The London School of Economics and Political Science

Essays on Macroeconomics

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Declaration

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Abstract

This thesis comprises three chapters on macroeconomics. Chapter 1 studies a paradox of precaution in international saving. In partial equilibrium, a small open economy that accumulates savings during good times can mitigate consumption falls during bad times. This chapter shows that, in general equilibrium, the opposite may be true if the amount of savings is large enough. More savings require more borrowing and higher leverage in the rest of the world, making it more prone to a financial crisis. Crises in the rest of the world then feed back to the saving economies and destabilize them. If the saving economies are collectively large but individually small, their national policymakers will not fully internalize the negative general equilibrium effect. Thus, in equilibrium, there will be excessive global imbalances and excessive volatility for the savers themselves.

Chapter 2 studies consumption-led growth. Investment is bounded by retained earnings for young firms relying on self-financing. The firms are underinvesting from the perspective of a constrained social planner who cannot inject funds to the self-financing firms directly, for two reasons. First, households do not internalize that additional consumption and labor supply increases the profits of the self-financing firms. Second, firms with credit access do not internalize that their expansion fueled by credit intensifies competition in the factor market, drives up factor prices, and squeezes the profits of self-financing firms. The social planner optimally chooses "pro-consumption" policies such as a consumption subsidy and a saving tax on the household to increase the consumption demand and the cost of credit. More consumption paradoxically leads to more investment and output for the self-financing firms.

Chapter 3 studies population aging with automation. This chapter develops a dynamic model that combines demographic transitions, as in Gertler (1999), with endogenous automation. Following Acemoglu and Restrepo (2018b), automation is modeled as the active replacement of labor with capital at the task level in response to a rise in the relative cost of labor to capital, leading to an endogenous increase in the capital share of output. It finds that allowing automation to react endogenously to demographic and productivity changes generates quantitatively relevant effects compared with the standard baseline where firms cannot respond through the automation margin.

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Chapter 1

When Being Thrifty is Risky: A Paradox of Precaution in International Saving

Abstract: In partial equilibrium, a small open economy that accumulates savings during good times can mitigate consumption falls during bad times. This chapter shows that, in general equilibrium, the opposite may be true if the amount of savings is large enough. More savings require more borrowing and higher leverage in the rest of the world, making it more prone to a financial crisis. Crises in the rest of the world then feed back to the saving economies and destabilize them. If the saving economies are collectively large but individually small, their national policymakers will not fully internalize the negative general equilibrium effect. Thus, in equilibrium, there will be excessive global imbalances and excessive volatility for the savers themselves.

1.1 Introduction

The emerging Asian economies recently became major contributors to the global imbalances. After the 1997 Asian financial crisis, they switched from running small current account deficits to large surpluses (Figure 1.1). While there are multiple explanations behind this "global saving glut" (Bernanke, 2005), the self-insurance motive is a key driving factor (Aizenman and Lee (2007); Calvo et al. (2012); Ghosh et al. (2014)). By contrast, the Anglo-American economies at the center of the international financial system have run large current account deficits. Their current account positions started to deteriorate significantly after 1997 and bottomed at -5% of their GDP in 2006 just before the global financial crisis. Thereafter, they borrowed an extra 2.5% of their GDP (approximately) each year, which is twice the pre-1997 level. In part, this could reflect their superior financial development, including a comparative advantage in producing riskless assets (Gourinchas et al., 2017). Their assets provide saving vehicles for consumption smoothing (Caballero et al., 2008) and insurance instruments for risk-sharing (Mendoza et al., 2009) to the less financially developed rest of the world.

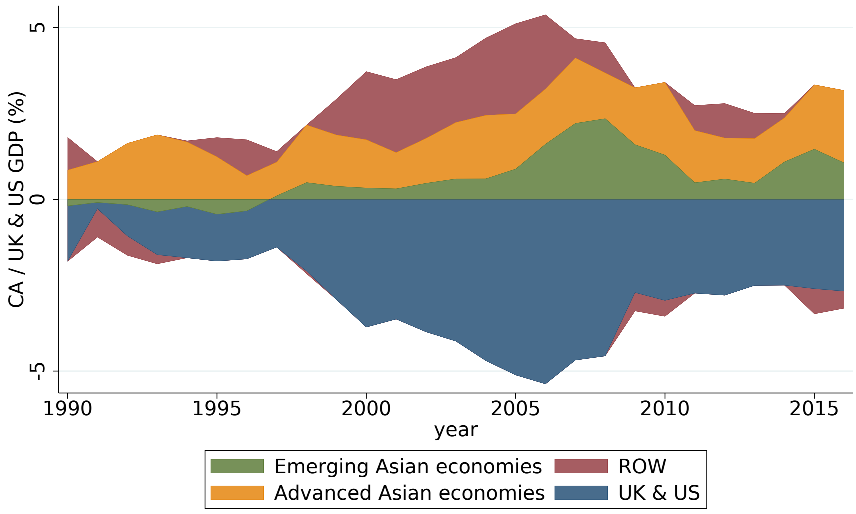

Figure 1.1: Global saving glut: surplus periphery and deficit center

This chapter asks the following question: to what extent is the strategy of saving for self-insurance effective in a world of large global imbalances? There are two competing forces when the savers are not small. First, more savings provide larger buffers against bad shocks and enhance economic stability directly. This is the positive direct effect as conventionally understood. Second, more savings require more borrowing and higher leverage in the rest of the world, rendering it more prone to a financial crisis. Crises in the rest of the world, when materialize, feed back to the saving economies and destabilize them indirectly. This negative general equilibrium (GE) effect, highlighted in this chapter, can overturn the positive direct effect of extra buffers. That is, higher savings can increase rather than decrease the volatility of the saving economies, making the self-insurance strategy self-defeating. I call this case a "paradox of precaution" in international saving.

This chapter leads to both positive and normative results. From a positive perspective, it shows that there is a paradox of precaution if the saving economies are not small. Intuitively, the negative GE effect is large if the saving economies are large, when their savings drive up leverage ratios in the rest of the world to a crisis-prone level. My quantitative model suggests that this might happen in the late 2010s.

From a normative perspective, it calls for international cooperation among the national policymakers in the surplus economies to rebalance their external positions for their own economic stability. An individual surplus economy such as China, Germany, or Japan only contributes to a fraction of the global imbalances. Therefore, each national policymaker does not fully internalize the negative GE effect. As a result, the saving glut economies acting individually saves more than acting collectively. The symptom is excessive global imbalances, and the consequence is excessive volatility for the savers themselves.

The 2008 global financial crisis could be a suggestive example of a large GE effect. As shown in Figure 1.2, Germany and Japan, which were among the main contributors to the global imbalances, fell into recessions in 2009. The recessions were even deeper than in the US, which was the epicenter of the global financial crisis. The growth rate of mainland China, which was another main contributor to the global imbalances, dropped by more than 5% from its pre-crisis peak. The European debt crisis in the early 2010s is yet another suggestive example of a large GE effect at work, even though the savers are more financially developed than the borrowers. Following the introduction of the Euro, Germany saved significantly in the southern European economies, expecting high returns. However, the consequent debt crisis in Greece and the uncertainty about the sustainability of the Eurozone drove Germany itself to the brink of recession during 2012 and 2013.

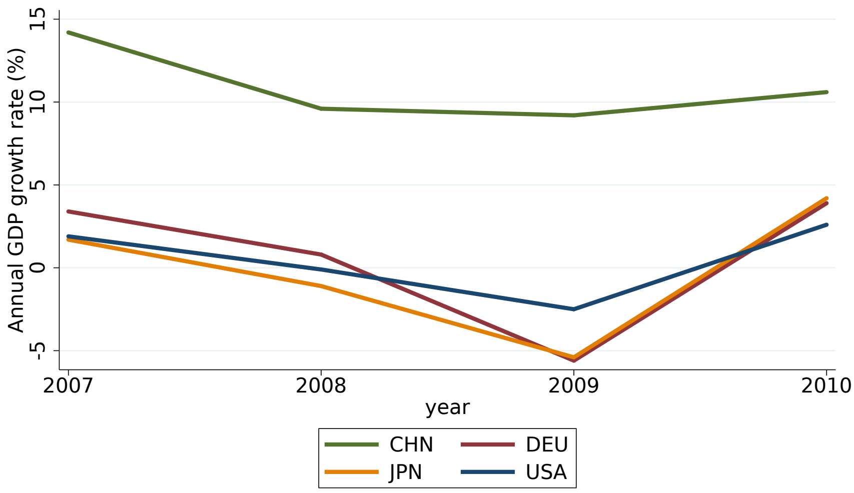

Figure 1.2: Growth rate for CHN, DEU, JPN and the USA in the global financial crisis

To begin with, this chapter presents an analytical two-country Lucas-tree model with collateral constraints to disentangle the direct and GE effects. The analytical model is then enriched with endogenous labor supply and production for quantitative analysis. In the model, financial crises in the borrowing economies, which come from the financial accelerator of the binding financial constraints reinforced by the fire sale of collateral *a la* Kiyotaki and Moore (1997), destabilize the saving economies through a contraction of the bond supply. More specifically, a decrease in the supply of bond prevents the saving economies from rolling over their savings. It disrupts consumption smoothing over time and across different states. The borrowing economies are more likely to deleverage in the first place, and in such a situation they deleverage more heavily, if additional savings from the saving economies drive up their average leverage ratios. As a result, higher savings increase the volatility of the saving economies through the general equilibrium effect. Consistently, Table 1.1 shows that there were large declines in the US riskless bond supplies around the global financial crisis when presumably riskless assets lost their "safe haven" status. The same applied to the early 2010s European debt crisis when even some government bonds were reconsidered as risky.

Table 1.1: A list of riskless assets: pre- and post-crisis in billions of US dollars

	Year	
	2007	2011
US Federal government debt held by public	5,136	10,692
Held by the Federal Reserve	736	1,700
Held by private investors	4,401	8,992
Government-sponsored enterprise (GSE) obligations	2,910	~~2,023~~
Agency- and GSE-backed mortgage pools	4,464	~~6,283~~
Private-issue asset-backed security (ABS)	3,901	~~1,277~~
Total safe asset held by private investors	15,676	8,992

Source: Barclays Capital (2012). The data come from Federal Reserve Flow of Funds, Haver Analytics and Barclays Capital. Numbers are struck through if the corresponding securities are believed to have lost their "safe haven" status after 2007.

It is worth noting that there are various spillback channels that count toward the negative GE effect other than the contraction of bond supply. For example, the collapse of trade, the flight of capital, the interruption of financial services provided by the core countries, or the loss from an outright default all transmit crises from the borrowing to the saving economies. Adding these additional spillback mechanisms will only enhance the GE effect and therefore make the paradox of precaution more likely, which strengthens my result. These channels are not included in this chapter for simplicity. Rather than emphasizing a specific new spillback channel *per se*, this chapter aims to highlight the possibility that the negative GE effect can dominate the positive direct effect.

This chapter contributes to four strands of literature. First, it extends the sudden-stop and capital control literature by going beyond the small open economy framework. The sudden-stop literature models infrequent crises using occasionally binding collateral constraint, which is also developed in this chapter. The literature generally draws the conclusion that the economies borrow too much or save too little ex-ante due to a fire-sale externality (for example, Bianchi (2011), Jeanne and Korinek (2010), Mendoza (2010). Benigno et al. (2013) and Jeanne and Korinek (2013) draw opposite conclusions by emphasizing ex-post policies.). For tractability, the literature uses the small open economy framework in which the international conditions are taken as given. This chapter shows that, the opposite may be true if the interactions between home and foreign economies are modeled explicitly.

Second, the chapter enriches the literature on global imbalances by explicitly studying government policies. The global imbalances literature proposes some explanations in which the imbalances are desirable outcomes from the asymmetry in the development of the financial system and other dimensions (for example, Caballero et al. (2008), Jin (2012), Mendoza et al. (2009)). In particular, the economies at the center of the international financial system provide insurance to the economies at the periphery, fulfilling an "exorbitant duty", in exchange for the "exorbitant privilege" of low financing cost (Gourinchas et al., 2017). In this chapter, a lack of international cooperation among the periphery economies leads to excessive savings and risks for themselves. The global imbalances are undesirably large for the savers.

Third, it supplements the paradox of global thrift literature by providing an alternative channel of self-defeating savings. In the literature following Keynes, large savings in a few economies lead to an undesirable global liquidity trap that reduces global output (Caballero et al. (2016), Caballero and Farhi (2017), Fornaro and Romei (2018)). This chapter does not involve the zero lower bound of nominal interest rates. It deviates from the literature further by focusing on the consumption volatility rather than the output levels, for the savers rather than the borrowers, as a consequence of additional savings.

Fourth, it complements the Triffin dilemma literature (Farhi and Maggiori (2017), Bordo and McCauley (2018)), which focuses on the deficit economies that provide reserve/safe assets to the rest of the world. A Triffin dilemma suggests that the global demand for reserve/safe assets will either remain dangerously unsatisfied or force excessive US money/debt, which is self-defeating for the reserve/safe asset status. My model argues that large savings are also self-defeating for the surplus economies. The more indebted the US is, the more likely it is to deleverage and reduce safe asset supply.

This chapter labels with a new term "paradox of precaution" the situation that the self-insurance by accumulating assets becomes self-defeating through the GE effect. It is a different concept to the seemingly related "paradox of prudence" coined in Brunnermeier and Sannikov (2016). The "paradox of prudence" refers to the situation that a fire sale of risky assets by the prudent investors endogenously increases the riskiness of the asset.

The rest of the chapter will be organized as follows: in Section Two, I introduce an analytical model and disentangle the direct and GE effects; in Section Three I construct a quantitative model and discuss the numerical method and calibration; in Section Four I conduct a positive analysis of the effect of a saving tax on economic volatility; in Section Five I conduct a normative analysis of the optimal saving tax, depending on whether the GE effect is ignored or internalized; and Section Six concludes.

1.2 Analytical model

This section builds a tractable model to disentangle the direct and general equilibrium (GE) effects of additional savings on the consumption smoothing of the saver. The environment is a two-country-Lucas-tree economy without any random shock. By assuming away shocks, the model becomes analytically tractable. This assumption will be relaxed in the next section to allow a generalization of the intuition from consumption smoothing to economic stabilizing.

The world economy consists of two countries: home and foreign. In each of the two countries, a representative consumer lives for three periods: today, tomorrow, and the future. The representative consumer makes consumption, domestic investment, and international saving decisions. She can invest in dividend-paying trees (with fixed supply) domestically. She can also trade a one-period riskless bond internationally. An international bond issuance must be backed by domestic tree collateral. Restricting the tree market to be domestic not only simplifies the analysis but also reflects the equity home bias in reality.

The two economies differ by the dividend processes of their trees. In particular, the foreign trees yield little dividend in the future, causing a potentially low future consumption that motivates savings today and tomorrow. To clear the market, the home economy borrows. This leads to a global imbalance between home and foreign. In reality, the fast aging economies such as China, Japan and Germany, and the oil exporters receiving windfall can be captured by such dividend processes. They are indeed large savers. The ultimate source of the global imbalances is, however, inessential for the mechanism highlighted in this section. I call the borrowing economy "home" and the saving economy "foreign" to follow the convention of calling the US "home".

1.2.1 Home consumer's problem

Formally, the representative home consumer solves the following problem.

$$\max \log C_1 + \beta \log C_2 + \beta^2 \log C_3$$

s.t.

$$C_1 + (K_1 - K_0)\, Q_1 + B_1 P_1 = K_0 d_1 + B_0 \tag{1.1}$$

$$C_2 + (K_2 - K_1)\, Q_2 + B_2 P_2 = K_1 d_2 + B_1 \tag{1.2}$$

$$- B_2 P_2 \leq \phi K_2 Q_2 \tag{1.3}$$

$$C_3 = K_2 d_3 + B_2 \tag{1.4}$$

Each tree yields d_t units of output at time t. The consumer uses the dividends from the tree $K_{t-1} d_t$ and the bond repayment B_{t-1} to consume C_t, to purchase new trees $K_t - K_{t-1}$ at price Q_t and to purchase new bond B_t at price P_t. The initial bond position is zero $B_0 = 0$ and the initial quantity of trees is K_0. If $B_t > 0$, the consumer purchases bond, and if $B_t < 0$, the consumer issues bond. The future period is the final period. Therefore, trees have zero residual value and bond cannot be issued.

The home economy cannot issue riskless bond $-B_2P_2$ tomorrow[1] more than a small[2] fraction ϕ of its tree collateral K_2Q_2. The collateral constraint comes from a limited enforcement problem *a la* Kiyotaki and Moore (1997). The home borrower cannot commit to repay so they pledge their trees as collateral. While foreign lenders cannot operate home trees, they can temporarily seize a fraction ϕ of the defaulting home borrower's trees and sell them at the market price. To ensure repayment, the foreign lenders will not lend more than the liquidation value of the collateral ϕK_2Q_2. In reality, safe bonds can be issued by the private sector and the government subject to explicit and implicit collateral constraints. I do not distinguish them in the model.

The trees are in fixed supply. Normalize the number of home trees to be 1. The tree market clears as follows.

$$K_t = 1, \forall t = 0, 1, 2.$$

The Euler equations for tree and bond tomorrow are the following, respectively.

$$Q_2 (1 - \phi\mu_2) = \beta \frac{C_2}{C_3} d_3 \tag{1.5}$$

$$P_2 (1 - \mu_2) = \beta \frac{C_2}{C_3} \tag{1.6}$$

$\mu_2 \geq 0$ is the shadow price of relaxing the collateral constraint (normalized by the shadow price of relaxing the budget constraint). If the collateral constraint is not binding, the shadow price is zero $\mu_2 = 0$. The asset prices $Q_2,, P_2$ are the discounted values of future payoffs. The future payoff for the trees only involves the future dividend d_3 because the trees have zero residual value after the final period. If the collateral constraint is binding, the shadow price is positive $\mu_2 > 0$. The asset prices are higher than the discounted future payoffs because an additional unit of tree or bond holding relaxes the collateral constraint.

When the collateral constraint is not binding, eliminating C_2 and C_3 from equa-

[1] For simplicity, assume there is no constraint for today's issuance. Alternatively, assume the constraint for today is not binding.

[2] The pledgeability ϕ cannot be too large. In this model, $\phi < 1/(1+\beta)$. Otherwise, an equilibrium might not exist when the legacy debt is large. To show this, assume the home collateral constraint is not binding and solve the bond position and the collateral price. It is a contradiction as the collateral constraint is violated. Intuitively, to roll over a large legacy debt, current borrowing has to be large, which is not feasible. Then assume the home collateral constraint is binding. However, the equilibrium Lagrangian multiplier μ_2 is negative. It is again a contradiction. Intuitively, the consumer values the collateral so much given its large pledgeability that the collateral constraint becomes slack.

tions (1.2), (1.4) and (1.6) using $\mu_2 = 0$, the bond position tomorrow relates to bond position today as follows.

$$-B_2P_2 = \frac{d_3P_2 + \beta(-B_1 - d_2)}{(1+\beta)} \tag{1.7}$$

From the expression, the economy issues more bond ($-B_2P_2$ larger) this period if it is more indebted ($-B_1$ larger) in the previous period. It rolls over the debt.

When the collateral constraint is binding and $\mu_2 > 0$, eliminating μ_2, Q_2, C_2, C_3 from equations (1.2), (1.3 with equality), (1.4), (1.5) and (1.6), the bond position tomorrow relates to bond position today as follows.

$$-B_2P_2 = \frac{\beta(d_2 - (-B_1))}{1/\phi - (1+\beta)} \tag{1.8}$$

The denominator is positive from the assumption that the pledgeability is not too large $\phi < 1/(1+\beta)$. So, the economy borrows less ($-B_2P_2$ smaller) this period if it is more indebted ($-B_1$ larger) in the previous period. It has a debt consolidation.

Intuitively, to pay back a larger maturing debt ($-B_1$ larger), the economy has to sell trees. But the quantity of the trees is fixed in equilibrium. Consequently, the tree price falls (Q_2 lower) and the collateral constraint tightens. Even less debt can be issued ($-B_2P_2$ smaller). It triggers another round of fire sale, so on and so forth. The economy is in a financial crisis *a la* Kiyotaki and Moore (1997). More specifically, from equations (1.2) and (1.8), when the collateral constraint is binding, tomorrow's consumption C_2 is a multiple $\frac{1/\phi-1}{1/\phi-(1+\beta)} > 1$ of resources available B_1+d_2.

$$C_2 = \frac{1/\phi - 1}{1/\phi - (1+\beta)}(B_1 + d_2)$$

Consumption falls more than one-to-one with a fall of dividend d_2 or resources available $B_1 + d_2$.

From equations (1.2), (1.3), (1.4), (1.5) and (1.6), the collateral constraint is binding when

$$d_2 < -B_1 + (1/\phi - (1+\beta))\,\phi/\beta P_2 d_3. \tag{1.9}$$

If the economy is more indebted ($-B_1$ larger), a deleverage is more likely (the constraint is binding when $-B_1$ is large enough from equation 1.9) and more heavily (the deleverage $1 - B_2P_2/B_1$ is larger from equation 1.8). This is a key building block of the GE effect.

1.2.2 Foreign consumer's problem

The representative foreign consumer solves a similar problem.

$$\max \log C_1^* + \beta \log C_2^* + \beta^2 \log C_3^*$$

s.t.

$$C_1^* + (K_1^* - K_0^*)\,Q_1^* + B_1^* P_1\,(1+\tau^*) = K_0^* d_1^* + B_0^* + T^* \tag{1.10}$$

$$C_2^* + (K_2^* - K_1^*)\,Q_2^* + B_2^* P_2 = K_1^* d_2^* + B_1^* \tag{1.11}$$

$$C_3^* = K_2^* d_3 + B_2^* \tag{1.12}$$

The variables of the foreign consumer are denoted with asterisk superscripts. The foreign problem resembles the home problem except for two differences. First, the consumer faces a saving tax that is fully rebated today.

$$B_1^* P_1 \tau^* = T^*.$$

The saving tax affects the equilibrium saving choice today B_1^* directly and the consumption in the future C_3^* indirectly. The instrument is introduced to allow the analysis of additional savings, which is ultimately induced by the tax. Second, as a saver, the consumer's collateral constraints are never binding and are therefore dropped.

The consumer saves tomorrow if the output d_3 is small in the future. Formally, a not very restrictive sufficient condition for $B_2^* > 0$ is $d_3^*/d_3 < \min(d_1^*/d_2, d_2^*/d_2)$. I assume so. If the foreign-to-home relative output in the future is lower than today and tomorrow, the foreign consumer should save at least tomorrow for the future. For ease of interpretation, I also assume that d_1^* is not too small so that the foreign consumer saves both today and tomorrow. Both assumptions will be relaxed in the quantitative model.

The trees are in fixed supply K^*. It captures the relative size of the foreign economy to the home economy, in which the quantity of trees is normalized to 1. The tree market clears as follows.

$$K_t^* = K^*, \forall t = 0, 1, 2$$

The Euler equation for bond tomorrow is the following.

$$P_2 = \beta \frac{C_2^*}{C_3^*} \tag{1.13}$$

Likewise, eliminating C_2^* and C_3^* from equations (1.11), (1.12) and (1.13), the bond position tomorrow relates to the bond position today as follows.

$$B_2^* P_2 = \frac{\beta\,(B_1^* + K^* d_2^*) - K^* d_3^* P_2}{(1+\beta)} \tag{1.14}$$

The world bond market clearing condition closes the model.

$$B_t + B_t^* = 0, \forall t = 1, 2. \tag{1.15}$$

For a given foreign saving tax τ^*, a general equilibrium is defined as a set of domestic tree prices Q_t, Q_t^* and international bond prices P_t such that (1) the consumption C_t, C_t^*, the bond positions B_t, B_t^* and the tree positions K_t, K_t^* solve the home and foreign consumers' problem; (2) the two domestic tree markets and the world bond market clears.

1.2.3 Results

The foreign savings today B_1^* are endogenous for an exogenously given τ^*. But it is equivalent and more convenient to treat τ^* as endogenous for an exogenously given B_1^*. The consumption smoothing involves saving today B_1^* and rolls over the savings tomorrow for future consumption C_3^*. This section shows the consequence of an additional unit of foreign bond B_1^* today (induced by τ^* change) on the future foreign consumption C_3^*.

Lemma 1. *The foreign economy cannot roll over its savings from today to tomorrow if the home economy deleverages and fails to roll over its debt* ($\mathrm{d}B_2^*/\mathrm{d}B_1^* < 0$ *if* $\mu_2 > 0$)*. The foreign economy can roll over its savings from today to tomorrow otherwise* ($\mathrm{d}B_2^*/\mathrm{d}B_1^* > 0$ *if* $\mu_2 = 0$)*.*

Proof. See Appendix 1.A.

□

The lemma reveals that more savings today lead to fewer savings tomorrow for the foreign economy if the home economy deleverages. The home deleverage prevents it from rolling over its debt. This, in turn, creates an asset shortage and prevents the foreign economy from rolling over the savings.

Proposition 2. *Additional savings today exacerbate the foreign economy consumption fall in the future if the home economy deleverages tomorrow* ($\mathrm{d}C_3^*/\mathrm{d}B_1^* < 0$ *if* $\mu_2 > 0$)*. Additional savings today mitigate the foreign economy consumption fall in the future otherwise* ($\mathrm{d}C_3^*/\mathrm{d}B_1^* > 0$ *if* $\mu_2 = 0$)*.*

Proof. From equation (1.12), $\mathrm{d}C_3^*/\mathrm{d}B_2^* > 0$. The results follow Lemma 1.

□

The proposition reveals the possibility of a foreign paradox of savings. Additional foreign savings today can lead to lower consumption in the future. It is self-defeating for the purpose of consumption smoothing.

Lemma 3. *In the laissez-faire equilibrium, the home economy deleverages tomorrow* ($\mu_2 > 0$) *if the foreign economy is relatively large* $K^* > \hat{K}^*$*, where* $\hat{K}^*$ *solves*

$$\frac{1}{1+\beta+\beta^2}\left((1+\beta) - \frac{(1+K^*d_3^*/d_3)}{(1+K^*d_1^*/d_1)} - \frac{\beta\,(1+K^*d_3^*/d_3)}{(1+K^*d_2^*/d_2)}\right) = \phi$$

Proof. See Appendix 1.A. □

If the foreign economy is large it demands lots of home assets. The home economy is, in turn, more indebted today and more likely to deleverage tomorrow.

Proposition 4. *Additional savings than the laissez-faire equilibrium today exacerbate the foreign economy consumption fall in the future if its size is relatively large ($\mathrm{d}C_3^*/\mathrm{d}B_1^* < 0$ if $K^* > \hat{K}^*$). Additional savings than the laissez-faire equilibrium today mitigate the foreign economy consumption fall in the future otherwise ($\mathrm{d}C_3^*/\mathrm{d}B_1^* > 0$ if $K^* < \hat{K}^*$).*

Proof. Combine Proposition 2 and Lemma 3. □

A paradox of saving appears when the foreign economy is relatively large. In reality, the savers are growing faster than the borrowers, as shown in Appendix 1.B. Following the proposition, a paradox of saving will be increasingly relevant in the future, if not already now.

Due to the rare nature of a financial crisis and identification problems, it is difficult to test the paradox empirically. For anecdotal evidence, the 2008 global financial crisis is a recent scenario in which the financial crises from borrowing economies (the US and the UK), to which the global saving glut (from China, Japan, Germany, and the oil exporters) contributed[3], destabilized the saving economies themselves.

Theorem 5. *The net effect of additional savings today on the foreign economy future consumption can be decomposed to a direct effect and a general equilibrium (GE) effect.*

$$\frac{\mathrm{d}C_3^*}{\mathrm{d}B_1^*} = \underbrace{\left.\frac{\partial C_3^*}{\partial B_1^*}\right|_{P_2}}_{\textit{direct effect}} + \underbrace{\frac{\partial C_3^*}{\partial B_2^*}\frac{\partial B_2^*}{\partial P_2}\frac{\mathrm{d}P_2}{\mathrm{d}B_1^*}}_{\textit{GE effect}}$$

The GE effect is negative if the home economy deleverages tomorrow and zero otherwise ($\frac{\partial C_3^*}{\partial B_2^*}\frac{\partial B_2^*}{\partial P_2}\frac{\mathrm{d}P_2}{\mathrm{d}B_1^*} < 0$ if $\mu_2 > 0$ and $\frac{\partial C_3^*}{\partial B_2^*}\frac{\partial B_2^*}{\partial P_2}\frac{\mathrm{d}P_2}{\mathrm{d}B_1^*} = 0$ if $\mu_2 = 0$). The direct effect is always positive ($\left.\frac{\partial C_3^*}{\partial B_1^*}\right|_{P_2} > 0$).

Proof. The decomposition results from applying the chain rule. $\frac{\partial C_3^*}{\partial B_2^*}\frac{\partial B_2^*}{\partial P_2} < 0$ comes from equations (1.12) and (1.14). The $\mathrm{d}P_2/\mathrm{d}B_1^* > 0$ for $\mu_2 > 0$ case results from eliminating B_2 and B_2^* from equations (1.8), (1.14) and (1.15). The $\mathrm{d}P_2/\mathrm{d}B_1^* = 0$ for $\mu_2 = 0$ case results from eliminating B_2 and B_2^* from equations (1.7), (1.14) and (1.15). $\partial C_3^*/\partial B_1^*|_{P_2} > 0$ results from equations (1.12) and (1.14). □

[3] The financial innovation and regulation failures also contributed to the crisis

The direct effect is the effect of additional savings with an infinitely elastic bond supply at a fixed bond price P_2. The direct effect always improves consumption smoothing. Intuitively, if the bond price is fixed, nothing discourages the foreign economy from rolling over its savings tomorrow.

The GE effect is the effect of additional savings from a potentially higher tomorrow bond price P_2 induced by the deleverage and contraction of asset supply in a financial crisis of the rest of the world. The GE effect is negative or zero. Intuitively, the lower tomorrow's interest rate is (P_2 larger), the smaller the savings (B_2^* smaller) are and hence the lower future consumption (C_3^* smaller) is. If the home economy deleverages tomorrow, the safe asset shortage drives up the bond price. Additional foreign savings today unintendedly trigger a more severe home deleverage tomorrow and hence a higher bond price. If the home economy does not deleverage tomorrow, the perfect consumption co-movements of the home and foreign economies between tomorrow and the future pins down the world bond price, regardless of the saving decision today.

The GE effect can overturn the conventional wisdom of saving for a rainy day. The net effect depends crucially on the relative magnitude of the direct and GE effects. In this model, the GE effect is so large that it always dominates the direct effect if the home economy deleverages tomorrow.

What's more, the decomposition reveals the possibility of excessive global imbalances. If the negative GE effect is ignored, a foreign policymaker thinks that additional savings always increase its future consumption. It saves more than that if it internalizes the negative GE effect[4]. In reality, the saving economies are collectively large but individually small. The major contributors of the global imbalances such as China, Germany, and Japan all have similar surpluses. The policymakers in these economies are likely to not fully internalize the negative GE effect, creating excessive global imbalances.

1.3 Quantitative model

I set up an international business cycle model with financial frictions. It extends the discussion from consumption smoothing to economic stabilizing, and quantifies the conditions for the paradox of precaution to appear.

The environment is a stochastic two-country-production economy with an infinite horizon. There is, again, one home economy and one foreign economy of which variables are denoted with asterisk superscripts. But they have infinite horizons. The output is not the exogenous dividend from the trees anymore, but is produced with land and labor. Land, like the tree in the analytical model, is traded domestically. A one-period riskless bond is traded internationally. The total debt cannot exceed a fraction of the land collateral for the same enforcement reasons. The two

[4]The home consumption smoothing is also interrupted from too much debt.

economies have similar consumption smoothing problems subject to similar budget constraints and credit constraints. They are calibrated asymmetrically to reflect the global imbalances from (1) tighter financial constraint in the foreign economy and (2) higher variance of productivity shock in the foreign economy with details in the calibration section.

1.3.1 Consumer-entrepreneur's problem

1.3.1.1 Home economy

The representative home consumer-entrepreneur solves the following problem.

$$V_t = \max \mathbb{E}_j \sum_{t=j}^{\infty} \beta^{t-j} \frac{\left(C_t - \chi \frac{(L_t^s)^{1+\omega}}{1+\omega}\right)^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$C_t + Q_t (K_t - K_{t-1}) + P_t B_t = W_t L_t^s + B_{t-1} + \left(Z_t K_{t-1}^{\alpha} \left(L_t^d\right)^{\gamma} - W_t L_t^d \right)$$

$$-P_t B_t + \theta W_t L_t^d \leq \phi Q_t K_t$$

The consumer-entrepreneur values consumption C_t and dislikes working L_t^s following the Greenwood–Hercowitz–Huffman specification (Greenwood et al. (1988)). This formulation of preferences removes the income effect on the labor supply by making the marginal rate of substitution between consumption and labor depends on the labor only. The consumer-entrepreneur receives income from working L_t^s at wage W_t, bond repayment B_{t-1}, and revenue from operating the firm $Z_t K_{t-1}^{\alpha} \left(L_t^d\right)^{\gamma} - W_t L_t^d$. The firm produces output using land K_{t-1} and labor L_t^d with a Cobb-Douglas production function of productivity Z_t. The proceeds are used to consume, purchase new land $K_t - K_{t-1}$ at price Q_t and purchase new bond B_t at price P_t. The intertemporal debt $-P_t B_t$, plus the intratemporal debt to finance the working capital requirement $\theta W_t L_t^d$, cannot exceed a fraction ϕ of the land value $Q_t K_t$. If the bond position is positive, the financial constraint should be interpreted as that the working capital debt is collateralized against the domestic land and the international reserves in bond $\theta W_t L_t^d \leq \phi Q_t K_t + P_t B_t$.

1.3.1.2 Foreign economy

Likewise, the representative foreign consumer-entrepreneur has a problem with the same structure as in the home economy.

$$V_t^* = \max \mathbb{E}_j \sum_{t=j}^{\infty} (\beta^*)^{t-j} \frac{\left(C_t^* - \chi^* \frac{(L_t^{s*})^{1+\omega^*}}{1+\omega^*}\right)^{1-\sigma^*} - 1}{1-\sigma^*}$$

s.t.

$$C_t^*+Q_t^*\left(K_t^*-K_{t-1}^*\right)+P_t\left(1+\tau^*\right)B_t^*=W_t^*L_t^{s*}+B_{t-1}^*+\left(Z_t^*\left(K_{t-1}^*\right)^{\alpha^*}\left(L_t^{d*}\right)^{\gamma^*}-W_t^*L_t^{d*}\right)+T_t^*$$

$$-P_tB_t^*+\theta^*W_t^*L_t^{d*}\leq\phi^*Q_t^*K_t^*$$

The only difference is that their savings are taxed at a constant rate τ^*, chosen by the government. A positive τ^* raises the effective price of bond and discourages saving. A negative τ^*, instead, reduces the effective price of bond and encourages saving. The tax revenue is fully rebated through a lump-sum transfer.

$$T_t^*=\tau^*P_tB_t^*.$$

In practice, a positive τ^* captures the effect of capital outflow controls. A negative τ^* captures the reserve accumulations by the central bank plus capital inflow controls. More specifically, the central bank issues papers or sells government bond to domestic agents and uses the proceeds to invest in international reserve assets. The interest rate of international reserve assets is usually lower than the domestic government bond or central bank liabilities. The difference is the saving subsidy τ^*. The capital inflow control rules out arbitrage.

1.3.1.3 Market clearing conditions

Each economy's land is in fixed supply. The land market clears in the home and foreign economies as follows.

$$K_t=K$$

$$K_t^*=K^*$$

The labor market clears in the home and foreign economies as follows.

$$L_t^s=L_t^d$$

$$L_t^{s*}=L_t^{d*}$$

The world bond market clears as follows.

$$B_t+nB_t^*=0,$$

where n is the population of the foreign economy as a multiple of the home economy.

1.3.1.4 Equilibrium conditions

The optimal conditions for the foreign economy are the following.

$$W_t^*=\chi^*\left(L_t^{s*}\right)^{\omega^*}$$

The foreign consumer supply labor until its disutility from working equals the utility from extra consumption. The labor supply increases with the wage.

$$\gamma^* Z_t^* \left(K_{t-1}^*\right)^{\alpha^*} \left(L_t^{d*}\right)^{\gamma^*-1} = \left(1+\mu_t^*\theta^*\right) W_t^*$$

The foreign entrepreneur's labor demand equals the marginal output and marginal cost of labor. $\mu_t^* \geq 0$ is the shadow price of relaxing the financial constraint (normalized by the shadow price of relaxing the budget constraint). The marginal cost of labor could be higher than wage because it tightens the financial constraint.

$$Q_t^* \left(1-\phi^*\mu_t^*\right) = \mathbb{E}_t\beta^* \left(\frac{C_{t+1}^* - \chi^* \frac{\left(L_{t+1}^{s*}\right)^{1+\omega^*}}{1+\omega^*}}{C_t^* - \chi^* \frac{\left(L_t^{s*}\right)^{1+\omega^*}}{1+\omega^*}} \right)^{-\sigma^*} \left(Q_{t+1}^* + \alpha^* Z_{t+1}^* \left(K_t^*\right)^{\alpha^*-1} \left(L_{t+1}^{d*}\right)^{\gamma^*}\right)$$

The land price equals the discounted future land price and rents if the financial constraint is not binding. It exceeds the discounted future payoffs if the financial constraint is binding because land provides additional benefit of relaxing the financial constraint.

$$P_t \left(1+\tau_t^* - \mu_t^*\right) = \mathbb{E}_t\beta^* \left(\frac{C_{t+1}^* - \chi^* \frac{\left(L_{t+1}^{s*}\right)^{1+\omega^*}}{1+\omega^*}}{C_t^* - \chi^* \frac{\left(L_t^{s*}\right)^{1+\omega^*}}{1+\omega^*}} \right)^{-\sigma^*}$$

The bond price equals discounted future unit repayment if the financial constraint is not binding. It exceeds the discounted future payoff if the financial constraint is binding because bond provides additional benefit of relaxing the financial constraint. A tax makes the bond less attractive and its price is lower.

The complementary slackness condition for the financial constraint is the following.

$$\mu_t^* \left(\phi^* Q_t^* K_t^* + P_t B_t^* - \theta^* W_t^* L_t^{d*}\right) = 0 \text{ and } \mu_t^* \geq 0$$

The shadow price is positive if the constraint is binding and is otherwise zero.

The home consumer-entrepreneur's maximizing problem yields the same sets of optimal conditions without a saving tax in the bond Euler equation.

$$W_t = \chi \left(L_t^s\right)^\omega$$

$$\gamma Z_t \left(K_{t-1}\right)^\alpha \left(L_t^d\right)^{\gamma-1} = \left(1+\mu_t\theta\right) W_t$$

$$Q_t \left(1-\phi\mu_t\right) = \mathbb{E}_t\beta \left(\frac{C_{t+1} - \chi \frac{\left(L_{t+1}^s\right)^{1+\omega}}{1+\omega}}{C_t - \chi \frac{\left(L_t^s\right)^{1+\omega}}{1+\omega}} \right)^{-\sigma} \left(Q_{t+1} + \alpha Z_{t+1} \left(K_t\right)^{\alpha-1} \left(L_{t+1}^d\right)^{\gamma}\right)$$

$$P_t\left(1-\mu_t\right)=\mathbb{E}_t\beta\left(\frac{C_{t+1}-\chi\frac{\left(L_{t+1}^s\right)^{1+\omega}}{1+\omega}}{C_t-\chi\frac{\left(L_t^s\right)^{1+\omega}}{1+\omega}}\right)^{-\sigma}$$

$$\mu_t\left(\phi Q_tK_t+P_tB_t-\theta W_tL_t^d\right)=0 \text{ and } \mu_t\geq 0$$

An equilibrium for a given saving tax τ^* is defined by a set of international bond price P_t, domestic land price Q_t, Q_t^* and wage W_t, W_t^* such that (1) the consumption C_t, C_t^*, the bond positions B_t, B_t^* the land positions K_t, K_t^*, the labor supply L_t^s, L_t^{s*}, and labor demand L_t^d, L_t^{d*}solve the problems of the consumer-entrepreneurs in the home and foreign economies; (2) the domestic land markets and labor markets clear; and (3) the international bond market clears.

1.3.2 Numerical Algorithm

The model is solved by a global, nonlinear solution method for two reasons. First, financial crises are infrequent events during which the binding financial constraint reinforces the initial fall in consumption and collateral price from a bad shock. Second, the consumer-entrepreneurs make portfolio choices between the safe bond and the risky land so that a nonstochastic steady state cannot be well defined.

The global solution is solved by iterating on the optimal conditions following Coleman (1990)'s time iteration algorithm. The algorithm starts by guessing the policy functions for the next period variables and solves the policy functions for current period variables. It then updates the policy functions for the next period variables using the solution and repeats the process until it converges. The details of the algorithm are described in Appendix 1.D.

For the tricky issue of two occasionally binding constraints, which leads to four binding-nonbinding combinations for each state in each iteration of the time iteration algorithm, I apply a transformation of the complementary slackness conditions to get rid of it. The complementary slackness condition in the form of $\mu=0, \mu\geq 0, X\geq 0$ is transformed to $\mu=\max(0,\hat{\mu}^3), X=\max(0,-\hat{\mu}^3)$ by introducing an auxiliary variable $\hat{\mu}\in(-\infty,\infty)$. The transformation satisfies the complementary slackness condition by construction and the new auxiliary variable does not have any restrictions of its value. The forward-looking equation system can then be treated as those without inequality conditions. It is worth noting that the auxiliary variable is raised to the cubic power to ensure that the second-order derivatives almost always exist. This facilitates root-finding using the fast Newton methods in each iteration.

I further implement an adaptive grid method to improve precision and use parallelization to improve speed. The model is hence solved fairly accurately according to the Euler equation error within a reasonable time. This allows me to proceed to the comparative static analysis, in which the model needs to be solved many times for different parameterizations. The details are in Appendix 1.D.

1.3.3 Calibration

I calibrate the model to the US and China annually. The home economy refers to the deficit economies represented by the US and the foreign economy refers to the surplus economies represented by China. Table 3.1 summarizes the calibration.

Table 1.2: Calibration

Parameter	Description	Value	Source/Target
β,β^*	discount rate	0.96	standard
σ, σ^*	relative risk aversion	2	standard
γ ,γ^*	labor share	2/3	standard
χ, χ^*	labor disutility coefficient	2/3	normalization
K, K^*	quantity of asset	1	normalization
α,α^*	asset share	0.05	normalization, US $QK/GDP = 1.25$
ω, ω^*	inverse Frisch elasticity of labor	1	Kimball and Shapiro (2008)
ϕ	pledgeability	0.29	tar. US freq. crisis 0.03
ϕ^*	pledgeability	0.1	tar. $NIIP^*/GDP^* = 0.4$
$\theta\gamma$	working capital coefficient	0.15	cal. $M_1/GDP = 0.15$
$\theta^*\gamma^*$	working capital coefficient	0.5	cal. $M_1^*/GDP^* = 0.5$
$\overline{Z}$	mean productivity	1	normalization
$\overline{Z}^*$	mean productivity	0.25	tar. $GDP^*/GDP = 0.5$
n	foreign population	4	cal. CN/US population
ρ	persistence of log productivity	0.55	tar. US log GDP autocorr 0.54
σ_Z	stderr of shock to log productivity	0.012	tar. US log GDP stderr 0.021
ρ^*	persistence of log productivity	0.79	tar. CN log GDP autocorr 0.73
σ_Z^*	stderr of shock to log productivity	0.0155	tar. CN log GDP stderr 0.030

A few parameters are standard in quantitative DSGE models. The discount rate β is set to 0.96, the relative risk aversion σ is set to 2, and the labor share γ is set to 2/3. The corresponding foreign variables β^*, σ^* and γ^* are set to the same value as home. The Frisch elasticity of labor supply $(1/\omega, 1/\omega^*)$ is set to equal to 1, in line with Kimball and Shapiro (2008).

A few parameters can be normalized. The labor disutility coefficients (χ, χ^*) are both normalized to the same as labor shares (γ, γ^*). The quantities of fixed assets (K, K^*) are both normalized to 1. Even if China have different disutility from working $(\chi^* \neq \chi)$ or different fixed assets for collateral $(K^* \neq K)$ in reality, the effects will be completely picked up through the calibration of the productivity processes (Z_t, Z_t^*). The fixed asset share coefficients (α, α^*) are set to be 0.05, so the mean fixed asset value to GDP (approximately $\alpha/(1-\beta)$ in the model) is about 1.25, which is roughly the value of residential housing collateral to GDP in the US. This calibration is neither essential because for any normalization, the effects will be fully absorbed through calibration of the pledgeability coefficient (ϕ, ϕ^*).

Four coefficients for the working capital and pledgeability constraints $(\theta, \theta^*, \phi, \phi^*)$ are specific to my model. The working capital coefficients (θ, θ^*) are calibrated so that the working capital to GDP ratios ($\theta\gamma, \theta^*\gamma^*$ in the model) match the cash equivalents M1 to GDP ratios, following Schmitt-Grohe and Uribe (2007). It captures the fact that firms need cash or credit lines to pay wage bills and suppliers in advance and the households need cash or credit cards to purchase goods and services. M1 includes currency or assets that can be quickly converted to cash and are therefore an appropriate proxy. In the US, the M1 to GDP ratio is around 0.15, and in China, it is around 0.5. The high demand for cash equivalents in China provides a reason why it is a net creditor.

The pledgeability coefficient ϕ^* for China is difficult to calibrate to micro-level data due to the high level of aggregation of the model and the wide dispersion in the loan-to-value restrictions in reality. Instead, it is chosen so that the mean bond to GDP ratio B^*/Y^* matches the net international investment position (NIIP) to GDP for China. It is around 0.4 in the early 2010s[5]. China's GDP is about half of the US in the early 2010s, this target implies that the US NIIP/GDP will be -0.2 in the absence of other economies. It is indeed the case for the US. The two-country model works well for the US and China. This approach leads to $\phi^* = 0.1$.

The pledgeability coefficient ϕ for the US suffers from the same difficulty to use micro-level data. Instead, it is calibrated to match the frequency of financial crises in the US as in Mendoza (2010). To be consistent with the empirical literature, a financial crisis is an event during which the financial constraint is binding, and the deleverage exceeds its one standard deviation. Over the last century, the US has encountered three major financial crises: the great depression, the savings and loan crisis, and the great recession. By targeting a frequency of crisis of 0.03, the value of ϕ is calibrated to be 0.29. It is consistent with the measures of the household and corporate leverage in the US, varying from 0.2 to 0.45.

The relative population of China to the US is 4 so $n = 4$. The mean productivity of the US $\overline{Z}$ is normalized to be 1 and the mean productivity of China $\overline{Z}^*$ is calibrated to be 0.25 to target a Chinese GDP as 1/2 of that of the US, which was the case in the early 2010s.

The log productivity processes are set to follow first-order autoregressive processes.

$$\log\left(Z_t - \overline{Z}\right) = \rho_Z \log\left(Z_t - \overline{Z}\right) + \epsilon_Z$$
$$\log\left(Z_t^* - \overline{Z}^*\right) = \rho_Z^* \log\left(Z_t^* - \overline{Z}^*\right) + \epsilon_Z^*$$

Each process is approximated using the quadrature procedure of Tauchen and Hussey (1991) with 3 nodes. The two shocks are assumed to be completely independent. The persistence and standard deviation are set to target the business cycle statistics of log GDP in both economies. The log GDP autocorrelation and standard deviation

[5]The NIIP of China under consideration sums the separately reported data from mainland China, the Hong Kong special administrative region of China, the Macau special administrative region of China, and the Taiwan province of China.

are calculated from HP filtered log real GDP data in a constant local currency unit. For the post-war period 1947-2017, the US log GDP has the autocorrelation 0.54 and standard deviation 0.021. For 1980-2017 the Chinese log GDP has the autocorrelation 0.73 and standard deviation 0.030. The pre-1980s data are not used as the Chinese economy was mostly central planning during that time. The targets lead to $\rho_z = 0.55, \sigma_z = 0.012, \rho_z^* = 0.79, \sigma_z^* = 0.0155$. The log output volatilities are larger than that of the shocks with the amplification from the financial frictions. The log output processes are, however, less persistent than that of the shocks because I abstract away from capital accumulation or capital adjustment cost.

The equilibrium equation system is summarized in Appendix 1.C. In Table 1.3, I summarize the model moments and the data moments from simulations. The simulated moments match the corresponding targets fairly well. I also report the relative volatility of consumption to output for a double-check. While the relative volatility does not closely follow the longest data series available, which was used as target of the productivity shock processes, they match the data fairly well for the recent three decades 1990-2017.

Table 1.3: Model and data moments

	Model	Data (detrended with HP filter)		
		1947-2017	1980-2017	1990-2017
$std\,(\log Y)$	0.0215	0.0212		0.0160
$autocorr\,(\log Y)$	0.5387	0.5375		0.5942
$std\,(\log Y^*)$	0.0302		0.0301	0.0266
$autocorr\,(\log Y^*)$	0.7341		0.7393	0.8588
$std\,(\log C)\,/std\,(\log Y)$	1.0283	0.8286		1.0393
$std\,(\log C^*)\,/std\,(\log Y^*)$	0.8664		1.1437	0.8233

The calibration leads to a mean international bond price of 0.965 and mean world interest rate of 3.6%. This is roughly in line with the long-run real interest rate.

1.4 Positive analyses

This section reports the effect of a saving tax on its consumption volatility for the foreign economy and how that changes with its relative size. The consumption volatility is calculated as the standard deviation of the log consumption adjusted with the disutility from working ($\hat{C}_t^* \equiv C_t^* - \chi\,(L_t^*)^{1+\omega^*}/(1+\omega^*)$) by simulating a long time series after solving the model globally[6]. I do not track the probability of a foreign financial crisis because the foreign economy is almost always a large creditor from the baseline calibration (NIIP/GDP=40%) and a crisis is hard to define.

[6]The effect on the unadjusted standard deviation of log consumption is similar. However, the adjusted ones are more appropriate targets for consumption smoothing in a model with endogenous labor supply.

1.4.1 Paradox of precaution

To understand the effect of a foreign saving tax, Figure 1.3 plots adjusted consumption volatility against different saving taxes, for two scenarios. The left panel is for a "foreign-small" scenario in which the average foreign/home GDP ratio is 0.5 as in the baseline calibration and the right panel is for a "foreign-large" scenario in which the average foreign/home GDP ratio is 0.8. There are two main results.

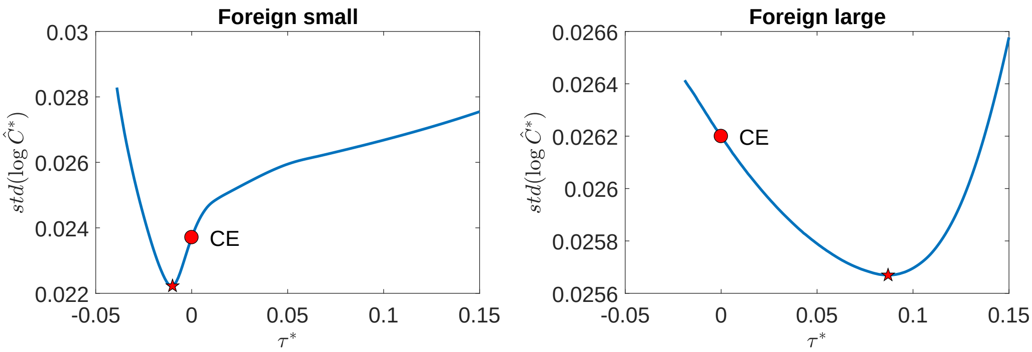

Figure 1.3: Effect of a foreign saving tax on foreign consumption volatility.
Left panel: foreign-small scenario, $\overline{Z}^*/\overline{Z} = 0.125$ so $nY^*/Y = 0.5$. Right panel: foreign-large scenario, $\overline{Z}^*/\overline{Z} = 0.342$ so $nY^*/Y = 0.8$. Circle marker: laissez-faire equilibrium. Star marker: the tax for volatility minimizing.

First, around the laissez-faire equilibrium (no tax, marked with a circle) the consumption volatility increases with the saving tax in the foreign-small scenario but decreases with the saving tax in the foreign-large scenario. The former is consistent with the conventional wisdom that a saving subsidy enhances economic stability. The latter shows a paradox of precaution: a saving subsidy increases economic volatility. This is consistent with the prediction of the analytical model that additional savings are bad for consumption smoothing when the saving economy is large.

Second, both curves are V-shaped. The consumption volatility is decreasing with the saving tax in the left-part of both panels, while it is increasing in the right-part. The lowest point of the curve is marked with a star in each panel. When the foreign economy is small, a saving subsidy of around 0.8 percentage points leads to the minimum volatility. When the foreign economy is large, a saving tax around 8.7 percentage points leads to the minimum volatility.

To understand the two results, Figures 1.4 and 1.5 augment Figure 1.3 by providing three additional statistics for the two scenarios: the average external saving position B^*/Y^*, the average bond price P, and the probability of a home crisis. To be consistent with the empirical literature, a home financial crisis is defined as an event when the financial constraint is binding and the deleverage $(\frac{B_{t-1}-B_t}{B_{t-1}})$ exceeds more than one standard deviation.

An increase of the saving tax reduces the external savings, as shown in the upper-right panel. The international bond price decreases consequently as in the lower-left panel and this induces lower leverage ratios in the home economy. Therefore, the probability of a home crisis decreases as shown in the lower-right panel.

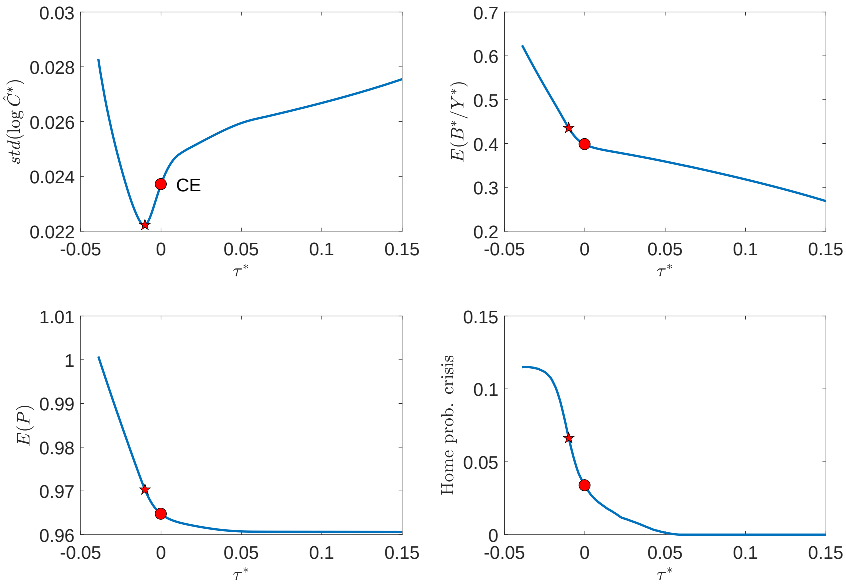

Figure 1.4: Effect of a foreign saving tax, foreign-small scenario
Circle marker: laissez-faire equilibrium. Star marker: the tax for volatility minimizing. Foreign small: $\overline{Z}^*/\overline{Z} = 0.125$ so $nY^*/Y = 0.5$.

In both figures, when the probability of a home crisis (as shown in the lower-right panel) is relatively small, which is true when the tax rate is relatively high and the foreign saving/GDP ratio (home debt/GDP ratio) is relatively low, the volatility-tax curve is upward-sloping (as shown in the upper-left panel). An increase in the savings (as shown in the upper-right panel), which is possible when the tax decreases, reduces the economic volatility (as shown in the upper-left panel). When the probability of a home crisis is large, which is true when the tax rate is relatively low and the foreign saving/GDP ratio (home debt/GDP ratio) is relatively high, the volatility-tax curve is downward-sloping. An increase in the savings, which is possible when the tax rate further decreases, increases economic volatility.

The probability of a home financial crisis (3% as calibrated) is small around the laissez-faire equilibrium when the foreign economy is small (GDP 50% of the home economy). The home debt/GDP ratio is 20% as calibrated. By contrast, the probability of a home financial crisis (13%) is large around the laissez-faire equilibrium when the foreign economy is large (GDP 80% of the home economy). The home debt/GDP ratio, which equals the foreign saving/GDP ratio divided by the foreign/home GDP ratio to clear the international bond market, is as high as 45%. Therefore, the volatility-tax curve is upward-sloping around the laissez-faire equilibrium with small home leverage and crisis probability. It is instead downward-sloping

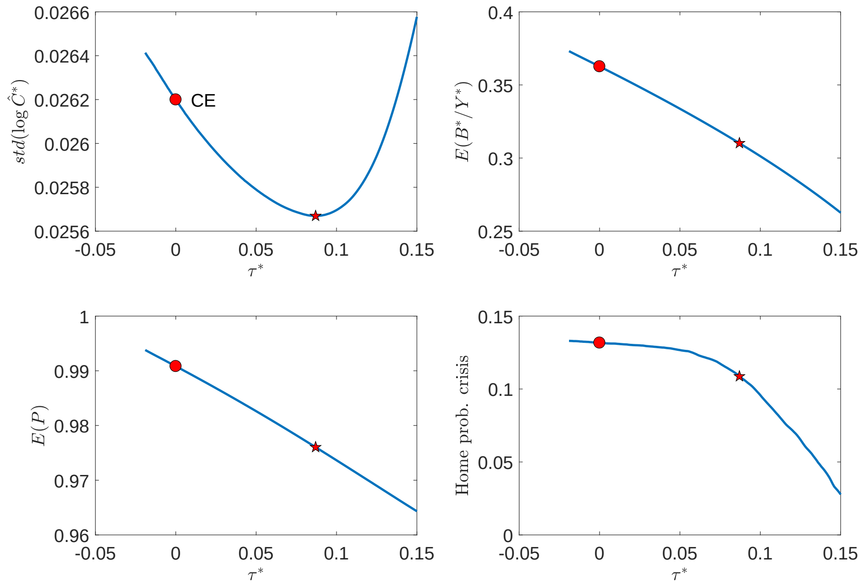

Figure 1.5: Effect of a foreign saving tax, foreign-large scenario
Circle marker: laissez-faire equilibrium. Star marker: the tax for volatility minimizing. Foreign large: $\overline{Z}^*/\overline{Z} = 0.342$ so $nY^*/Y = 0.8$

around the laissez-faire equilibrium with large home leverage and crisis probability.

Following the insights from the analytical model, the probability of a home financial crisis is large if the foreign economy is large, or a low tax (a large subsidy) induces the foreign economy to save a lot. There is a large spillover effect from additional foreign savings to home financial stability when the home leverage is already high. The destabilizing spillback effect is large, in turn, when the probability of a home financial crisis is large. As a result, the GE effect is large if the global imbalances are large and the home economy is crisis-prone.

Why is the magnitude of the spillback effect closely related to the probability of a home financial crisis? First, a home deleverage event prevents the foreign saver from rolling over its savings for consumption smoothing. The foreign economy is forced to increase consumption immediately but reduce consumption thereafter. Second, the reduced buffers leave the foreign economy vulnerable to the realizations of its shocks. Third, low saving and low output reinforce each other in the foreign economy when its financial constraint is binding. The foreign economy, which maintains about 40% external savings to GDP in the baseline calibration and 36% when its GDP is 80% of the home economy, is effectively using both the land and the international savings as collateral for its large working capital requirement by calibration ($\theta^* W_t^* L_t^{d*} \leq \phi^* Q_t^* K_t^* + P_t B_t^*$). A low supply of asset from the home economy tightens the working capital constraint and depresses the output in the foreign economy. Being more indebted, the home economy is more likely to deleverage and in such a situation deleverages more heavily. As a result, the spillback effect is larger.

Figure 1.6 plots the ergodic distribution of the home debt position, normalized by its mean for the foreign-large scenario (the foreign-small scenario is similar). The normalization facilitates the comparison of the two distributions. Light blue bars refer to the probability density distribution with the foreign saving tax $\tau^* = 0.087$ and light red bars refer to the probability distribution for the laissez-faire equilibrium $\tau^* = 0$. The foreign saving tax is the tax that minimizes foreign consumption volatility.

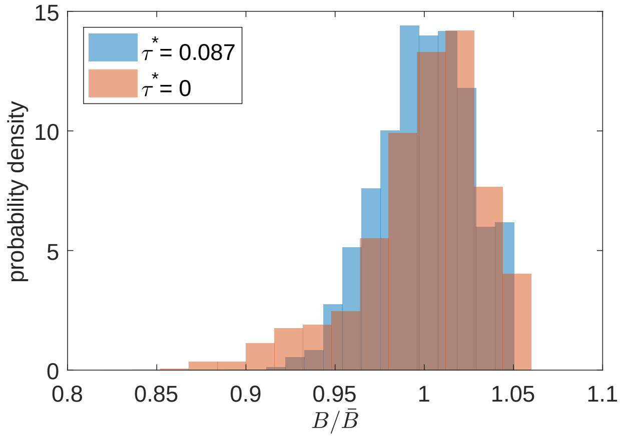

Figure 1.6: Distribution of normalized bond position, foreign-large scenario
Light blue: saving tax $\tau^* = 0.087$. Light red: laissez-faire equilibrium.

The distribution of the normalized home debt position is skewed. Its fat tail comes from the home deleverages reinforced by the fall of consumption and land price. The deleverages lead to an acute shortage of safe assets and it interrupts foreign consumption smoothing. The volatility of the home asset supply introduces volatility into the foreign economy.

Importantly, the variance of the normalized home bond position is smaller with a foreign saving tax. From Figure 1.5, the foreign saving tax reduces the foreign saving to GDP ratios from 36% to 31%. This means that the home leverage ratio reduces from 45% to 39%. As a result, a home deleverage is less likely and less heavy, which improves foreign economic stability. In the foreign-large scenario, an 8.7 percentage saving tax reduces the volatility of home asset supply to such a degree that it dominates the adverse direct effect of lower buffers, and the foreign volatility reduces from 0.0262 to 0.0257.

To further understand how a negative home shock destabilizes the foreign economy around the laissez-faire equilibrium and the equilibrium with a foreign saving tax, I conduct an event study for the foreign-large scenario. The event window covers two years before and after the event. In the calibration, the realizations of the productivity Z and Z^* are each discretized to three nodes. So, naturally, there are good, normal, and bad states for the home economy. I define negative home shock events as those in the simulations when (1) the state of the current period is bad and (2) the state of the previous period is either normal or good. The events account for 9.8% of all simulations. The medians of the selected variables are reported in Figure 1.7.

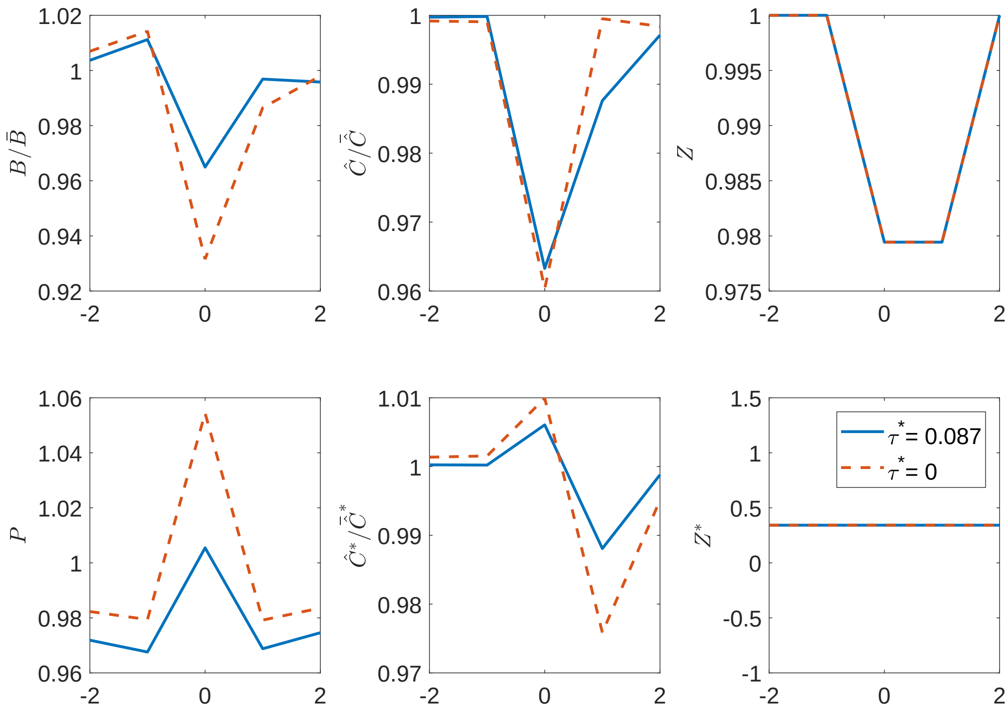

Figure 1.7: Event study: home bad shock at year 0, foreign-large scenario
Solid blue: saving tax $\tau^* = 0.087$. Dash red: laissez-faire equilibrium.

The right panel tracks the median productivity of the economies. From the upper-right panel, the event is indeed a bad shock to the home economy, starting from a normal state and lasting for one period (in median). The lower-right panel shows that the median productivity state for the foreign economy around the event is normal. This comes from the assumption that both shocks are perfectly uncorrelated.

The dashed red line represents the laissez-faire equilibrium. From the upper-left panel, the home economy accumulates debt gradually prior to the bad shock and deleverage heavily for about 8% when the bad shock hits. The home economy starts to increase borrowing again after the shock. In line with the bond deleverage at the shock, the home consumption drops for about 4% from the upper-middle panel. From the lower-left panel, the bond price shoots up at the shock due to the limited asset supply from the home deleverage.

The foreign consumption rises slightly at the shock and drops close to 3% one year after the shock, from the lower-middle panel. The slight rise of consumption results from low asset supply after the home deleverage. Without enough assets to postpone consumption, the foreign economy has to consume more. But why does foreign consumption drop significantly one year later in the absence of a domestic shock? First, it reflects the mechanical effect to restore bond positions to the previous level.

Second, the low level of external assets tightens the foreign working capital constraint and generates an output loss. In other words, the shortage of safe assets exacerbates foreign financial frictions and amplifies the shock transmission.

The solid blue line is for the same low-home productivity event, but the foreign saving tax is fixed at $\tau^* = 0.087$. Compared with the responses in the laissez-faire economy, the home economy is less indebted during good times and deleverages less at the shock. As a result, the consumption fall in the home economy is smaller, the bond price is lower, and the consumption adjustment for the foreign economy is also shallower. In one word, the GE effect is smaller.

To further understand the effect of the relative country size, Figure 1.8 plots the foreign consumption volatility deviation from the laissez-faire equilibrium by a 1% foreign saving tax, for different foreign/home GDP ratios.

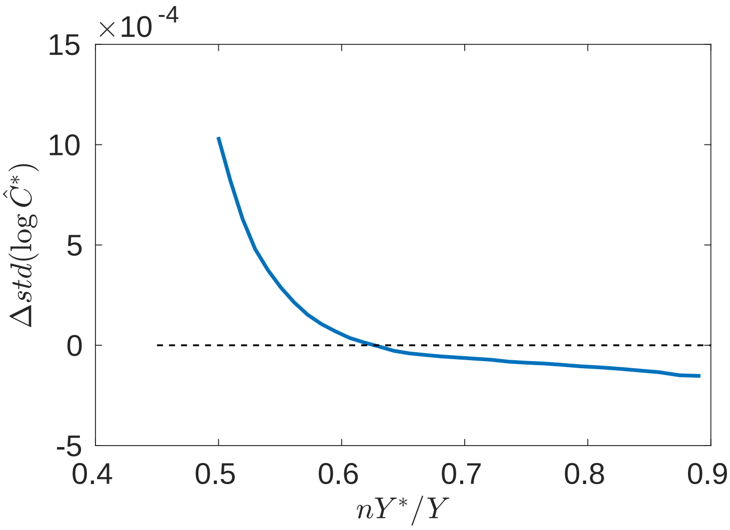

Figure 1.8: Effect of a foreign saving tax $\tau^* = 0.01$ on foreign consumption volatility for different relative GDP

The figure confirms that a paradox of precaution appears when the foreign economy is relatively large. What's more, the turning point is when the foreign economy is about 63% of home GDP. This is approximately China's relative GDP to the US in 2017. Therefore, a paradox of precaution might appear by the late 2010s.

The late 2010s estimation can be radical. The Chinese financial system is improving, and the US also enhances financial regulations. Mapping to the model, the working capital coefficient for China θ^* is decreasing, the pledgeability for China ϕ^* is increasing, and the US might have imposed a prudential tax themselves on borrowing.

The late 2010s estimation can, however, also be conservative. First, the model abstracts away other forceful channels of crisis contagion from the home economy to the foreign economy, for example, through the collapse of trade, the flight of capital, the interruption of financial services provided by the core countries, or the loss from an outright default. The GE effect should be stronger in reality than in the model. Second, the GE effect can also be stronger with reproducible capital that this model abstracts away. Reproducible capital propagates financial crises in the home economy. This exacerbates the overall asset shortage problem from a foreign perspective. The shortage of safe assets further forces the foreign economy to increase in its portfolio the risky domestic capital.

The net bias is unclear. A safe conclusion is that we are increasingly likely to see a paradox of precaution over time as the economies that save, which comprise many financially underdeveloped emerging markets, generally grow faster than the economies that borrow, which comprise mainly financially developed advanced economies.

1.4.2 Paradox of precaution - borrower version

I briefly discuss in this section the effect of a home borrowing tax. When the foreign economy is large, the home borrowing tax reduces home volatility around the laissez-faire equilibrium, consistent with the conventional wisdom. However, when the foreign economy is small, the home borrowing tax increases home volatility around the laissez-faire equilibrium. It also comes from the GE effect dominating the direct effect. More borrowing increases the probability of a home crisis from home shocks directly. But it also increases the asset supply to the foreign economy. The foreign economy becomes safer from its shocks and is less likely to withdraw its savings in large amounts. As a result, the home economy has a more stable source of finance and becomes less vulnerable. This is a borrower's version of the paradox of precaution.

1.5 Normative analyses

The laissez-faire equilibrium is hardly optimal. There are inefficiencies both at the domestic level and the international level. However, an individual surplus economy such as China, Germany, or Japan only contributes to a fraction of the global imbalances, so a national policymaker does not fully internalize the negative GE effect.

I restrict the policy instrument to be a constant saving tax. It is feasible in practice through capital control and/or reserve accumulation. To study the optimal policies for the savers, I consider two extremes of the foreign policymaker's problems.

First, the policymaker completely ignores the GE effect. For this purpose, I extend the model to assume a continuum of identical atomistic foreign economies. There is a policymaker in each of them. The policymaker in economy i chooses τ_i^* to maximize the unconditional expected welfare $\mathbb{E}V_i^*$, taking the world bond price P_t processes as given. The detailed policymaker's problem ignoring the GE effect is described in Appendix 1.E, as well as the numerical algorithm for this complex problem. In a symmetric equilibrium, $\tau_i^* = \tau^*$.

The setting resembles the policymaker's problem in a small open economy with one key difference: the GE effect exists but is ignored. The policymakers take the P_t processes as given and choose τ_i^* optimally. But the P_t processes are an equilibrium outcome affected by τ^*.

From the policymaker i's perspective, there is a domestic pecuniary externality that matters because the consumer-entrepreneur takes the land price $Q_{i,t}^*$ and wage $W_{i,t}^*$ as given but these prices affect the tightness of the collateral constraint when it is binding. In other words, the consumer-entrepreneur fails to internalize the fire-sale externality when they borrow. The policymaker knows that a saving subsidy induces more precautionary savings at good times. It mitigates the consumption and land price fall in bad times. The higher land price, in turn, relaxes the financial constraint and make the economy better off. The policymaker, therefore, wants to subsidize savings, as in Bianchi and Mendoza (2011); Bianchi (2011); Bianchi and Mendoza (2018).

Second, the policymaker completely internalizes the GE effect. For this purpose, the original model in section 1.3.1 is used. This is the case when a single saver is dominatingly large or the foreign policymakers cooperate. A foreign policymaker who internalizes the GE effect chooses τ^* to maximize the unconditional expected welfare $\mathbb{E}V^*$ of the competitive equilibrium. From the perspective of a foreign policymaker that fully internalizes the GE effect, there are two additional inefficiencies internationally. Firstly, the foreign policymaker understands that a saving tax reduces foreign savings. It reduces the home leverage and the home economy deleverages less frequently and heavily. So, the home supply of safe assets and the equilibrium bond price P_t are less volatile, which is good for the foreign economy. The foreign policymaker, therefore, would like to tax savings to internalize this GE effect I emphasized. Secondly, the foreign policymaker also knows that it can use its monopoly power on the supply of savings. A saving tax reduces foreign savings and raises the interest rate. The foreign policymaker, therefore, would like to tax savings to extract the monopoly rents from the home economy. This extra GE effect that I do not emphasize also drives the optimal choice toward a saving tax for nonstability reasons. The net effect is ambiguous in theory.

Figure 1.9 adds two markers to the previous Figure 1.3. The square marker shows the foreign policymaker's choice if the GE effect is fully ignored. The triangle marker shows the foreign policymaker's choice if the GE effect is fully internalized. The results are quite different for the two extremes of the policymakers problem.

First, a foreign policymaker subsidizes saving if the GE effect is ignored because the choice is driven solely by domestic concerns. However, it leads to an even worse than the laissez-faire equilibrium for the foreign economies from the ignored negative GE effect. When foreign economies are collectively half the size of the home as shown

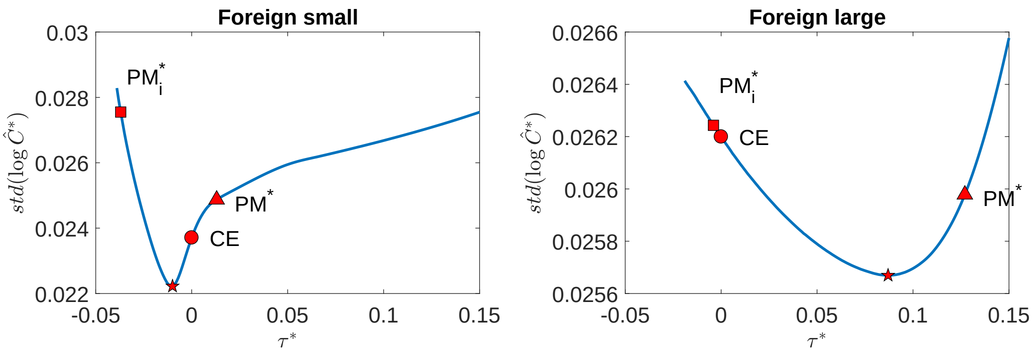

Figure 1.9: Effect of a foreign saving tax on foreign consumption volatility

Left: foreign-small scenario, $\overline{Z}^*/\overline{Z} = 0.125$ so $nY^*/Y = 0.5$. Right: foreign-large scenario, $\overline{Z}^*/\overline{Z} = 0.342$ so $nY^*/Y = 0.8$. Circle marker: laissez-faire equilibrium. Star marker: the tax for volatility minimizing. Square marker: policymaker's choice ignoring GE effect ignored. Triangle marker: policymaker's choice considering GE effect.

in the left panel, the standard deviation of log adjusted consumption $std\left(\log \hat{C}^*\right)$ increases from 0.024 in laissez-faire to 0.028 with intervention by policymakers. The expected external saving position B^*/Y^* increases from 0.4 in laissez-faire to 0.6. This means the debt-to-GDP ratio for the home economy increases from 0.2 in laissez-faire to 0.3, a 50% increase. Foreign welfare, not shown in the figure, is actually lower. When the foreign economy is large, the pattern is similar but the magnitude is more limited. This is because, facing a low interest rate the policymaker does not want to subsidize saving too much.

The result has strong policy implications. Global imbalances are not necessarily desirable outcomes. It can be a symptom of a lack of cooperation between policymakers in the surplus economies. The consequence is excessive volatility for the savers themselves. While it is a Pareto improvement for the surplus economies to reduce savings collectively, no policymaker wants to do this individually. Cooperation between the surplus economies is thus necessary.

Second, the foreign policymaker taxes saving if the GE effect is fully internalized. The saving tax is larger than that minimizes the consumption volatility, reflecting the extra GE effect from a higher interest rate. For the foreign-small scenario, the policymaker would rather choose a saving tax to reap the monopoly rents at a higher interest rate despite the economy being more volatile than the laissez-faire. The result echoes the Lucas criticism on the small welfare costs of economic volatility in this kind of model.

The reality likely lies closer to the first result that the GE effect is completely ignored. In addition to the fact that the surplus economies are individually small but collectively large, policymakers in reality may also ignore the GE effect emphasized in this chapter, simply because they are unaware of it when they choose policies.

1.6 Conclusion

This chapter studies two competing forces of additional savings on economic stability in a two-country framework. The direct effect of a larger buffer tends to enhance the saving economy's consumption smoothing and reduce its consumption volatility. The general equilibrium (GE) effect does the opposite to the saving economy by increasing the probability and depth of financial crises in the borrowing economy.

The policy implications of the chapter are twofold[7]. First, it calls for global rebalances when saving economies are so large that a paradox of precaution appears. The quantitative model suggests that, for the stability of their economies, policymakers in saving economies should switch from encouraging saving to discouraging saving from the late 2010s, given their size. While the estimation is approximate given the model and parameter uncertainty, it is safe to say that the paradox of precaution is increasingly likely over time because the developing savers tend to grow faster than the developed borrowers. Policymakers in the saving economies should switch from the saving subsidy to a saving tax in the foreseeable future, if not now. Second, the chapter calls for international cooperation between the policymakers in the surplus economies. Without global cooperation, there are excessive global imbalances and excessive volatility for the savers themselves.

[7]In practice, two extra policies are worth consideration for global rebalances beyond the model. First, better global safety nets from a more powerful global lender of last resort and bilateral currency swap arrangements reduce the need for precautionary savings. Second, liberalizing the trade of service to also helps economies at the center of the international financial system to narrow down their current account deficits.

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Appendices

1.A Lemma 1 and 3

To prove Lemma 1, eliminate B_2 and P_2 from equations (1.8), (1.14) and (1.15) for the $\mu_2 > 0$ case and from equations (1.7), (1.14) and (1.15) for the $\mu_2 = 0$ case. In the $\mu_2 > 0$ case,

$$\frac{\beta\left(d_2 - B_1^*\right)}{1/\phi - (1+\beta)} = \frac{\beta\left(B_1^* + K^* d_2^*\right) - K^* d_3 \beta\left(d_2 - B_1^*\right) / \left(\left(1/\phi - (1+\beta)\right) B_2^*\right)}{1+\beta}.$$

In equilibrium, $B_1^* < d_2$ as $B_2^* > 0$ from the global imbalance assumption. If $B_1^* \uparrow \Rightarrow B_2^* \uparrow$, the left-hand-side is smaller but the right-hand-side is larger, which is a contradiction. In the $\mu_2 = 0$ case, P_2 is a constant $\beta\left(d_2 + K^* d_2^*\right) / \left(d_3 + K^* d_3^*\right)$. From equation (1.14), $B_1^* \uparrow \Rightarrow B_2^* \uparrow$.

To prove Lemma 3, solve the laissez-faire equilibrium with $\tau^* = 0$ assuming the collateral constraint is not binding. From the goods market clearing condition $C_t + C_t^* = d_t + K^* d_t^*, \forall t = 1, 2, 3$ and the bond Euler equations of both economies for today

$$P_1 = \beta \frac{C_1}{C_2} = \beta \frac{C_1^*}{C_2^*}$$

and tomorrow

$$P_2 = \beta \frac{C_2}{C_3} = \beta \frac{C_2^*}{C_3^*},$$

the bond pricing equations are straightforward as follows.

$$P_1 P_2 = \beta^2 \frac{d_1 + K^* d_1^*}{d_3 + K^* d_3^*} \tag{1.16}$$

$$P_2 = \beta \frac{d_2 + K^* d_2^*}{d_3 + K^* d_3^*} \tag{1.17}$$

From the home bond Euler equations and the budget constraints (1.1), (1.2), and (1.4)

$$B_2 = \frac{\beta^2}{1+\beta+\beta^2}\left(\frac{d_1}{P_1 P_2} + \frac{d_2}{P_2} + d_3\right) - d_3 \tag{1.18}$$

From the above three equations

$$B_2 = \frac{d_3}{1+\beta+\beta^2}\left(\frac{(1+K^*d_3^*/d_3)}{(1+K^*d_1^*/d_1)} + \frac{\beta\,(1+K^*d_3^*/d_3)}{(1+K^*d_2^*/d_2)} - (1+\beta)\right)$$

From the home bond and tree Euler equations, the collateral constraint is not binding if

$$-B_2 < \phi d_3. \tag{1.19}$$

Subsitute with the expression of B_2,

$$\frac{1}{1+\beta+\beta^2}\left((1+\beta) - \frac{(1+K^*d_3^*/d_3)}{(1+K^*d_1^*/d_1)} - \frac{\beta\,(1+K^*d_3^*/d_3)}{(1+K^*d_2^*/d_2)}\right) < \phi.$$

The left-hand side is increasing in K^* when $d_3^*/d_3 < \min\left(d_1^*/d_2, d_2^*/d_2\right)$, which I assume to generate the global imbalances in the first place.

1.B Size of the savers

The economic growth rate is generally faster in the financially underdeveloped e-merging markets, which run overall current account surpluses, than the financially developed advanced economies, which run overall current account deficits. As a result, savers are likely to gain relative importance compared with borrowers over time, as shown in Figure 1.10.

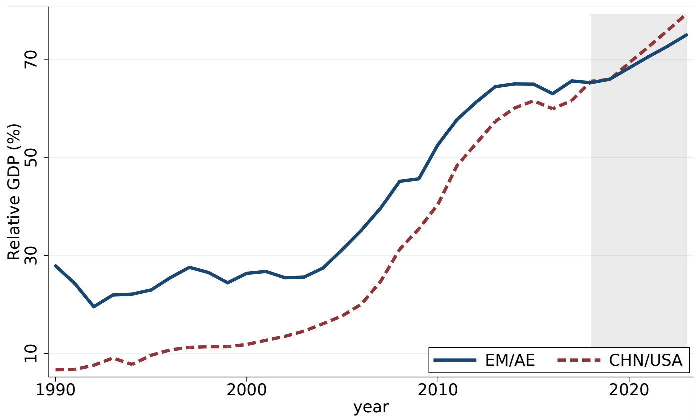

Figure 1.10: Relative size of emerging markets to advanced economies
Data and definition of emerging markets and advanced economies from IMF World Economic Outlook 2018. Shaded area: projections.

While a small open economy has previously been a useful framework to study the emerging markets, it is less apparent recently. The home economy mainly refers to the US, and, to some extent, the UK, which are at the center of the international financial system with the comparative advantage to provide safe assets for the rest of the world. The US international investment position is evidently deteriorating over time as shown in Figure 1.11. The same is true for the UK. While a financial crisis is not necessarily imminent, the risk of adjustment is increasing.

Figure 1.11: Net international investment position for the US
Data from FRED

1.C Equation system

The normalization of land to unit discussed in the calibration section is already implemented. The complementary slackness conditions are transformed with new variables $\hat{\mu}_t, \hat{\mu}_t^* \in (-\infty, \infty)$ as discussed in the numerical algorithm section.

$$V_t = \frac{\left(C_t - \chi \frac{L_t^{1+\omega}}{1+\omega}\right)^{1-\sigma} - 1}{1-\sigma} + \mathbb{E}_t \beta V_{t+1} \tag{1.20}$$

$$C_t + P_t B_t = Z_t L_t^{\gamma} + B_{t-1} \tag{1.21}$$

$$\gamma Z_t = (1 + \mu_t \theta) \chi L_t^{\omega+1-\gamma} \tag{1.22}$$

$$Q_t (1 - \phi \mu_t) = \mathbb{E}_t \beta \left(\frac{C_{t+1} - \chi \frac{L_{t+1}^{1+\omega}}{1+\omega}}{C_t - \chi \frac{L_t^{1+\omega}}{1+\omega}} \right)^{-\sigma} \left(Q_{t+1} + \alpha Z_{t+1} L_{t+1}^{\gamma}\right) \tag{1.23}$$

$$P_t (1 - \mu_t) = \mathbb{E}_t \beta \left(\frac{C_{t+1} - \chi \frac{L_{t+1}^{1+\omega}}{1+\omega}}{C_t - \chi \frac{L_t^{1+\omega}}{1+\omega}} \right)^{-\sigma} \tag{1.24}$$

$$P_t B_t + \phi Q_t - \theta \chi L_t^{\omega+1} = \max\left(0, -\hat{\mu}^3\right) \tag{1.25}$$

$$\mu_t = \max\left(0, \hat{\mu_t}^3\right) \tag{1.26}$$

$$V_t^* = \frac{\left(C_t^* - \chi^* \frac{(L_t^*)^{1+\omega^*}}{1+\omega^*}\right)^{1-\sigma^*} - 1}{1-\sigma^*} + \mathbb{E}_t \beta^* V_{t+1}^* \tag{1.27}$$

$$C_t^* + P_t B_t^* = Z_t^* (L_t^*)^{\gamma^*} + B_{t-1}^* \tag{1.28}$$

$$\gamma^* Z_t^* = (1 + \mu_t^* \theta^*) \chi^* (L_t^*)^{\omega^*+1-\gamma^*} \tag{1.29}$$

$$Q_t^* \left(1-\phi^* \mu_t^*\right) = \mathbb{E}_t \beta^* \left(\frac{C_{t+1}^* - \chi^* \frac{\left(L_{t+1}^*\right)^{1+\omega^*}}{1+\omega^*}}{C_t^* - \chi^* \frac{\left(L_t^*\right)^{1+\omega^*}}{1+\omega^*}} \right)^{-\sigma^*} \left(Q_{t+1}^* + \alpha^* Z_{t+1}^* \left(L_{t+1}^*\right)^{\gamma^*} \right) \tag{1.30}$$

$$P_t \left(1+\tau^* - \mu_t^*\right) = \mathbb{E}_t \beta^* \left(\frac{C_{t+1}^* - \chi^* \frac{\left(L_{t+1}^*\right)^{1+\omega^*}}{1+\omega^*}}{C_t^* - \chi^* \frac{\left(L_t^*\right)^{1+\omega^*}}{1+\omega^*}} \right)^{-\sigma^*} \tag{1.31}$$

$$P_t B_t^* + \phi^* Q_t^* - \theta^* \chi^* \left(L_t^*\right)^{\omega^*+1} = \max\left(0, -\left(\hat{\mu}^*\right)^3\right) \tag{1.32}$$

$$\mu_t^* = \max\left(0, \left(\hat{\mu_t}^*\right)^3\right) \tag{1.33}$$

$$B_t + n B_t^* = 0 \tag{1.34}$$

1.D Numerical algorithm

Time iteration algorithm

I use a time iteration algorithm as in Coleman (1990), which directly operates on first-order conditions, modified to address occasionally binding constraints. To solve the dynamic equation system $\mathbb{E}_t f\left(X_{t+1}, s_{t+1}, X_t, s_t, X_{t-1}, \tau^*\right) = 0$ defined as equations 1.20 to 1.34 where X_t are the endogenous variables, $s_t \in Z_t \times Z_t^*$ are the shock variables, do the following:

1, Generate a discrete grid for the economy's state variable X_{t-1} and current shock variable s_t. I will use adaptive grids to be explained in more detail.

2, Make an initial guess of the time-invariant policy function $X_t = \mathcal{X}^0\left(X_{t-1}, s_t\right)$. The policy functions are approximated with the finite element method to take care of the kinks arising from the occasionally binding constraints. Linear interpolation is used.

3, Iteration m

(a) Solve X_t^m such that $\mathbb{E}_t f\left(\mathcal{X}^{m-1}\left(X_t^m, s_{t+1}\right), s_{t+1}, X_t^m, s_t, X_{t-1}, \tau^*\right) = 0$ for each grid point.

(b) Update the guess of policy function $\mathcal{X}^m$ so that $X_t^m = \mathcal{X}^m\left(X_{t-1}, s_t\right)$

(c) Go to (a) if the update is larger than a threshold $\left|\left|\mathcal{X}^m - \mathcal{X}^{m-1}\right|\right| > \epsilon$ and set $m = m+1$.

Adaptive grids

The adaptive grids are implemented in the following way.

1, The model is solved using a coarse equidistant grid with few points.

2, The accuracy of the solution is then evaluated according to the Euler equation errors at the middle of any two grid points.

3, New grid points are placed at those inaccurate parts according to the following the two rules.

(a) The number of new grid points is proportional to the Euler equation errors between existing grid points.

(b) No new grid points will be placed on inaccurate parts that will not be visited in the ergodic distribution of variables. The ergodic states are calculated directly from the policy functions.

4, Steps 1-3 are repeated in several rounds.

More specifically, the initial grid points are $80 \times 3 \times 3 = 720$, in which 80 are for the continuous state B_{t-1} and 3 for home productivity and 3 for foreign productivity. I add two rounds of extra points $80 \times 3 \times 3$ and $160 \times 3 \times 3$ to those states that are not accurate. The maximum Euler equation error is usually below 10^{-4} from the solution. The adaptive grid method significantly accelerates the solution and improves the precision for a given number of final grid points.

Simulation

The solutions are policy functions. To simulate, generate a series of productivity shocks and choose an initial state. Evaluate the policy function to get new states and other variables of interest. Repeat this to simulate forward. Discard the initial 20% of simulations with the potential transition from the choice of the initial state. The simulations are then used to generate statistics for the ergodic distribution and for event analysis.

Parallelization

The time iteration step 3(a) is the most costly because roots need to be found for a non-linear equation system for a large number of states. The task is parallelized to accelerate. The policy functions can be solved fairly quickly and accurately. To give some sense, for one set of parameterizations the model can be solved in 10 minutes with 8 CPU cores on the LSE Fabian high-performance computing platform.

1.E Foreign policymaker's problem ignoring GE effect

For simplicity, assume foreign economies receive the same productivity shock Z_t^*. The welfare V_i^* for a given state solves the foreign representative consumer-entrepreneur's utility maximization problem, taking the foreign policymaker's choice τ_i^* as given.

$$V_i^*\left(B_{i,t-1}^*, \overline{B}_{t-1}^*, Z_t^*, Z_t\right) = \max \frac{\left(C_{i,t}^* - \chi \frac{\left(L_{i,t}^{s*}\right)^{1+\omega^*}}{1+\omega^*}\right)^{1-\sigma^*} - 1}{1-\sigma^*} + \mathbb{E}_t V_i^*\left(B_{i,t}^*, \overline{B}_t^*, Z_{t+1}^*, Z_{t+1}\right)$$

s.t.

$$
\begin{aligned}
& C^*_{i,t} + Q^*_{i,t}\left(K^*_{i,t} - K^*_{i,t-1}\right) + P_t\left(1+\tau^*_i\right) B^*_{i,t} \\
& = W^*_{i,t} L^{s*}_{i,t} + B^*_{i,t-1} + \left(Z^*_t \left(K^*_{i,t-1}\right)^{\alpha^*} \left(L^{d*}_{i,t}\right)^{\gamma^*} - W^*_{i,t} L^{d*}_{i,t} \right) + T^*_{i,t}
\end{aligned}
$$

$$
-P_t B^*_{i,t} + \theta^* W^*_{i,t} L^{d*}_{i,t} \leq \phi^* Q^*_{i,t} K^*_{i,t}
$$

The state variables for the foreign economy i's problem are $B^*_{i,t-1}, \overline{B}^*_{t-1}, Z^*_t, Z_t$. The representative consumer-entrepreneur takes as given the rest of the foreign economies' bond policy function $\overline{B}^*_t = \overline{\mathcal{B}}^*\left(\overline{B}^*_{t-1}, Z^*_t, Z_t\right)$ and the bond price from the GE $P_t = \mathcal{P}\left(\overline{B}^*_{t-1}, Z^*_t, Z_t\right)$. Her individual choices such as $B^*_{i,t}$ have no effect on the international bond prices.

The saving tax is fully rebated $\tau^*_i P_t B^*_{i,t} = T^*_{i,t}$ through lump-sum transfers. The domestic land market clears with normalization $K^*_{i,t} = 1$ and domestic labor market clears $L^{s*}_{i,t} = L^{d*}_{i,t}$.

The policymaker maximizes the unconditional expected welfare $\mathbb{E}V^*_i$ of the representative consumer-entrepreneur by choosing the saving tax τ_i.

General equilibrium

In a symmetric equilibrium $B^*_{i,t} = \overline{B}^*_t = B^*_t$ and $\tau^*_i = \tau^*$. The policy functions for economy i degenerates. For example, the bond policy function $B^*_{i,t} = \mathcal{B}^*_i\left(B^*_{i,t-1}, \overline{B}^*_{t-1}, Z^*_t, Z_t\right)$ degenerate to $B^* = \mathcal{B}^*\left(B^*_{t-1}, Z^*_t, Z_t\right)$. Although the individual consumer-entrepreneur's problem in economy i takes both the individual and aggregate states, the two states become identical in a symmetric equilibrium.

The rest of the foreign economies' bond policy function $\overline{B}^*_t = \overline{\mathcal{B}}^*\left(\overline{B}^*_{t-1}, Z^*_t, Z_t\right)$ and the bond price from the general equilibrium $P_t = \mathcal{P}\left(\overline{B}^*_{t-1}, Z^*_t, Z_t\right)$ solve the general equilibrium of competitive equilibrium defined in section 1.3.1, given the optimal choice of the saving tax τ^* by symmetric foreign policymakers.

Numerical algorithm

The problem can be solved by repeating two steps. First, for given τ^* solve the problem in section 1.3.1 for the processes of P_t and B_t. Second, given P_t and B_t processes, solve the policymaker's problem in section 1.E for τ_i and set $\tau = \tau_i$. Repeat the two steps until it converges. The algorithm within each step follows section 1.D.

The second step involves an extra continuous state. To mitigate the curse of dimension, I use a set of efficient grids for the aggregate state $\overline{B}^*_{t-1}$. More specifically, one grid point is placed exactly at the state where the home constraint is marginally binding, and the rest of the grid points are placed sparingly. By doing this, a few grid points are sufficient to capture the aggregate policy function well.

Chapter 2

Consumption-led Growth

Abstract: Investment is bounded by retained earnings for young firms relying on self-financing. The firms are underinvesting from the perspective of a constrained social planner who cannot inject funds to the self-financing firms directly, for two reasons. First, households do not internalize that additional consumption and labor supply increases the profits of the self-financing firms. Second, firms with credit access do not internalize that their expansion fueled by credit intensifies competition in the factor market, drives up factor prices, and squeezes the profits of self-financing firms. The social planner optimally chooses "pro-consumption" policies such as a consumption subsidy and a saving tax on the household. More consumption paradoxically leads to more investment and output for the self-financing firms.

2.1 Introduction

What shall a government do to accelerate economic growth in a developing economy with poor financial development? The "pro-business" policies, such as a wage suppression to increase firm profit and a credit subsidy to encourage investment, seem to be one answer (see e.g. Itskhoki and Moll (2019)). This chapter argues instead that the optimal policies can be "pro-consumption". A consumption subsidy increases the firm profit as well, and a saving tax mitigates the capital misallocation between the self-financing firms and the firms with credit access (see e.g. Song et al. (2014)).

Central to the theory are the firms that are young, small, investing in risky projects, investing in innovative projects, and/or not guaranteed by the government. Although being more productive, they tend to be very likely denied bank credit. Equity finance is also quite limited from the poorly developed venture capital or stock market in developing economies. As a result, they rely on self-financing and cannot invest more than their retained earnings. When the constraint is binding, they remain too small for too long.

The competitive equilibrium is inefficient from two market failures in the economy. First, the households do not internalize that additional consumption and labor supply lead to higher output and profit level of the self-financing firms. To improve, a social planner subsidizes household consumption to distort the household consumption–leisure choice toward consumption. This drives up the consumption demand and boosts firm profits[1].

Second, firms with credit access do not internalize that their expansion fueled by credit out-competes self-financing firms. They intensify the competition on the factor market, such as labor, office space, energy, and intermediary goods. The pecuniary externality from higher factor prices matters because it squeezes the profits of the constrained self-financing firms. The competitive economy is dynamically inefficient. To improve, a social planner taxes household saving to make borrowing costlier by the external-financing firms. The external-financing firms shrink, release resources to the self-financing firms, and they expand. While the resulting consumption increase crowds out aggregate investment, it crowds in investment by the more productive self-financing firms and mitigates capital misallocation.

In the model, the self-financing firms with exogenous entry and exit dynamics make hiring, investment, and dividend decisions, for the interest of a representative household. They rely on self-financing because they cannot borrow or finance with equity in the form of paying a negative dividend. The finite horizon prevents the firms from growing to the point that all desired investment can be financed with retained earnings. The assumption of common objectives for the interest of the household minimizes the heterogeneity to be tracked. The firms can be thought of as being operated by the members of the household with full consumption-insurance who pool their resources at the end of each period to consume. In short, the firms are modeled as the banks in Gertler and Karadi (2011), except that they use their own funds. The framework allows great tractability because the self-financing firms are easily aggregated to a constrained representative firm.

I study the Ramsey problem analytically for a constrained social planner who cannot inject funds directly to the self-financing firms. The social planner's solution can be decentralized with a consumption subsidy and a saving tax.

The model is then brought to numerical analysis by calibrating to China. The pro-consumption policies can increase welfare by as much as increasing consumption by 1% permanently while keeping labor supply unchanged from the steady state of the competitive equilibrium. The long-run aggregate consumption is 5% higher, the long-run investment is 13% lower, and the long-run aggregate output is 4% higher. The optimal consumption subsidy reduces from about 16% in the short run to 14% in the long run and the optimal saving tax rises from about 4% to 4.5%.

If only constant non-negative taxes are allowed, the optimal policy is a constant saving tax at 3.67%. The consumption-equivalent of welfare gain is still positive at 0.34%, although the long-run aggregate output is 2% lower. But the 2% lower long-

[1]The consumption subsidy can be interpreted as a working subsidy. It drives up the labor supply and boosts firm profits. The household ends up with more consumption.

run output is achieved with 16% lower aggregate capital and slightly lower labor, suggesting a large improvement of the total factor productivity.

This chapter contributes to the economic policy debate by showing that pro-consumption policies paradoxically lead to an expansion of the self-financing firms, and increase economic welfare as a whole. It also shows that the best combination of the pro-consumption policies leads to aggregate output growth both in the short run and in the long run, consistent with the at-first-glance puzzling idea of consumption-led growth. Finally, it provides a very tractable framework for self-financing firms.

Three closely related papers are Song et al. (2014), Ventura and Voth (2015), and Itskhoki and Moll (2019). The first studies the growth effect of exchange rate policy, interest rate policy, and deposit rate policy. In particular, it shows that high interest rate mitigates the disadvantage of financially constrained firms, reduces wages, and increases the speed of transition from low- to high-productivity firms. Its analysis is positive. This chapter highlights this channel as a second reason for pro-consumption policies by building a tractable model for self-financing firms and further develops it normatively by solving a constrained social planner's problem. Ventura and Voth (2015) provides a historical account that the expansion of British government borrowing during the industrial revolution crowded out agricultural investments by the nobility and freed resources for the growth of self-financing industries like textile. Itskhoki and Moll (2019) advocates pro-business policies such as labor and saving subsidy for a developmental government. The consumption subsidy in this chapter shares the same insight with the labor subsidy in their paper. By contrast, the social planner in this chapter optimally chooses a saving tax, rather than a saving subsidy as in their paper, for the additional insight in Song et al. (2014) and Ventura and Voth (2015). The optimal policies under one umbrella are instead pro-consumption.

This chapter complements the larger literature on capital misallocation and productivity. A low interest rate can worsen capital misallocation through various other channels as in Liu et al. (2017), Caggese and Perez-Orive (2017), Bleck and Liu (2018). In the international context, the financial resource curse can also be understood with various other channels of misallocation as in Reis (2013), Benigno and Fornaro (2014), Gopinath et al. (2017).

This chapter differs crucially from the Keynesian framework that the output is demand-dependent in the short-run. First, there is insufficient investment by the financially constrained firms rather than insufficient demand. Second, the pro-consumption policies have a long-run effect if the self-financing firms are constrained at the steady state. In some Keynesian models with secular stagnation (see e.g. Benigno and Fornaro (2017); Eggertsson et al. (2019)), demand policies can have a long-run effect by driving the economy out of a permanent liquidity trap. The inefficiency in this literature comes from the zero-lower bound of nominal interest rate, which is absent in my real framework.

The concept of consumption-led growth in this chapter differs crucially from

Brunnermeier et al. (2018). In their paper, it means a consumption boom financed by a current account deficit stimulates the non-tradable sector and its innovation productivity. In this chapter, it is about a developmental government actively choosing pro-consumption policies, leading to higher output or higher welfare at least.

The rest of this chapter will be organized as follows: in Section Two, I introduce the self-financing firms and study the optimal consumption subsidy; in Section Three I extend the model to two sectors allowing firms with access to credit and study optimal consumption subsidy and saving tax; in Section Four I conduct numerical analysis. Section Five concludes.

2.2 Model

In this section, I introduce a one-sector model of self-financing firms. There are three types of agents: a representative household, self-financing firms, and a government. The household makes consumption, saving, and labor supply decisions. The self-financing firms, owned by the household, make labor demand, investment, and dividend decisions, for the interest of the household. The government collects taxes on consumption and saving and transfers the revenue back to the household in a lump sum.

2.2.1 Household's problem

The representative household makes consumption C_t, saving B_t and labor supply L_t decisions to maximize its sum of discounted utility subject to period budget constraints as follows.

$$\max_{C_t,B_t,L_t} \sum_{t=1}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\chi}}{1+\chi} \right)$$

s.t.

$$\left(1+\tau_t^C\right) C_t + \left(1+\tau_t^B\right) P_t B_t = W_t L_t + D_t + B_{t-1} + T_t$$

The household derives utility from consumption and disutility from labor supply. The intertemporal elasticity of substitution is $1/\gamma$, the inverse Frisch elasticity is χ, and the discount factor is β. It earns wage W_t, receives dividends from firms D_t, gets repayment from maturing one-period riskless bond B_{t-1} and is rebated government revenue T_t in a lump sum. The household uses the proceeds to consume and make bond investment B_t at the market price P_t. The government imposes a proportional consumption tax τ_t^c (subsidy if it is negative) and a saving tax τ_t^B (subsidy if it is negative).

The marginal utility for an additional unit of consumption is $\beta^t C_t^{-\gamma}$ and the marginal disutility from an additional unit of labor supply is $\beta^t L_t^{\chi}$. From each unit of additional labor supply, the wage W_t can be used to purchase $W_t/\left(1+\tau_t^C\right)$ unit of consumption goods. The household chooses to supply labor until the additional utility from consumption equals the marginal disutility of labor supply.

$$W_t / \left(1+\tau^C\right) C_t^{-\gamma} = L_t^{\chi} \tag{2.1}$$

The effective after-tax bond price is $\left(1+\tau_t^B\right) P_t$. The household purchases bonds until the effective bond price equals its discount factor.

$$\left(1+\tau_t^B\right) P_t = \beta \frac{C_t^{\gamma}}{C_{t+1}^{\gamma}} \tag{2.2}$$

2.2.2 Firms' problem

There is a continuum of measure one firms with exogenous entry and exit dynamics. Each firm decides how much labor to demand and produces goods with capital and labor using the same constant return to scale production function. A firm exits with an independent probability $1-\sigma$ every period after the production. If the firm exits, it brings its profits back to the household as a final dividend payment. On average, a firm operates $1/(1-\sigma)$ periods. In total, $1-\sigma$ firms exit. The same measure of new firms enter to replace each exiting firm, keeping the total firms fixed. The new firms receive a start-up fund as a small proportion $\eta < 1-\sigma$ of the total profits of the firms from the household. Each of the remaining and new firms makes a decision on how much capital to buy and how much discretionary dividend to pay, for the interest of the household. In short, the firms are modeled as the banks in Gertler and Karadi (2011) except that they finance themselves.

The firm i's problem is the following.

$$\max_{l_{i,t}^c, k_{i,t}^c, d_{i,t}^c} \sum_{t=1}^{\infty} \beta^t C_t^{-\gamma} \sigma^{t-1} \left((1-\sigma)\pi_{i,t}^c + \sigma d_{i,t}^c\right)$$

s.t.

$$\pi_{i,t}^c = \left(A^c \left(k_{i,t-1}^c\right)^{\alpha} \left(l_{i,t}^c\right)^{1-\alpha} + (1-\delta) k_{i,t-1}^c - W_t l_{i,t}^c\right)$$

$$k_{i,t}^c + d_{i,t}^c = \pi_{i,t}^c$$

$$d_{i,t}^c \geq 0$$

A firm has the probability $\sigma^{t-1}(1-\sigma)$ to exit just after the period t, and it brings its profits $\pi_{i,t}^c$ to the household as a final dividend. It has the probability $\sigma^{t-1}\sigma$ to remain active after the period t and it chooses a discretionary dividend payment d_t^c to the household. Therefore, the expected total dividend at time t is $\sigma^{t-1}\left((1-\sigma)\pi_{i,t}^c + \sigma d_{i,t}^c\right)$. It discounts its expected future dividend by the household's marginal utility of consumption $\beta^t C_t^{-\gamma}$.

At the beginning of the period, the firm decides how much labor to demand $l_{i,t}^c$, given its capital stock $k_{i,t-1}^c$. It produces using a Cobb-Douglas production function with common productivity A^c, capital share α and labor share $1-\alpha$. It pays market wage W_t. Capital depreciates with rate δ. Therefore, the firm's profit is $\pi^c_{i,t} = \left(A^c \left(k^c_{i,t-1}\right)^{\alpha} \left(l^c_{i,t}\right)^{1-\alpha} + (1-\delta)k^c_{i,t-1} - W_t l^c_i\right)$. If the firm remains active after the production (or it is a new firm), it determines the investment $k^c_{i,t}$ and the discretionary dividend $d^c_{i,t}$ out of the profit $\pi^c_{i,t}$ (or its start-up fund). The firm cannot pay any negative dividend $d_{i,t} \geq 0$. Without equity finance or debt finance, the firm has to rely on self-financing.

Denote $\xi^c_{i,t}, \nu^c_{i,t}, \nu^c_{i,t}\mu^c_{i,t}$ the shadow prices to relax the profit constraint, the budget constraint and the no equity finance constraint, the first-order conditions (FOCs) are the following.

$$\text{FOC } l^c_{i,t}: \ W^c_t = (1-\alpha)A^c \left(k^c_{i,t-1}\right)^{\alpha} \left(l^c_{i,t}\right)^{-\alpha}$$

The firm demands labor until its marginal product of labor equals the wage.

$$\text{FOC } \pi^c_{i,t}: \ \beta^t C_t^{-\gamma} \sigma^{t-1} (1-\sigma) - \xi^c_{i,t} + \nu^c_{i,t} = 0$$

The firm wants to increase its profit as each unit increases the household utility by $\beta^t C_t^{-\gamma} \sigma^{t-1} (1-\sigma)$, but it tightens the profit-constraint at the shadow price $\xi^c_{i,t}$ and relaxes the budget constraint at the shadow price $\nu^c_{i,t}$. For optimality, the net marginal return is zero.

$$\text{FOC } k^c_{i,t}: \ -\nu^c_{i,t} + \xi^c_{i,t+1} \left(\alpha A^c \left(k^c_{i,t}\right)^{\alpha-1} \left(l^c_{i,t+1}\right)^{1-\alpha} + (1-\delta)\right) = 0$$

An additional unit of capital investment tightens the budget constraint at the shadow price $\nu^c_{i,t}$ but relaxes the profit constraint in the next period by $\alpha A^c \left(k^c_{i,t}\right)^{\alpha-1} \left(l^c_{i,t+1}\right)^{1-\alpha} + (1-\delta)$ at the shadow price $\xi^c_{i,t+1}$. For optimality, the net marginal return is zero.

$$\text{FOC } d^c_{i,t}: \ \beta^t C_t^{-\gamma} \sigma^{t-1} \sigma = (1-\mu^c_{i,t})\nu^c_{i,t}$$

An additional unit of discretionary dividend increases the household utility by $\beta^t C_t^{-\gamma} \sigma^{t-1} \sigma$. It tightens the budget constraint at the shadow price $\nu^c_{i,t}$ but relaxes the no equity finance constraint at the shadow price $\nu^c_{i,t}\mu^c_{i,t}$. For optimality, the net marginal return is zero. The associated complementary slackness condition is the following.

$$d^c_{i,t} \geq 0, \mu^c_{i,t} \geq 0, d^c_{i,t}\mu^c_{i,t} = 0$$

Eliminate the shadow prices $\xi^c_{i,t}, \nu^c_{i,t}$, the capital Euler equation for the firm is

$$1 = \left(\alpha A^c \left(k^c_{i,t}\right)^{\alpha-1} \left(l^c_{i,t+1}\right)^{1-\alpha} + (1-\delta)\right) \beta \frac{C_t^{\gamma}}{C_{t+1}^{\gamma}} \left((1-\sigma)\left(1-\mu^c_{i,t}\right) + \sigma \frac{1-\mu^c_{i,t}}{1-\mu^c_{i,t+1}}\right).$$

The price of capital, which is unit, equals the marginal return of capital multiplying the discount factor used by the firm. The discount factor used by the firm differs from the household, with an additional term coming from the random exit. The additional term equals to 1 if the firm is unconstrained in investment $\mu_{i,t}^c = 0$. In this case the firm's discount factor is identical to the household. If the firm is constrained in investment $\mu_{i,t}^c > 0$, the firm's discount factor is usually smaller than the household. The "impatient" firm requires a higher return of capital, and as a result, invests less amount of capital than that an unconstrained firm would like to choose.

From the FOC of labor, the firms choose the same labor-capital ratio. Combine with the Euler equation, their shadow value of relaxing the non-negative dividend constraints (normalized by the shadow price of relaxing the budget constraint) must also be identical.

$$\mu_{i,t}^c = \mu_t$$

As a result, they make the same discretionary dividend and investment decisions proportional to their profits. Denote the total dividend D_t, total discretionary dividend D_t^c, total capital K_t, and total labor L_t. The firms' optimal choices can be easily aggregated. The optimal labor demand, optimal capital investment, budget constraint, dividend payment, and complementary slackness conditions are the following, respectively.

$$W_t = (1-\alpha)A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{-\alpha}. \tag{2.3}$$

$$1 = \left(\alpha A^c \left(K_t^c\right)^{\alpha-1} \left(L_{t+1}^c\right)^{1-\alpha} + (1-\delta)\right) \beta \frac{C_t^\gamma}{C_{t+1}^\gamma} \left((1-\sigma)(1-\mu_t) + \sigma \frac{1-\mu_t}{1-\mu_{t+1}}\right) \tag{2.4}$$

$$K_t^c + D_t^c = (\sigma+\eta)\left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c\right) \tag{2.5}$$

$$D_t = (1-\sigma-\eta)\left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c\right) + D_t^c \tag{2.6}$$

$$D_t^c \geq 0, \mu_t \geq 0, D_t^c \mu_t = 0 \tag{2.7}$$

The continuing firms have in total $\sigma\left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c\right)$ profits and the new firms bring in another $\eta\left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c\right)$ start-up funds. The firms use the proceeds to make total investment K_t^c and total discretionary dividend D_t^c. The total dividend from the firms equals the profits from the exiting firms, subtracting the start-up fund, and adding the total discretionary dividend.

If the non-negative dividend constraint is not binding $\mu_t = 0$, the investment K_t^c is determined by equation (2.4). If the constraint is binding $\mu_t^c > 0$, the discretionary dividend $D_t^c = 0$ and the investment K_t^c evolves as in equation (2.5). The self-financing firms' aggregate solution is identical to the solution to the problem of a representative self-financing firm facing an additional investment constraint as follows.

$$\max_{L_t^c, K_t^c, D_t} \sum_{t=1}^{\infty} \beta^t C_t^{-\gamma} D_t$$

s.t.

$$K_t^c + D_t = \alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c$$

$$K_t^c \leq (\sigma + \eta) \left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c\right)$$

Intuitively, the exit of firms is like paying a compulsory dividend of a fraction $1-\sigma$ of the profits, which squeezes the retained-earnings the firms have for investment. If the firm horizon is short (σ small) and the start-up fund is small (η small), the firms will not be able to grow to the point that all desired investment can be financed with profits.

In reality, firms that are young, small, investing in risky projects, investing in innovative projects, and/or not guaranteed by the government are likely to rely on self-financing. They are also more likely to exit business over time.

2.2.3 Competitive equilibrium

The bond market clears, the labor market clears, and the government runs a balanced budget so that the taxes are fully rebated in lump-sum transfers.

$$B_t = 0 \tag{2.8}$$

$$L_t = L_t^c \tag{2.9}$$

$$\tau_t^C C_t + \tau_t^B P_t B_t = T_t \tag{2.10}$$

For given taxes $\left\{\tau_t^C, \tau_t^B\right\}_{t=1}^{\infty}$ and initial aggregate capital stock K_0^c, a competitive equilibrium is characterized by the labor supply equation (2.1), the labor demand equation (2.3), the labor market clearing condition (3.7), the capital Euler equation (2.4), the firms budget constraint (2.5), the complementary slackness condition (2.7), and the resource constraint

$$C_t + K_t^c = A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\alpha} + (1-\delta) K_{t-1}^c.$$

The transfers T_t, the bond position B_t, the dividend D_t and the bond price P_t can then be derived from equation (2.10), (2.8), (2.6), and (2.2), respectively. The household budget constraint is satisfied by Walras law.

2.2.4 Optimal taxes

A constrained social planner cannot inject funds directly to the self-financing firms. It maximizes the household objective, subject to the resource constraint and the aggregate investment constraint as follows.

$$\max_{C_t, L_t^c, K_t^c} \sum_{t=1}^{\infty} \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{(L_t^c)^{1+\chi}}{1+\chi} \right)$$

s.t.

$$C_t + K_t^c = A^c \left(K_{t-1}^c\right)^{\alpha} (L_t^c)^{1-\alpha} + (1-\delta) K_{t-1}^c$$

$$K_t^c \leq (\sigma + \eta) \left(\alpha A^c \left(K_{t-1}^c\right)^{\alpha} (L_t^c)^{1-\alpha} + (1-\delta) K_{t-1}^c \right)$$

Denote the shadow prices for the two constraints λ_t and $\lambda_t \omega_t$, respectively. Define wage W_t as in the labor demand equation (2.3) in the competitive equilibrium and r_t^c the marginal return of capital $r_t^c = \alpha A^c \left(K_{t-1}^c\right)^{\alpha-1} (L_t^c)^{1-\alpha}$. The FOCs for consumption, labor, and capital are the following.

$$0 = \frac{\partial \mathcal{L}_t}{\partial C_t} = C_t^{-\gamma} - \lambda_t$$

The shadow price of relaxing the resource constraint equals the marginal utility of consumption.

$$\frac{\partial \mathcal{L}_t}{\partial L_t} = -(L_t^c)^{\chi} + \lambda_t W_t + (\sigma + \eta) \alpha \omega_t \lambda_t W_t = 0$$

The marginal disutility from working equals the wage multiplying the marginal utility of consumption plus an additional benefit to be specified. For each unit of additional labor supply, the social planner understands that the output increases by W_t, which is the marginal productivity of labor. The firms earn additional profits αW_t. For the firms they have $(\sigma + \eta) \alpha W_t$ additional retained earnings to invest. As the marginal return from investment is $\omega_t \lambda_t$, we have the additional return in utility terms $(\sigma + \eta) \alpha \omega_t \lambda_t W_t$.

$$0 = \frac{\partial \mathcal{L}_t}{\partial K_t^c} = -\lambda_t + \lambda_{t+1} \left(r_{t+1}^c + (1-\delta)\right) - \lambda_t \omega_t + (\sigma + \eta) \left(\alpha r_{t+1}^c + (1-\delta)\right) \omega_{t+1} \lambda_{t+1}$$

An additional unit of capital investment tightens the resource constraint at the shadow price λ_t, but it relaxes the next period resource constraint by $r_{t+1}^c + (1-\delta)$ at the shadow price λ_{t+1}. Moreover, the additional unit of capital investment tightens the investment constraint at the shadow price $\lambda_t \omega_t$, but it relaxes the next period investment constraint by $(\sigma + \eta) \left(\alpha r_{t+1}^c + (1-\delta)\right)$ at the shadow price $\omega_{t+1} \lambda_{t+1}$. The associated complementary slackness condition is the following.

$$\left((\sigma + \eta) \left(\alpha A^c \left(K_{t-1}^c\right)^{\alpha} (L_t^c)^{1-\alpha} + (1-\delta) K_{t-1}^c \right) - K_t^c \right) \omega_t = 0, \omega_t \geq 0.$$

The social planner's choice of labor supply is therefore,

$$W_t \left(1 + \alpha (\sigma + \eta) \omega_t\right) C_t^{-\gamma} = (L_t^c)^{\chi}. \tag{2.11}$$

Comparing with the labor supply equation in the competitive equilibrium (2.1), the constrained social planner chooses to work more and consume more because it internalizes that the additional consumption and labor supply increase the self-financing firms' profits, allowing them to make more investment. The social cost from consumption is $1/\left(W_t\left(1+\alpha\left(\sigma+\eta\right)\omega_t\right)\right)$ unit of labor. It is less than $1/W_t$ unit of labor[2].

The constrained social planner's solution can be decentralized with the consumption tax. The optimal consumption tax solves

$$\frac{1}{1+\tau_t^C}=1+\alpha\left(\sigma+\eta\right)\omega_t.$$

As a result, $\tau_t^C<0$ if $\omega_t>0$. The constrained planner indeed imposes a consumption subsidy if the investment constraint is binding. The subsidy (-τ_t^C) is higher if the shadow price of relaxing the investment constraint (normalized by the marginal utility of consumption) is higher.

The consumption subsidy drives up the consumption demand. It induces the firms to hire more workers, boosts profits of the self-financing firms and allows them to make more investment[3].

The consumption subsidy not only increases output in the current period with additional labor, but also increases output in the next period with additional investment. More consumption paradoxically leads to more output in this period and the next period for the self-financing firms.

If the self-financing firms have a low exit rate ($1-\sigma$ small), or in other words, the investment constraint is slack (σ large), the investment constraint will not be binding at the steady state. But it is binding during the early stage of transition when the aggregate capital stock is so low that the desired investment is relatively large. The constrained social planner then imposes the consumption subsidy during the early stage of transition, which decreases to zero over time. This accelerates the transition to the steady state. The growth rate is higher. In this sense, this pro-consumption policy generates consumption-led growth during the transition.

If the self-financing firms instead have a high exit rate ($1-\sigma$ large), or in other words, the investment constraint is tight (σ small), the investment constraint will be binding at the steady state. The constrained social planner then imposes the consumption subsidy even at the steady state. Such a consumption subsidy not only accelerates the transition to the steady state but also increases the steady-state output itself. In this sense, this pro-consumption policy generates consumption-led growth both during the transition and to the higher steady state.

[2]Put it in another way, the social return from working is larger than W unit of consumption.

[3]Put it in another way, the consumption subsidy drives up the labor supply and boosts firm profits. The consumption subsidy has the same effect as a working subsidy.

2.3 Two-sector extension

In this section, I introduce a two-sector extension. In addition to a continuum of measure one of self-financing firms in one sector, there is also a continuum of measure one firms with full credit access in another sector. The setting of a firm's problem is identical to the specification in section 2.2.2 except for two modifications. First, each firm operates the same *decreasing return to scale* production function. Second, the self-financing firms can trade bonds with each other within their sector while the firms with credit access can also trade bonds with the household.

All intuition goes through during the transition without making the two modifications, but one sector will disappear at the steady-state. The decreasing return to scale captures the scarcity of managerial or organization capital in reality, and it is convenient for modeling two sectors without the complication of two goods. The inter-firm bond market within the sector allows for easy aggregation.

2.3.1 Competitive equilibrium

In the self-financing sector, the firm i's problem is the following.

$$\max_{l^c_{i,t}, k^c_{i,t}, d^c_{i,t}, b^c_{i,t}} \sum_{t=1}^{\infty} \beta^t C_t^{-\gamma} \sigma^{t-1} \left((1-\sigma)\, \pi^c_{i,t} + \sigma d^c_{i,t} \right)$$

s.t.

$$\pi^c_{i,t} = \left(A^c \left(k^c_{i,t-1}\right)^{\alpha} \left(l^c_{i,t}\right)^{1-\theta} + (1-\delta) k^c_{i,t-1} - W^c_t l^c_{i,t} \right) + b^c_{i,t-1}$$

$$k^c_{i,t} + d^c_{i,t} + P^c_t b^c_{i,t} = \pi^c_{i,t}$$

$$d^c_{i,t} \geq 0$$

The firm's production function is decreasing return to scale $\alpha < \theta$. The firm can buy $b^c_{i,t}$ bond at the market price P^c_t in an inter-firm market within the self-financing sector. This market clears when the net bond position is zero.

$$\int_0^1 b^c_{i,t} = 0$$

With the two modifications, each firm hires the same amount of labor and invests the same amount of capital in the sector. The law of motion of the aggregate variables for the self-financing firms looks like section 2.2.2 with minimum changes (details in Appendix 2.A) as follows. And they are identical if $\theta = \alpha$.

$$W_t = (1-\theta) A^c \left(K^c_{t-1}\right)^{\alpha} \left(L^c_t\right)^{-\theta} \tag{2.12}$$

$$1 = \left(r^c_{t+1} + (1-\delta)\right) \beta \frac{C_t^{\gamma}}{C_{t+1}^{\gamma}} \left((1-\sigma)(1-\mu_t) + \sigma \frac{1-\mu_t}{1-\mu_{t+1}} \right) \tag{2.13}$$

$$r_t^c = \alpha A^c \left(K_{t-1}^c\right)^{\alpha-1} \left(L_t^c\right)^{1-\theta} \tag{2.14}$$

$$K_t^c + D_t^c = (\sigma + \eta) \left(\alpha A^c \left(K_{t-1}^c\right)^{\alpha} \left(L_t^c\right)^{1-\theta} + (1-\delta) K_{t-1}^c\right) \tag{2.15}$$

$$D_t = (1 - \sigma - \eta) \left(\theta A^c \left(K_{t-1}^c\right)^{\alpha} \left(L_t^c\right)^{1-\theta} + (1-\delta) K_{t-1}^c\right) + D_t^c \tag{2.16}$$

$$D_t^c \geq 0, \mu_t \geq 0, D_t^c \mu_t = 0 \tag{2.17}$$

The equations are the optimal labor demand, optimal capital investment, capital return, budget constraint, dividend payment, and complementary slackness conditions, respectively.

In the unconstrained sector with full access to the credit market, the firm i's problem is the following.

$$\max_{l_{i,t}^u, k_{i,t}^u, d_{i,t}^u, b_{i,t}^u} \sum_{t=1}^{\infty} \beta^t C_t^{-\gamma} \sigma^{t-1} \left((1-\sigma)\, \pi_{i,t}^u + \sigma d_{i,t}^u\right)$$

s.t.

$$\pi_{i,t}^u = A^u \left(k_{i,t-1}^u\right)^{\alpha} \left(l_{i,t}^u\right)^{1-\theta} + (1-\delta) k_{i,t-1}^u - W_t^u l_{i,t}^u + b_{i,t-1}^u$$

$$k_{i,t}^u + d_{i,t}^u + P_t b_{i,t}^u = \pi_{i,t}^u$$

$$d_{i,t}^u \geq 0$$

The firm can trade bond $b_{i,t}^u$ at the market price P_t with the other firms in this sector and with the household. The bond market clears when the net bond position between this sector and the household is zero.

$$\int_0^1 b_{i,t}^u + B_t = 0$$

Denote capitalized variables the aggregate variable in this sector, the optimality conditions can be derived likewise. The labor demand equation takes the same form.

$$W_t = (1-\theta) A^u \left(K_{t-1}^u\right)^{\alpha} \left(L_t^u\right)^{-\theta} \tag{2.18}$$

The abitrage between the bond investment and capital investment equalizes the return.

$$1/P_t = r_{t+1}^u + (1-\delta)$$

where the capital return for the unconstrained sector r^u is defined similarly as

$$r_t^u = \alpha A^u \left(K_{t-1}^u\right)^{\alpha-1} \left(L_t^u\right)^{1-\theta}. \tag{2.19}$$

Combine with the household bond Euler equation (2.2), the investment of the unconstrained firms will be affected by the saving tax[4].

$$1+\tau_t^B=\left(r_{t+1}^u+(1-\delta)\right)\beta\frac{C_t^\gamma}{C_{t+1}^\gamma} \tag{2.20}$$

The labor market clearing condition closes the model. The labor demand by the two sectors of firms equals the labor supply by the household.

$$L_t^c+L_t^u=L_t$$

From the two labor demand equations (2.12) and (2.18) from the two sectors, labor is allocated proportionally to the capital stock adjusted by productivities between the two sectors.

$$L_t^c=\frac{\left(A^c\right)^{1/\theta}\left(K_{t-1}^c\right)^{\alpha/\theta}}{\left(A^c\right)^{1/\theta}\left(K_{t-1}^c\right)^{\alpha/\theta}+\left(A^u\right)^{1/\alpha}\left(K_{t-1}^u\right)^{\alpha/\theta}}L_t \tag{2.21}$$

$$L_t^u=\frac{\left(A^u\right)^{1/\alpha}\left(K_{t-1}^u\right)^{\alpha/\theta}}{\left(A^c\right)^{1/\theta}\left(K_{t-1}^c\right)^{\alpha/\theta}+\left(A^u\right)^{1/\theta}\left(K_{t-1}^u\right)^{\alpha/\theta}}L_t \tag{2.22}$$

2.3.2 Optimal taxes

A constrained social planner cannot inject funds directly to the self-financing firms. It maximizes the household objective subject to the resource constraint and the aggregate investment constraint for the self-financing sector as following.

$$\max_{C_t,L_t,K_t^u,K_t^c}\sum_{t=1}^{\infty}\beta^t\left(\frac{C_t^{1-\gamma}}{1-\gamma}-\frac{L_t^{1+\chi}}{1+\chi}\right)$$

s.t.

$$C_t+K_t^u+K_t^c=A^u\left(K_{t-1}^u\right)^\alpha\left(L_t^u\right)^{1-\theta}+(1-\delta)K_{t-1}^u+A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{1-\theta}+(1-\delta)K_{t-1}^c$$

$$K_t^c\leq(\sigma+\eta)\left(\theta A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{1-\theta}+(1-\delta)K_{t-1}^c\right)$$

where L^c and L^u are defined as in equation (2.21) and (2.22). The constrained social planner cannot allocate labor arbitrarily between the two types of firms but follows the market equilibrium.

Denote the shadow prices for the two constraints λ_t, $\omega_t\lambda_t$. Define wage W_t and return of capital in the unconstrained sector r_t^u as equations (2.1) and (2.19) in the competitive equilibrium. The FOCs (details in Appendix 2.A) for consumption,

[4] A negative saving tax (saving subsidy) cannot be an equilibrium as the firm will borrow to pay dividend.

labor, and capital are the following.

$$0 = \frac{\partial \mathcal{L}_t}{\partial C_t} = C_t^{-\gamma} - \lambda_t$$

The shadow price of relaxing the resource constraint equals the marginal utility of consumption.

$$0 = \frac{\partial \mathcal{L}_t}{\partial L_t} = -L_t^{\chi} + \lambda_t W_t + (\sigma + \eta)\, \theta L_t^c / L_t \omega_t \lambda_t W_t \tag{2.23}$$

The marginal disutility from working equals the wage multiplying the marginal utility of consumption plus an additional benefit to be specified. For each unit of additional labor supply, the social planner understands that the employment of the self-financing firms increases proportionally by L_t^c/L_t. This increases output in this sector by $L_t^c/L_t W_t$. Those firms earn additional profits $\theta L_t^c/L_t W_t$. Therefore, they have $(\sigma + \eta)\, \theta L_t^c/L_t W_t$ additional retained earnings to invest. As the marginal return from investment is $\omega_t \lambda_t$, we have the additional return in utility terms $(\sigma + \eta)\, \theta L_t^c/L_t \omega_t \lambda_t W_t$.

$$\begin{aligned} 0 = \frac{\partial \mathcal{L}_t}{\partial K_t^u} &= -\lambda_t + \left(r_{t+1}^u + 1 - \delta\right) \lambda_{t+1} \\ &\quad - (\sigma + \eta)\, L_{t+1}^c / L_{t+1} \,(1 - \theta)\, r_{t+1}^u \omega_{t+1} \lambda_{t+1} \end{aligned} \tag{2.24}$$

The marginal disutility from reducing one unit of consumption for investment at this period is $-\lambda_t$ and this leads to an additional $r_{t+1}^u + 1 - \delta$ unit of return valued at the price λ_{t+1}, subtracting an additional cost to be specified. For an additional unit of capital in the unconstrained sector, it reduces total labor available for the existing firms by $(1 - \theta)\, r_{t+1}^u / (\theta W_t)$ unit. Multiply this with the return from one unit additional labor $(\sigma + \eta)\, \theta L_t^c/L_t \omega_t \lambda_t W_t$ as analyzed from the optimal labor supply condition, we get the additional disutility term $-(\sigma + \eta)\, L_{t+1}^c/L_{t+1}\,(1 - \theta)\, r_{t+1}^u \omega_{t+1} \lambda_{t+1}$. The competitive equilibrium is dynamic inefficient as the social return of investment is smaller (from the additional cost $(\sigma + \eta)\, L_{t+1}^c/L_{t+1}\,(1 - \theta)\, r_{t+1}^u \omega_{t+1} \lambda_{t+1}$) than the private return in the unconstrained sector.

The associated complementary slackness condition is the following.

$$\left((\sigma + \eta)\left(\theta A^c \left(K_{t-1}^c\right)^{\alpha} \left(L_t^c\right)^{1-\theta} + (1 - \delta)\, K_{t-1}^c\right) - K_t^c\right) \omega_t = 0, \omega_t \geq 0$$

The social planner's labor supply is the following.

$$W_t \left(1 + (\sigma + \eta)\, \theta L_t^c / L_t \omega_t\right) C_t^{-\gamma} = L_t^{\chi}$$

It chooses to supply more labor and consume more than the competitive equilibrium (2.12) with the same reason as in section 2.2.4.

The social planner's investment decision is the following.

$$1 = \left(r_{t+1}^u + 1 - \delta\right) \beta \frac{C_t^\gamma}{C_{t+1}^\gamma} \left(1 - \frac{(\sigma + \eta) L_{t+1}^c / L_{t+1} (1 - \theta) r_{t+1}^u \omega_{t+1}}{r_{t+1}^u + 1 - \delta}\right)$$

It chooses to invest less in the unconstrained sector as it demands a higher return r_{t+1}^u than the competitive equilibrium (2.20). It internalizes the expansion in the unconstrained sector out-competes the self-financing firms in the factor market and squeezes their profits.

The constrained social planner's solution can be implemented by choosing τ_t^C and τ_t^Bas follows.

$$\frac{1}{1 + \tau_t^C} = 1 + \theta (\sigma + \eta) L_t^c / L_t \omega_t$$

$$\frac{1}{1 + \tau_t^B} = 1 - \frac{(\sigma + \eta) L_{t+1}^c / L_{t+1} (1 - \theta) r_{t+1}^u \omega_{t+1}}{r_{t+1}^u + 1 - \delta}$$

In addition to a saving subsidy $\tau_t^C < 0$, it imposes an additional saving tax $\tau_t^B > 0$ if the investment constraint for the self-financing firms is binding $\omega_t > 0$. The saving tax is higher if the shadow price of relaxing the investment constraint (normalized by the marginal utility of consumption) is higher.

The saving tax drives up the required return of investment in the unconstrained sector by $1/\left(1 + \tau_t^B\right)$. It induces the unconstrained sector to invest less and releases labor to the self-financing firms at the next period. This boosts the next-period profits of the self-financing firms and allows them to make more investment then. While the resulting consumption increase from the saving tax crowds out investment in the unconstrained sector, it crowds in investment of the constrained firms in the next period.

If the self-financing firms have a high exit rate ($1-\sigma$ large), or in other words, the investment constraint is tight (σ small), the investment constraint is binding at the steady-state. The constrained social planner imposes both the consumption subsidy and the saving tax at the steady-state. The unconstrained sector shrinks and the consrained sector expands. The net effect on aggregate consumption and output in the long run is ambiguous. But the pro-consumption policies lead to unambiguously higher welfare by internalizing the two externalities.

The pro-consumption policies work differently to a generic demand policy in the Keynesian framework as it aims to increase the profits of the constrained self-financing firms. In particular, lowering the interest rate the firms facing is counter-productive here because it exacerbates the misallocation on the supply side. The policymaker instead inserts a wedge between the saving and borrowing rates.

2.4 Numerical Analysis

To quantitatively evaluate the net effect of the pro-consumption policies, I calibrate the two-sector extension of the model to China. Moreover, I also study optimal simple taxes, which rule out any subsidy or time-varying tax. The social planner starts to intervene from the steady-state of the competitive equilibrium without any tax.

2.4.1 Calibration

The model is calibrated to China annually. The self-financing sector can be interpreted as the private sector and the sector with credit access can be interpreted as the state-owned sector. Normalizing the productivity for the self-financing sector $A^c = 1$, there are 9 parameters to calibrate. Table 3.1 summarizes the calibration.

Table 2.1: Calibration

Parameter	Value	Description
α	1/3	Capital share
δ	0.1	Depreciation rate
γ	2	$1/\gamma$ the intertemporal elasticity of substitution
χ	1	Inverse Frisch elasticity
β	0.96	Discount factor
θ	0.5	$1-\theta$ labor share, China is 50%.
σ	2/3	$1/(1-\sigma)$ average private-sector firm life, China is 3
η	0.0152	Start-up fund ratio, tar. $L^c/L = 80\%$ and $I^c/I = 60\%$
A^u	0.5724	Uncon. firm productivity, tar. $L^c/L = 80\%$ and $I^c/I = 60\%$

I begin with the 6 conventional parameters. The capital share α is set to 1/3. The depreciation rate δ is set to 0.1. The intertemporal elasticity of substitution $1/\gamma$ is set to 0.5. The inverse Frisch elasticity of labor χ is set to 1. The discount factor β is set to 0.96. The Chinese labor share is around 50% so $1-\theta$ is set to 0.5.

Three parameters (σ, η, A^u) are specific to the model. The choice is meant to be suggestive. The average life of the private-sector firms in China is 3 years, so $1/(1-\sigma)$ is set to 3. The short life span of the private-sector firms provides one rationale for why it is so difficult for them to obtain bank credit. The state-owned firms can have a significantly longer expected life. But a different calibration for them is unnecessary as they are not financially constrained.

The start-up fund to profit ratio η and the productivity of the unconstrained firms A^u are deep parameters. To pin down the two, I target the Chinese private-sector employment share L^c/L and investment share I^c/I at the steady-state of the competitive equilibrium[5]. The recent private-sector employment share is 80%, and the private-sector investment share is 60% in China. I use these two values. As a

[5]The net investment $I = \delta K$ and $I^c = \delta K^c$ at the steady state

result, $A^u = 0.5724$ and $\eta = 0.0152$. The state-owned firms are much less efficient than the private sector, and the private sector receives little start-up fund.

2.4.2 Optimal taxes

The model is solved globally using the relaxation algorithm. Figure 2.1 shows the constrained social planner's solution in blue solid lines and the competitive equilibrium in red dashed lines. The red dash lines are horizontal because the economy starts from the steady state of the competitive equilibrium. The self-financing firms' employment as a fraction of total is about 80%, and capital investment in total investment is about 60%. These are exactly the targets for the calibration.

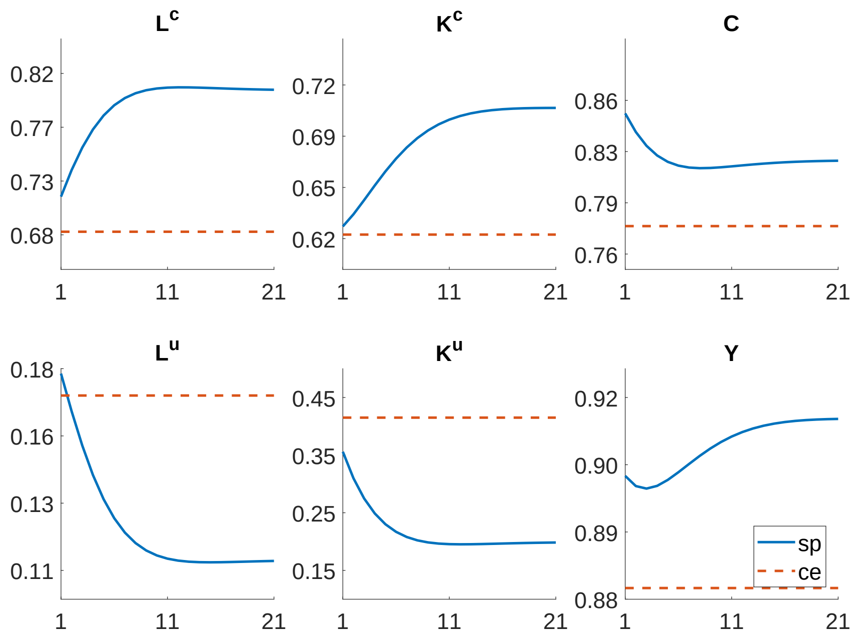

Figure 2.1: Social planner vs. competitive equilibrium

The social planner expands the self-financing sector at the cost of the unconstrained sector. Both labor and capital reallocate to the self-financing sector over 20 years. The aggregate consumption and output are higher. The labor demands from the two sectors adjust slowly towards the new steady state because the capital stocks in the two sectors evolve slowly. The labor demand in the short run in the unconstrained sector is even higher than the competitive equilibrium. This is possible because the social planner increases the aggregate employment a lot in response to the boom from higher consumption demand.

For ease of comparison, Figure 2.2 shows the deviation of the social planner's choice from the competitive equilibrium for the four aggregate variables: the labor, capital, consumption, and output. The social planner chooses a higher employment. The employment increases overtime to reach about 7% more in the long run. The

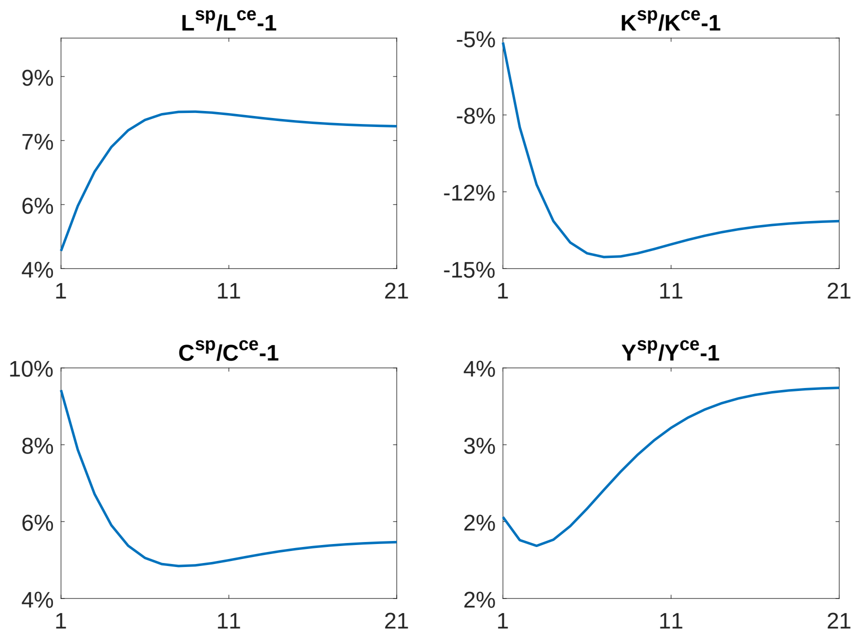

Figure 2.2: The social planner's choice, in percentage deviation

(roughly) upward trend mainly reflects the reduction of capital mis-allocation and the increase in total factor productivity. The capital stock drops immediately in the short run, and it further drops to 13% lower in the long run. The social planner prefers a lower capital stock, suggesting that the economy as a whole is dynamic inefficient. The consumption jumps to nearly 10% higher in the short run, benefiting from the reduction in the investment and the additional labor supply. It is more than 5% higher in the long run. Output increases by 2% in the short run and eventually grows to nearly 4% higher. The results are consistent with the idea of consumption-led growth.

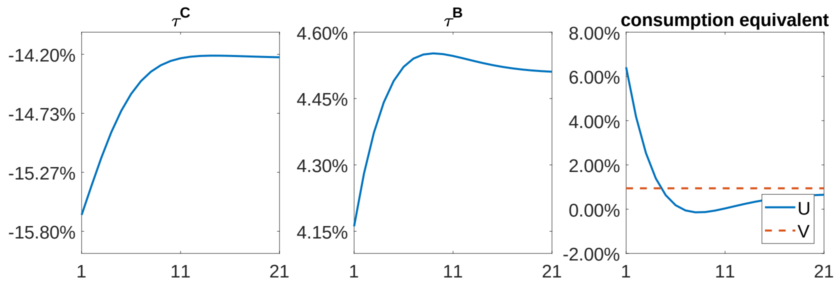

Figure 2.3: Tax and welfare

The social planner's solution can be implemented with a time-varying consumption subsidy and a time-varying saving tax, as shown in Figure 2.3. The optimal consumption subsidy reduces from about 16% in the short run to 14% in the long run and the optimal saving tax rises from about 4% to 4.5%. The right panel shows the welfare effect in terms of equivalent consumption change to the competitive equilibrium, keeping the labor supply unchanged. The pro-consumption policies increase economic welfare by that equals about 1% more consumption forever than the competitive equilibrium. The red dashed line shows the welfare gain in consumption-equivalent for each period. The household benefits a lot in the short run, it loses a little in the medium-run and benefits about 0.5% higher welfare in consumption-equivalent per year in the long run.

2.4.3 Optimal constant non-negative taxes

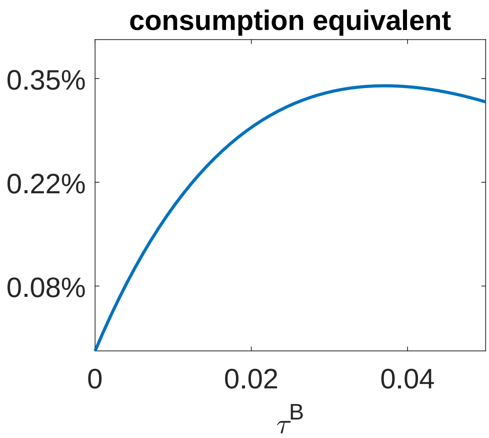

Figure 2.4: Welfare effect of the constant saving tax

The time-varying subsidy and tax can be difficult to implement in practice. In particular, there is no non-distortionary lump-sum tax to finance a large subsidy[6]. I study in this section the optimal constant non-negative taxes the social planner would like to choose, still from the steady state of the competitive equilibrium.

The social planner will not choose any consumption tax but will choose a constant saving tax at 3.67%. Therefore, the long-run interest rate for the unconstrained firms doubles from 4% to 8%. This leads to a consumption-equivalent of welfare gain at 0.34%, about 1/3 of that in the last section. Figure 2.4 shows how the welfare gain in consumption-equivalent changes with different constant saving tax. The curve is upward-sloping when the tax is small as it improves capital allocation. The curve becomes downward-sloping when the tax is large as it distorts the unconstrained sector investment so much that the benefit from crowding in investment in the self-financing sector is dominated.

Figure 2.5 shows the social planner's solution with the 3.67% constant saving tax in blue solid lines against the competitive equilibrium in red dashed lines. While the pattern of sectoral reallocation is still there, as in the social planner's solution with the time-varying consumption subsidy and saving tax, consumption is higher

[6]The revenue from the saving tax is insufficient to cover the consumption subsidy.

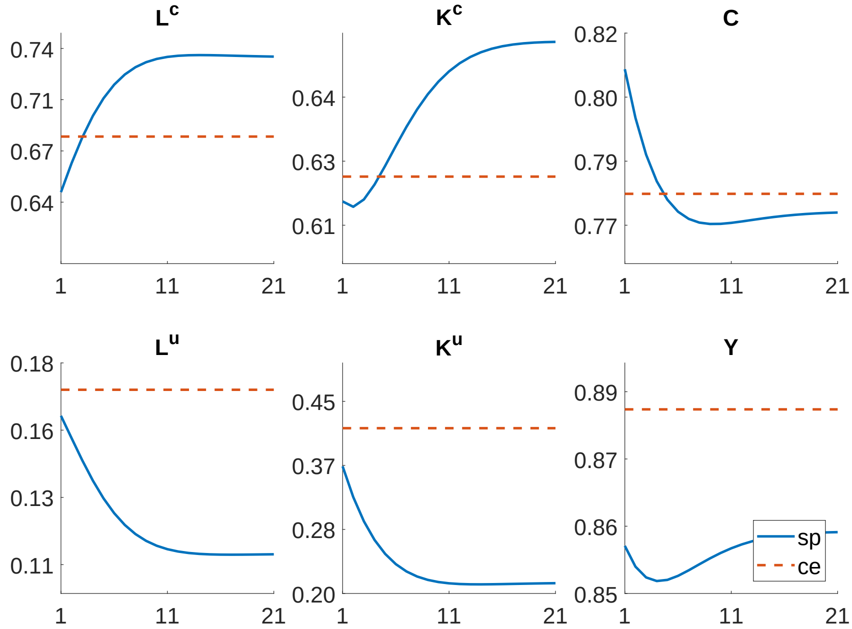

Figure 2.5: Social planner vs. competitive equilibrium (constant saving tax)

in the short run but lower in the long run. Moreover, the aggregate output is lower throughout the transition. This is because, without the consumption subsidy, the total employment is lower.

For ease of comparison, Figure 2.6 shows the deviation of the social planner's choice from the competitive equilibrium for the four aggregate variables. While the pattern of the variables is still there as in the social planner's solution with the time-varying consumption subsidy and saving tax, the long-run implications are quite different. The employment and consumption are both slightly smaller in the long run. The output is about 2% lower. But the 2% lower long-run output is achieved with 16% lower aggregate capital and slightly lower labor, suggesting a large improvement of the total factor productivity.

The restricted pro-consumption policy, namely a constant saving tax at 3.67%, leads to lower aggregate output both in the short run and long run. Consumption does not lead to growth in output, though it leads to growth in productivity and welfare.

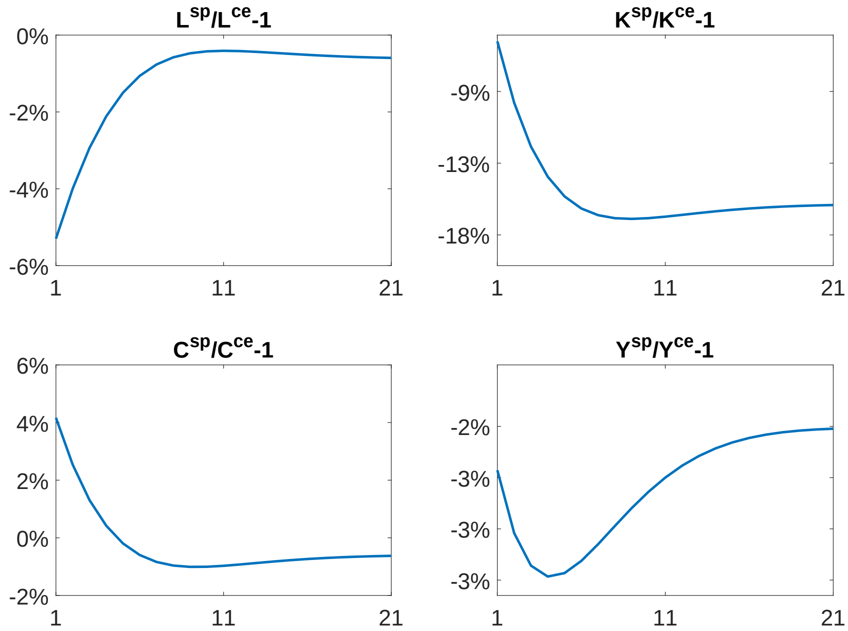

Figure 2.6: The social planner's choice (constant saving tax), in percentage deviation

2.5 Conclusion

This chapter highlights two market failures in the presence of self-financing firms that justify the consumption subsidy and saving tax. The pro-consumption policies tend to increase consumption as well as output. Restricting the policies to include constant non-negative tax only, the social planner will choose a constant saving tax. The pro-consumption policy leads just to growth in welfare, not in output anymore.

For future work, another closely related channel that the unconstrained sector fails to internalize is worth exploring. If the two sectors produce different but highly substitutable goods, the expansion of the unconstrained firms steals market share from the self-financing firms and squeezes their profits. Moreover, the insights and the modeling framework in this chapter are also easy to generate to an open economy. It provides a rationale for capital inflow control.

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Appendices

2.A Solution details

Modified self-financing firm's problem in section 2.3

Denote $\xi_{i,t}^c, \nu_{i,t}^c, \nu_{i,t}^c \mu_{i,t}^c$ the shadow prices for the three constraints, respectively, the optimality conditions are the following.

$$\text{FOC } l_{i,t}^c: \; W_t^c = (1-\theta) A^c \left(k_{i,t-1}^c\right)^{\alpha} \left(l_{i,t}^c\right)^{-\theta}$$

$$\text{FOC } \pi_{i,t}^c: \; \beta^t C_t^{-\gamma} \sigma^{t-1} (1-\sigma) - \xi_{i,t}^c + \nu_{i,t}^c = 0$$

$$\text{FOC } k_{i,t}^c: \; -\nu_{i,t}^c + \xi_{i,t+1}^c \left(\alpha A^c \left(k_{i,t}^c\right)^{\alpha-1} \left(l_{i,t+1}^c\right)^{1-\theta} + (1-\delta) \right) = 0$$

$$\text{FOC } d_{i,t}^c: \; \beta^t C_t^{-\gamma} \sigma^{t-1} \sigma = (1-\mu_{i,t}^c) \nu_{i,t}^c$$

$$\text{FOC } b_{i,t}^c: \; -\nu_{i,t}^c P_t^c + \xi_{i,t+1}^c = 0$$

$$\text{Complementarity slackness condition } d_{i,t}^c \geq 0, \mu_{i,t}^c \geq 0, d_{i,t}^c \mu_{i,t}^c = 0$$

There is an additional FOC with regard to $b_{i,t}$, in addition to $k_{i,t}^c$. From no-arbitrage, the return from capital and bond must equal each other.

$$\alpha A^c \left(k_{i,t}^c\right)^{\alpha-1} \left(l_{i,t+1}^c\right)^{1-\theta} + (1-\delta) = 1/P_t$$

From this no-arbitrage condition and the FOC with regard to $l_{i,t}^c$,

$$k_{i,t}^c = \left(\frac{1}{A^c} \left(\frac{W_t}{1-\theta} \right)^{1-\theta} \left(\frac{1/P_t - (1-\delta)}{\alpha} \right)^{\theta} \right)^{1/(\alpha-\theta)}$$

$$l_{i,t}^c = \left(\frac{1}{A^c} \left(\frac{W_t}{1-\theta} \right)^{1-\alpha} \left(\frac{1/P_t - (1-\delta)}{\alpha} \right)^{\alpha} \right)^{1/(\alpha-\theta)}$$

Therefore, each firm hires the same amount of workers and invests the same amount of capital.

$$k_{i,t}^c = k_t^c$$

$$l_{i,t}^c = l_t^c$$

Eliminate $\xi_{i,t}^c, \nu_{i,t}^c$. The Euler equation for the firm is

$$1 = \left(\alpha A^c \left(k_{i,t}^c\right)^{\alpha-1} \left(l_{i,t+1}^c\right)^{1-\theta} + (1-\delta)\right) \beta \frac{C_t^\gamma}{C_{t+1}^\gamma} \left((1-\sigma)\left(1-\mu_{i,t}^c\right) + \sigma \frac{1-\mu_{i,t}^c}{1-\mu_{i,t+1}^c}\right)$$

Their shadow value of relaxing the non-negative dividend constraints must be identical.

$$\mu_{i,t}^c = \mu_t$$

As a result, they make the same discretionary dividend and investment decisions proportional regardless of their profits. Denote the total dividend D_t,total discretionary dividend D_t^c, total capital K_t and total labor L_t. Clearing the inter-firm bond market, the aggregate variables evolve in the following way.

$$W_t = (1-\theta) A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{-\theta}$$

$$1 = \left(\alpha A^c \left(K_t^c\right)^{\alpha-1} \left(L_{t+1}^c\right)^{1-\theta} + (1-\delta)\right) \beta \frac{C_t^\gamma}{C_{t+1}^\gamma} \left((1-\sigma)(1-\mu_t) + \sigma \frac{1-\mu_t}{1-\mu_{t+1}}\right)$$

$$K_t^c + D_t^c = (\sigma+\eta)\left(\alpha A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\theta} + (1-\delta) K_{t-1}^c\right)$$

$$D_t = (1-\sigma-\eta)\left(\theta A^c \left(K_{t-1}^c\right)^\alpha \left(L_t^c\right)^{1-\theta} + (1-\delta) K_{t-1}^c\right) + D_t^c$$

Complementarity slackness condition $D_t^c \geq 0, \mu_t \geq 0, D_t^c \mu_t = 0$

Constrained social planner's problem in section 2.3

The social planner maximizes

$$\max \sum \beta^t \left(\frac{C_t^{1-\gamma}}{1-\gamma} - \frac{L_t^{1+\chi}}{1+\chi}\right)$$

s.t.

$$C_t + K_t^c + K_t^u = Y_t^u + (1-\delta) K_{t-1}^u + Y_t^c + (1-\delta) K_{t-1}^c$$

$$K_t^c \leq (\sigma+\eta)\left(\theta Y_t^c + (1-\delta) K_{t-1}^c\right)$$

after eliminating L_t^c and L_t^u using equations (2.21) and (2.22),

$$Y_t^u = (A^u)^{1/\theta} \left(K_{t-1}^u\right)^{\alpha/\theta} L_t^{1-\theta} \left((A^c)^{1/\theta} \left(K_{t-1}^c\right)^{\alpha/\theta} + (A^u)^{1/\theta} \left(K_{t-1}^u\right)^{\alpha/\theta}\right)^{\theta-1}$$

$$Y_t^c = (A^c)^{1/\theta} \left(K_{t-1}^c\right)^{\alpha/\theta} L_t^{1-\theta} \left((A^c)^{1/\theta} \left(K_{t-1}^c\right)^{\alpha/\theta} + (A^u)^{1/\theta} \left(K_{t-1}^u\right)^{\alpha/\theta}\right)^{\theta-1}$$

To begin with, derive the FOCs of Y_t^u with regard to L_t^u, K_t^u and K_t^c

$$\begin{aligned}\frac{\partial Y_t^u}{\partial L_t} &= (1-\theta)\frac{Y_t^u}{L_t^u}\frac{L_t^u}{L_t} \\ &= W_t L_t^u / L_t\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_{t+1}^u}{\partial K_t^u} &= \frac{\alpha}{\theta}\frac{Y_{t+1}^u}{K_t^u} + \frac{\alpha}{\theta}\frac{Y_{t+1}^u}{K_t^u}\frac{(A^u)^{1/\theta}(K_t^u)^{\alpha/\theta}(\theta-1)}{(A^c)^{1/\theta}(K_t^c)^{\alpha/\theta} + (A^u)^{1/\theta}(K_t^u)^{\alpha/\theta}} \\ &= r_{t+1}^u/\theta + r_{t+1}^u L_{t+1}^u / L_{t+1}(1-1/\theta)\end{aligned}$$

$$\begin{aligned}\frac{\partial Y_{t+1}^u}{\partial K_t^c} &= \frac{\alpha}{\theta}\frac{Y_{t+1}^c}{K_t^c}\frac{(A^u)^{1/\theta}(K_t^u)^{\alpha/\theta}(\theta-1)}{(A^c)^{1/\theta}(K_t^c)^{\alpha/\theta} + (A^u)^{1/\theta}(K_t^u)^{\alpha/\theta}} \\ &= r_{t+1}^c L_{t+1}^u / L_{t+1}(1-1/\theta)\end{aligned}$$

The FOCs of Y_t^c are symmetric.

Denote the shadow prices λ_t, $\omega_t\lambda_t$ for the two constraints, respectively. The FOCs

$$\begin{aligned}0 = \frac{\partial \mathcal{L}_t}{\partial L_t} &= -L_t^\chi + \frac{\partial Y_t^u}{\partial L_t}\lambda_t + (1+(\sigma+\eta)\theta\omega_t)\frac{\partial Y_t^c}{\partial L_t}\lambda_t \\ &= -L_t^\chi + W_t\frac{L_t^u}{L_t}\lambda_t + (1+(\sigma+\eta)\theta\omega_t)W_t\frac{L_t^c}{L_t}\lambda_t \\ &= -L_t^\chi + \lambda_t W_t + (\sigma+\eta)\theta\frac{L_t^c}{L_t}\omega_t\lambda_t W_t\end{aligned}$$

$$\begin{aligned}0 = \frac{\partial \mathcal{L}_t}{\partial K_t^u} &= -\lambda_t + \lambda_{t+1}\left(\frac{\partial Y_{t+1}^u}{\partial K_t^u} + (1-\delta)\right) + \lambda_{t+1}\frac{\partial Y_{t+1}^c}{\partial K_t^u} + \omega_{t+1}\lambda_{t+1}(\sigma+\eta)\theta\frac{\partial Y_{t+1}^c}{\partial K_t^u} \\ &= -\lambda_t + \lambda_{t+1}\left(r_{t+1}^u/\theta + r_{t+1}^u L_{t+1}^u/L_{t+1}(1-1/\theta) + (1-\delta)\right) \\ &\quad + \lambda_{t+1}\left(r_{t+1}^u L_{t+1}^c/L_{t+1}(1-1/\theta)\right) \\ &\quad + \omega_{t+1}\lambda_{t+1}(\sigma+\eta)\left(\theta r_{t+1}^u L_{t+1}^c/L_{t+1}(1-1/\theta)\right) \\ &= -\lambda_t + \left(r_{t+1}^u + (1-\delta)\right)\lambda_{t+1} - (\sigma+\eta)L_{t+1}^c/L_{t+1}(1-\theta)r_{t+1}^u\omega_{t+1}\lambda_{t+1}\end{aligned}$$

$$
\begin{aligned}
0 = \frac{\partial \mathcal{L}_t}{\partial K_t^c} &= -\lambda_t - \omega_t\lambda_t + \lambda_{t+1}\frac{\partial Y_{t+1}^u}{\partial K_t^c} + \lambda_{t+1}\left(\frac{\partial Y_{t+1}^c}{\partial K_t^c} + (1-\delta)\right) \\
&+ \omega_{t+1}\lambda_{t+1}(\sigma+\eta)\left(\theta\frac{\partial Y_{t+1}^c}{\partial K_t^c} + (1-\delta)\right) \\
&= -\lambda_t + \lambda_{t+1}\left(r_{t+1}^c L_{t+1}^u / L_{t+1}(1-1/\theta)\right) \\
&+ \lambda_{t+1}\left(r_{t+1}^c/\theta + r_{t+1}^c L_{t+1}^c/L_{t+1}(1-1/\theta) + 1 - \delta\right) \\
&- \omega_t\lambda_t + \omega_{t+1}\lambda_{t+1}(\sigma+\eta)\left(\theta\left(r_{t+1}^c/\theta + r_{t+1}^c L_{t+1}^c/L_{t+1}(1-1/\theta)\right) + (1-\delta)\right) \\
&= -\lambda_t + \lambda_{t+1}\left(r_{t+1}^c + 1 - \delta\right) \\
&- \omega_t\lambda_t + (\sigma+\eta)\left(\left(1 - L_{t+1}^c/L_{t+1}(1-\theta)\right)r_{t+1}^c + (1-\delta)\right)\omega_{t+1}\lambda_{t+1}
\end{aligned}
$$

2.B Equation system for the two-sector extension

When the constraint is binding, the competitive equilibrium is characterized by 7 variables $W_t, K_t^c, K_t^u, L_t^c, L_t^u, L_t, C_t$ satisfying the following equations.

$$W_t = (1-\theta)A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{-\theta}$$

$$W_t = (1-\theta)A^u\left(K_{t-1}^u\right)^\alpha\left(L_t^u\right)^{-\theta}$$

$$W_t\left(1-\tau_t^L\right)/\left(1+\tau^C\right)C_t^{-\gamma} = L_t^\chi$$

$$1+\tau_t^B = \left(\alpha A^u\left(K_t^u\right)^{\alpha-1}\left(L_{t+1}^u\right)^{1-\theta} + (1-\delta)\right)\beta\frac{C_t^\gamma}{C_{t+1}^\gamma}$$

$$K_t^c = (\sigma+\eta)\left(\theta A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{1-\theta} + (1-\delta)K_{t-1}^c\right)$$

$$C_t + K_t^u + K_t^c = A^u\left(K_{t-1}^u\right)^\alpha\left(L_t^u\right)^{1-\theta} + (1-\delta)K_{t-1}^u + A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{1-\theta} + (1-\delta)K_{t-1}^c$$

$$L_t^c + L_t^u = L_t$$

The constrained social planner's equilibrium is characterized by 10 variables $W_t, K_t^c, K_t^u, L_t^c, L_t^u, L_t, C_t, \omega_t, r_t^c, r_t^u$ satisfying the following equations.

$$W_t = (1-\theta)A^c\left(K_{t-1}^c\right)^\alpha\left(L_t^c\right)^{-\theta}$$

$$W_t = (1-\theta)A^u\left(K_{t-1}^u\right)^\alpha\left(L_t^u\right)^{-\theta}$$

$$W_t\left(1+(\sigma+\eta)\theta L_t^c/L_t\omega_t\right)C_t^{-\gamma} = L_t^\chi$$

$$1 = \left(r_{t+1}^u + 1 - \delta\right)\beta\frac{C_t^\gamma}{C_{t+1}^\gamma}\left(1 - \frac{(\sigma+\eta)L_{t+1}^c/L_{t+1}(1-\theta)r_{t+1}^u\omega_{t+1}}{r_{t+1}^u + 1 - \delta}\right)$$

$$r_t^u = \alpha A^u\left(K_{t-1}^u\right)^{\alpha-1}\left(L_t^u\right)^{1-\theta}$$

$$1+\omega_t = \left(\left(r_{t+1}^c + (1-\delta)\right) + (\sigma+\eta)\left(\left(1 - L_{t+1}^c/L_{t+1}(1-\theta)\right)r_{t+1}^c + (1-\delta)\right)\omega_{t+1}\right)\beta\frac{C_t^\gamma}{C_{t+1}^\gamma}$$

$$r_t^c = \alpha A^c\left(K_{t-1}^c\right)^{\alpha-1}\left(L_t^c\right)^{1-\theta}$$

$$K_t^c = (\sigma + \eta)\left(\theta A^c\left(K_{t-1}^c\right)^{\alpha}\left(L_t^c\right)^{1-\theta} + (1-\delta)\, K_{t-1}^c\right)$$

$$C_t + K_t^u + K_t^c = A^u\left(K_{t-1}^u\right)^{\alpha}\left(L_t^u\right)^{1-\theta} + (1-\delta)\, K_{t-1}^u + A^c\left(K_{t-1}^c\right)^{\alpha}\left(L_t^c\right)^{1-\theta} + (1-\delta)\, K_{t-1}^c$$

$$L_t^c + L_t^u = L_t$$

Chapter 3

Aging with Automation

Abstract: This chapter develops a dynamic model that combines demographic transitions, as in Gertler (1999), with endogenous automation. Following Acemoglu and Restrepo (2018b), automation is modeled as the active replacement of labor with capital at the task level in response to a rise in the relative cost of labor to capital, leading to an endogenous increase in the capital share of output. It finds that allowing automation to react endogenously to demographic and productivity changes generates quantitatively relevant effects compared with the standard baseline where firms cannot respond through the automation margin.

3.1 Introduction

The pressing issue of population aging in the industrialized economies stimulates a large literature. However, few works incorporate its potential interactions with automation, which is a widely discussed topic on its own. Rising life expectancy and declining birth rate result in fewer workers supporting more retirees. Insufficient workers are the concern[1]. At the same time, robots and artificial intelligence are replacing human beings from manual tasks such as bolt tightening in an assembly line to cognitive tasks such as real-time language translation. Workers outcompeted by machines is the concern[2].

This chapter aims to study the consequence of demographic transitions and automation as a whole. Firms actively choose to use capital to replace workers in more tasks if the relative cost of labor to capital increases from aging. This endogenous automation alleviates the negative impact of aging on economic growth and fiscal sustainability. Aging, in turn, alleviates the downward pressure on wage growth of automation from technology progress.

The framework is a flexible overlapping-generations model developed by Gertler

[1]For example, Gruber and Wise (1998); Blake and Mayhew (2006); Aslanyan (2014); Cooley and Henriksen (2018); Docquier et al. (2019)

[2]For example, Autor et al. (2003); Goos and Manning (2007); Autor and Dorn (2013); Michaels et al. (2014); Arntz et al. (2016); Graetz and Michaels (2018); Gregory et al. (2018)

(1999). By introducing random transition from work to retirement and from retirement to death, realistic life cycle and demographic transitions can be easily calibrated with the transition probabilities. We introduce endogenous automation into this framework and contrast the result to that when the automation channel is off.

The endogenous automation follows the task-based approach developed by Acemoglu and Restrepo (2018b). More specifically, the final output is produced by combining the outputs of various tasks. Tasks differ in their relative efficiency in using capital and labor. For given relative cost of factors, some tasks are chosen by the firms to produce with capital and the others with labor. Automation is the process in which more tasks are produced by capital rather than labor when capital becomes relatively cheaper than labor[3].

The model is calibrated to the US to study the economic dynamics over the next 50 years. We find that leaving out automation underestimates the annual per capita economic growth rate by 0.2 percentage points and overestimates the pension to GDP obligations by 1 percentage point in 50 years. Automation does slow down wage growth rate slightly in the beginning, but it accelerates wage growth later. The capital share is expected to rise about 6 percentage points in 50 years, higher than the 4 percentage points over the past 50 years[4], with population aging and technology progress.

This chapter contributes to two strands of largely independently developed literature. The framework follows Gertler (1999) and recent developments by Ferrero (2010); Carvalho et al. (2016). It is more flexible than conventional overlapping-generations models with a fixed life cycle. The automation part builds on the microfoundation in Acemoglu and Restrepo (2018b), which originates from Zeira (1998) and Acemoglu and Autor (2011). With minimum simplification and functional assumption, we make the automation channel quite tractable for quantitative analysis. Abeliansky and Prettner (2017), Acemoglu and Restrepo (2017b), and Acemoglu and Restrepo (2018a) are three closely related papers on demographics and automation. In their papers, the model is not brought to calibration directly but means to provide insights for empirical tests. We complement this literature by providing a calibrated dynamic model.

The rest of the chapter will be organized as follows: in Section Two, we set up the model; in Section Three, we calibrate the model; in Section Four we discuss the numerical results. Section Five concludes.

[3]For example, Lewis (2011); Manuelli and Seshadri (2014); Clemens et al. (2018)

[4]The declining of labor share over the past decades have been well documented by, for example, Elsby et al. (2013); Karabarbounis and Neiman (2013).

3.2 Model

The model comprises of three building blocks: the demographic block, the automation block, and the pension block. The demographic block is based on Gertler (1999). The automation block is based on Acemoglu and Restrepo (2018b) with two simplifications to facilitate analytical intuition and numerical calibration. The pension block is simply about government financing pension with a payroll tax.

3.2.1 Demographics

Individuals evolve through two distinct stages of life: worker and retiree. A worker retires with probability $1-\omega_t$ at time t and becomes a retiree. A retiree dies with probability $1-\gamma_t$. The retirement and death probabilities are independent across the individuals. Denote variables of the workers with a superscript w and variables of the retirees with a superscript r. The mass of workers at time t are L_t^w and retirees are L_t^r. Each period $n_t L_{t-1}^w$ mass of new workers join the working force as newborn or immigrants. The law of motion of the workers and the retirees is the following.

$$L_t^w = \omega_t L_{t-1}^w + n_t L_{t-1}^w,$$

$$L_t^r = \gamma_t L_{t-1}^r + (1-\omega_t) L_{t-1}^w.$$

The expected years of working is $1/(1-\omega_t)$, and the expected life after retirement is $1/(1-\gamma_t)$. In this framework, an increase in longevity is a decrease in the mortality rate $1-\gamma_t$. A decrease in newborns/immigrants is a decrease in n_t. An increase in the mandatory retirement age is a decrease in the retirement rate $1-\omega_t$ and an increase in the death probability $1-\gamma_t$, keeping the expected total life of working and retirement constant $1/(1-\omega_t)+1/(1-\gamma_t)$.

Denote the old-age-dependency ratio $\Psi_t \equiv L_t^r/L_t^w$, it evolves as follows.

$$(\omega_t + n_t)\Psi_t = \gamma \Psi_{t-1} + (1-\omega_t)$$

The problem for a worker i consists of choosing consumption $c_{i,t}$ and asset $a_{i,t}$ to solve

$$V_{i,t}^w = \max_{c_{i,t}^w, a_{i,t}^w} \left[\left(c_{i,t}^w\right)^\rho + \beta \left(\omega_{t+1} V_{i,t+1}^w + (1-\omega_{t+1}) V_{i,t+1}^r\right)^\rho \right]^{1/\rho}$$

subject to

$$c_{i,t}^w + a_{i,t}^w = R_t a_{i,t-1}^w + W_t^w (1-\tau_t)$$

or

$$a_{i,t-1}^w = 0$$

if she is a new worker.

The preference is Epstein-Zin without risk-aversion on the retirement event. This not only is convenient but also reflects the fact that the retirement event is highly predictable in reality, although it is an artificial risk in the model. She derives utility from current consumption $c^w_{i,t}$. Her value is a CES aggregation of current consumption $c^w_{i,t}$ and discounted future value $\omega_{t+1}V^w_{i,t+1} + (1-\omega_{t+1})\, V^r_{i,t+1}$, as a worker or retiree. The elasticity of intertemporal substitution is $1/(1-\rho)$.

The asset can be either physical capital or government bond. The two assets pay the same gross return R_t by no-arbitrage if there is no risk in the capital return. We assume so because we study the life-cycle consumption-saving decision. Denote r_t the rental rate of capital and δ the capital depreciation rate. The no-arbitrage condition for time t is the following.

$$R_{t+1} = r_{t+1} + (1-\delta)$$

Each worker supplies 1 unit of labor inelastically. She receives a market wage W_t and pays a payroll tax τ_t. If she is a new worker starting at period t, she carries no assets $a^w_{i,t-1} = 0$. She chooses between consumption $c^w_{i,t}$ and asset position $a^w_{i,t}$ with her income.

The consumption-saving choice for a retiree i, likewise, consists of choosing consumption $c^r_{i,t}$ and assets $a^r_{i,t}$ to solve

$$V^r_{i,t} = \max_{c^r_{i,t}, a^r_{i,t}} \left[\left(c^r_{i,t}\right)^\rho + \beta\gamma_{t+1}\left(V^r_{i,t+1}\right)^\rho\right]^{1/\rho}$$

subject to

$$c^r_{i,t} + a^r_{i,t} = \frac{R_t}{\gamma_t} a^r_{i,t-1} + W^r_t$$

or

$$c^r_{i,t} + a^r_{i,t} = \frac{R_t}{\gamma_t} a^w_{i,t-1} + W^r_t$$

if she retires in the current period.

A retiree does not work and survives with probability γ_{t+1} to next period. She receives a pension W^r_t when alive. The return on her investment is $\frac{R_{t+1}}{\gamma_{t+1}}$ where R_{t+1} is the interest rate. In essence, the retirees turn their wealth over to a perfectly competitive mutual fund industry which invests the proceeds and pays back a premium over the market return to compensate for the probability of death, as Blanchard (1985) and Yaari (1965).

In Appendix 3.A we present the complete solution to the two optimization problems with a guess and verify approach. The propensity of consumption out of total wealth turns out to be the same among the individuals of the same type, facilitating aggregation greatly.

The retiree's consumption is a fraction of ξ_t^r of the sum of wealth $\frac{R_t}{\gamma_t} a_{i,t-1}^r$ (or $\frac{R_t}{\gamma_t} a_{i,t-1}^w$ if she is just retired) and human capital H_t^r.

$$c_{i,t}^r = \xi_t^r \left(\frac{R_t}{\gamma_t} a_{i,t-1}^r + H_t^r \right)$$

The retiree's human capital is her pension plus the discounted future human capital.

$$H_t^r = W_t^r + \frac{\gamma_{t+1} H_{t+1}^r}{R_{t+1}}$$

The consumption propensity ξ_t^r evolves in the following way.

$$\frac{1}{\xi_t^r} = 1 + \gamma_t \beta^{1/(1-\rho)} R_{t+1}^{\rho/(1-\rho)} \frac{1}{\xi_{t+1}^r}$$

Likewise, the worker's consumption is a fraction ξ_t^w of the sum of wealth $R_t a_{i,t-1}^w$ and human capital H_t^w.

$$c_{i,t}^w = \xi_t^w \left(R_t a_{i,t-1}^w + H_t^w \right)$$

The worker's human capital is her after-tax wage plus the discounted future human capital as a worker or retiree, weighted by an adjusted probability accounting for the two different stages.

$$H_t^w = W_t^w (1 - \tau_t) + \frac{\left(\omega_{t+1} / \Omega_{t+1}^w \right) H_{t+1}^w + \left(1 - \omega_{t+1} / \Omega_{t+1}^w \right) H_{t+1}^r}{R_{t+1}}$$

where

$$\Omega_{t+1}^w = \omega_{t+1} + (1 - \omega_{t+1}) \left(\frac{\xi_{t+1}^w}{\xi_{t+1}^r} \right)^{1/\rho - 1}$$

The consumption propensity ξ_t^w evolves in the following way.

$$\frac{1}{\xi_t^w} = 1 + \beta^{1/(1-\rho)} \left(\Omega_{t+1}^w R_{t+1} \right)^{\rho/(1-\rho)} \frac{1}{\xi_{t+1}^w}$$

Denote $C_t^w, A_t^w, C_t^r, A_t^r$ the total consumption and asset position for the workers and retirees respectively. The total wealth of the current workers are those who remain workers from the last period $\omega_t R_t A_{t-1}^w$ and the total human capital for both the remaining and new workers $H_t^w L_t^w$. Given the same propensity of consumption, the law of motion for the aggregate consumption and asset position of the workers is the following.

$$C_t^w = \xi_t^w \left(\omega_t R_t A_{t-1}^w + H_t^w L_t^w \right)$$

$$A_t^w = \omega_t R_t A_{t-1}^w + W_t^w L_t^w (1 - \tau_t) - C_t^w$$

The total wealth of a retiree comprises three parts. The first part is the asset of the surviving retirees $\gamma_t A_{t-1}^r$ multiplying the interest rate R_t / γ_t from the life insurance scheme. The second part is the wealth from those workers who retire this period $(1-\omega) R_t A^w_{t-1}$. The third part is the human capital for the surviving retirees $H^r_t L^r_t$. Given the common propensity of consumption, the law of motion for the aggregate consumption and asset position of the retirees is the following.

$$C^r_t = \xi^r_t \left(R_t A^r_{t-1} + (1-\omega_t) R_t A^w_{t-1} + H^r_t \Psi_t L^w_t \right)$$

$$A^r_t = R_t A^r_{t-1} + (1-\omega_t) R_t A^w_{t-1} + W^r_t \Psi_t L^w_t - C^r_t$$

3.2.2 Automation

There is a representative firm. The final output Y_t is produced by combining the outputs from a continuum of measure one tasks $Y_{j,t}$, $j \in [0,1]$ with unit elasticity of substitution.

$$\ln Y_t = \int_0^1 \ln Y_{j,t} \mathrm{d}j$$

If the marginal cost of production for task j is $p_{j,t}$, using the final output as numeraire, the optimal task input is

$$Y_{j,t} = Y_t / p_{j,t} \tag{3.1}$$

and the cost satisfy

$$0 = \int_0^1 \ln p_{j,t}. \tag{3.2}$$

All the tasks are technologically automatable. A task can be produced either with human labor $l_{j,t}$ or machines $k_{j,t}$. It is worth noting that being technologically automatable is not identical to being actually automated. The firm might not choose to do so for economic reasons. The production function for task j is the following.

$$Y_{j,t} = \nu_j Z^L_t l_{j,t} + \eta_j Z^K_t k_{j,t}$$

The productivity of labor comprises a task-specific term ν_j and a common labor augmenting technology Z^L_t. Likewise, the productivity of capital comprises a task-specific term η_j and a common capital augmenting technology Z^K_t.

The firm hires labor at the market wage W_t and rents capital at the market rate r_t. The tasks are ranked such that the relative productivity of labor to capital ν_j/η_j is increasing in its index j. So, there exists a threshold J_t such that above the threshold, all the tasks are produced with labor, and below the threshold, all the tasks are produced with machines. The tasks produced with machines are actually automated. Formally,

$$Y_{j,t} = \begin{cases} \eta_j Z^K_t k_{j,t} & \text{if } j \leq J_t \\ \nu_j Z^L_t l_{j,t} & \text{if } j > J_t \end{cases} \tag{3.3}$$

and J_t solves

$$\frac{\nu_{J_t}}{\eta_{J_t}} = \frac{W_t/Z_t^L}{r_t/Z_t^K}. \tag{3.4}$$

J_t is increasing in $\frac{W_t/Z_t^L}{r_t/Z_t^K}$ because ν_j/η_j is increasing in j. Intuitively, if the cost of effective unit of labor (W_t/Z_t^L) increases relative to the usage cost of effective unit of capital (r_t/Z_t^K), the firm wants to replace labor with machines in more tasks. More tasks are automated if there is a scarcity in the effective unit of labor $(Z_t^L L_t)$ or an abundance in the effective units of capital $(Z_t^K K_t)$.

The marginal cost of production for each task j is

$$p_{j,t} = \begin{cases} \frac{r_t}{\eta_j Z_t^K} & \text{if } j \leq J_t \\ \frac{W_t}{\nu_j Z_t^L} & \text{if } j > J_t. \end{cases} \tag{3.5}$$

From the optimal task input (3.1), the optimal task production (3.3), and the marginal cost of production for each task (3.5), eliminate $p_{j,t}$ and $Y_{j,t}$ and integrate, we have the aggregate capital and labor used in production.

$$K_{t-1} = \frac{Y_t}{r_t} J_t \tag{3.6}$$

$$L_t = \frac{Y_t}{W_t}(1 - J_t) \tag{3.7}$$

The total capital available for production in current period is predetermined $\int_0^1 k_{j,t} = K_{t-1}$.

Eliminate $p_{j,t}, r_t, W_t$ from the price condition (3.2), the marginal cost of production for each task (3.5), the total capital and labor used (3.6) and (3.7), we have the aggregate production function in Cobb-Douglas form.

$$Y_t = Z_t K_{t-1}^{J_t} L_t^{1-J_t} \tag{3.8}$$

where

$$\ln Z_t = \int_0^{J_t} \ln\left(\frac{\eta_j Z_t^K}{J_t}\right) \mathrm{d}j + \int_{J_t}^1 \ln\left(\frac{\nu_j Z_t^L}{1 - J_t}\right) \mathrm{d}j. \tag{3.9}$$

The measure of tasks automated J_t coincides with the capital share. Automation has two interpretations in this model. At the micro-level, automation means more tasks produced with capital. At the macro-level, automation means higher capital share.

The TFP Z_t is a complicated function of the measure of tasks automated J_t following equation (3.9). However, it is not a perfect measure of productivity between production functions with different J_t, the measure of automation. From the firm's optimization problem, J_t is the optimal choice of automation. Therefore, $Z(J_t) K_{t-1}^{J_t} L_t^{1-J_t} > Z(J) K_{t-1}^{J} L_t^{1-J}, \forall J \neq J_t$. Even an alternative choice of J might increase the TFP term $Z(J) > Z(J_t)$, the output must be lower.

That all tasks are technologically automatable is our first simplification to Acemoglu and Restrepo (2018b). In their framework, the measure of technologically automatable tasks could also grow, creating an extensive margin of automation. However, the process is difficult to calibrate, and we leave it out.

From equation (3.4), the extent to which the measure of automated tasks J_t increases with the relative labor cost W_t/r_t (or more generally the relative effective cost, adjusted the by the relative common term of productivity Z_t^L/Z_t^K) depends on the distribution of the relative task-specific productivity for each tasks ν_j/η_j. The elasticity of automation can be defined as follows.

$$\frac{\partial J_t}{\partial (W_t/r_t)} \Bigg/ \frac{1-J_t}{W_t/r_t}$$

It captures the fraction of tasks to be automated out of the existing manual tasks $1-J_t$ for a marginal rise of the relative labor cost W_t/r_t.

The automation elasticity is a constant if the relative task-specific capital to labor productivity term has the functional form $\eta_j/\nu_j = (1-j)^{1/\alpha}$ where $\alpha > 0$. We assume so and will explain what it means.

From equation (3.4), the threshold is solved.

$$J_t = 1 - \left(\frac{W_t/Z_t^L}{r_t/Z_t^K}\right)^{-\alpha} \tag{3.10}$$

The elasticity of automation is the constant α, fixing the common labor and capital augmenting productivity Z_t^L and Z_t^K.

$$\frac{\partial J_t}{\partial (W_t/r_t)} \Bigg/ \frac{1-J_t}{W_t/r_t} = \alpha$$

To understand the functional assumption, think about two cases: ν_j is a constant or η_j is a constant. If $\nu_j = 1$, the measure of tasks with task-specific capital productivity below η is η^α and the density of tasks with task-specific capital productivity η is $\alpha\eta^{\alpha-1}$. For $\alpha < 1$, the density is higher for tasks with low task-specific capital productivity. Otherwise, the density is higher for tasks with high task-specific capital productivity. Alternatively, if $\eta_j = 1$, the measure of tasks with task-specific labor productivity below ν is $1-(1/\nu)^{1/\alpha}$. It is a Pareto distribution with a coefficient $1/\alpha$.

The functional assumption does not impose any restrictions on the functional form of ν_j or η_j itself. Following the assumption and using the equation (3.9), we have

$$\ln Z_t = J_t \ln\left(Z_t^K/J_t\right) + (1-J_t)\ln\left(Z_t^L/(1-J_t)\right) + 1/\alpha \int_0^{J_t} \ln(1-j)\,\mathrm{d}j + \int_0^1 \ln \nu_j \mathrm{d}j.$$

For any distribution of ν_j (which indirectly determines η_j from $\eta_j/\nu_j = (1-j)^{1/\alpha}$), it only enters the TFP term as a constant.

Importantly, the system is observationally equivalent to that assuming $\nu_j = 1$. To see this, we only need to normalize the common labor and capital augmenting productivity term by a time-invariant constant $\tilde{Z}_t^K = Z_t^K / e^{\int_0^1 \ln \nu_j \mathrm{d}j}$ and $\tilde{Z}_t^L = Z_t^L / e^{\int_0^1 \ln \nu_j \mathrm{d}j}$. The TFP term becomes

$$\ln Z_t = J_t \ln \left(\tilde{Z}_t^K / J_t \right) + (1 - J_t) \ln \left(\tilde{Z}_t^L / (1 - J_t) \right) + 1/\alpha \int_0^{J_t} \ln (1 - j) \, \mathrm{d}j. \quad (3.11)$$

For ease of notation, we omit the tilde as long as it does not cause any confusion.

The functional form assumption is our second simplification to Acemoglu and Restrepo (2018b) framework. We make this assumption in order to bring their theoretical framework to numerical analysis.

Three equations can summarize the representative firm's aggregate production function with automation: the Cobb-Douglas production function (3.8), the automation channel (3.10) and the TFP expression (3.11).

If the automation channel is shut down by imposing a further constraint $J_t = J$ on the firm, the aggregate production function is still Cobb-Douglas with expression (3.8), (3.11) and the binding constraint $J_t = J$. It degenerates to a conventional specification of the production function.

3.2.3 Pension and market clearing conditions

The government runs a balanced budget to finance the repayment of debt $R_t B_{t-1}$and the pension $W_t^r \Psi_t L_t^w$ obligation from the issuance of new bond B_t and payroll tax τ_t on the wage income[5] from the workers $W_t L_t^w$.

$$B_t + \tau_t W_t L_t^w = R_t B_{t-1} + W_t^r \Psi_t L_t^w$$

The government maintains its debt at a fixed fraction of GDP.

$$B_t = \zeta^B Y_t$$

A retiree receives the state pension as a fixed fraction of the worker's net wage. ζ^r is the net pension replacement rate.

$$W_t^r = \zeta^r (1 - \tau_t) W_t^w$$

[5] It is identical to a lump sum tax as the labor supply is exogenous.

The asset market clears[6].

$$A_t^w + A_t^r = K_t + B_t$$

By Walras Law, the goods market clears.

$$Y_t = C_t^w + C_t^r + G_t + K_t - (1-\delta) K_{t-1}$$

The equation system that characterizes the equilibrium is summarized in Appendix 3.B.

3.3 Calibration

The model is calibrated to the US annually. The depreciation rate δ is 10%. The coefficient ρ is set to -1, and this makes the inter-temporal elasticity of substitution to be 0.5. Low value like this is standard in the public finance literature.

The elasticity of automation is calibrated indirectly through the elasticity of substitution between capital and labor. From the transformation $(\mathrm{d}J)/(1-J) = J\mathrm{d}\ln((1-J)/J)$ and the Cobb-Douglas production function property $(1-J_t)/J_t = (W_tL_t)/(r_tK_t)$, the elasticity of automation relates to the elasticity of the substitution between capital and labor in the following way.

$$\alpha = J_t(\sigma_t - 1)$$

where σ_t is the elasticity of substitution $\sigma_t \equiv \partial \ln(L_t/K_t)/\partial \ln(r_t/W_t)$.

The functional assumption requiring $\alpha > 0$ is only consistent with a larger than unit elasticity of substitution between capital and labor $\sigma > 1$. For the purpose of calibrating the automation elasticity indirectly, the proper elasticity of substitution between capital and labor σ should be the long-run elasticity, and the capital should include highly labor-substitutable robots. It could be larger than many estimates of σ in the literature. Nonetheless, there are a few recent papers conclude that the elasticity is larger than one $\sigma > 1$. Karabarbounis and Neiman (2013) estimates $\sigma \approx 1.2 \sim 1.4$. The median 1.3 is used in the numerical analysis. The corresponding automation elasticity $\alpha = 40\% \times (1.3 - 1)$ as the US capital share is 40%.

The US net pension replacement rate ζ^r is 47%. The national debt to GDP ratio ζ^B is set to 33%. While at a first glance the US national debt to GDP is close to 100% as of 2018, about 2/3 were held by the Fed, US government and foreigners.

The demographic transition lasts 50 years. Initially, the probability of retiring $1-\omega$ is 1/45. There are 45 years of working on average. If an average worker starts working at 20, he/she retires at the age of 65, which is typical for the US. The

[6] There will be missing one equation in the system using the (current period) asset market clearing condition rather than the goods market-clearing condition

mortality rate $1-\gamma$ is calibrated to 1/15. The worker lives for another 15 years on average after retirement. This makes the total life to be 80, which is slightly higher than the recent US life-expectancy 79. The sum of newborns (directly to working age) and immigrants are set to $1-\omega+1\%$ of the worker population. This leads to a steady-state population growth rate of 1% as in the past decades. In the end of next 50 years, the probability of retiring is unchanged. But the mortality rate $1-\gamma$ becomes 1/20. On average, there are 5 years more life. The sum of newborns and immigrants are set to $1-\omega+0.1\%$, so the worker population only grows at 0.1% annually. The transition is smooth such that the value of ω, γ, n changes linearly over the 50 years.

The growth rate of the labor augmenting technology (Z_t^L) and the capital augmenting technology (Z_t^K) are calibrated to yield a 2% steady-state output per capita growth if the capital share is fixed at 40%. The capital augmenting technology growth rate is set to be 1%. Then the labor augmenting technology growth is 1.33%. This means that capital-augmenting technology growth contributes to about 1/3 of the overall growth. It is a conservative number consistent with the low end of the estimated contribution of the investment-specific technological progress to the overall growth in Greenwood and Krusell (2007).

The model does not admit a balanced growth path unless the automation channel is shut down. To approximate the initial (detrended) capital stock and asset position, we assume the economy starts from the detrended steady state of an artificial economy without automation. In the artificial economy, the representative firm faces an extra constraint $J_t = 40\%$, which is the recent value for the US, and the demographic coefficients are at the initial level as aforementioned, which are also the recent value for the US. The discount factor $\beta = 0.998$ is then calibrated by targeting a 4% annual real interest rate in the US on the balanced growth path[7]. The initial level of the relative size of the capital and labor augmenting technology $Z_1^K/Z_1^L = 3.92$ is calibrated to target the 40% capital share on the balanced growth path.

The calibration is summarized in Table 3.1.

[7] A value β close to unity is consistent with the micro evidence in Hurd (1990).

Table 3.1: Calibration

Parameter	Value	Description
δ	10%	Depreciation rate
$1/(1-\rho)$	0.5	Intertemporal Elasticity of substitution
ζ^r	47%	Net pension replacement rate
ζ^B	33%	US national debt to GDP, by US private investors
g^{Z^K}	1%	Capital-augmenting technology growth rate
g^{Z^L}	1.33%	Labor-augmenting technology growth rate
α	0.12	Elasticity of automation
$1-\omega$	1/45	Probability of retiring
$1-\gamma_t$	$1/15 \rightarrow 1/20$	Mortality rate
$\omega + n_t - 1$	$1\% \rightarrow 0.1\%$	Growth rate of the worker population
β	0.998	Discount rate, targeting $R = 1.04$
Z_1^K/Z_1^L	3.92	Initial relative productivity, targeting $J = 40\%$

3.4 Numerical Analysis

The perfect foresight model is solved globally using a slightly modified relaxation algorithm, with details in Appendix 3.D. Figure 3.1 shows the transition dynamics over 50 years driven by aging, capital-augmenting technology progress and labor-augmenting technology progress. The blue solid lines are for the aging economy with automation. For comparison, the red dashed lines are for the aging economy with the automation channel shut down by imposing an additional constraint $J_t = 40\%$ for the representative firm's problem. The upper panel presents the transition dynamics of the real interest rate, capital share, pension obligation as a fraction of GDP. The lower panel presents the log output per capita, log capital stock per capita, and log wage, all removing a 2% trend.

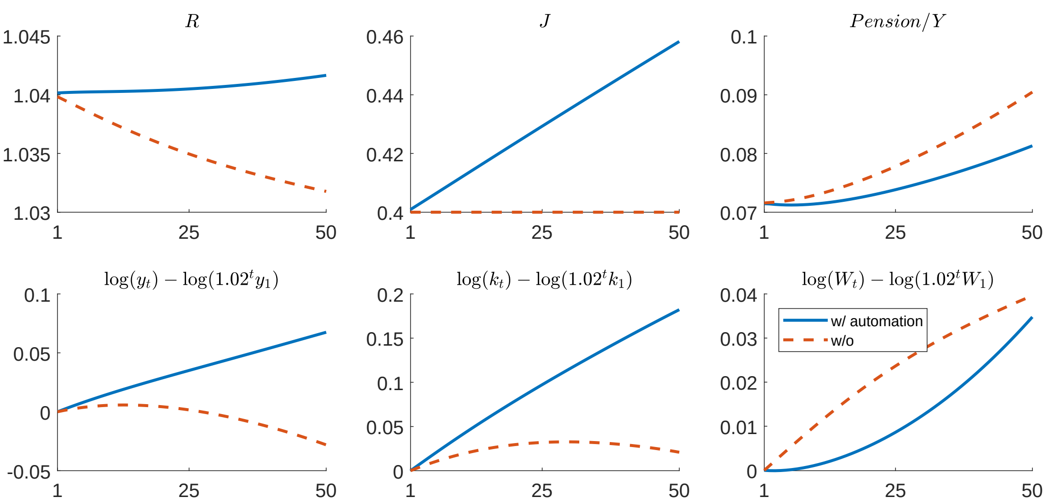

Figure 3.1: Transition dynamics

For the aging economy without automation (shown in red dashed lines), the interest rate falls by slightly less than 1 percentage point over 50 years, driven by the desire to save for the longer retirement. The capital share is fixed as constant by definition. The pension obligation rises by more than 2 percentage points of GDP as there are more pension receivers relative to workers from population aging over time. The output per capita grows a little bit faster than 2% initially while eventually falls below. The same is true for the capital. There are two counteracting forces here. Each worker is expected to live longer, so they would like to increase investment, which boost capital per worker and output per worker. However, the retiree population increases over time relative to the worker population. It drives down per capita values of output and capital. The second force only kicks in gradually. Therefore, the output and capital per capita rises above trend initially and drops below later. The wage rises by about 4 percentage points above trend over 50 years. The upward trend comes from the fact that longevity and low birth rate lead to higher capital to labor ratio, and hence, labor becomes relatively scarce.

Taking automation into consideration (shown in blue solid lines) yields different transition dynamics. The aging and capital-augmenting technology change induce the firm to replace labor with machines in more tasks. Capital share is 6 percentage points higher. As capital is more useful, the interest rate is 1 percentage point higher. This induces more investments and the capital per capita is 20 percentage points higher. Output per capita is 10 percentage points higher with more capital and more efficient usage of the shrunk labor force in the unautomated tasks. Wage is less than 1 percentage point lower with automation as labor is less scarce. The pension obligation as a fraction of GDP is 1 percentage points lower as the economy is larger.

The differences are quantitatively relevant. By incorporating automation, the interest rate increases slightly, suggesting that aging with automation cannot explain much about a secular fall in the real interest rate. The increase of the capital share in the next 50 years is even larger in magnitude than the 4 percentage points over the past 50 years. Dividing the 10-percentage point difference in output per capita by 50 years, missing automation underestimates the output per capita annual growth rate by 0.2 percentage points. Likewise, the capital per capita annual growth rate is underestimated by 0.4 percentage points leaving out automation.

The capital-augmenting technology growth and aging reinforce each other in driving the difference. Figure 3.2 plots for either channel the log of output per capita deviation of the economy modeling automation from that not modeling. The capital augmenting technology growth alone leads to 7 percentage points higher output per capita over the 50 years, as shown in the red dashed line. Automation allows the economy to exploit more of the productivity gain of capital by using it more intensively in more tasks. Aging alone leads to less than 1 percentage point output per capita difference, as shown in the yellow dotted line. Combining the two

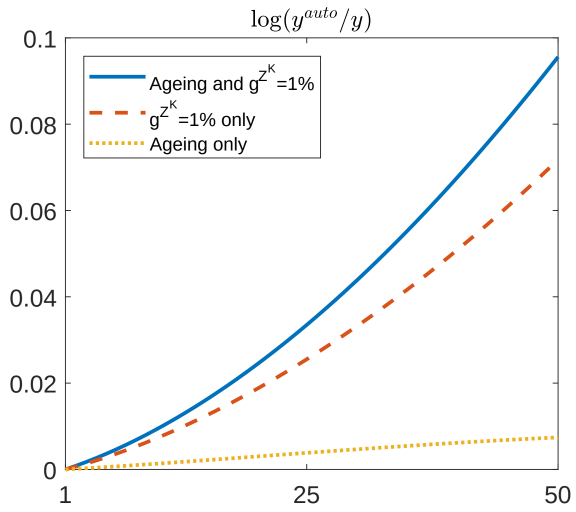

Figure 3.2: Effect of automation for different exogenous shocks

channels generates a larger effect than the sum of the two effects. On the one hand, aging further increases the degree of automation, making the output more sensitive to capital augmenting technology. On the other hand, automation driven by capital augmenting technology growth further mitigates the labor shortage from aging and improves output.

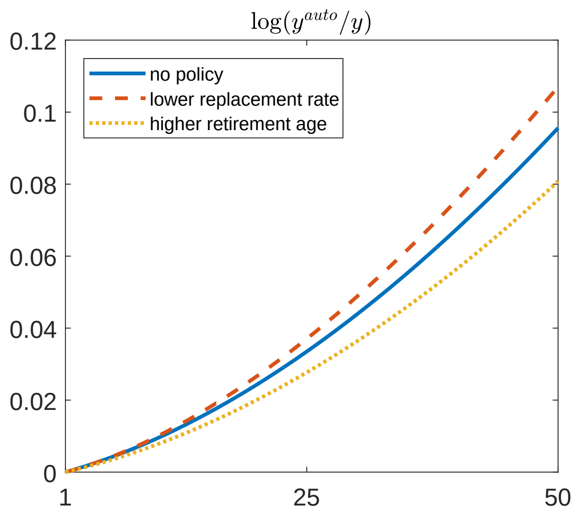

Figure 3.3: Effect of automation for different policies

The rising pension obligation motivates policies such as the transition from the pay as you go system to a fully funded system or an increase in the retirement age. Figure 3.3 plots for the two different policies the log of output per capita deviation of the economy modeling automation from that not modeling. More specifically, the moving from the pay as you go system to the fully funded system is modeled as a decline in the replacement rate of pension financed by the payroll tax. The replacement rate decreases to half of the original linearly over 50 years. In our calibration, it decreases from 40% to 20%. It results in higher savings by the household. As we do not differentiate between the government saving on behalf of the household and the household saving directly, it could also be interpreted as a partial replacement by the fully funded pension system. The rising mandatory retirement age is modeled with a decrease in the probability of retirement $1 - \omega$ and an increase in the probability of mortality $1 - \gamma$, while keeping the average total working and retirement life $1/(1-\omega) + 1/(1-\gamma)$ constant. The policy increases ω linearly from $1 - 1/45$ to $1 - 1/50$ while keeping the years spent in retirement fixed.

The partial transition from the pay as you go to the fully funded pension system results in 1 percentage point higher output per capita deviation from automation. By contrast, the higher mandatory retirement age narrows the deviation by 1 percentage point. This is because the first policy increases the saving demand and drives up automation by 1 more percentage point, and the second reduces the saving demand so that it slows down automation slightly. Modeling automation is thus also relevant for policy evaluations with regard to aging.

The results are robust to different specification of the elasticity of automation and the capital augmenting technology growth rate. Roughly speaking, cut the elasticity of automation by half or cut the capital augmenting technology growth rate by half leads to about half the effects. They are still quantitatively relevant.

3.5 Conclusion

This chapter provides a convenient framework to combine rich demographic transitions with microfounded automation. We find that allowing automation to react endogenously to demographic and productivity changes is quantitatively relevant for economic forecasting. Aging research missing the automation channel can underestimate economic growth and overestimate the fiscal burden. Automation research missing aging can overstate the impact on wage. This framework may also serve as a laboratory for policy evaluations.

Several directions of further research appear fruitful. First, an elastic labor supply can be incorporated into the model. This allows a sharper analysis of the net impact on employment. Second, the automation elasticity is a key parameter worth further exploration. From the theoretical side, allowing the elasticity of substitution between task outputs to be different than one leads to an aggregate production function of the CES form. The elasticity of automation can be calibrated separately to the elasticity of substitution. From the empirical side, direct identification of the automation elasticity is yet to be done.

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Appendices

3.A Solution to the demographic block

The retiree's problem

The first-order condition with respect to asset accumulation for a retiree is

$$\left(c_{i,t}^{r}\right)^{\rho-1}=\beta\gamma\left(V_{i,t+1}^{r}\right)^{\rho-1}\frac{\partial V_{i,t+1}^{r}}{\partial a_{i,t}^{r}}.$$

The envelope condition is

$$\frac{\partial V_{i,t}^{r}}{\partial a_{i,t-1}^{r}}=\left(V_{i,t}^{r}\right)^{1-\rho}\left(c_{i,t}^{r}\right)^{\rho-1}\frac{R_t}{\gamma}.$$

The resulting Euler equation takes the standard form

$$c_{i,t}^{r}=\left(\beta R_{t+1}\right)^{-\epsilon}c_{i,t+1}^{r},$$

where $\epsilon\equiv(1-\rho)^{-1}$ is the elasticity of intertemporal substitution.

Guess that the consumption is a fraction of total wealth,

$$c_{i,t}^{r}=\xi_t^{r}\left(\frac{R_t}{\gamma}a_{i,t-1}^{r}+H_t^{r}\right),$$

where H_t^r represent the present discounted value of current and future pension that is independent of individual-specific characteristics:

$$H_t^{r}=W_t^{r}+\frac{\gamma H_{t+1}^{r}}{R_{t+1}}.$$

Substitute the guess into the Euler equation yields a law of motion for the marginal propensity to consume of a retiree ξ_t^r,

$$\xi_t^{r}\left(\frac{R_t}{\gamma}a_{i,t-1}^{r}+H_t^{r}\right)=\left(\beta R_{t+1}\right)^{-\epsilon}\xi_{t+1}^{r}\left(\frac{R_{t+1}}{\gamma}a_{i,t}^{r}+H_{t+1}^{r}\right).$$

Substitute the guess into the budget constraint of a retiree and use the recursive expression of H_t^r to eliminate W_t^r leads to express assets as

$$a_{i,t}^{r}+\frac{\gamma H_{t+1}^{r}}{R_{t+1}}=(1-\xi_{t}^{r})\left(\frac{R_t}{\gamma}a_{i,t-1}^{r}+H_{t}^{r}\right).$$

Combining the last two expressions gives a non-linear first-order difference equation for the marginal propensity to consume of the form

$$\frac{1}{\xi_t^r}=1+\gamma\beta^{\epsilon}R_{t+1}^{\epsilon-1}\frac{1}{\xi_{t+1}^{r}}.$$

Moreover, guess that the value function is linear in consumption according to

$$V_{i,t}^{r}=\Delta_t^r c_{i,t}^{r}.$$

Then, from the Bellman equation,

$$\left(\Delta_t^r c_{i,t}^r\right)^{\rho}=\left(c_{i,t}^{r}\right)^{\rho}+\gamma\beta\left(\Delta_{t+1}^{r}c_{i,t+1}^{r}\right)^{\rho}.$$

Substituting for the consumption Euler equation and simplifying yields

$$\left(\Delta_t^r\right)^{\rho}=1+\gamma\beta^{\epsilon}R_{t+1}^{\epsilon-1}\left(\Delta_{t+1}^{r}\right)^{\rho}.$$

From the difference equation of consumption propensity,

$$\xi_t^r=\left(\Delta_t^r\right)^{-\rho}.$$

The worker's problem

The first-order condition with respect to asset accumulation for a worker is

$$\left(c_{i,t}^{w}\right)^{\rho-1}=\beta\left[\omega V_{i,t+1}^{w}+(1-\omega)V_{i,t+1}^{r}\right]^{\rho-1}\left(\omega\frac{\partial V_{i,t+1}^{w}}{\partial a_{i,t}^{w}}+(1-\omega)\frac{\partial V_{i,t+1}^{r}}{\partial a_{i,t}^{w}}\right).$$

The envelope conditions are

$$\frac{\partial V_{i,t}^{w}}{\partial a_{i,t-1}^{w}}=\left(V_{i,t}^{w}\right)^{1-\rho}\left(c_{i,t}^{w}\right)^{\rho-1}R_t$$

and

$$\frac{\partial V_{i,t}^{r}}{\partial a_{i,t-1}^{w}}=\left(V_{i,t}^{r}\right)^{1-\rho}\left(c_{i,t}^{r}\right)^{\rho-1}R_t.$$

The return for a new retiree is R_t rather than $\frac{R_t}{\gamma^a}$.

Guess that the old worker's value function has the same form

$$V_{i,t}^{w}=\Delta_t^w c_{i,t}^{w}.$$

The Euler equation then becomes

$$\left(\beta\Omega_{t+1}^{w}R_{t+1}\right)^{\epsilon}c_{i,t}^{w}=\omega c_{i,t+1}^{w}+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)c_{i,t+1}^{r}.$$

where

$$\Omega_{t+1}^{w}=\omega+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)^{1-\rho}.$$

Substitute the guesses into the Bellman equation

$$\left(\Delta_{t}^{w}c_{i,t}^{w}\right)^{\rho}=\left(c_{i,t}^{w}\right)^{\rho}+\beta\left[\omega\Delta_{t+1}^{w}c_{i,t+1}^{w}+(1-\omega)\Delta_{t+1}^{r}c_{i,t+1}^{r}\right]^{\rho}.$$

Combining it with the Euler equation gives

$$\left(\Delta_{t}^{w}\right)^{\rho}=1+\beta^{\epsilon}\left(\Omega_{t+1}^{w}R_{t+1}\right)^{\epsilon-1}\left(\Delta_{t+1}^{w}\right)^{\rho}.$$

Guess that consumption of the worker is also a fraction of total wealth.

$$c_{i,t}^{w}=\xi_{t}^{w}\left(R_{t}a_{i,t-1}^{w}+H_{t}^{w}\right)$$

where H_t^w represent the present discounted value of current and future human wealth that is independent of individual-specific characteristics:

$$H_{t}^{w}=W_{t}^{w}\left(1-\tau_{t}\right)+\frac{\omega H_{t+1}^{w}+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)^{1-\rho}H_{t+1}^{r}}{\Omega_{t+1}^{w}R_{t+1}}.$$

Substitute the guesses and the decision rule for a retiree into the Euler equation gives

$$\begin{aligned}
&\left(\beta\Omega_{t+1}^{w}R_{t+1}\right)^{\epsilon}\xi_{t}^{w}\left(R_{t}a_{i,t-1}^{w}+H_{t}^{w}\right)\\
&=\omega\xi_{t+1}^{w}\left(R_{t+1}a_{i,t}^{w}+H_{t+1}^{w}\right)+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)\left(\Delta_{t+1}^{r}\right)^{-\rho}\left(R_{t+1}a_{i,t}^{w}+H_{t+1}^{r}\right)\\
&=\Omega_{t+1}^{w}R_{t+1}\xi_{t+1}^{w}\left(a_{i,t}^{w}+\frac{\omega H_{t+1}^{w}+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)^{1-\rho}H_{t+1}^{r}}{\Omega_{t+1}^{w}R_{t+1}}\right)\\
&+(1-\omega)\left(\frac{\Delta_{t+1}^{r}}{\Delta_{t+1}^{w}}\right)^{1-\rho}\left(\left(\Delta_{t+1}^{w}\right)^{-\rho}-\xi_{t+1}^{w}\right)\left(R_{t+1}a_{i,t}^{w}+H_{t+1}^{r}\right),
\end{aligned}$$

where the definition of Ω_{t+1}^{w} is used in the last equation.

Substitute the guesses into the budget constraint gives

$$a^w_{i,t} + \frac{\omega H^w_{t+1} + (1-\omega)\left(\frac{\Delta^r_{t+1}}{\Delta^w_{t+1}}\right)^{1-\rho} H^r_{t+1}}{\Omega^w_{t+1} R_{t+1}} = (1-\xi^w_t)\left(R_t a^w_{i,t-1} + H^w_t\right).$$

Combining the last two equations,

$$\frac{1}{\xi^w_t} = 1+\beta^\epsilon \left(\Omega^w_{t+1}R_{t+1}\right)^{\epsilon-1} \frac{1}{\xi^w_{t+1}} - \frac{(1-\omega)\left(\frac{\Delta^r_{t+1}}{\Delta^w_{t+1}}\right)^{1-\rho}\left(\left(\Delta^w_{t+1}\right)^{-\rho} - \xi^w_{t+1}\right)\left(R_{t+1}a^w_{i,t} + H^r_{t+1}\right)}{\Omega^w_{t+1}R_{t+1}\xi^w_{t+1}\xi^w_t\left(R_t a^w_{i,t-1} + H^w_t\right)}.$$

Comparing with the difference equation of Δ^w_t, the propensity of consumption is simply

$$\xi^w_t = (\Delta^w_t)^{-\rho}.$$

3.B System of equations

Given exogenous Z^K_t, Z^L_t, an equilibrium is characterized by the following 22 equations.

The workers

$$C^w_t = \xi^w_t\left(\omega R_t A^w_{t-1} + H^w_t L^w_t\right) \tag{3.12}$$

$$A^w_t = \omega R_t A^w_{t-1} + W^w_t L^w_t (1-\tau_t) - C^w_t \tag{3.13}$$

$$H^w_t = W^w_t (1-\tau_t) + \frac{\omega H^w_{t+1} + (1-\omega)\left(\frac{\Delta^r_{t+1}}{\Delta^w_{t+1}}\right)^{1-\rho} H^r_{t+1}}{\Omega^w_{t+1}R_{t+1}} \tag{3.14}$$

$$\Omega^w_{t+1} = \omega + (1-\omega)\left(\frac{\Delta^r_{t+1}}{\Delta^w_{t+1}}\right)^{1-\rho} \tag{3.15}$$

$$(\Delta^w_t)^\rho = 1 + \beta^\epsilon\left(\Omega^w_{t+1}R_{t+1}\right)^{\epsilon-1}\left(\Delta^w_{t+1}\right)^\rho \tag{3.16}$$

$$\xi^w_t = (\Delta^w_t)^{-\rho} \tag{3.17}$$

The retirees

$$C^r_t = \xi^r_t\left(R_t A^r_{t-1} + (1-\omega) R_t A^w_{t-1} + H^r_t \Psi_t L^w_t\right) \tag{3.18}$$

$$A^r_t = R_t A^r_{t-1} + (1-\omega) R_t A^w_{t-1} + W^r_t \Psi_t L^w_t - C^r_t \tag{3.19}$$

$$H^r_t = W^r_t + \frac{\gamma H^r_{t+1}}{R_{t+1}} \tag{3.20}$$

$$(\Delta^r_t)^\rho = 1 + \gamma\beta^\epsilon R^{\epsilon-1}_{t+1}\left(\Delta^r_{t+1}\right)^\rho \tag{3.21}$$

$$\xi^r_t = (\Delta^r_t)^{-\rho} \tag{3.22}$$

The firms

$$R_{t+1} = Z_{t+1} J_{t+1} \left(K_t\right)^{J_{t+1}-1} \left(L_{t+1}^w\right)^{1-J_{t+1}} + 1 - \delta \tag{3.23}$$

$$r_t = Z_t J_t \left(K_{t-1}\right)^{J_t - 1} \left(L_t^w\right)^{1-J_t} \tag{3.24}$$

$$W_t^w = Z_t \left(1 - J_t\right) \left(K_{t-1}\right)^{J_t} \left(L_t^w\right)^{-J_t} \tag{3.25}$$

$$Y_t = Z_t K_{t-1}^{J_t} \left(L_t^w\right)^{1-J_t} \tag{3.26}$$

The automation

$$J_t = 1 - \left(\frac{W_t / Z_t^L}{r_t / Z_t^K}\right)^{-\alpha} \tag{3.27}$$

$$\ln Z_t = J_t \ln\left(\frac{Z_t^K}{J_t}\right) + \left(1 - J_t\right) \ln\left(\frac{Z_t^L}{1 - J_t}\right) - \frac{1}{\alpha}\left(\left(1 - J_t\right) \ln\left(1 - J_t\right) + J_t\right) \tag{3.28}$$

The government budget constraint

$$B_t = R_t B_{t-1} + W_t^r \Psi_t L_t^w - \tau_t W_t^w L_t^w \tag{3.29}$$

$$B_t = \zeta^B Y_t \tag{3.30}$$

$$W_t^r = \zeta^r \left(1 - \tau_t\right) W_t^w \tag{3.31}$$

The resource constraint

$$Y_t = C_t^w + C_t^r + K_t - \left(1 - \delta\right) K_{t-1} \tag{3.32}$$

The labor market dynamics

$$\left(\omega + n\right) \Psi_t = \gamma \Psi_{t-1} + \left(1 - \omega\right) \tag{3.33}$$

3.C Existence of a balanced growth path

There exists a balanced growth path if $g_t^{Z^K} \to 0$ asymptotically. To see this, from equation (3.27), (3.28), (3.24), (3.25)

$$g^W = g^{Z^L} - g^{Z^K}$$

$$g^Z = J g^{Z^K} + \left(1 - J\right) g^{Z^L}$$

$$g^Z = \left(1 - J\right) \left(g^K - g^L\right)$$

$$g^W = g^Z + J \left(g^K - g^L\right)$$

So,

$$g^Z = 0$$

$$g^W = g^K - g^L = g^{Z^L}$$

$$g^Z = (1-J)\, g^{Z^L}$$

If a balanced growth path exists, we can detrend the variables so that they converge to a steady-state. More specifically, define the effective labor augmenting productivity $\mathcal{Z}_t \equiv (Z_t)^{1/(1-J_t)}$. Denote variables with a check the variables detrended by the number of workers$\check{X}_t \equiv X_t/Z_t^L$. Denote variables with a hat the variables detrended by the number of workers and the labor augmenting productivity growth $\hat{X}_t \equiv X_t/\left(Z_t^L L_t^w\right)$. The equilibrium is characterized alternatively by the 22 detrended variables $\hat{C}_t^w, \hat{A}_t^w, \xi_t^w, \Delta_t^w, \Omega_t^w,\ \hat{C}_t^r, \hat{A}_t^r, \xi_t^r, \Delta_t^r,\ R_t, r_t, \check{W}_t^w, \check{W}_t^r, \check{H}_t^w, \check{H}_t^r,$ $\hat{K}_t, \Psi_t, \hat{Y}_t, \hat{B}_t, \tau_t,\ J_t, \check{Z}_t^{\mathcal{L}}$ that satisfy the following 22 equations.

The workers

$$\hat{C}_t^w = \xi_t^w \left(\frac{\omega_t R_t \hat{A}_{t-1}^w}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \check{H}_t^w \right) \tag{3.34}$$

$$\hat{A}_t^w = \frac{\omega_t R_t \hat{A}_{t-1}^w}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \check{W}_t^w (1-\tau_t) - \hat{C}_t^w \tag{3.35}$$

$$\check{H}_t^w = \check{W}_t^w (1-\tau_t) + \frac{\omega_{t+1}\check{H}_{t+1}^w + (1-\omega_{t+1})\left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho} \check{H}_{t+1}^r}{\Omega_{t+1}^w R_{t+1}} \left(1+g^{Z^L}\right) \tag{3.36}$$

$$\Omega_{t+1}^w = \omega_{t+1} + (1-\omega_{t+1}) \left(\frac{\Delta_{t+1}^r}{\Delta_{t+1}^w}\right)^{1-\rho} \tag{3.37}$$

$$(\Delta_t^w)^\rho = 1 + \beta^\epsilon \left(\Omega_{t+1}^w R_{t+1}\right)^{\epsilon-1} \left(\Delta_{t+1}^w\right)^\rho \tag{3.38}$$

$$\xi_t^w = (\Delta_t^w)^{-\rho} \tag{3.39}$$

The retirees

$$\hat{C}_t^r = \xi_t^r \left(\frac{R_t \hat{A}_{t-1}^r}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \frac{(1-\omega_t) R_t \hat{A}_{t-1}^w}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \check{H}_t^r \Psi_t \right) \tag{3.40}$$

$$\hat{A}_t^r = \frac{R_t \hat{A}_{t-1}^r}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \frac{(1-\omega_t) R_t \hat{A}_{t-1}^w}{\left(1+g^{Z^L}\right)(\omega_t + n_t)} + \check{W}_t^r \Psi_t - \hat{C}_t^r \tag{3.41}$$

$$\check{H}_t^r = \check{W}_t^r + \frac{\gamma_{t+1} \check{H}_{t+1}^r \left(1+g^{Z^L}\right)}{R_{t+1}} \tag{3.42}$$

$$(\Delta_t^r)^\rho = 1 + \gamma_{t+1} \beta^\epsilon R_{t+1}^{\epsilon-1} \left(\Delta_{t+1}^r\right)^\rho \tag{3.43}$$

$$\xi_t^r = (\Delta_t^r)^{-\rho} \tag{3.44}$$

The firms

$$R_{t+1} = J_{t+1}\left(\frac{\hat{K}_t}{\left(1+g^{Z^L}\right)\left(\omega_{t+1}+n_{t+1}\right)}\right)^{J_{t+1}-1}\left(\check{\mathcal{Z}}_{t+1}\right)^{1-J_{t+1}} + 1 - \delta \tag{3.45}$$

$$r_t = J_t\left(\frac{\hat{K}_{t-1}}{\left(1+g^{Z^L}\right)\left(\omega_t+n_t\right)}\right)^{J_t-1}\left(\check{\mathcal{Z}}_t\right)^{1-J_t} \tag{3.46}$$

$$\check{W}_t^w = \left(1-J_t\right)\left(\frac{\hat{K}_{t-1}}{\left(1+g^{Z^L}\right)\left(\omega_t+n_t\right)}\right)^{J_t}\left(\check{\mathcal{Z}}_t\right)^{1-J_t} \tag{3.47}$$

$$\hat{Y}_t = \left(\frac{\hat{K}_{t-1}}{\left(1+g^{Z^L}\right)\left(\omega_t+n_t\right)}\right)^{J_t}\left(\check{\mathcal{Z}}_t\right)^{1-J_t} \tag{3.48}$$

The automation

$$J_t = 1 - \left(\frac{\check{W}_t^w}{r_t/Z_t^K}\right)^{-\alpha} \tag{3.49}$$

$$\left(1-J_t\right)\ln\check{\mathcal{Z}}_t = J_t \ln\left(\frac{Z_t^K}{J_t}\right) + \left(1-J_t\right)\ln\left(\frac{1}{1-J_t}\right) - \frac{1}{\alpha}\left(\left(1-J_t\right)\ln\left(1-J_t\right)+J_t\right) \tag{3.50}$$

The government budget constraint

$$\hat{B}_t = \frac{R_t\hat{B}_{t-1}}{g^{Z^L}\left(\omega_t+n_t\right)} + \check{W}_t^r\Psi_t - \tau_t\check{W}_t^w \tag{3.51}$$

$$\hat{B}_t = \zeta^B\hat{Y}_t \tag{3.52}$$

$$\check{W}_t^r = \zeta^r\left(1-\tau_t\right)\check{W}_t^w \tag{3.53}$$

The resource constraint

$$\hat{Y}_t = \hat{C}_t^w + \hat{C}_t^r + \hat{K}_t - \frac{\left(1-\delta\right)\hat{K}_{t-1}}{\left(1+g^{Z^L}\right)\left(\omega_t+n_t\right)} \tag{3.54}$$

The labor market dynamics

$$\left(\omega_t+n_t\right)\Psi_t = \gamma_t\Psi_{t-1} + \left(1-\omega_t\right) \tag{3.55}$$

3.D Solution algorithm

The equilibrium can be solved globally by the relaxation algorithm. For perfect foresight model, the equation system such as 3.B can be summarized as $f_t\left(X_{t+1}, X_t, X_{t-1}\right) = 0$ where X_t is the vector of endogenous variables at time t. If X_1 and X_{T+1} are known, the full transition paths X_2 to X_T can be solved from the large equation system $f_t(X_{t+1}, X_t, X_{t-1}) = 0$ for $t = 2$ to T. There are efficient ways to solve the large equation system. X_{T+1} is known for large T if the system converges to a steady-state or a balanced growth path.

In our original problem $g_t^{Z^K} = g^{Z^K} > 0, \forall t$. A balanced growth path does not exist according to section 3.C. To use the relaxation method, we solve an alternative problem such that for large T the capital augmenting technology stops growing $g_t^{Z^K} = 0, \forall t \geq T+1$. A balanced growth path exists for the alternative problem. The solution to the alternative problem using the relaxation method can be arbitrarily close to the original problem if T is large enough.