might be socially desirable if the investors have good non-fundamental signals. However, our next proposition established that this is not the case when discount factor is high.

**Proposition 1.3 (Inefficiency from learning NF information III)** *If the fundamental signal precision is sufficiently high and the discount factor is strictly less than one, then*

1. $\lim_{\hat{\tau}_u \to 0} var(\delta p_{t+1} + d_t - p_t) > \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t - p_t)$,
2. $\lim_{\hat{\tau}_u \to 0} var(\delta p_{t+1} + d_t | I_{it}, p_t) > \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t | I_{it}, p_t)$,

3. $\lim_{\hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_t} > \lim_{\hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t}$,

4. $\lim_{\hat{\tau}_u \to 0} var(\theta_t|p_t) < var(\theta_t|p_t)$ *for any* $\hat{\tau}_u \in \mathcal{R}^+$ *if the discount factor is sufficiently high.*

From the proposition, if the investors obtain good non-fundamental signals, then stock volatility, perceived asset risk, and price sensitivity to noise factor $\frac{\partial p_t}{\partial u_t}$ are lower than the level which the investors have no NF information. However, price informativeness will always be lower than the level which the investors do not have NF information if discount factor is sufficiently high. This is because the price is more sensitive to future non-fundamental shock, but the price sensitivity to current non-fundamental shock marginally declines due to the dynamic feedback effect. Figure 1.3 displays simulation results of an increase in non-fundamental

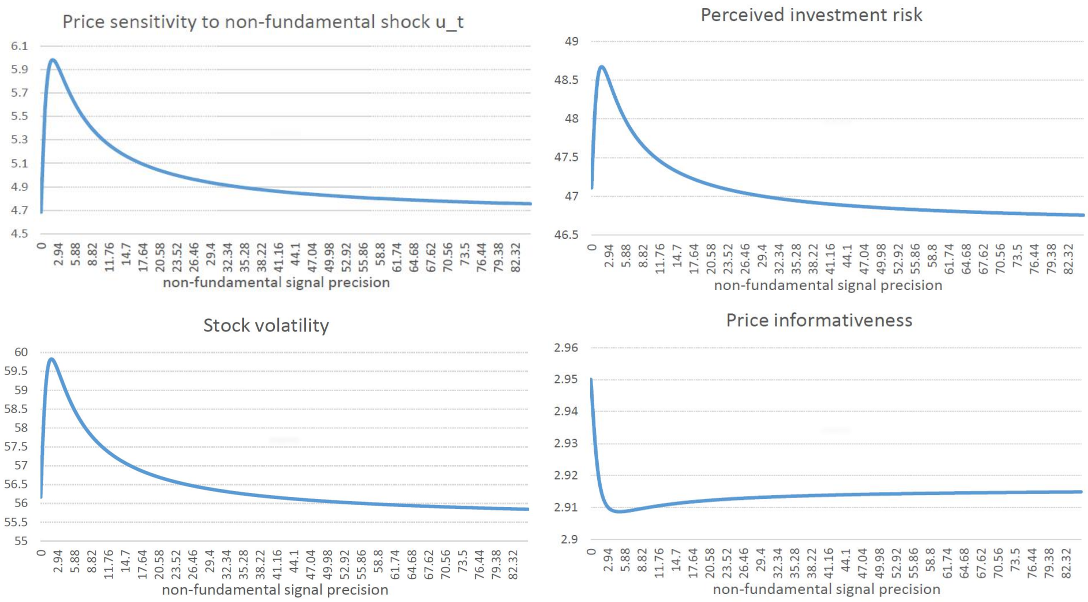


Figure 1.3: Simulation results of an increase in non-fundamental signal precision $\hat{\tau}_u \in [0, 85]$ with $\tau_u = 3$, $\rho = 0.085$, $\hat{\tau}_\theta = 100$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 3$.

signal precision when the discount factor is 0.97. From the figure, price sensitivity to current non-fundamental shock, asset volatility, and perceived investment risk increase initially before declining to a new lower level. Price informativeness sharply declines and converges to a lower value than the level which the investors do not have NF information.

Indeed, the discount factor partly determines magnitude of the adverse dynamic feedback effect. As shown in the next proposition, we found that asset volatility, perceived investment risk, and price sensitivity to current non-fundamental shock converge to the same level which the investors do not have NF information when the discount factor is one. Intuitively, when the investors care about future price immensely, they will be heavily affected by the adverse dynamic feedback effect from higher future NF information risk which can offset the benefit from learning more NF information completely.

**Proposition 1.4 (Inefficiency from learning NF information IV)** *If* $\hat{\tau}_\theta$ *is sufficiently high and discount factor is one, then*

1. $\lim_{\hat{\tau}_u \to 0} var(\delta p_{t+1} + d_t - p_t) = \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t - p_t)$,

2. $\lim_{\hat{\tau}_u \to 0} var(\delta p_{t+1} + d_t | I_{it}, p_t) = \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t | I_{it}, p_t)$,

3. $\lim_{\hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_t} = \lim_{\hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t}$

As shown in Figure 1.4, price sensitivity to current non-fundamental shock, asset volatility, and perceived investment risk are always higher than the level which the investors do not have NF information. Price informativeness will always be suboptimal when the investors learn NF information. Indeed, we also found that price efficiency will be optimal when the investors do not learn NF information if the discount factor approaches one, as stated in the following remark.

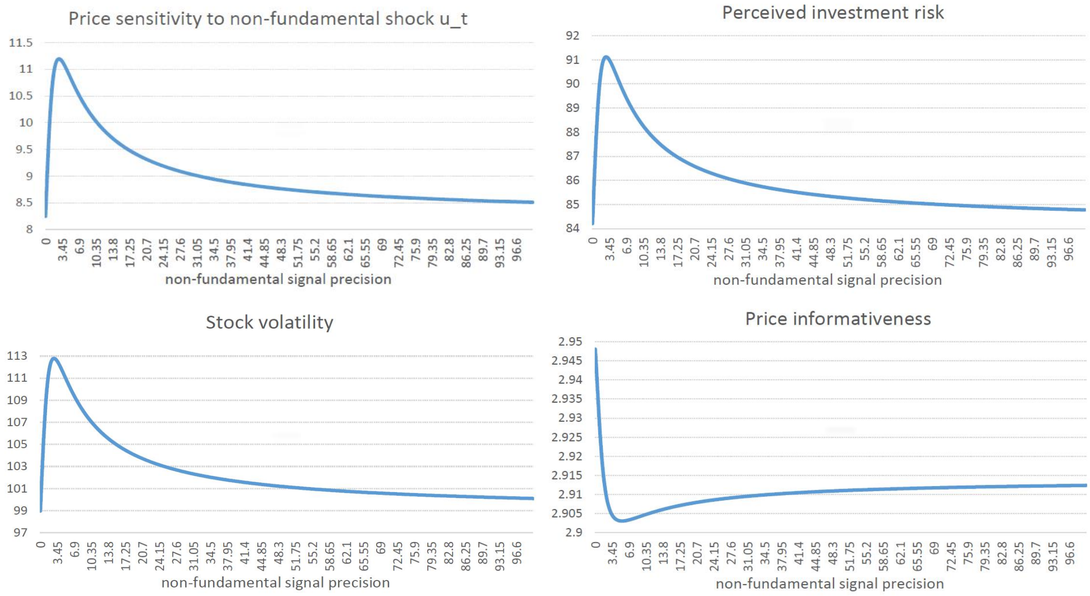


Figure 1.4: Simulation results of an increase in non-fundamental signal precision $\hat{\tau}_u \in [0, 100]$ with $\tau_u = 3$, $\rho = 0.085$, $\hat{\tau}_\theta = 100$, $\alpha = 0.94$, $\delta = 1$, $\tau_\theta = 3$.

**Remark 1.4** *If discount factor is one and fundamental signal precision is sufficiently high, then price informativeness and market liquidity are maximized if* $\hat{\tau}_u = 0$. *Specifically, for any* $\hat{\tau}_u \in \mathcal{R}^+$,

1. $var(\theta_t | p_t) > \lim_{\hat{\tau}_u \to 0} var(\theta_t | p_t)$,

2. $\frac{\partial p_t}{\partial u_{t+1}} > \lim_{\hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_{t+1}}$,

3. $\frac{\partial p_t}{\partial u_t} \geq \lim_{\hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_t}$.

So far, the main inefficiency from NF information production comes from the negative dynamic feedback effect of future NF information risk. One might wonder whether allowing the investors to learn future NF information that their future counterparts learn would solve this problem. That is, what if the investors in period $t$ can acquire information about $u_{t+2}$? We found that the inefficiency problem still remains. In fact, price informativeness can deteriorate even more. To see this, we now assume that the investors have perfect signals about non-fundamental factors of several periods ahead as follows.

**Assumption 1.2** *Private signal profile is* $s_{it} = \{\theta_t\} \cup \bigcup_{k=1}^{n} \{u_{t+k}\}$

Consider the following proposition.

**Proposition 1.5 (Learning far-future NF information)** *Provided that Assumption 1.2 is true, price impact of current noise factor* ($\frac{\partial p_t}{\partial u_t}$) *is non-zero for any finite* $n \geq 1$. *Also, if* $n$ *increases, then*

1. *asset volatility, perceived asset risk, and price impact of current noise factor* ($\frac{\partial p_t}{\partial u_t}$) *decrease if discount factor is lower than one.*
2. *asset volatility, perceived asset risk, and price impact of current noise factor* ($\frac{\partial p_t}{\partial u_t}$) *do not change if discount factor is one.*
3. *price informativeness is strictly lower if discount factor is sufficiently high.*

This proposition confirms that learning information about non-fundamental shocks of several periods ahead can deteriorate price efficiency, especially if the investors care a lot about the future (i.e. high discount factor). The reason is that 1) future information uncertainty still remains in the stock prices and 2) the negative feedback dynamic effect from NF information uncertainty has significant impact on the investors. As a result, the price sensitivity to current non-fundamental shock will decline marginally when the investors obtain more non-fundamental signals. To illustrate, suppose that the investors in every period $t$ obtain perfect noise signals about $\{u_{t+1}, u_{t+2}\}$. As written mathematically below, asset price in period $t+1$ now contains $u_{t+3}$ which is unknown to investors in period $t$.

$$
\begin{aligned}
p_t &= f(d_{t-1}, \theta_t, u_{t+1}, u_{t+2}, u_t) \\
p_{t+1} &= f(d_t, \underbrace{\theta_{t+1}, u_{t+3}}_{\text{unknown}}, u_{t+2}, u_{t+1})
\end{aligned}
$$

Thus, even though the investors obtain more future non-fundamental information, the asset price will always be volatile to non-fundamental shock $u_t$. In addition, the asset price is now sensitive to new future non-fundamental shock $u_{t+2}$, causing price informativeness to decline further. Overall, an attempt to learn new NF information of future periods, instead of improving efficiency, can turn out to harm market liquidity and price informativeness tremendously.

This finding provides an important insight for market regulators into how the benefits of learning NF information can be overrated. Typically, we believe that the investors can reduce their investment risks by learning more NF information which could improve price efficiency. However, our model suggests that stock prices could become more volatile, less informative, less liquid, and harder to estimate for all the investors. To promote price efficiency, perhaps a better way is to promote fundamental information disclosure to allow the investors to properly assess the firm prospects, both for the short term and the long-term. This can reduce the future fundamental information risk and improve the quality of stock prices.

### 1.4.3 What if the investors can observe historical prices?

So far, we have assumed that investors cannot observe past prices when updating their posteriors. This assumption, while giving us a tractable model, might raise concerns on the result robustness. In this section, we will discuss how relaxing this assumption may affect our findings.

Why do investors value historical prices? Recall that when investors learn NF information, asset prices will be informative about future noise factors. Therefore, the historical prices will be another valuable source of learning information about **current** noise factor $u_t$, helping the investors in period $t$ extract relevant information from current asset price more efficiently. To clarify, below is the mathematical expression of asset price in period $t$ and $t-1$.

$$p_t = f(d_{t-1}, \theta_t, u_{t+1}, u_t)$$
$$p_{t-1} = f(d_{t-2}, \theta_{t-1}, u_t, u_{t-1}).$$

The investors in period $t$ would optimally use $p_t$ to learn about $\theta_t$ and $u_{t+1}$ to forecast their expected returns $\delta p_{t+1} + d_t$. Therefore, learning $u_t$ from price $p_{t-1}$ would increase their information quality obtained from price $p_t$ even further. How much information about $u_t$ that the investors can extract from $p_{t-1}$ depends on the investor's information about past dividend shocks, past non-fundamental shocks, and the length of observable past price sequence. Note that observing longer periods of past prices should increase the information quality about $u_t$. To illustrate, consider $p_{t-2} = f(d_{t-3}, \theta_{t-2}, u_{t-1}, u_{t-2})$ which is informative about $u_{t-1}$. Indeed, $p_{t-2}$ reveals information about $u_{t-1}$, helping the investors infer the information about $u_t$ from $p_{t-1}$ more accurately.

To conclude, the impact of NF information production missing from our previous analysis is the impact from more information about **current** noise factor from past prices. In other words, there is endogenous increase in public information about current noise factor when the non-fundamental signal precision increases. This implies that most of our analysis should remain true, particularly the findings based on the investors who obtain perfect signals about $\theta_t$ and $u_{t+1}$. This is because the investors already have perfect private information, and thus additional information from past prices will be irrelevant.

However, one may wonder whether this assumption will significantly affect our analysis when the investors have imperfect signals. With imperfect signals, the investors still depend on the information they could learn from asset prices. Therefore, observing historical prices would improve their information quality. This positive effect might dominate the negative dynamic feedback effect which might compromise some of our findings.

To have a preliminary investigation on this conjecture while maintaining tractability of our model, we now assume that the investors also obtain a private signal about current noise factor $u_t$ with identical signal precision as if they can observe $n$ periods of past prices. From the equilibrium price function, we know that $\frac{1}{k_3}(k_4 p_{t-1} - k_1 d_{t-2} - k_2 \theta_{t-1} - k_5 \bar{\theta}) = u_t + \frac{1}{k_3} u_{t-1}$. Provided that the investors know all past dividend shocks, this implies that the investors who observe $\{p_{t-1}, p_{t-2}, \ldots, p_{t-n}\}$ will obtain the information equivalent to observing a signal about $u_t$ with signal precision $k_3^2 \tau_u \left(\frac{1-k_3^{2n}}{1-k_3^2}\right)$. Let $s_{up,it} = u_t + \sqrt{\frac{1}{\hat{\tau}_{up}}} \epsilon_{up,it}$ in which $\epsilon_{up,it}$ is an i.i.d. white noise normally-distributed random variable and $\hat{\tau}_{up} = k_3^2 \tau_u \left(\frac{1-k_3^{2n}}{1-k_3^2}\right)$. We make the following assumption.

**Assumption 1.3** *Private signal profile is $s_{it} = \{s_{\theta,it}, s_{u,it}, s_{up,it}\}$*

This assumption states that in addition to the fundamental and non-fundamental information, the investors also receive a private signal about current non-fundamental shock $u_t$ which is informationally equivalent to observing $n$ periods of past prices. Figure 1.5 compares two simulation results when the investors cannot observe private signal $s_{up,it}$ and when $n = 4$ (equivalent to observing $\{p_{t-1}, p_{t-2}, p_{t-3}, p_{t-4}\}$). From the figure, allowing the investors who have imperfect fundamental signals to observe past prices improves price efficiency compared to the case of no past price observation. Since the precision of current non-fundamental signal is endogenously increasing, the effect from observing past prices will start taking place when the investors have sufficiently good non-fundamental signals. Indeed, the investors also obtain better NF information from the asset price, but this might contribute positively to price efficiency. From our previous analysis, when the investors have good non-fundamental signals, the positive effect of learning more NF information will start dominating the negative dynamic feedback effect. Therefore, having better quality of NF information from the market price can positively affect price efficiency at this stage. However, note that the negative dynamic feedback effect which causes price inefficiency still exists.

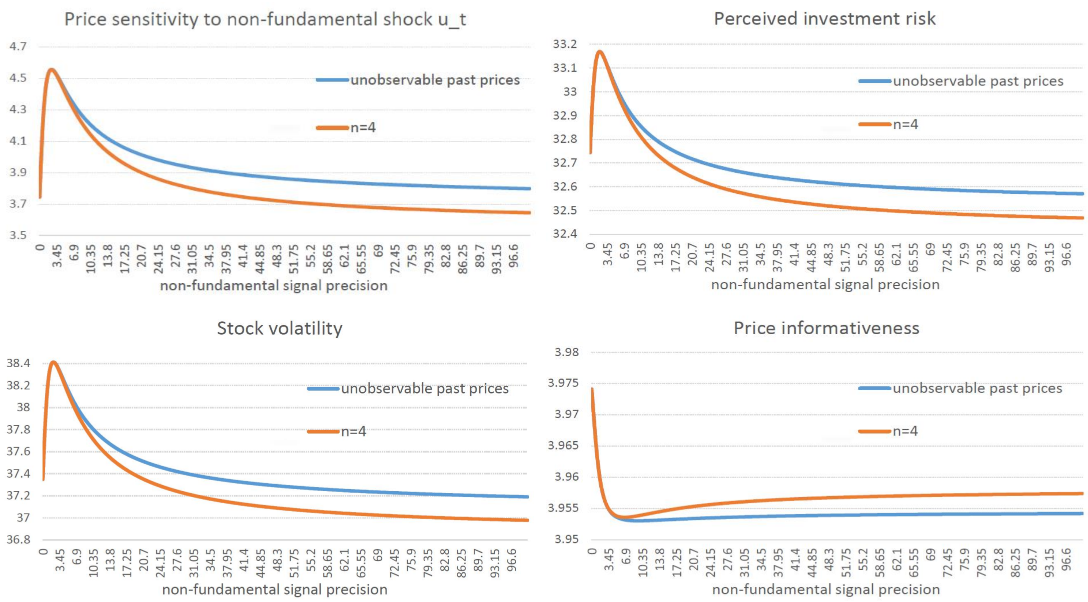


Figure 1.5: Simulation results of an increase in non-fundamental signal precision $\hat{\tau}_u \in [0, 100]$ when the investors cannot observe $s_{up,it}$ (i.e unobservable past prices) and when they can observe $s_{up,it}$ for $n = 4$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\hat{\tau}_\theta = 100$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

In contrast, if the investors already have perfect fundamental signals, the type of information the investors would learn from the price would be only the NF information. Thus, the assumption on past price observations should affect only the speed of convergence of price efficiency when there is an increase in non-fundamental signal precision. This is confirmed in the following proposition.

**Proposition 1.6 (Inefficiency from learning NF information V)** *If private signal profile satisfies Assumption 1.3 and the discount factor is sufficiently high, then for any $\hat{\tau}_u > 0$,* *1.* $\lim_{\hat{\tau}_\theta \to \infty} var(\theta_t|p_t) > \lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to 0} var(\theta_t|p_t)$,

*2.* $\lim_{\hat{\tau}_\theta \to \infty} \frac{\partial p_t}{\partial u_{t+1}} > \lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_{t+1}}$*, and*

*3.* $\lim_{\hat{\tau}_\theta \to \infty} \frac{\partial p_t}{\partial u_t} \geq \lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to 0} \frac{\partial p_t}{\partial u_t}$ *if the discount factor is one.*

From the proposition, our findings on price efficiency from the previous section remain true. When the investors have perfect fundamental signals and high discount factor, the price informativeness will always be suboptimal. Also, when the discount factor is one, both price informativeness and market liquidity are maximized when the investors do not have NF information, because the investors do not need the informational benefit from past prices to learn the fundamental information.

## 1.5 Impact of learning information about current non-fundamental shock

In reality, apart from learning information about future noise shocks, the investors also process order-flow data to estimate past or current non-fundamental shocks. This section will look into the impact of learning information about current non-fundamental shock $u_t$ on price informativeness and examine whether the interacting effects of learning information about both current-period and future-period noise shocks can deteriorate price efficiency.

How would acquisition of the current-noise information affect price formation? In general, there are two sources of financial information that investors in any period $t$ can learn from: private signal $s_{it}$ and public signal $p_t$. Since better information about noise factor $u_t$ helps the investors extract relevant information from asset price $p_t$ more accurately, price informativeness can improve. However, as documented in standard literature, price informativeness deterioration might come from herd behavior of the investors; they depend too much on information from the price and too little on their private information, resulting in less degree of information aggregation in asset prices.

In this section, we will show that price informativeness can decrease when the investors obtain better current-noise information from a different angle. That is, the investors might extract more NF information from asset price when they have better current-noise information which can destroy price informativeness. Formally, we assume that investors in period $t$ can observe a private signal about noise factor $u_t$ in addition to their private signals about fundamental factor $\theta_t$ and future non-fundamental shock $u_{t+1}$. Define $s_{uc,it} = u_t + \sqrt{\frac{1}{\hat{\tau}_{uc}}}\epsilon_{uc,it}$ in which $\epsilon_{uc,it} \sim N(0,1)$ is an i.i.d. idiosyncratic disturbance. We make the following assumption.

**Assumption 1.4** *Private signal profile is* $s_{it} = \{s_{\theta,it}, s_{u,it}, s_{uc,it}\}$

Applying this assumption to our model, we obtain the following proposition.

**Proposition 1.7 (Current-period noise information)** *Provided that Assumption 1.4 is satisfied, asset price dynamics when the investors have perfect signal about* $\theta_t$ *and* $u_{t+1}$ *does not depend on* $\hat{\tau}_{uc}$*. Also, for* $X = \{d_{t-1}, \theta_t, u_t, u_{t+1}\}$,

$$\lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_{uc} \to \infty} p_t(X) = \lim_{\hat{\tau}_u \to \infty, \hat{\tau}_{uc} \to \infty} p_t(X) = \lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to \infty} p_t(X).$$

This proposition emphasizes the role of current-noise information. First, the current-noise information is redundant when investors already have perfect private signals. Second, when the investors have perfect information about $u_t$, the asset price function will be equivalent to when the investors have perfect private signals. This is because the investors with imperfect private signals, either fundamental or non-fundamental signals, now have a perfect public signal – the asset price – to learn from. Note that this intuition works only if the investors already have perfect signal of one information type so that the investors can infer the other information type from the asset price perfectly.

There are two implications on price informativeness from this proposition. First, when the investors already have good information about future noise factor $u_{t+1}$, more information about current noise factor $u_t$ will allow the investors to obtain more fundamental information from the asset price which can improve price informativeness. This is shown in Figure 1.6 which illustrates how an exogenous increase in signal precision of $s_{uc}$, the signal about $u_t$, can improve price efficiency.

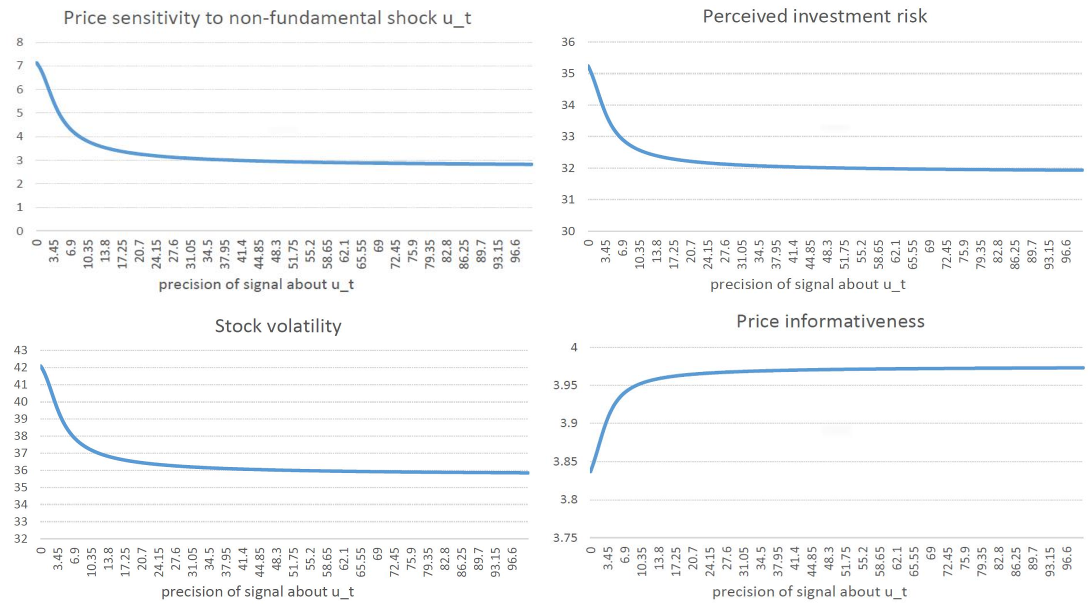


Figure 1.6: Simulation results of an increase in signal precision $\hat{\tau}_{uc} \in [0, 100]$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\hat{\tau}_\theta = 3$, $\hat{\tau}_u = 2000$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

However, when the investors already have good private signal about fundamental factor, the investors will instead use the asset price to obtain more NF information, resulting in possible deterioration of price informativeness as stated in the following remark.

**Remark 1.5** *If 1) $\hat{\tau}_\theta$ and discount factor are sufficiently high and 2) $\hat{\tau}_u$ is sufficiently low, then*

$$\lim_{\hat{\tau}_{uc} \to 0} var(\theta_t | p_t) < \lim_{\hat{\tau}_{uc} \to \infty} var(\theta_t | p_t).$$

Figure 1.7 illustrates the impact of an increase in precision of current-noise signal when the investors have good fundamental signals. Indeed, the evolution of price efficiency is similar to the case of an increase in non-fundamental signal precision. From the figure, stock volatility and price sensitivity to noise factor $u_t$ increases initially before gradually decreasing. Price informativeness non-monotonically decreases to a lower level.

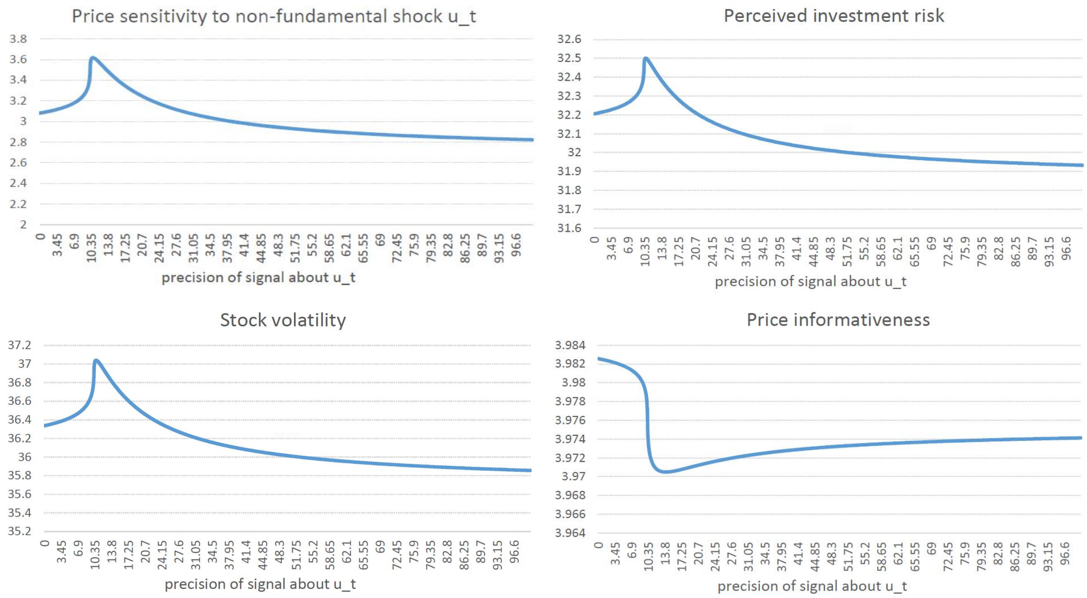


Figure 1.7: Simulation results of an increase in signal precision $\hat{\tau}_{uc} \in [0, 100]$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\hat{\tau}_\theta = 5000$, $\hat{\tau}_u = 0.5$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

This finding provides one important insight into how post-trade market transparency (i.e. public disclosure of past order flows) can adversely affect stock prices. Depending on what information is in high demand from the investors, price informativeness which regulators would like to promote can instead decrease in response to higher degree of post-trade transparency.

## 1.6 Endogenous information choice and information technology growth.

So far, we have seen how learning information about future non-fundamental shock can cause inefficiency to stock prices. However, given the assumption on exogenous non-fundamental signal precision, we cannot see clearly whether the investors will indeed acquire more NF information. Our next task is to study whether the investors will actually acquire NF information. If so, under what conditions such NF information acquisition can deteriorate price inefficiency. To this end, this section extends the model to incorporate endogenous information acquisition.

### 1.6.1 Model setting: information acquisition stage

We will now incorporate endogenous information acquisition stage to the model, allowing the investors to strategically choose their private signal precisions. Denote $s_{\theta,it} = \theta_t + \sqrt{\frac{1}{\hat{\tau}_{\theta,it}}}\epsilon_{\theta,it}$ in which $\epsilon_{\theta,it}$ is an i.i.d. random variable from normal distribution $N(0,1)$ with signal precision $\hat{\tau}_{\theta,it}$, and $s_{u,it} = u_{t+1} + \sqrt{\frac{1}{\hat{\tau}_{u,it}}}\epsilon_{u,it}$ in which $\epsilon_{u,it}$ is an i.i.d. random variable from normal distribution $N(0,1)$ with signal precision $\hat{\tau}_{u,it}$. Define $\hat{\tau}_{it} = \{\hat{\tau}_{\theta,it}, \hat{\tau}_{u,it}\}$. Prior to trading in period $t$, all investors who are born in period $t$ can choose their private signal precisions $\hat{\tau}_{it}$ subject to the following information technology constraint:

$$\hat{\tau}_{\theta,it} + \gamma\hat{\tau}_{u,it} \leq \Gamma$$

where $\gamma \in \mathcal{R}^+$ is a relative cost of processing NF information and $\Gamma \in \mathcal{R}^+$ is the information-processing capacity. Assume that private signal profile is $s_{it} = \{s_{\theta,it}, s_{u,it}\}$ in which signal precision $\hat{\tau}_{it}$ is now endogenously determined. The equilibrium definition which now includes optimality condition for the information acquisition decision is as follows.

**Definition 1.7 (Competitive rational expectation equilibrium)** *An equilibrium consists of information acquisition choices $\{\hat{\tau}_{it}\}_{\forall i,t}$, market price functions $\{p_t(\cdot)\}_{\forall t}$, and demand functions of all investors $X(p)$ such that*

1. *each investor chooses signal precisions optimally*

$$\hat{\tau}_{it} \in \arg\max_{\hat{\tau}_{it}} U_i(\pi_{it}) \quad s.t. \quad \hat{\tau}_{\theta,it} + \gamma\hat{\tau}_{u,it} \leq \Gamma$$

2. *each investor chooses demand schedule optimally.*

$$x_{it}(p_t) \in \arg\max_{x} U_i(\pi_{it}(x)|I_{it})$$

3. *market-clearing condition of the asset market is satisfied.*

$$\int_i x_{it} di = 0$$

Also, the equilibrium selection criteria now includes symmetric and stationary information choice of the investors as below.

**Definition 1.8 (Equilibrium selection)** *The equilibrium is such that*

1. *equilibrium signal precision is symmetric and stationary.*

$$\hat{\tau}_{it} = \hat{\tau} = \{\hat{\tau}_\theta, \hat{\tau}_u\}$$

2. *market price function is linear and stationary.*

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

3. *provided that multiple linear equilibria exist, choose the most informative linear equilibrium.*

### 1.6.2 Optimal information choice

First, let us examine how investors decide between processing fundamental and non-fundamental information. Obviously, both types of information reduce forecasting error of different stochastic components in future asset prices. Comparing to standard goods-consumption model, both information types are comparable to differentiated goods which increase expected utility of the investors. However, the investors have limited resources (i.e limited information-processing capacity to process financial data) and must choose signal precision of each information type in an optimal way. The following lemma describes optimal information choice of investors in the equilibrium.

**Lemma 1.1 (Optimal information choice)** *An equilibrium information choice* $(\hat{\tau}_\theta, \hat{\tau}_u)$ *of all interior solutions must satisfy*

$$\frac{1}{\gamma} = \left(\frac{a_1(\tau_u + \hat{\tau}_u) + \tau_u k_3(k_3 a_1 - k_2 a_2)}{a_2(\tau_\theta + \hat{\tau}_\theta) - \tau_u k_2(k_3 a_1 - k_2 a_2)}\right)^2$$

*in which* $a_1 = \delta\frac{k_1}{k_4} + 1$ *and* $a_2 = \frac{\delta}{k_4}$.

This is a standard optimality condition in which relative marginal cost of processing more fundamental information (left side) equals to its relative marginal benefit (right side). Both $a_1$ and $a_2$ are the coefficients of $\theta_t$ and $u_{t+1}$ in expected return term $E(\delta p_{t+1} + d_t)$, respectively.[6] The marginal cost term is directly derived from information technology constraint. The marginal benefit term depends on 1) what information type the investors consider more significant for their future price forecasts reflected by $a_1$ and $a_2$, 2) which factors are more volatile and thus more valuable to learn as measured by $\tau_u$ and $\tau_\theta$, and 3) which information type the investors can free-ride from the asset price as captured by $|k_3 a_1 - k_2 a_2|$ weighted by information quality of the asset price $\tau_u$, the inverse volatility of current-noise factor $u_t$. For instance, when $\frac{k_3}{k_2} < \frac{a_2}{a_1}$, a case of NF information shortage in the asset price, the investors are more likely to process more NF information on their own.

### 1.6.3 Information-processing capacity advancement

Now, let us analyze the impact of information-processing capacity development on equilibrium information choice and price informativeness. Generally, better information-processing capacity shall improve price informativeness. However, this depends on the type of financial information that investors choose to process. If the investors process non-fundamental information heavily when the information technology improves, price informativeness might in fact decrease. The following proposition establishes that, if the NF information is costly to process, a permanent decline in price informativeness is possible when the capacity is sufficiently good.

**Proposition 1.8 (NF information acquisition)** *If* $\gamma$ *and* $\delta$ *are sufficiently high, then there exists* $\Gamma^* \in \mathcal{R}^+$ *such that*

1. $\hat{\tau}_u > 0$ *if* $\Gamma > \Gamma^*$ *and* $\hat{\tau}_u = 0$ *otherwise,*
2. *there exists* $\bar{\Gamma} > \Gamma^*$ *such that for any* $\Gamma \geq \bar{\Gamma}$,

$$var(\theta_t|p_t) > \lim_{\Gamma \to \Gamma^*} var(\theta_t|p_t)$$

From the proposition, when the investors care a lot about the future and when NF information is relatively hard to process, they will process cheaper fundamental information when the processing capacity is low. This is because financial information is scarce, and market price which is the public source of financial information is still uninformative.

However, once the processing capacity is high, there is abundance of fundamental information both from private signals and asset price. Therefore, at some point all investors will

[6] Note that $E(\delta p_{t+1} + d_t) = (\delta\frac{k_1}{k_4} + 1)\theta_t + \frac{\delta}{k_4}u_{t+1} + \cdots = a_1\theta_t + a_2 u_{t+1} + \ldots$.

start processing more NF information to beat the market. This is shown in Figure 1.8 which plots simulation results of an increase in information-processing capacity when the cost of processing NF information is high. Even though the investors may also process more fundamental

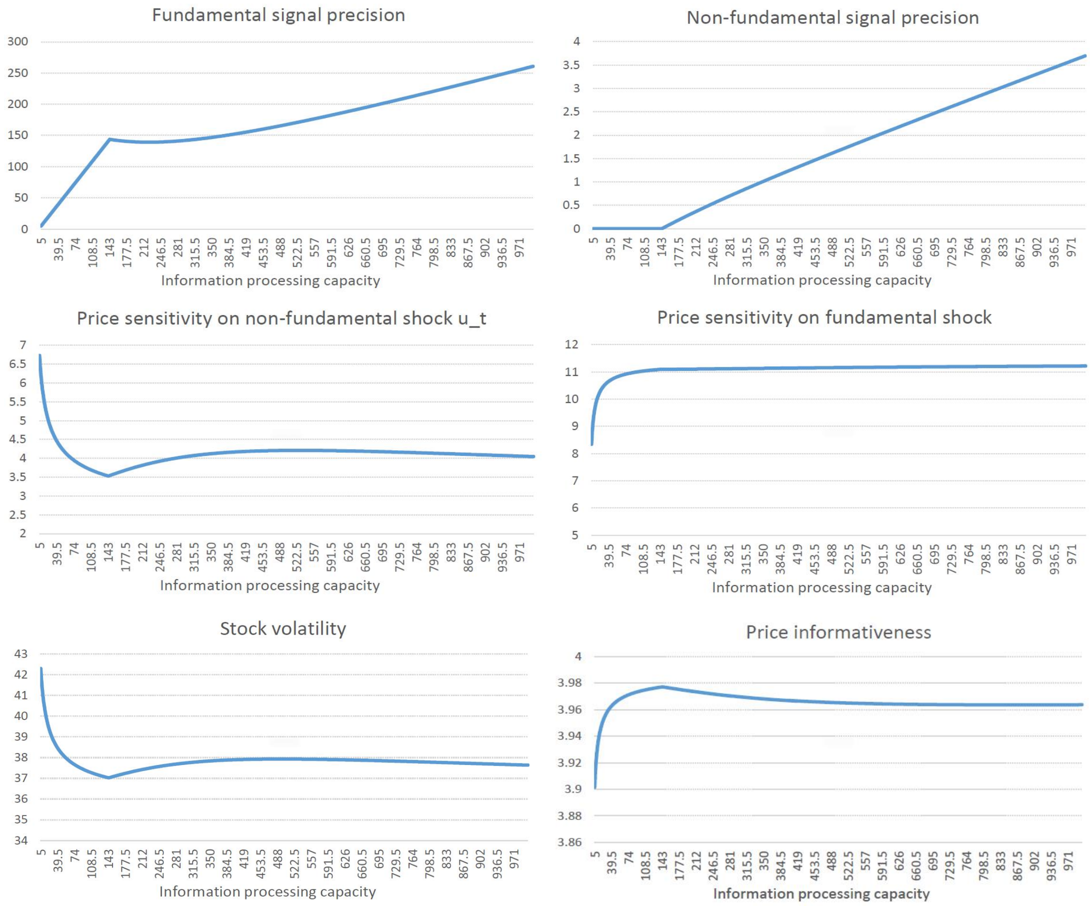


Figure 1.8: Simulation results of an increase in information processing capacity $\Gamma$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\gamma = 200$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

information at this stage, it will marginally improve their fundamental information quality as they already have good fundamental information. From the proposition, the efficiency effect of the capacity increase is ambiguous in the neighborhood of $\Gamma^*$, the capacity threshold that the investors start processing NF information. This is because the NF information is highly costly, and thus the negative impact of NF information acquisition might not dominate the positive impact of fundamental information acquisition at this point. However, once the market price aggregates more NF information when the capacity is higher, the investors can now learn NF information from the price, resulting in the deterioration of price informativeness.

Another important implication from the proposition is that the deterioration in price informativeness from NF information production is bound to happen. This finding is counterintuitive at first glance. Intuitively, if the relative cost of NF information is sufficiently high and the capacity is good, the investors will not process NF information as the asset price should be (almost) fully revealing. However, recall from our previous section that stock prices will never be fully revealing because of future fundamental information risk. Intuitively, investment risks of the investors are not fully eliminated as they cannot learn perfectly the fundamental information that their future counterparts learn. As a consequence, the growing processing capacity will eventually drive the investors to process the costly non-fundamental information once trading on fundamental information no longer beats the market in due course.

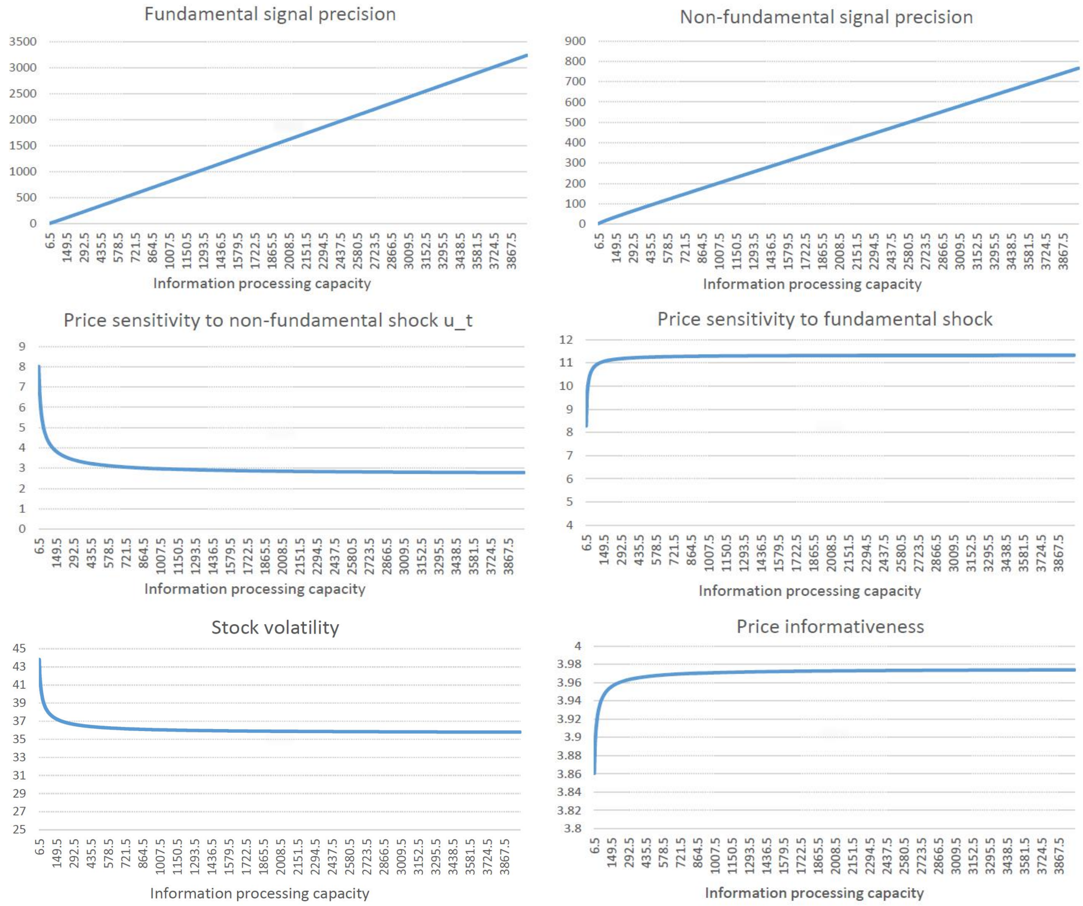


Figure 1.9: Simulation results of an increase in information processing capacity $\Gamma$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\gamma = 1$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

It is worth noting that more NF information production does not imply that price informativeness must decline. Remember that the processing capacity advancement is essential for investors to process more fundamental information. Therefore, if the investors are in shortage of the fundamental information, price informativeness can increase even though the investors also process more NF information. An example is when the cost of processing NF information is relatively low as shown in Figure 1.9. In this scenario, the investors will process more of both information types, and the negative effect of the NF information acquisition is dominated by the positive effect of the fundamental information production.

Another note is that, if the cost of processing NF information is moderate, the deterioration of price informativeness can be temporary. In this scenario, the investors might start processing the NF information when the quality of their fundamental information is still imperfect. Therefore, improvement in information processing capacity is still necessary for the investors to process fundamental information later on. As such, price informativeness can bounce up to a higher level when the processing capacity is more developed as shown in Figure 1.10.

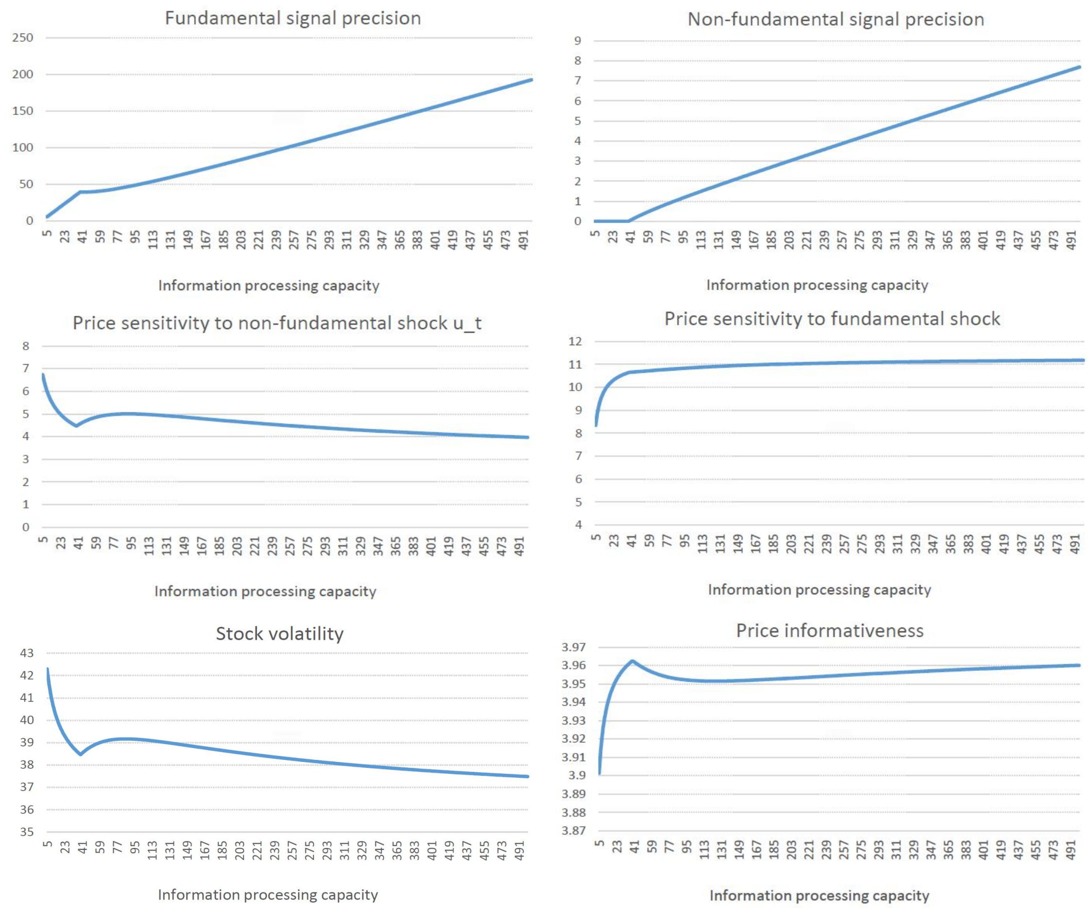


Figure 1.10: Simulation results of an increase in information processing capacity $\Gamma$ with parameter value of $\tau_u = 4$, $\rho = 0.085$, $\gamma = 40$, $\alpha = 0.94$, $\delta = 0.97$, $\tau_\theta = 4$.

## 1.7 Concluding remark

This paper studies the impact of learning information about future non-fundamental shocks on stock price formation and efficiency and provides an insight into how information technology growth can be detrimental to stock markets. Our main finding is that, contradicting to conventional wisdom on rational speculative trade, an increase in non-fundamental information quality can increase stock volatility, decrease market liquidity, and reduce price informativeness. Moreover, advancement in information-processing capacity can harm price informativeness if the investors exploit its benefit toward processing more NF information. Our results point out an important challenge faced by regulators on how to regulate stock markets and promote efficiency in the current era of fast-paced information technology advancement.

Our paper can be extended to study a few related issues in stock markets. First, one can relax the assumption about exogenous dividend payments and incorporate decision making of firms to the model. This will allow us to see how a change in stock price dynamics from learning NF information affects the firm's decision making and its feedback effect toward the stock traders. Second, one can consider endogenous acquisition of NF information from a different perspective. That is, instead of imposing the information technology constraint, one can consider costly information acquisition in which the investors face an increasing cost of acquiring NF information. In this way, we can analyze whether NF information acquisition in the presence of dynamic feedback effect will exhibit strategic complementarity or substitutability among the investors. Because of the dynamic feedback effect which endogenously increases stock return volatility, it is possible to obtain strategic complementarity result along with its consequence of over-acquisition of NF information.

## 1.8 Appendix

### 1.8.1 Equilibrium characterization

In this section, we will solve for equilibrium price function assuming that the investors obtain private signal of $s_{it} = \{s_{\theta,it}, s_{u,it}, s_{uc,it}\}$. Recall that $s_{\theta,it} = \theta_t + \sqrt{\frac{1}{\hat{\tau}_\theta}}\epsilon_{\theta,it}$ in which $\epsilon_{\theta,it}$ is an i.i.d. random variable drawn from normal distribution $N(0,1)$ and $\hat{\tau}_\theta$ is precision of fundamental signal, $s_{u,it} = u_{t+1} + \sqrt{\frac{1}{\hat{\tau}_u}}\epsilon_{u,it}$ in which $\epsilon_{u,it}$ is an i.i.d. random variable drawn from normal distribution $N(0,1)$ and $\hat{\tau}_u$ is precision of non-fundamental signal, and $s_{uc,it} = u_t + \sqrt{\frac{1}{\hat{\tau}_{uc}}}\epsilon_{uc,it}$ in which $\epsilon_{uc,it} \sim N(0,1)$ is an i.i.d. idiosyncratic disturbance. Suppose that equilibrium price function at period $t$ is

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

in which the set of coefficients $\{k_1, k_2, k_3, k_4, k_5\}$ will be determined using market-clearing condition. The standard utility-maximizing trading strategy of each investor from optimization problem is

$$x_{it} - u_{it} = \frac{E(\delta p_{t+1} + d_t|I_{it}) - p_t}{Var(\delta p_{t+1} + d_t|I_{it})}.$$

The next step is to find the posterior belief $\delta p_{t+1} + d_t|I_{it}$ of investor $i$ given signal precision $\{\hat{\tau}_u, \hat{\tau}_\theta\}$. Denote a random variable $z_t = a_1\theta_t + a_2 u_{t+1}$. Our objective is to find the posterior $z_t|p_t, s_{it}, u_{it}$. We first update stochastic components in $z_t$ using private signal $s_{it}$ which yields

$$\theta_t|s_{\theta,it} = N\left(\frac{\hat{\tau}_{\theta,it}s_{\theta,it}}{\tau_\theta + \hat{\tau}_\theta}, (\tau_\theta + \hat{\tau}_\theta)^{-1}\right)$$

$$u_{t+1}|s_{u,it} = N\left(\frac{\hat{\tau}_{u,it}s_{u,it}}{\tau_u + \hat{\tau}_u}, (\tau_u + \hat{\tau}_u)^{-1}\right)$$

which , by independence, implies that

$$z_t|s_{\theta,it}, s_{u,it} = N\left(a_1\frac{\hat{\tau}_{\theta,it}s_{\theta,it}}{\tau_\theta + \hat{\tau}_\theta} + a_2\frac{\hat{\tau}_{u,it}s_{u,it}}{\tau_u + \hat{\tau}_u}, a_1^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2(\tau_u + \hat{\tau}_u)^{-1}\right)$$

Also, the posterior belief about noise factor $u_t$ assuming that $\tau_{ui} = 0$ is

$$u_t|s_{uc,it}, u_{it} = N\left(\frac{\hat{\tau}_{uc}}{\tau_u + \hat{\tau}_{uc}}s_{uc,it}, (\tau_u + \hat{\tau}_{uc})^{-1}\right)$$

The last step is to update the posterior belief using $p_t$. Denote $\hat{p}_t \equiv k_4 p_t - k_1 d_{t-1} - k_5\bar{\theta} = k_2\theta_t + k_3 u_{t+1} + u_t$ which is informationally equivalent to $p_t$ about $z_t$. That is, $z_t|p_t = z_t|\hat{p}_t$. To find $z_t|\hat{p}_t$, we find conditional probability distribution of $\hat{p}_t|z_t$ and apply Bayes' rule to obtain $z_t|\hat{p}_t$. Rewrite $\hat{p}_t$ as $\hat{p}_t = \beta z_t + r_t$ which contains relevant information $z_t$ and a stochastic residual term $r_t$. A natural solution is to find optimal $\beta$ which minimizes $var(r_t)$, find $\hat{p}_t|z_t$, and apply Bayes' rule. However, $z_t$ and $r_t$ might be dependent since both factors can share the same random variables $\theta_t$ and $u_{t+1}$. To proceed, we assume that the investors ignore the correlation between $z_t$ and $r_t$. To find the optimal $\beta \in \arg\min_\beta\ E(\hat{p}_t - \beta z_t|s_{it}, u_{it})^2$, we first substitute $\hat{p}_t$ and $z_t$ in the objective function which gives

$$\beta \in \arg\min_\beta\ (k_2 - \beta a_1)^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + (k_3 - \beta a_2)^2(\tau_u + \hat{\tau}_u)^{-1} + (\tau_u + \hat{\tau}_{uc})^{-1}.$$

Taking first-order condition yields

$$-(k_2-\beta a_1)a_1(\tau_\theta+\hat{\tau}_\theta)^{-1}+-(k_3-\beta a_2)a_2(\tau_u+\hat{\tau}_u)^{-1}=0,$$

which can be easily proved that second-order derivative is always positive and thus second-order condition is satisfied. Rearranging the condition gives

$$\beta=\frac{k_2a_1(\tau_u+\hat{\tau}_u)+k_3a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}$$

which implies that

$$E(r_t|s_{it},u_{it},z_t)=(k_2-\beta a_1)\frac{\hat{\tau}_\theta s_\theta}{\hat{\tau}_\theta+\tau_\theta}+(k_3-\beta a_2)\frac{\hat{\tau}_u s_u}{\hat{\tau}_u+\tau_u}+\frac{\hat{\tau}_{uc}s_{uc}}{\hat{\tau}_{uc}+\tau_u}$$

$$var(r_t|s_{it},u_{it},z_t)=\frac{(k_2a_2-k_3a_1)^2}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}+(\tau_u+\hat{\tau}_{uc})^{-1}$$

where the first term noise comes from the mismatch in the composition of $\theta_t$ and $u_{t+1}$ and the second term is the noise from $u_t$ in $\hat{p}_t$. Define $\hat{\hat{p}}_t=\frac{1}{\beta}(\hat{p}_t-E(r_t|s_{it},u_{it},z_t))$. Substituting $E(r_t|s_{it},u_{it},z_t)$ yields

$$\hat{\hat{p}}_t=\frac{\hat{p}_t-\left((k_2-\beta a_1)\frac{\hat{\tau}_\theta s_\theta}{\hat{\tau}_\theta+\tau_\theta}+(k_3-\beta a_2)\frac{\hat{\tau}_u s_u}{\hat{\tau}_u+\tau_u}+\frac{\hat{\tau}_{uc}s_{uc}}{\hat{\tau}_{uc}+\tau_u}\right)}{\beta}$$

which, given that $\frac{1}{\beta}\hat{p}_t=z_t+\frac{1}{\beta}r_t$, implies that

$$\hat{\hat{p}}_t|z_t=N\left(z_t,\frac{var(r_t|s_{it},u_{it},z_t)}{\beta^2}\right)$$

Denote $\tau_P=\beta^2 var(r_t|s_{it},u_{it},z_t)^{-1}$ which is the signal precision of $\hat{\hat{p}}_t$. By standard Bayes rule, we obtain

$$z_t|s_{it},u_{it},p_t=N\left(\frac{\tau_Z\left(a_1\frac{\hat{\tau}_\theta s_\theta}{\hat{\tau}_\theta+\tau_\theta}+a_2\frac{\hat{\tau}_u s_u}{\hat{\tau}_u+\tau_u}\right)+\tau_P\hat{\hat{p}}_t}{\tau_Z+\tau_P},(\tau_Z+\tau_P)^{-1}\right)$$

where $\tau_Z=(a_1^2(\tau_\theta+\hat{\tau}_\theta)^{-1}+a_2^2(\tau_u+\hat{\tau}_u)^{-1})^{-1}$ which gives the following lemma.

**Lemma 1.2 (Posterior updating)** *Let $z_t=a_1\theta_t+a_2u_{t+1}$, then*

$$z_t|s_{it},u_{it},p_t=N\left(\frac{\tau_Z\left(a_1\frac{\hat{\tau}_\theta s_{\theta,it}}{\hat{\tau}_\theta+\tau_\theta}+a_2\frac{\hat{\tau}_u s_{u,it}}{\hat{\tau}_u+\tau_u}\right)+\tau_P\hat{\hat{p}}_t}{\tau_Z+\tau_P},(\tau_Z+\tau_P)^{-1}\right)$$

*where*

$$\beta=\frac{k_2a_1(\tau_u+\hat{\tau}_u)+k_3a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}$$

$$\hat{\hat{p}}_t=\frac{1}{\beta}\left(k_4p_t-k_1d_{t-1}-k_5\bar{\theta}-(k_2-\beta a_1)\frac{\hat{\tau}_\theta s_\theta}{\hat{\tau}_\theta+\tau_\theta}-(k_3-\beta a_2)\frac{\hat{\tau}_u s_u}{\hat{\tau}_u+\tau_u}-\frac{\hat{\tau}_{uc}s_{uc,it}}{\hat{\tau}_{uc}+\tau_u}\right)$$

$$\tau_Z=a_1^2(\tau_\theta+\hat{\tau}_\theta)^{-1}+a_2^2(\tau_u+\hat{\tau}_u)^{-1}$$

$$\tau_P=\beta^2\left(\frac{(k_2a_2-k_3a_1)^2}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}+(\tau_u+\hat{\tau}_{uc})^{-1}\right)^{-1}$$

Next, we use the lemma to characterize equilibrium price using market-clearing condition. We first characterize general form which dictates both stationary and non-stationary equilibrium. Then, we impose stationary condition to obtain our stationary equilibrium. From market-clearing condition $\int_i x_{it} di = 0$, we obtain

$$\int_i \frac{E(\delta p_{t+1} + d_t | I_t) - p_t}{\rho Var(\delta p_{t+1} + d_t | I_t)} di + u_t = 0$$

because $\int u_{it} di = u_t$. Rearranging the condition gives

$$\int_i E(\delta p_{t+1} + d_t | I_t) di + \rho Var(\delta p_{t+1} + d_t | I_t) u_t = p_t.$$

Recall that

$$p_t = \frac{1}{k_{4t}}(k_{1t} d_{t-1} + k_{2t}\theta_t + k_{3t} u_{t+1} + u_t + k_{5t}\bar{\theta}).$$

Notice the time subscript in all coefficients in the price equation. However, to have clear distinction between coefficients and random variables, we will drop subscript t and use $'$ to indicate coefficients in period $t+1$ instead as shown below.

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

and

$$p_{t+1} = \frac{1}{k'_4}(k'_1 d_t + k'_2\theta_{t+1} + k'_3 u_{t+2} + u_{t+1} + k'_5\bar{\theta})$$

First, we know that

$$\begin{aligned}
\delta p_{t+1} + d_t &= (\delta\frac{k'_1}{k'_4} + 1)d_t + \delta\frac{k'_2}{k'_4}\theta_{t+1} + \delta\frac{k'_3}{k'_4}u_{t+2} + \frac{\delta}{k'_4}u_{t+1} + \delta\frac{k'_5}{k'_4}\bar{\theta} \\
&= (\delta\frac{k'_1}{k'_4} + 1)(\alpha d_{t-1} + \theta_t) + \delta\frac{k'_2}{k'_4}\theta_{t+1} + \delta\frac{k'_3}{k'_4}u_{t+2} + \frac{\delta}{k'_4}u_{t+1} + (\delta\frac{k'_5}{k'_4} + \delta\frac{k'_1}{k'_4} + 1)\bar{\theta}.
\end{aligned}$$

Notice that

$$E(\delta p_{t+1} + d_t | I_t) = \alpha(\delta\frac{k'_1}{k'_4} + 1)d_{t-1} + (\delta\frac{k'_5}{k'_4} + \delta\frac{k'_1}{k'_4} + 1)\bar{\theta} + E((\delta\frac{k'_1}{k'_4} + 1)\theta_t + \frac{\delta}{k'_4}u_{t+1} | I_t)$$

$$Var(\delta p_{t+1} + d_t | I_t) = Var(\delta\frac{k'_2}{k'_4}\theta_{t+1} + \delta\frac{k'_3}{k'_4}u_{t+2}) + Var((\delta\frac{k'_1}{k'_4} + 1)\theta_t + \frac{\delta}{k'_4}u_{t+1} | I_t).$$

Let $a_1 = \delta\frac{k'_1}{k'_4} + 1$, $a_2 = \frac{\delta}{k'_4}$ and $z_t = a_1\theta_t + a_2 u_{t+1}$ . Then, we have

$$E(\delta p_{t+1} + d_t | I_t) = \alpha(\delta\frac{k'_1}{k'_4} + 1)d_{t-1} + (\delta\frac{k'_5}{k'_4} + \delta\frac{k'_1}{k'_4} + 1)\bar{\theta} + E(z_t | I_t)$$

$$Var(\delta p_{t+1} + d_t | I_t) = Var(\delta\frac{k'_2}{k'_4}\theta_{t+1} + \delta\frac{k'_3}{k'_4}u_{t+2}) + Var(z_t | I_t).$$

Using the market-clearing condition, we obtain

$$\begin{aligned}
p_t = \alpha(\delta\frac{k'_1}{k'_4} + 1)d_{t-1} + (\delta\frac{k'_5}{k'_4} + \delta\frac{k'_1}{k'_4} + 1)\bar{\theta} + \int_i E(z_t | I_{it}) di \\
+ \left(Var(\delta\frac{k'_2}{k'_4}\theta_{t+1} + \delta\frac{k'_3}{k'_4}u_{t+2}) + Var(z_t | I_{it})\right) u_t \quad (*)
\end{aligned}$$

Based on Lemma 1.2, we know that

$$\int_i E(z_t|I_{it})di = \int_i \frac{\tau_Z}{\tau_Z+\tau_P}\left(a_1\frac{\hat{\tau}_\theta s_\theta}{\hat{\tau}_\theta+\tau_\theta}+a_2\frac{\hat{\tau}_u s_u}{\hat{\tau}_u+\tau_u}\right)+\frac{\tau_P}{\tau_Z+\tau_P}\hat{\hat{p}}_t\, di$$
$$= \frac{\tau_Z}{\tau_Z+\tau_P}\left(a_1\frac{\hat{\tau}_\theta \theta_t}{\hat{\tau}_\theta+\tau_\theta}+a_2\frac{\hat{\tau}_u u_{t+1}}{\hat{\tau}_u+\tau_u}\right)+\frac{\tau_P}{\tau_Z+\tau_P}\int_i \hat{\hat{p}}_{it}\, di \quad (**)$$

and that

$$\int_i \hat{\hat{p}}_{it} di = \frac{1}{\beta}\left(k_4 p_t - k_1 d_{t-1} - k_5\bar{\theta} - (k_2-\beta a_1)\frac{\hat{\tau}_\theta \theta_t}{\hat{\tau}_\theta+\tau_\theta} - (k_3-\beta a_2)\frac{\hat{\tau}_u u_{t+1}}{\hat{\tau}_u+\tau_u} - \frac{\hat{\tau}_{uc} u_t}{\hat{\tau}_{uc}+\tau_u}\right)$$

Substituting $\int_i \hat{\hat{p}}_{it} di$ into (**) and simplifying the expression yields

$$\int_i E(z_t|I_{it})di = \theta_t\left(a_1 - \frac{\tau_P}{\tau_Z+\tau_P}(\frac{k_2}{\beta})\right)\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta+\tau_\theta}\right)+$$
$$u_{t+1}\left(a_2 - \frac{\tau_P}{\tau_Z+\tau_P}(\frac{k_3}{\beta})\right)\left(\frac{\hat{\tau}_u}{\hat{\tau}_u+\tau_u}\right) - u_t\left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(\frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc}+\tau_u}\right)\left(\frac{1}{\beta}\right)+$$
$$\left(\frac{\tau_P}{\tau_Z+\tau_P}\right)(\frac{k_4 p_t - k_1 d_{t-1} - k_5\bar{\theta}}{\beta}).$$

Last step is to plug in $\int_i E(z_t|I_{it})di$ and $var(z_t|I_{it})$ into the market-clearing condition (*) and compare the coefficients with the price equation we conjectured from the beginning. Let $var(\delta\frac{k_2'}{k_4'}\theta_{t+1}+\delta\frac{k_3'}{k_4'}u_{t+2}) = V_t$ and note that $var(z_t|I_{it}) = (\tau_Z+\tau_P)^{-1}$. Once comparing the coefficients, we obtain that equilibrium price function in any linear and symmetric equilibrium given signal precision $(\hat{\tau}_u, \hat{\tau}_\theta.\hat{\tau}_{uc})$ must be such that

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

in which the set of coefficients must satisfy the following system of equations.

$$\frac{k_1}{k_4} = \alpha a_1$$

$$k_2 = \frac{a_1\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta+\tau_\theta}\right)}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right)+\left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta+\tau_\theta}-\frac{\tau_{ui}}{\tau_{ui}+\tau_u}\right)\frac{1}{\beta}}$$

$$k_3 = \frac{a_2\left(\frac{\hat{\tau}_u}{\hat{\tau}_u+\tau_u}\right)}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right)+\left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(\frac{\hat{\tau}_u}{\hat{\tau}_u+\tau_u}-\frac{\tau_{ui}}{\tau_{ui}+\tau_u}\right)\frac{1}{\beta}}$$

$$k_4 = \frac{1}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right)+\left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(1-\frac{\tau_{ui}}{\tau_{ui}+\tau_u}\right)\frac{1}{\beta}}$$

$$\frac{k_5}{k_4} = \delta\frac{k_5'}{k_4'} + a_1$$

where

$$a_1 = \delta\frac{k_1'}{k_4'}+1, \quad a_2 = \frac{\delta}{k_4'}, \quad V_t = \delta^2\left(\left(\frac{k_2'}{k_4'}\right)^2\sigma_\theta^2 + \left(\frac{k_3'}{k_4'}\right)^2\sigma_u^2\right)$$

$$\beta = \frac{k_2 a_1(\tau_u+\hat{\tau}_u)+k_3 a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2(\tau_u + \hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2 \left( \frac{(k_2 a_2 - k_3 a_1)^2}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)} + (\tau_u + \hat{\tau}_{uc})^{-1} \right)^{-1}$$

Lastly, we impose stationarity condition and finalize our equilibrium characterization. Consider $\frac{k_1}{k_4} = \alpha a_1$, $\frac{k_5}{k_4} = \delta \frac{k_5'}{k_4'} + a_1$ and, $a_1 = \delta \frac{k_1'}{k_4'} + 1$. Imposing stationary condition $\{k_1 = k_1', k_4 = k_4', k_5 = k_5'\}$, we obtain that.

$$\frac{k_1}{k_4} = \frac{\alpha}{1 - \alpha\delta} \quad \text{and} \quad \frac{k_5}{k_4} = \frac{1}{(1-\delta)(1-\alpha\delta)}$$

Combining all the results proves the following proposition.

**Proposition 1.9 (Equilibrium characterization)** *Any stationary equilibrium price given signal precision $\{\hat{\tau}_\theta, \hat{\tau}_u, \hat{\tau}_{uc}\}$ is such that*

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

*in which*

$$\frac{k_1}{k_4} = \frac{\alpha}{1-\alpha\delta}$$

$$k_2 = \frac{a_1\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta}\right)}{\rho\left((\tau_Z + \tau_P)^{-1} + V_t\right) + \left(\frac{\tau_P}{\tau_Z + \tau_P}\right)\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u}\right)\frac{1}{\beta}}$$

$$k_3 = \frac{a_2\left(\frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u}\right)}{\rho\left((\tau_Z + \tau_P)^{-1} + V_t\right) + \left(\frac{\tau_P}{\tau_Z + \tau_P}\right)\left(\frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u}\right)\frac{1}{\beta}}$$

$$k_4 = \frac{1}{\rho\left((\tau_Z + \tau_P)^{-1} + V_t\right) + \left(\frac{\tau_P}{\tau_Z + \tau_P}\right)\left(1 - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u}\right)\frac{1}{\beta}}$$

$$\frac{k_5}{k_4} = \frac{1}{(1-\delta)(1-\alpha\delta)}$$

*where*

$$a_1 = \frac{1}{1-\alpha\delta}, \quad a_2 = \frac{\delta}{k_4}, \quad V_t = \delta^2\left(\left(\frac{k_2}{k_4}\right)^2\sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2\sigma_u^2\right)$$

$$\beta = \frac{k_2 a_1(\tau_u + \hat{\tau}_u) + k_3 a_2(\tau_\theta + \hat{\tau}_\theta)}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2(\tau_u + \hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2 \left( \frac{(k_2 a_2 - k_3 a_1)^2}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)} + (\tau_u + \hat{\tau}_{uc})^{-1} \right)^{-1}$$

Denote $\hat{k}_4 = \frac{1}{k_4}$. From the proposition, we can easily substitute $(\hat{\tau}_\theta, \hat{\tau}_u, \hat{\tau}_{uc})$ further and obtain the following corollary.

**Corollary 1.1** *Provided that $\hat{\tau}_{uc} = 0$ and $\hat{\tau}_\theta \to \infty$, then*

*1. if $\hat{\tau}_u = 0$, then*

$$\frac{k_1}{k_4} = \frac{\alpha}{1-\alpha\delta}, \frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, k_3 = 0, \hat{k}_4 = \rho\delta^2 \left( \left(\frac{1}{1-\alpha\delta}\right)^2 \sigma_\theta^2 + \hat{k}_4^2 \sigma_u^2 \right)$$

*2. if $\hat{\tau}_u \to \infty$, then*

$$\frac{k_1}{k_4} = \frac{\alpha}{1-\alpha\delta}, \frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, k_3 = \delta, \hat{k}_4 = \rho\delta^2 \left( \left(\frac{1}{1-\alpha\delta}\right)^2 \sigma_\theta^2 + \delta^2 \hat{k}_4^2 \sigma_u^2 \right).$$

### 1.8.2 Omitted proofs

**Proof of Remark 1.1:** To prove this remark, we first obtain equilibrium characterization of the baseline model by substituting $\hat{\tau}_u = 0$ and $\hat{\tau}_{uc} = 0$ into Proposition 1.9, which yields

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + u_t + k_5\bar{\theta})$$

where

$$\frac{k_1}{k_4} = \frac{\alpha}{1-\alpha\delta}, \qquad \frac{k_2}{k_4} = \left(\frac{\hat{\tau}_\theta + k_2^2\tau_u}{\tau_\theta + \hat{\tau}_\theta + k_2^2\tau_u}\right)\left(\frac{1}{1-\alpha\delta}\right)$$

$$\frac{k_5}{k_4} = \frac{1}{(1-\delta)(1-\alpha\delta)}, \qquad k_4 = \left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + k_2^2\tau_u}\right)\frac{1}{K}$$

$$K = \rho\left[\frac{1}{(1-\delta\alpha)^2(k_2^2\tau_u + \hat{\tau}_\theta + \tau_\theta)} + \delta^2\left(\left(\frac{k_2}{k_4}\right)^2\frac{1}{\tau_\theta} + \left(\frac{1}{k_4}\right)^2\frac{1}{\tau_u}\right)\right].$$

Next, we simplify the system of equations by substituting $\frac{k_2}{k_4}$ and $K$ into the equation $k_4 = \left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + k_2^2\tau_u}\right)\frac{1}{K}$ which yields

$$\underbrace{\frac{k_2}{\hat{\tau}_\theta}\left(1 + \frac{\delta^2(\hat{\tau}_\theta + k_2^2\tau_u)^2}{\tau_\theta + k_2^2\tau_u + \hat{\tau}_\theta}\left[\frac{1}{k_2^2\tau_u} + \frac{1}{\tau_\theta}\right]\right)}_{\text{LHS}} = \underbrace{\frac{1}{\rho}\left(\frac{1}{1-\alpha\delta}\right)}_{\text{RHS}} \tag{1.8.2.1}$$

which is now a function of only $k_2$. Note that

$$k_4 = (1-\alpha\delta)\left(\frac{\tau_\theta + \hat{\tau}_\theta + k_2^2\tau_u}{\hat{\tau}_\theta + k_2^2\tau_u}\right)k_2$$

is a function of $k_2$ which will always exist if $k_2$ exists. Thus, we obtain the following lemma.

**Lemma 1.3** *A baseline equilibrium exists iff there exists $k_2 \in \mathcal{R}^+$ such that*

$$\frac{k_2}{\hat{\tau}_\theta}\left(1 + \frac{\delta^2(\hat{\tau}_\theta + k_2^2\tau_u)^2}{\tau_\theta + k_2^2\tau_u + \hat{\tau}_\theta}\left[\frac{1}{k_2^2\tau_u} + \frac{1}{\tau_\theta}\right]\right) = \frac{1}{\rho}\left(\frac{1}{1-\alpha\delta}\right).$$

Next, we prove existence condition by finding a condition when $k_2$ exists. Considering the left term (LHS) from Equation 1.8.2.1, we can find that $\frac{\partial LHS}{\partial k_2} \geq 0$ if and only if

$$3\delta^2\tau_u^4 k_2^8 + 6\left(\delta^2\tau_\theta + \frac{7}{6}\delta^2\hat{\tau}_\theta + \frac{1}{6}\tau_\theta\right)\tau_u^3 k_2^6 + 3k_2^4\tau_u^2\left((\delta^2 + \frac{2}{3})\tau_\theta^2 + \frac{\hat{\tau}_\theta}{3}(7\delta^2+2)\tau_\theta + 5\delta^2\hat{\tau}_\theta^2\right)$$
$$+ \tau_u k_2^2\left(\tau_\theta^3 + 2\tau_\theta^2\hat{\tau}_\theta(\delta^2+1) + \tau_\theta\hat{\tau}_\theta^2 + \delta^2\hat{\tau}_\theta^3\right) \geq \delta^2\tau_\theta\hat{\tau}_\theta^2\left(\tau_\theta + \hat{\tau}_\theta\right) \quad (1.8.2.2)$$

which implies that there exists $k_2^*$ such that $\frac{\partial LHS}{\partial k_2} \geq 0$ if $k_2 \geq k_2^*$ and $\frac{\partial LHS}{\partial k_2} \leq 0$ if $k_2 \leq k_2^*$ as plotted in Figure 1.11. This implies that $k_2$ exists if and only if the right term (RHS) in Equation 1.8.2.1 is sufficiently high to have an intersection. That is, when $\rho$ is sufficiently low. This proves the first result in the remark. Note that we choose the most informative equilibrium, the equilibrium with the higher $k_2$ which is on the right side as marked in the figure.

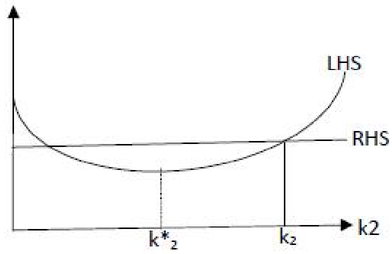


Figure 1.11: A diagram illustrating how $k_2$ of the most informative equilibrium is determined from an intersection of LHS and RHS.

To perform comparative statics on price informativeness, recall that price informativeness depends on $var(\theta_t|p_t) = (\tau_\theta + k_2^2\tau_u)^{-1}$. Therefore, unless we analyze the effect of a change in $\tau_\theta$, we can just analyze the effect on $k_2$ using the equation 1.8.2.1. A higher $k_2$ would imply that price informativeness is higher.

First, an increase in $\rho$ decreases $k_2$ since RHS shifts downward. To see how an increase in $\hat{\tau}_\theta$ affects $k_2$, we first obtain that $\frac{\partial LHS}{\partial \hat{\tau}_\theta} \leq 0$ iff

$$\delta^2\tau_u^4 k_2^8 + 2\left(\delta^2(\tau_\theta + \hat{\tau}_\theta) + \frac{\tau_\theta}{2}\right)\tau_u^3 k_2^6 + k_2^4(\tau_\theta + \hat{\tau}_\theta)(\delta^2(\tau_\theta + \hat{\tau}_\theta) + 2\tau_\theta)\tau_u^2$$
$$+ k_2^2\tau_u\tau_\theta(\tau_\theta + \hat{\tau}_\theta)^2 \geq \delta^2\tau_\theta\hat{\tau}_\theta^2 \quad (1.8.2.3)$$

This finding implies that there exists $\hat{k}_2$ such that the LHS curve will shift down for all values of $k_2 \geq \hat{k}_2$ and shift up for $k_2 \leq \hat{k}_2$ when $\hat{\tau}_\theta$ increases. Given $k_2^*$ in equation 1.8.2.2 and $\hat{k}_2$ in equation 1.8.2.3, we know that a sufficient condition for the most informative equilibrium to exhibit $\frac{\partial k_2}{\partial \hat{\tau}_\theta} \geq 0$ is when the equilibrium $k_2 \geq max\{k_2^*, \hat{k}_2\}$. This condition guarantees that the most informative equilibrium $(k_2 \geq k_2^*)$ is increasing in $\hat{\tau}_\theta$. Therefore, a sufficient condition for the most informative $k_2$ to be increasing in $\hat{\tau}_\theta$ is that $RHS$ is sufficiently high (such that $k_2 \geq max\{k_2^*, \hat{k}_2\}$). This can be achieved when $\rho$ is sufficiently low.

Lastly, to prove that the market price is never fully revealing, it is sufficient to prove that $\lim_{\hat{\tau}_\theta \to \infty} k_4$ is finite. Recall that

$$\frac{k_2}{k_4} = \left(\frac{\hat{\tau}_\theta + k_2^2\tau_u}{\tau_\theta + \hat{\tau}_\theta + k_2^2\tau_u}\right)\left(\frac{1}{1-\alpha\delta}\right) \quad \text{and} \quad k_4 = \left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + k_2^2\tau_u}\right)\frac{1}{K}$$

$$\text{where} \quad K = \rho \left[ \frac{1}{(1-\delta\alpha)^2(k_2^2\tau_u + \hat{\tau}_\theta + \tau_\theta)} + \delta^2 \left( \left(\frac{k_2}{k_4}\right)^2 \frac{1}{\tau_\theta} + \left(\frac{1}{k_4}\right)^2 \frac{1}{\tau_u} \right) \right].$$

Taking the limit, we obtain

$$\lim_{\hat{\tau}_\theta \to \infty} \frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}$$

$$\lim_{\hat{\tau}_\theta \to \infty} k_4 = \frac{1}{\lim_{\hat{\tau}_\theta \to \infty} K}$$

$$\lim_{\hat{\tau}_\theta \to \infty} K = \rho \left[ \delta^2 \left( \left(\frac{1}{1-\alpha\delta}\right)^2 \frac{1}{\tau_\theta} + \left(\frac{1}{\lim_{\hat{\tau}_\theta \to \infty} k_4}\right)^2 \frac{1}{\tau_u} \right) \right].$$

Substitute $\lim_{\hat{\tau}_\theta \to \infty} K$ into $\lim_{\hat{\tau}_\theta \to \infty} k_4$ and solve for $\lim_{\hat{\tau}_\theta \to \infty} k_4$ in the most informative equilibrium yields

$$\lim_{\hat{\tau}_\theta \to \infty} k_4 = \frac{1 + \sqrt{1 - \frac{4\rho^2\delta^4\sigma_\theta^2\sigma_u^2}{(1-\alpha\delta)^2}}}{\frac{2\rho\delta^2\sigma_\theta^2}{(1-\alpha\delta)^2}}$$

which is finite. This proves the third result in the remark.

**Proof of Remark 1.2:** Let signal profile be $s_{it} = \{\theta_k\}_{k=t}^{t+n}$. Suppose that equilibrium price function is

$$p_t = k_1 d_{t-1} + k_{2,1}\theta_t + k_{2,2}\theta_{t+1} + \cdots + k_{2,n+1}\theta_{t+n} + k_4 u_t \tag{1.8.2.4}$$

Recall that the trading strategy of each investor $i$ is

$$x_{it} - u_{it} = \frac{E(\delta p_{t+1} + d_t | I_{it}) - p_t}{Var(\delta p_{t+1} + d_t | I_{it})}$$

which, once combining with market-clearing condition, implies that

$$p_t = \int_i E(\delta p_{t+1} + d_t | I_{it}) di + \rho Var(\delta p_{t+1} + d_t | I_{it}) u_t. \tag{1.8.2.5}$$

We know that

$$E(\delta p_{t+1} + d_t | I_{it}) = (\delta k_1 + 1) d_t + \delta(k_{2,1}\theta_{t+1} + k_{2,2}\theta_{t+2} + \cdots + k_{2,n}\theta_{t+n})$$

$$var(\delta p_{t+1} + d_t | I_{it}) = \delta^2 var(k_{2,n+1}\theta_{t+n+1} + k_4 u_{t+1} | I_{it}) = \delta^2(k_{2,n+1}^2\sigma_\theta^2 + k_4^2\sigma_u^2).$$

Substituting into Equation 1.8.2.5 yields

$$p_t = (\delta k_1 + 1) d_t + \delta(k_{2,1}\theta_{t+1} + k_{2,2}\theta_{t+2} + \cdots + k_{2,n}\theta_{t+n}) + \rho\delta^2(k_{2,n+1}^2\sigma_\theta^2 + k_4^2\sigma_u^2) u_t$$

Comparing the coefficients with Equation 1.8.2.4 proves the following lemma

**Lemma 1.4** *Given signal profile of $s_{it} = \{\theta_k\}_{k=t}^{t+n}$, the equilibrium price is*

$$p_t = \frac{\alpha}{1-\alpha\delta} d_{t-1} + \frac{\theta_t}{1-\alpha\delta} + \delta \frac{\theta_{t+1}}{1-\alpha\delta} + \delta^2 \frac{\theta_{t+2}}{1-\alpha\delta} + \cdots + \delta^n \frac{\theta_{t+n}}{1-\alpha\delta} + k_4 u_t$$

*where* $k_4 = \rho\delta^2 \left[ \frac{\delta^{2n}}{(1-\alpha\delta)^2}\sigma_\theta^2 + k_4^2\sigma_u^2 \right]$

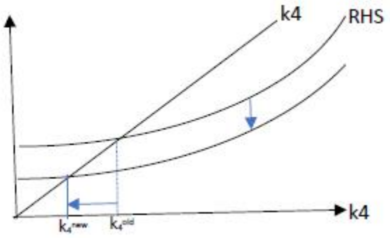


Figure 1.12: A diagram illustrating how an increase in $n$ reduces $k_4$ in the equilibrium.

It is immediate to see that the price is not fully revealing as $k_4$ cannot be zero for any finite value of $n$. This proves the first result in the remark.

Next, we will analyze the effect of an increase in $n$ on $k_4$. From $k_4 = \underbrace{\rho\delta^2 \left[\frac{\delta^{2n}}{(1-\alpha\delta)^2}\sigma_\theta^2 + k_4^2\sigma_u^2\right]}_{\text{RHS}}$, one can plot a diagram as shown in Figure 1.12.

As shown in Figure 1.12, an increase in $n$ causes RHS to shift down, resulting in lower equilibrium $k_4$. Also, when discount factor is one, we obtain $k_4 = \rho\left[\frac{1}{(1-\alpha)^2}\sigma_\theta^2 + k_4^2\sigma_u^2\right]$ which is no longer depending on $n$. This proves the second and third finding in the remark.

**Proof of Remark 1.3:** First, we characterize the equilibrium in the static setting when $\hat{\tau}_\theta \to \infty$. Suppose that equilibrium price in period 1 is

$$p_1 = k_1\theta + k_2u_2 + k_3u_1.$$

Combine the standard trading strategy in period 1

$$x_i - u_i = \frac{E(\theta + u_2|I_i) - p_1}{var(\theta + u_2|I_i)},$$

with the market-clearing condition yields

$$p_1 = E(\theta + u_2|I_i) + \rho var(\theta + u_2|I_i)u_1.$$

Define $\hat{p} \equiv \frac{p_1 - k_1\theta}{k_2} = u_2 + \frac{k_3}{k_2}u_1$ and $\tau_p = \left(\frac{k_2}{k_3}\right)^2\tau_u$. We obtain that

$$u_2|p_1, s_{it} \sim N\left(\frac{\hat{\tau}_u s_{u,i} + \tau_p\hat{p}}{\hat{\tau}_u + \tau_p + \tau_u}, (\hat{\tau}_u + \tau_p + \tau_u)^{-1}\right).$$

Substituting into the market-clearing price condition gives

$$p_1 = \theta + \frac{\hat{\tau}_u s_{u,i} + \tau_p\hat{p}}{\hat{\tau}_u + \tau_p + \tau_u} + \rho(\hat{\tau}_u + \tau_p + \tau_u)^{-1}u_1.$$

Comparing the coefficients, we obtain

$$k_1 = 1, \quad k_2 = \frac{\hat{\tau}_u + \tau_p}{\hat{\tau}_u + \tau_p + \tau_u}, \quad k_3 = \frac{\tau_p\frac{k_3}{k_2} + \rho}{\hat{\tau}_u + \tau_p + \tau_u}.$$

which we can solve to get $\frac{k_2}{k_3} = \frac{\hat{\tau}_u}{\rho}$ and that

$$k_2 = \frac{\hat{\tau}_u + \left(\frac{\hat{\tau}_u}{\rho}\right)^2\tau_u}{\hat{\tau}_u + \left(\frac{\hat{\tau}_u}{\rho}\right)^2\tau_u + \tau_u}, \quad k_3 = \frac{\rho + \left(\frac{\hat{\tau}_u}{\rho}\right)\tau_u}{\hat{\tau}_u + \left(\frac{\hat{\tau}_u}{\rho}\right)^2\tau_u + \tau_u}.$$

From the formula, we obtain $\lim_{\hat{\tau}_u \to \infty} k_2 = 1$ and $\lim_{\hat{\tau}_u \to \infty} k_3 = 0$ which implies that $\lim_{\hat{\tau}_u \to \infty} var(\theta + u_2 - p_1) = 0$. Also, it is immediate to see that $\frac{\partial k_2}{\partial \hat{\tau}_u} > 0$ For $k_3$, first-order derivative implies that $\frac{\partial k_3}{\partial \hat{\tau}_u} > 0$ if and only if $\hat{\tau}_u^2 \tau_u^2 + 2\tau_u \rho^2 \hat{\tau}_u + \rho^4 - \tau_u^2 \rho^2 > 0$. Solving this quadratic equation for positive value of $\hat{\tau}_u$, we obtain that $\hat{\tau}_u > \frac{\rho(\tau_u - \rho)}{\tau_u}$. Therefore, $\frac{\partial k_3}{\partial \hat{\tau}_u} > 0$ iff $\hat{\tau}_u > \frac{\rho(\tau_u - \rho)}{\tau_u}$. This also immediately proves that $\frac{\partial\ var(\theta + u_2 - p_1)}{\partial \hat{\tau}_u} < 0$ if $\hat{\tau}_u > \frac{\rho(\tau_u - \rho)}{\tau_u}$ as $k_2$ increases while $k_3$ decreases. From perceived asset risk $var\theta + u_2 | p_1, I_i = (\tau_u + \tau_p + \hat{\tau}_u)^{-1}$, it must decrease when $\hat{\tau}_u$ increases as $\frac{\partial \tau_p}{\partial \hat{\tau}_u} > 0$.

For the price informativeness, we know that $var(\theta | p_1) = (\tau_\theta + (k_2^2 + k_3^2)^{-1} \tau_u)^{-1}$. Since $var(\theta | p_1, \hat{\tau}_u = 0) = (\tau_\theta + (\frac{\tau_u}{\rho})^2 \tau_u)^{-1}$ and $\lim_{\hat{\tau}_u \to \infty} var(\theta | p_1) = (\tau_\theta + \tau_u)^{-1}$, this implies that price informativeness must increase for some $\hat{\tau}_u$ if $\rho > \tau_u$, because $var(\theta | p_1, \hat{\tau}_u = 0) > \lim_{\hat{\tau}_u \to \infty} var(\theta | p_1)$ if $\rho > \tau_u$. Moreover, considering the first-order derivative $\frac{\partial k_2^2 + k_3^2}{\partial \hat{\tau}_u}$, we obtain

$$sgn(\frac{\partial (k_2^2 + k_3^2)}{\partial \hat{\tau}_u}) = sgn\left( \left( \frac{\hat{\tau}_u}{\rho} \right)^3 \tau_u + \frac{\tau_u}{\rho} (\tau_u - \hat{\tau}_u) - \rho \right)$$

which will always be positive if $\tau_u > \rho$ and $|\tau_u - \rho|$ is sufficiently high, and will be negative if $\tau_u < \rho$ and $\hat{\tau}_u$ is sufficiently small. This proves the remark.

**Proof of Proposition 1.1:** First we substitute $\hat{\tau}_{uc} = 0$ and $\hat{\tau}_\theta \to \infty$ into equilibrium characterization in Proposition 1.9 which gives the following lemma.

**Lemma 1.5** *Provided that $\hat{\tau}_{uc} = 0$ and $\frac{1}{\hat{\tau}_\theta} = 0$, the equilibrium price is such that*

$$k_3 = \delta \frac{\hat{\tau}_u + k_3^2 \tau_u}{\hat{\tau}_u + k_3^2 \tau_u + \tau_u}$$

$$\frac{k_2}{k_4} = \frac{1}{1 - \alpha\delta}$$

$$k_4 = \frac{1 - \frac{\delta}{k_3} \left( \frac{k_3^2 \tau_u}{\hat{\tau}_u + k_3^2 \tau_u + \tau_u} \right)}{\rho \delta^2 \left( \left( \frac{k_2}{k_4} \right)^2 \sigma_\theta^2 + \left( \frac{k_3}{k_4} \right)^2 \sigma_u^2 + \frac{1}{k_4^2} (\hat{\tau}_u + k_3^2 \tau_u + \tau_u)^{-1} \right)}$$

From this characterization, we know that when $\hat{\tau}_u = 0$, then

$$k_3 = 0 \quad \text{and} \quad k_4 = \frac{1}{\rho \delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \frac{1}{k_4^2} \sigma_u^2 \right)}$$

Let $\hat{k}_4 = \frac{1}{k_4}$ be the price sensitivity to noise factor $u_t$. Rewriting the condition of $k_4$ gives

$$\hat{k}_4 = \rho \delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \hat{k}_4^2 \sigma_u^2 \right) \quad \to \quad \hat{k}_4 = \frac{1 - \sqrt{1 - 4\rho^2 \delta^4 \hat{k}_2^2 \sigma_\theta^2 \sigma_u^2}}{2\rho \delta^2 \sigma_u^2}.$$

To prove the first result in the remark, taking first-order derivative with respect to $\hat{\tau}_u$ around $\hat{\tau}_u = 0$ on $k_3$ and $\hat{k}_4$ gives

$$\left. \frac{\partial k_3}{\partial \hat{\tau}_u} \right|_{\hat{\tau}_u = 0} = \frac{1}{\tau_u} \quad \text{and} \quad \left. \frac{\partial \hat{k}_4}{\partial \hat{\tau}_u} \right|_{\hat{\tau}_u = 0} = \frac{\delta^2 \hat{k}_4 (\tau_u - \rho \hat{k}_4)}{\tau_u (\tau_u - 2\rho \hat{k}_4 \delta^2)}.$$

For the sign of $\left. \frac{\partial \hat{k}_4}{\partial \hat{\tau}_u} \right|_{\hat{\tau}_u = 0}$, it is positive if

1. $\tau_u - \rho\hat{k}_4 \geq 0$ and $\tau_u - 2\rho\hat{k}_4\delta^2 \geq 0$, or

2. $\tau_u - \rho\hat{k}_4 \leq 0$ or $\tau_u - 2\rho\hat{k}_4\delta^2 \leq 0$.

This implies that if $\delta$ is sufficiently high, $\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}\Big|_{\hat{\tau}_u=0} \geq 0$ if $\tau_u \geq 2\rho\hat{k}_4\delta^2$ or $\tau_u \leq \rho\hat{k}_4$. Checking the condition $\tau_u \geq 2\rho\hat{k}_4\delta^2$ by substituting $\hat{k}_4$ yields

$$1 \geq 1 - \sqrt{1 - \frac{4\rho^2\delta^4(\frac{1}{1-\alpha\delta})^2\sigma_\theta^2}{\tau_u}}$$

which is always true. This proves that, if $\delta$ is sufficiently high, then

$$\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}\bigg|_{\hat{\tau}_u=0} = \frac{\delta^2\hat{k}_4(\tau_u - \rho\hat{k}_4)}{\tau_u(\tau_u - 2\rho\hat{k}_4\delta^2)} \geq 0.$$

Also, since price sensitivity to next-period noise factor $u_{t+1}$ is $\frac{\partial p_t}{\partial u_{t+1}} = \frac{k_3}{k_4} = k_3\hat{k}_4$, we immediately obtain that $\frac{\partial p_t}{\partial u_{t+1}}$ is increasing in $\hat{\tau}_u$ when $\hat{\tau}_u = 0$. This proves the first result in the proposition.

Next, we check the sign of $\frac{\partial var(\delta p_{t+1} + d_t - p_t)}{\partial \hat{\tau}_u}$. Taking first-order derivative with respect to $\hat{\tau}_u$ and simplifying the condition gives

$$\frac{\partial var(\delta p_{t+1} + d_t - p_t)}{\partial \hat{\tau}_u}\bigg|_{\hat{\tau}_u=0} = 2\hat{k}_4\left((\delta^2+1)\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u} - \frac{\delta k_4}{\tau_u}\right)$$

which, once substituting $\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}$, will be positive if

$$\tau_u(\delta(\delta^2+1) - 1) > \rho k_4(1-\delta)^2$$

which is true if $\delta$ is sufficiently high. This proves the second result in the proposition.

It is immediate to see that price informativeness is decreasing when $\hat{\tau}_u = 0$ since the $\frac{k_2}{k_4}$ remains unchanged, and $\frac{k_3}{k_4}$ and $\frac{1}{k_4}$ are both increasing. This proves the third result in the proposition.

Lastly, we check the sign of $\frac{\partial var(\delta p_{t+1} + d_t | I_{it})}{\partial \hat{\tau}_u}$. First, note that $var(\delta p_{t+1} + d_t | I_{it}) = \delta^2\left(\left(\frac{k_2}{k_4}\right)^2\sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2\sigma_u^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)$. The first-order derivative with respect to $\hat{\tau}_u$ implies that

$$\frac{\partial var(\delta p_{t+1} + d_t | I_{it})}{\partial \hat{\tau}_u}\bigg|_{\hat{\tau}_u=0} \geq 0 \quad \text{iff} \quad \frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}\bigg|_{\hat{\tau}_u=0} \geq \frac{\hat{k}_4}{2\tau_u}.$$

Substituting $\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}\Big|_{\hat{\tau}_u=0}$ into the condition $\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u}\Big|_{\hat{\tau}_u=0} \geq \frac{\hat{k}_4}{2\tau_u}$ and simplifying the inequality condition gives the condition $\delta^2 \geq \frac{1}{2}$ which is true if $\delta$ is sufficiently high. This proves the last result in the proposition.

**Proof of Proposition 1.2:** Recall from Lemma 1.5 that, provided that $\hat{\tau}_{uc} = 0$ and $\hat{\tau}_\theta \to \infty$, the equilibrium price is such that

$$k_3 = \delta\frac{\hat{\tau}_u + k_3^2\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}$$

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta} = \hat{k}_2$$

$$k_4 = \frac{1 - \frac{\delta}{k_3}\left(\frac{k_3^2\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right)}{\rho\delta^2\left(\left(\frac{k_2}{k_4}\right)^2\sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2\sigma_u^2 + \frac{1}{k_4^2}(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)}$$

Denote $\lim_{\hat{\tau}_u\to\infty} k_3 = k_{3\infty}$, $\frac{1}{k_4} = \hat{k}_4$, and $\lim_{\hat{\tau}_u\to\infty}\hat{k}_4 = \hat{k}_{4\infty}$. When $\hat{\tau}_u \to \infty$, we obtain that $k_{3\infty} = \delta$ and $\hat{k}_{4\infty} = \rho\delta^2(\hat{k}_2^2\sigma_\theta^2 + \delta^2\sigma_u^2\hat{k}_{4\infty}^2)$ which can be solved to get $\hat{k}_{4\infty} = \frac{1-\sqrt{1-4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2}}{2\rho\delta^4\sigma_u^2}$. Now, to find $\hat{k}_4$ for arbitrary $\hat{\tau}_u$, consider the equation of $\hat{k}_4$.

$$\hat{k}_4 = \frac{\rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)}{1 - \frac{\delta}{k_3}\left(\frac{k_3^2\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right)}$$

which is equivalent to

$$\hat{k}_4 = \delta\hat{k}_4\left(\frac{k_3\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right) + \rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)$$

Comparing with the equation of $\hat{k}_{4\infty}$, we obtain that $\hat{k}_4 > \hat{k}_{4\infty}$ if and only if at $\hat{k}_4 = \hat{k}_{4\infty}$,

$$\delta\hat{k}_4\left(\frac{k_3\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right) + \rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right) > \rho\delta^2(\hat{k}_2^2\sigma_\theta^2 + \delta^2\sigma_u^2\hat{k}_4^2). \tag{1.8.2.6}$$

This is directly proved from the left panel in Figure 1.13 which shows that the RHS term which determines $\hat{k}_{4\infty}$ must be lower than the RHS term which determines $\hat{k}_4$ evaluated at $\hat{k}_{4\infty}$ to achieve $\hat{k}_4 \geq \hat{k}_{4\infty}$. Also, since a) $RHS$ is continuous in $\hat{\tau}_u$, b) $\hat{k}_4$ when $\hat{\tau}_u = 0$ is weakly higher than $\hat{k}_{4,\infty}$ which can be proved easily by substituting $\hat{\tau}_u = 0$ into Lemma 1.5, and c) $\frac{\partial \hat{k}_4}{\partial \hat{\tau}_u} > 0$ from Proposition 1.1, the condition 1.8.2.6 is sufficient to rule out the other case when all the intersections of $\hat{k}_4$ are on the left side of $\hat{k}_{4\infty}$. This case is shown in the right panel in Figure 1.13.

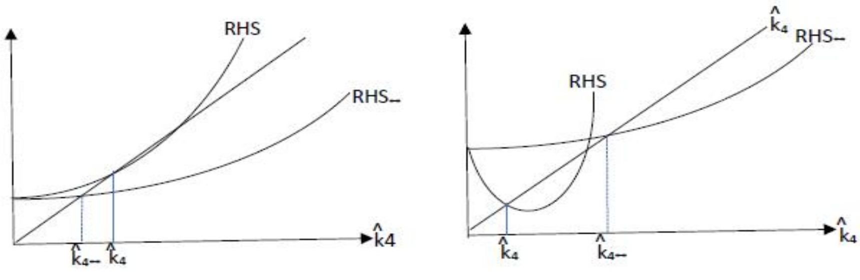


Figure 1.13: The diagram (left) illustrating that $RHS$ which determines $\hat{k}_{4\infty}$ must be below RHS which determines $\hat{k}_4$ to achieve $\hat{k}_4 \geq \hat{k}_{4\infty}$. The right panel illustrates another possible case which yields $\hat{k}_4 \leq \hat{k}_{4\infty}$.

Simplifying Equation 1.8.2.6 yields

$$\rho\delta^4\sigma_u^2\hat{k}_4^2 < \delta\hat{k}_4\left(\frac{k_3\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right) + \rho\delta^2\left(k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)$$

which is always true if

$$\rho\delta^2\sigma_u^2\hat{k}_4^2 < \delta\hat{k}_4\left(\frac{k_3\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right) + \rho\delta^2\left(k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right).$$

because $\delta \leq 1$. Rearranging and simplifying the condition, given that $\hat{\tau}_u \neq 0$ yields

$$\frac{\rho\hat{k}_4}{\tau_u}\left((1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u\right) < \tau_u.$$

This proves the following lemma.

**Lemma 1.6** *For any $\hat{\tau}_u \in \mathcal{R}^+$, $\hat{k}_{4\infty} < \hat{k}_4$ if and only if $\frac{\rho\hat{k}_{4\infty}}{\tau_u}\left((1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u\right) < \tau_u$.*

Indeed, $\hat{k}_{4\infty} < \hat{k}_4$ is always true for any positive $\hat{k}_4$ if $\delta$ is sufficiently high. To see this, substituting $\hat{k}_{4\infty}$ into $\frac{\rho\hat{k}_{4\infty}}{\tau_u}\left((1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u\right) < \tau_u$ gives

$$\left(\frac{1-\sqrt{1-4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2}}{2\rho\delta^4\sigma_u^2}\right)\left(\frac{\rho}{\tau_u}\right)\left((1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u\right) < \tau_u.$$

which is equivalent to

$$1-\sqrt{1-4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2} < \frac{2\tau_u\delta^4}{(1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u}$$

which is true if $\delta$ is sufficiently high. This proves the proposition.

**Proof of Proposition 1.3 and 1.4:** Recall from Lemma 1.5 that if $\hat{\tau}_{uc} = 0$ and $\hat{\tau}_\theta \to \infty$, the equilibrium price is such that

$$k_3 = \delta\frac{\hat{\tau}_u + k_3^2\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}$$

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}$$

$$k_4 = \frac{1 - \frac{\delta}{k_3}\left(\frac{k_3^2\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right)}{\rho\delta^2\left(\left(\frac{k_2}{k_4}\right)^2\sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2\sigma_u^2 + \frac{1}{k_4^2}(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right)}$$

which, given that $\hat{k}_4 = \frac{1}{k_4}$, is equivalent to

$$\hat{k}_4 = \delta\hat{k}_4\left(\frac{k_3\tau_u}{\hat{\tau}_u + k_3^2\tau_u + \tau_u}\right) + \rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + k_3^2\sigma_u^2\hat{k}_4^2 + \hat{k}_4^2(\hat{\tau}_u + k_3^2\tau_u + \tau_u)^{-1}\right).$$

From the previous equilibrium conditions, when $\hat{\tau}_u = 0$, we obtain

$$k_3 = 0, \quad \hat{k}_2 = \frac{1}{1-\alpha\delta}, \quad \hat{k}_4 = \rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + \hat{k}_4^2\sigma_u^2\right) \quad (*)$$

Also, when $\hat{\tau}_u \to \infty$, we obtain

$$k_3 = \delta, \quad \hat{k}_2 = \frac{1}{1-\alpha\delta}, \quad \hat{k}_4 = \rho\delta^2\left(\hat{k}_2^2\sigma_\theta^2 + \delta^2\hat{k}_4^2\sigma_u^2\right) \quad (**).$$

Denote $\hat{k}_{4,0} = \hat{k}_4$ at $\hat{\tau}_u = 0$ and $\hat{k}_{4,\infty} = \lim_{\hat{\tau}_u \to \infty} \hat{k}_4$. From Equation $(*)$ and $(**)$, solving for the most informative equilibrium $\hat{k}_{4,0}$ and $\hat{k}_{4,\infty}$ yields

$$\hat{k}_{4,0} = \frac{1 - \sqrt{1 - 4\rho^2\delta^4\hat{k}_2^2\sigma_\theta^2\sigma_u^2}}{2\rho\delta^2\sigma_u^2}, \quad \text{and} \quad \hat{k}_{4,\infty} = \frac{1 - \sqrt{1 - 4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2}}{2\rho\delta^4\sigma_u^2}$$

which implies that

$$\hat{k}_{4,0} > \hat{k}_{4,\infty}.$$

Furthermore, one can prove that

$$\begin{aligned} var(\delta p_{t+1} + d_t - p_t | \hat{\tau}_u = 0) &= \delta^2\hat{k}_2^2\sigma_\theta^2 + \delta^2\hat{k}_{4,0}^2\sigma_u^2 + \hat{k}_{4,0}^2\sigma_u^2 \\ &> \delta^2\hat{k}_2^2\sigma_\theta^2 + \delta^4\hat{k}_{4,\infty}^2\sigma_u^2 + \hat{k}_{4,\infty}^2\sigma_u^2 \\ &= \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t - p_t) \end{aligned}$$

$$\begin{aligned} var(\delta p_{t+1} + d_t | I_{it}, p_t, \hat{\tau}_u = 0) &= \delta^2\hat{k}_2^2\sigma_\theta^2 + \delta^2\hat{k}_{4,0}^2\sigma_u^2 \\ &> \delta^2\hat{k}_2^2\sigma_\theta^2 + \delta^4\hat{k}_{4,\infty}^2\sigma_u^2 \\ &= \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t | I_{it}, p_t). \end{aligned}$$

Also, when $\delta = 1$ which implies that $\hat{k}_{4,0} = \hat{k}_{4,\infty}$, it is immediate to see that

$$\begin{aligned} var(\delta p_{t+1} + d_t - p_t | \hat{\tau}_u = 0) &= \hat{k}_2^2\sigma_\theta^2 + \hat{k}_{4,0}^2\sigma_u^2 + \hat{k}_{4,0}^2\sigma_u^2 \\ &= \hat{k}_2^2\sigma_\theta^2 + \hat{k}_{4,\infty}^2\sigma_u^2 + \hat{k}_{4,\infty}^2\sigma_u^2 \\ &= \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t - p_t) \end{aligned}$$

$$\begin{aligned} var(\delta p_{t+1} + d_t | I_{it}, p_t, \hat{\tau}_u = 0) &= \hat{k}_2^2\sigma_\theta^2 + \hat{k}_{4,0}^2\sigma_u^2 \\ &= \hat{k}_2^2\sigma_\theta^2 + \hat{k}_{4,\infty}^2\sigma_u^2 \\ &= \lim_{\hat{\tau}_u \to \infty} var(\delta p_{t+1} + d_t | I_{it}, p_t) \end{aligned}$$

For the price informativeness, recall that $\hat{k}_{4,0} = \hat{k}_{4,\infty} \leq \hat{k}_4$ when $\delta = 1$ by Proposition 1.2 and $\hat{k}_3 > 0$. Therefore, if $\delta$ is sufficiently high, then

$$\begin{aligned} var(\theta_t | p_t, \hat{\tau}_u = 0) &= \left( \tau_\theta + \left( \frac{\hat{k}_2^2}{\hat{k}_{4,0}^2} \right) \tau_u \right)^{-1} \\ &< \left( \tau_\theta + \left( \frac{\hat{k}_2^2}{\hat{k}_3^2 + \hat{k}_4^2} \right) \tau_u \right)^{-1} = var(\theta_t | p_t) \end{aligned}$$

This proves Proposition 1.3 and 1.4.

**Proof of Remark 1.4:** This is directly proved from Proposition 1.2, 1.3, and 1.4.

**Proof of Proposition 1.5:** Let signal profile be $s_{it} = \{u_k\}_{k=t+1}^{t+n}$. Suppose that equilibrium price function is

$$p_t = k_1 d_{t-1} + k_2\theta_t + k_{3,1}u_{t+1} + k_{3,2}u_{t+2} + \cdots + k_{3,n}u_{t+n} + k_4 u_t \tag{1.8.2.7}$$

Recall that the trading strategy of each investor $i$ is

$$x_{it} - u_{it} = \frac{E(\delta p_{t+1} + d_t | I_{it}) - p_t}{Var(\delta p_{t+1} + d_t | I_{it})}$$

which, by market-clearing condition, implies that

$$p_t = \int_i E(\delta p_{t+1} + d_t | I_{it}) di + \rho Var(\delta p_{t+1} + d_t | I_{it}) u_t. \tag{1.8.2.8}$$

We know that

$$E(\delta p_{t+1} + d_t | I_{it}) = (\delta k_1 + 1) d_t + \delta(k_{3,1} u_{t+2} + k_{3,2} u_{t+3} + \cdots + k_{3,n-1} u_{t+n} + k_4 u_{t+1})$$

$$var(\delta p_{t+1} + d_t | I_{it}) = \delta^2 var(k_2 \theta_{t+1} + k_{3,n} u_{t+n+1} | I_{it}) = \delta^2 (k_2^2 \sigma_\theta^2 + k_{3,n}^2 \sigma_u^2).$$

Substituting into Equation 1.8.2.8 yields

$$p_t = (\delta k_1 + 1) d_t + \delta(k_{3,1} u_{t+2} + k_{3,2} u_{t+3} + \cdots + k_{3,n-1} u_{t+n} + k_4 u_{t+1}) + \rho \delta^2 (k_2^2 \sigma_\theta^2 + k_{3,n}^2 \sigma_u^2) u_t.$$

Comparing the coefficients with Equation 1.8.2.7 yields the following lemma

**Lemma 1.7** *Given signal profile of $s_{it} = \{u_k\}_{k=t+1}^{t+n}$, the equilibrium price is*

$$p_t = \frac{\alpha}{1 - \alpha\delta} d_{t-1} + \frac{\theta_t}{1 - \alpha\delta} + \delta k_4 u_{t+1} + \delta^2 k_4 u_{t+2} + \cdots + \delta^n k_4 u_{t+n} + k_4 u_t$$

*where* $k_4 = \rho \delta^2 \left[ \left( \frac{1}{1-\alpha\delta} \right)^2 \sigma_\theta^2 + \delta^{2n} k_4^2 \sigma_u^2 \right]$.

From the lemma, it is immediate to see that the price is not fully revealing as $k_4$ cannot be zero for any finite value of $n$. Next, we will analyze the effect of an increase in $n$ on $k_4$. From $k_4 = \underbrace{\rho \delta^2 \left[ \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^{2n} k_4^2 \sigma_u^2 \right]}_{\text{RHS}}$, one can plot a diagram as shown in Figure 1.14

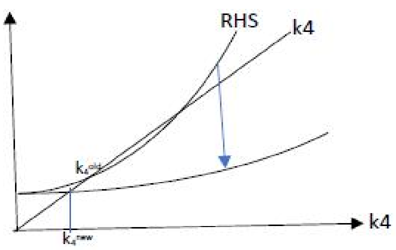


Figure 1.14: A diagram illustrating how an increase in $n$ causes RHS to tilt downward which causes the equilibrium $k_4$ to decrease.

From Figure 1.14, an increase in $n$ cause RHS to tilt downward, resulting in lower equilibrium $k_4$. To conclude, $k_4$ is decreasing in $n$ when discount factor is less than one. From this finding, we can further prove that

$$var(\delta p_{t+1} + d_t - p_t) = \delta^2 \sigma_\theta^2 \left( \frac{1}{1 - \alpha\delta} \right)^2 + \sigma_u^2 k_4^2 \left( 1 + \delta^{2(n+1)} \right)$$

$$var(\delta p_{t+1} + d_t | I_{it}, p_t) = \delta^2 \sigma_\theta^2 \left( \frac{1}{1 - \alpha\delta} \right)^2 + \sigma_u^2 k_4^2 \delta^{2(n+1)}$$

are both decreasing when $n$ increases when $\delta < 1$.

When $\delta = 1$, we know that $k_4$ is independent in $n$ as $k_4 = \rho \left[ \left( \frac{1}{1-\alpha} \right)^2 \sigma_\theta^2 + k_4^2 \sigma_u^2 \right]$. Also, substituting $\delta = 1$ into $var(\delta p_{t+1} + d_t - p_t)$ and $var(\delta p_{t+1} + d_t | I_{it}, p_t)$ gives

$$var(\delta p_{t+1} + d_t - p_t) = \sigma_\theta^2 \left( \frac{1}{1 - \alpha} \right)^2 + 2\sigma_u^2 k_4^2$$

$$var(\delta p_{t+1} + d_t | I_{it}, p_t) = \sigma_\theta^2 \left( \frac{1}{1 - \alpha} \right)^2 + \sigma_u^2 k_4^2$$

which does not depend on $n$ as well. However, price informativeness must be decreasing when $n$ increases as

$$var(\theta_t | p_t) = \left( \tau_\theta + \left( \frac{1}{1 - \alpha} \right)^2 \left( \frac{1}{k_4^2 (n+1)} \right) \tau_u \right)^{-1}$$

which is increasing in $n$. This proves the proposition.

**Proof of Proposition 1.6:** First, we will prove that $\lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t} \leq \lim_{\hat{\tau}_\theta \to \infty} \frac{\partial p_t}{\partial u_t}$. Recall from Proposition 1.9 that any stationary equilibrium price with signal precision $\{\hat{\tau}_\theta, \hat{\tau}_u, \hat{\tau}_{uc}\}$ is such that

$$p_t = \frac{1}{k_4} (k_1 d_{t-1} + k_2 \theta_t + k_3 u_{t+1} + u_t + k_5 \bar{\theta})$$

in which

$$\frac{k_1}{k_4} = \frac{\alpha}{1 - \alpha\delta}$$

$$k_2 = \frac{a_1 \left( \frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta} \right)}{\rho \left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( \frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$k_3 = \frac{a_2 \left( \frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u} \right)}{\rho \left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( \frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$k_4 = \frac{1}{\rho \left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( 1 - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$\frac{k_5}{k_4} = \frac{1}{(1 - \delta)(1 - \alpha\delta)}$$

where

$$a_1 = \frac{1}{1 - \alpha\delta}, \quad a_2 = \frac{\delta}{k_4}, \quad V_t = \delta^2 \left( \left( \frac{k_2}{k_4} \right)^2 \sigma_\theta^2 + \left( \frac{k_3}{k_4} \right)^2 \sigma_u^2 \right)$$

$$\beta = \frac{k_2 a_1 (\tau_u + \hat{\tau}_u) + k_3 a_2 (\tau_\theta + \hat{\tau}_\theta)}{a_1^2 (\tau_u + \hat{\tau}_u) + a_2^2 (\tau_\theta + \hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2 (\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2 (\tau_u + \hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2 \left( \frac{(k_2 a_2 - k_3 a_1)^2}{a_1^2 (\tau_u + \hat{\tau}_u) + a_2^2 (\tau_\theta + \hat{\tau}_\theta)} + (\tau_u + \hat{\tau}_{uc})^{-1} \right)^{-1}$$

Let $\hat{\tau}_{uc} = \hat{\tau}_{up}$. Denote $\lim_{\hat{\tau}_u \to \infty} k_3 = k_{3\infty}$, $\hat{k}_4 = \frac{1}{k_4}$, and $\lim_{\hat{\tau}_u \to \infty} k_4 = k_{4\infty}$. Taking limit $\hat{\tau}_\theta \to \infty$ and simplifying the equilibrium characterization yield the following equilibrium conditions:

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta} = \hat{k}_2$$

$$k_3 = \delta \left( \frac{k_3^2(\tau_u + \hat{\tau}_{up}) + \hat{\tau}_u}{\hat{\tau}_u + \tau_u + k_3^2(\tau_u + \hat{\tau}_{up})} \right)$$

$$\hat{k}_4 = \delta \hat{k}_4 \left( \frac{k_3 \tau_u}{\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u} \right) + \rho\delta^2 \left( \hat{k}_2^2 \sigma_\theta^2 + k_3^2 \sigma_u^2 \hat{k}_4^2 + \hat{k}_4^2 (\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u)^{-1} \right)$$

Also, when considering $\hat{\tau}_u \to \infty$, we obtain $k_{3\infty} = \delta$ and $\hat{k}_{4\infty} = \rho\delta^2(\hat{k}_2^2\sigma_\theta^2 + \delta^2\sigma_u^2\hat{k}_{4\infty}^2)$. Following the same argument in the proof of Proposition 1.2 (to obtain Lemma 1.6), we know that $\hat{k}_4 > \hat{k}_{4\infty}$ if and only if, at $\hat{k}_4 = \hat{k}_{4\infty}$,

$$\rho\delta^4\sigma_u^2\hat{k}_4^2 < \delta \hat{k}_4 \left( \frac{k_3 \tau_u}{\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u} \right) + \rho\delta^2 \left( \hat{k}_2^2 \sigma_\theta^2 + k_3^2 \sigma_u^2 \hat{k}_4^2 + \hat{k}_4^2 (\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u)^{-1} \right)$$

which is always true if

$$\rho\delta^2\sigma_u^2\hat{k}_4^2 < \delta \hat{k}_4 \left( \frac{k_3 \tau_u}{\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u} \right) + \rho\delta^2 \left( \hat{k}_2^2 \sigma_\theta^2 + k_3^2 \sigma_u^2 \hat{k}_4^2 + \hat{k}_4^2 (\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up}) + \tau_u)^{-1} \right)$$

because $\delta \leq 1$. Rearranging and simplifying the condition, given that $\hat{\tau}_u \neq 0$ yields

$$\frac{\rho\hat{k}_4}{\tau_u} \left( (1-\delta^2)(\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up})) + \tau_u \right) < \tau_u.$$

Indeed, $\hat{k}_{4\infty} < \hat{k}_4$ is always true if $\delta$ is sufficiently high and $n$ is finite (to guarantee that $\hat{\tau}_{up}$ is finite). To see this, substituting $\hat{k}_4 = \hat{k}_{4\infty}$ into $\frac{\rho\hat{k}_4}{\tau_u} \left( (1-\delta^2)(\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up})) + \tau_u \right) < \tau_u$. gives

$$\left( \frac{1 - \sqrt{1 - 4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2}}{2\rho\delta^4\sigma_u^2} \right) \left( \frac{\rho}{\tau_u} \right) \left( (1-\delta^2)(\hat{\tau}_u + k_3^2(\tau_u + \hat{\tau}_{up})) + \tau_u \right) < \tau_u.$$

which is equivalent to

$$1 - \sqrt{1 - 4\rho^2\delta^6\hat{k}_2^2\sigma_\theta^2\sigma_u^2} < \frac{2\tau_u\delta^4}{(1-\delta^2)(\hat{\tau}_u + \kappa_3^2\tau_u) + \tau_u}$$

which is always true if $\delta$ is sufficiently high and $n$ is finite (to guarantee that $\hat{\tau}_{up}$ is finite). Also, substituting $\hat{\tau}_u = 0$ into the equilibrium condition, we obtain that $\hat{k}_4 \geq \hat{k}_{4\infty}$ at $\hat{\tau}_u = 0$. From the proof, we obtain the following lemma

**Lemma 1.8** *Provided that Assumption 1.3 is true, then*

1. $\lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t} < \lim_{\hat{\tau}_\theta \to \infty} \frac{\partial p_t}{\partial u_t}$ *for any* $\hat{\tau}_u \in \mathcal{R}^+$.
2. $\lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to \infty} \frac{\partial p_t}{\partial u_t} \leq \lim_{\hat{\tau}_\theta \to \infty} \frac{\partial p_t}{\partial u_t}$ *at* $\hat{\tau}_u = 0$.

Next, we will prove Proposition 1.6. First note that $\lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0}\hat{k}_4 = \lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to\infty}\hat{k}_4$ if $\delta = 1$ from Proposition 1.4. Note that Proposition 1.4 remains valid even though $\hat{\tau}_{uc} = \hat{\tau}_{up} > 0$ since the signal about $u_t$ does not affect the limit values in which $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$ of all coefficients in the equilibrium price function. Also, from the equilibrium condition, $k_3 = 0$ if $\hat{\tau}_u = 0$ and $k_3 > 0$ if $\hat{\tau}_u > 0$. Therefore, by Lemma 1.8, we immediately obtain that

$$\lim_{\hat{\tau}_\theta\to\infty}\frac{\partial p_t}{\partial u_t} \geq \lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0}\frac{\partial p_t}{\partial u_t} \quad \text{if} \quad \delta = 1$$

$$\lim_{\hat{\tau}_\theta\to\infty}\frac{\partial p_t}{\partial u_{t+1}} > \lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0}\frac{\partial p_t}{\partial u_{t+1}}.$$

Lastly, from $\lim_{\hat{\tau}_\theta\to\infty}\frac{\partial p_t}{\partial u_t} \geq \lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0}\frac{\partial p_t}{\partial u_t}$ when $\delta = 1$ and $\lim_{\hat{\tau}_\theta\to\infty}\frac{\partial p_t}{\partial u_{t+1}} > 0$. Therefore, if $\delta$ is sufficiently high, then

$$\begin{aligned}\lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0} var(\theta_t|p_t) &= \left(\tau_\theta + \left(\frac{\hat{k}_2^2}{\lim_{\hat{\tau}_\theta\to\infty,\hat{\tau}_u\to 0}\hat{k}_4^2}\right)\tau_u\right)^{-1} \\ &< \left(\tau_\theta + \left(\frac{\hat{k}_2^2}{\lim_{\hat{\tau}_\theta\to\infty}(\hat{k}_3^2+\hat{k}_4^2)}\right)\tau_u\right)^{-1} = \lim_{\hat{\tau}_\theta\to\infty} var(\theta_t|p_t)\end{aligned}$$

This proves the proposition.

**Proof of Proposition 1.7:** Recall from Proposition 1.9 that any stationary equilibrium price with signal precision $\{\hat{\tau}_\theta, \hat{\tau}_u, \hat{\tau}_{uc}\}$ is such that

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

in which

$$\frac{k_1}{k_4} = \frac{\alpha}{1-\alpha\delta}$$

$$k_2 = \frac{a_1\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta+\tau_\theta}\right)}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right) + \left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(\frac{\hat{\tau}_\theta}{\hat{\tau}_\theta+\tau_\theta} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc}+\tau_u}\right)\frac{1}{\beta}}$$

$$k_3 = \frac{a_2\left(\frac{\hat{\tau}_u}{\hat{\tau}_u+\tau_u}\right)}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right) + \left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(\frac{\hat{\tau}_u}{\hat{\tau}_u+\tau_u} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc}+\tau_u}\right)\frac{1}{\beta}}$$

$$k_4 = \frac{1}{\rho\left((\tau_Z+\tau_P)^{-1}+V_t\right) + \left(\frac{\tau_P}{\tau_Z+\tau_P}\right)\left(1 - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc}+\tau_u}\right)\frac{1}{\beta}}$$

$$\frac{k_5}{k_4} = \frac{1}{(1-\delta)(1-\alpha\delta)}$$

where

$$a_1 = \frac{1}{1-\alpha\delta}, \quad a_2 = \frac{\delta}{k_4}, \quad V_t = \delta^2\left(\left(\frac{k_2}{k_4}\right)^2\sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2\sigma_u^2\right)$$

$$\beta = \frac{k_2 a_1(\tau_u+\hat{\tau}_u) + k_3 a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u) + a_2^2(\tau_\theta+\hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2(\tau_u + \hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2 \left( \frac{(k_2 a_2 - k_3 a_1)^2}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)} + (\tau_u + \hat{\tau}_{uc})^{-1} \right)^{-1}$$

**Case I: when $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$**

Taking limits of $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$ gives $\tau_Z \to \infty$. Also, one can easily simplify the characterization into

$$k_2 = \frac{a_1}{\rho V_t}, \quad k_3 = \frac{a_2}{\rho V_t}, \quad k_4 = \frac{1}{\rho V_t}$$

which implies that

$$\frac{k_2}{k_4} = a_1 = \frac{1}{1 - \alpha\delta}, \quad \frac{k_3}{k_4} = a_2 = \frac{\delta}{k_4} \to k_3 = \delta$$

Denote $\hat{k}_2 = \frac{k_2}{k_4}$, $\hat{k}_3 = \frac{k_3}{k_4}$, $\hat{k}_4 = \frac{1}{k_4}$. From the equation $k_4 = \frac{1}{\rho V_t}$, we can rewrite it after substituting $\hat{k}_2$ and $\hat{k}_3$ as follows:

$$\hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

To conclude, if $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$, then

$$\hat{k}_2 = \frac{1}{1 - \alpha\delta}, \quad k_3 = \delta, \quad \hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

which immediately proves that price efficiency of all measures does not depend on $\hat{\tau}_{uc}$.

**Case II: when $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_{uc} \to \infty$**

Taking limit $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_{uc} \to \infty$ yields $\tau_P \to \infty$, $\tau_Z = \frac{\tau_u + \hat{\tau}_u}{a_2^2}$ and $\beta = \frac{k_3}{a_2}$. Substituting $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_{uc} \to \infty$, $\beta = \frac{k_3}{a_2}$, $\tau_Z = \frac{\tau_u + \hat{\tau}_u}{a_2^2}$, and $\tau_P \to \infty$ into the characterization and simplifying the system of equations yield

$$k_2 = \frac{a_1}{\rho V_t}, \quad k_3 = \frac{a_2}{\rho V_t}, \quad k_4 = \frac{1}{\rho V_t}$$

which implies that

$$\frac{k_2}{k_4} = a_1 = \frac{1}{1 - \alpha\delta}, \quad \frac{k_3}{k_4} = a_2 = \frac{\delta}{k_4} \to k_3 = \delta$$

Denote $\hat{k}_2 = \frac{k_2}{k_4}$, $\hat{k}_3 = \frac{k_3}{k_4}$, $\hat{k}_4 = \frac{1}{k_4}$. From the equation $k_4 = \frac{1}{\rho V_t}$, we can rewrite it after substituting $\hat{k}_2$ and $\hat{k}_3$ as follows:

$$\hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

To conclude, if $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_{uc} \to \infty$, then

$$\hat{k}_2 = \frac{1}{1 - \alpha\delta}, \quad k_3 = \delta, \quad \hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

**Case III: when $\hat{\tau}_u \to \infty$ and $\hat{\tau}_{uc} \to \infty$**

Taking limit $\hat{\tau}_u \to \infty$ and $\hat{\tau}_{uc} \to \infty$ in the equilibrium characterization yields $\tau_P \to \infty$, $\tau_Z = \frac{\tau_\theta + \hat{\tau}_\theta}{a_1^2}$ and $\beta = \frac{k_2}{a_1}$. Substituting $\hat{\tau}_u \to \infty$, $\hat{\tau}_{uc} \to \infty$, $\beta = \frac{k_2}{a_1}$, $\tau_Z = \frac{\tau_\theta + \hat{\tau}_\theta}{a_1^2}$, and $\tau_P \to \infty$ into the characterization and simplifying the system of equations yield

$$k_2 = \frac{a_1}{\rho V_t}, \quad k_3 = \frac{a_2}{\rho V_t}, \quad k_4 = \frac{1}{\rho V_t}$$

which implies that

$$\frac{k_2}{k_4} = a_1 = \frac{1}{1 - \alpha\delta}, \quad \frac{k_3}{k_4} = a_2 = \frac{\delta}{k_4} \to k_3 = \delta.$$

Denote $\hat{k}_2 = \frac{k_2}{k_4}$, $\hat{k}_3 = \frac{k_3}{k_4}$, $\hat{k}_4 = \frac{1}{k_4}$. From the equation $k_4 = \frac{1}{\rho V_t}$, we can rewrite it after substituting $\hat{k}_2$ and $\hat{k}_3$ as follows:

$$\hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

To conclude, if $\hat{\tau}_u \to \infty$ and $\hat{\tau}_{uc} \to \infty$, then

$$\hat{k}_2 = \frac{1}{1 - \alpha\delta}, \quad k_3 = \delta, \quad \hat{k}_4 = \rho\delta^2 \left( \left( \frac{1}{1 - \alpha\delta} \right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

From the three cases above, we obtain that, given that $X = \{d_{t-1}, \theta_t, u_t, u_{t+1}\}$,

$$\lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_{uc} \to \infty} p_t(X) = \lim_{\hat{\tau}_u \to \infty, \hat{\tau}_{uc} \to \infty} p_t(X) = \lim_{\hat{\tau}_\theta \to \infty, \hat{\tau}_u \to \infty} p_t(X).$$

as all the coefficients in the equilibrium price equation are identical across three cases. This proves the proposition.

**Proof of Remark 1.5:** Recall from Proposition 1.9 that any stationary equilibrium price with signal precision $\{\hat{\tau}_\theta, \hat{\tau}_u, \hat{\tau}_{uc}\}$ is such that

$$p_t = \frac{1}{k_4}(k_1 d_{t-1} + k_2\theta_t + k_3 u_{t+1} + u_t + k_5\bar{\theta})$$

in which

$$\frac{k_1}{k_4} = \frac{\alpha}{1 - \alpha\delta}$$

$$k_2 = \frac{a_1 \left( \frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta} \right)}{\rho\left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( \frac{\hat{\tau}_\theta}{\hat{\tau}_\theta + \tau_\theta} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$k_3 = \frac{a_2 \left( \frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u} \right)}{\rho\left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( \frac{\hat{\tau}_u}{\hat{\tau}_u + \tau_u} - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$k_4 = \frac{1}{\rho\left( (\tau_Z + \tau_P)^{-1} + V_t \right) + \left( \frac{\tau_P}{\tau_Z + \tau_P} \right) \left( 1 - \frac{\hat{\tau}_{uc}}{\hat{\tau}_{uc} + \tau_u} \right) \frac{1}{\beta}}$$

$$\frac{k_5}{k_4} = \frac{1}{(1-\delta)(1-\alpha\delta)}$$

where

$$a_1 = \frac{1}{1-\alpha\delta}, \quad a_2 = \frac{\delta}{k_4}, \quad V_t = \delta^2 \left( \left(\frac{k_2}{k_4}\right)^2 \sigma_\theta^2 + \left(\frac{k_3}{k_4}\right)^2 \sigma_u^2 \right)$$

$$\beta = \frac{k_2 a_1(\tau_u + \hat{\tau}_u) + k_3 a_2(\tau_\theta + \hat{\tau}_\theta)}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2(\tau_\theta + \hat{\tau}_\theta)^{-1} + a_2^2(\tau_u + \hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2 \left( \frac{(k_2 a_2 - k_3 a_1)^2}{a_1^2(\tau_u + \hat{\tau}_u) + a_2^2(\tau_\theta + \hat{\tau}_\theta)} + (\tau_u + \hat{\tau}_{uc})^{-1} \right)^{-1}$$

**Case I: when $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} = 0$.**

Substituting $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} = 0$ into the characterization gives

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, \quad k_3 = 0, \quad k_4 = \frac{1}{K}$$

where $K = \rho\delta^2 \left( \left(\frac{k_2}{k_4}\right)^2 \frac{1}{\tau_\theta} + \left(\frac{1}{k_4}\right)^2 \frac{1}{\tau_u} \right)$. Denote $\hat{k}_4 = \frac{1}{k_4}$. Then, we have

$$\hat{k}_4 = \rho\delta^2 \left( \left(\frac{1}{1-\alpha\delta}\right)^2 \sigma_\theta^2 + \sigma_u^2 \hat{k}_4^2 \right).$$

**Case II: when $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} \to \infty$.**

Substituting $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} \to \infty$ into the characterization gives

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, \quad k_3 = \delta, \quad k_4 = \frac{1}{K}$$

where $K = \rho\delta^2 \left( \left(\frac{k_2}{k_4}\right)^2 \frac{1}{\tau_\theta} + \delta^2 \left(\frac{1}{k_4}\right)^2 \frac{1}{\tau_u} \right)$. Denote $\hat{k}_4 = \frac{1}{k_4}$. Then, we have

$$\hat{k}_4 = \rho\delta^2 \left( \left(\frac{1}{1-\alpha\delta}\right)^2 \sigma_\theta^2 + \delta^2 \sigma_u^2 \hat{k}_4^2 \right).$$

Comparing two cases above, we see that when $\delta = 1$, $k_2$ and $k_4$ are identical across two cases. Consider $var(\theta_t|p_t) = \left(\tau_\theta + \left(\frac{k_2^2}{k_3^2+1}\right)\tau_u\right)^{-1}$. When $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} = 0$, we have $var(\theta_t|p_t) = \left(\tau_\theta + k_2^2\tau_u\right)^{-1}$. When $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} \to \infty$, we have $var(\theta_t|p_t) = \left(\tau_\theta + \left(\frac{k_2^2}{\delta^2+1}\right)\tau_u\right)^{-1}$. Therefore, when discount factor is one, $var(\theta_t|p_t)$ given that $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} = 0$ must be lower than that of which $\hat{\tau}_\theta \to \infty$, $\hat{\tau}_u = 0$, and $\hat{\tau}_{uc} \to \infty$. By continuity, this inequality is valid if $\delta$ is sufficiently high. This proves the remark.

**Proof of Lemma 1.1:** First, we find the ex-ante payoff of each investor $i$. Consider the ex-ante utility function

$$E(U_{it}(\pi_{it}|I_{it})) = E(E(\delta p_{t+1}+d_t-p_t|I_{it})(x_t-u_{it})) - \frac{\rho}{2}E\left(Var((\delta p_{t+1}+d_t-p_t)(x_t-u_{it})|I_{it})\right)$$

Substituting $x_{it} - u_{it} = \frac{E(\delta p_{t+1}+d_t|I_{it})-p_t}{Var(\delta p_{t+1}+d_t|I_{it})}$ yields

$$\begin{aligned}
E(U_{it}(\pi_{it}|I_{it})) &= E\left(\frac{E(\delta p_{t+1}+d_t-p_t|I_{it})^2}{\rho Var(\delta p_{t+1}+d_t|I_{it})}\right) - \frac{1}{2\rho}E\left(\frac{E(\delta p_{t+1}+d_t-p_t|I_{it})^2}{Var(\delta p_{t+1}+d_t|I_{it})}\right)\\
&= E\left(\frac{E(\delta p_{t+1}+d_t-p_t|I_{it})^2}{2\rho Var(\delta p_{t+1}+d_t|I_{it})}\right)\\
&= \frac{Var\left(E(\delta p_{t+1}+d_t-p_t|I_{it})\right)+E(E(\delta p_{t+1}+d_t-p_t|I_{it}))}{2\rho Var(\delta p_{t+1}+d_t|I_{it})}
\end{aligned}$$

Since for any random variable $X$,

$$Var(X) = Var(E(X|I)) + E(Var(X|I))$$

together with law of iterated expectation, we obtain

$$\begin{aligned}
E(U_{it}(\pi_{it}|I_{it})) &= \frac{Var\left(\delta p_{t+1}+d_t-p_t\right) - Var(\delta p_{t+1}+d_t|I_{it}) + E(\delta p_{t+1}+d_t-p_t)}{2\rho Var(\delta p_{t+1}+d_t|I_{it})}\\
&= \frac{Var\left(\delta p_{t+1}+d_t-p_t\right) + E(\delta p_{t+1}+d_t-p_t)}{2\rho Var(\delta p_{t+1}+d_t|I_{it})} - \frac{1}{2\rho}
\end{aligned}$$

which proves the following lemma

**Lemma 1.9** *Ex-ante payoff of each investor is*

$$E(U_{it}(\pi_{it}|I_{it})) = \frac{Var\left(\delta p_{t+1}+d_t-p_t\right) + E(\delta p_{t+1}+d_t-p_t)}{2\rho Var(\delta p_{t+1}+d_t|I_{it})} - \frac{1}{2\rho}$$

To find optimal information choice, it is easy to see that investors will choose optimal information choice to minimize $Var(\delta p_{t+1}+d_t|I_{it})$ to maximize the ex-ante utility. This is because $(\hat{\tau}_{u,it}, \hat{\tau}_{\theta,it})$ does not enter into $E(\delta p_{t+1}+d_t-p_t)$ and that investors do not take into account their influence of unconditional variance $Var\left(\delta p_{t+1}+d_t-p_t\right)$ in the competitive market setting. Since $Var(\delta p_{t+1}+d_t|I_{it}) = V_t + (\tau_Z+\tau_P)^{-1}$, optimization problem of any investor $i$ amounts to

$$(\hat{\tau}_{u,it}, \hat{\tau}_{\theta,it}) \in \arg\max_{\hat{\tau}_{uit},\hat{\tau}_{\theta it}} \tau_Z + \tau_P$$

subject to the technological constraint

$$\hat{\tau}_{\theta,it} + \gamma\hat{\tau}_{u,it} \leq \Gamma$$

Since we characterize only symmetric and stationary equilibrium information choice $(\hat{\tau}_\theta, \hat{\tau}_u)$ , we drop subscript $it$ and set Lagrangian function as follows.

$$L = \tau_Z + \tau_P + \lambda(\Gamma - \hat{\tau}_\theta - \gamma\hat{\tau}_u)$$

Recall that when $\hat{\tau}_{uc} = 0$

$$\beta = \frac{k_2a_1(\tau_u+\hat{\tau}_u) + k_3a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u) + a_2^2(\tau_\theta+\hat{\tau}_\theta)}$$

$$\tau_Z = (a_1^2(\tau_\theta+\hat{\tau}_\theta)^{-1} + a_2^2(\tau_u+\hat{\tau}_u)^{-1})^{-1}$$

$$\tau_P = \beta^2\left(\frac{(k_2a_2-k_3a_1)^2}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)} + (\tau_u)^{-1}\right)^{-1}.$$

We obtain

$$\frac{\partial \tau_Z}{\partial \hat{\tau}_\theta} = \frac{a_1^2(\tau_u+\hat{\tau}_u)^2}{(a_1^2(\tau_u+\hat{\tau}_u)+\ a_2^2(\tau_\theta+\hat{\tau}_\theta))^2} \qquad \frac{\partial \tau_Z}{\partial \hat{\tau}_u} = \frac{a_2^2(\tau_\theta+\hat{\tau}_\theta)^2}{(a_1^2(\tau_u+\hat{\tau}_u)+\ a_2^2(\tau_\theta+\hat{\tau}_\theta))^2}$$

$$\frac{\partial \beta}{\partial \hat{\tau}_\theta} = \frac{a_1a_2(k_3a_1-k_2a_2)(\tau_u+\hat{\tau}_u)}{(a_1^2(\tau_u+\hat{\tau}_u)+\ a_2^2(\tau_\theta+\hat{\tau}_\theta))^2} \qquad \frac{\partial \beta}{\partial \hat{\tau}_u} = \frac{a_1a_2(k_2a_2-k_3a_1)(\tau_\theta+\hat{\tau}_\theta)}{(a_1^2(\tau_u+\hat{\tau}_u)+\ a_2^2(\tau_\theta+\hat{\tau}_\theta))^2}$$

$$\frac{\partial(\tau_P/\beta^2)}{\partial \hat{\tau}_\theta} = \left(\frac{\tau_P}{\beta^2}\right)^2\left(\frac{k_2a_2-k_3a_1}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}\right)^2 a_2^2$$

$$\frac{\partial(\tau_P/\beta^2)}{\partial \hat{\tau}_u} = \left(\frac{\tau_P}{\beta^2}\right)^2\left(\frac{k_2a_2-k_3a_1}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}\right)^2 a_1^2$$

and the first-order condition for any interior solution of $(\hat{\tau}_u, \hat{\tau}_\theta)$ is

$$\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_u} - 2\lambda\gamma = 0$$

$$\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_\theta} - 2\lambda = 0$$

$$\hat{\tau}_\theta + \gamma\hat{\tau}_u = \Gamma$$

which can be simplified to the two following equations.

$$\frac{\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_\theta}}{\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_u}} = \frac{\frac{\partial \tau_Z}{\partial \hat{\tau}_\theta}+2\beta\frac{\partial \beta}{\partial \hat{\tau}_\theta}(\frac{\tau_P}{\beta^2})+\beta^2\frac{\partial(\tau_P/\beta^2)}{\partial \hat{\tau}_\theta}}{\frac{\partial \tau_Z}{\partial \hat{\tau}_u}+2\beta\frac{\partial \beta}{\partial \hat{u}_\theta}(\frac{\tau_P}{\beta^2})+\beta^2\frac{\partial(\tau_P/\beta^2)}{\partial \hat{u}_\theta}} = \frac{1}{\gamma}$$

$$\text{and} \quad \hat{\tau}_\theta + \gamma\hat{\tau}_u = \Gamma.$$

Let

$$B = \frac{\tau_P}{\beta^2} = \left(\frac{(k_2a_2-k_3a_1)^2}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)} + (\tau_u)^{-1}\right)^{-1}.$$

Substituting all the expressions into the optimality condition gives

$$\frac{1}{\gamma} = \frac{(a_1(\tau_u+\hat{\tau}_u)+\beta B(k_3a_1-k_2a_2)a_2)^2}{(a_2(\tau_\theta+\hat{\tau}_\theta)+\beta B(k_2a_2-k_3a_1)a_1)^2}$$

which also shows that $\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_\theta} \geq 0$ and $\frac{\partial \tau_Z+\tau_P}{\partial \hat{\tau}_u} \geq 0$. Also,

$$\begin{aligned}\beta B &= \frac{k_2a_1(\tau_u+\hat{\tau}_u)+k_3a_2(\tau_\theta+\hat{\tau}_\theta)}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)}\left(\frac{(k_2a_2-k_3a_1)^2}{a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)} + (\tau_u)^{-1}\right)^{-1} \\ &= \frac{k_2a_1(\tau_u+\hat{\tau}_u)+k_3a_2(\tau_\theta+\hat{\tau}_\theta)}{(k_2a_2-k_3a_1)^2+(\tau_u)^{-1}X}\end{aligned}$$

where $X = a_1^2(\tau_u+\hat{\tau}_u)+a_2^2(\tau_\theta+\hat{\tau}_\theta)$. Note that

$$a_1(\tau_u+\hat{\tau}_u)+\beta B(k_3a_1-k_2a_2)a_2 = \frac{\left(k_3(k_3a_1-k_2a_2)+a_1(\tau_u+\hat{\tau}_u)(\tau_u)^{-1}\right)X}{(k_2a_2-k_3a_1)^2+(\tau_u)^{-1}X}$$

$$a_2(\tau_\theta+\hat{\tau}_\theta)+\beta B(k_2a_2-k_3a_1)a_1 = \frac{\left(k_2(k_2a_2-k_3a_1)+a_2(\tau_\theta+\hat{\tau}_\theta)(\tau_u)^{-1}\right)X}{(k_2a_2-k_3a_1)^2+(\tau_u)^{-1}X}$$

and thus substituting $\beta X$ into the optimality condition yields

$$\frac{1}{\gamma} = \left(\frac{k_3(k_3a_1-k_2a_2)+a_1(\tau_u+\hat{\tau}_u)(\tau_u)^{-1}}{k_2(k_2a_2-k_3a_1)+a_2(\tau_\theta+\hat{\tau}_\theta)(\tau_u)^{-1}}\right)^2$$

This simplified optimality condition is similar to equating slope of indifference curve of investors with price ratio of the technological constraint. The right hand side term is absolute value of MRS of choosing information bundle $(\hat{\tau}_\theta, \hat{\tau}_u)$ while the left hand side is the price ratio which is somewhat depend on the signal precision itself due to the non-linear nature of technological constraint.

Last step is to check second-order condition whether the function $\tau_P + \tau_Z$ is quasi-concave. A sufficient way to prove this is to show that (negative) slope of the indifference curve is decreasing in $\hat{\tau}_\theta$. To see this, let

$$M_U = k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u)(\tau_u + \tau_{ui})^{-1}$$

$$M_D = k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)(\tau_u + \tau_{ui})^{-1}$$

$$\text{and} \quad M = \frac{M_U}{M_D}.$$

Differentiate the slope of the indifference curve gives

$$\begin{aligned}\frac{\partial M^2}{\partial^2 \hat{\tau}_\theta} &= -\frac{2M^2}{M_D^2}(\tau_u)^{-1}\left((k_3a_1 - k_2a_2)^2 + a_1^2(\tau_u + \hat{\tau}_u)(\tau_u)^{-1} + a_2^2(\tau_\theta + \hat{\tau}_\theta)(\tau_u)^{-1}\right) \\ &< 0\end{aligned}$$

which is sufficient to prove that the second-order condition is satisfied. Thus, we obtain that, in any symmetric equilibrium, optimal information choice $(\hat{\tau}_u, \hat{\tau}_\theta)$ for any interior solution must satisfy

$$\frac{1}{\gamma} = \left(\frac{\tau_u k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u)}{\tau_u k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)}\right)^2$$

which proves the lemma.

**Proof of Proposition 1.8:** Recall that any equilibrium information choice of an interior solution must satisfy

$$\frac{1}{\gamma} = \left(\frac{\tau_u k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u)}{\tau_u k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)}\right)^2.$$

However, if $\gamma$ is sufficiently high such that

$$\frac{1}{\gamma} < \left(\frac{a_1\tau_u}{\tau_u k_2^2 a_2 + a_2(\tau_\theta + \Gamma)}\right)^2,$$

Then, a corner solution of $\hat{\tau}_u = 0$ and $\hat{\tau}_\theta = \Gamma$ is possible.

Next, we will prove that $\hat{\tau}_u > 0$ if and only if $\Gamma > \Gamma^*$. Let $\Gamma^*$ satisfy

$$\frac{1}{\gamma} = \left(\frac{a_1\tau_u}{\tau_u \bar{k}_2^2 a_2 + a_2(\tau_\theta + \Gamma^*)}\right)^2.$$

where $\bar{k}_2$ is evaluated at $(\hat{\tau}_\theta = \Gamma, \hat{\tau}_u = 0)$.

Now, consider $\Gamma < \Gamma^*$. Suppose that there is an interior solution $(\hat{\tau}_\theta, \hat{\tau}_u)$ such that $\hat{\tau}_u > 0$ in which

$$\frac{1}{\gamma} = \left(\frac{\tau_u k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u)}{\tau_u k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)}\right)^2.$$

Then, we can increase $\gamma$ such that

$$\frac{1}{\gamma} < \left(\frac{\tau_u k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u)}{\tau_u k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)}\right)^2.$$

for any $(\hat{\tau}_\theta, \hat{\tau}_u)$. That is, set $\gamma$ to be sufficiently high to surpass the lowerbound of $\left(\frac{\tau_u k_3(k_3a_1-k_2a_2)+a_1(\tau_u+\hat{\tau}_u)}{\tau_u k_2(k_2a_2-k_3a_1)+a_2(\tau_\theta+\hat{\tau}_\theta)}\right)$ The lowerbound exists and must be positive because 1) $k_2$, $k_3$, and $k_4$ are positive finite value in the equilibrium which causes $\tau_u k_2(k_2a_2 - k_3a_1) + a_2(\tau_\theta + \hat{\tau}_\theta)$ to be finite and 2) $\tau_u k_3(k_3a_1 - k_2a_2) + a_1(\tau_u + \hat{\tau}_u) > 0$ when $\gamma$ is sufficiently high. The latter condition is true since $\hat{\tau}_u \leq \frac{\Gamma^*}{\gamma}$ in any equilibrium and that $\frac{\Gamma^*}{\gamma} = \left(\frac{a_1\tau_u\sqrt{\Gamma^*}}{\tau_u\bar{k}_2^2a_2+a_2(\tau_\theta+\Gamma^*)}\right)^2$ which implies that $\hat{\tau}_u$ (and $k_3$) are approximately zero when $\gamma$ is sufficiently high. Therefore, if $\gamma$ is sufficiently high, then only corner solution where $\hat{\tau}_u = 0$ exists.

When $\Gamma > \Gamma^*$, the corner solution is not possible since

$$\frac{1}{\gamma} = \left(\frac{a_1\tau_u}{\tau_u\bar{k}_2^2a_2 + a_2(\tau_\theta + \Gamma^*)}\right)^2 > \left(\frac{a_1\tau_u}{\tau_u\ k_2^2\big|_{\hat{\tau}_u=0,\hat{\tau}_\theta=\Gamma}\ a_2 + a_2(\tau_\theta + \Gamma)}\right)^2.$$

as $k_2^2\big|_{\hat{\tau}_u=0,\hat{\tau}_\theta=\Gamma} > \bar{k}_2$ based on Remark 1.1 proved in the baseline equilibrium setting. This proves the first result in the proposition

To prove the second statement in the proposition, consider

$$\frac{1}{\gamma} = \left(\frac{a_1\tau_u}{\tau_u\bar{k}_2^2a_2 + a_2(\tau_\theta + \Gamma^*)}\right)^2.$$

If $\gamma$ is high, then $\Gamma^*$ must be high to achieve the break-even point. Recall from our previous proof that if $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u = 0$, then

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, \quad k_3 = 0, \quad k_4 = \frac{1}{K}$$

where $K = \rho\delta^2\left(\left(\frac{k_2}{k_4}\right)^2\frac{1}{\tau_\theta} + \left(\frac{1}{k_4}\right)^2\frac{1}{\tau_u}\right)$. Denote $\hat{k}_4 = \frac{1}{k_4}$. Then, we have

$$\hat{k}_4 = \rho\delta^2\left(\left(\frac{1}{1-\alpha\delta}\right)^2\sigma_\theta^2 + \sigma_u^2\hat{k}_4^2\right).$$

By continuity, we obtain that if $\Gamma^*$ is sufficiently large due to high $\gamma$, then the equilibrium price at $\Gamma^*$ is such that

$$\frac{k_2}{k_4} \approx \frac{1}{1-\alpha\delta}, \quad k_3 = 0, \quad \hat{k}_4 \approx \rho\delta^2\left(\left(\frac{1}{1-\alpha\delta}\right)^2\sigma_\theta^2 + \sigma_u^2\hat{k}_4^2\right).$$

Now, consider $\Gamma \to \infty$ which implies that $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$ in the interior solution. Recall from the previous proof that if $\hat{\tau}_\theta \to \infty$ and $\hat{\tau}_u \to \infty$, then

$$\frac{k_2}{k_4} = \frac{1}{1-\alpha\delta}, \quad k_3 = \delta, \quad \hat{k}_4 = \rho\delta^2\left(\left(\frac{1}{1-\alpha\delta}\right)^2\sigma_\theta^2 + \delta^2\sigma_u^2\hat{k}_4^2\right).$$

This immediately proves that if discount factor is sufficiently high which causes $\hat{k}_4$ (and $k_2$) to be approximately identical when $\Gamma = \Gamma^*$ and when $\Gamma \to \infty$, then

$$var(\theta_t|p_t, \Gamma = \Gamma^*) = (\tau_\theta + k_2^2\tau_u)^{-1} < (\tau_\theta + \frac{k_2^2}{1+\delta^2}\tau_u)^{-1} = \lim_{\Gamma\to\infty} var(\theta_t|p_t)$$

By continuity, there exists $\bar{\Gamma} > \Gamma^*$ such that for any $\Gamma \geq \bar{\Gamma}$, $var(\theta_t|p_t) > \lim_{\Gamma\to\Gamma^*} var(\theta_t|p_t)$. This proves the second statement and the proposition.

# Chapter 2

# Periphery Dealers in Over-the-counter Markets

## 2.1 Introduction

Over-the-counter (OTC) markets are off-exchange decentralized markets where investors search for trading counterparties and privately negotiate to settle trades. In the past years, OTC markets have been important segments for trading wide-ranging financial products such as interbank loans, derivatives, and fixed-income securities. Due to non-standardized nature of these financial products, OTC markets are illiquid and highly dependent on dealers, the market makers who quote prices and provide liquidity to investors using their inventories. To design appropriate regulatory framework and ensure market efficiency within OTC markets, understanding trading strategies of all market participants is thus crucial for policymakers.

The objective of this paper is to study trading behavior of buy-side investors, particularly on their dealer choices, and discuss the implications on market efficiency and stability. Our research question comes from recent empirical findings about business models of dealers locating at different positions in OTC dealer network. Several empirical studies on OTC secondary markets have documented persistent core-periphery dealer network – few highly interconnected dealers constitute the core and several sparsely connected dealers constitute the periphery.[1] Based on Li and Schürhoff (2019), dealers at the core are the main suppliers of liquidity (i.e. immediacy provision), executing incoming order flows using their own inventories to provide immediacy and offloading their positions later with another end-user investor. In contrast, dealers at the periphery are distributors of liquidity (immediacy) from core dealers to end-user investors. Compared to core dealers, periphery dealers are more likely to pre-arrange trades between a more central dealer and an end-user investor rather than matching trades between two end-user

[1]Examples of empirical literatures on OTC trading network include Li and Schürhoff (2019) for municipal bonds market, Di Maggio et al (2017) for corporate bonds market, Bech and Atalay (2010) and Afonso and Lagos (2014) on interbank market, and Hollifield et al (2017) for securitization market.

investors or taking positions to provide liquidity.[2] According to Li and Schürhoff (2019),

> "...consistent with central dealers providing more immediate execution and getting compensated for doing so, we find that central dealers match buyers with sellers more directly. After purchasing a bond from an investor, a central dealer is more likely to sell it to the end-buyer than to off-load it to another dealer...central dealers are more likely to offer immediacy by trading on a principal basis, that is, by taking bonds into inventory, than to prearrange trades between a buyer and a seller, which takes time to execute...we find that bonds flow from periphery to center and partially back. Dealers in the middle of a chain are more central than either the dealer purchasing the bond from the customer or the dealer ultimately selling the bond to the customer. Central dealers thus act as hubs by redistributing the order flow."

This finding raises a few interesting theoretical questions. Without a physical barrier that prohibits investors to contact core dealers directly and shorten the intermediation chain,[3] why do some investors choose to trade with periphery dealers? To be specific, given superior dealer network position and an ability to exercise price discrimination of core dealers, why can periphery dealers compete and co-exist with core dealers? We aim to shed light on this issue by providing a new theory on strategic dealer choice of buy-side investors which explains why periphery dealers can compete for order flows with core dealers.

Our theory is based on a premise that buy-side investors form a non-binding long-term trading relationship with sell-side dealers to obtain liquidity (immediacy). Relationship establishment among market participants is common in financial markets due to their repeated interactions.[4] Because OTC markets are search-based markets which involve investors searching for a dealer (or vice versa) who is willing to trade at the best price, an ability to maintain contact with customers is valuable for dealers.[5] One competitive strategy of dealers is to form a non-binding long-term relationship with investors by offering special benefits, such as price discount or inside information, to the investors who frequently trade with them.[6] This paper considers a type of long-term relationship in which dealers offer liquidity insurance – a promise to execute trade orders on the spot at an affordable price (i.e. immediacy provision) during illiquid periods – to their loyal clients. We will show that investors might strategically choose periphery dealers, because they expect to receive higher future benefits from trading with the periphery dealers than trading with core dealers.

We develop a game-theoretic infinite-period model of a dealer market for liquidity service, the service to execute trade order on the spot (i.e. immediacy). In short, the model consists of *a principal dealer* who can produce indivisible units of costly liquidity service, *a competitive*

[2] Similar to Li and Schürhoff (2019), Hollifield et al (2017) documented that periphery dealers are more likely to pre-arrange trades compared to core dealers who mostly execute principal trades in securitization market.

[3] Note that this situation is different from public exchanges, where only exchange members can participate in trading, thus the investors must trade via brokerage firms which justify the existence of brokers.

[4] A few examples include order preferencing agreement between brokers and specialists (Benveniste et al (1992), Harris (2002)) or among dealers in NASDAQ (Dutta and Madhavan (1997)), information sharing between brokers and investors (Di Maggio et al (forthcoming)), relationship lending in interbank markets (Afonso et al (2014)) and repo markets (Han and Nikolaou (2016)).

[5] As mentioned in Li and Schürhoff (2019), "Locating bonds and potential buyers thus requires that financial intermediaries have active relationships with various types of investors as well as with other dealers."

[6] Hendershott et al (2017) also found evidence of client-dealer relationships in corporate bonds market and confirmed the impact of long-term relationship on execution costs.

*principal dealer* who always provides liquidity service at a competitive price, *a group of homogeneous investors* who face i.i.d. random liquidity shocks and will demand one unit of liquidity service from a principal dealer, and *an agency dealer* who cannot produce liquidity service but can intermediate trades between principal dealers and investors.[7] Indeed, the principal dealer represents dealers at the core and the agency dealer represents dealers at the periphery in the dealer network. There are two random states which determine the principal dealers' costs of providing liquidity service: good states in which the cost is low and bad states in which the cost is high. All agents are long-lived and can form a non-binding relationship among each other. We apply the Folk theorem to study incentive of all players to enter a non-binding long-term relationship and characterize optimal dealer choice of the investors.

Our setup exhibits two main features which capture trade frictions in OTC markets. The first one is cash constraint problem, in which the investors do not have enough cash to pay for costly liquidity service in bad states. In reality, when providing immediacy, a principal dealer may incur high inventory risk in some periods, particularly for trade orders of older, seasoned, or illiquid security. Therefore, even though investors might value immediacy much higher than liquidity cost of the principal dealer, the investors are unable to get immediacy if they do not have enough cash to pay now. The second feature is imperfect information problem, in which the investors do not have any information about trade or liquidity shocks of other investors, capturing the current state of opacity within OTC markets.[8]

The model gives three main insights. First, the investors must have frequent liquidity shocks to successfully form long-term relationship with the principal dealer directly. Specifically, due to the cash constraint problem, the investor and the principal dealer make a non-binding agreement, in which the dealer will provide liquidity service at an affordable price in bad states as long as the investor continues trading with the dealer and pays a premium in good states. If the relationship formation is successful, both parties will obtain higher surplus: the investor secures liquidity in future bad states and the principal dealer earns higher profit from future order flows. However, to form a relationship, the investor must bring sufficiently high future benefits (i.e. high frequency of liquidity needs) to the principal dealer, so that the dealer would commit to incur an upfront loss of liquidity provision in bad states. Similarly, the investor must also face frequent liquidity shocks to willingly trade with the principal dealer in good states. If the investor has infrequent liquidity shocks, either 1) the dealer will strategically refuse to provide costly liquidity when a bad state comes or 2) the investor will not trade with the principal dealer when a good states come. This two-sided commitment problem creates a non-physical barrier against the investors to establish the relationship with the principal dealer and obtain liquidity in bad states if they have infrequent liquidity shocks.

The second insight is that an agency dealer can attenuate the commitment problem and sustain the relationship between the investors and the principal dealer. Intuitively, the agency dealer can form non-binding relationships with *several* investors and aggregate frequency of

[7]This assumption on exogenous business model of dealers allows us to examine the dealer choice of the buy-side investors in a tractable framework. There are several literatures that focus on the sell-side behavior and try to explain who will emerge as core or periphery dealers. See literature review for more details.

[8]Currently, several OTC markets require only dissemination of post-trade information *without trader identity* to the public. Thus, the investors are unable to identify trading patterns of other investors. Note that this informational friction prevents the investors from forming relationships among each other.

liquidity needs to be sufficiently high. Then, the agency dealer forms another non-binding relationship with the principal dealer on behalf of all the investors and agrees on *liquidity quota,* the minimum per-period amount of liquidity service that the principal dealer must provide in bad states. Setting low liquidity quota will increase the principal dealer's incentive to commit to costly liquidity provision in bad states. This is because, to continue receiving future order flows of *all* the investors who connect with the agency dealer, the principal dealer only needs to incur relatively low upfront cost of costly liquidity provision in bad states when the quota is low. Therefore, if the investors have infrequent trading needs, they will trade with the agency dealer to obtain the benefit of liquidity insurance rather than trading with the principal dealer for a one-period benefit.

Our third insight is that, when connecting with agency dealer, investors can still face liquidity shortage if too many investors simultaneously demand liquidity from the agency dealer in bad states. This is because the agreed liquidity quota must be low enough for the principal dealer to commit to the relationship. This finding sheds light on an empirical finding on a tradeoff between execution cost and trading speed (Li and Schürhoff (2019)). From our model, there is a tradeoff between execution cost and the degree of liquidity insurance coverage when an investor selects a dealer: a relationship with principal dealer offers full liquidity insurance with expensive premium while a relationship with agency dealer offers partial liquidity insurance with cheaper premium. In appendix, we extend the model to include investors with heterogeneous frequency of liquidity shocks and show that this tradeoff is applicable to only frequent-liquidity-need investors who can form direct relationship with principal dealer. However, due to the commitment problem, investors who have infrequent liquidity shocks can obtain only the latter deal, granting market power to the agency dealer (at the periphery) to co-exist with principal dealer (at the core).

Our finding has important implications on market efficiency and stability within OTC secondary markets. Contradicting to traditional view on long intermediation chain as a source of allocative inefficiency, our model suggests that an agency dealer who intermediates trade between a main liquidity-providing principal dealer and a group of investors can improve allocative efficiency and market liquidity. However, in terms of market stability as measured by the likelihood of first-trigger event of systemic crisis, the impact of agency dealers is ambiguous. On the one hand, they reduce liquidity risk of existing small investors who cannot obtain liquidity in bad states without agency dealers. On the other hand, the presence of agency dealers can attract more small investors to enter the market and start investing in the security. However, these investors are highly subjected to liquidity shortage when several investors demand liquidity simultaneously. Depending on further knock-on effects from the first-round liquidity dry-up among the investors, emergence of agency dealers can cause an alarming concern to market regulators.

### Related literature

Our paper relates to the literature of endogenous intermediation in over-the-counter markets. A group of literatures adopted search and bargaining model which involves atomistic players randomly matched among each other to trade assets.[9] To illustrate, Hugonnier et al (2016) considered a search model in which investors have heterogenous valuation and found that those with moderate valuation will emerge as intermediaries. Üslü (2019) considered a case when investors have heterogenous search intensity and established that those with high search intensity will emerge as intermediaries. Farboodi et al (2017) relaxed the assumption of exogenous search intensity. Afonso and Lagos (2015) developed a search model to study fed funds market and show that the equilibrium exhibits endogenous intermediation even though all players are identical. While these literatures can capture several important features of over-the-counter asset markets, this random matching model might not be suitable for studying relationship-based OTC markets in which market participants repeatedly interact with each other. This paper fills in the gap by studying impact of long-term relationship between dealers and end-user clients on OTC market structure.

Our paper also belongs to a growing literature of financial network formation. Farboodi (2017) studied formation of credit lending network in which players form a link to manage their counterpart risks. The mechanism is different from our paper which focuses on liquidity management in secondary asset markets. Chang and Zhang (2019) developed a dynamic matching model in which the matching process is endogenous and traders have heterogeneous volatility of private asset valuation. They established that the equilibrium network exhibits a multi-layered hierarchy core-periphery structure in which investors with relatively low volatility emerge as intermediaries and investors with relatively high volatility act as end-user clients. This is different from our paper which focuses on strategic dealer choice of end-user investors through the lens of long-term relationships. Similar to ours, Hollifield et al (2017) developed a theoretical model to study strategic dealer choice of buy-side investors. Different from our findings, they established that some investors choose periphery dealers because the investors are indifferent choosing between core and periphery dealers; the dealers have full bargaining power and capture the whole trade surplus. Wang (2017) considered strategic long-term relationship formation among dealers who have repeated interactions and explained how competition and inventory management strategy among dealers can endogenously create core-periphery dealer network. In Wang (2017), all dealers have exogenous customer bases. While this assumption can represent some wholesale markets such as interbank markets well, the effect of dealer choice of end-user investors on market structure in some secondary asset markets, such as fixed-income security markets, is unclear. We contribute to this literature by studying strategic dealer choice of end-user investors.

The closest literature to our paper is Neklyudov and Sambalaibat (2017), which developed a matching model consisting of dealers and end-user clients. Similar to this paper, Neklyudov and Sambalaibat (2017) found that traders with frequent liquidity needs choose core dealers to obtain immediacy in the future. Traders with infrequent liquidity needs choose periphery dealers since they can obtain better price. However, the underlying theory in Neklyudov and Sambalaibat (2017) is different from ours. The main assumptions in Neklyudov and Sambalaibat (2017) are 1) that end-user clients must always trade with the dealer whom they choose at the beginning, 2) that pricing is non-strategic (Nash bargaining) and 3) that either a) order matching

[9]A few literatures on over-the-counter asset markets with random matching model include Duffie et al (2005,2007), Lagos et al (2011), Vayanos and Weill (2008), Lagos and Rocheteau (2009).

capability of dealers when the match involves two dealers is sufficiently more efficient than that with one dealer or(and) b) bargaining power of the clients when trading via longer dealer chain is sufficiently larger than that when trading via one dealer. Under these assumptions, the periphery dealers can survive because surplus that the clients receive from the periphery dealer can be higher than trading with core dealers. This is different from our paper mainly in two ways. First, we apply the Folk theorem which allows the cost of forming client-dealer relationships to be endogenous and explain why the investors must trade with specific dealers repeatedly. Second, our result suggests that investors with infrequent liquidity needs choose periphery dealers because they cannot form relationships with core dealers due to commitment problem.

In finance literature, the theory of financial intermediary and social pressure in repeated game setting with commitment limitation has been applied to credit payment enforcement.[10] Babus and Hu (2017) and Fainmesser (2019) discussed the role of financial intermediaries on solving commitment problem of unsecured debt contract, in which borrowers may strategically default or not pay the debt. In their models, the financial intermediary can solve the commitment issue by threatening to exclude bad borrowers from future interactions, prohibiting them to meet several lenders who are clients of the intermediary in the future. Our paper contributes to this literature by studying the role of intermediary who solves commitment problem of costly liquidity provision in secondary asset markets.

### Structure of the paper

The rest of the paper is structured as follows. Section 2 outlines the basic model. Section 3 discusses the equilibrium without long-term relationship. Section 4 introduces a notion of liquidity insurance relationship and characterizes existence condition. Section 5 introduces an agency dealer to the basic model. Section 6 discusses the role of agency dealer in the equilibrium. Section 7 provides additional discussion of the results. Section 8 concludes. An analysis of the model with heterogeneous investors and omitted proofs are in appendix.

## 2.2 Basic model

Consider an infinite-period game consisting of a principal dealer $P$ who can provide indivisible units of liquidity service, a competitive principal dealer $P'$ who non-strategically provides liquidity service at a competitive price, and a set of homogeneous investors $I = \{1, 2, \dots, N\}$ with $|I| = n$ who face random liquidity demands.[11] The liquidity service refers to dealer service to execute a trade order on the spot by taking positions (i.e. immediacy). Assume that $n$ is common knowledge. Time is discrete and all players have discount factor $\delta$.

There are binary observable market states $\theta_t \in \{G, B\}$ with $Pr(\theta_t = G) = p$ which determine the cost of liquidity provision of both principal dealers $\{P, P'\}$. Specifically, the

[10] See Greif et al (1994) and Fainmesser (2019) for more references.

[11] An implicit assumption in the model is that there is no asymmetric information about the asset fundamentals across all players.

per-unit cost of liquidity provision of all principal dealers at time $t$ is

$$C_t = \begin{cases} 0 & \text{if } \theta_t = G \\ C & \text{otherwise.} \end{cases}$$

This assumption on the cost structure reflects liquidity condition in the dealer market in each period. When $\theta_t = G$ (i.e. good state), principal dealers expect to find another counterparty to offload their positions without much effort, and thus the cost of providing liquidity is null. The opposite case is when $\theta_t = B$ (i.e. bad state) in which the dealers expect to incur high inventory cost or high effort cost when offloading their positions.

In each period, investors face i.i.d. stochastic demand of liquidity service $l_{it} \in \{0, 1\}$ with $Pr(l_{it} = 1) = q$. Denote $Q = \frac{q\delta}{1-\delta}$. Private valuation of liquidity service of investor $i$ is

$$V_{it} = \begin{cases} 0 & \text{if } l_{it} = 0 \\ V & \text{otherwise.} \end{cases}$$

Note that this valuation does not reflect the asset fundamental value per-se. Instead, it indicates investors' private benefit of obtaining liquidity service (i.e. the benefit of fast trade execution). In each period, all investors have cash endowment of $V_L$ which is not transferable across periods. We make the following assumption.

**Assumption 2.1 (Trade friction)** $V > C > V_L$

This assumption specifies that trade should always take place as the investors' valuation of liquidity service is always higher than liquidity-providing cost of dealers. However, the investors do not have enough cash to cover high cost of liquidity provision of dealers in bad states.

## Timeline

For every period $t \geq 1$, the game runs as follows. At the beginning of period $t$, all players observe market state $\theta_t$ and investors observe their own liquidity shocks $l_{it}$. Then, the investors contact principal dealer $P$ who will quote $\beta_{it} \in \mathcal{R}_+$ to all investors $i \in I$. After observing their own quotes, all investors decide whether to accept or reject the quote, denoted by $\gamma_{it} \in \{0, 1\}$. If investor $i$ accepts the quote ($\gamma_{it} = 1$), dealer $P$ will provide liquidity service to investor $i$ and receive cash $\beta_{it}$. If rejecting the quote, investor $i$ will obtain competitive price from dealer $P'$ which will give investor $i$ an outside option of $max\{0, V_{it} - C_t\}$.[12] After trade settlement, dealer $P$ observes liquidity shock of all investors $l_t = \{l_{it}\}_{i=1}^{N}$, and the game moves to the next period. Assume that the investors cannot observe neither actions, principal dealer's quotes, nor liquidity shocks of other investors throughout the game, reflecting the current limited market transparency situation of several OTC markets.[13] Formally, per-period payoff of investor $i$ ($u_{it}$) and principal dealer $P$ ($\pi_t$) for any period $t \geq 1$ satisfy

$$u_{it} = \gamma_{it}(V_{it} - \beta_{it}) + (1 - \gamma_{it})max\{0, V_{it} - C_t\}, \qquad \pi_t = \sum_{i \in I} \gamma_{it}(\beta_{it} - C_t).$$

[12] This assumption implies that 1) market condition $\theta_t$ is homogeneous across dealers which represents frictionless inter-dealer market where principal dealers can easily search and obtain liquidity among each other to realize full trade surplus and 2) investors have perfect information about which principal dealers have liquidity in each period.

[13] This is equivalent to the standard assumption of private monitoring in the repeated game literature.

### Histories, information sets, and strategies

Let $\beta_t = \{\beta_{it}\}_{i=1}^N$ and $\gamma_t = \{\gamma_{it}\}_{i=1}^N$ be the sequence of actions of principal dealer $P$ and all investors, respectively. Let the set of histories at the beginning of period $t$ be

$$H_t = \bigcup_{j=1}^{t-1} [l_j \times \theta_j \times \beta_j \times \gamma_j]$$

which consists of market states, liquidity shock realization of all investors, price quotes of the dealer to all investors, and the decision of investors up to period $t-1$. The information set of principal dealer $P$ $(h_{Pt})$ and investor $i \in I$ $(h_{it})$ at the beginning of period $t$ are

$$h_{Pt} = H_t \qquad \text{and} \qquad h_{it} = \bigcup_{j=1}^{t-1} [l_{ij} \times \theta_j \times \beta_{ij} \times \gamma_{ij}] \, .$$

The strategy of principal dealer $P$ and all investors $i \in I$ are the functions which map the information set at the beginning of period $t$ and additional information observed in period $t$ to an action set such that

$$\beta_t(\theta_t, h_{Pt}) \in \mathcal{R}_+^n \qquad \text{and} \qquad \gamma_{it}(\theta_t, l_{it}, \beta_{it}, h_{it}) \in \{0,1\}.$$

The solution concept in this paper is the standard perfect Bayesian equilibrium.

## 2.3 Static equilibrium

To understand the role of long-term relationship, I will first characterize an equilibrium in a static setting. It is trivial to see that there exists an equilibrium in which trade occurs only in good states as established in the following proposition.

**Proposition 2.1 (Static equilibrium)** *There exists an equilibrium in which for any investor $i \in I$,*

$$\beta_{it}^* = \begin{cases} 0 & \textit{if } \theta_t = G \\ C & \textit{otherwise} \end{cases}, \qquad \gamma_{it}^*(\beta_{it}) = \begin{cases} 1 & \textit{if } \beta_{it} < C_t < V_{it} \\ 0 & \textit{otherwise} \end{cases}$$

This result highlights a cash constraint problem in bad states. As shown in Figure 2.1, the investors cannot buy liquidity service in bad states due to limited cash. In reality, there are two possible scenarios, depending on whether the investors would like to buy or sell an asset. If investors would like to buy an asset, they will not be able to buy immediately, causing allocative inefficiency. However, if investors would like to sell an asset, they are still able to sell immediately but at a cheap price. To illustrate, if $F$ is a fair price of the asset, then the dealer would buy at the maximum of $F - c$ to compensate for costly liquidity service. However, the investor would like to sell at the minimum of $F - V_L$ to meet his urgent need of cash. In this situation, the investor has two options: 1) do not sell the asset or 2) sell the asset at $F - c$ now. The former choice creates allocative inefficiency while the latter choice is known for creating fire-sale phenomenon which is a threat to market stability. In any event, the possible failure of obtaining liquidity service motivates investors to form a long-term relationship with the principal dealer, inter-temporally hedging against their liquidity needs.


Figure 2.1: Static equilibrium with no trade during bad states.

## 2.4 Liquidity insurance relationship

This section will discuss what liquidity insurance relationship is, why it exists, and how it works. The following definition formally describes the notion of liquidity insurance relationship.

**Definition 2.1 (Liquidity insurance relationship)** *Provided that $max\{x_G, x_B\} \leq V_L$, an equilibrium exhibits liquidity insurance relationship between investor $i$ and principal dealer $P$, if the equilibrium outcome for every $t$ is such that*

$$\beta^*_{it} = \begin{cases} x_G & \text{if } \theta_t = G \\ x_B & \text{if } \theta_t = B \end{cases}, \qquad \gamma^*_{it}(\beta_{it}) = \begin{cases} 1 & \text{if } \beta_{it} \leq \beta^*_{it} \leq V_{it} \\ 0 & \text{otherwise.} \end{cases}$$

By definition, liquidity insurance relationship requires the dealer to provide liquidity service at price $x_B \leq V_L$ during bad states in exchange of all future trade orders of the investor, as illustrated in Figure 2.2. To compensate for the dealer loss in bad periods, the investor pays a premium $x_G$ to the dealer when the state is good. Indeed, this informal relationship allows the investor to hedge his liquidity needs across periods. In this way, both parties can obtain higher surplus; the dealer earns more profits in good states and the investor obtains liquidity service in bad states. This relationship will continue as long as 1) the dealer continues providing liquidity as agreed in every period and 2) the investor continues buying liquidity service from the dealer as agreed in every period.[14] Any deviation will result in reversion to a no-relationship outcome in subsequent periods in which both players obtain their outside options.


Figure 2.2: Liquidity provision of $P$ under liquidity insurance relationship.

Under what conditions would the players successfully form the relationship then? The answer lies upon whether they can reach an agreement on $(x_G, x_B)$ which determines allocation of the trade surplus. Because of no legal enforcement on the agreement, the relationship can exist if both parties obtain sufficiently high surplus in the relationship that they are willing to commit to the relationship. For the dealer, $x_G$ must be sufficiently high that he is willing to incur upfront loss in bad states to continue reaping future benefits in good states of the relationship.

[14]Note that there are several possible punishment strategies to sustain the relationship. However, this trigger strategy is the most severe one, which will give us the weakest condition for the relationship equilibrium to exist.

On the other hand, $x_G$ must be sufficiently low that the investor is willing to pay the premium during good states, instead of strategically opting out from the relationship and choose $P'$, to continue the relationship and obtain liquidity insurance during bad periods. This two-sided commitment problem is shown in the following lemma.

**Lemma 2.1 (Insurance premium)** *In any equilibrium,*

1. *principal dealer $P$ will commit to the liquidity insurance relationship if and only if*

$$x_G \geq \frac{1+Q(1-p)}{Qp}(C - x_B),$$

2. *investor $i$ will commit to the liquidity insurance relationship if and only if*

$$x_G \leq \max\left\{\frac{Q(1-p)}{1+Qp}(V - x_B), V_L\right\}.$$

The lemma highlights how feasible range of the premium depends on $Q$, the investor's trading frequency weighted by discount factor. On the one hand, when the investor has low future trading needs, reflected by low $Q$, he will request lower insurance premium since he would not get much benefit from future liquidity coverage. Also, such premium must be feasible to pay during good states (i.e. $x_G \leq V_L$). On the other hand, the dealer will demand high premium when the investor has low future trading needs, because the investor will rarely bring future order flows during good states. Therefore, the investor must have sufficiently high liquidity needs to successfully reach the agreement with the dealer, as confirmed in the following proposition.

**Proposition 2.2 (Equilibrium existence)** *A liquidity insurance equilibrium between investor $i$ and principal dealer $P$ exists if and only if $V_L > (1-p)C$ and*

$$Q \geq max\left\{\frac{1+\sqrt{1+4p(1-p)(V-C)}}{2p(1-p)(V-C)}, \frac{C - V_L}{V_L - (1-p)C}\right\}.$$

This proposition is intuitive. When the investors have low trading needs, they will obtain low benefit from the relationship and will not commit to paying the premium to the dealer. For the same reason, the dealer will not commit to providing costly liquidity during bad states as he expects to reap low future benefit from the investor. This implies that the investors suffer from the commitment problem and are excluded by the principal dealer, despite possibility of higher payoffs for both parties under the agreement, when the investors have infrequent liquidity needs.

## 2.5 The model with an agency dealer

As discussed previously, lack of legal enforcement on the relationship contract and the commitment problem cause relationship failure when the investors have infrequent liquidity needs. This commitment problem exists because a threat by individual investors to terminate the relationship is not significant enough to affect the principal dealer. However, what if the investors collectively form the relationship with a principal dealer as a group? That is, if the dealer violates the agreement with an investor in the coalition, *all* the investors would collectively terminate the relationship. As shown in panel (b) in Figure 2.3, when the investors form a coalition, they can pool their liquidity needs, which will enlarge contract space that they can negotiate with the principal dealer, and collectively punish the principal dealer when needed. With larger set of contracting space and collective punishment, possibility of reaching a long-term agreement increases.

Unfortunately, such coalition formation is not possible due to lack of necessary information for collective punishment. To form a coalition, the investors must have information of other investors' trading activities to monitor and collectively punish the principal dealer. However, the investors cannot observe neither actions, liquidity shocks, nor the principal dealer's quotes of other investors in the OTC markets.

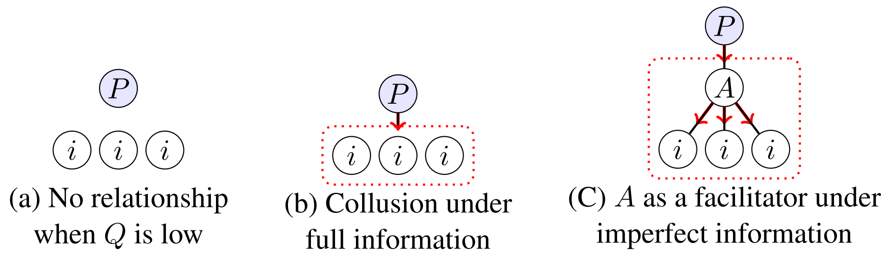


Figure 2.3: Illustration of the role of an agency dealer

This gives rise to emergence of an agency dealer who can aggregate liquidity needs of the investors, act on behalf of all the investors and the principal dealer, and punish the principal dealer for all the investors when needed. This is shown in panel (c) in Figure 2.3. In this case, the investors only need to monitor and maintain a long-term relationship with the agency dealer which does not require the investors to have perfect information about trades of other investors. Also, a threat imposed by the agency dealer towards the principal dealer to terminate the relationship would represent a collective threat from all the investors. Therefore, if all players can reach an agreement which provides correct incentives to the agency dealer, the relationship formation can be successful.

In this paper, we will prove that agency dealer can indeed solve the commitment problem and help investors obtain costly liquidity in bad states when the investors have infrequent liquidity needs. We start with formal description of the model with agency dealer in this section. In the next section, we introduce the notion of liquidity insurance relationship via agency dealer, characterize existence condition and highlight important implications of market fragmentation on market efficiency and stability.

### Model setting

Consider an infinite-period baseline model which now consists of an agency dealer $A$ who can intermediate trades between a principal dealer and the investors. The agency dealer has no cost of intermediation but will incur indefinite inventory cost if producing liquidity service. To solve for the interesting case which the agency dealer solves the commitment problem and not the problem of cash constraint in good states, we assume that $V_L > (1-p)C$ and $Q \geq \frac{C-V_L}{V_L-(1-p)C}$.

### Timeline

For every period $t \geq 1$, the game runs as follows. At the beginning of period $t$, all players observe market state $\theta_t$ and all investors observe their own liquidity shocks. All investors cannot observe liquidity shocks of other investors. Let $I_t = \{i|l_{it=1}\}$ be the set of active investors with $|I_t| = n_t$. All active investors $I_t$ contact agency dealer $A$ who in turn contacts principal dealer $P$ to get a quote.[15] Inactive investors $i \notin I_t$ realize payoff $0$.

First, principal dealer $P$ quotes $(\beta_{At}, m_t) \in \mathcal{R}_+ \times \mathcal{I}_+$ to agency dealer $A$ which specifies per-unit upstream price of liquidity $\beta_{At}$ and maximum quantity of liquidity $m_t$ that he is willing to supply at price $\beta_{At}$. If $m_t < n_t$, principal dealer $P$ will accommodate the residuals $n_t - m_t$ at competitive price $C_t$.

After obtaining the quote, agency dealer $A$ chooses $(d_{At}, \beta_t) \in \{0,1\} \times \mathcal{R}_+^{n_t}$ which decides whether to accept the quote ($d_{At} = 1$) from principal dealer $P$ and what prices to quote to all active investors denoted by $\beta_t = \{\beta_{it}|\forall i \in I_t\}$. If rejecting the principal dealer's offer ($d_{At} = 0$), he will get a competitive upstream price quote $C_t$ from principal dealer $P'$ and obtain an additional outside option $W_{At}$.[16] Let $\hat{\beta}_{At} = d_{At}\beta_{At} + (1 - d_{At})C_t$ be the actual price that agency dealer $A$ pays to principal dealer $P$ for trade orders covered within the supply quantity $m_t$.

After observing retail price quote $\beta_{it}$, each investor $i \in I_t$ chooses $\gamma_{it} \in \{0,1\}$ to indicate whether to accept the offer. If rejecting the offer, the investors obtain outside option of $max\{0, V - C_t\}$. Afterwards, agency dealer $A$ reports final trade demands $\sum_{i \in I_t} \gamma_{it}$ to principal dealer $P$ to settle all trades.[17]

At the end of period $t$, after trade settlement, principal dealer $P$ observes liquidity shocks and retail prices of all investors $\{l_t, \beta_t\}$ and all active investors $I_t$ observe the actual upstream price $\hat{\beta}_{At}$, and the game continues to the next period.[18] Formally, the per-period payoff of active investor $i \in I_t$ ($\pi_{it}$), of principal dealer $P$ ($\pi_{Pt}$), and of agency dealer $A$ ($\pi_{At}$) satisfy

$$\pi_{Pt} = d_{At} \left( \sum_{i \in I_t} \gamma_{it} \right) (\beta_{At} - C_t)$$

$$\pi_{At} = \sum_{i \in I_t} \gamma_{it}\beta_{it} - \min \left\{ m_t, \sum_{i \in I_t} \gamma_{it} \right\} \hat{\beta}_{At} - \max \left\{ 0, \sum_{i \in I_t} \gamma_{it} - m_t \right\} C_t + (1 - d_{At})W_{At}$$

$$\pi_{it} = \gamma_{it}(V - \beta_{it}) + (1 - \gamma_{it})max\{0, V - C_t\}.$$

[15] This implies that agency dealer $A$ knows the liquidity shock realization of all investors before contacting $P$.

[16] Note that this assumption is without loss of generality, as it is always optimal for agency dealer $A$ to fully route the order to either $P$ or $P'$. When routing some orders to $P'$ and some to $P$, $P$ can detect that deviation in the next stage which will trigger punishment in subsequent periods. Thus, it is always optimal for $A$ to fully route the orders to $P'$ if deciding to defect in the first place. Also, note that $A$ has no incentive to renegotiate with $P$ to get higher payoff by lowering liquidity insurance, as the investor will also detect such renegotiation ex-post through a higher markup.

[17] This simplifies the game as in reality $A$ can strategically report the wrong number of liquidity demand during good state. However, relaxing this assumption does not matter, as agency dealesr $A$ has no incentive to direct partial trade orders to $P'$ anyway.

[18] The assumption on ex-post $l_t$ observability of principal dealer $P$ simplifies our analysis to the case of perfect monitoring on the behaviour of all investors.[19] Also, this assumption of ex-post trading price observability reflects the current state of limited post-trade transparency in OTC markets, which allows all players to observe additional information after trade settlement. We will discuss in detail about this information structure in the next part.

### Histories, information sets, and strategies

Denote $A_{kt}$ the set of actions taken by player $k \in \{P, A\} \cup I_t$ in period $t$. Information set of each player at the beginning of period $t$ consists of a sequence of observable past actions and state realization up until period $t-1$. We assume that all players can observe only their neighbours' actions who have direct interaction with them at the time of trading, but they can observe additional information ex-post.[20] Specifically, let $h_{kt} \subset H_t$ be the information set of player $k$ at the beginning of period $t$ such that

$$h_{kt} = \begin{cases} \bigcup_{j=1}^{t-1} \left[\times_{k\in\{P,A\}} A_{kj} \times \left\{\sum_{i\in I_j} \gamma_{ij}\right\} \times [l_j \times \theta_j]\right] & \text{if } k = P \\ \bigcup_{j=1}^{t-1} \left[\times_{k\in\{P,A\}\cup I_j} A_{jt} \times [l_j \times \theta_j]\right] & \text{if } k = A \\ \bigcup_{j\in T_{kt}} \left[A_{kj} \times \beta_{kj} \times \hat{\beta}_{Aj}\right] \times \bigcup_{j=1}^{t-1} [l_{kj} \times \theta_j] & \text{if } k \in I \end{cases}$$

where $T_{kt} = \{j | k \in I_j, \ \forall j \le t-1\}$ is the set of active periods of investor $k$ up to period $t-1$.

In simple words, everyone knows all past realizations of market state. Principal dealer $P$ knows 1) past actions of himself and of the agency dealer, 2) past liquidity shocks of all investors, and 3) total number of executed trades ($\sum_{i\in I_j} \gamma_{ij}$) in the past. The agency dealer has perfect information about the game. The investor $i$ knows 1) his own past liquidity shocks, 2) his own past actions, 3) past quotes he obtained from agency dealer $A$ when he is active, and 4) past upstream prices of his own trades.

We consider only pure strategy equilibrium. Specifically, the strategy of principal dealer $P$, agency dealer $A$, and investor $i \in I$ for any period $t$ maps the information that each player knows at the time of trading to an action set such that

1. for principal dealer $P$ who quotes upstream price and sets supply quantity:

$$(\beta_{At}, m_t)(\theta_t, h_{Pt}) \in \mathcal{R}_+ \times \mathcal{I}_+$$

2. for agency dealer $A$ who decides whether to accept $P$'s offer and determines downstream prices:

$$(d_{At}, \beta_t)(\theta_t, \beta_{At}, m_t, h_{At}, l_t) \in \{0, 1\} \times \mathcal{R}_+^{n_t}$$

3. for investor $i \in I$ who decides whether to accept $A$'s offer:

$$\gamma_{it}(\beta_{it}, l_{it}, \theta_t, h_{it}) \in \{0, 1\}.$$

The solution concept used in this paper is the standard perfect Bayesian equilibrium.

### Discussion on the model

Recall that the main cause, which prevents investors to pool their liquidity demands and negotiate as a group with the principal dealer directly, is lack of information about other investors' trade. This is reflected in the assumption that the investors can only observe information of their own

[20] This is similar to the current state of OTC markets, as post-trade transparency regulation which requires publication of past trading details (price and quantity) allows the market participants to obtain the additional information ex-post.

transactions but not others. For the collusion among investors to succeed, all the investors must be able to **simultaneously** punish the principal dealer when appropriate. Without the information to monitor the principal dealer properly, collective punishment is impossible.

From the model, there are two important features when trading via agency dealer. First, a threat by the agency dealer toward the principal dealer represents a threat from all the investors. Such threat can substitute the collective punishment of all the investors, provided that the agency dealer has the incentive to punish the principal dealer. Second, even though investors do not have information about other investors, they now have information about upstream prices of their trades ex post. If the upstream price is contingent on whether the principal dealer and the agency dealer are still in the agreement, then information about upstream price is sufficient for the investors to monitor both dealers and assess whether the relationship is still ongoing. These two features, as we will see in the next section, contribute to the success of relationship formation.

## 2.6 Relationship via an agency dealer

We will begin our analysis by formally defining the notion of liquidity insurance relationship via agency dealer. Different from our baseline case, it is a non-binding agreement between a principal dealer and a group of investors with an intervention of agency dealer who can make profits from his intermediation service, as described in the following definition.

**Definition 2.2 (Liquidity insurance equilibrium via agency dealer)** *Provided that* $max\{x_G+F_{iG}, x_B+F_{iB}\} \leq V_L$ *for all* $i \in I$ *and* $n^* \leq n$*, an equilibrium exhibits liquidity insurance relationship among* $\{P, A, I\}$ *if the equilibrium outcome in every period* $t \geq 1$ *is such that*

$$(\beta^*_{At}, m^*_t) = \begin{cases} (x_G, n) & \text{if } \theta_t = G \\ (x_B, n^*) & \text{otherwise} \end{cases}$$

$$(d^*_{At}, \beta^*_t) = \begin{cases} (1, \{x_G + F_{iG}\}_{i=1}^{n_t}) & \text{if } \theta_t = G \\ (1, \{x_B + F_{iB}\}_{i=1}^{n_t}) & \text{if } \theta_t = B \text{ and } n_t \leq n^* \\ (1, \{x_B + F_{iB}\}_{i=1}^{n^*} \cup \{C + F_i\}_{i=n^*+1}^{n_t}) & \text{if } \theta_t = B \text{ and } n_t > n^* \end{cases}$$

$$\gamma^*_{it} = \begin{cases} 1 & \text{if } \beta_{it} \leq \beta^*_{it} \leq V_L \\ 0 & \text{otherwise} \end{cases}$$

From the definition, investors form a relationship with agency dealer $A$, who will in turn form liquidity insurance relationship on their behalf with principal dealer $P$. In the agreement, principal dealer $P$ trades with agency dealer $A$ at per-unit price $x_G$ in good states for unlimited quantity and at per-unit price $x_B$ for maximum quantity $n^* \leq n$ in bad states. Thus, $n^*$ determines the extent of insurance coverage from principal dealer $P$. If $n^* = n$, we say that the insurance is *unconstrained* which guarantees *full* insurance to all investors. If $n^* < n$, the insurance is *constrained* in which the investors get *partial* insurance coverage and face liquidity shortage in some bad periods.[21] Agency dealer $A$ accepts the offer and quotes the agreed retail

[21]Note that if one assumes arbitrary distribution of liquidity shock, $n^* < n$ can provide full insurance to all investors if the probability that more than $n^*$ investors would demand liquidity at the same time is zero.

price $\beta^*_{it}$ to investors with price markup $(F_{iG}, F_{iB})$. For the investors, they always accept offer from agency dealer $A$, unless the price exceeds $V_L$. This idea is illustrated in Figure 2.4.

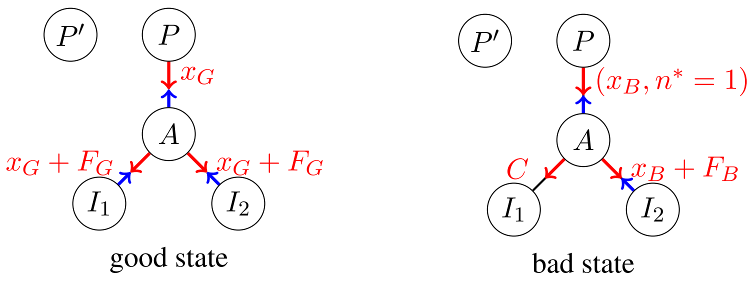


Figure 2.4: Illustration of an equilibrium outcome under liquidity insurance relationship via agency dealer when $n^* = 1$.

It is worth noting that pooling liquidity needs of all the investors enlarges contract space of choosing $m^*_t$, the liquidity quota, against the number of investors. With bilateral relationship between $P$ and $i$, liquidity quota in both states is fixed (i.e. $m^*_t = n = 1$). However, when there are several investors, the choice of liquidity quota is richer (i.e. $m^*_t(\theta_t = G) = n$ and $m^*_t(\theta_t = B) = n^* \leq n$). The ability to adjust the liquidity quota $n^*$ (and $\frac{n^*}{n}$), which affects incentives to form the relationship of both $P$ and $I$, from liquidity pooling is the main key for successful relationship formation.

Lastly, to reflect how principal dealers in relationship-based OTC markets compete for relationships with agency dealers in practice, we make an additional assumption on outside option of agency dealer $A$. Denote $\pi^*_{At} = \pi_{At}(\beta^*_{At}, m^*_t, d^*_{At}, \beta^*_t, \gamma^*_t)$ the payoff of agency dealer in period $t$ under the liquidity insurance equilibrium outcome in Definition 2.2. Assume that the outside option of agency dealer $A$ in period $t$ is $W_{At} = \pi^*_{At} - \pi_{At}(d_{At} = 1)$ if there exists $k \leq t - 2$ such that

1. history of action sets of $\{P, A, I\}$ of all periods up to period $k$ coincides with the liquidity insurance relationship equilibrium outcome,

$$(\beta_{Aj}, m_j, d_{Aj}, \beta_j, \gamma_j) = (\beta^*_{Aj}, m^*_j, d^*_{Aj}, \beta^*_j, \gamma^*_j), \quad \forall j \leq k$$

2. market state in period $k + 1$ is bad,

$$\theta_{k+1} = B$$

3. agency dealer $A$ has been rejecting offers from principal dealer $P$ since period $k + 1$,

$$d_{Aj} = 0, \quad \forall j \text{ where } k + 1 \leq j \leq t - 1.$$

Otherwise, the outside option $W_{At} = 0$.

In other words, if in the past

1. agency dealer $A$ had successfully formed a long-term relationship with principal dealer $P$ and all investors $I$, and

2. agency dealer $A$ rejected an offer from principal dealer $P$ in a bad state and has been rejecting all offers since then,

then the agency dealer will obtain a payoff identical to the payoff under the relationship equilibrium outcome if rejecting offer from principal dealer $P$ in period $t$ (i.e. $\pi_{At}(d_{At} = 0) = \pi^*_{At}$). Intuitively, this assumption allows agency dealer to establish a new relationship with other principal dealers in the market if the relationship with principal dealer $P$ is broken. As we will see later, the possibility to establish a new relationship will incentivise the agency dealer to punish principal dealer $P$ if principal dealer $P$ deviates from the agreement in bad states. Without this assumption, the agency dealer might have an incentive not to punish principal dealer $P$ in bad states, as the agency dealer will lose future intermediation profits. Note from the assumption that agency dealer $A$ must have had relationships with all the investors up until period $k$ to entitle for this outside-option relationship payoff. [22]

### 2.6.1 Trigger strategy via an agency dealer

Next, we propose a strategy profile of all players that can support the relationship outcome in the equilibrium. There are many possible strategy profiles that one can propose. However, the strategy must ensure that every player will be monitored and punished by some players if deviating. We propose the following strategy.

**Definition 2.3 (Trigger strategy via an agency dealer)** *The modified trigger strategy consists of*

1. *a strategy of principal dealer $P$ in which*

   a) $(\beta_{At}, m_t) = (\beta^*_{At}, m^*_t)$ *if the history set $h_{Pt}$ satisfies, for all $j \leq t-1$,*

      i. *$P$ never deviates in the past, $(\beta_{Aj}, m_j) = (\beta^*_{Aj}, m^*_j)$*

      ii. *$A$ never deviates in the past, $(d_{Aj}, \beta_j) = (d^*_{Aj}, \beta^*_j)$*

      iii. *$I$ never deviate in the past, $\sum_{i \in I_j} \gamma_{ij} \geq \sum_{i \in I_j} \gamma^*_{ij}$*

   b) $(\beta_{At}, m_t) = (C_t, n)$ *if otherwise*

2. *a strategy of agency dealer $A$ in which*

   a) $(d_{At}, \beta_t) = (d^*_{At}, \beta^*_t)$ *if the history set $h_{At}$ satisfies, for all $j \leq t-1$,*

      i. *$P$ never deviates in the past, $(\beta_{Aj}, m_j) = (\beta^*_{Aj}, m^*_j)$*

      ii. *$A$ never deviates in the past, $(d_{Aj}, \beta_j) = (d^*_{Aj}, \beta^*_j)$*

      iii. *and $P$ does not deviate in period $t$, $(\beta_{At}, m_t) = (\beta^*_{At}, m^*_t)$*

   b) $(d_{At}, \beta_t) = (0, \beta^*)$ *if otherwise.*

3. *a strategy of every investor $i \in I$ in which*

[22] This gives an incentive for $A$ to maintain the relationship with every $I$. Without this assumption, the agency dealer might have incentive in some cases to terminate the agreement with a few investors. Relaxing this assumption will introduce one more equilibrium condition which requires that the agency dealer cannot connect with too many investors; that is, the size of agency dealer must be small enough.

*a)* $\gamma_{it} = \gamma^*_{it}$ *if the history set* $h_{it}$ *satisfies, for all* $j \in T_{it}$*,*

*i.* $P$ *and* $A$ *never deviate in good state in the past,* $\hat{\beta}_{Aj} = \beta^*_{Aj}$

*ii.* $A$ *never unilateral deviate with* $i$ *in the past,* $\beta_{ij} = \beta^*_{ij}$

*iii.* $i$ *never deviates in the past,* $\gamma_{ij} = \gamma^*_{ij}$

*iv. and* $A$ *does not deviate in period* $t$*,* $\beta_{it} = \beta^*_{it}$

*b)* $\gamma_{it} = 0$ *if otherwise.*

This strategy follows the rationale of standard trigger strategy. Every player will continue choosing the agreed actions as long as they do not detect any deviation. If they detect a deviation, they will terminate the relationship and revert to the no-relationship equilibrium strategy forever. From the definition, every player reverts to the static Nash equilibrium outcome after detecting deviation, except agency dealer $A$ who will reject offer from principal dealer $P$ but still quote downstream price $\beta^*_t$ after the deviation, as it is weakly dominant to do so.[23] Note that investors can monitor agency dealer $A$ and principal dealer $P$ (imperfectly) but not other investors.

The punishment scheme in this proposed strategy is as follows. When $P$ deviates in a bad state by not providing costly liquidity, $A$ will reject the offer and terminate the relationship with $P$. This threat is credible, because $A$, if punishing, can obtain outside option equivalent to the equilibrium relationship payoff in all subsequent periods.[24] When $P$ deviates in a good state by quoting upstream price higher than the agreed level, active investors, who can observe the change in upstream price in the subsequent period, and agency dealer $A$ will terminate the relationship. The threat by agency dealer $A$ is also credible, because agency dealer $A$ can obtain (weakly) lower upstream price by choosing $P'$ in the deviating period and in all subsequent periods.

When any $i \in I$ deviates in a good state by rejecting offer, $P$ will terminate the relationship with $A$.[25] When $A$ deviates in a good state, $P$ will terminate the relationship, triggering active investors who can observe the changing upstream price to terminate the relationship with $A$. The resulting termination by the investors will threat $A$ not to deviate, since $A$ will no longer obtain the equilibrium relationship payoff (i.e. $W_{At} = 0$ for all subsequent periods). Note that $A$ and $I$ have no incentive to deviate during bad states.

### 2.6.2 Commitment problem and incentive constraints

Next, we will examine the commitment problem and construct incentive constraints of all players. This is not as obvious as our baseline case, because it now involves an agency dealer who has a private interest of maximizing his intermediation profit.

[23] This is because the agency dealer can still make positive profits even if $W_{At} = 0$. Thus, the agency dealer has an incentive to quote the price at $\beta^*$ to deceive remaining investors that the relationship is still on.

[24] Note that the investors themselves cannot detect the deviation of the principal dealer in bad states when there is partial insurance ($n^* < n$), because the investors do not know whether the no-trade outcome is due to the principal dealer's deviation. Therefore, $A$ must be the one who punishes $P$.

[25] This implies that deviation from one investor will break down the whole relationship. Note that, if investor $i$ deviates, $A$ may have an incentive *not to punish* the investor. This is because $A$ obtains more profits when there are more trades to intermediate, thus he prefers more investors to contact him. Therefore, $P$ must be the one who punishes investor $i$ by stopping providing liquidity to $A$. In practice, principal- and agency dealer might resolve this problem through renegotiation. We assume that the renegotiation is not possible.

### I: Commitment problem of principal dealer $P$ and investors $I$

Similar to baseline case, $P$ must commit to providing liquidity in bad states and investors must commit to trading with $A$ instead of seeking cheaper prices from $P'$ in good states.[26] Therefore, $(x_G, x_B)$ must be sufficiently high to incentivize $P$, and $x_G + F_{iG}$ must be sufficiently low to incentivize every investor $i \in I$.

To see this formally, let $\alpha_i(n^*, n_t) : \mathcal{I}_+ \times \mathcal{I}_+ \to [0, 1]$ be a liquidity allocation function to investor $i$ which indicates the probability that investor $i$ will obtain liquidity in period $t$, given liquidity quota $n^*$ and aggregate liquidity demand $n_t$. If $\alpha_i = 1$, investor $i$ obtains liquidity with probability one. Let $f(k) = Pr(\sum_{j \in I} l_{jt} = k)$ and $f_{-i}(k) = Pr(\sum_{j \in I-i} l_{jt} = k)$ be the probability distribution of aggregate liquidity shocks when including and excluding investor $i$, respectively. Denote $\kappa(n^*) = \sum_{k \le n^*} f(k)(k) + \sum_{n \ge k > n^*} f(k)(n^*)$ and $y_i(n^*) = \sum_{k \le n^*-1} f_{-i}(k) + \sum_{k > n^*-1} f_{-i}(k)\alpha_i(n^*, k)$. Intuitively, $\kappa(n^*)$ is the expected amount of liquidity that $P$ must provide to $A$ during bad states and $y_i(n^*)$ is the probability that investor $i$ obtains liquidity in bad states. Consider the following lemma.

**Lemma 2.2 (Equilibrium price)** *In any equilibrium*

1. *principal dealer $P$ will commit to liquidity insurance agreement if*

$$x_G \ge \frac{n^* + \frac{\delta}{1-\delta}(1-p)\kappa(n^*)}{nQp}(C - x_B)$$

2. *investor $i \in I$ will commit to liquidity insurance agreement if*

$$x_G + F_{iG} \le max\left\{\frac{Q(1-p)y_i(n^*)}{1+Qp}(V - (x_B + F_{iB})), V_L\right\}$$

This lemma highlights how changing liquidity quota $n^*$ affects incentive of principal dealer $P$ and investors. For the principal dealer, a decrease in liquidity quota $n^*$ increases his incentive to commit to the relationship, as reflected in lower requirement on premium $x_G$. Setting low $n^*$ decreases the relationship-continuation cost of $P$, because $P$ just needs to provide costly liquidity (at most) $n^*$ units in bad states to continue the relationship and obtain future order flows from agency dealer. On the other hand, a decrease in $n^*$ reduces incentive of the investors to commit to the agreement. Intuitively, lower $n^*$ decreases the likelihood of obtaining liquidity in bad states $y_i(n^*)$, resulting in lower willingness of the investors to trade with agency dealer in good states.

### II: Commitment problem of agency dealer $A$

Ideally, principal dealer $P$ and investors would like to trade via an intermediating person who has no interest misalignment. However, it is unrealistic as actions taken by agency dealer $A$ is strategic and can only be detected with delay. In the relationship, a good agency dealer $A$ must have incentive to 1) route trade orders of all investors in every period to principal dealer $P$ and

[26] Note that principal dealer has no commitment problem in good states. If $P$ deviates in good states, $A$ will reject the offer immediately.

2) acquire liquidity from principal dealer $P$ and pass it to demanding investors. The second agency problem is not our concern, as the payoff structure of agency dealer $A$ induces him to intermediate as many transactions as possible. In fact, agency dealer $A$ has no incentive to withhold liquidity in any period. This is one of the main features of agency dealer $A$ attributing to successful relationship formation: by delegating the power to a third party who has no misaligned interest, all the investors do not have to worry about collective punishment by all the investors on agency dealer $A$.

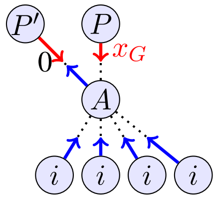


Figure 2.5: Illustration of strategic deviation of $A$ when choosing whom to send trade orders to.

The first problem is of our interest. Recall that investors can monitor agency dealer $A$ by looking at upstream price after trade settlement, but this will cause delay in punishment if agency dealer $A$ deviates. As illustrated in Figure 2.5, delay in punishment creates an opportunity for agency dealer $A$ to strategically route orders to $P'$, pay upstream price 0 (instead of $x_G$), and get upfront higher intermediation profits in the deviating period. Once agency dealer $A$ pulls the trigger, principal dealer $P$ will terminate the relationship. Active investors who observe abnormal upstream price will stop trading with agency dealer $A$ in subsequent periods. Therefore, principal dealer $P$ and investors must leave agency dealer $A$ a sufficiently large trade surplus, in the form of intermediation fees $(F_{iG}, F_{iB})$, so that he will commit to sending trade orders to $P$ in good states and continue collecting future intermediation fees. Denote $\sigma(n^*) = \frac{\kappa(n^*)}{\kappa(n)}$ which is the degree of liquidity insurance. Consider the following lemma which is derived from incentive constraint of agency dealer $A$.

**Lemma 2.3 (Intermediation fee)** *A pair of intermediation fee* $(F_B, F_G)$ *in any symmetric equilibrium satisfies*

$$Q((1-p)\sigma(n^*)F_B + pF_G) \geq x_G.$$

From the lemma, the extent of agency cost depends on upstream price $x_G$. This is intuitive, as the upfront deviating benefit of agency dealer $A$ depends on price gap between what $P$ and $P'$ charge, and this gap is the premium $x_G$. Also, the more frequently investors trade (higher $Q$), the more frequently agency dealer $A$ obtains intermediation fees, and thus lower fees per transaction is required to incentivize agency dealer $A$.

### 2.6.3 Equilibrium analysis: effectiveness of agency dealer

Our last step is to find out when agency dealer $A$ can successfully help the investors, *who otherwise would have been excluded from principal dealer* $P$, eventually obtain costly liquidity in bad states from principal dealer $P$. Let $Q_P$ be the minimum $Q$ for the liquidity insurance equilibrium between principal dealer $P$ and investor $i$ to exist, and $Q_A$ be the minimum $Q$ for the liquidity insurance equilibrium between $\{P, A, I\}$ to exist. Therefore, if $Q_P > Q_A$ and if $Q \in [Q_A, Q_P)$, then the investors who cannot form a relationship with $P$ can now do so via agency dealer $A$. The following proposition provides the condition which agency dealer $A$ can successfully help the investors get liquidity insurance from the principal dealer.

**Proposition 2.3 (Effectiveness of agency dealer)** *$Q_A < Q_P$ if, for any $i \in I$, there exists $n^* \in \mathcal{I}^+$ such that*

$$\frac{1 + Q(1-p)}{\frac{n^*}{n} + Q(1-p)\sigma(n^*)} > \left(1 + \frac{1}{Qp}\right)\left(\frac{1}{y_i(n^*)}\right).$$

This condition compares the benefit of successful relationship formation by trading via $A$ with its cost. The benefit is the change in incentive of principal dealer $P$ to enter the relationship as reflected in the left term. The right term accounts for the cost of trading via $A$ (i.e. the intermediation fee) and any negative impact on the incentive of investors.[27]

$$\underbrace{\frac{1 + Q(1-p)}{\frac{n^*}{n} + Q(1-p)\sigma(n^*)}}_{\substack{\text{principal dealer} \\ \text{IC}}} > \underbrace{\left(1 + \frac{1}{Qp}\right)}_{\substack{\text{intermediation} \\ \text{fee}}}\underbrace{\left(\frac{1}{y_i(n^*)}\right)}_{\substack{\text{investor } i \\ \text{IC}}}$$

When will this condition be satisfied? The answer relies on whether there exists a real-valued liquidity quota $n^*$ that can satisfy this condition. Recall that liquidity quota $n^*$ is the minimum per-period amount of liquidity that principal dealer $P$ must provide in every bad states. A reduction in $n^*$ will effectively reduce the *upfront* cost of sustaining the relationship of the principal dealer relative to its benefit of relationship continuation of obtaining future order flows from all investors. This is shown in the ratio $\frac{n}{n^*}$ in the incentive constraint of the principal dealer in the left side. When the relationship-continuation cost is lower (i.e. lower $n^*$) or the relationship-continuation benefit is higher (i.e. higher $n$), the principal dealer would be more willing to sustain the relationship. Also, a decrease in $n^*$ reduces the degree of liquidity insurance (i.e. lower $\sigma(n^*)$). As a result, a decrease in $n^*$ results in higher likelihood of successful relationship formation via agency dealer. Note that, from the proposition, the liquidity quota must be less than the number of investors (i.e. $n^* < n$) for the condition to be satisfied. This implies that the investors will obtain partial insurance when trading with agency dealer.

**Remark 2.1** *If $Q_A < Q_P$, $n^* < n$ must be true*

Unfortunately, decreasing $n^*$ also has a negative effect on incentive of the investors, as it reduces the extent of liquidity insurance. Specifically, a decline in $n^*$ reduces the probability that investors will obtain liquidity $y_i(n^*)$, decreasing their willingness to commit to the relationship. This idea is graphically illustrated in Figure 2.6. Overall, whether the relationship formation will be successful depends on which effect dominates. This implies that relationship formation via agency dealer $A$ cannot always solve the commitment problems.

[27] Note that in the case of perfect monitoring (i.e when investors can perfectly observe other investor's actions, liquidity shocks, and the principal dealer's response to every investors), then investors would no longer need $A$. This implies that the intermediation fees will disappear from this condition. That is, $\frac{1+Q(1-p)}{\frac{n^*}{n}+Q(1-p)\sigma(n^*)} \geq \frac{1}{y_i(n^*)}$.


Figure 2.6: Graphical illustration of how high probability of simultaneous liquidity shocks by several investors can cause relationship failure.

When would agency dealer $A$ be most efficient at facilitating the relationship then? Ideally, one should be able to reduce $n^*$ to minimize the relationship-continuation cost of principal dealer $P$ with minimal impact on liquidity insurance coverage of the investors. This is achievable only when investors *rarely* acquire liquidity at the same time: the probability that several investors face liquidity shocks in the same period is low. An extreme case is when the probability of liquidity shock $q$ is almost zero, implying that it is almost impossible that more than one investor would demand liquidity at the same time. To clarify this point, consider the following corollary.

**Corollary 2.1** *If* $q \in \mathcal{N}_\epsilon(0)$, *then*

1. $y_i(n^*) \approx 1$ *and* $\sigma(n^*) \approx 1$ *for any* $n^* \in \mathcal{I}^+$

2. $Q_A < Q_P$ *if* $p > \frac{1}{2}$ *and*
$$\frac{n}{n^*} > \frac{1 + pQ}{Q(2p - 1)}$$

The first condition states that even $n^* = 1$ can guarantee almost full insurance to all investors when the probability of liquidity shock $q$ is almost zero. This is because it is very rare that more than one investor would acquire liquidity at the same time. In this case, one can set the quota $n^* = 1$ to minimize the relationship-continuation cost with trivial effect on the degree of liquidity insurance coverage, resulting in the highest likelihood of successful relationship formation with (almost) full insurance coverage towards the investors. The second condition states that, if the number of investors is sufficiently large, agency dealer $A$ will have adequate negotiation power to connect investors with the principal dealer. An implication from this finding is that, in practice, an agency dealer will be ideal for investors who 1) rarely face liquidity shocks (i.e. low $q$) but 2) are reasonably risk-averse about liquidity risk (i.e. moderate $Q$).

### Result discussion

Our finding sheds light on the strategic dealer choice of investors in relationship-based OTC markets. When the investors choose whom to trade with, they will take into account all possible future benefits that they can obtain from the chosen dealer. In the relationship-based market, when principal dealers attempt to compete with agency dealers and shorten the intermediation chain, by quoting a one-period cheaper price to the low-trading-need investors, the investors still prefer their agency dealers. This is because the investors know that maintaining informal relationships with the agency dealer can bring higher benefits from future liquidity insurance, the benefit that principal dealer cannot offer due to commitment problem. This can explain why the agency dealers, despite locating at inferior network positions, can still attract clients and survive in the OTC markets.

Another interpretation of this finding relates to empirical findings on a tradeoff between execution cost and trading speed when investors choose a dealer (Li and Schürhoff (2019)). In relationship-based market, it might be more appropriate to state that there is a tradeoff between execution cost ($x_G$) and the degree of future liquidity insurance ($y_i(n^*)$) when choosing a dealer. That is, principal dealers will offer full liquidity insurance with expensive premium (i.e. high $x_G$ and $y_i(n^*) = 1$), while agency dealers will offer partial liquidity insurance with cheaper premium (i.e. low $x_G$ and $y_i(n^*) < 1$). When the investors have low trading needs, they cannot choose the former agreement, granting market power to agency dealers to co-exist with the principal dealers.

### 2.6.4 Implications on equilibrium prices

One interesting empirical prediction of our study is the relationship between equilibrium price and length of intermediation chain in relationship-based markets. Our model suggests that retail prices quoted by agency dealers can be cheaper than that quoted by principal dealers, because the price accounts for future liquidity insurance benefits that their respective clients would obtain. This contradicts to traditional belief that retail prices from longer intermediation chain is higher due to higher markups. Consider the following proposition.

**Proposition 2.4 (Price comparison)** *If $Q_A \leq Q < Q_P$, a downstream price quoted by agency dealer $A$ in any liquidity insurance equilibrium among $\{P, A, I\}$ must be lower than any price that would have induced principal dealer $P$ to form a direct relationship with an investor $i \in I$.*

From Figure 2.7, without the intervention of agency dealer $A$, principal dealer $P$ will charge expensive price $\underline{x}_G$ which is too high for investor $i$ to commit to the relationship. However, when trading via agency dealer, the investor can get cheaper prices due to lower degree of future liquidity insurance. In the equilibrium, the size of coalition must be high enough for the agency dealer to secure good price $x_G + F_G$ from the principal dealer. As a result, the final price $x_G + F_G$ will be lower than the quote that the investors would have obtained directly from the principal dealer, despite combining with the intermediation markup.


Figure 2.7: Graphical illustration of price comparison in which $\underline{x}_G > x_G + F_G$ if $Q_A \leq Q < Q_P$.

### 2.6.5 Discussion on market efficiency and stability

In terms of market efficiency, our finding provides an important insight into conventional wisdom about "long intermediation chain as a source of allocative inefficiency". In our model, agency dealers who act as financial intermediaries between principal dealers and end-user investors improve allocative efficiency. With their presence in the market, low-trading-need investors, such as retail investors, are more likely to obtain liquidity (i.e. immediacy) even though they are not active players in secondary OTC markets. This results in improvement on allocative efficiency and reduction in liquidity risk of OTC-based assets. Moreover, the presence of agency dealers can enhance market participation of low-trading-need investors. Since the agency dealers reduce liquidity risk and increase total surplus of the low-trading-need investors, presence of the agency dealers can incentivize more low-trading-need investors, who might not invest in the asset when there is no agency dealers, to enter the market.[28] The resulting increase in market participation can improve market efficiency and liquidity further.

However, in terms of financial stability as measured by the likelihood of first-trigger event of systemic crisis, it is ambiguous to conclude whether agency dealers will be socially desirable. Remember that trading with agency dealers solves the commitment problem because of lower liquidity quota in bad states. Therefore, low-trading-need investors are still not fully insured against extreme events when several investors demand liquidity simultaneously, as illustrated in Figure 2.8. The liquidity shortage among several low-trading-need investors in extreme events, despite highly unlikely, is still possible. Unfortunately, in some extreme events such as when the investors would like to sell the asset simultaneously to meet their cash demands, liquidity shortage can cause market disruption and trigger systemic crisis from asset fire-sale phenomenon.

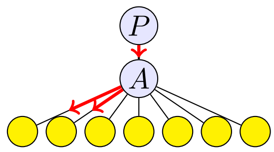


Figure 2.8: Graphical illustration of liquidity shortage in an extreme event when all the investors demand liquidity simultaneously when $n^* = 2$.

Consequently, presence of agency dealers has ambiguous effect on market stability. On the one hand, they improve liquidity condition of **existing investors with low trading needs** who would have invested in the asset, regardless of their presence. In this case, agency dealers improve market stability. On the other hand, agency dealers can also deteriorate market stability by attracting more low-trading-need investors to enter the market and start investing in the security. However, the low-trading-need investors are highly subjected to liquidity shortage during extreme events. Depending on further knock-on effects from the first-round liquidity dry-up among the low-trading-need investors, emergence of agency dealers can cause an alarming concern to the market regulators in terms of market stability.

## 2.7 Result discussion

### 2.7.1 Will investors with high trading needs choose agency dealers?

The assumption of homogeneous investors in the model so far allows us to understand the role of agency dealers, but the equilibrium outcome when there are several types of investors in

[28]To see this, one can easily extend our model by allowing the investors to decide whether to enter the market in period 0. If entering, they have to pay participation cost $c$.

the market remains unclear. Given high bargaining power of obtaining good price of agency dealers, would high-trading-need investors who can form direct relationship with principal dealers choose agency dealers instead?

In appendix, we extend our model to include two types of investors: an investor with high trading needs (i.e. the high type) and a group of investors with low trading needs (i.e. the low type). We found that the high-type investor might indeed choose agency dealer, if the size of agency dealer (i.e. the number of low-type investors connecting with the agency dealer) is sufficiently large. This is because the high-type can obtain reasonably low price even if he might be able to get only partial liquidity insurance from the agency dealer. However, if the agency dealer does not have sufficiently large low-type clientèle, the high-type investor will prefer direct relationship with the principal dealer to avoid intermediation fees and obtain full liquidity insurance.

The implication on market efficiency when the high-type switches to agency dealer is ambiguous. On the one hand, the low-type investors may benefit from the resulting higher bargaining power of the agency dealer. On the other hand, when the high-type chooses the agency dealer, either 1) the high-type will crowd out liquidity from the low-type in bad states if liquidity quota does not increase in response to the joining of the high-type or 2) the low-type might face higher price due to higher upstream price and agency cost if liquidity quota increases in response to the joining of the high-type. Moreover, it is possible that some low-type investors will no longer be able to form a relationship with the agency dealer upon the joining of the high-type investor. In our model, we cannot examine this effect, since we assume that the agency dealer can obtain outside option equivalent to the equilibrium relationship payoff only if the agency dealer still connects with **all** the (low-type) investors in the game. Under this assumption, every investor matters to the agency dealer, thus the agency dealer can commit to every relationships with the investors. Relaxing this assumption might cause the agency dealer to lose a few low-type investors upon the joining of the high-type investor, resulting in lower allocative efficiency.

### 2.7.2 Possibility of a principal dealer executing agency trade.

In our model, we assume that only agency dealer can intermediate trades between a principal dealer and investors. However, one can argue that in reality a principal dealer can take both roles: providing liquidity by taking positions for their own clients and intermediating transactions between low-trading-need investors and another principal dealer. Our conjecture is that intermediation via an agency dealer who do not take positions and do not have their own high-trading-need client base is preferable for both the investors and the upstream principal dealer. This is because of higher agency cost when trading via the principal dealer. Unlike an agency dealer, a principal dealer has an incentive to withhold trade orders from the investors during good times. This agency problem supports the theory that vertical market fragmentation (i.e existence of periphery layer) can still emerge despite the presence of multiple principal dealers.

### 2.7.3 Direct relationship with partial insurance coverage

In our model, an agency dealer can sustain relationship between a principal dealer and several investors when the investors have infrequent liquidity shocks, by reducing liquidity quota to induce the principal dealer to eventually commit to the relationship. In the equilibrium, the investors will only obtain partial insurance coverage from the agency dealer. From this finding, one can argue that the investors can just form *partial* insurance relationship directly with a principal dealer, by requiring the principal dealer to provide liquidity with some positive probability. This type of direct relationship will save the intermediation fees and improve the surplus of both the investor and the principal dealer.

Unfortunately, this type of direct relationship with partial insurance between a principal dealer and an investor involves a new *imperfect monitoring* problem: the investor cannot monitor perfectly whether the principal dealer is still randomly providing liquidity or he strategically reneges the agreement in bad states. From our model, trading via an agency dealer can eliminate this imperfect monitoring problem, because investors do not have to monitor the agency dealer who, unlike the principal dealer, has no incentive to withhold liquidity in bad states. Moreover, trading via agency dealer allows investors to pool their liquidity shocks and enlarge contracting space, increasing the likelihood of relationship formation. Therefore, intermediated relationship can be preferable to direct relationship with partial insurance.

### 2.7.4 Multiple agency dealers and competition

Throughout the paper, we consider the model with one agency dealer. So far, our result suggests that a market with only one agency dealer might be socially optimal (i.e. highest likelihood for successful relationship), because contracting space is largest and a threat that the agency dealer can impose on the principal dealer is maximized. Also, when we allow our model to contain multiple agency dealers, equilibrium market structure is likely to be a market which only one agency dealer forms the relationship with all the investors. However, the market structure with one agency dealer might no longer be an equilibrium when we consider a more general setting.

Firstly, our model assumes that there is perfect competition among sell-side dealers. Therefore, it is not surprising to get the equilibrium in which the market contains only one agency dealer. However, if we consider competition structure and surplus allocation in a more realistic setting, an equilibrium with several agency dealers can emerge endogenously. This is because the investors need to establish relationships with several dealers to increase their outside options.[29]

Secondly, our model assumes that other principal dealers in the market will consider forming relationship with the agency dealer only if 1) the agency dealer terminates the relationship with previous principal dealer **in a bad state** and 2) **all** investors are still in the relationship with the agency dealer. Under this assumption, the agency dealer can commit to relationships with all the investors in the game. If relaxing this assumption, a commitment problem of agency dealer will arise when agency dealer is too big, because punishment from each individual investor might not be strong enough to threat the agency dealer to commit to the relationship in good states. That is, in periods when the number of active investors is small and the market state is good,

[29] See Wang (2017) for more details on relationship formation and market competition

the agency dealer will choose the other principal dealer $P'$, get cheaper upstream price in the deviating period, and establish relationship with a new principal dealer in subsequent periods. If the size of agency dealer is large enough, relationship termination from a few active investors in the deviating period will not be enough to prevent the agency dealer from forming a new relationship later. Thus, in the relationship equilibrium, each individual investor must be able to effectively punish the agency dealer. Indeed, relaxing this assumption will introduce one more equilibrium condition on the size of agency dealer. Due to the size limit on agency dealer, equilibrium with multiple agency dealers will naturally exist.

## 2.8 Concluding remark.

To summarize, this paper studies strategic dealer choice of buy-side investors in over-the-counter (OTC) asset markets and provides a new insight into why periphery dealers, despite locating at inferior positions in OTC dealer network, can survive and co-exist with core dealers. Our main finding is that investors will trade with periphery dealers to obtain the benefit of long-term relationship instead of trading with core dealers on a one-period basis when the investors have infrequent liquidity needs, allowing periphery dealers to co-exist with core dealers.

Our finding on long-term relationships opens up a challenging question for policy makers on designing appropriate marketplaces for OTC-traded products. Long-term relationships and dealer network have been the bone of over-the-counter markets for several years due to the feature of OTC-traded products which are non-standardized, traded in large lots, and highly illiquid. However, recent financial crisis has put this traditional OTC market mechanism into question. What should be a suitable policy direction for regulators or what regulatory framework should be put in place are important questions we are yet to discover.

## 2.9 Appendix

### 2.9.1 Additional material: the model with heterogeneous investors

In this section, we extend our model to have two types of investors: a high-trading-need investor who can form direct relationship with principal dealers and a group of low-trading-need investors who cannot do so, and characterize the conditions when the high-trading-need investor chooses the agency dealer (pooling equilibrium) instead of the principal dealer (separating equilibrium).


Figure 2.9: Pooling equilibrium (left) and separating equilibrium (right).

#### Model extension

Consider the model with an agency dealer and two types of investors: a high-type investor $H$ with probability of liquidity shock $q_H$ and a group of low-type investors $I_L = \{1, 2, \cdots, N\}$ with probability of liquidity shock $q_L$ and $|I_L| = n$. Define $Q_i = \frac{q_i \delta}{1-\delta}$. We limit our analysis to a special case under the following assumptions.

**Assumption 2.2**

*1.* $q_L \in \mathcal{N}_\epsilon(0)$

*2.* $Q_H \geq Q_P \geq Q_L$

*3.* $n \geq \frac{1}{Q_L} max \left\{ 2Y - Q_H, \frac{2p(1+Q_L p)^2(C-V_L)}{(1-p)((Q_L p)^2(V-V_L)-(1+Q_L p)^2(C-V_L))} - Q_H \right\}$

*4.* $\{P, A\}$ *obtain their lowest possible payoffs.*

The first assumption assumes that the low-type investors rarely face liquidity shocks, implying that they can obtain almost full insurance in the equilibrium. This assumption allows us to draw an insight about equilibrium market structure without technical complications about a change in the low-type investors' incentive from endogenous adjustment in the degree of liquidity insurance coverage.

The second and third assumption specify characteristics of the low-type and the high-type. The second assumption states that only the high-type investor can form the relationship directly with the principal dealer. The third assumption states that the number of the low-type investors is sufficiently large that they can form the relationship with the agency dealer without the inclusion of the high-type investor; that is, a separating equilibrium can exist. The last assumption specifies equilibrium surplus allocation of all players in a competitive dealer market, where the sell-side dealers compete for relationship formation in a large decentralized market. This also implies that we will consider only **the most-likely-to-exist equilibrium**, in which the low-type investors obtain their highest possible payoffs.

### Timeline

The timeline of the game is identical to our main setting, except that the investors must choose whom they want to form the relationship with in period $t = 0$. Specifically, all investors $i \in I$ decide whom they want to form the long-term relationship in period 0, denoted by $d_{i0} = \{A, P\}$. Assume that every player can observe all the investors' dealer choice, implying that the investors know the coalition size of the agency dealer. Denote $N(A) = \{i|d_{i0} = A\}$ and $N(P) = \{i|d_{i0} = P\}$ the set of investors who choose $A$ and $P$ in period 0, respectively.

### The equilibrium

Before characterizing the equilibrium, we first formally define two types of the equilibrium.

**Definition 2.4 (Equilibrium definition)** *A liquidity insurance equilibrium is*

1. *a pooling equilibrium if*
   a) $d_{i0} = A$ *for all* $i \in I$
   b) $F_{iG} = \frac{x_G}{pQ_i}$ *and* $F_{iB} = 0$
2. *a separating equilibrium if*
   a) $d_{i0} = A$ *for all* $i \in I_L$
   b) $d_{H0} = P$
   c) $n^* = 1$

From the definition, we rule out the possibility of cross-subsidy across investors under pooling equilibrium in condition (1.b). Specifically, the investors are responsible to pay for only the agency cost incurred from their own transactions, and the agency dealer $A$ cannot strategically place higher fees to some investors to cover other investors' agency costs. We also consider only the separating equilibrium when liquidity quota $n^*$ is one. This is intuitive given that the probability of liquidity shocks of the low-type investors is almost zero.

Our next step is to characterize existence condition of both the pooling and separating equilibrium. The condition will depend on the decision of the high-type investor, because the low-type investors have no other options but to form the relationship with the agency dealer. Therefore, we only need to compare the expected payoff that the high-type would get under separating equilibrium with his expected payoff under pooling equilibrium.

What is the key factor that determines the dealer choice of the high-type investor? Obviously, one is the equilibrium price that he could obtain from both dealers. However, the high-type investor must also consider his probability of obtaining liquidity in the future (i.e. his insurance coverage $y_i(n^*)$), if he chooses the agency dealer. To illustrate this point, we will consider two cases of pooling equilibrium: the case of $n^* = 1$ and the case of $n^* = 2$. In the first case, the agency dealer does not alter the liquidity quota to cover the inclusion of the high-type investor; therefore, the high-type investor faces higher liquidity risk (i.e. lower $y_i(n^*)$). The second case is when the agency dealer adjusts the liquidity quota to $n^* = 2$ to cover for the high-type investor. In this case, the high type still obtains (almost) full insurance when choosing the agency dealer.

**Case I: liquidity quota $n^* = 2$ under pooling equilibrium.**

The first case is when the liquidity quota rises by one unit to cover frequent liquidity needs of the high-type investor. As shown in Figure 2.10, a unit increase in the quota will guarantee the high-type investor almost full insurance coverage. Therefore, the decision of the high-type investor will depend on the final price that he would be charged by the agency dealer. Consider the following proposition.

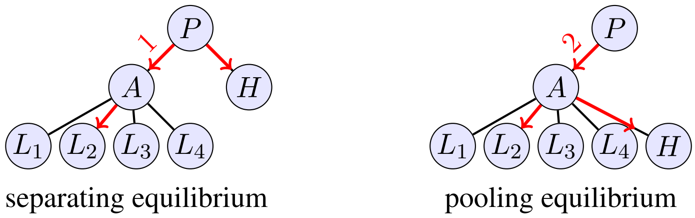


Figure 2.10: Graphical illustration of a unit increase in liquidity quota when the high-type investor chooses $A$.

**Proposition 2.5 (Equilibrium characterization I)** *Provided that $n^* = 2$ under pooling equilibrium and that Assumption 2.2 is satisfied,*

1. *pooling equilibrium exists if $p > \frac{1}{2}$* ***and***

$$NQ_L \geq \frac{2 + Q_H}{2p - 1}$$

2. *separating equilibrium exists if $p \leq \frac{1}{2}$,* ***or*** *if $p > \frac{1}{2}$ and*

$$NQ_L \leq \frac{2 + Q_H}{2p - 1}.$$

Interpretation of this proposition is straightforward. The high-type investor would prefer $A$ if the price quoted by $A$ is cheaper than the price quoted by $P$. Therefore, when coalition size of the low-type $nQ_L$ is large enough relative to his own size $Q_H$, the high-type investor would arbitrage by choosing the agency dealer.

Interestingly, the pooling equilibrium is more efficient than the separating equilibrium due to positive externality from the high-type investor: the amount of liquidity provision in bad states is slightly *higher* than the separating equilibrium. In the pooling, the low-type investors have higher chance of obtaining liquidity in bad states, because they, as a group, can obtain two units of liquidity instead of one unit when the high-type investor does not have liquidity shock. However, will pooling equilibrium be preferable to all players? According to the next proposition, the low-type investors in fact receive lower payoff if the high-type investor chooses the agency dealer.

**Proposition 2.6 (Surplus allocation)** *In the pooling equilibrium, the low-type investors obtain lower payoff compared to the case which the high-type investor chooses the principal dealer.*

This finding is surprising, given that the inclusion of the high-type investor should help the low-type investors get lower price due to larger group size. In addition, liquidity insurance coverage of the low-type investors increases. The answer is because the upstream price depends not only on the number of investors but also on the liquidity quota that the agency dealer requests from the principal dealer. Upon the participation of the high-type investor, the upstream price quoted to the agency dealer in fact increases which can hurt the low-type investors. Moreover, the extra trade surplus goes to the agency dealer who also charges higher intermediation fees to the low-type investors in response to higher upstream price.[30] This finding implies that agency dealer might improve market liquidity and efficiency, but he can have too much bargaining power over low-type investors who have no other options but to trade with him.

**Case II: liquidity quota $n^* = 1$ under pooling equilibrium.**

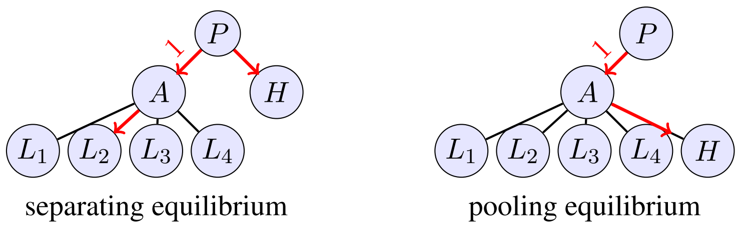


Figure 2.11: Graphical illustration of no adjustment in liquidity quota when the high-type investor chooses $A$.

The second type of pooling equilibrium is when the agency dealer does not request more liquidity quota from the principal dealer to cover liquidity needs of the high-type investor in the pooling equilibrium. Obviously, this is an inefficient pooling equilibrium, as the high-type investor crowds out liquidity coverage from the low-type investors. In this situation, the dealer choice of the high-type investor depends on both the price and the degree of his insurance coverage he can obtain from $A$. Let $x_{PA}$ be the price that $P$ offers to $A$ when $H$ chooses $A$ and and $x_{PH}$ be the price that $P$ offers to $H$ directly. The following lemma provides a sufficient condition for the high-type investor to prefer $A$.[31]

**Lemma 2.4** *In period 0, $d_{H0} = A$ if*

$$x_{PH} - x_{PA} \geq \frac{1}{Q_{HP}} x_{PA} + \left(\frac{1-p}{p}\right)(nq_L)(V - V_L)$$

From the lemma, even though $A$ can get lower price from $P$, choosing $A$ is not always an optimal decision for the high-type investor because of two reason because of two reasons: positive intermediation fees (i.e. $\frac{1}{Q_{HP}} x_{PA}$) and lower probability of obtaining liquidity in bad states (i.e. $\left(\frac{1-p}{p}\right)(nq_L)(V - V_L)$). Indeed, the larger the number of low-type investors is, the

[30]Recall that the agency cost (i.e. minimum intermediation fees required by the agency dealer) depends directly on the upstream price. When the upstream price increases, the agency dealer has more incentive to send trade orders to the competitive principal dealer, resulting in higher agency cost.

[31]Note that this lemma considers a strategy profile with the lowest possible probability that the high-type investor will get liquidity in bad states to obtain a sufficient condition.

lower the chance that the high-type investor would get obtain liquidity in bad states is. Thus, the price quoted by $A$ must be sufficiently low to compensate for the lower insurance coverage.

Can a pooling equilibrium exist? From the lemma, the number of low-type investors must be so high that $A$ can get a very attractive upstream price for a pooling equilibrium to exist. The following proposition provides a sufficient condition for existence of the pooling equilibrium.

**Proposition 2.7 (Equilibrium characterization II)** *Provided that 1) $n^* = 1$ under pooling equilibrium, 2) $p > \frac{1}{2}$, and 3) $q_H$ is sufficiently low, there is a real interval $[\underline{Q}, \bar{Q}] \in \mathcal{R}^+$ in which a pooling equilibrium exists if $NQ_L \subset [\underline{Q}, \bar{Q}]$*

This proposition is straightforward. If 1) the frequency of liquidity needs of the high-type is sufficiently low that his benefit of choosing $A$ is relatively large and 2) the number of the low-type investors is also large enough to bring the upstream price down significantly, then the high-type investor will choose the agency dealer. However, the main difference from the previous type of pooling equilibrium is the restriction on upperbound of frequency of aggregate liquidity needs of all the low-type investors $(nq_L)$. This is because, unlike the previous type of pooling equilibrium, the high-type investor faces higher likelihood of liquidity shortage when the whole group of low-type investors demand liquidity more frequently.

### Discussion

Overall, what do we learn from this section? For one thing, we know that agency dealers and principal dealers are not direct competitors. From the model, an agency dealer attracts trade orders from low-trading-need investors who are not the main target clients of principal dealers. However, if agency dealers are bigger, they can start competing with principal dealers by attracting the high-type investors. Such competition can be undesirable to both low-type investors and principal dealers. From the model, under the case of efficient pooling equilibrium ($n^* = 2$) in which a high-type investor can bring more liquidity to low-type investors, the market player who gains is the agency dealer at the expense of the low-type investors. In a nutshell, agency dealers should be sufficiently big to effectively negotiate with principal dealers, but they can become too big and harm the main market participants – main liquidity-providing supplier (principal dealers) and end-user investors.

### 2.9.2 Omitted proofs

First, the list of a few notations used in the following proofs are as follows.

- $f(k) = Pr(\sum_{j\in I} l_{jt} = k)$
- $f_{-i}(k) = Pr(\sum_{j\in I-i} l_{jt} = k)$
- $\kappa(n^*) = \sum_{k\leq n^*} f(k)(k) + \sum_{n\geq k>n^*} f(k)(n^*)$
- $\sigma(n^*) = \frac{\kappa(n^*)}{\kappa(n)}$
- $y_i(n^*) = \sum_{k\leq n^*-1} f_{-i}(k) + \sum_{k>n^*-1} f_{-i}(k)\alpha_i(n^*, k)$

Also, note that $\kappa(n) = \sum_{k\leq n} Pr\left(\sum_{i\in I} l_{it} = k\right) k = nq$ since it is the mean of binomial distribution.

**Proof of Lemma 2.1:** To prove the lemma, we have to consider both "if" and "only if" condition. For the "if" condition, we follow the standard trigger strategy and use the static equilibrium strategy as a threat on deviation of both players. That is, if someone deviates, both of them will receive their respective outside options in all subsequent periods. For the "only if" part, it is always true, because both players must obtain higher payoff under equilibrium relationship outcome than their outside options when they choose to form the relationship. In an equilibrium, the following incentive constraints must be true.

1. the investor is willing to pay insurance premium $x_G$ and continue the relationship instead of contacting $P'$ in good states

$$-x_G + Q[p(V - x_G) + (1-p)(V - x_B)] \geq QpV \quad [E1.1]$$

2. the insurance premium $x_G$ is feasible to pay by the investor

$$x_G \leq V_L \quad [E1.2]$$

3. the principal dealer is willing to provide costly liquidity immediacy and continue the relationship rather than withholding the liquidity in bad states

$$x_B - C + Q(px_G + (1-p)(x_B - C)) \geq 0 \quad [E2]$$

Rearranging [E1.1] and combining with [E1.2] yields

$$x_G \leq max\left\{\frac{Q(1-p)}{1+Qp}(V - x_B), V_L\right\} \quad [E3]$$

Rearranging [E2] gives

$$x_G \geq \frac{1+Q(1-p)}{Qp}(C - x_B) \quad [E4]$$

which proves the lemma.

**Proof of Proposition 2.2:** To prove the existence condition, we find the condition in which there exists $(x_G, x_B) \in [0, V_L] \times [0, V_L]$ such that [E3] and [E4] are satisfied. From Lemma 2.1, [E3] and [E4] are both satisfied when

$$\frac{1+Q(1-p)}{Qp}(C - x_B) \leq \frac{Q(1-p)}{1+Qp}(V - x_B) \quad [E5]$$

and

$$\frac{1+Q(1-p)}{Qp}(C - x_B) \leq V_L \quad [E6].$$

Rearranging [E5] gives

$$\frac{1+Q(1-p)}{Qp}C - \frac{Q(1-p)}{1+Qp}(V) \leq \frac{1+Q(1-p)+Qp}{Qp(1+Qp)}x_B.$$

From this condition, it is obvious that $x_B = V_L$ will maximize the chance of equilibrium existence. This is also intuitive as it reduces the distortion in bad states. Substituting $x_B = V_L$ into [E5] and [E6] gives

$$\frac{1+Q(1-p)}{Qp}(C - V_L) \leq \frac{Q(1-p)}{1+Qp}(V - V_L) \quad [E5']$$

$$\frac{1+Q(1-p)}{Qp}(C - V_L) \leq V_L \quad [E6']$$

Solving $[E5']$ for $Q \in \mathcal{R}_+$ gives

$$Q \geq \frac{1+\sqrt{1+4p(1-p)(V-C)}}{2p(1-p)(V-C)}$$

and rearranging $[E6']$ gives

$$Q \geq \frac{C - V_L}{V_L - (1-p)C} \text{ provided that } V_L > (1-p)C$$

which proves the proposition.

**Proof of Lemma 2.2:** To prove this lemma, we consider the following incentive constraints of the principal dealer and all investor $i \in I$:

1. principal dealer $P$ must have incentive to provide liquidity in bad states and continue the relationship

$$n^*(x_B - C) + \frac{\delta}{1-\delta}\left(nqpx_G - (1-p)(C - x_B)\kappa(n^*)\right) \geq 0 \quad [A1]$$

2. investor $i$ has no incentive to choose $P'$ during good state

$$-(x_G + F_G) + Q(p(V - (x_G + F_G)) + (1-p)y_i(n^*)(V - (x_B + F_B)) \geq QpV \quad [A2]$$

3. investor $i$ can afford to purchase liquidity service in all periods.

$$x_G + F_{iG} \leq V_L \quad [A3]$$

$$x_B + F_{iB} \leq V_L \quad [A4].$$

Rearranging $[A1]$ yields

$$x_G \geq \frac{n^* + \frac{\delta}{1-\delta}(1-p)\kappa(n^*)}{nQp}(C - x_B) \quad [E1].$$

Rearranging $[A2]$ and combining with $[A3]$ give

$$x_G + F_{iG} \leq max\left\{\frac{Q(1-p)y_i(n^*)}{1+Qp}(V - (x_B + F_{iB})), V_L\right\} \quad [E2]$$

which proves the lemma.

**Proof of Lemma 2.3:** To prove this lemma, we consider incentive constraint of the agency dealer. The reason we restrict our attention to only symmetric equilibrium in which all investors face same allocation rule $\alpha_i(n^*, n_t) = \alpha_j(n^*, n_t)$ which implies that, for a given $n^*$, $y_i(n^*) = y_j(n^*)$ for all $i, j \in I$ and face same intermediation fees in every state $(F_{iB}, F_{iG}) = (F_{jB}, F_{jG}) = (F_B, F_G)$, is that it maximizes the likelihood of an equilibrium to exist. Since for an equilibrium to exist, **all** investors must agree to the relationship, including the ones who face the most restrictive incentive constraint. Under symmetric equilibrium, incentive constraint of the agency dealer is

$$(1-p)F_B\left[\sum_{k\leq n^*}\left[Pr\left(\sum_{i\in I} l_{it} = k\right)k\right] + \sum_{n\geq k>n^*}\left[Pr\left(\sum_{i\in I} l_{it} = k\right)n^*\right]\right] \\ + pF_G\sum_{k\leq n}\left[Pr\left(\sum_{i\in I} l_{it} = k\right)k\right] \geq \frac{(1-\delta)nx_G}{\delta}$$

This is in fact the most restrictive incentive constraint of the agency dealer, which is when he chooses $P'$ when **all** investors demand liquidity in good states and earns maximum deviating payoff $nx_G$.[32] Substituting $\kappa(n) = nq$ and rearranging the constraint give

$$Q((1-p)\sigma(n^*)F_B + pF_G) \geq x_G \quad [I]$$

which proves the lemma.

**Proof of Proposition 2.3:** To prove the proposition, we find the existence condition of the liquidity insurance equilibrium among $\{P, A, I\}$ and compare it to the existence condition of the direct relationship equilibrium between $\{P, i\}$. To characterize the existence condition, we find the minimum value of $Q$ in which there exists $\{x_G, x_B, F_G, F_B\}$ such that incentive constraints of all players $([E1], [E2], [I])$ and feasibility conditions $([A3], [A4])$ are satisfied.

First,

$$F_B = V_L - x_B \quad [C1]$$

[32] We confirmed this again in the proof of Proposition 2.3

must be true or else we can increase $x_B$ or $F_B$ which will relax other constraints and get the lower $Q$. Next, from $[A3]$,

$$F_B = max\left\{\frac{\frac{x_G}{Q} - pF_G}{(1-p)\sigma(n^*)}, 0\right\} \quad [C2]$$

or else we can decrease $F_B$ and relax other constraints to get lower $Q$. Considering the interior solution, we obtain from [C1] and [C2] that

$$x_B = V_L - \frac{\frac{x_G}{Q} - pF_G}{(1-p)\sigma(n^*)} \quad [C3].$$

Next, from $[E1]$, we obtain

$$(1 + \frac{n}{n^*}QA)(C - x_B) \leq \frac{n}{n^*}Qpx_G$$

where $A = (1-p)\sigma(n^*) < 1$. Next, substituting $x_B$ from $[C3]$ into this condition gives

$$(1 + \frac{n}{n^*}QA)\left(C - \left(V_L - \frac{\frac{x_G}{Q} - pF_G}{A}\right)\right) \leq \frac{n}{n^*}Qpx_G \quad [E1'].$$

Rearranging $[E1']$ gives

$$(1 + \frac{n}{n^*}QA)\,(C - V_L) \leq \frac{n}{n^*}Qp(x_G + F_G) - \left(\frac{n}{n^*} + \frac{1}{QA}\right)x_G + \frac{p}{A}F_G \quad [E1''].$$

Consider $[E2]$. Substituting $F_B = V_L - x_B$ from $[C1]$ and rearranging $[E2]$ gives

$$x_G + F_G \leq \frac{Q(1-p)y_i(n^*)}{1+Qp}(V - V_L) \quad [E2'].$$

From both $[E1'']$ and $[E2']$, to relax the constraint and minimize the value of $Q$, we must set $x_G$ to its lowest possible value while setting $F_G$ to its highest possible value to relax the constraint $[E1'']$ while unchanging the left term of $[E2']$. Since $F_B$ cannot go below zero, we obtain the following lemma.

**Lemma 2.5** $x_G = pQF_G$ *and* $F_B = 0$

This lemma is intuitive. Because there is restriction during the bad state on the maximum price that the principal dealer can charge, further distortion by setting $F_B$ higher will deteriorate profit of $P$ even more. Lastly, substituting $x_G = pQF_G$ into $[E1'']$ yields

$$(1 + \frac{n}{n^*}QA)\,(C - V_L) \leq \frac{n}{n^*}Q^2p^2F_G$$

which can be simplified into

$$\frac{(1 + \frac{n}{n^*}QA)\,(C - V_L)}{\frac{n}{n^*}Q^2p^2} \leq F_G \quad [E1'''].$$

Also, substituting $x_G = pQF_G$ into $[E2']$ gives

$$F_G \leq \frac{(1-p)Qy_i(n^*)(V - V_L)}{(1+Qp)^2} \quad [E2''].$$

Therefore, the equilibrium exist if

$$\frac{(1+\frac{n}{n^*}QA)\,(C-V_L)}{\frac{n}{n^*}Q^2p^2} \leq \frac{(1-p)Qy_i(n^*)(V-V_L)}{(1+Qp)^2}$$

which is equivalent to

$$\frac{(1+\frac{n}{n^*}QA)\,(C-V_L)\,(1+Qp)}{\frac{n}{n^*}Q^2p^2y_i(n^*)} \leq \frac{Q(1-p)(V-V_L)}{1+Qp} \quad [E3.A]$$

Next, we compare the equilibrium existence condition $[E3.A]$ with the existence condition from Proposition 2.2. Recall the condition $[E5']$ from the proof in Proposition 2.2 that liquidity insurance equilibrium between $\{i, P\}$ for any $i \in I$ exists if and only if

$$\frac{(1+Q(1-p))(C-V_L)}{Qp} \leq \frac{Q(1-p)(V-V_L)}{1+Qp} \quad [E3.P]$$

Let $Q_P$ be the minimum $Q$ that satisfies liquidity insurance existence condition between $\{P, i\}$ in the basic model and $Q_A$ be the minimum $Q$ that satisfies liquidity insurance existence condition between $\{P, A, I\}$. From [$E3.A$] and [$E3.P$], $Q_A < Q_P$ if

$$\frac{(1+\frac{n}{n^*}QA)\,(C-V_L)\,(1+Qp)}{\frac{n}{n^*}Q^2p^2y_i(n^*)} \leq \frac{(1+Q(1-p))(C-V_L)}{Qp}.$$

Substituting $A = (1-p)\sigma(n^*)$ into the condition and simplifying give

$$\left(\frac{1+Q(1-p)}{\frac{n^*}{n}+Q(1-p)\sigma(n^*)}\right) \geq \left(1+\frac{1}{Qp}\right)\left(\frac{1}{y_i(n^*)}\right).$$

Lastly, we need to verify that, given that $Q_P > Q > Q_A$,

1. the maximum agency cost of $A$ is indeed $nx_G$ which will validate the incentive constraint of $A$ in Lemma 2.3.
2. $x_G + F_G \leq V_L$

To prove the first claim, we need to show that, when $pQF_G = x_G$,

$$nx_G > zx_G + (n-z)\frac{qp\delta}{1-(1-q)\delta}(x_G+F_G)$$

where the right term of the inequality is the expected payoff of the agency dealer when he decides to choose $P'$ when there are $z$ liquidity orders in a good period. Substituting $pQF_G = x_G$ into the inequality condition yields

$$1 > \frac{qp\delta}{1-(1-q)\delta}(1+\frac{1}{pQ})$$

which, once being simplified, becomes

$$1 > p$$

which is always true. This validates the incentive constraint of $A$ in Lemma 2.3.

To prove our second claim, it is sufficient to prove that $x_F + F_G$ is lower than any price $x_G$ offered by $P$ to $i$ under direct liquidity insurance relationship between $\{P, i\}$. First, the direct liquidity insurance relationship equilibrium between $\{P, i\}$ does not exist (i.e. $Q < Q_P$) when any $x_G$ which satisfies the following constraint

$$x_G \geq \frac{1 + Q(1-p)}{Qp}(C - x_B)$$

violates the incentive constraint of investor $i$ such that

$$x_G > \frac{Q(1-p)}{1+Qp}(V - x_B) \quad [Z1].$$

However, any equilibrium price $x_G + F_G$ under liquidity insurance equilibrium between $\{P, A, I\}$ must satisfy the incentive constraint of investor $i$, which is

$$x_G + F_G \leq \frac{Q(1-p)y_i(n^*)}{1+Qp}(V - (x_B + F_B)) \quad [Z2].$$

Let $\underline{x}_G = \frac{Q(1-p)}{1+Qp}(V - x_B)$ be the minimum price that the principal dealer quotes to investor $i$ in any direct relationship equilibrium. Since $y_i(n^*) \leq 1$, we obtain

$$x_G + F_G \leq \frac{Q(1-p)y_i(n^*)}{1+Qp}(V - (x_B + F_B)) < \underline{x}_G.$$

Since $\underline{x}_G \leq V_L$ at $x_B = V_L$ by assumption, $x_G + F_G \leq V_L$ must be true. This proves the second claim and the proposition.

**Proof of Proposition 2.4:** Recall from the proof of Proposition 2.3 that

$$x_G + F_G \leq \frac{Q(1-p)y_i(n^*)}{1+Qp}(V - (x_B + F_B)) < \underline{x}_G.$$

This immediately proves the proposition.

**Proof of Corollary 2.1:** From the first-order Taylor approximation, $lim_{q\to 0}\kappa(n^*) = nq(1-q)^{n-1} = nq$. Also, $lim_{q\to 0}f_{-i}(0) \to 1$. Therefore, $lim_{q\to 0}y_i(n^*) = 1$, $\sigma(n^*) = \frac{\kappa(n^*)}{\kappa(n)}$, and $lim_{q\to 0}\,\sigma(n^*) = 1$. This proves the first condition in the corollary.

Next, from Proposition 2.3, $Q_A < Q_P$ if

$$\frac{1 + Q(1-p)}{\frac{n^*}{n} + Q(1-p)\sigma(n^*)} \geq \left(1 + \frac{1}{Qp}\right)\left(\frac{1}{y_i(n^*)}\right).$$

Substituting $y_i(n^*) = 1$ and $\sigma(n^*) = 1$ into the condition gives

$$\frac{1 + Q(1-p)}{\frac{n^*}{n} + Q(1-p)} \geq 1 + \frac{1}{Qp}$$

which will be satisfied if $p > \frac{1}{2}$ and

$$\frac{n}{n^*} \geq \frac{1+pQ}{Q(2p-1)}.$$

This proves the corollary.

**Proof of Proposition 2.5:** To prove this proposition, we 1) compare the expected payoff of the high-type investor if he chooses $A$ with his expected payoff if he chooses $P$ and 2) check that the low-type can form the relationship under both types of equilibrium. Let $x_P$ be the price that the principal dealer would charge the high-type investor under separating equilibrium and $x_A + F_H$ be the equilibrium price that the agency dealer would charge the high-type investor under pooling equilibrium. From Lemma 2.2, when $n^* = 2$ under pooling equilibrium and $\{P, A\}$ obtains lowest possible payoff by Assumption 2.2, we obtain

$$x_P = \left(\frac{1}{Q_H p} + \frac{1-p}{p}\right)(C - V_L)$$

and

$$x_A = \left(\frac{2}{(Q_H + nQ_L)p} + \frac{1-p}{p}\right)(C - V_L) \quad \text{and} \quad F_H = \frac{x_A}{pQ_H}.$$

Therefore, the high-type investor would choose $A$ if

$$\left(\frac{2}{(Q_H + nQ_L)p} + \frac{1-p}{p}\right)(1 + \frac{1}{Q_H p}) \leq \frac{1}{Q_H p} + \frac{1-p}{p}.$$

Simplifying the condition gives

$$nQ_L \geq \frac{2+Q_H}{2p-1} \quad \text{provided that} \quad p > \frac{1}{2} \quad [C1].$$

Next,
pooling equilibrium. From Lemma 2.2, the low-type investors would commit to the relationship in the pooling equilibrium if

$$\left(\frac{2}{nQ_L + Q_H} + \frac{1-p}{p}\right)\left(\frac{Q_L p + 1}{Q_L p}\right) \leq \frac{Q_L(1-p)(V - V_L)}{(1 + Q_L p)(C - V_L)}.$$

Rearranging the condition yields

$$n \geq \frac{1}{Q_L}\left(\frac{2p(1+Q_L p)^2(C - V_L)}{(1-p)((Q_L p)^2(V - V_L) - (1 + Q_L p)^2(C - V_L))} - Q_H\right)$$

which is always true by Assumption 2.2. Similarly, from Lemma 2.2, the low-type investors would commit to the relationship in the separating equilibrium if

$$\left(\frac{1}{nQ_L} + \frac{1-p}{p}\right)\left(\frac{Q_L p + 1}{Q_L p}\right) \leq \frac{Q_L(1-p)(V - V_L)}{(1 + Q_L p)(C - V_L)}$$

which is equivalent to

$$n \geq \frac{1}{Q_L}\left(\frac{p(1+Q_L p)^2(C - V_L)}{(1-p)((Q_L p)^2(V - V_L) - (1 + Q_L p)^2(C - V_L))}\right).$$

This is always satisfied by Assumption 2.2. This proves the proposition.

**Proof of Proposition 2.6:** Let $b_s$ be the price that $A$ charges to the low-type investors under separating equilibrium and $b_p$ be the price that $A$ charges to the low-type investors under pooling equilibrium. Notice that $b_s < b_p$ if $[C1]$ is true. To see this, from Lemma 2.2, $b_s < b_p$ if

$$\frac{1}{nQ_Lp} + \frac{1-p}{p} < \frac{2}{(Q_H + nQ_L)p} + \frac{1-p}{p}.$$

Rearranging the condition gives

$$nQ_L > Q_H.$$

which is always true when $[C1]$ is true. This implies that the payoff of the low-type investors can be lower in the pooling equilibrium compared to the case when the high-type investor chooses $P$. Note that this is also true despite relaxing the restriction on no cross-subsidy across investors. To see this, first note that the payoff of the principal dealer from the relationship does not change (which is $2(C - V_L)$). Next, when the high-type investor chooses $A$, it will drive the upstream price upward. Therefore, the amount of surplus that all players must leave to $A$ must be strictly higher by at least the new upstream price $x_G$. However, upon choosing $A$, the high-type investor increases total trade surplus by almost zero since $q_L \to 0$. This proves Proposition 2.6.

**Proof of Lemma 2.4:** To prove the lemma, we compare the payoff of the high-type investor when he chooses $A$ against his payoff when he chooses $P$. Let $x_{PA}$ be the price that $P$ offers to $A$ when $H$ chooses $A$. Let $x_{PH}$ be the price that $P$ offers to $H$ when $H$ chooses $P$. Given that $F_H = \frac{1}{Q_Hp}$, the payoff when $H$ chooses $A$ is higher than the payoff when he chooses $P$ if

$$Q_H(p(V - X_{PA} - F_H) + (1-p)M(V - V_L)) \geq Q_H(p(V - x_{PH}) + (1-p)(V - V_L))$$

where $M$ is the probability that the high-type investor will get liquidity in bad states if joining $A$. This term will be at the lowest value when $M = 1 - nq_L$, which is when $A$ gives priority to the low-type investors. Substituting $M = 1 - nq_L$ and $F_H = \frac{1}{Q_Hp}$ into the condition, we obtain

$$x_{PH} - x_{PA} \geq \frac{1}{Q_Hp}x_{PA} + \left(\frac{1-p}{p}\right)(nq_L)(V - V_L)$$

which proves the lemma.

**Proof of Proposition 2.7:** From Lemma 2.2, we obtain

$$x_{PH} = \left(\frac{1}{Q_Hp} + \frac{1-p}{p}\right)(C - V_L)$$

and that

$$x_{PA} = \left(\frac{1}{(Q_Ln + Q_H)p} + \left(\frac{1-p}{p}\right)\left(\frac{nQ_L + Q_H - nQ_Lq_H}{nQ_L + Q_H}\right)\right)(C - V_L).$$

Let $\hat{Q} = nQ_L$ and $Q_N = Q_H + \hat{Q}$. Plugging $x_{PH}$ and $x_{PA}$ into

$$x_{PH} - x_{PA} \geq \frac{1}{Q_H p} x_{PA} + \left(\frac{1-p}{p}\right)(nq_L)(V - V_L)$$

yields

$$\frac{1}{Q_H p} + \frac{1-p}{p} - \left(1 + \frac{1}{Q_H p}\right)\left(\frac{1}{Q_N p} + \left(\frac{1-p}{p}\right)\left(1 - \frac{nq_L q_H}{\hat{q}}\right)\right) \geq \hat{q}\left(\frac{1-p}{p}\right)\left(\frac{V - V_L}{C - V_L}\right)$$

which must be true if

$$\frac{1}{Q_H p} + \frac{1-p}{p} - \left(1 + \frac{1}{Q_H p}\right)\left(\frac{1}{Q_N p} + \frac{1-p}{p}\right) \geq \hat{q}\left(\frac{1-p}{p}\right)\left(\frac{V - V_L}{C - V_L}\right).$$

Rearranging this condition gives

$$\hat{Q} - \frac{1}{p} - (1-p)Q_N \geq \hat{Q} Q_N q_H (1-p)\left(\frac{V - V_L}{C - V_L}\right)$$

which is identical to

$$\hat{Q}^2 q_H (1-p)\left(\frac{V - V_L}{C - V_L}\right) + \hat{Q}\left[q_H^2 (\frac{\delta}{1-\delta})(1-p)\left(\frac{V - V_L}{C - V_L}\right) - p\right] + \frac{1}{p} + (1-p)Q_H \leq 0.$$

The left side is the standard quadratic function of $\hat{Q}$. From the above inequality, a solution for $\hat{Q}$ exists if

$$q_H^2 (\frac{\delta}{1-\delta})(1-p)\left(\frac{V - V_L}{C - V_L}\right) - p < 0$$

and

$$\left(p - q_H^2 (\frac{\delta}{1-\delta})(1-p)\left(\frac{V - V_L}{C - V_L}\right)\right)^2 - 4q_H(1-p)\left(\frac{V - V_L}{C - V_L}\right)\left(\frac{1}{p} + (1-p)Q_H\right) \geq 0.$$

Both of them will be true if $q_H$ is sufficiently low. This proves Proposition 2.7.

# Chapter 3

# Strategic Formation of Financial Network

## 3.1 Introduction

In globalization era, financial interconnectedness becomes a prominent feature of global financial landscape via all kinds of connection - equities, debts, derivatives, and other types of financial obligations. While financial interlinkage brings a benefit of risk-sharing across financial entities, it also generates financial fragility, or the so-called systemic risk, due to cascading failure from interdependencies within the financial system. The materialization of systemic risk in the recent financial crisis highlights a necessity to understand formation of financial network and possible build-up of systemic risk.

While the literatures have unveiled systemic consequence of cascading failures of a fixed financial network,[1] the story of financial network formation which is critical for regulatory design remains mystifying to regulators. In this spirit, this paper develops a game-theoretic model to study strategic formation of financial network. The main objective is to identify equilibrium pattern of financial network emerged from strategic decision of all self-interested agents who can trade forward contracts (i.e. assets) to obtain fractions of investment return of other players in a trading network. We contribute to the literature by shedding light on how the equilibrium asset allocation depends on structural properties of trading network.

In the model, there is a finite number of risk-averse agents investing in a risky project and holding unlimited amount of cash. To diversify the risk, agents can issue forward contracts (i.e. asset) which guarantee a fraction of their project returns and sell to whom they have trading connection with. All trades are bilateral, each involving two parties privately agreeing on asset price and quantity. All trading relationships are represented by a directed network, in which

[1] A few examples include Allen and Gale (2000), Elliott et al (2014), Acemoglu et al (2015), Cabrales et al (2014).

vertices represent all agents in the economy and arcs indicate possibilities of bilateral trading among agents.[2]

Asset prices are endogenously determined in two stages. In the first stage, agents simultaneously quote asset prices to all neighbouring buyers with a promise to sell unlimited amount of assets at the quoted prices. In the second stage, after observing the quotes, all agents decide quantities of asset demand. Asset resale is prohibited throughout the paper. We characterize the unique subgame perfect equilibrium for an arbitrary trading network under standard assumptions of CARA utility function and normal distribution of asset returns.

The main feature of the model is endogeneity of asset demand function, which depends on topology of the trading network and asset return correlations. Intuitively, a quantity change in asset demand of an agent from a price change causes his neighbour, his neighbour's neighbours, and so on to optimally adjust their portfolio compositions accordingly. This in turn can cause non-trivial change in asset demand of all agents despite the agents being indirectly connected through a sequence of counterparties. Such information is summarized in *quantity impact matrix*, the coefficient matrix of asset demand functions determined in the second stage, which indicates the sensitivity of demand quantity to a price change of all assets.

To examine how equilibrium asset allocation depends on structure of trading network, we use the concept of line graph transformation to obtain *a network of asset flows* and identify *trading-link* positions. Conceptually, the line graph transformation converts links of all agents in the original network to become vertices in a new network, the network of asset flows, and edges between each pair of new nodes represent the agents that tie the asset flows together. Theoretically, the line graph is a network representation of local interactions of all players, so its centrality plays a critical role in determining the equilibrium outcome.

There are three main insights from the model. First, the quantity impact matrix corresponds to contribution matrix of a generalized Bonacich centrality measure of an asset-flow network, in which the links between asset-flow nodes represent the buying parties of the asset flows. The weight assigned to each link in the asset-flow network depends on opportunity of the buyer in the flow to find substitutable assets and to sell his own asset. Intuitively, if a flow is central, a price change in the flow will have significant impact on asset demand quantity of other asset flows in the network.

The second insight is that the equilibrium asset allocation corresponds to a generalized Bonacich centrality index of a network of asset flows, in which the links between asset-flow nodes represent the buying and selling parties of the flows. Intuitively, demand quantity in a flow is large (small) if the flow is adjacent to large-quantity flows with positive (negative) weight in the asset flow network. We found that the weight assigned to each link in the asset flow network relies on 1) the opportunity of the buying agent in the flow to buy other assets or to sell his asset, 2) the opportunity of the selling agent to sell the asset to other neighbours, and 3) the cost of network externality from a price increase in a flow towards asset demand of other flows that belong to the same seller. Intuitively, the selling agent will internalize the externality cost by strategically increasing trading quantity (by reducing the price) in the asset flow that generates positive externality toward his other asset outflows.

[2]Note that the assumption on directed network implies that some agents might be only the seller or the buyer in the relationship.

The third insight is that a constrained efficient allocation, the allocation which maximizes the Utilitarian welfare subject to the trade friction of incomplete network, corresponds to a generalized Bonacich centrality of an asset flow network, in which the links between asset-flow nodes represent the agents whose utilities are affected by a quantity change in the corresponding pair of asset-flow nodes. The main difference between the constrained efficient outcome and the equilibrium outcome is the weight assigned to the links in the asset flow network. Specifically, the weight for the constrained efficiency outcome depends on whether a quantity increase in both connecting asset-flow nodes increases the utility of the agent corresponding to the link between the two nodes.

We also found that the equilibrium asset allocation in a complete network is not efficient in general due to monopolistic price-setting friction. An exception is when asset correlations converge to one or become sufficiently negative. In both cases, the (inverse) asset demand function becomes perfectly elastic; an increase in asset price can cause significant impact on the demand quantity. Interestingly, when the correlation is sufficiently negative, market efficiency can also be achieved even though the assets are not perfectly substitutable. The intuition is as follows. With negative correlation, agents are willing to purchase high volume of assets despite facing high asset price due to hedging benefits. When the correlation is highly negative, the sellers can quote (almost) a zero-risk-premium asset price without losing the demand quantity. However, a price increase above this level will significantly reduce demand quantity since it exceeds the return mean. Since all agents cannot extract more rents by raising asset prices and limiting demand quantities, the market inefficiency from the monopolistic friction disappears.

To obtain more insights, we further examine two specific cases: the case of no correlation and the case of identical correlation in a star network and a core-periphery network. The no-correlation assumption simplifies the model to a monopoly problem with differentiated goods, as the quantity impact matrix becomes diagonal. We found that equilibrium asset allocation and risk premium depend conversely on the degree centrality of selling parties, as predicted by the standard monopoly problem.

The model provides a few interesting insights when examining the case of negative correlations. Specifically, in the star network, central agent holds larger proportion of his own asset than peripheral agents when the correlation is sufficiently negative. Moreover, when the number of peripheral agents increases, the central agent sells less proportion of his own asset away despite facing higher asset demands. The explanation is that, when the correlation is negative, virtual value of assets to an agent depends on the variety of assets that the agent is holding due to hedging benefits, resulting in larger amount of asset holdings of the hub in the equilibrium.

Considering equilibrium payoff, in the case of no asset correlation, trading with well-connected neighbours is less beneficial than with stand-alone neighbours because of higher competition. However, when the correlation is negative, connecting to well-connected agent can be more beneficial. In the star network, if the asset correlation is negative, the utility of the peripherals increases when the number of peripherals increases. This is because the benefit from higher virtual asset valuation to the hub, of which the peripherals can take advantage by charging higher prices, outweighs the loss from higher competition.

### Related literature.

This paper belongs to a vast growing literature in network theory in economics. Bramoullé et al (2016) provided detailed discussion on the literature. Our paper is closely related to a small literature of network games which map equilibrium outcome to Bonacich centrality in a linear-quadratic framework. Ballester et la (2006) was the first to find the linkage between Nash equilibrium and Bonacich centrality. Specifically, they found that Nash equilibrium of noncooperative games with finite number of players, local complementarity, and linear-quadratic interdependent utility is proportional to players' Bonacich centrality in the network when the direct effect is small; that is, when the equilibrium is interior and unique. Ballester and Calvó-Armengol (2010) generalized the theory to cover broader class of games with local substitutability in which local complementarity can be induced by a specific transformation. Bramoullé and Kranton (2007) considered a game of public goods provision which has large direct effects. Bramoullé et al (2014) generalized the findings in Ballester et al (2006) and Bramoullé and Kranton (2007) by considering arbitrary level of direct effects and concluded that equilibrium outcome depends on the lowest eigenvalue.

A few papers have developed network models with linear-quadratic framework and small direct effects to study specific settings. For example, Calvó-Armengol et al (2009) developed a network model to study peer effects in education. Denbee et al (2018) developed a model of interbank network to study reserve holding decisions of banks. Candogan et al (2012) considered a trading game in which a monopolist can sell a divisible good to consumers who face local peer effects in social network from the goods consumption. They found that equilibrium prices of the monopolist charged to each consumer correspond to Bonacich centrality of the consumer's position in the social network. The most closely related literature to this paper is Bimpikis et al (forthcoming). Specifically, Bimpikis et al (forthcoming) considered a Cournot game with homogenous goods in bipartite network and applied the concept of line graph to show that the equilibrium production quantity corresponds to Bonacich centrality of the line graph. Our paper contributes to this literature by considering an asset trading game and relating the equilibrium asset allocation to the Bonacich centrality of the asset flow network.

There is a growing literature of trading in decentralized markets which can be divided mainly into two groups based on assumptions of market environment. In the first group, studies of asset trading were conducted in a search and bargaining model in which atomistic traders are matched randomly in large markets (e.g. Duffie et al (2005,2007), Lagos et al (2011), Vayanos and Weill (2008), Lagos and Rocheteau (2009), Afonso and Lagos (2015)). In the other group, trades occur on a fixed network (e.g. Kranton and Minehart (2001), Gale and Kariv (2007), Blume et al (2009), Manea (2011), Candogan et al (2012), Gofman (2014), Nava (2015), Condorelli et al (2017), Choi et al (2017), Malamud and Rostek (2017), Kondor and Babus (2018), Bimpikis et al (forthcoming)). Our paper belongs to the latter group. To the best of our knowledge, this paper is the first to show the relationship between equilibrium asset allocation and centrality of asset flows in decentralized asset markets.

Lastly, this paper also relates to a literature of financial network formation. While the theory literature focused mainly on financial network formation from credit lending which can create systemic crisis due to a sequence of counterparty risk exposure (e.g. Farboodi (2017), Babus

(2016), Acemoglu et al (2015), Zawadowski (2013)), a small literature discussed systemic risk from common asset holdings (e.g. Wagner (2008, 2010), Raffestin (2014)). Allen et al (2012) developed a model with endogenous asset holding to analyze how an interplay between common asset holding and short-term debt can generate systemic risk. We contribute to the literature by characterizing strategic asset holdings of all agents in a general trading network which could be further generalized to analyze the degree of systemic risk.

### Structure of the paper.

The rest of the paper is organized as follows. Section 2 describes the model setting. Section 3 discusses the notion of line graph transformation. Section 4 examines equilibrium outcome and discusses how it relates to the network centrality. Section 5 considers a constrained efficient outcome and implications on market efficiency. Section 6 considers a few special cases of trading network. Section 7 concludes. All proofs are in appendix.

## 3.2 The model

Consider a one-period economy with a finite number of risk-averse agents described by a set of players $N = \{1, 2, ..., n\}$. Each agent invests in a risky project and holds unlimited amount of cash. The project returns follow a joint normal distribution $N(d, \Sigma)$ in which $\Sigma$ is positive definite with all diagonal elements of one ($\sigma_i^2 = 1; \forall i \in N$).[3] Denote off-diagonal element $\sigma_{ij} = \Sigma_{ij}$. All agents have CARA utility function with identical degree of risk aversion $\alpha$.

To diversify the risk, each agent can trade forward contract, a claim on the issuer's project return.[4] Equivalently, all agents are endowed with one divisible unit of risky forward contract (i.e. asset), which is a claim to their own project returns, and they can trade the assets among each other. Denote forward contract of agent $i$ as asset $i$. Assume that $d_i \geq \alpha\sigma_i^2, \forall i \in N$ to ensure that every agent prefers holding one unit of their own assets at the outset.[5]

To describe the trading environment, let a graph $g$ represent bilateral trading relationships among agents, in which $ij \in g$ indicates the possibility of agent $i$ selling asset $i$ to agent $j$. The trading network is directed (i.e. $ij \in g \not\rightarrow ji \in g$) and common knowledge. Throughout the paper, flow $ij \in g$ refers to trade link $ij$. Let $B(i) = \{j \in N, ij \in g\}$ and $S(i) = \{k \in N, ki \in g\}$ be the set of buyers and sellers of agent $i$, respectively. Denote $\bar{S}(i) = S(i) \cup \{i\}$ and $\bar{B}(i) = B(i) \cup \{i\}$. Let $q_{ij}$ be asset quantity that agent $i$ sells to agent $j$ and $q_{ii} = 1 - \sum\limits_{j \in B(i)} q_{ij}$

[3] This implies that all the project returns have homogeneous volatility which is normalized to one, but the pairwise correlation across agents can be different.

[4] Note that out setting fits in two possible scenarios. The first is when the project ownership is transferred. This is just a standard asset trading game. The other case is when the project ownership is not transferable. In this case, the agents may refuse to pay the project return to the contract buyers when exposing to losses from other assets in the portfolio. Consequently, while sharing the idiosyncratic risk amongst agents, trading forward contracts also originates the network of risk exposure, thereby generating systemic risk. Our setting can explain this situation if the agents do not realize the counterparty risk when trading assets.

[5] The marginal benefit of holding more units of asset exceeds the marginal cost when the amount of asset holding is less than $\frac{d_i}{\alpha\sigma_i^2}$. To see this, note that the first-order condition when the agent cannot trade is $\frac{du_i}{dq_i i} = d_i - \alpha\sigma_i^2 q_i = 0$ which gives the optimal asset holding $q_i = \frac{d_i}{\alpha\sigma_i^2}$. Therefore, if $\frac{d_i}{\alpha\sigma_i^2} < 1$, holding one unit of assets is be optimal. This assumption is analogous to the classical non-satiation assumption within the set $[0, 1]^n$. If this assumption is violated, the agents would prefer discarding a part of initial asset holding, as holding one unit of asset is not optimal.

is the net asset $i$ holding of agent $i$ after trading. The CARA utility function of agent $i$ can be simplified to the following mean-variance utility function:

$$U_i = d_i - \sum_{j \in B(i)} q_{ij}(d_i - p_{ij}) + \sum_{k \in S(i)} q_{ki}(d_k - p_{ki}) - \frac{\alpha}{2}[\sum_{k \in \bar{S}(i)} \sum_{l \in \bar{S}(i)} \sigma_{kl} q_{ki} q_{li}].$$

The asset prices are endogenously determined in two stages. In the first stage, every agent $i \in N$ quotes (per unit) asset prices, denoted by $p_{ij}$, to all his potential buyers $j \in B(i)$. The price-quoting process for all transactions takes place simultaneously. In the second stage, upon receiving the price offers, all agents $j \in B(i)$ decide how much to buy from agent $i$, denoted by $q_{ij}$. Afterwards, all trades are settled in which all agents $j \in B(i)$ pay $p_{ij}q_{ij}$ to agent $i$ and obtain asset $i$ at the quantity $q_{ij}$. For simplicity, assume that agents cannot sell other agents' assets. We follow the standard notion of subgame perfect equilibrium as follows.

**Definition 3.1 (Subgame Perfect Nash Equilibrium)** *The equilibrium strategy profile consists of a vector of asset prices $P$ and a collection of demand function $Q = \{q_j : P \to \mathbb{R}\}$ such that*

$$q_j = \arg\max_{q_j \in Q_j} u_j(Q, P); \forall j \in N$$

$$p_i = \arg\max_{p_i \in \mathbb{R}^{n(i)}} u_i(q(P), P); \forall i \in N$$

*where $Q_j = \{q_j : P^{\mathbb{R}_+^{N(j)}} \to \mathbb{R}^{n(j)}\} \quad \forall j \in N$ and $q(P) = \{q_j(P), \forall j \in N\}$*

This is the standard definition of the subgame perfect equilibrium which could be solved by backward induction. First, we obtain asset demand functions of all players in the second stage by solving the system of best-response functions given an arbitrary price vector $P$. Given the asset demand functions, we then solve for equilibrium prices in the first stage.

## 3.3 Notion of line graph transformation

This section will discuss the concept of line graph transformation. Instead of considering the position of decision-making agents, we will consider the position of *asset flows* using the notion of line graph. Essentially, the line graph transformation is a means of constructing the network of relationships, by converting all links of the original graph into vertices of the new graph and assigning links to the new nodes accordingly. Consider the formal definition of the line graph in Definition 3.2.

**Definition 3.2 (Line graph)** *Consider a network $G = (V, E)$. The line graph of G, $L(G) = (V', E')$, is such that*

*1) each vertex of $L(G)$ describes an edge of G.*

*2) a pair of vertices of $L(G)$ are connected if and only if the corresponding edges in G are incident.*

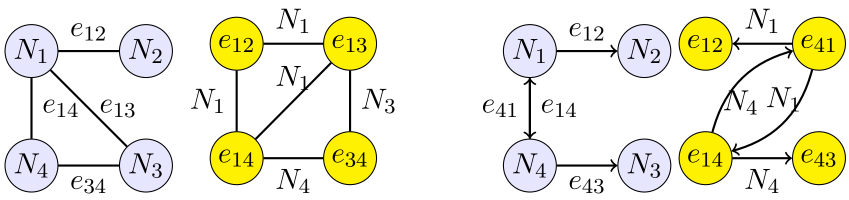


Figure 3.1: Graphical illustration of line graph transformation for undirected network (left) and directed network (right)

Simply put, a line graph of $G$ describes the relationships between edges of $G$. By transforming every edge of $G$ to be vertices of $L(G)$, two nodes of $L(G)$ are linked if the corresponding edges in $G$ are neighbouring by a node in $G$. The line graph transformation is straightforward for the case of undirected network. The left panel in Figure 3.1 displays a line graph for an undirected network in which all the links in the original network becomes the vertices in the new network.

The definition can be applied to the case of directed network. However, a complication arises when assigning links and link directions to the line graph. In the literature, a directed line graph $L(G) = (V', E')$ is the transformation of the directed network $G = (V, E)$ when 1) all the arcs $ij \in E$ are transformed to be nodes $ij \in V'$ of the directed line graph $L(G)$ and 2) two vertices which represent directed edges from $i_1$ to $j_1$ and from $i_2$ to $j_2$ in $G$ are connected in the line graph if $j_1 = i_2$. This is shown in the right panel of Figure 3.1. From the figure, the new nodes represent trading relationships between agents and the new edges represent the players who knot the trading links together. The direction of the new edges indicates the direction of asset flowing within the original trading network.

In our paper, the transformed line graph will consist of four types of linkages. Analogous to traditional network theory, we define four types of link adjacency matrices which summarize the information about different types of connection between two flows as follows.[6]

**Definition 3.3 (Link adjacency matrix)** *In a line graph L(G), the entries of its corresponding link adjacency matrices $W_B^{out}$, $W_B^{in}$, $W_S^{in}$, $W_S^{out}$ are*

$$W^{in}_{S,(i_1j_1,i_2j_2)} = \begin{cases} 1 & \text{if } i_1 = j_2 \\ 0 & \text{if otherwise} \end{cases} \qquad W^{out}_{S,(i_1j_1,i_2j_2)} = \begin{cases} 1 & \text{if } i_1 = i_2 \text{ and } j_1 \neq j_2 \\ 0 & \text{if otherwise} \end{cases}$$

$$W^{out}_{B,(i_1j_1,i_2j_2)} = \begin{cases} 1 & \text{if } j_1 = i_2 \\ 0 & \text{if otherwise} \end{cases} \qquad W^{in}_{B,(i_1j_1,i_2j_2)} = \begin{cases} 1 & \text{if } j_1 = j_2 \text{ and } i_1 \neq i_2 \\ 0 & \text{if otherwise} \end{cases}$$

From the definition, $W_B^{out}$ and $W_S^{in}$ are the standard out-degree and in-degree adjacency matrices for a directed line graph in the literature. Intuitively, $W_B^{out}$ identifies which flows depart from the same buyer while $W_S^{in}$ indicates which flows are pointing toward the same seller in the original trading network. Consider the left panel in Figure 3.2. For the flow $ij$, the matrix $W_B^{out}$ will identify the asset outflows of buyer $j$ while the matrix $W_S^{in}$ will identify the asset inflows

[6] In the literature, the adjacency matrix for a directed line graph is the outward adjacency matrix $W_B^{out}$. Note that the link adjacency matrix is not necessary symmetric.

of seller $i$. Similarly, the other two matrices $W_B^{in}$ and $W_S^{out}$ will indicate which flows have the same buyer and the same seller, respectively. For flow $ij$ in Figure 3.2, $W_B^{in}$ will identify all the other flows that agent $j$ is the buyer, while $W_S^{out}$ will identify all the other flows that agent $i$ is the seller.

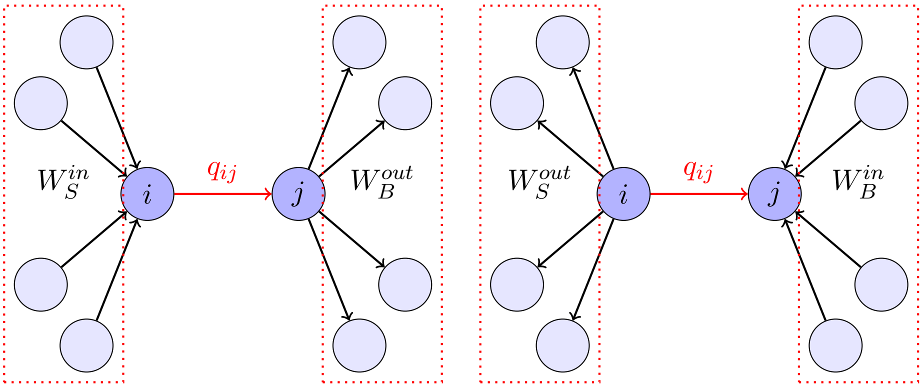


Figure 3.2: Graphical illustration of the matrix $W_S^{in}$ and $W_B^{out}$ (left) and the matrix $W_S^{out}$ and $W_B^{in}$ (right).

The usage difference among these four adjacency matrices might not be evident at this point, but all of them play important parts in determining centrality index of asset flows in later sections. In short, $W_B^{in}$ and $W_B^{out}$ will matter for characterizing asset demand function in the second stage, when the agents must decide how much to buy assets for a given price. Meanwhile, $W_S^{in}$ and $W_S^{out}$ will matter when characterizing equilibrium price and asset allocation in the first stage, when the agents must decide what prices to set.

## 3.4 Equilibrium outcome and network structure

This section will analyze how equilibrium outcome depends on structural properties of trading network. We will proceed by considering the definition of generalized Bonacich centrality measure and proving that quantity impact matrix, asset allocation, and risk premium in the equilibrium are all closely related to the generalized Bonacich centrality of asset flow network.

To begin, we will list down all relevant notations in our analysis. Let the row (column) $ik$ of a matrix be the row (column) corresponding to the flow $ik$ and $n(g) = |g|$ be the number of transactions. Define the following notations.

1. $Q_{n(g)\times 1}$ is a quantity vector of all transactions $ij \in g$.
2. $P_{n(g)\times 1}$ is a price vector of all transaction $ij \in g$.
3. $\Sigma_L$ is a $n(g) \times n(g)$ correlation matrix in which $\Sigma_{L,(i_1k_1,i_2k_2)} = \sigma_{i_1i_2}$.
4. $d$ is a $n(g) \times 1$ vector of asset return mean where $d_{ik} = d_i$.
5. $z$ is a $n(g) \times 1$ vector of asset correlations where $z_{ik} = \sigma_{ik}$.

Now, consider the following definition of generalized Bonacich centrality for a line graph.[7]

[7]See Jackson (2008) for more details on Bonacich centrality.

**Definition 3.4 (Generalized Bonacich Centrality)** *Let $K$ be an $n \times 1$ real-valued vector and $A$ be a weighted adjacency matrix of a line graph $L(G)$. The vector of the generalized Bonacich centrality $C(A, K)$ of $L(G)$ is such that*

$$C(A, K) = (I - A)^{-1} K$$

*where $(I - A)^{-1}$ is a contribution matrix and $K$ is a vector of base values.*

In the literature, the simplest centrality measure is the degree which counts the number of ties. In contrast, the generalized Bonacich centrality quantifies a node importance that depends not only on the number of its neighbours but also on the *importance* of its neighbours. A node with *less degree* might be more (less) important as measured by the Bonacich centrality if it is surrounded by important nodes when the weight is positive (negative).[8]

The information about asset flow centrality is contained in the contribution matrix $(I - A)^{-1}$: row $i_1 j_1$ of $(I - A)^{-1}$ represents the impact of flow $i_1 j_1$ to the network and the column $i_2 j_2$ represents the impact of other flows in the network toward flow $i_2 j_2$. Vector $K$ identifies a base value of each node, capturing the extent to which the centrality of each node is affected by some exogenous factors. Multiplying this contribution matrix with the vector $K$ gives a generalized Bonacich centrality measure; that is, centrality of flow $i_1 j_1$ is the weighted sum of the contributions of flow $i_1 j_1$ toward the network.

In our trading game, the line graph is a network representation of local interactions of all players, in which a node can represent trading decision in a transaction and a link between two nodes represents the player who makes strategic trading decision of the corresponding nodes. Hence, an equilibrium outcome emerged from strategic local interactions of the entire trading network will somehow related to the generalized Bonacich centrality index, taking into account both direct effects (toward adjacent asset-flow nodes) and indirect effects (toward other non-adjacent asset-flow nodes). Also, the weight for adjusting adjacency matrix will indicate the degree of strategic interdependence. That is, a positive (negative) weight indicates the degree of strategic complementarity (substitution).

### 3.4.1 Asset demand function

We will start the equilibrium analysis by characterizing an equilibrium demand function. Let $\circ$ be the standard Hadamard product matrix operator. That is, for any $m \times n$ matrix $A$ and $B$, $(A \circ B)_{ij} = A_{ij} B_{ij}$. Consider the following lemma.

**Lemma 3.1 (Equilibrium demand function)** *The equilibrium asset demand function is such that*

$$Q = \Phi(d - \alpha z) - \Phi P$$

*where $\Phi = \frac{1}{\alpha}(I - (\Sigma_L \circ W_B^{out} - \Sigma_L \circ W_B^{in}))^{-1}$ is a quantity impact matrix.*

From the lemma, the magnitude of quantity impact of $p_{ij}$ relates to Bonacich centrality of flow $ij$ in an asset flow network. When asset returns are correlated, marginal utility of buying

[8] See Bonacich (1987) and Bonacich and Lloyd (2004) for the interpretation of negative weights.

assets of an investor depends on his portfolio composition. Therefore, if an agent decides to buy less amount of his neighbour's asset, it will affect his neighbour's trading decision which in turn affects his neighbours' neighbour's decision and so on, resulting in the asset demand being dependent on asset prices of all transactions. Information about the global effect of a price change throughout the network is summarized in the quantity impact matrix $\Phi$.

Indeed, the quantity impact matrix is the contribution matrix of the asset flow network with weighted adjacency matrix $A = \Sigma_L \circ W_B^{out} - \Sigma_L \circ W_B^{in}$. This implies that a sensitivity of demand quantity to $p_{i_1 j_1}$ depends on the contribution value of flow $i_1 j_1$ toward other flows embedded in the asset flow network. If flow $i_1 j_1$ is in a central position, in which a change in the quantity of asset flow $q_{i_1 j_1}$ significantly interrupts other flows in the network, the impact of change in $p_{i_1 j_1}$ will be influential. Figure 3.3 illustrates how a change in $p_{31}$ can affect asset demands of all agents. When asset demand $q_{31}$ changes due to a change in price $p_{31}$, this will directly affect $q_{13}$ which will further affect other flows. If flow 31 is central, the impact towards other flows can be significant.

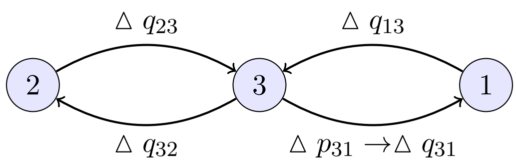


Figure 3.3: Graphical illustration in the original trading network of how a change in $p_{31}$ can affect asset demands of all agents.

Note that we assign non-zero weights to the link adjacency matrices which identifies all pairs of asset flows linked by the same buyers. This is intuitive, as asset demands depend on strategic buying decision of agents. In particular, $W_B^{out}$ and $W_B^{in}$ capture how buying decision of an agent will be affected by demand quantity of his asset outflows and his other asset inflows, respectively. This is graphically illustrated in Figure 3.4. Also, the link adjacency matrix $W_B^{out}$ is positively weighted while $W_B^{in}$ is negatively weighted by asset correlations to reflect the asset substitutability degree. Intuitively, when the asset correlation is positive, the agent's asset demand in a flow will be positively (negatively) affected by a quantity increase in another asset outflow (inflow).

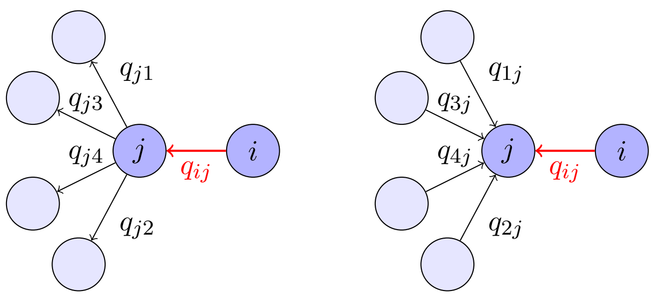


Figure 3.4: Graphical illustration of how buying decision of agent $j$ in flow $ij$ is affected by demand quantity in his other asset inflows (right) and asset outflows (left).

### 3.4.2 Asset allocation and prices

This section will discuss how equilibrium asset allocation depends on structure of asset-flow network. We will show that the equilibrium asset allocation corresponds to a generalized Bonacich centrality of network of asset flows with appropriate weight assignment to each link.

First, we must ensure that our interior solution in the first stage game satisfies second-order condition. Consider an arbitrary $n \times n$ matrix $E$. A submatrix of $E$, denoted by $E[i_1, i_2, \cdots, i_n | j_1, j_2, \cdots, j_k]$, is the matrix consisting of intersection of rows $\{i_1, i_2, \cdots, i_n\}$ and columns $\{j_1, j_2, \cdots, j_n\}$. Let $M_i^{out} = \{ik; \forall k \in B(i)\}$ be the set of outflows of agent $i$ and $M_i^{in} = \{ki; \forall k \in S(i)\}$ be the set of inflows of agent $i$. Denote $E_i^{a,b}$ be the submatrix of $E[M_i^a, M_i^b]$ when $a, b \in \{in, out\}$ which contains the intersection of rows $M_i^a$ and columns $M_i^b$. To illustrate, $\Phi_i^{out,in}$ is the submatrix of $\Phi$ in which all the rows correspond to all asset outflows of agent $i$ and all the columns correspond to all asset inflows of agent $i$. We make the following assumption.

**Assumption 3.1 (Second-order condition)** *For all $i \in N$,*

1. $\Phi_i^{out,out} + \Phi_i^{out,out'}$ *is positive definite*
2. $\Phi_i^{in,out'} \Sigma_{L,i}^{in,in} \Phi_i^{in,out} - [\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi_i^{in,out} + (\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi_i^{in,out})'] + \sigma_i^2 \Phi_i^{out,out'} \Phi_i^{out,out}$ *is positive definite*

This assumption gives a sufficient condition to guarantee that an interior price solution derived from first-order condition in the first stage is utility-maximizing.[9] Intuitively, the assumption puts a restriction on asset demand function to be well-behaved. The first restriction is analogous to downward-sloping demand function in the standard profit maximization problem. The second restriction takes into account a change in marginal utility from risk component of risk-averse sellers. Consider the following proposition regarding the equilibrium asset allocation.

**Proposition 3.1 (Equilibrium asset allocation)** *In a trading network $G$, provided that Assumption 3.1 is true, the equilibrium asset allocation corresponds to a generalized Bonacich centrality index of a line graph of $G$. Specifically, there exists a real-valued diagonal matrix $V = V(\Phi, W_S^{out})$ and a real-valued off-diagonal matrix $F = F(\Phi, W_S^{out})$ such that*

$$Q = (I - V\bar{A})^{-1} V(\hat{\mathbf{1}} - z)$$

*where $\bar{A} = \Sigma_L \circ (W_B^{out} + W_S^{in} - W_B^{in} - W_S^{out}) + F \circ W_S^{out}$.*

Intuitively, the adjacency matrix $\bar{A}$ summarizes local interactions of all selling parties, who indirectly choose demand quantity of all his asset outflows by strategically setting prices in the first stage. To interpret this proposition, I decompose the adjacency matrix $\bar{A}$ into three components as follows:

$$\bar{A} = \underbrace{\Sigma_L \circ (W_B^{out} - W_B^{in})}_{(1)} + \underbrace{\Sigma_L \circ (W_S^{in} - W_S^{out})}_{(2)} + \underbrace{F \circ W_S^{out}}_{(3)}.$$

[9] Specifically, the assumption guarantees that $u_i(q(P), P)$, is concave in prices $p_i$ so that the first-order (linear) condition is indeed the best response function for all sellers.

The first component (1) identifies how every pair of asset flows in the asset-flow network is connected by the buying parties in the flows. Specifically, the matrices capture the opportunity of the buying party in each flow to sell their own assets to their neighbors ($W_B^{out}$) and to buy assets from other neighbors ($W_B^{in}$). All links in the asset-flow network are weighted by asset correlation to adjust for asset substitutability. Intuitively, this component identifies the asset demand function. Notice that this matrix component is identical to the adjacency matrix that determines the quantity impact matrix $\Phi$, the coefficient matrix of asset demand function, as established in Lemma 3.1.

The second and third components (2,3) identify how every pair of asset flows is connected by the selling parties in the flows. Specifically, the second component (2) captures the opportunity of the selling party in each flow to sell their assets to other neighbors ($W_S^{out}$) and to buy other assets from their neighbors ($W_S^{in}$), weighted by correlation matrix to account for asset substitutability.

Interestingly, the last component (3) identifies how every pair of asset flows can be affected by a cost of network externality, an externality of a price change in a flow which can affect demand quantity in other outflows of the same seller. Intuitively, a price change in flow $ij$ will not only affect demand quantity $q_{ij}$ but also indirectly affect demand quantity in other outflows of the same selling agent ($q_{ik}; \forall k \in B(i)$), even though prices in other outflows remain unchanged. The extent of network externality is indicated in the *off-diagonal* matrix $F(\Phi, W_S^{out})$. When flow $ij$ has positive externality toward other outflows of agent $i$ (i.e. a quantity increase in flow $ij$ increases demand quantity in other outflows), agent $i$ will strategically increase $q_{ij}$, by setting low price $p_{ij}$, to earn extra profits in other asset outflows from the externality.

Another weighting matrix that affects the centrality of asset flows is the *diagonal* real-valued matrix $V(\Phi, W_S^{out})$. Intuitively, it identifies own-price network effect. To illustrate, when agent $i$ increases price $p_{ij}$, there is direct effect on quantity demand $q_{ij}$. However, there is also indirect network effect on quantity demand $q_{ij}$. Because demand quantity in other flows will be affected from the first-round change in $q_{ij}$, asset demand $q_{ij}$ will in turn change in response to endogenous change of portfolio composition of agent $j$.

Lastly, regarding equilibrium asset price, consider the asset demand function.

$$\underbrace{\frac{1}{\alpha}(I - (\Sigma_L \circ W_B^{out} - \Sigma_L \circ W_B^{in}))^{-1}}_{\text{Quantity impact matrix } \Phi}(d - P - \alpha r) = \underbrace{(I - V\bar{A})^{-1} V(\hat{\mathbf{1}} - z)}_{\text{Equilibrium asset allocation Q}}$$

Recall that the quantity impact matrix $\Phi$ is the contribution matrix corresponding to a generalized Bonacich centrality of the asset flow network with weighted adjacency matrix $\Sigma_L \circ W_B^{out} - \Sigma_L \circ W_B^{in}$. As we have discussed, the equilibrium asset allocation $Q$ is the generalized Bonacich centrality index of asset flow network with weighted adjacency matrix $V\bar{A}$ and base vector $V(\hat{\mathbf{1}} - z)$. Therefore, the implied risk premium adjusted for the degree of risk aversion and asset correlation is the vector of base value that equates these two centrality measures.

## 3.5 Implications on market efficiency

There are two main frictions in this economy that cause market inefficiency. One is trade friction from incomplete trading network, in which the agents are restricted to trade with only the connecting agents. The other friction is trade friction from monopolistic price-setting behavior. To shed light on market efficiency, this section will characterize the constrained optimal allocation, which maximizes Utilitarian social welfare subject to the trade friction from incomplete trading network, and discuss the conditions when the decentralized outcome is constrained efficient.[10] Let $W(Q, g) = \sum_{i \in N} U_i$ be the Utilitarian welfare function of the economy with network structure $g$. Consider the following definition.

**Definition 3.5 (Constrained efficiency)** *A constrained efficient allocation $Q^{SB}(g)$ is such that*

$$Q^{SB}(g) = \underset{Q \in \mathbb{R}^{n(g)}}{\arg\max}\, W(Q, g)$$

From the definition, a constrained efficient allocation maximizes the sum of utilities of all agents given that a social planner can only allocate the goods between agents who are connected in the trading network. Note that the allocation is Pareto efficient when network $g$ is complete. Given the definition, we can characterize a constrained efficient allocation as follows.

**Proposition 3.2 (Constrained efficient allocation)** *In a trading network $g$, the constrained efficient allocation is*

$$Q^{SB} = \frac{1}{2}\left(I - \frac{1}{2}A^{SB}\right)^{-1}(\hat{\mathbf{1}} - z)$$

*where* $\quad A^{SB} = \Sigma_L \circ (W_S^{in} + W_B^{out} - W_S^{out} - W_B^{in}).$

From the proposition, the constrained efficient allocation is also associated with a generalized Bonacich centrality, in which the weight takes into account only the asset correlation of adjacent asset flows. Intuitively, when a social planner chooses optimal asset allocation, he considers how an asset quantity change in a flow affects portfolio risk of the agents involved. Unlike the allocation from the decentralized system, there is no quantity impact matrix involved in the weighting matrix or the base-value vector. This is because the social planner decides how to allocate assets by internalizing how quantity change in one asset flow affects welfare, in contrast to the decentralized outcome in which each agent will exercise his bargaining power by limiting trading quantities and charging higher prices. For a decentralized economy to be constrained efficient, consider the next proposition.

**Proposition 3.3 (Constrained efficiency)** *The decentralized equilibrium allocation $Q^E$ is constrained efficient if and only if*

$$\left(\Phi' \circ (W_S^{out} + I)\right)^{-1} Q^E \to \hat{0}.$$

[10]Note that the allocation which maximizes Utilitarian social welfare function is the set of Pareto efficient allocations, as the utility function is quasi-linear without restrictions on monetary transfer across agents. Such allocation is also minimizing the aggregate portfolio risk of the whole economy.

This proposition implies that a decentralized equilibrium can be efficient when (inverse) demand function of all sellers are almost perfectly elastic; that is, a slight change in a price has significant impact on demand quantity in all asset outflows of the same seller. An obvious example is when the asset is (almost) perfectly substitutable among all the agents such that no seller could afford a significant loss in demand by increasing asset prices. In this case, the economy is converging towards a price-taking economy, where the decision of all agents when choosing demand quantity in the second stage has no effect on equilibrium prices, resulting in no distortion in the decentralized outcome.

To see this clearly, consider an economy with identical asset correlation. Let $r^* \in \{-\frac{1}{|N|-1}, 1\}$. The following corollary provides an example when the decentralized outcome is efficient.

**Corollary 3.1** *If asset returns are identically correlated and the trading network is complete, then the asset allocation is efficient iff $r \to r^*$.*

This corollary shows that when there is no trade friction from incomplete trading network, the monopolistic inefficiency does not dissipate away due to monopolistic behavior of agents. When the correlation is positive, an increase in the number of agents does not eliminate the market inefficiency as the assets are always heterogeneous. Figure 3.5 compares the equilibrium allocation (left) with the efficient allocation (right) when there is no asset correlation.[11] From the figure, agents hold excessive amount of their assets and buy too few of other assets in the decentralized outcome.

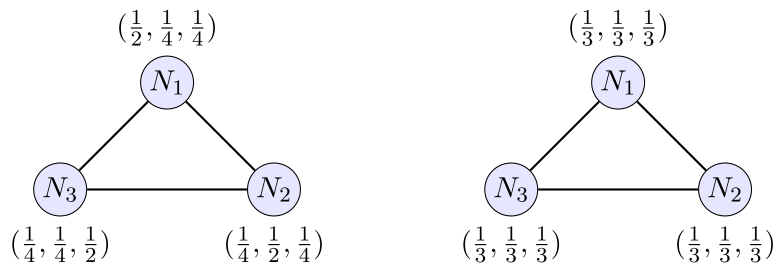


Figure 3.5: Equilibrium asset allocation (left) and efficient allocation(right) when there is no asset return correlation

Interestingly, when the correlation is sufficiently negative, the decentralized outcome will converge towards the efficient allocation. Specifically, the marginal utility of holding assets is increasing in the quantity of all assets in the pool due to hedging benefit when the correlation is negative. When the marginal benefit of hedging is relatively high, the buyers are willing to purchase high volume of assets despite facing high asset prices, resulting in elastic (inverse) demand function when the correlation is sufficiently negative or when the number of agents is sufficiently high. Therefore, optimal pricing strategy of the seller is to quote the highest possible level of prices such that the buyers are still willing to buy assets. This outcome is equivalent to the price-taking economy, in which the asset prices are fixed at the zero-risk-premium prices.

[11] Note that this allocation is also Pareto optimal for any correlation value regardless of the difference in the mean of asset returns, as there is no restriction on monetary transfers

At this price, all agents will choose to trade assets at the efficient level (full hedging) despite the persistence of monopolistic friction, as illustrated in Figure 3.6.

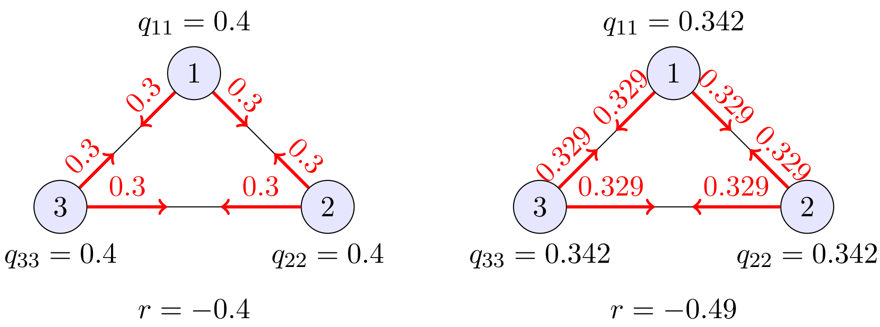


Figure 3.6: Numerical result of asset allocation when $r \to -\frac{1}{|N|-1}$

## 3.6 Special cases

To obtain more insights, this last section will analyse the equilibrium outcome of a few special cases. Throughout this section, assume that trading network is undirected; that is, if $ij \in g$, then $ji \in g$. Also, all numerical examples assume that $\alpha = 1$ and $d_i = 1$ for all agents. We will first consider a case of no asset correlation. Then we consider a case of identical asset correlations with two structures of trading network: the star network and the core-periphery network.

### 3.6.1 No asset correlation

To begin, we make the following assumption to restate that there is no pairwise correlation between assets; the asset returns are i.i.d. normally distributed.

**Assumption 3.2** $\sigma_{ij} = 0 \quad \forall i, j \in N$ *and* $i \neq j$

The following corollary describes the equilibrium.

**Corollary 3.2 (Equilibrium outcome I)** *In an arbitrary network $g$, if Assumption 3.2 is satisfied,* $\forall ij \in g$

*1)* $q_{ij} = \frac{1}{2+n(i)}$
*2)* $p_{ij} = d_i - \frac{\alpha}{2+n(i)}$
*3)* $u_i = d_i - \frac{\alpha}{2+n(i)} + \frac{\alpha}{2} \sum\limits_{j \in N(i)} (\frac{1}{(2+n(j))^2})$
*4)* $q_{ii} = \frac{2}{2+n(i)}$.

Under the assumption of independent return distributions, the situation collapses to a classical monopoly problem. In our context, each agent is both a monopolist of his own asset and a buyer of his neighbours' assets. From the corollary, an asset price depends on return mean of the asset $d_i$ adjusted for risk premium and price markup. When the agent is more risk-averse, the buying agent will demand higher risk premium, pushing the price downward. Also, higher number of buyers of agent $i$ (i.e. higher $n(i)$) increases the equilibrium price since agent $i$ attains more monopoly power.

Interestingly, asset prices depend only on the degree of the sellers and not of the buyers; that is $p_{ij} = p_{ik}, \forall j, k \in N(i)$.[12] This is because all agents could raise asset price without affecting the buyers' demand of other assets when the asset return is uncorrelated. As predicted by standard monopoly problem, well-connected sellers can charge high prices and limit the amount of assets sold to each neighbour to gain higher profits. Also, the well-connected agents will trade away larger fraction of their own assets due to higher asset demands.

It is true that the utility from selling assets of agents depends on the number of neighbors of the agents. However, the agent's net utility also depends on his neighbours' degree. This is because the agents are also buying assets from the neighbors. Therefore, the marginal utility of connecting to one more agent decreases if this agent is well-connected. Let $n$ be the degree centrality of agent $i$ and $n(j)$ be the degree centrality of agent $j$. The marginal utility of agent $i$ connecting to agent $j$ is

$$u_i(N(i) + j) - u_i(N(i)) = \frac{\alpha}{(2 + n + 1)(2 + n)} + \frac{\alpha}{2(2 + n(j))}$$

From the above equation, the marginal utility of connecting to $j$ diminishes when the degree of agent $j$ increases. This finding implies that agents prefer forming a relationship with low-degree agents to gain the highest bargaining power when considering endogenous trading network formation.

### 3.6.2 Identical asset correlation

This subsection will analyse equilibrium outcome when asset return correlations are identical. Consider the following assumption.

**Assumption 3.3** $\sigma_{ij} = \sigma_{kl} = r \in (-\frac{1}{|N|-1}, 1)$, $\forall ij \in g$ *and* $i \neq j$.[13]

#### Star network

A star network is the network structure in which there is one node (hub) at the centre and a few nodes (periphery) connecting to the hub. Therefore, the centre node has superior competitive power over the peripheral nodes. Let $i$ be the agent at the centre and $j$ be an agent at the periphery. Denote $n = n(j)$. Consider the following remark.

**Remark 3.1** *If Assumption 3.3 is satisfied and trading network is a star network, then*
*1)* $q_{ii} \geq q_{jj}$ *if and only if* $r \leq \frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)} < 0$
*2)* $q_{ij} \leq q_{ji}$
*3)* $\lim_{r \to -\frac{1}{n}} q_{ij} = 0$, $\lim_{r \to -\frac{1}{n}} q_{ji} = 1$

In the star network, the hub is both the monopolist of his own asset and the monopsonist of his neighbouring assets. When the correlation increases, the hub will buy less of his neighbouring assets, as the marginal benefit drops significantly from holding large unit of all assets in the pool.

[12] When there is no asset return correlation, the utility function is separable in the quantity of asset holdings by assumption of CARA utility function. Note that the quantity impact matrix is diagonal.

[13] To guarantee the positive definiteness of the variance-covariance matrix, the correlation must be greater than $-\frac{1}{|N|-1}$.

Meanwhile, as the marginal benefit of selling the asset increases when the correlation increases, the hub will sell more units of asset to the peripherals by quoting lower prices, resulting in $q_{ii} \leq q_{jj}$ when the correlation is positive. Based on the remark, regardless of the correlation value, quantity of asset outflow from the hub is always lower than quantity of asset inflow to the hub (i.e. $q_{ij} \leq q_{ji}$) due to monopolistic and monopsonistic power of the hub. This is illustrated in the right panel of Figure 3.7.

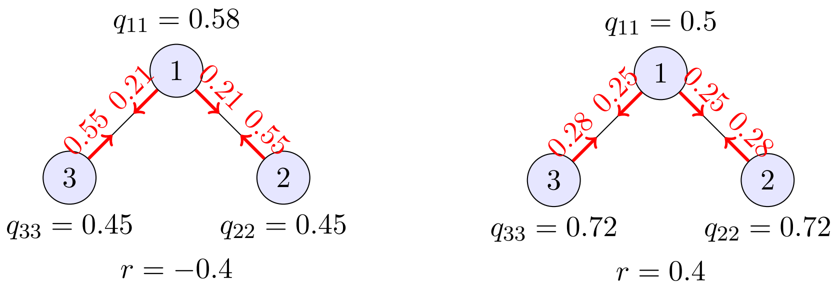


Figure 3.7: Numerical result of asset allocation when $q_{ii} > q_{jj}$ (left) and $q_{ii} < q_{jj}$ (right)

However, the scenario can be different when the correlation is negative. In this case, the opportunity to buy a variety of assets decreases the willingness to sell the asset of the hub due to hedging motives. In fact, the hub is willing to buy more of neighboring assets at the higher price. The peripherals, in spite of keening to buy more assets from the hub, cannot do so as the hub also would like to hold his own asset and will request expensive prices from the peripherals who would like to buy. As a result, the peripherals will demand less of the hub's asset compared to the case of no asset correlation. When the correlation is sufficiently negative, the asset outflow quantity from the hub is so low that the proportion of the hub's asset in his portfolio is greater than the proportion of the peripherals' own asset holding, $q_{ii} \geq q_{jj}$, as illustrated in the left panel in Figure 3.7. An extreme case is when the correlation goes to $-\frac{1}{n}$, in which the hub fully exploits the hedging opportunity by buying all assets from his neighbors and selling none to them (full hedging).

The next question is how the equilibrium outcome would change when the number of periphery agents increases. Consider the following remark.

**Remark 3.2** *Consider two star-network economies with the number of the periphery agents $n$ and $n'$ such that $n' > n$. If Assumption 3.3 is satisfied, then*

*1) $\exists r < 0$ such that $q_{jj}(n') < q_{jj}(n)$*

*2) $\exists r < 0$ such that $q_{ii}(n') > q_{ii}(n)$*

*3) $\exists r < 0$ such that $u_j(n') > u_j(n)$*

This remark highlights the role of asset correlation toward the competition structure in the network. Generally, when the correlation is positive, more number of the peripherals means that there are more substitutable assets for the hub to buy. Therefore, the hub will demand less unit of assets from each periphery agent, and the utility of the periphery will decrease due to more aggressive competition.

Interestingly, when the asset correlation is negative, the outcome could be different. From the remark, it is possible that the proportion of the peripherals' own asset holding decreases and

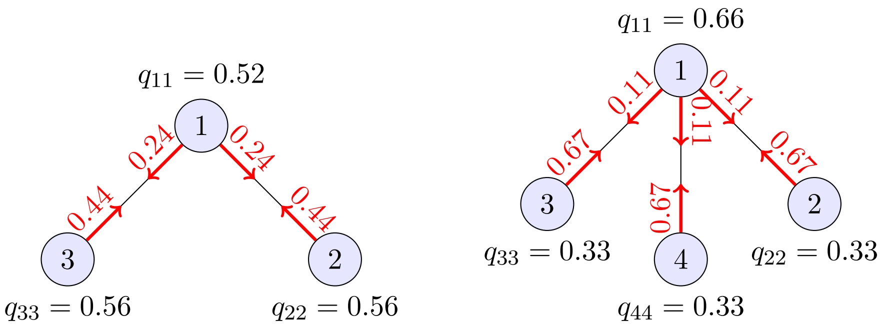


Figure 3.8: Numerical example of asset allocation when r = -0.3

that of the hub increases when the number of the periphery agents increases, as numerically illustrated in Figure 3.8. Since the marginal utility of the hub to buy assets depends positively on the variety of assets in possession when the correlation is negative, the hub will want to buy larger amount of assets from each agent at the periphery when the number of periphery agents increases. Meanwhile, the marginal benefit of holding own asset of the hub is so high that the proportion of hub's own asset in the portfolio also increases, despite higher demand from more buyers.

The last point in the remark sounds counter-intuitive at first. It is natural to think that payoffs of the periphery agents will decline when adding more periphery agents due to higher competition. However, the virtual value of the peripherals' assets toward the hub depends crucially on the number of periphery agents. When the correlation is negative, the more variety of assets the hub is holding, the more valuable the peripheral assets are to the hub due to hedging benefits, and thus the higher prices the peripherals could charge to the hub. Therefore, if the correlation is sufficiently negative, the benefit from higher virtual asset value outweighs the costs from more competition, resulting in higher utility of all the peripherals, as numerically illustrated in Figure 3.9.

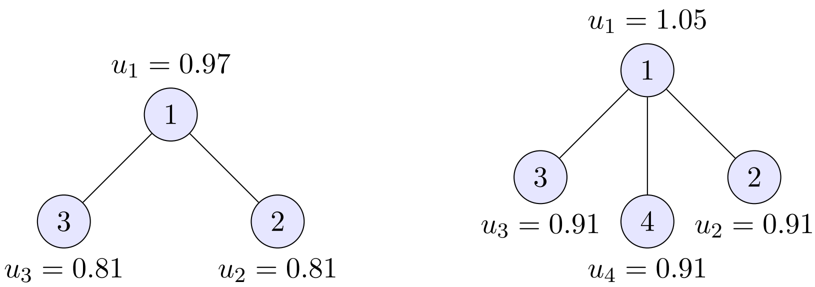


Figure 3.9: The utility of agents when r = -0.3

### Core-periphery network

The core-periphery structure describes a topology in which the cores refer to the central and densely connected set of nodes and the peripherals are the sparsely connected set of nodes who connect to only the cores. This section examines the equilibrium outcome in a case of two core agents, each connecting with two peripheral agents, as shown in Figure 3.10.

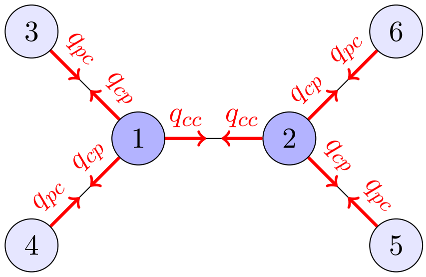


Figure 3.10: An example of a core-periphery network

Let $q_{cc}, q_{cp}, q_{pc}$ $(p_{cc}, p_{cp}, p_{pc})$ denote equilibrium quantities and prices of transactions between the cores, from the cores to the peripherals, and from the peripherals to the cores, respectively. Consider the following proposition.

**Remark 3.3** *If Assumption 3.3 is satisfied and the economy is the core-periphery network with* $n(core) = 2$ *and* $n(periphery) = 2$*, then*

*1)* $\exists k \in \mathcal{R}^+$ *such that* $\left.\frac{\partial q_{cc}}{\partial r}\right|_{r=k} > 0$.

*2)* $q_{cp} < q_{pc}$ *and* $p_{cp} - d_c > p_{pc} - d_p$.

*3)* $p_{cp} < p_{cc}$ *and* $q_{cp} < q_{cc}$ *when* $r < 0$.

Intuitively, when the correlation is positive, an increase in correlation should reduce the volume of asset trading between any pair of agents. However, this is not always the case for the core-periphery network. The remark shows that the core agents may exchange more unit of assets with the other core agent when the correlation increases. The main explanation relies on the cores' network position, where they can sell their assets relatively easy compared to the peripherals. Therefore, the cores are more willing to buy additional assets from others compared to the peripherals.

The second statement in the remark arises from the friction of imperfect competition between the cores and the peripherals. Similar to the case of the star network, the cores are the sole buyers and sellers of the peripheral assets in the core-periphery network. Therefore, the cores will quote higher prices to the peripherals and limit the trading quantity in the equilibrium, as illustrated in Figure 3.11.

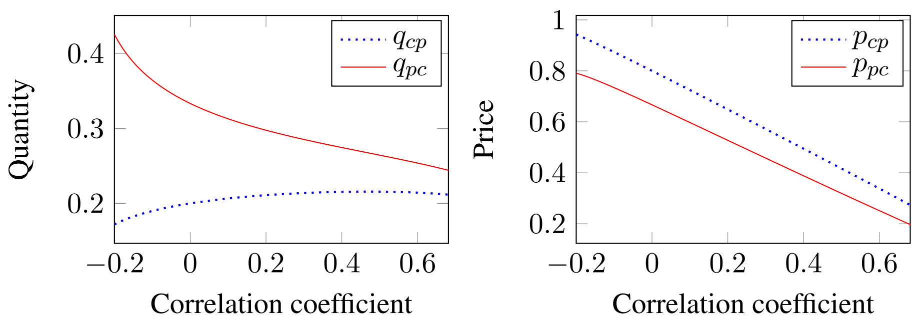


Figure 3.11: Asset flow and prices between the cores and the peripherals

Interestingly, based on the remark, the equilibrium price and trading quantity from the cores to the peripherals are lower than the trading quantities among the cores when the correlation is negative. When the asset is complementary, the richer variety of assets the cores are holding, the higher the marginal benefit of the cores to buy more assets. Thus, the cores could charge higher prices to their core neighbour, who value the asset greatly, and still receive higher asset demand in the equilibrium. This is numerically illustrated in Figure 3.12

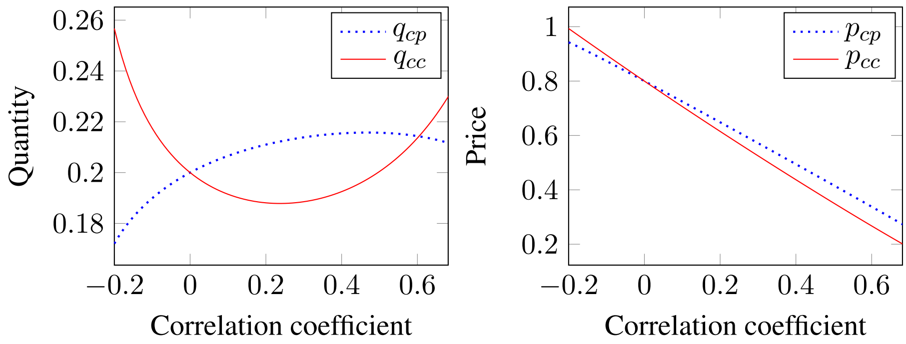


Figure 3.12: Trading quantities and prices between the cores and the peripherals

## 3.7 Concluding remark

To summarize, this paper develops a theoretical model to study the formation of financial network driven by risk diversification motives. Using the concept of line graph transformation, we found that equilibrium asset allocation corresponds to a generalized Bonacich centrality of the network of asset flows.

It is worth pointing out that our main results are restrictive to linearity properties of best-response functions based on the assumption of CARA utility function and joint normal distribution of asset returns. With the linear best responses, unique equilibrium trading outcome coincides with the Bonacich centrality of the line graph in the same manner with Ballester et al (2006) and Bimpikis et al (forthcoming). Deviating from this assumption causes the best-response function to contain high-order non-linear terms which can compromise our conclusion. However, it seems intuitive to confirm that Bonacich centrality of the line graph is the right solution for any decentralized trading game with linear trading strategies.

The natural step from here is to study the implications of systemic risk from trade friction. One can use this model to explore how topology of trading network shapes the fragility of financial network. This model can also serve as a tool for policymakers to study regulatory effect of limiting market participation of financial institutions to limit financial interconnectedness. Possible future extensions of the paper include:

1. Moral hazard. This paper abstracts from risk-taking behaviour of agents by the assumption on exogeneity of project riskiness. Relaxing this assumption is also the next step to study the risk-taking channel, in which the moral hazard is the key to propagate and amplify the systemic risk in the financial network. Risk-sharing is the double-edge sword as it also generates the excessive risk-taking behaviour when agents partially bear the risk from their actions. The extension of this model by endogenizing the choice of investment in the project is another challenge for future research.

2. Endogenous trading network. In this paper, one main assumption is the exogenous trading network. Relaxing this assumption to incorporate endogenous trading network formation can provide more interesting insights on the formation of financial network.

## 3.8 Appendix

### 3.8.1 Summary of matrix notations

Let the row (column) $ik$ of a matrix be the row (column) corresponding to the transaction $ik$. Let $n(i) = |N(i)|$ and $n(g) = |g|$ Define the following notations

1. The link adjacency matrix

   a) $W^{in}_{S,(i_1k_1,i_2k_2)} = \begin{cases} 1 & \text{if } i_1 = j_2 \\ 0 & \text{if otherwise} \end{cases}$

   b) $W^{out}_{B,(i_1k_1,i_2k_2)} = \begin{cases} 1 & \text{if } j_1 = i_2 \\ 0 & \text{if otherwise} \end{cases}$

2. The adjustment matrix

   a) $W^{out}_{S,(i_1k_1,i_2k_2)} = \begin{cases} 1 & \text{if } i_1 = i_2 \text{ and } j_1 \neq j_2 \\ 0 & \text{if otherwise} \end{cases}$

   b) $W^{in}_{B,(i_1k_1,i_2k_2)} = \begin{cases} 1 & \text{if } j_1 = j_2 \text{ and } i_1 \neq i_2 \\ 0 & \text{if otherwise} \end{cases}$

3. $Q_{n(g)\times 1}$ is the vector of quantity for all transaction $ij \in g$.

4. $P_{n(g)\times 1}$ is the vector of prices for all transaction $ij \in g$.

5. $\Sigma_L$ is the $n(g) \times n(g)$ matrix in which $\Sigma_{L,(i_1k_1,i_2k_2)} = \sigma_{i_1i_2}$.

6. Let F be the $n(g) \times n(g)$ matrix, then

   a) $\hat{F}^k_m$ = $F \circ W^k_m$ for $k \in \{in, out\}$ and $m \in \{S, B\}$ where $W^k_m$ is the link adjacency matrix

   b) $F_D = F \circ I$ where $I$ is the identity $n(g) \times n(g)$ matrix

   c) $F^{out,out}_i$ is the $n(i) \times n(i)$ submatrix of $F[ik; \forall k \in N(i)|ij; \forall j \in N(i)]$.

   d) $F^{in,out}_i$ is the $n(i) \times n(i)$ submatrix of $F[ki; \forall k \in N(i)|ij; \forall j \in N(i)]$.

   e) $F^{in,in}_i$ is the $n(i) \times n(i)$ submatrix of $F[ki; \forall k \in N(i)|ji; \forall j \in N(i)]$.

   f) $F^{out,in}_i$ is the $n(i) \times n(i)$ submatrix of $F[ik; \forall k \in N(i)|ji; \forall j \in N(i)]$.

7. $[k]^{-D}_{j\times j}$, $k \in \mathbb{R}$ be the $j \times j$ matrix with all zeros at the diagonal and all $k$ off the diagonals.

8. d is the $n(g) \times 1$ vector of asset return where $d_{ik} = d_i$

9. z is the $n(g) \times 1$ vector of asset correlation where $z_{ik} = \sigma_{ik}$

### 3.8.2 Equilibrium characterization

To characterize an equilibrium, I use the backward induction method. Consider buyer's maximization problem in which agent $j \in N$ will choose asset demand vector $q_j$ to maximize his utility given any arbitrary price vector $P$ as follows

$$q_j = \arg\max_{q_j \in Q_j} u_j(Q, P)$$

where $Q_j = \{q_j : P^{\mathbb{R}_+^{N(j)}} \to \mathbb{R}^{n(j)}\}$.

Recall the mean-variance utility function $\forall i \in N$

$$U_j = d_j - \sum_{k \in N(j)} q_{jk}(d_j - p_{jk}) + \sum_{k \in N(j)} q_{kj}(d_k - p_{kj}) - \frac{\alpha}{2}[\sum_{k \in \bar{N}(j)} \sum_{l \in \bar{N}(j)} \sigma_{kl} q_{kj} q_{lj}].$$

We obtain the first-order condition for buyer $j$ as follows.

$$\frac{du_j}{dq_{ij}} = d_i - p_{ij} - \alpha\sigma_{ij}(1 - \sum_{k \in N(j)} q_{jk}) - \alpha \sum_{m \in N(j)} \sigma_{mi} q_{mj} = 0; \quad \forall i \in N(j).$$

Since the Hessian matrix, where $j_k \in N(j)$ for all k, is

$$H(u_j)(q_j) = -\alpha \begin{bmatrix} \frac{\partial^2 u_j}{\partial q_{j_1 j}^2} & \frac{\partial^2 u_j}{\partial q_{j_1 j} q_{j_2 j}} & \cdots & \frac{\partial^2 u_j}{\partial q_{j_1 j} q_{j_{n(j)} j}} \\ \frac{\partial^2 u_j}{\partial q_{j_2 j} q_{j_1 j}} & \frac{\partial^2 u_j}{\partial q_{j_2 j}^2} & \cdots & \frac{\partial^2 u_j}{\partial q_{j_2 j} q_{j_{n(j)} j}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 u_j}{\partial q_{j_{n(j)} j} q_{j_1 j}} & \frac{\partial^2 u_j}{\partial q_{j_{n(j)} j} q_{j_2}} & \cdots & \frac{\partial^2 u_j}{\partial^2 q_{j_{n(j)} j}} \end{bmatrix}$$

$$= -\alpha \begin{bmatrix} \sigma_{j_1 j_1} & \sigma_{j_1 j_2} & \cdots & \sigma_{j_1 j_{n(j)}} \\ \sigma_{j_2 j_1} & \sigma_{j_2 j_2} & \cdots & \sigma_{j_2 j_{n(j)}} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{j_{n(j)} j_1} & \sigma_{j_{n(j)} j_2} & \cdots & \sigma_{j_{n(j)} j_{n(j)}} \end{bmatrix}$$

which is negative definite as the correlation matrix is positive definite by assumption. First-order condition gives the demand function for asset $i$ as follows:

$$\begin{aligned} p_{ij} &= d_i - \alpha\sigma_{ij}(1 - \sum_{k \in N(j)} q_{jk}) - \alpha \sum_{m \in N(j)} \sigma_{mi} q_{mj} \\ &= d_i - \alpha\sigma_{ij} + \alpha[\sigma_{ij} \sum_{k \in N(j)} q_{jk} - \sum_{m \in N(j)} \sigma_{mi} q_{mj}]. \end{aligned}$$

Using the matrix notations, the demand function can be written in matrix form as

$$P = d - \alpha z + \alpha[\Sigma_L \circ (W_B^{out} - W_B^{in}) - \Sigma_D]Q.$$

Rearranging the terms gives

$$Q = (\Sigma_L \circ (W_b^{out} - W_b^{in}))Q + \frac{1}{\alpha}\Sigma_D^{-1}[d - P - \alpha z].$$

Define $\bar{A} = \Sigma_L \circ (W_b^{out} - W_b^{in})$. We obtain that

$$Q = \frac{1}{\alpha}(I - \bar{A})^{-1}[d - P - \alpha z].$$

Let $\Phi = \frac{1}{\alpha}(I - \bar{A})^{-1}$ be the price impact matrix. Substituting into the asset demand function gives

$$q(P) = \Phi[d - P - \alpha z].$$

As a result, we obtain the following lemma.

**Lemma 3.2** *The equilibrium asset demand function is such that*

$$Q = \Phi(d - \alpha z) - \Phi P$$

*where* $\Phi = \frac{1}{\alpha}(I - (\Sigma_L \circ W_B^{out} - \Sigma_L \circ W_B^{in}))^{-1}$.

Moving to the first stage, agent $i$ chooses price vector $p_i$ such that

$$p_i = \arg\max_{p_i \in \mathbb{R}^{+n(i)}} u_i(q(P), P)$$

subject to asset demand function $q(P) = \Phi[d - P - \alpha z]$.

The first-order condition of agent $i$ for transaction $ij$ is

$$\frac{\partial u_i}{\partial p_{ij}} = q_{ij} + \sum_{m \in N(i)} (p_{im} - d_i) \frac{\partial q_{im}}{\partial p_{ij}} - \sum_{m \in N(i)} (p_{mi} - d_m) \frac{\partial q_{mi}}{\partial p_{ij}} - \alpha \sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} = 0.$$

To verify that the first-order condition gives the maximum solution, it suffices to prove that $u_i(q(P), P)$ is concave in $p_i$. Since

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial p_{ij}^2} &= 2\frac{\partial q_{ij}}{\partial p_{ij}} - \alpha \sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} \frac{\partial q_{ki}}{\partial p_{ij}} \frac{\partial q_{li}}{\partial p_{ij}} \\
&= 2\frac{\partial q_{ij}}{\partial p_{ij}} - \alpha[\sum_{k \in N(i)} \sum_{l \in N(i)} \sigma_{kl} \frac{\partial q_{ki}}{\partial p_{ij}} \frac{\partial q_{li}}{\partial p_{ij}} - 2 \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} \sum_{l \in N(i)} \sigma_{il} \frac{\partial q_{li}}{\partial p_{ij}} + \sigma_i^2 (\sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}})^2]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 u_i}{\partial p_{ij} \partial p_{ih}} &= \frac{\partial q_{ij}}{\partial p_{ih}} + \frac{\partial q_{ih}}{\partial p_{ij}} - \alpha \sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} \frac{\partial q_{ki}}{\partial p_{ih}} \frac{\partial q_{li}}{\partial p_{ij}} \\
&= \frac{\partial q_{ij}}{\partial p_{ih}} + \frac{\partial q_{ih}}{\partial p_{ij}} - \alpha[\sum_{k \in N(i)} \sum_{l \in N(i)} \sigma_{kl} \frac{\partial q_{ki}}{\partial p_{ih}} \frac{\partial q_{li}}{\partial p_{ij}} - (\sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} \sum_{l \in N(i)} \sigma_{il} \frac{\partial q_{li}}{\partial p_{ih}} \\
&\quad + \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ih}} \sum_{l \in N(i)} \sigma_{il} \frac{\partial q_{li}}{\partial p_{ij}}) + \sigma_i^2 (\sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}})(\sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ih}})]
\end{aligned}$$

which can be written in terms of Hessian matrix as

$$\begin{aligned}
H(u_i)(p_i) = &-(\Phi_i^{out,out} + \Phi_i^{out,out'}) - \alpha\{(\Phi_i^{in,out'} \Sigma_{L,i}^{in,in} \Phi_i^{in,out} \\
&- [\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi_i^{in,out} + (\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi_i^{in,out})'] + \sigma_i^2 \Phi_i^{out,out'} \Phi_i^{out,out})\}
\end{aligned}$$

where

1. $\Phi_i^{out,out}$ is the $n(i) \times n(i)$ submatrix $\Phi[ik; \forall k \in B(i)|ij; \forall j \in B(i)]$

2. $\Phi_i^{in,out}$ is the $n(i) \times n(i)$ submatrix $\Phi[ki; \forall k \in S(i)|ij; \forall j \in B(i)]$

3. $\Sigma_{L,i}^{in,in}$ is the $n(i) \times n(i)$ submatrix $\Sigma_L[ki; \forall k \in S(i)|ji; \forall j \in S(i)]$

4. $\Sigma_{L,i}^{out,in}$ is the $n(i) \times n(i)$ submatrix $\Sigma_L[ik; \forall k \in B(i)|ji; \forall j \in S(i)]$.

Then I obtain the following Lemma.

**Lemma 3.3** *$u(q(P), P)$ is concave in $p_i$ if $H(u_i)(p_i)$ is positive definite.*

By Assumption 3.1, $u(q(P), P)$ is concave. Thus, the first-order condition is utility maximizing. Recall that the first-order condition is

$$\frac{\partial u_i}{\partial p_{ij}} = q_{ij} + \sum_{m \in N(i)} (p_{im} - d_i) \frac{\partial q_{im}}{\partial p_{ij}} - \sum_{m \in N(i)} (p_{mi} - d_m) \frac{\partial q_{mi}}{\partial p_{ij}} - \alpha \sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} = 0.$$

Consider the last term $\sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}}$, one can expand this term into

$$\begin{aligned}
\sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} &= \sum_{k \in N(i)} \sum_{l \in N(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} + \sum_{l \in N(i)} \sigma_{il} q_{ii} \frac{\partial q_{li}}{\partial p_{ij}} + \sum_{k \in N(i)} \sigma_{ki} q_{ki} \frac{\partial q_{ii}}{\partial p_{ij}} + \sigma_i^2 q_{ii} \frac{\partial q_{ii}}{\partial p_{ij}} \\
&= \sum_{k \in N(i)} \sum_{l \in N(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} + \sum_{l \in N(i)} \sigma_{il} \frac{\partial q_{li}}{\partial p_{ij}} - \sum_{l \in N(i)} \sum_{k \in N(i)} \sigma_{il} q_{ik} \frac{\partial q_{li}}{\partial p_{ij}} \\
&\quad - \sum_{k \in N(i)} \sigma_{ki} q_{ki} \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} - \sigma_i^2 \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} + \sigma_i^2 \sum_{k \in N(i)} q_{ik} \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}}.
\end{aligned}$$

Simplifying the terms gives

$$\begin{aligned}
\sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}} &= \sum_{k \in N(i)} \sum_{l \in N(i)} \frac{\partial q_{li}}{\partial p_{ij}} (\sigma_{kl} q_{ki} - \sigma_{il} q_{ik}) + \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} \sum_{k \in N(i)} (\sigma_i^2 q_{ik} - \sigma_{ki} q_{ki}) \\
&\quad - \sigma_i^2 \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} + \sum_{l \in N(i)} \sigma_{li} \frac{\partial q_{li}}{\partial p_{ij}}.
\end{aligned}$$

Substituting $\sum_{k \in \bar{N}(i)} \sum_{l \in \bar{N}(i)} \sigma_{kl} q_{ki} \frac{\partial q_{li}}{\partial p_{ij}}$ into the first-order condition gives

$$\begin{aligned}
&q_{ij} - \alpha[\sum_{k \in N(i)} \sum_{l \in N(i)} \frac{\partial q_{li}}{\partial p_{ij}} (\sigma_{kl} q_{ki} - \sigma_{il} q_{ik}) + \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} \sum_{k \in N(i)} (\sigma_i^2 q_{ik} - \sigma_{ki} q_{ki})] + \\
&[\sum_{m \in N(i)} \frac{\partial q_{im}}{\partial p_{ij}} (p_{im} - d_i) - \sum_{m \in N(i)} \frac{\partial q_{mi}}{\partial p_{ij}} (p_{mi} - d_m)] + \alpha[\sigma_i^2 \sum_{k \in N(i)} \frac{\partial q_{ik}}{\partial p_{ij}} - \sum_{l \in N(i)} \sigma_{li} \frac{\partial q_{li}}{\partial p_{ij}}] = 0
\end{aligned}$$

which can be written in matrix form as follows.

$$\begin{aligned}
(I + \alpha[\hat{\Phi}_B^{out'} (\hat{\Sigma}_{LB}^{in} + \Sigma_D - \hat{\Sigma}_{LB}^{out}) + (\hat{\Phi}_S^{out'} + \Phi'_D)(\hat{\Sigma}_{LS}^{out} + \Sigma_D - \hat{\Sigma}_{LS}^{in})])Q - (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D)(P - d) \\
- \alpha(\Sigma_D(\hat{\Phi}_S^{out'} + \Phi'_D)\hat{\mathbf{1}} - \hat{\Phi}_B^{out'} z) = 0
\end{aligned}$$

where

1. $\hat{\Phi}_i^k = \Phi \circ W_m^k$ for $k \in \{in, out\}$ and $m \in \{S, B\}$ where $W_m^k$ is the link adjacency matrix.

2. $\hat{\Sigma}_{Lm}^k = \Sigma_L \circ W_m^k$ for $k \in \{in, out\}$ and $m \in \{S, B\}$

3. $\Sigma_{D,(n\times n)} = \Sigma_{L,(n\times n)} \circ I_{(n\times n)}$ ; $n = n(g)$

4. $\Phi_{D,(n\times n)} = \Phi_{(n\times n)} \circ I_{(n\times n)}$.

The last step is to use the demand function to solve for the equilibrium price. Define

$$\Gamma = I + \alpha[\hat{\Phi}_B^{out'}(\hat{\Sigma}_{LB}^{in} + \Sigma_D - \hat{\Sigma}_{LB}^{out}) + (\hat{\Phi}_S^{out'} + \Phi'_D)(\hat{\Sigma}_{LS}^{out} + \Sigma_D - \hat{\Sigma}_{LS}^{in})]$$
$$\Theta = -\alpha(\Sigma_D(\hat{\Phi}_S^{out'} + \Phi'_D)\hat{\mathbf{1}} - \hat{\Phi}_B^{out'}z)$$

Given $\Gamma$ and $\Theta$, the first-order condition in matrix form can be simplified into

$$\Gamma Q - (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D)(P - d) + \Theta = 0.$$

Substituting the demand function $q(p) = \Phi[d - p - \alpha z]$ gives

$$\Gamma\Phi[d - P - \alpha z] - (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D)(P - d) + \Theta = 0$$
$$-(\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D + \Gamma\Phi)(P - d) + (\Theta - \alpha\Gamma\Phi z) = 0.$$

Therefore, the equilibrium price is

$$P = d + (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D + \Gamma\Phi)^{-1}[\Theta - \alpha\Gamma\Phi z]$$

with the equilibrium asset allocation

$$Q = \Phi[(\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi'_D + \Gamma\Phi)^{-1}[\alpha\Gamma\Phi z - \Theta] - \alpha z].$$

Therefore, we obtain the following proposition.

**Proposition 3.4 (Equilibrium characterization)** *If Assumption 3.1 is satisfied, the equilibrium $(P, q(P))$ that solves the trading game is such that*

$$P = d + (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi_D + \Gamma\Phi)^{-1}[\Theta - \alpha\Gamma\Phi z]$$
$$q(P) = \Phi[d - P - \alpha z]$$

*Where* $\Theta = \alpha(\hat{\Phi}_B^{out'}z - (\hat{\Phi}_S^{out'} + \Phi_D)\hat{\mathbf{1}})$

$$\Gamma = I + \alpha[\hat{\Phi}_B^{out'}(\hat{\Sigma}_{LB}^{in} + I - \hat{\Sigma}_{LB}^{out}) + (\hat{\Phi}_S^{out'} + \Phi_D)(\hat{\Sigma}_{LS}^{out} + I - \hat{\Sigma}_{LS}^{in})].$$

Next, we can further simplify the equilibrium asset allocation. From the demand function $Q = \Phi(d - P - \alpha z)$, I obtain $d - P = \Phi^{-1}Q + \alpha z$. Substituting this into the equilibrium condition in the proposition gives

$$\underbrace{(\Gamma + (\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi_D)(\Phi^{-1}))Q}_{\text{LHS}} = \underbrace{-\alpha(\hat{\Phi}_S^{out'} - \hat{\Phi}_B^{out'} + \Phi_D)z - \Theta}_{\text{RHS}} \tag{3.8.2.1}$$

Substituting $\Theta$ and $\Sigma_D = I$ into the right hand side (RHS) gives

$$RHS = \alpha(\hat{\Phi}_S^{out'} + \Phi_D)(\hat{\mathbf{1}} - z).$$

For the left-hand side (LHS), substituting $\Gamma$ and $\Phi$ gives

$$LHS = [I + \alpha(\hat{\Phi}_S^{out'} + \Phi_D)[(I + \hat{\Sigma}_{LB}^{in}) + (I + \hat{\Sigma}_{LS}^{out})]] - \alpha(\hat{\Phi}_S^{out'} + \Phi_D)(\hat{\Sigma}_{LB}^{out} + \hat{\Sigma}_{LS})]Q.$$

Define $A = \hat{\Sigma}_{LB}^{out} + \hat{\Sigma}_{LS}$. Rewrite Equation 3.8.2.1 as

$$[I + \alpha(\hat{\Phi}_S^{out'} + \Phi_D)[(I + \hat{\Sigma}_{LB}^{in}) + (I + \hat{\Sigma}_{LS}^{out})]] - \alpha(\hat{\Phi}_S^{out'} + \Phi_D)A]Q = \alpha(\hat{\Phi}_S^{out'} + \Phi'_D)(\hat{\mathbf{1}} - z) \tag{3.8.2.2}$$

Define $W = \alpha[I + \alpha(\hat{\Phi}_S^{out'} + \Phi_D)[(I + \hat{\Sigma}_{LB}^{in}) + (I + \hat{\Sigma}_{LS}^{out})]]^{-1}(\hat{\Phi}_S^{out'} + \Phi_D)$. We can rewrite Equation 3.8.2.2 as

$$(I - WA)Q = W(\hat{\mathbf{1}} - z) \quad \rightarrow \quad Q = (I - WA)^{-1}W(\hat{\mathbf{1}} - z)$$

which gives the next corollary.

**Corollary 3.3** *The equilibrium asset allocation is*

$$Q = (I - WA)^{-1}W(\hat{\mathbf{1}} - z)$$

*where*

1. $W = \alpha[I + \alpha(\hat{\Phi}_S^{out'} + \Phi_D)[(I + \hat{\Sigma}_{LB}^{in}) + (I + \hat{\Sigma}_{LS}^{out})]]^{-1}(\hat{\Phi}_S^{out'} + \Phi_D)$
2. $A = \hat{\Sigma}_{LB}^{out} + \hat{\Sigma}_{LS}$.

### 3.8.3 Positive definiteness of correlation matrix

This section will find a condition to guarantee positive definiteness of asset correlation matrix when the asset return correlations are identical. Mathematically, the correlation among random variables are not pairwise independent. If $R$ is a correlation matrix, then the correlations must at least satisfy the condition $det(R) > 0$. This section will prove that the condition $1 > r > -\frac{1}{n-1}$ when $n$ is the number of assets is sufficient to guarantee positive definiteness of the correlation matrix.

Given a correlation matrix

$$R = \begin{bmatrix} 1 & r & r \\ r & \ddots & r \\ r & r & 1 \end{bmatrix}_{n\times n},$$

one can rewrite the matrix to be $R = rA - (r-1)I$ when $A$ is the matrix with all entries of 1 and $I$ is the diagonal matrix. Next, one can find that

$$det(R) = r^n det(A - \frac{r-1}{r}I).$$

Let $\lambda = \frac{r-1}{r}$. Substituting $\lambda$ into the above equation gets

$$det(R) = r^n det(A - \lambda I) \tag{3.8.3.1}$$

where $det(A - \lambda I)$ is the characteristic polynomials of matrix $A$. From

$$det(A - \lambda I) = f(\lambda) = (-1)^n[\lambda^n - a_1\lambda^{(n-1)} + \cdots + (-1)^n a_n]$$

where $a_1 = tr(A), a_n = det(A), a_i$ is the sum of the row-i diagonal minors of $A$, implying that

$$a_1 = tr(A) = n$$
$$a_n = det(A) = 0$$
$$a_i = 0,$$

substituting all the conditions into the characteristic polynomials gives

$$det(A - \lambda I) = f(\lambda) = (-1)^n \lambda^{(n-1)}(\lambda - n).$$

Substituting $\lambda = \frac{r-1}{r}$ and $det(A - \lambda I)$ into Equation 3.8.3.1 gives

$$det(R) = (1-r)^{(n-1)}(1 + r(n-1)).$$

Therefore, the necessary and sufficient condition for the correlation matrix $R$ to be positive definite is $r > -\frac{1}{n-1}$.

### 3.8.4 Omitted proofs

**Proof of Lemma 3.1:** See the proof of equilibrium characterization.

**Proof of Proposition 3.1:** From Corollary 3.3, we know that

$$(W^{-1} - A)Q = \hat{\mathbf{1}} - z$$

where

$$W = \alpha[I + \alpha(\hat{\Phi}_S^{out'} + \Phi_D)[(I + \hat{\Sigma}_{LB}^{in}) + (I + \hat{\Sigma}_{LS}^{out})]]^{-1}(\hat{\Phi}_S^{out'} + \Phi_D)$$
$$A = \hat{\Sigma}_{LB}^{out} + \hat{\Sigma}_{LS}.$$

Substituting $W$ and $A$ gives

$$(\frac{1}{\alpha}(\hat{\Phi}_S^{out'} + \Phi_D)^{-1} + [2I + \hat{\Sigma}_{LB}^{in} + \hat{\Sigma}_{LS}^{out}] - \hat{\Sigma}_{LB}^{out} - \hat{\Sigma}_{LS}^{in})Q = \hat{\mathbf{1}} - z \quad (*).$$

Let $V = [2I + \frac{1}{\alpha}(\hat{\Phi}_S^{out'} + \Phi_D)_D^{-1}]^{-1}$ and $\bar{A} = \hat{\Sigma}_{LB}^{out} + \hat{\Sigma}_{LS}^{in} - \frac{1}{\alpha}(\hat{\Phi}_S^{out'} + \Phi_D)_{off}^{-1} - \hat{\Sigma}_{LB}^{in} - \hat{\Sigma}_{LS}^{out}$ where $(\hat{\Phi}_S^{out'} + \Phi_D)_{off}^{-1}$ is the matrix collecting only the off-diagonal terms (i.e. $(\hat{\Phi}_S^{out'} + \Phi_D)^{-1} \circ ([1]_{n(g)\times n(g)} - I)$) and $(\hat{\Phi}_S^{out'} + \Phi_D)_D^{-1}$ is the matrix collecting only the diagonal terms (i.e $(\hat{\Phi}_S^{out'} + \Phi_D)^{-1} \circ I$). Substituting K and $\bar{A}$ into the equation gives

$$Q = (I - V\bar{A})^{-1}V(\hat{\mathbf{1}} - z).$$

This proves the proposition.

**Proof of Proposition 3.2:** To prove the proposition, we find first-order condition of $\frac{\partial W}{\partial q_{ij}} = \frac{\partial U_i + U_j}{\partial q_{ij}}$ for all $ij \in g$. The first-order condition written in matrix form for all $ij \in g$ is $2Q = A^{SB}Q + (\hat{1} - z)$ where $A^{SB} = \Sigma^{in}_{LS} + \Sigma^{out}_{LB} - \Sigma^{out}_{LS} - \Sigma^{in}_{LB}$. This implies that

$$Q = \frac{1}{2}\left(I - \frac{1}{2}A^{SB}\right)^{-1}(\hat{\mathbf{1}} - z).$$

Note that the second-order condition is always satisfied as the correlation matrix is positive definite. This proves the proposition.

**Proof of Proposition 3.3:** Recall Equation $(*)$ that the equilibrium asset allocation satisfies

$$(\frac{1}{\alpha}(\hat{\Phi}^{out'}_S + \Phi_D)^{-1} + [2I + \hat{\Sigma}^{in}_{LB} + \hat{\Sigma}^{out}_{LS}] - \hat{\Sigma}^{out}_{LB} - \hat{\Sigma}^{in}_{LS})Q^{SB} = \hat{\mathbf{1}} - z.$$

From Proposition 3.2, the constrained efficient allocation is

$$Q^E = \frac{1}{2}\left(I - \frac{1}{2}A^{SB}\right)^{-1}(\hat{\mathbf{1}} - z)$$

where $A^{SB} = \Sigma^{in}_{LS} + \Sigma^{out}_{LB} - \Sigma^{out}_{LS} - \Sigma^{in}_{LB}$. This implies that, when $Q^E = Q^{SB} = Q$,

$$(\frac{1}{\alpha}(\hat{\Phi}^{out'}_S + \Phi_D)^{-1} + [2I + \hat{\Sigma}^{in}_{LB} + \hat{\Sigma}^{out}_{LS}] - \hat{\Sigma}^{out}_{LB} - \hat{\Sigma}^{in}_{LS})Q = (2I - A^{SB})Q$$

must be true. Simplifying the terms give $(\hat{\Phi}^{out'}_S + \Phi_D)^{-1}Q = 0$. This proves the proposition.

**Proof of Corollary 3.1:** To prove this proposition, we find the quantity impact matrix $\Phi$ of a complete network. Let $n$ be the number of agents in the economy. Let the $i^{th}$ n-1 rows/columns associate with the transaction **outflow** from the $i^{th}$ agent as shown below

$$\begin{array}{c c}
 & \begin{array}{cccccc} 12 & \cdots & 1n & 21 & \cdots & n(n-1) \end{array} \\
\begin{array}{c} 12 \\ \vdots \\ 1n \\ 21 \\ \vdots \\ n(n-2) \\ n(n-1) \end{array} &
\begin{bmatrix}
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa \\
aa & aa & aa & aa & aa & aa
\end{bmatrix}
\end{array}$$

and the correlation of asset flow is

$$\Sigma_L = \begin{bmatrix}
[1]_{(n-1)\times(n-1)} & [r]_{(n-1)\times(n-1)} & [r]_{(n-1)\times(n-1)} \\
[r]_{(n-1)\times(n-1)} & \ddots & [r]_{(n-1)\times(n-1)} \\
[r]_{(n-1)\times(n-1)} & [r]_{(n-1)\times(n-1)} & [1]_{(n-1)\times(n-1)}
\end{bmatrix}_{n(n-1)\times n(n-1)}$$

From Proposition 3.4 that $\bar{A} = \Sigma_L \circ (W^{out}_B - W^{in}_B)$, the elements of matrix $(I - \bar{A})$ is

$$(I - \bar{A})_{(i_1k_1, i_2k_2)} = \begin{cases} 1 & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ -r & \text{if } j_1 = i_2 \text{ and } j_1 \neq j_2 \\ r & \text{if } j_1 = j_2 \text{ and } i_1 \neq i_2 \\ 0 & \text{if otherwise.} \end{cases}$$

Now, consider the following matrix $B$ defined as follows.

$$B_{(i_1k_1,i_2k_2)} = \begin{cases} a = -\frac{(n-2)r+1}{K} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ c = \frac{r}{K} & \text{if } i_1 \neq i_2 \text{ and } j_1 = j_2 \\ b = -\frac{r}{K} & \text{if } j_1 = i_2 \text{ and } j_1 \neq j_2 \\ 0 & \text{if otherwise} \end{cases}$$

where $K = (n-1)r^2 - (n-2)r - 1$. In what follows, I argue that $B$ is the inverse matrix of $(I - \bar{A})$ which can be easily verified by checking that $(I - \bar{A})B = I$. Thus, the quantity impact matrix $\Phi = \frac{1}{\alpha}(I - \bar{A})^{-1}$ is

$$\Phi_{(i_1j_1,i_2j_2)} = \begin{cases} -\frac{(n-2)r+1}{\alpha K} & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ \frac{r}{\alpha K} & \text{if } i_1 \neq i_2 \text{ and } j_1 = j_2 \\ -\frac{r}{\alpha K} & \text{if } j_1 = i_2 \text{ (and } j_1 \neq j_2) \\ 0 & \text{if otherwise} \end{cases}$$

where $K = (n-1)r^2 - (n-2)r - 1$. From $\Phi$, we can obtain that

$$\hat{\Phi}_S^{out} = 0 \quad \text{and} \quad \Phi_D = -\left(\frac{(n-2)r+1}{\alpha K}\right) I_{n(n-1)\times n(n-1)}.$$

Therefore,

$$(\hat{\Phi}_S^{out} + \Phi_D)^{-1} = \Phi_D^{-1} = -\left(\frac{\alpha K}{(n-2)r+1}\right) I_{n(n-1)\times n(n-1)}$$

which implies that all elements in $(\hat{\Phi}_S^{out} + \Phi_D)^{-1}$ will be zero if $K = 0$. This is true when $r \in \{-\frac{1}{n-1}, 1\}$.[14] This proves the proposition.

**Proof of Corollary 3.2:** First, reorder the rows/columns associate with the transaction for all matrices as $\{12, 13, ..., 1n(1), 21, 23, ..., 2n(2), ...., nn(n)\}$, which can be illustrated as below

$$\begin{array}{c c} & \begin{array}{cccccc} 12 & \cdots & 1n(1) & 21 & \cdots & nn_n \end{array} \\ \begin{array}{c} 12 \\ \vdots \\ 1n_1 \\ 21 \\ \vdots \\ nn_{n-1} \\ nn_n \end{array} & \begin{bmatrix} aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \\ aaa & aaa & aaa & aaa & aaa & aaa \end{bmatrix} \end{array}$$

Also, given $\sigma_{ij} = 0; \forall i, j \in N$, the correlation matrix of the asset flow or $\Sigma_L$ becomes

$$\Sigma_{L,(i_1k_1,i_2k_2)} = \begin{cases} \sigma_i^2 & \text{if } i_1 = i_2 \\ 0 & \text{if otherwise} \end{cases}$$

[14] Note that one can go further to obtain equilibrium characterization by substituting all matrices of the relevant parameter value and verifying the second-order condition). One will obtain that $q_{ij} = \frac{(n-2)r+1}{(n+1)((n-2)r+1)+r}$ and $p_{ij} = d_{ij} - \alpha\frac{((n-1)r+1)^2}{(n+1)((n-2)r+1)+r}$.

Also, by definition of $W_b^{out}$ and $W_b^{in}$, all the elements where $i_1 = i_2$ are zero. Thus, the quantity impact matrix $\Phi$ is simplified to

$$\Phi = \frac{1}{\alpha}(I - \bar{A})^{-1}\Sigma_D^{-1} = \frac{1}{\alpha}\Sigma_D^{-1} = \Phi_D$$

where the last equality is from $\Phi_D = \Phi \circ I = \Phi$ as the matrix $\Phi$ is a diagonal matrix.

To find the equilibrium price, recall from Proposition 3.4 that

$$\Gamma = I + \alpha[\hat{\Phi}_B^{out'}(\hat{\Sigma}_{LB}^{in} + \Sigma_D\hat{\Sigma}_{LB}^{out}) + (\hat{\Phi}_S^{out'} + \Phi_D')(\hat{\Sigma}_{LS}^{out} + \Sigma_D - \hat{\Sigma}_{LS}^{in})]$$
$$\Theta = -\alpha(\Sigma_D(\hat{\Phi}_S^{out'} + \Phi_D')\hat{\mathbf{1}} - \hat{\Phi}_B^{out'}z).$$

Since 1) $\Phi = \Phi_D$ and 2) the elements on the diagonal of $W_B^{out}$, $W_S^{in}$ and $W_S^{out}$ are zero, then

$$\hat{\Phi}_B^{out} = \Phi \circ W_B^{out} = 0$$
$$\hat{\Phi}_S^{out} = \Phi \circ W_S^{out} = 0$$
$$\hat{\Phi}_S^{in} = \Phi \circ W_S^{in} = 0.$$

Therefore, we obtain that

$$\Gamma = I + \alpha\Phi_D'(\hat{\Sigma}_{LS}^{out} + \Sigma_D - \hat{\Sigma}_{LS}^{in}).$$

Since $\hat{\Sigma}_{LS}^{in} = \Sigma_L \circ W_S^{in} = 0$ and $\Phi_D' = \Phi_D$, substituting these two conditions gives

$$\Gamma = 2I + \Sigma_{Ls}^{out} \quad \text{and} \quad \Theta = -\hat{\mathbf{1}}.$$

The last step is to plug in all the coefficient matrices into the equilibrium condition. From Proposition 3.4, we obtain that

$$\frac{1}{\alpha\sigma_i^2}A_i(p_i - d_i) = -\hat{1} \quad for \quad i \in N$$

where

$$A_i = \underset{n(i)\times n(i)}{\begin{bmatrix} 3 & 1 & \cdots \\ 1 & \ddots & 1 \\ \vdots & 1 & 3 \end{bmatrix}} \quad \text{and} \quad p_i - d_i = \underset{n(i)\times 1}{\begin{bmatrix} \vdots \\ p_{ij} - d_i \\ \vdots \end{bmatrix}}.$$

To solve for $p_i$, by symmetry of $p_{ij} = p_{ik}$ for $j, k \in N(i)$, I obtain

$$p_{ij} = d_i - \alpha\frac{\sigma_i^2}{2 + n(i)}.$$

To find the equilibrium quantity, substituting $p$ into the demand function $q(P) = \Phi[d - P - \alpha z]$ gives

$$q_{ij} = \frac{1}{2 + n(i)}.$$

Also,

$$q_{ii} = 1 - \sum_{k\in N(i)} q_{ik} = 1 - \frac{n(i)}{2 + n(i)} = \frac{2}{2 + n(i)}$$

and

$$
\begin{aligned}
U_i &= d_i - \sum\nolimits_{j\in N(i)} q_{ij}(d_i - p_{ij}) + \sum\nolimits_{k\in N(i)} q_{ki}(d_k - p_{ki}) - \frac{\alpha}{2}[\sum\nolimits_{k\in \bar{N}(i)} \sum\nolimits_{l\in \bar{N}(i)} \sigma_{kl} q_{ki} q_{li}] \\
&= d_i - \sum\nolimits_{j\in N(i)} q_{ij}(d_i - p_{ij}) + \sum\nolimits_{k\in N(i)} q_{ki}(d_k - p_{ki}) - \frac{\alpha}{2}[\sum\nolimits_{k\in \bar{N}(i)} \sigma_k^2 q_{ki}^2] \\
&= d_i - \alpha\frac{\sigma_i^2}{2+n(i)} + \frac{\alpha}{2}\sum_{j\in N(i)} (\frac{\sigma_j^2}{(2+n(j))^2}).
\end{aligned}
$$

Note that the second-order condition is satisfied as all diagonal elements of $\Phi$ are positive. This proves the corollary.

**Proof of Remark 3.1:** First, we find the equilibrium characterization of a star network based on Proposition 3.4. Let $n$ be the number of peripherals $\{1, 2, 3..., n\}$ and denote the center node (hub) $h$. Let the first $n$ rows/columns associate with transaction **outflows** from the hub and the last $n$ rows/columns associate with transaction **inflows** to the hub. Then the link adjacency matrices are as follows:

$$
W_B^{out} = \begin{bmatrix} [0]_{n\times n} & I_{n\times n} \\ [1]_{n\times n} & [0]_{n\times n} \end{bmatrix} \qquad W_B^{in} = \begin{bmatrix} [0]_{n\times n} & [o]_{n\times n} \\ [0]_{n\times n} & [1]_{n\times n}^{-D} \end{bmatrix}
$$

$$
W_S^{out} = \begin{bmatrix} [1]_{n\times n}^{-D} & [0]_{n\times n} \\ [0]_{n\times n} & [0]_{n\times n} \end{bmatrix} \qquad W_S^{in} = \begin{bmatrix} [0]_{n\times n} & [1]_{n\times n} \\ I_{n\times n} & [0]_{n\times n} \end{bmatrix}
$$

and the correlation of asset flow is

$$
\Sigma_L = \begin{bmatrix} [1]_{n\times n} & [r]_{n\times n} \\ [r]_{n\times n} & I + [r]_{n\times n}^{-D} \end{bmatrix} \qquad \text{and} \quad \Sigma_D = \begin{bmatrix} I_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & I_{n\times n} \end{bmatrix}.
$$

Therefore, we can obtain the quantity impact matrix $\Phi$ of a star network as follows.

$$
\Phi = \frac{1}{\alpha}\left[\begin{array}{ccc|ccc}
c & d & d & re & rf & rf \\
d & \ddots & d & rf & \ddots & rf \\
d & d & c & rf & rf & re \\
\hline
g & \cdots & g & e & f & f \\
\vdots & \ddots & \vdots & f & \ddots & f \\
g & \cdots & g & f & f & e
\end{array}\right]
$$

where $c = 1 - \frac{r^2}{(nr+1)(r-1)}$, $d = \frac{-r^2}{(nr+1)(r-1)}$, $e = \frac{-(r(n-1)+1)}{(nr+1)(r-1)}$, $f = \frac{r}{(nr+1)(r-1)}$, $g = \frac{-r}{(nr+1)(r-1)}$. Next step is to find all the expressions for the coefficients in the equilibrium condition. Let $[k]^{-D}$ for $k \in \mathcal{R}$ be the matrix with all zeros at the diagonal and all $k$ off the diagonals. Given the quantity impact matrix $\Phi$ above, we can obtain

$$\hat{\Phi}_B^{out} = \Phi \circ W_B^{out} = \frac{1}{\alpha}\begin{bmatrix} [0]_{n\times n} & (re)I_{n\times n} \\ [g]_{n\times n} & [0]_{n\times n} \end{bmatrix} \qquad \hat{\Phi}_S^{out} = \Phi \circ W_B^{out} = \frac{1}{\alpha}\begin{bmatrix} [d]_{n\times n}^{-D} & [0]_{n\times n} \\ [0]_{n\times n} & [0]_{n\times n} \end{bmatrix}$$

$$\hat{\Sigma}_{LB}^{in} = \Sigma_L \circ W_B^{in} = \frac{1}{\alpha}\begin{bmatrix} [0]_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & [r]_{n\times n}^{-D} \end{bmatrix} \qquad \hat{\Sigma}_{LB}^{out} = \Sigma_L \circ W_B^{out} = \frac{1}{\alpha}\begin{bmatrix} [0]_{n\times n} & rI_{n\times n} \\ [r]_{n\times n} & [0]_{n\times n} \end{bmatrix}$$

$$\hat{\Sigma}_{LS}^{in} = \Sigma_L \circ W_S^{in} = \frac{1}{\alpha}\begin{bmatrix} [0]_{n\times n} & [r]_{n\times n} \\ rI_{n\times n} & [0]_{n\times n} \end{bmatrix} \qquad \hat{\Sigma}_{LS}^{out} = \Sigma_L \circ W_S^{out} = \frac{1}{\alpha}\begin{bmatrix} [1]_{n\times n}^{-D} & [0]_{n\times n} \\ [0]_{n\times n} & [0]_{n\times n} \end{bmatrix}$$

$$\Sigma_{LD} = \Sigma_L \circ I = \frac{1}{\alpha}\begin{bmatrix} I_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & I_{n\times n} \end{bmatrix} \qquad \Phi_D = \Phi_D' = \Phi \circ I = \frac{1}{\alpha}\begin{bmatrix} cI_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & eI_{n\times n} \end{bmatrix}$$

where $c = 1 - \frac{r^2}{(nr+1)(r-1)}$, $d = \frac{-r^2}{(nr+1)(r-1)}$, $e = \frac{-(r(n-1)+1)}{(nr+1)(r-1)}$, $f = \frac{r}{(nr+1)(r-1)}$, $g = \frac{-r}{(nr+1)(r-1)}$. Also, from Proposition 3.4, we can further obtain the expression for $\Gamma$ and $\Theta$ as follows. Let $K = (nr+1)(r-1)$, To find $\Gamma$, substituting all relevant variables found above gives

$$\Gamma = \begin{bmatrix} I_{n\times n} + [1]_{n\times n} & [0]_{n\times n} \\ [0]_{n\times n} & (1+B)I_{n\times n} \end{bmatrix} \text{where} \quad B = \frac{(r(n-1)+1)(r^2-1)}{K}$$

Similarly, we can obtain

$$\Theta = -\begin{bmatrix} \frac{nr^2-(n-1)r-1}{K} \\ \frac{(r(n-1)+1)(r^2-1)}{K} \end{bmatrix} = -\begin{bmatrix} [1]_{n\times 1} \\ [B]_{n\times 1} \end{bmatrix}.$$

The last step is to plug in all the expressions into the equilibrium condition in Proposition 3.4 and impose symmetric conditions. We can obtain the following expression for the equilibrium asset price and allocation.

$$\begin{bmatrix} \frac{p_h-d}{\alpha} \\ \frac{p_p-d}{\alpha} \end{bmatrix} = \begin{bmatrix} -r - \frac{(r-1)(nr-r+1)(-n^2r^2+nr^2-4nr+r^2+2r-3)}{X} \\ -1 - \frac{(r-1)(n^3(3r^3+2r^2)-6n^2r^3+7n^2r^2+4n^2r+2nr^3-10nr^2+8nr+2n+r^2-5r+4)}{X} \end{bmatrix}$$

$$\begin{bmatrix} q_h \\ q_p \end{bmatrix} = \begin{bmatrix} \frac{(nr+1)((n-1)r+1)((2n-3)r+3)}{X} \\ \frac{((n-1)r+1)(n^2(r^2+r)-n(r^2-2r)+n-r^2-r+2)}{X} \end{bmatrix}$$

where $X = n^3(r^3+5r^2)+n^2(-5r^3+5r^2+7r)+n(r^3-12r^2+10r+3)+r^2+r^3+n^4r^3-8r+6$. Note that the second-order condition is also satisfied. To see this, first consider the second-order condition for trading strategy of the peripherals.

$$\frac{\partial^2 u_p}{\partial p_{ph}^2} = 2\frac{\partial q_{ph}}{\partial p_{ph}} - \alpha[\frac{\partial q_{ph}}{\partial p_{ph}}^2 + \frac{\partial q_{hp}}{\partial p_{ph}}^2 - 2r\frac{\partial q_{ph}}{\partial p_{ph}}\frac{\partial q_{hp}}{\partial p_{ph}}] \tag{3.8.4.1}$$

Also, recall that

$$\Phi = \frac{1}{\alpha}\left[\begin{array}{ccc|ccc} c & d & d & re & rf & rf \\ d & \ddots & d & rf & \ddots & rf \\ d & d & c & rf & rf & re \\ \hline g & \cdots & g & e & f & f \\ \vdots & \ddots & \vdots & f & \ddots & f \\ g & \cdots & g & f & f & e \end{array}\right]$$

where $c = 1 - \frac{r^2}{(nr+1)(r-1)}$, $d = \frac{-r^2}{(nr+1)(r-1)}$, $e = \frac{-(r(n-1)+1)}{(nr+1)(r-1)}$, $f = \frac{r}{(nr+1)(r-1)}$, $g = \frac{-r}{(nr+1)(r-1)}$. Given the expression of $\Phi$ , we obtain

$$\frac{\partial q_{ph}}{\partial p_{ph}} = -\frac{e}{\alpha} \quad \text{and} \quad \frac{\partial q_{hp}}{\partial p_{ph}} = -\frac{re}{\alpha}.$$

Substituting into Equation 3.8.4.1 gives

$$\frac{\partial^2 u_p}{\partial p_{ph}^2} = -2\frac{e}{\alpha} - \frac{e^2}{\alpha} + \frac{r^2 e^2}{\alpha} = \frac{-2e - (1-r^2)e^2}{\alpha}.$$

Since $e = \frac{r(n-1)+1}{(nr+1)(1-r)} > 0$ as $r \in (\frac{-1}{n}, 1)$, then $\frac{\partial^2 u_p}{\partial p_{ph}^2} < 0$ which satisfies the maximization problem.

To check the second-order condition for the hub, recall that the second-order condition for the first stage is satisfied if $H(u_i)(p_i)$ is positive definite from Lemma 3.3.

For the first term $\Phi_i^{out,out} + \Phi_i^{out,out'}$, recall that

$$\Phi_i^{out,out} = \begin{bmatrix} c & d & d \\ d & \ddots & d \\ d & d & c \end{bmatrix}$$

where $c = 1 - \frac{r^2}{(nr+1)(r-1)} = 1 + d, \quad d = \frac{-r^2}{(nr+1)(r-1)}$. To check that $\Phi_i^{out,out}$ is positive definite, first define

$$K = \begin{vmatrix} 1+d & d & d \\ d & \ddots & d \\ d & d & 1+d \end{vmatrix}_{k\times k}, \quad \forall k \leq n(g)$$

which implies that

$$K = (1+d)^k (1 - \frac{d}{1+d})^{(k-1)} (1 + \frac{d}{1+d}(k-1)).$$

Since $d = \frac{-r^2}{(nr+1)(r-1)} \geq 0$ as $r \in (\frac{-1}{n}, 1)$, then $K > 0,\ \forall k \leq n(g)$. Therefore, $\Phi_i^{out,out}$ is positive definite and thus $\Phi_i^{out,out} + \Phi_i^{out,out'}$ is positive definite.

Consider the leftover term in $H(u_i)(p_i)$. Since $\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi_i^{in,out}$ is symmetric, it is sufficient to prove that

$$\Phi_i^{in,out'} \Sigma_{L,i}^{in,in} \Phi_i^{in,out} - 2\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi^{in,out} + +\sigma_i^2 \Phi_i^{out,out'} \Phi_i^{out,out}$$

is positive definite. Given the expression of quantity impact matrix $\Phi$ we found earlier, one can obtain

$$\Phi_i^{in,out} = [g]_{n\times n},\ \Phi_i^{out,out} = cI_{n\times n} + [d]_{n\times n}^{-D},\ \Sigma_{L,i}^{in,in} = I_{n\times n} + [r]_{n\times n}^{-D},\ \Sigma_{L,i}^{out,in} = [r]_{n\times n}.$$

Therefore, we obtain that

$$\begin{aligned} &\Phi_i^{in,out'} \Sigma_{L,i}^{in,in} \Phi_i^{in,out} - 2\Phi_i^{out,out'} \Sigma_{L,i}^{out,in} \Phi^{in,out} + \sigma_i^2 \Phi_i^{out,out'} \Phi_i^{out,out} \\ &\qquad = (1 + 2d + nd^2 + N) I_{n\times n} + (2d + nd^2 + N)[1]_{n\times n}^{-D} \end{aligned}$$

where $N = n(g^2(1+(n-1)r) - 2gr(c+d(n-1)))$. To see that this term is positive definite, define

$$L = \begin{vmatrix} 1+A & A & A \\ A & \ddots & A \\ A & A & 1+A \end{vmatrix}_{l\times l}$$

where $A = 2d + nd^2 + n(g^2(1+(n-1)r) - 2gr(c+d(n-1)))$. Therefore, we just need to prove that $A$ is positive definite. Since we know that

$$L = (1+A)^l(1-\frac{A}{1+A})^{(l-1)}(1+\frac{A}{1+A}(l-1))$$

Substituting all the relevant parameter values into $A$ gives

$$A = \frac{nr^4}{(nr+1)^2(r-1)^2} - \frac{2r^2}{(nr+1)(r-1)} - \frac{nr^2(r(n-1)+1)}{3(nr+1)(r-1)} \geq 0.$$

The last inequality is from the assumption that $r \in (\frac{-1}{n}, 1)$. Therefore, the second-order condition of the hub is satisfied. Up to this point, we can obtain the following corollary.

**Corollary 3.4 (Equilibrium characterization of star trading network)** *Let $i$ be the agent at the central (hub) and $j$ be at the periphery. If Assumption 3.3 is satisfied and the network is the star network, then*

$$\begin{bmatrix} q_{ij} \\ q_{ji} \end{bmatrix} = \begin{bmatrix} \frac{(nr+1)((n-1)r+1)((2n-3)r+3)}{X} \\ \frac{((n-1)r+1)(n^2(r^2+r)-n(r^2-2r-1)-r^2-r+2)}{X} \end{bmatrix}$$

*where* $X = n^4r^3+n^3(r^3+5r^2)+n^2(-5r^3+5r^2+7r)+n(r^3-12r^2+10r+3)+r^2+r^3-8r+6.$

To prove the first statement in the remark, first note that $X > 0$. To see this, consider

$$\frac{\partial X}{\partial r} = r^2[3n^4+3n^3-15n^2+3n+3] + r[10n^3+10n^2-24n+2] + 7n^2+10n-8.$$

Since $[3n^4+3n^3-15n^2+3n+3] > 0$ and $[10n^3+10n^2-24n+2] > 0$ when $n \geq 2$, then $\inf_{r\in(-\frac{1}{n},1)}[\frac{\partial X}{\partial r}] = 3n+1+\frac{1}{n}+\frac{3}{n^2} > 0$. Therefore, $min_r X = X(r = -\frac{1}{n})$ which is

$$\begin{aligned} X(r=-\frac{1}{n}) &= -n-1+5n+\frac{5}{n}+5-7n+-\frac{1}{n^2}-\frac{12}{n}-10+3n+(\frac{1}{n})^2-(\frac{1}{n})^3+\frac{8}{n}+6 \\ &= \frac{1}{n}-\frac{1}{n^3} > 0. \end{aligned}$$

Therefore, $X > 0$. Consider the condition $q_{ii} \geq q_{jj}$, this is true if and only if $q_{ji} \geq nq_{ij}$. Substituting $q_{ji}$ and $q_{ij}$ into the inequality condition and simplifying the terms give $r^2(2n^2 - 2n - 1) + r(4n-1) + 2 \leq 0$, which will be satisfied if

$$r \in (\frac{-(4n-1)-\sqrt{8n+9}}{2(2n^2-2n-1)}, \frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)}).$$

It is easy to check that $\frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)} \leq 0$ as $(4n-1)^2 \geq 8n+9$ if and only if $2n^2-2n-1 \geq 0$ By assumption that $r \in (-\frac{1}{n}, 1)$, there exists the range of correlation such that $q_{ii} \leq q_{jj}$ if

$-\frac{1}{n} \leq \frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)}$, which is always satisfied when $n \geq 2$. Also, $\frac{-(4n-1)-\sqrt{8n+9}}{2(2n^2-2n-1)} < -\frac{1}{n}$ is always satisfied, since rearranging the condition gives $n \geq -\frac{1}{3+\sqrt{8n+9}}$ which always holds. Therefore, $q_{ii} \geq q_{jj}$ iff

$$r \in (-\frac{1}{n}, 1) \cap (\frac{-(4n-1)-\sqrt{8n+9}}{2(2n^2-2n-1)}, \frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)})$$

which is

$$r \in (-\frac{1}{n}, \frac{-(4n-1)+\sqrt{8n+9}}{2(2n^2-2n-1)}).$$

This proves the first statement in the remark.

To prove the second statement in the remark, suppose that $q_{ji} \geq q_{ij}$. Substituting $q_{ji}$ and $q_{ij}$ and simplifying the terms gives

$$((n-1)r+1)(r-1) \leq 0$$

which is always true as $r \in (-\frac{1}{n}, 1)$. This proves the statement.

To prove the last statement in the remark, substituting $r \in \{-\frac{1}{n}, 1\}$ into the characterization in Lemma 3.4 immediately proves the statement. This proves the statement and the remark.

**Proof of Remark 3.2:** Let $n' = n+1$. To prove the first statement in the remark, consider the equilibrium outcome when $r \to -\frac{1}{n'}$. By Corollary 3.4, one can obtain $lim_{r \to -\frac{1}{n'}} q_{ji} = 1$. Also, substituting $r = -\frac{1}{n'}$ into the analytical solution for $q_{ji}(n)$ gives

$$q_{ji}(n, r = -\frac{1}{n'}) = \frac{2}{3} - \frac{4n + \frac{16}{3}}{6n^2 + 15n + 14} < 1.$$

By continuity of $q_{ji}$ in r, there exists $r < 0$ such that $q_{ji}(n') > q_{ji}(n)$. Since $q_{jj}(n') < q_{jj}(n)$ if and only if $q_{ji}(n') > q_{ji}(n)$, this proves the statement.

To prove the second statement in the remark, consider the equilibrium outcome when $r \to -\frac{1}{n'}$. By Corollary 3.4, one can obtain $n'q_{ij}(n', r \to -\frac{1}{n'}) = 0$. Substituting $r = -\frac{1}{n'}$ into the analytical solution for $q_{ij}(n)$ gives

$$nq_{ij}(n, r = -\frac{1}{n'}) = \frac{2n(n+6)}{6n^2 + 15n + 14} > 0.$$

By continuity of $q_{ij}$ in r, there exists $r < 0$ such that $nq_{ij}(n) > n'q_{ij}(n')$. Since $q_{ii}(n') > q_{ii}(n)$ if and only if $nq_{ij}(n) > n'q_{ij}(n')$, this proves the second statement in the remark.

To prove condition 3, note that the utility of the periphery agent $j$ is

$$u_j = d_j - (p_{ij} - d_i)q_{ij} + (p_{ji} - d_j)q_{ji} - \frac{\alpha}{2}(q_{ij}^2 + (1 - q_{ji})^2 + 2rq_{ij}(1 - q_{ji})).$$

By Corollary 3.4, one can obtain

$$\lim_{r \to -\frac{1}{n'}} u_j(n') = d_j + \alpha \left[ \frac{11}{5(n+1)} - \frac{86\frac{n}{5} + \frac{124}{5}}{6n^2 + 15n + 14} \right].$$

Also, we obtain

$$\lim_{r\to -\frac{1}{n'}} u_j(n) = d_j - \alpha \left[ \frac{24n^4 + \frac{314n^3}{3} + \frac{376n^2}{3} + \frac{88n}{9} - \frac{512}{9}}{(n+1)(6n^2+15n+14)^2} + \frac{1}{18} \right].$$

Therefore, I obtain

$$\lim_{r\to -\frac{1}{n'}} (u_j(n') - u_j(n)) = \frac{1}{18} - \left[ \frac{\frac{28n^3}{3} + \frac{89n^2}{3} + \frac{236n}{9} - \frac{244}{9}}{(n+1)(6n^2+15n+14)^2} \right]$$

which is always greater than 0. To see this, suppose that it is not, then

$$(n+1)(6n^2+15n+14)^2 \le 18 \left[ \frac{28n^3}{3} + \frac{89n^2}{3} + \frac{236n}{9} - \frac{244}{9} \right]$$

which gives

$$36n^5 + 216n^4 + 405n^3 + 279n^2 + 144n + 684 \le 0$$

which is impossible. By continuity of $u_j$ in $r$, there exists $r < 0$ such that $u_j(n') > u_j(n)$. This proves the last statement in the remark.

**Proof of Remark 3.4:** First, we find the equilibrium outcome for the core-periphery network when there are two agents at the core and two agents at the periphery. Let agent 1 and 2 be the cores, agent 3 and 4 be the peripherals connecting to 1, and agent 5 and 6 be the peripherals connecting to 2. Let the row and column of all matrices are ordered as $\{12, 13, 14, 21, 25, 26, 31, 41, 52, 62\}$.

First, we find the expression for $\Phi$. Since this is the symmetric network, I can only consider the elements in column 12, 13, and 31, as the elements in other columns can be found by symmetric properties. Let $K_1 = -2r^2 + 3r + 1$, $K_2 = -2r^2 + 2r + 1$, $K_3 = (r-1)K_1$, and $M = (2r+1)K_3$. From Proposition 3.4, one can obtain the elements of column 12,13,31 in $\Phi$ as follows:

$$\Phi(:,12)' = \frac{1}{\alpha}\left[ \left(\tfrac{r+0.5}{K_1} - \tfrac{1}{2(r-1)}\right) \quad \tfrac{-r^2}{K_3} \quad \tfrac{-r^2}{K_3} \quad \tfrac{-r}{K_3} \quad \tfrac{r^2}{K_3} \quad \tfrac{r^2}{K_3} \quad \tfrac{-r}{K_3} \quad \tfrac{-r}{K_3} \quad \tfrac{r}{K_3} \quad \tfrac{r}{K_3} \right]$$
$$\Phi(:,13)' = \frac{1}{\alpha}\left[ -\tfrac{r^2}{M} \quad -\tfrac{-(2r^4-6r^3+4r+1)}{M} \quad -\tfrac{r^2K_2}{M} \quad -\tfrac{rK_2}{M} \quad -\tfrac{r^3}{M} \quad -\tfrac{r^3}{M} \quad -\tfrac{rK_2}{M} \quad -\tfrac{rK_2}{M} \quad -\tfrac{r^2}{M} \quad -\tfrac{r^2}{M} \right]$$
$$\Phi(:,31)' = \frac{1}{\alpha}\left[ \tfrac{r^2}{M} \quad -r\tfrac{(-2r^3+2r^2+4r+1)}{M} \quad \tfrac{r^2K_2}{M} \quad \tfrac{rK_2}{M} \quad \tfrac{r^2}{M} \quad \tfrac{r^2}{M} \quad -\tfrac{(-2r^3+2r^2+4r+1)}{M} \quad \tfrac{rK_2}{M} \quad \tfrac{r^2}{M} \quad \tfrac{r^2}{M} \right]$$

Using this matrix $\Phi$, one can obtain other coefficient matrices in Proposition 3.4 as follows.

$$\Gamma = \begin{bmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_4 \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} a_1 & a_1 & a_1 \\ a_2 & a_3 & a_1 \\ a_2 & a_2 & a_3 \end{bmatrix}, \quad \begin{cases} a_1 = \frac{3r+1}{K_1} + 1 \\ a_2 = \frac{(3r+1)K_2}{(2r+1)K_1} \\ a_3 = 2.5 - \frac{(2r^2+2.5r+0.5)}{(2r+1)K_1} \\ a_4 = \frac{2(-r^4-2r^3+5r^2+5r+1)}{(2r+1)K_1} \end{cases}$$

$$\Theta = \left[ -\frac{3r+1}{K_1} \quad -\frac{(3r+1)K_2}{(2r+1)K_1} \quad -\frac{(3r+1)K_2}{(2r+1)K_1} \quad -\frac{3r+1}{K_1} \quad -\frac{(3r+1)K_2}{(2r+1)K_1} \quad -\frac{(3r+1)K_2}{(2r+1)K_1} \right.$$
$$\left. \underbrace{\cdots \quad -\frac{(r+1)(-2r^3+2r^2+4r+1)}{(2r+1)K_1} \quad \cdots}_{\times 4} \right]$$

Let $p_{cc}, p_{cp}, p_{pc}(q_{cc}, q_{cp}, q_{pc})$ denote the price(quantity) of transaction between the cores, that from the cores to the peripherals, and that from the peripherals to the cores, respectively. Substituting all the relevant parameters into Proposition 3.4 and imposing symmetric condition, one can obtain the equilibrium outcome as follows.

$$\begin{bmatrix} p_{cc} - d_c \\ p_{cp} - d_c \\ p_{pc} - d_p \end{bmatrix} = -\alpha \begin{bmatrix} \frac{184r^8+142r^7-866r^6-822r^5+475r^4+874r^3+411r^2+82r+6}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \\ \frac{168r^8+98r^7-768r^6-656r^5+441r^4+758r^3+363r^2+76r+6}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \\ \frac{152r^8+150r^7-694r^6-794r^5+293r^4+804r^3+455r^2+110r+10}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \end{bmatrix}$$

and

$$\begin{bmatrix} q_{cc} \\ q_{cp} \\ q_{pc} \end{bmatrix} = \begin{bmatrix} -\frac{-400r^8+52r^7+1898r^6+312r^5-1700r^4-871r^3+85r^2+120r+18}{2(-216r^8+138r^7+1090r^6-398r^5-1331r^4-86r^3+535r^2+238r+30)} \\ -\frac{-320r^8-40r^7+1570r^6+460r^5-1428r^4-879r^3+25r^2+108r+18}{2(-216r^8+138r^7+1090r^6-398r^5-1331r^4-86r^3+535r^2+238r+30)} \\ \frac{-336r^8+36r^7+1554r^6+256r^5-1336r^4-731r^3-3r^2+64r+10}{2(r-1)(216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30)} \end{bmatrix}$$

The last step is to check whether the second-order condition in the first stage is satisfied. Consider the second-order condition for the peripheral agent:

$$\frac{\partial^2 u_p}{\partial p_{pc}^2} = 2\frac{\partial q_{pc}}{\partial p_{pc}} - \alpha[\frac{\partial q_{pc}}{\partial p_{pc}}^2 + \frac{\partial q_{cp}}{\partial p_{pc}}^2 - 2r\frac{\partial q_{pc}}{\partial p_{pc}}\frac{\partial q_{cp}}{\partial p_{pc}}]$$

Substituting the first-order derivative value using the $\Phi$ matrix as previously found, I obtain the following equation:

$$\frac{\partial^2 u_p}{\partial p_{pc}^2} = \frac{(-2r^3+2r^2+4r+1)(2r^4-8r^3+2r^2+5r+1)}{\alpha(2r+1)^2(r-1)(-2r^2+3r+1)^2}$$

which is always negative as $-2r^3+2r^2+4r+1 > 0$ and $2r^4-8r^3+2r^2+5r+1 > 0$ when $r \in (-\frac{1}{5}, 1)$. To check the second-order condition for the cores, we must prove that $H(u_i)(p_i)$ from Lemma 3.3 must be positive definite. Since

$$\Phi_i^{out,out} = \begin{bmatrix} \frac{r+0.5}{K_1} - \frac{1}{2(r-1)} & -\frac{r^2}{M} & -\frac{r^2}{M} \\ -\frac{r^2}{K_3} & -\frac{2r^4-6r^3+4r+1}{M} & -\frac{r^2K_2}{M} \\ -\frac{r^2}{K_3} & -\frac{r^2}{M} & -\frac{2r^4-6r^3+4r+1}{M} \end{bmatrix} \quad \Sigma_{L,i}^{in,in} = \begin{bmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{bmatrix}$$

$$\Phi_i^{in,out} = \begin{bmatrix} -\frac{r}{K_3} & -\frac{rK_2}{M} & -\frac{rK_2}{M} \\ -\frac{r}{K_3} & -\frac{rK_2}{M} & -\frac{rK_2}{M} \\ -\frac{r}{K_3} & -\frac{rK_2}{M} & -\frac{rK_2}{M} \end{bmatrix} \quad \Sigma_{L,i}^{out,in} = \begin{bmatrix} r & r & r \\ r & r & r \\ r & r & r \end{bmatrix}.$$

Substituting into the Hessian matrix $H$ and checking all the principal minors, one can find that $H$ is positive definite if $r \in (-\frac{1}{5}, 0.6813)$. Therefore, the equilibrium outcome is

$$\begin{bmatrix} p_{cc} - d_c \\ p_{cp} - d_c \\ p_{pc} - d_p \end{bmatrix} = -\alpha \begin{bmatrix} \frac{184r^8+142r^7-866r^6-822r^5+475r^4+874r^3+411r^2+82r+6}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \\ \frac{168r^8+98r^7-768r^6-656r^5+441r^4+758r^3+363r^2+76r+6}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \\ \frac{152r^8+150r^7-694r^6-794r^5+293r^4+804r^3+455r^2+110r+10}{216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30} \end{bmatrix}$$

and

$$\begin{bmatrix} q_{cc} \\ q_{cp} \\ q_{pc} \end{bmatrix} = \begin{bmatrix} -\frac{-400r^8+52r^7+1898r^6+312r^5-1700r^4-871r^3+85r^2+120r+18}{2(-216r^8+138r^7+1090r^6-398r^5-1331r^4-86r^3+535r^2+238r+30)} \\ -\frac{-320r^8-40r^7+1570r^6+460r^5-1428r^4-879r^3+25r^2+108r+18}{2(-216r^8+138r^7+1090r^6-398r^5-1331r^4-86r^3+535r^2+238r+30)} \\ \frac{-336r^8+36r^7+1554r^6+256r^5-1336r^4-731r^3-3r^2+64r+10}{2(r-1)(216r^7+78r^6-1012r^5-614r^4+717r^3+803r^2+268r+30)} \end{bmatrix}$$

if $r \in (-\frac{1}{5}, 0.6813)$.

To prove the first statement in the remark, using the equilibrium asset quantity above, one can obtain $\frac{\partial q_{cc}}{\partial r}(r = 0.5) = 2.1025$. By continuity, $\exists k \in \mathbf{R}^+$ such that $\frac{\partial q_{cc}}{\partial r}(r = k) > 0$. This proves the statement.

To prove the second statement in the remark, using the equilibrium condition above, I obtain

$$q_{cp} - q_{pc} = \frac{2(r-1)\overbrace{\left(4r^6 - 11r^5 - 22r^4 + 18r^3 + 35r^2 + 15r + 2\right)}^{A}}{\underbrace{216r^7 + 78r^6 - 1012r^5 - 614r^4 + 717r^3 + 803r^2 + 268r + 30}_{B}}$$

$$(p_{cp} - d_c) - (p_{pc} - d_p) = \frac{-2\alpha(2r+1)(r-1)\overbrace{\left(4r^6 - 11r^5 - 22r^4 + 18r^3 + 35r^2 + 15r + 2\right)}^{A}}{\underbrace{216r^7 + 78r^6 - 1012r^5 - 614r^4 + 717r^3 + 803r^2 + 268r + 30}_{B}}$$

To see that $A > 0$ and $B > 0$ when $r \in (-\frac{1}{5}, 1)$, set $A = 0$ and $B = 0$ and solve for $r$ yields

1. if $r \in \{-1.3146, -0.5449, 1.5846, 3.7135, 0.0096i - 0.3442, -0.0096i - 0.3442\}$, then $A = 0$.
2. if $r \in \{-1.8088, -0.5887, -0.4562, -0.3983, -0.3013, 1.1899, 2.0023\}$, then $B = 0$.

Since $A > 0$ and $B > 0$ when $r = 0$, then $A > 0$ and $B > 0$ when $r \in (-\frac{1}{5}, 1)$ by continuity. Therefore, $q_{cp} < q_{pc}$ and $p_{cp} - d_c > p_{pc} - d_p$. This proves the statement.

To prove the last statement, using the equilibrium condition previously found, I obtain

$$p_{cp} - p_{cc} = \frac{-2r\alpha(2r+1)(r^2-1)\overbrace{\left(-4r^4 - 9r^3 + 25r^2 + 18r + 3\right)}^{C}}{(216r^7 + 78r^6 - 1012r^5 - 614r^4 + 717r^3 + 803r^2 + 268r + 30)}$$

$$q_{cp} - q_{cc} = \frac{2r\overbrace{\left(20r^6 - 3r^5 - 85r^4 - 48r^3 + 20r^2 + 18r + 3\right)}^{D}}{(216r^7 + 78r^6 - 1012r^5 - 614r^4 + 717r^3 + 803r^2 + 268r + 30)}$$

Note that $C > 0$ when $-\frac{1}{5} < r < 0$ since $-4r^4 - 9r^3 > 0$ and $25r^2 + 18r + 3 > 0$. To see that $D > 0$, setting $D = 0$ and solving for $r$ give

$$r \in \{-1.4831, -0.5776, -0.3896, -0.3211, 0.6041, 2.3172\}$$

Since $D > 0$ when $r = 0$. By continuity, $D > 0$ if $r \in (-\frac{1}{5}, 0)$. Therefore, $p_{cp} < p_{cc}$ and $q_{cp} < q_{cc}$ when $r$ is negative. This proves the statement and the remark.

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