Rough Volatility and Portfolio Optimisation under Small Transaction Costs

A thesis submitted for the degree of Doctor of Philosophy

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of 104 pages.

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Abstract

The first chapter of the thesis presents the study of the linear-quadratic ergodic control problem of fractional Brownian motion. Ergodic control problems arise naturally in the context of small cost asymptotic expansion of utility maximisation problems with frictions. The optimal solution to the ergodic control problem is derived through the use of an infinite dimensional Markovian representation of fractional Brownian motion as a superposition of Ornstein-Uhlenbeck processes. This solution then allows to compute explicit formulas for the minimised objective value through the variance of the stationary distribution of the Ornstein-Uhlenbeck processes.

Building on the first chapter, the second chapter of the thesis presents the main result. This is motivated by the problem an agent faces when trying to minimise her utility loss in the presence of quadratic trading costs in a rough volatility model. Minimising the utility loss amounts to studying a tracking problem of a target that depends on the rough volatility process. This tracking problem is minimised at leading order by an asymptotically optimal strategy that is closely linked to the ergodic control problem of fractional Brownian motion. This asymptotically optimal strategy is explicitly derived. Moreover, the leading order of the small cost expansion is shown to depend only on the roughest part of the considered target. It therefore depends on the Hurst parameter.

The third chapter is devoted to a numerical analysis of the utility loss studied in the second chapter. For this, we compare the utility loss in a rough volatility model to a semimartingale stochastic volatility model. The parameter values for both models are fitted to match frictionless utility for realistic values. By applying the result obtained in the second chapter of the thesis, the difference between leading order of utility loss can be explicitly compared.

Contents

Introduction

Motivation and Related Literature

Transaction costs such as bid-ask spreads or price impact are a key obstacle for the successful implementation of trading strategies. Indeed, many initially promising strategies produce losses unless the effects of the trading costs are mitigated by an appropriate implementation that balances the gains and costs of trading.

The stochastic control problems describing how to do this optimally are challenging, but much progress has recently been made in the practically relevant limiting regime of *small* transaction costs. In this case, the trading friction can be viewed as a singular perturbation of a frictionless baseline problem. The leading-order correction terms can in turn be described in terms of sensitivity parameters derived from this frictionless model and the corresponding optimal trading strategy.

To wit, if the frictionless optimiser is an (sufficiently regular) Itô process then the adjustment due to small transaction costs is described by an ergodic tracking problem for Brownian motion, where transaction costs are traded off against average-squared deviations. On each small time interval, the original problem only enters locally through the volatility of the Brownian motion, which corresponds to the volatility of the frictionless target strategy. This connection to ergodic control of Brownian motion can be made precise using analytic methods as in Soner and Touzi (2013); Altarovici *et al.* (2015); Moreau *et al.* (2017); Bayraktar *et al.* (2019) or probabilistic arguments Kallsen and Muhle-Karbe (2017); Ahrens and Kallsen (2015); Cai *et al.* (2017b,a); Herdegen *et al.* (2019), leading to explicit asymptotic formulas for the optimal trading strategies and their performance in rather general settings.

Much less is known about frictionless target strategies whose local behavior does not resemble a Brownian motion. Using probabilistic techniques, Rosenbaum and Tankov (2011, 2014) study tracking problems for general pure-jump targets. They show that for small transaction costs, there is again a link to an ergodic control problem where the target Brownian motion is replaced by an α-stable process matching the local behavior of the frictionless optimiser. This limiting process still has independent increments, but since pure-jump processes display less fluctuations than Brownian motion (as measured by their Blumenthal-Getoor index $\alpha \in (1, 2)$), the impact of transaction costs is of a higher asymptotic order in these settings. A limiting case are deterministic target strategies which can be tracked with much less trading, compare, e.g., the equilibrium models of Vayanos (1998) and Weston (2018).

In contrast to these models where state variables fluctuate less than Brownian motion, recent empirical Gatheral *et al.* (2018); Bayer *et al.* (2016) and theoretical evidence Jaisson and Rosenbaum (2015, 2016); Fukasawa (2011); El Euch *et al.* (2018) documents that “volatility is rough”, in that its local degree of activity is much higher than for models driven by Brownian motion. To wit, its local behaviour corresponds to the one of fractional Brownian motion with a Hurst index H substantially smaller than the value $1/2$ for standard Brownian motion. As a result, frictionless optimal trading strategies that depend on the volatility process then naturally display the same high degree of activity, raising the question how (small) transaction costs should be taken into account in this context. The present thesis tackles this problem using probabilistic techniques.

Framework

We now outline our framework and main results. We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, where the filtration is generated by a standard Brownian motion $(W_t)_{t\geq 0}$. “Rough” processes (such as volatilities and corresponding target positions) are modelled using a (Riemann-Liouville) fractional Brownian (fBM) adapted to the filtration $\mathbb{F}$, with Hurst parameter $H \in (0, 1/2)$:[1]

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} dW_s, \quad t \geq 0,$$

where $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$ is the Gamma function.

Financial Market To illustrate how rough trading strategies naturally arise in frictionless markets with rough volatility, consider a financial market with two assets. The first one is “safe”, with price normalised to one. The second one is risky, with constant risk premium $\mu > 0$ as in Chacko and Viceira (2005) and rough volatility as in Gatheral *et al.* (2018):

$$dS_t = \mu dt + \sigma_t dW_t. \tag{1}$$

Here, the volatility process

$$\sigma_t = \exp(Y_t^H)$$

is the exponential of a fractional Ornstein-Uhlenbeck (fOU) process as in Gatheral *et al.* (2018). That is, as in Cheridito *et al.* (2003), the dynamics of Y^H are

$$dY_t^H = \kappa(\theta - Y_t^H)dt + \eta dW_t^H, \quad \text{for constants } \kappa, \eta, \theta > 0.$$

Frictionless Trading Without transaction costs, trading strategies are naturally parametrised by the number φ_t of risky shares held at each time $t \in [0, T]$. Starting from a fixed initial endowment x, the wealth process generated by a strategy $(\varphi_t)_{t\geq 0}$ is then given by the stochastic integral $X_\cdot^\varphi = x + \int_0^\cdot \varphi_t dS_t$.

[1] See Chapter 1 for more details and properties. Our results also apply to the classical fBM case so we will use RLfBM for this framework.

As in Kallsen (2002); Martin and Schöneborn (2011); Gârleanu and Pedersen (2013, 2016), we consider the simplest and most tractable frictionless optimisation problem, where the agent maximises one-period expected returns penalised for the corresponding variances. The continuous-time version of this criterion is

$$\mathbb{E}\left[X_T^\varphi - \frac{\gamma}{2}\langle X^\varphi\rangle_T\right] = x + \mathbb{E}\left[\int_0^T \left(\mu\varphi_t - \frac{\gamma\sigma_t^2}{2}\varphi_t^2\right)dt\right] \to \max_{(\varphi_t)_{t\in[0,T]}}! \tag{2}$$

Here, $T > 0$ is the agent's finite planning horizon and $\gamma > 0$ is her constant (absolute) risk aversion, which trades off the relative importance of expected returns and the risk penalty. Pointwise maximisation of the integrand immediately yields the frictionless optimiser, which inherits the roughness of the volatility process:

$$\hat{\varphi}_t = \frac{\mu}{\gamma\sigma_t^2}, \quad t \in [0,T]. \tag{3}$$

Trading with Frictions Large trades executed quickly adversely affect the corresponding execution prices; as in Almgren and Chriss (2001); Gârleanu and Pedersen (2016); Moreau *et al.* (2017); Guasoni and Weber (2017) we assume for tractability that the impact relative to the unaffected price S_t is linear in trade size and speed. To wit, the execution price when trading $\Delta\varphi_t$ shares over a time interval $[t, t+\Delta t]$ is

$$S_t + \lambda\frac{\Delta\varphi_t}{\Delta t},$$

where $\lambda > 0$ describes the magnitude of the price impact. The associated trading cost relative to the frictionless value $S_t\Delta\varphi_t$ is

$$\lambda\left(\frac{\Delta\varphi_t}{\Delta t}\right)^2\Delta t.$$

In the continuous-time limit, the frictional wealth process in turn complements its frictionless counterpart by a quadratic cost on the trading rate:

$$X_T^{\varphi,\lambda} = x + \int_0^T \varphi_t dS_t - \lambda\int_0^T \dot{\varphi}_t^2 dt,$$

where the position φ_t starts from an initial allocation x_0 and is adjusted at rate $\dot{\varphi}_t$:[1]

$$\varphi_t = x_0 + \int_0^t \dot{\varphi}_s ds, \quad t \in [0,T].$$

In direct analogy to the frictionless case (2) and as in Gârleanu and Pedersen (2013, 2016), the agent maximises expected returns penalised for risk and trading costs:

$$\mathbb{E}\left[X_T^{\varphi,\lambda} - \frac{\gamma}{2}\langle X^{\varphi,\lambda}\rangle_T\right] = \mathbb{E}\left[\int_0^T \left(\mu\varphi_t - \frac{\gamma\sigma_t^2}{2}\varphi_t^2 - \lambda\dot{\varphi}_t^2\right)dt\right] \to \max_{(\dot{\varphi}_t)_{t\in[0,T]}}! \tag{4}$$

[1] With superlinear trading costs, the restriction to such absolutely continuous trading strategies is without loss of generality, compare Guasoni and Rásonyi (2015).

Using the representation (3) for the frictionless optimiser, we can rewrite this goal functional as

$$\mathbb{E}\left[\int_0^T \left(\mu\hat{\varphi}_t - \frac{\gamma\sigma_t^2}{2}\hat{\varphi}_t^2\right) dt\right] - \mathbb{E}\left[\int_0^T \left(\frac{\gamma\sigma_t^2}{2}(\varphi_t - \hat{\varphi}_t)^2 + \lambda\dot{\varphi}_t^2\right) dt\right]. \tag{5}$$

Here, the first term is the frictionless performance of the corresponding optimiser. The second term collects the performance losses due to transaction costs that the agent seeks to minimise. This is a tradeoff between the (average squared) displacement of the actual position from the frictionless optimum and the corresponding trading costs. Accordingly, the agent solves a linear-quadratic "tracking problem" as in Kohlmann and Tang (2002); Ankirchner and Kruse (2015); Cai *et al.* (2017a); Bank *et al.* (2017); Bank and Voß (2018):[1]

$$\mathbb{E}\left[\int_0^T \left\{\bar{\gamma}_t(\varphi_t - \hat{\varphi}_t)^2 + \lambda(\dot{\varphi}_t)^2\right\} dt\right] \to \min_{(\dot{\varphi}_t)_{t\in[0,T]}} ! \qquad \text{where } \bar{\gamma}_t = \frac{\gamma\sigma_t^2}{2}. \tag{6}$$

Here, the target in the tracking problem is given by the frictionless optimiser $\hat{\varphi}$, which typically is a rough process for rough volatility models such as (1).

Problems of the form of (6) can generally be solved in terms of BSDEs, see Kohlmann and Tang (2002); Ankirchner and Kruse (2015); Bank and Voß (2018). This leads to explicit formulas if both trading costs λ and the risk penalty γ_t are constant Bank *et al.* (2017). In contrast, for stochastic γ_t the optimal trading rate is expressed in terms of the solution of a backward-stochastic Riccati equation, which makes it hard to infer comparative statics and also sufficiently slows down the numerical implementation.

We therefore perform a small-cost analysis (6), and exhibit "asymptotically optimal trading strategies" that minimise the performance losses relative to the frictionless case at the leading order for small λ. We also provide a closed-form expression for the corresponding performance losses. Similarly as in the literature for Itô process targets Cai *et al.* (2017a), this is achieved by relating the small-cost limit of the tracking problem (6) to an ergodic control problem. In the present context with rough target strategies, the limiting target turns out to be a fractional Brownian motion. In particular, this target process does not have independent increments and is in fact not even Markovian. However, by exploiting the linearity of the ergodic control problem and the fact that fBM can be represented as an integral of standard OU processes, we show in Chapter 1 that the ergodic control problem can be solved by a superposition of the corresponding explicit solutions for standard OU processes. It extends earlier results of Kleptsyna *et al.* (2005) for the case $H > 1/2$, where fBM is more regular than BM, to general $H \in (0, 1)$.[2]

Subsequently, in Chapter 2, we prove that a suitable concatenation of the solutions of the ergodic control problems from Chapter 1 indeed leads to asymptotically optimal

[1] This holds exactly for the local mean-variance preferences considered here. For more general preferences, the same equivalence remains true at the leading-order for small transaction costs, compare Rogers (2004); Janeček and Shreve (2004); Kallsen and Muhle-Karbe (2017); Moreau *et al.* (2017); Cayé *et al.* (2018).

[2] This parallels results of Carr and Madan (1999) and Hubalek *et al.* (2006); Černý (2007) for option pricing and mean-variance hedging, where integral representations of general options also allow to construct solutions for more complex models as superpositions of simpler ones.

strategies for tracking problems of the form (6). We start by working directly with the BSDE solutions to the tracking problem (6) and are able to study their convergence under appropriate rescaling. In the absence of classical stochastic calculus techniques, we replace the use of Itô's formula by a Taylor expansion of the target on small intervals. Moreover, independence of increments is replaced by the mixing property of increments of fBM. After rescaling, we are able to show the convergence of the processes in (6) towards their stationary Gaussian counterparts appearing in the ergodic control problem. Using the formulas obtained in Chapter 1, this permits us to recover leading order formulas obtained in the case $H = 1/2$ in Cai *et al.* (2017a).

Finally, in Chapter 3, we investigate the quantitative properties of the asymptotic formulas from Chapter 2. More specifically, we study whether the effects of transaction costs are more or less pronounced for rough volatility models compared to their classical counterparts. Three potentially competing effects are at work here: on the one hand, the paths of fractional Brownian motion with Hurst index $H < 1/2$ fluctuate more wildly than their counterparts for standard Brownian motion. Whence, conversely to the less wildly fluctuating target strategies studied in Rosenbaum and Tankov (2014), the asymptotic order of the performance loss due to small transaction costs is magnified for rough targets. Accordingly, performance losses are higher than for classical models *if* transaction costs are "sufficiently small". On the other hand, however, fBM with Hurst index $H < 1/2$ exhibits negative autocorrelation, which can be exploited for tracking such targets unlike for standard Brownian motion. Finally, for a given level of transaction costs, results for fBM and standard BM depend on the parameters of the respective stochastic volatility processes, which should be estimated from the same dataset for a fair comparison.

To address this, we match the moments of a classical stochastic volatility model to their counterparts in the rough volatility model estimated from an equity time series in Gatheral *et al.* (2018). We then compare the corresponding approximate performance losses for small transaction costs and find that the utility loss for the rough models is smaller for values of the trading costs parameter in line with empirical estimates in Gârleanu and Pedersen (2013); Cartea and Jaimungal (2016). However, when the market becomes more liquid and transaction costs get smaller by a factor ten, the utility losses of the two models coincide. We also test that these results are not artefacts of our asymptotic analysis, by confirming through a simulation study that the explicit formulas for small transaction costs provide an excellent approximation of the ones actually realised.

Chapter 1

Ergodic Control of Fractional Brownian Motion

This chapter is based on joint work with Dr. Christoph Czichowsky.

As we have already explained, our goal is to provide an asymptotic expansion of the utility of the local mean-variance portfolio optimisation problem with rough volatility for small quadratic costs. Because of the H-self-similarity of fractional Brownian motion, the leading order coefficient in this expansion will be determined in terms of the optimal value of the ergodic linear-quadratic tracking problem of fractional Brownian motion. In this chapter, we provide the solution to this problem for all Hurst parameters $H \in (0, 1)$.

In the case $H > 1/2$, where fractional Brownian motion is more regular than Brownian motion, the solution to the ergodic linear-quadratic tracking problem of fractional Brownian motion has already be obtained by Kleptsyna *et al.* (2005). Their argument uses explicit computations with the kernel representation of fractional Brownian motion for $H > 1/2$. In the case $H < 1/2$, the kernel representation of fBM makes it harder to carry over their argument. However, the formulas obtained in the paper are extended to $H < 1/2$. We prove this by combining two things: the linear-quadratic structure of the optimisation problem and an infinite-dimensional Markovian representation of fBM. This Markovian representation goes back to Carmona and Coutin (1998) and represents fBM as the superposition of Ornstein-Uhlenbeck processes with different speeds of mean reversion with respect to a Borel measure on $(0, \infty)$; see (1.2). It has been more recently considered in connection with affine representations of fractional processes (Harms and Stefanovits (2019)) and multi-factor approximations of rough volatility models (Abi Jaber and El Euch (2019)).

For each OU process, we only have to solve a standard Markovian optimal control problem that can be solved via the HJB equation. This does not need any non-semimartingale stochastic calculus. It turns out that the solution to the linear-quadratic ergodic control problem for each OU process is again given in terms of OU processes with adjusted speed of mean reversion. Moreover, the optimal value of the control problem corresponds to the variance of their stationary distributions. The solution for fractional Brownian motion is then simply given by the superposition of the solutions to the problems of the OU processes. This indicates that our results would carry over to processes that can be represented as superpositions of OU processes with different Borel measures on $(0, \infty)$ as long as they integrate the covariances of the stationary distribution of the adjusted OU processes.

1.1 Fractional Brownian motion

We start this chapter by reminding key facts on fractional Brownian motion. Most of the content of this section is extracted from Nourdin (2012).

Definition 1.1.1. *For $H \in [0,1]$, the fractional Brownian motion (fBM) of Hurst parameter H is a centered continuous Gaussian process $B^H = (B_t^H)_{t\geq 0}$ with covariance function*

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

for $t, s > 0$.

In the case where $H = 1/2$, we recover the usual covariance structure of Brownian motion. In addition, in the case where $H > 1/2$, we have positively correlated increments of fBM, and when $H < 1/2$, the increments are negatively correlated. The next proposition states key properties of fBM as exposed in Nourdin (2012), Proposition 1.6, Proposition 2.2.

Proposition 1.1.2. *The fBM satisfies the following properties*

- *For $a > 0$, $(a^{-H} B_{at}^H)_{t\geq 0} \stackrel{d}{=} (B_t^H)_{t\geq 0}$. (Self-similarity)*
- *For $h > 0$, $(B_{t+h}^H - B_h^H)_{t\geq 0} \stackrel{d}{=} (B_t^H)_{t\geq 0}$. (Stationarity of increments)*
- *The sample paths of B^H are for any $\alpha \in (0, H)$, α-Hölder continuous on each compact set. (Hölder continuity)*

Hence, in the case when $H < 1/2$, fBM has rougher paths than Brownian motion whereas for $H > 1/2$, fBM has smoother paths. Typical path realisations of fBM for different values of H can be seen on Figure 1.1.

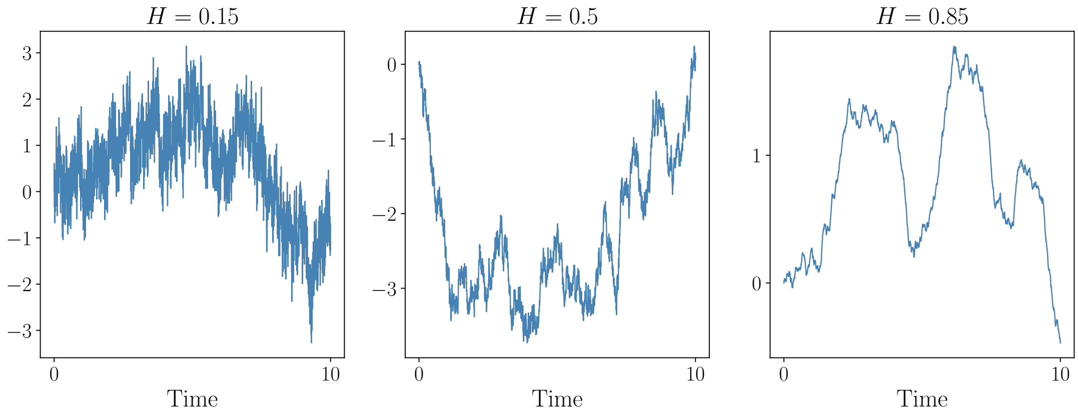

Figure 1.1: Path realisation of fBM for different Hurst parameter H values.

Moreover, as previously mentioned, in the case where $H \neq 1/2$, we have that fBM is not a semimartingale (Nourdin, Theorem 2.2) and it is not a Markov process (Nourdin, Theorem 2.3).

We have the following original Mandelbrot and Van Ness (1968) representation of fBM, for $H \in (0, 1/2) \cup (1/2, 1)$,

$$B_t^H = \frac{1}{c_H} \left(\int_{-\infty}^{0} \left\{ (t-u)^{H-1/2} - (-u)^{H-1/2} \right\} dW_u + \int_0^t (t-u)^{H-1/2} dW_u \right), \tag{1.1}$$

where

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \left\{ (1+u)^{H-1/2} - u^{H-1/2} \right\}^2 du} < \infty$$

and $(W_t)_{t\in\mathbb{R}}$ is a two-sided Brownian motion, i.e., $W_t = W_t^1$ for $t \geq 0$, $W_t = W_{-t}^2$ for $t < 0$, for $(W_t^1)_{t\geq 0}, (W_t^2)_{t\geq 0}$ two independent classical Brownian motions. The integral on the positive real line in (1.1) is referred to as (in general up to some multiplicative constant)

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} dW_s,$$

known as the Riemann-Liouville fractional Brownian motion (RLfBM). It is worth noticing that increments of RLfBM are not stationary. Nonetheless, RLfBM shares many properties with fBM and is often used in place of fBM in rough volatility models. In the asymptotic framework, both objects behave very similarly, see for example Lim and Sithi (1995).

Alternatively, from Norros *et al.* (1999), we also have the following representation of fBM in terms of integral with respect to a Brownian motion on the positive real line,

$$B_t^H = \int_0^t K_H(t,s) dW_s,$$

where for $H < 1/2$, $t > s$,

$$K_H(t,s) = b_H \left\{ \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-1/2} - \left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} du \right\}$$

with

$$b_H = \sqrt{\frac{2H}{(1-2H)B(1-2H, H+1/2)}},$$

where $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx$, is the Beta function. With this representation, fBM generates the same filtration as the underlying BM.

From Carmona and Coutin (1998), we have a useful representation of RLfBM as an infinite dimensional Markov process. More precisely, we have

$$W_t^H = \int_0^\infty Y_t^\gamma \mu(d\gamma), \tag{1.2}$$

for

$$\mu(d\gamma) = \frac{1}{\Gamma(H+1/2)\Gamma(1/2-H)} \frac{1}{\gamma^{1/2+H}} d\gamma,$$

and the Ornstein-Uhlenbeck processes Y_t^γ, $\gamma > 0$, satisfying the dynamics

$$\begin{aligned} dY_t^\gamma &= -\gamma Y_t^\gamma dt + dW_t \\ Y_0^\gamma &= 0 \end{aligned}$$

or explicitely given by

$$Y_t^\gamma = \int_0^t e^{-\gamma(t-s)} dW_s,$$

being Ornstein-Uhlenbeck processes sharing the same noise with different speed of mean-reversion. A similar representation holds for fBM as in Harms and Stefanovits (2019) Theorem 3.5 but requires a random initial condition. Exploiting representation (1.2) will be of great use when solving the ergodic control problem of RLfBM.

1.2 Ergodic control of fractional Brownian motion

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ generated by a Brownian Motion $(W_t)_{t\geq 0}$. We start by studying the linear-quadratic ergodic regular control problem of a RLfBM target,

$$\begin{aligned} &\min_{u\in\mathcal{A}} J^W(u;\alpha), \\ &J^W(u;\alpha) := \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(X_t - \alpha W_t^H)^2 dt + \int_0^T r(u_t)^2 dt\right], \end{aligned} \tag{1.3}$$

with $q, r > 0$ and $\alpha \in \mathbb{R}$,

$$X_t = x_0 + \int_0^t u_s ds,$$

and $\mathcal{A}$ the set of adapted strategies u such that $J^W(u, \alpha) < \infty$. These type of strategies are called regular control. As mentioned at the beginning of this chapter, we will use the representation (1.2) of RLfBM for the proof of the main statement of this chapter. We will then make the connection with the ergodic control of fBM. We therefore start by considering the problem of tracking an Ornstein-Uhlbeck target.

In that case, the regular linear-quadratic ergodic control problem is quite easily solved using standard Markovian stochastic control theory. With the notation introduced above, we have the following result. This result is probably known but we did not find any reference for it.

Proposition 1.2.1. *Fix $\gamma > 0$, the linear-quadratic ergodic control problem of the Ornstein-Uhlenbeck process Y^γ is given by*

$$\min_{u \in \mathcal{A}} J^{OU}(u; \alpha),$$
$$J^{OU}(u; \alpha) := \limsup_{T \to \infty} \frac{1}{T} J_T^{OU}(u; \alpha) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T q(X_t - \alpha Y_t^\gamma)^2 dt + \int_0^T r(u_t)^2 dt\right],$$

where the set $\mathcal{A}$ of adapted controls u such that $J^{OU}(u; \alpha) < \infty$. This control problem admits an optimal solution u^γ characterised by the feedback form,

$$u_t^\gamma = \delta\left(\alpha \frac{\delta}{\delta + \gamma} Y_t^\gamma - X_t^\gamma\right),$$
$$X_t^\gamma = x_0^\gamma + \int_0^t u_s^\gamma ds,$$

with $\delta = \sqrt{q/r}$.

Proof. The proof of this proposition is given in the technical section 1.3.1. □

By linearity of the optimiser in the target, the previous result can also be extended to a target being a linear combination of Ornstein-Uhlenbeck processes sharing the same noise as presented in the next result.

Proposition 1.2.2. *For $N \in \mathbb{N}^*$ consider a vector $(\gamma_1, \ldots, \gamma_N)$, with $\gamma_i > 0$ for $i = 1, \ldots, N$, a sequence $(\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$ and the Ornstein-Uhlenbeck processes $Y_t^{\gamma_i}$ satisfying,*

$$dY_t^{\gamma_i} = -\gamma_i Y_t^{\gamma_i} dt + dW_t$$
$$Y_0^{\gamma_i} = 0,$$

for all $i = 1, \ldots, N$. Consider the target given by

$$Y_t^N := \sum_{i=1}^{N} \alpha_i Y_t^{\gamma_i},$$

and the linear-quadratic ergodic control problem of this target Y^N, i.e., minimise

$$J^{OU,N}(u) := \limsup_{T\to\infty} \frac{1}{T} J_T^{OU,N}(u) = \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T q(X_t - Y_t^N)^2 dt + \int_0^T r(u_t)^2 dt\right],$$

over the set $\mathcal{A}$ of adapted controls such that $J^{OU,N}(u) < \infty$. Then, the optimal solution to this problem is characterised by the following feedback form,

$$\begin{aligned} u_t^N &= \delta\left(\sum_{i=1}^{N} \alpha_i \frac{\delta}{\delta + \gamma_i} Y_t^{\gamma_i} - X_t^N\right), \\ X_t^N &= x_0^N + \int_0^t u_s^N ds, \end{aligned}$$

with $\delta = \sqrt{q/r}$.

Proof. The proof of this proposition is given in the technical section 1.3.1. □

Remark 1.2.3. *Motivated by the explicit solution for the finite time horizon linear-quadratic control problem in Bank* et al. *(2017), we provide a useful representation of the solution first pointed out in Gârleanu and Pedersen (2013). The optimal rate in Proposition 1.2.1 can be rewritten as*

$$\begin{aligned} u_t^\gamma &= \delta(\hat{\xi}_t^\gamma - X_t^\gamma), \\ \hat{\xi}_t^\gamma &= \frac{\delta}{\delta + \gamma} \alpha Y_t^\gamma = \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} \alpha Y_s^\gamma ds \,\middle|\, \mathcal{F}_t\right], \end{aligned}$$

since for $s > t$,

$$Y_s^\gamma = e^{-\gamma(t-s)} Y_t^\gamma + \int_t^s e^{-\gamma(s-u)} dW_u.$$

We call $\hat{\xi}^\gamma$ the signal process*. The motivation for this terminology comes from the fact that the signal process is in essence a convex combination of future values of the target. This process becomes in particular useful when the target to track is non-Markovian. As in the finite time horizon case, we expect the signal process to be linear in the target it tracks. In the case of Proposition 1.2.2, we have the signal process given by*

$$\hat{\xi}_t^N = \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} Y_s^N ds \,\middle|\, \mathcal{F}_t\right].$$

which exhibits the linearity of the signal.

In the following lemma, we compute some covariance limits involving the signal $\hat{\xi}^\gamma$ that will be necessary for our limiting argument and explicit computations.

Lemma 1.2.4. *In the notation introduced above, we have,*

$$\begin{aligned}
\lim_{t\to\infty} \mathbb{E}\left[Y_t^{\gamma} Y_t^{\bar{\gamma}}\right] &= \frac{1}{\gamma+\bar{\gamma}}, \\
\lim_{t\to\infty} \mathbb{E}\left[\hat{\xi}_t^{\gamma} Y_t^{\bar{\gamma}}\right] &= \frac{\delta}{\delta+\gamma}\frac{1}{\gamma+\bar{\gamma}} \\
\lim_{t\to\infty} \mathbb{E}\left[\hat{\xi}_t^{\gamma} \hat{\xi}_t^{\bar{\gamma}}\right] &= \frac{\delta}{\delta+\gamma}\frac{\delta}{\delta+\bar{\gamma}}\frac{1}{\gamma+\bar{\gamma}} \\
\lim_{t\to\infty} \mathbb{E}\left[X_t^{\gamma} Y_t^{\bar{\gamma}}\right] &= \frac{1}{\gamma+\bar{\gamma}}\frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})}, \\
\lim_{t\to\infty} \mathbb{E}\left[X_t^{\gamma} \hat{\xi}_t^{\bar{\gamma}}\right] &= \frac{1}{\gamma+\bar{\gamma}}\frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})}\frac{\delta}{\delta+\bar{\gamma}}, \\
\lim_{t\to\infty} \mathbb{E}\left[X_t^{\gamma} X_t^{\bar{\gamma}}\right] &= \frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})}\frac{1}{\gamma+\bar{\gamma}}\frac{\delta}{2}\left(\frac{1}{\delta+\gamma}+\frac{1}{\delta+\bar{\gamma}}\right).
\end{aligned}$$

Proof. The proof of the statement is found in the technical section 1.3.2. □

The previous results in Proposition 1.2.2 motivates our solution to the RLfBM ergodic control problem (1.3). Indeed, the representation (1.2) of RLfBM as an infinite integral of Ornstein-Uhlenbeck processes allows us to guess the optimal rate for the ergodic control problem of RLfBM. The following result provides the rigorous statement.

Proposition 1.2.5. *The minimising solution to the linear-quadratic ergodic control problem of RLfBM* (1.3)*,*

$$\min_{u\in\mathcal{A}} J^W(u;\alpha),$$

is given by

$$\begin{aligned}
\hat{u}_t &= \delta\left\{\alpha\int_0^\infty \frac{\delta}{\delta+\gamma} Y_t^{\gamma}\mu(d\gamma) - \hat{X}_t\right\}, \\
\hat{X}_t &= x_0 + \int_0^t \hat{u}_s ds,
\end{aligned}$$

with $\delta = \sqrt{q/r}$*.*

Proof. Without loss of generality, we assume $\alpha = 1$. We start by considering the linear-quadratic ergodic control of a general target ξ and we derive a necessary and sufficient condition for optimality similar to what is obtained in Bank *et al.* (2017) and Kleptsyna *et al.* (2005). The problem consists of minimising

$$J(u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(X_t^u - \xi_t)^2 dt + \int_0^T r(u_t)^2 dt\right], \tag{1.4}$$

over a set $\mathcal{A}$ of admissible controls u, i.e., adapted processes, for which, $J(u) < \infty$. An admissible control process $\bar{u} \in \mathcal{A}$ is then optimal if it satisfies

$$J(u) \geq J(\bar{u})$$

for any $u \in \mathcal{A}$. Denoting $X = X^u, \bar{X} = X^{\bar{u}}$, we express the difference as

$$
\begin{aligned}
J(u) - J(\bar{u}) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T 2q(X_t - \bar{X}_t)(\bar{X}_t - \xi_t)dt + \int_0^T 2r\bar{u}_t(u_t - \bar{u}_t)\right] \\
+ \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(X_t - \bar{X}_t)^2 dt + \int_0^T r(u_t - \bar{u}_t)^2\right] \qquad (1.5) \\
\geq \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T 2q(X_t - \bar{X}_t)(\bar{X}_t - \xi_t)dt + \int_0^T 2r\bar{u}_t(u_t - \bar{u}_t)\right],
\end{aligned}
$$

where the inequality follows from the fact that the second term in (1.5) is always non-negative. Hence, for an admissible strategy $\bar{u} \in \mathcal{A}$ to be optimal, it is sufficient for it to satisfy

$$
\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T 2q(X_t - \bar{X}_t)(\bar{X}_t - \xi_t)dt + \int_0^T 2r\bar{u}_t(u_t - \bar{u}_t)\right] = 0, \qquad (1.6)
$$

for any $u \in \mathcal{A}$. We call (1.6) the *first order condition* of the problem (1.4). Denoting by $w = u - \bar{u}$ and using Fubini's theorem, we can further rewrite the first order condition as,

$$
\begin{aligned}
\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T 2q\left(\int_0^t w_s ds\right)(\bar{X}_t - \xi_t)dt + \int_0^T 2r\bar{u}_t w_t dt\right] = 0 \\
\Leftrightarrow \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T w_s \left\{(2q)\int_s^T (\bar{X}_t - \xi_t)dt + (2r)\bar{u}_s\right\} ds\right] = 0, \qquad (1.7)
\end{aligned}
$$

for all $w = u - \bar{u}$, with $u \in \mathcal{A}$.

In particular, we know from Proposition 1.2.1 that the above first order condition (1.7) is satisfied by the optimal solution u^γ when the target $\xi_t = Y_t^\gamma$ is an Ornstein-Uhlenbeck.

In the case of RLfBM as target,

$$
\xi_t = W_t^H = \int_0^\infty Y_t^\gamma \mu(d\gamma),
$$

we need to check that the first order condition (1.7) holds for our following proposed solution

$$
\begin{aligned}
\hat{u}_t &= \int_0^\infty u_t^\gamma \mu(d\gamma), \\
\hat{X}_t &= x_0 + \int_0^t \hat{u}_s ds = \int_0^\infty X_t^\gamma \mu(d\gamma),
\end{aligned}
$$

where X_t^γ is the optimal position in the problem with target Y^γ given by

$$
X_t^\gamma = x_0^\gamma + \int_0^t u_s^\gamma ds,
$$

with

$$
x_0^\gamma = \Gamma(H + 1/2)e^{-\gamma} x_0.
$$

To see this, we recall that for $t > 0$, the measure μ satisfies

$$\frac{t^{H-1/2}}{\Gamma(H+1/2)} = \int_0^\infty e^{-\gamma t}\mu(d\gamma).$$

Choosing $t = 1$ allows us to write

$$x_0 = \int_0^\infty x_0^\gamma \mu(d\gamma).$$

We show in Proposition 1.2.7 that with the defined rate $\hat{u}$ and corresponding position $\hat{X}$, we have

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(\hat{X}_t - W_t^H)^2 + \int_0^T r(\hat{u}_t)^2 dt\right] < \infty,$$

and therefore $\hat{u}$ is admissible. In fact, we compute the corresponding objective value $J(\hat{u})$ explicitely. Let us now fix $u \in \mathcal{A}$ an arbitrary competing trading rate, and denote $w = u - \hat{u}$.

Plugging now $\hat{u}, \hat{X}$ in place of $\bar{u}, \bar{X}$ in the limit of (1.7) for the target W^H gives

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T w_s \left\{(2q)\int_s^T \int_0^\infty (X_t^\gamma - Y_t^\gamma)\mu(d\gamma)dt + (2r)\int_0^\infty u_s^\gamma \mu(d\gamma)\right\} ds\right] \\
&= \limsup_{T\to\infty} \int_0^\infty \frac{1}{T}\mathbb{E}\left[\int_0^T w_s \left\{(2q)\int_s^T (X_t^\gamma - Y_t^\gamma)dt + (2r)u_s^\gamma\right\} ds\right]\mu(d\gamma), \qquad (1.8)
\end{aligned}$$

by Fubini's theorem. Provided we can interchange the limit and the integral in the above expression, we would obtain

$$\int_0^\infty \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T w_s \left\{(2q)\int_s^T (X_t^\gamma - Y_t^\gamma)dt + (2r)u_s^\gamma\right\} ds\right]\mu(d\gamma) = 0,$$

as the first order condition (1.7) is satisfied for every Ornstein-Uhlenbeck target Y^γ for any perturbation w, in this case given by $w = u - \hat{u}$. Modulo the interchangiblity argument, this shows the rate $\hat{u}$ satisfies the first order condition (1.7) and it is therefore optimal.

We devote the rest of the proof to the justification for the interchangeability of the limit and the integral in (1.8). We start by using the tower property of conditional expectation to rewrite

$$\begin{aligned}
f_T(\gamma) &:= \frac{1}{T}\mathbb{E}\left[\int_0^T w_s \left\{(2q)\int_s^T (X_t^\gamma - Y_t^\gamma)dt + (2r)u_s^\gamma\right\} ds\right] \\
&= \frac{1}{T}\int_0^T \mathbb{E}\left[w_s\left\{(2q)\mathbb{E}\left[\int_s^T (X_t^\gamma - Y_t^\gamma)dt \,\middle|\, \mathcal{F}_s\right] + (2r)u_s^\gamma\right\}\right] ds.
\end{aligned}$$

From Proposition 1.2.1, we recall

$$u_t^\gamma = \delta\left(\frac{\delta}{\delta+\gamma}Y_t^\gamma - X_t^\gamma\right).$$

By Itô's formula, u^γ satisfies the following dynamics,

$$du_t^\gamma = \delta^2(X_t^\gamma - Y_t^\gamma)dt + \frac{\delta^2}{\delta+\gamma}dW_t,$$
$$u_0^\gamma = -\delta x_0^\gamma.$$

Intregrating we have, for $0 < s \leq T$,

$$u_T^\gamma = u_s^\gamma + \int_s^T \delta^2(X_t^\gamma - Y_t^\gamma)dt + \frac{\delta^2}{\delta+\gamma}\int_s^T dW_t.$$

Hence, we can write, for $0 < s \leq T$,

$$u_s^\gamma = \mathbb{E}\left[u_T^\gamma \,|\, \mathcal{F}_s\right] - \delta^2 \mathbb{E}\left[\int_s^T (X_t^\gamma - Y_t^\gamma)dt \,\middle|\, \mathcal{F}_s\right],$$

and $f_T(\gamma)$ becomes,

$$f_T(\gamma) = \frac{1}{T}\mathbb{E}\left[\int_0^T w_s(2r)\mathbb{E}\left[u_T^\gamma \,|\, \mathcal{F}_s\right] ds\right].$$

Using Cauchy-Schwarz inequality, Jensen's inequality and Fubini's theorem, we estimate $f_T(\gamma)$ by

$$\begin{aligned}
f_T(\gamma) &\leq 2r\frac{1}{T}\mathbb{E}\left[\int_0^T w_s^2 ds\right]^{1/2} \mathbb{E}\left[\int_0^T \mathbb{E}\left[u_T^\gamma \,|\, \mathcal{F}_s\right]^2 ds\right]^{1/2} \\
&\leq 2r\left(\frac{1}{T}\mathbb{E}\left[\int_0^T w_s^2 ds\right]\right)^{1/2}\left(\frac{1}{T}\int_0^T \mathbb{E}\left[(u_T^\gamma)^2\right] ds\right)^{1/2} \\
&\leq 2r\left(\frac{1}{T}\mathbb{E}\left[\int_0^T w_s^2 ds\right]\right)^{1/2} E\left[(u_T^\gamma)^2\right]^{1/2}.
\end{aligned}$$

Since $w = u - \hat{u}$ with $u, \hat{u} \in \mathcal{A}$, we have

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T w_s^2 ds\right] < \infty.$$

and therefore for T large enough, we can assume the existence of a constant $C > 0$, such that

$$\frac{1}{T}\mathbb{E}\left[\int_0^T w_s^2 ds\right] \leq C.$$

Using C as an absorbing constant, we bound $f_T(\gamma)$ as

$$f_T(\gamma) \leq CE\left[(u_T^\gamma)^2\right] =: g_T(\gamma).$$

Using the fact that

$$u_T^\gamma = \delta \left(\frac{\delta}{\delta+\gamma} Y_T^\gamma - X_T^\gamma \right),$$

we can compute

$$E\left[(u_T^\gamma)^2\right] = \delta^2 \left\{ \left(\frac{\delta}{\delta+\gamma}\right)^2 \mathbb{E}\left[(Y_T^\gamma)^2\right] - \frac{2\delta}{\delta+\gamma}\mathbb{E}\left[Y_T^\gamma X_T^\gamma\right] + \mathbb{E}\left[(X_T^\gamma)^2\right] \right\}$$

Using (1.32),(1.34),(1.35) in the proof of Lemma 1.2.4 we obtain the explicit expressions of $\mathbb{E}\left[(Y_T^\gamma)^2\right], \mathbb{E}\left[Y_T^\gamma X_T^\gamma\right], \mathbb{E}\left[(X_T^\gamma)^2\right]$. In particular, we notice that $E\left[(u_T^\gamma)^2\right]$ can be expressed as a sum of components that are monotone with respect to T. Using then the monotone convergence theorem, we obtain

$$\begin{aligned}
&\lim_{T\to\infty} \int_0^\infty g_T(\gamma)\mu(d\gamma) = \int_0^\infty \lim_{T\to\infty} g_T(\gamma)\mu(d\gamma) \\
&= \delta^2 \int_0^\infty \left\{ \left(\frac{\delta}{\delta+\gamma}\right)^2 \frac{1}{2\gamma} - \frac{2\delta}{\delta+\gamma}\frac{1}{2\gamma}\frac{\delta^2}{(\delta+\gamma)^2} + \frac{\delta^2}{(\delta+\gamma)^2}\frac{\delta}{\delta+\gamma}\frac{1}{2\gamma} \right\} \mu(d\gamma) \\
&= \int_0^\infty \frac{\delta^4}{2(\delta+\gamma)^3}\mu(d\gamma) < \infty.
\end{aligned}$$

We therefore use the generalised Lebesgue dominated convergence theorem to justify the interchangeability of limit and integration in equation (1.8) given by

$$\lim_{T\to\infty} \int_0^\infty f_T(\gamma)\mu(d\gamma) = \int_0^\infty \lim_{T\to\infty} f_T(\gamma)\mu(d\gamma) = 0,$$

which concludes the proof. □

Remark 1.2.6. *Echoing Remark 1.2.3, we express the solution to the ergodic control of RLfBM in terms of the signal process. The process towards which the optimal rate reverses can be rewritten as,*

$$\hat{\xi}_t = \int_0^\infty \hat{\xi}_t^\gamma \mu(d\gamma) = \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} W_s^H ds \,\middle|\, \mathcal{F}_t \right], \tag{1.9}$$

where we get the second equality by using the representation

$$W_s^H = \int_0^\infty Y_s^\gamma \mu(d\gamma).$$

The conditional expectation expression in (1.9) *clearly exhibits the key feature of the signal process. In essence, the optimal rate is a feedback towards a weighted sum of future values of the RLfBM target. This is quite intuitive as we would expect the optimal solution to exploit the autocorrelation of the target. In the case of RLfBM and fBM with $H < 1/2$, the autocorrelation is negative. We provide further interpretation and graphs with the path realisation of a signal process in Chapter 3.*

Proposition 1.2.7. *The minimised ergodic objective of the linear-quadratic ergodic control of RLfBM is given by*

$$J(\alpha) = \alpha^2 \frac{q}{\delta^{2H}} \frac{1}{2} \left\{ 1 + \frac{1}{\sin(\pi H)} \right\}.$$

Proof. The proof of the proposition can be found in the technical section 1.3.3. □

1.2.1 Ergodic control of fBM and RLfBM

Our aim is now to extend our result from the ergodic control of RLfBM to the ergodic control of fBM. Recall from Mandelbrot and Van Ness (1968) that we have the following representation of fBM,

$$\begin{aligned} B_t^H &= \frac{1}{c_H} \left(\int_{-\infty}^0 \left\{ (t-u)^{H-1/2} - (-u)^{H-1/2} \right\} dW_u + \int_0^t (t-u)^{H-1/2} dW_u \right) \\ &=: \frac{1}{c_H} \Gamma(H+1/2) \left(I_t + W_t^H \right), \end{aligned} \tag{1.10}$$

with the constant

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \{(1+u)^{H-1/2} - u^{H-1/2}\}^2 du.}$$

We want to show that up to the multiplicative constant appearing in (1.10), RLfBM and fBM provide the same minimised ergodic control problem objective and the structure of the optimal solution for RLfBM carries over to fBM. We first need to show that the process I_t has no effect in the long run and has therefore no impact on the objective value.

We have, for $t > s \geq 0$,

$$I_t - I_s = \frac{1}{\Gamma(H+1/2)} \int_{-\infty}^0 \left\{ (t-u)^{H-1/2} - (s-u)^{H-1/2} \right\} dW_u,$$

and therefore,

$$\begin{aligned} \mathbb{E}\left[(I_t - I_s)^2 \right] &= \frac{1}{\Gamma(H+1/2)^2} \int_0^\infty \left\{ (t+u)^{H-1/2} - (s+u)^{H-1/2} \right\}^2 du \\ &= \frac{1}{\Gamma(H+1/2)^2} (t-s)^{2H} \int_{\frac{s}{t-s}}^\infty \left\{ (1+u)^{H-1/2} - u^{H-1/2} \right\}^2 du, \end{aligned} \tag{1.11}$$

and we have the following result.

Lemma 1.2.8. *The function g defined by*

$$g(s) = \int_s^\infty \left\{ (1+u)^{H-1/2} - u^{H-1/2} \right\}^2 du,$$

for $H < 1/2$ and $s \geq 0$, satisfies for $s > 0$,

$$g(s) \leq \left(\frac{1}{2} - H \right)^2 \frac{s^{2H-2}}{2-2H},$$

and note that

$$g(s) \leq g(0) < \infty,$$

for every $s > 0$.

Proof. We know that for $H < 1/2$, the function $f(x) = x^{H-1/2}$, is convex for $x > 0$. Its derivative is $f'(x) = (H - 1/2)x^{H-3/2} < 0$ for all $x > 0$. Moreover, for $x > y$, we have $f(x) < f(y)$, and

$$f(x) \geq f(y) + f'(y)(x - y).$$

Hence, for $x > y$ we obtain

$$f(y) - f(x) \leq -f'(y)(y - x),$$

which in our case, translates to

$$u^{H-1/2} - (1 + u)^{H-1/2} \leq \left(\frac{1}{2} - H\right) u^{H-3/2}.$$

Hence, we have the required estimate,

$$g(s) \leq \int_s^\infty \left(\frac{1}{2} - H\right)^2 u^{2H-3} du = \left(\frac{1}{2} - H\right)^2 \frac{s^{2H-2}}{2 - 2H}.$$

□

The next result indicates that the contribution to the ergodic control problem objective of the process I in the decomposition (1.10) is zero.

Proposition 1.2.9. *For the process* I *from* (1.10), *we define*

$$\begin{aligned} \hat{\xi}_t^I &= \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} I_s ds \,\middle|\, \mathcal{F}_t\right], \\ u_t^I &= \delta(\hat{\xi}_t^I - X_t^I), \\ X_t^I &= x_0 + \int_0^t u_s^I ds, \end{aligned}$$

for which we have,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \left\{ q\mathbb{E}\left[(X_t^I - I_t)^2\right] + r\mathbb{E}\left[(u_t^I)^2\right] \right\} dt = 0.$$

Proof. Since for $t > 0$, I_t is $\mathcal{F}_0$-measurable, we know that

$$\hat{\xi}_t^I = \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} I_s ds \,\middle|\, \mathcal{F}_t\right] = \int_t^\infty \delta e^{-\delta(s-t)} I_s ds,$$

and

$$u_t^I = \delta(\hat{\xi}_t^I - X_t^I).$$

We start by showing that the second moments of the processes $X_t^I - I_t$ and u_t^I are bounded from above by a continuous function in t. Once the bounds are established, we can use the mean-value theorem and consider the limiting value of these second moments. This amounts to computing the long-term variance of the two Gaussian processes.

Hence, assuming without loss of generality $x_0 = 0$, we start by studying the limit of $\mathbb{E}\left[(X_t^I - I_t)^2\right]$ as $t \to \infty$, where the solution for X_t^I is given by

$$X_t^I = \int_0^t \delta e^{-\delta(t-s)} \hat{\xi}_s^I ds.$$

We then rewrite

$$X_t^I - I_t = (\hat{\xi}_t^I - I_t) + (X_t^I - \hat{\xi}_t^I),$$

and start by computing the limit of $\mathbb{E}\left[(\hat{\xi}_t^I - I_t)^2\right]$ as $t \to \infty$.

To this effect, we notice

$$\hat{\xi}_t^I - I_t = \int_t^\infty \delta e^{-\delta(s-t)}(I_s - I_t)ds,$$

and by Jensen's inequality,

$$\mathbb{E}\left[(\hat{\xi}_t^I - I_t)^2\right] \leq \int_t^\infty \delta e^{-\delta(s-t)} \mathbb{E}\left[(I_s - I_t)^2\right] ds.$$

Using Lemma 1.2.8, we obtain for $s > t > 0$, and some absorbing constant $C > 0$,

$$\mathbb{E}\left[(I_s - I_t)^2\right] \leq C(s-t)^2 t^{2H-2},$$

and therefore

$$\mathbb{E}\left[(\hat{\xi}_t^I - I_t)^2\right] \leq C\Gamma(3)t^{2H-2} \to 0, \tag{1.12}$$

as $t \to \infty$ because $H < 1$.

Next, we consider the limit as $t \to \infty$ of $\mathbb{E}\left[(X_t^I - \hat{\xi}_t^I)^2\right]$. We rewrite,

$$X_t^I - \hat{\xi}_t^I = \int_0^t \delta e^{-\delta(t-s)}(\hat{\xi}_s^I - \hat{\xi}_t^I)ds + \hat{\xi}_t^I(\eta_t - 1), \tag{1.13}$$

where $\eta_t = 1 - \exp(-\delta t)$. We focus first our attention on the first term of (1.13). For this, we consider for $0 < s < t$,

$$\begin{aligned}\hat{\xi}_t^I - \hat{\xi}_s^I &= \int_t^\infty \delta e^{-\delta(u-t)} I_u du - \int_s^\infty \delta e^{-\delta(u-s)} I_u du \\ &= \int_0^\infty \delta e^{-\delta v} \left(I_{t+v} - I_{s+v}\right) dv.\end{aligned}$$

Using again Lemma 1.2.8, we obtain the estimates

$$\begin{aligned}\mathbb{E}\left[(I_{t+v} - I_{s+v})^2\right] &\leq C_1(t-s)^2(v+s)^{2H-2}, \\ \mathbb{E}\left[(I_{t+v} - I_{s+v})^2\right] &\leq C_2(t-s)^{2H},\end{aligned}$$

for $C_1, C_2 > 0$ absorbing constants. By Jensen's inequality, it follows

$$\begin{aligned}\mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] &\leq C_1(t-s)^2 \int_0^\infty \delta e^{-\delta v}(v+s)^{2H-2} dv, \\ \mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] &\leq C_2(t-s)^{2H}.\end{aligned} \tag{1.14}$$

Using $\eta_t \leq 1$ together with Jensen's inequality, we obtain

$$\mathbb{E}\left[\left\{\int_0^t \delta e^{-\delta(t-s)}(\hat{\xi}_s^I - \hat{\xi}_t^I) ds\right\}^2\right] \leq \int_0^t \delta e^{-\delta(t-s)} \mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] ds.$$

Combining this with the previous estimates yields

$$\begin{aligned}&\int_0^t \delta e^{-\delta(t-s)} \mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] ds \\ &= \int_0^{t/2} \delta e^{-\delta(t-s)} \mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] ds + \int_{t/2}^t \delta e^{-\delta(t-s)} \mathbb{E}\left[\left(\hat{\xi}_t^I - \hat{\xi}_s^I\right)^2\right] ds \\ &\leq C_2 e^{-\delta t/2} t^{2H} + C_1 \int_{t/2}^t \delta e^{-\delta(t-s)} (t-s)^2 \left\{\int_0^\infty \delta e^{-\delta v}(v+s)^{2H-2} dv\right\} ds \\ &\leq C_2 e^{-\delta t/2} t^{2H} + C_1 \int_{t/2}^t e^{-\delta(t-s)} (t-s)^2 s^{2H-2} ds \\ &\leq C_2 e^{-\delta t/2} t^{2H} + C_1 \Gamma(3) t^{2H-2} \to 0,\end{aligned}$$

as $t \to \infty$. This allows to conclude that the first term in (1.13) vanishes in $L^2(\mathbb{P})$ as $t \to \infty$. For the second term of (1.13) we can use estimate (1.14) with $s = 0$ to obtain for some constant $C > 0$

$$\mathbb{E}\left[\left\{\hat{\xi}_t^I(\eta_t - 1)\right\}^2\right] \leq C e^{-2\delta t} t^{2H} \to 0,$$

as $t \to \infty$. It follows that $\mathbb{E}\left[(X_t^I - \hat{\xi}_t^I)^2\right]$ converges to 0 as $t \to \infty$. Combining this with (1.12), we conclude that $\mathbb{E}\left[(X_t^I - I_t)^2\right] \to 0$ as $t \to 0$. Moreover, since

$$\mathbb{E}\left[(u_t^I)^2\right] = \delta^2 \mathbb{E}\left[(X_t^I - \hat{\xi}_t^I)^2\right] \to 0,$$

as $t \to \infty$, this concludes the proof. $\square$

The previous result allows us to obtain the optimal solution to the ergodic control problem of fBM.

Corollary 1.2.10. *Consider the linear-quadratic ergodic control of fBM described by the minimisation of*

$$\min_{u \in \mathcal{A}} J(u, \alpha)$$

with

$$J(u, \alpha) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T q(X_t - \alpha B_t^H)^2 dt + \int_0^T r(u_t)^2 dt \right],$$

over the set $\mathcal{A}$ of admissible strategies, i.e., adapted strategies such that $J(u, \alpha) < \infty$. This problem admits an optimal solution,

$$\begin{aligned} \hat{\xi}_t^B &= \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} \alpha B_s^H ds \,\middle|\, \mathcal{F}_t \right], \\ u_t^B &= \delta(\hat{\xi}_t^B - X_t^B), \\ X_t^B &= x_0 + \int_0^t u_s^B ds, \end{aligned}$$

for $\delta = \sqrt{q/r}$. Moreover, the minimised objective is given by

$$J(u; \alpha) = \alpha^2 \frac{q\Gamma(2H+1)}{\delta^{2H}} \left\{ \frac{1 + \sin(\pi H)}{2} \right\}.$$

Proof. Without loss of generality, we consider $\alpha = 1$. From the representation (1.10), we have

$$W_t^H = \frac{c_H}{\Gamma(H + 1/2)} B_t^H - I_t,$$

We know from Proposition 1.2.7 that the optimal solution to the ergodic control of RLfBM is characterised by the signal

$$\begin{aligned} \hat{\xi}_t^W &= \mathbb{E}\left[\int_t^\infty \delta e^{-\delta(s-t)} \alpha W_s^H ds \,\middle|\, \mathcal{F}_t \right] \\ &= \frac{c_H}{\Gamma(H + 1/2)} \hat{\xi}_t^B - \hat{\xi}_t^I, \end{aligned}$$

where we use representation (1.10) and the linearity of the signal in the target to separate it into two signals. We denote $a_H = c_H / \Gamma(H + 1/2)$, and the decomposition of the signal $\hat{\xi}^W$ allows to decompose the optimal rate $\hat{u} = u^W$ and the position $\hat{X} = X^W$ as

$$\begin{aligned} u_t^W &= a_H u_t^B - u_t^I \\ X_t^W &= a_H X_t^B - X_t^I. \end{aligned}$$

Considering subproblems corresponding to B^H and I separately, we have seen in Proposition 1.2.9 that the subproblem linked to the process I is not relevant in the ergodic control problem. More precisely, we know

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(X_t^W - W_t^H)^2 dt + \int_0^T r(u_t^W)^2 dt\right] < \infty.$$

Hence, we have,

$$\begin{aligned}\mathbb{E}\left[(X_t^W - W_t^H)^2\right] &= \mathbb{E}\left[(a_H X_t^B - a_H B_t^H)^2\right] + \mathbb{E}\left[(I_t - X_t^I)^2\right] \\ &\quad + 2\mathbb{E}\left[(a_H X_t^B - a_H B_t^H)(I_t - X_t^I)\right], \end{aligned} \tag{1.15}$$

$$\mathbb{E}\left[(u_t^W)^2\right] = \mathbb{E}\left[(a_H u_t^B)^2\right] + \mathbb{E}\left[(u_t^I)^2\right] - 2a_H \mathbb{E}\left[u_t^B u_t^I\right]. \tag{1.16}$$

For both equalities above, we know that their limit is finite as $t \to \infty$. Combining

$$\begin{aligned}\left|\mathbb{E}\left[(a_H X_t^B - a_H B_t^H)(I_t - X_t^I)\right]\right| &\le \mathbb{E}\left[(a_H X_t^B - a_H B_t^H)^2\right]^{1/2} \mathbb{E}\left[(I_t - X_t^I)^2\right]^{1/2}, \\ \left|\mathbb{E}\left[u_t^B u_t^I\right]\right| &\le \mathbb{E}\left[(u_t^B)^2\right]^{1/2} \mathbb{E}\left[(u_t^I)^2\right]^{1/2},\end{aligned}$$

together with the fact from Proposition 1.2.9 that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[(I_t - X_t^I)^2\right] &= 0, \\ \lim_{t\to\infty} \mathbb{E}\left[(u_t^I)^2\right] &= 0,\end{aligned}$$

it must be that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left[(a_H X_t^B - a_H B_t^H)^2\right] &< \infty, \\ \lim_{t\to\infty} \mathbb{E}\left[(u_t^B)^2\right] &< \infty,\end{aligned}$$

as this would otherwise contradict the finiteness of the limits in (1.15) and (1.16) as $t \to \infty$. This also allows to see that the component I does not matter for the ergodic control problem of B_t^H and the optimality of the processes $\hat{\xi}_t^B$, u_t^B and X_t^B is verified.

Moreover, the minimised objective value is obtained directly from the minimised objective value for the RLfBM ergodic control as in Proposition 1.2.7, by multiplying with the constant a_H^{-2}. Hence, we have

$$\begin{aligned}J(u^B;1) &= \left\{\frac{\Gamma(H+1/2)}{c_H}\right\}^2 \frac{q}{\delta^{2H}} \frac{1}{2}\left(1 + \csc(\pi H)\right) \\ &= \frac{q}{\delta^{2H}}\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\},\end{aligned}$$

where the last equality is detailed in the technical section 1.3.5. □

Remark 1.2.11. *The result in Corollary 1.2.10 was derived for the case* $H > 1/2$ *in Kleptsyna* et al. *(2005). The argument exposed in the paper relies on the Volterra rep-*

resentation of fBM and can also be adapted to the case $H < 1/2$. This requires working with the Volterra representation for $H < 1/2$ and update computations to account for the kernel difference. For convenience we recall the Volterra representation for $H < 1/2$,

$$B_t^H = \int_0^t K_H(t,s)dW_s,$$

where for $t > s$,

$$K_H(t,s) = b_H \left\{ \left(\frac{t}{s}\right)^{H-1/2} (t-s)^{H-1/2} - \left(H - \frac{1}{2}\right) s^{1/2-H} \int_s^t (u-s)^{H-1/2} u^{H-3/2} du \right\},$$

for some constant $b_H > 0$.

In Chapter 3, we provide further interpretation of the formula obtained in Corollary 1.2.10 and highlight the differences with the case $H = 1/2$.

1.3 Technical Section

1.3.1 Proof of Propositions 1.2.1 and 1.2.2

Proof of Proposition 1.2.1. Without loss of generality, we set $\alpha = 1$ and denote $J^{OU}(u;\alpha)$ by $J^{OU}(u)$. To accomodate with usual notation in the existing literature such as Arapostathis *et al.* (2012), we rewrite the above ergodic control problem by introducing the process $Z_t = (Z_t^1, Z_t^2)^T \in \mathbb{R}^2$, satisfying

$$\begin{aligned} dZ_t &= \begin{pmatrix} -\gamma Z_t^1 \\ u_t \end{pmatrix} dt + \begin{pmatrix} 1 \\ 0 \end{pmatrix} dW_t, \\ Z_0 &= \begin{pmatrix} 0 \\ x_0^\gamma \end{pmatrix}. \end{aligned} \tag{1.17}$$

The first component of Z corresponds to the Ornstein-Uhlenbeck process while the second is simply the controlled process. In other words, we have $Z^1 \equiv Y^\gamma$ and $Z^2 \equiv X^\gamma$. The objective $J^{OU}(u)$ is rewritten as

$$J^{OU}(u) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \left\{ q(Z_t^1 - Z_t^2)^2 + r(u_t)^2 \right\} dt \right]. \tag{1.18}$$

We know from Arapostathis *et al.* (2012), that the HJB equation for ergodic control problems is derived by considering the HJB equation of the infinite horizon discounted problem and letting the discount factor go to 0. Hence, the HJB equation related to ergodic control problems described by (1.17) and (1.18) corresponds to finding a pair (V, η), where V is a real function, $V(z)$ for $z = (z_1, z_2)^T \in \mathbb{R}^2$, and $\eta \geq 0$, such that,

$$\min_u \left\{ q(z_1 - z_2)^2 + ru^2 - \eta + V_{z_1}(z)(-\gamma z_1) + V_{z_2}(z)u + \frac{1}{2}V_{z_1 z_1}(z) \right\} = 0. \tag{1.19}$$

In particular, a candidate for optimal control in the feedback form is given by

$$u(z) = \frac{-V_{z_2}(z)}{2r}.$$

From the linear-quadratic nature of the problem, we know the function $V(z)$ should exhibit a dependence on the squared deviation between linear functions of the target and the controlled position. Hence, it should be of the form

$$V(z) = a(z_1 - bz_2)^2 + c,$$

for some $a, b, c \in \mathbb{R}$. We therefore make the following Ansatz for the function $V(z)$,

$$V(z) = a(z_1)^2 + b(z_2)^2 + cz_1 z_2 + d, \tag{1.20}$$

for $a, b, c, d \in \mathbb{R}$, the equation (1.19) rewrites as

$$a - \eta + \left(q - \frac{c^2}{4r} - 2a\gamma\right) z_1^2 + \left(q - \frac{b^2}{r}\right)(z_2)^2 + \left(-2q - \frac{bc}{r} - c\gamma\right) z_1 z_2 = 0,$$

for all $z \in \mathbb{R}^2$. Solving for a, b, c, the solution triplet is given by

$$a = \frac{2q^{3/2}\sqrt{r} + qr\gamma}{2(\sqrt{q} + \sqrt{r}\gamma)^2}, \quad b = \sqrt{qr}, \quad c = -\frac{2q\sqrt{r}}{\sqrt{q} + \sqrt{r}\gamma},$$

and

$$\eta = a.$$

In particular, the feedback control is given by

$$u(z) = \frac{-2bz_2 - cz_1}{2r} = \frac{\delta^2}{(\delta + \gamma)} z_1 - \delta z_2. \tag{1.21}$$

We now proceed to the verification of our Ansatz. Let u be any admissible control, in the sense that $J^{OU}(u) < \infty$. Let (V, η) be a solution to the HJB equation (1.19) with V a function in the form (1.20). By Itô's formula, we have for the process $V(Z_t)$,

$$\begin{aligned} dV(Z_t) = &\left[V_{z_1}(Z_t)(-\gamma Z_t^1) + V_{z_2}(Z_t)u_t + \frac{1}{2}V_{z_1 z_1}(Z_t)\right] dt \\ &+ V_{z_1}(Z_t)dW_t. \end{aligned} \tag{1.22}$$

Let us now write,

$$\begin{aligned}
&\int_0^T \left[q(Z_t^1 - Z_t^2)^2 + r(u_t)^2\right] dt \\
&= \int_0^T \left[q(Z_t^1 - Z_t^2)^2 + r(u_t)^2 - \eta + V_{z_1}(Z_t)(-\gamma Z_t^1) + V_{z_2}(Z_t)u_t + \frac{1}{2}V_{z_1 z_1}(Z_t)\right] dt \\
&\quad + \eta T + \int_0^T V_{z_1}(Z_t)dW_t + V(Z_0) - V(Z_T).
\end{aligned}$$

But since (V, η) satisfies the HJB equation (1.19), we obtain the inequality

$$\int_0^T \left[q(Z_t^1 - Z_t^2)^2 + r(u_t)^2\right] dt \geq \eta T + V(Z_0) - V(Z_T) + \int_0^T V_{z_1}(Z_t)dW_t. \tag{1.23}$$

By admissibility of u, we have

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{E}\left[(Z_t^1 - Z_t^2)^2\right] dt < \infty, \tag{1.24}$$

which implies,

$$\lim_{T\to\infty} \mathbb{E}\left[(Z_T^1 - Z_T^2)^2\right] < \infty, \quad \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(Z_T^1 - Z_T^2)^2\right] = 0. \tag{1.25}$$

Moreover, since Z^1 equals the Ornstein-Uhlenbeck process Y^γ,

$$\begin{aligned}
dZ_t^1 &= -\gamma Z_t^1 dt + dW_t, \\
Z_0^1 &= 0,
\end{aligned}$$

with mean-reversion speed γ, we have the following long-term behavior,

$$\lim_{T\to\infty} \mathbb{E}\left[(Z_T^1)^2\right] < \infty, \quad \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(Z_T^1)^2\right] = 0. \tag{1.26}$$

Using (1.25) and (1.26), we also obtain

$$\begin{aligned}
\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(Z_T^2)^2\right] &= \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[(Z_T^2 - Z_T^1 + Z_T^1)^2\right] \\
&\leq \lim_{T\to\infty} \frac{1}{T}\left\{2\mathbb{E}\left[(Z_T^1 - Z_T^2)^2\right] + 2\mathbb{E}\left[(Z_T^1)^2\right]\right\} = 0. \qquad (1.27)
\end{aligned}$$

Since,

$$\mathbb{E}\left[V(Z_T)\right] = a\mathbb{E}\left[(Z_T^1)^2\right] + b\mathbb{E}\left[(Z_T^2)^2\right] + c\mathbb{E}\left[Z_T^1 Z_T^2\right] + d,$$

and

$$\left|\mathbb{E}\left[Z_T^1 Z_T^2\right]\right| \leq \mathbb{E}\left[(Z_T^1)^2\right]^{1/2} \mathbb{E}\left[(Z_T^2)^2\right]^{1/2},$$

we conclude, using (1.26) and (1.27), that

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[V(Z_T)\right]=0. \tag{1.28}$$

Moreover, by (1.24), we have

$$\mathbb{E}\left[\int_0^T (Z_t^1-Z_t^2)^2dt\right]<\infty,\quad \forall T>0,$$

and similarly to (1.27), together with the fact that Z^1 is an Ornstein-Uhlenbeck process, we obtain

$$\mathbb{E}\left[\int_0^T (Z_t^2)^2dt\right]\le 2\mathbb{E}\left[\int_0^T (Z_t^1-Z_t^2)^2dt\right]+2\mathbb{E}\left[\int_0^T (Z_t^1)^2dt\right]<\infty,\quad \forall T>0.$$

Since $V_{z_1}(Z_t)=2aZ_t^1+cZ_t^2$, we can conclude

$$\mathbb{E}\left[\int_0^T V_{z_1}(Z_t)^2dt\right]\le 2\mathbb{E}\left[\int_0^T 4a^2(Z_t^1)^2dt\right]+2\mathbb{E}\left[\int_0^T c^2(Z_t^2)^2dt\right]<\infty. \tag{1.29}$$

Taking now expectation in (1.23) and using (1.29), we see that the stochastic integral is a true martingale. This implies

$$\mathbb{E}\left[\int_0^T \left\{q(Z_t^1-Z_t^2)^2+r(u_t)^2\right\}dt\right]\ge \eta T+V(Z_0)-\mathbb{E}\left[V(Z_T)\right].$$

Dividing by T and taking the limit, together with (1.28), we have

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \left\{q(Z_t^1-Z_t^2)^2+r(u_t)^2\right\}dt\right]\ge\eta.$$

Hence, we have shown that $J^{OU}(u)\ge\eta$, for any admissible control u. Considering our feedback control $u(Z_t)$ from (1.21), we obtain (1.23) with equality. Therefore, we need to verify (1.28) and (1.29) to conclude our proof.

To this end, since $Z_t^1=Y_t^\gamma$, we notice that

$$u(Z_t)=\delta\left(\frac{\delta}{\delta+\gamma}Y_t^\gamma-Z_t^2\right).$$

But since $dZ_t^2=u(Z_t)dt$, we can solve explicitly and obtain

$$Z_T^2=e^{-\delta T}x_0^\gamma+\int_0^T \delta e^{-\delta(t-s)}\frac{\delta}{\delta+\gamma}Y_s^\gamma ds.$$

In what follows, we can assume $x_0^\gamma=0$, without loss of generality. We estimate, by

Jensen's inequality,

$$\mathbb{E}\left[(Z_T^2)^2\right] \leq (1-e^{-\delta T})\left(\frac{\delta}{\delta+\gamma}\right)^2 \int_0^T \delta e^{-\delta(T-s)}\mathbb{E}\left[(Y_s^\gamma)^2\right] ds.$$

Introducing for simplification an absorbing constant $C > 0$ and using the variance of the Ornstein-Uhlenbeck process,

$$\mathbb{E}\left[(Y_s^\gamma)^2\right] = \frac{1}{2\gamma}\left(1-e^{-2\gamma s}\right),$$

we get

$$\mathbb{E}\left[(Z_T^2)^2\right] \leq C\left[\frac{1-e^{-\delta T}}{\delta} - \frac{e^{-2\gamma T}-e^{-\delta T}}{\delta-2\gamma}1_{\{\delta\neq 2\gamma\}} - e^{-\delta T}T1_{\{\delta=2\gamma\}}\right], \tag{1.30}$$

and therefore,

$$\lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left[(Z_T^2)^2\right] = 0. \tag{1.31}$$

Using (1.31) and (1.26), we obtain (1.28) similarly as for any admissible control u.

To show that (1.29) holds, again assuming without loss of generality $x_0^\gamma = 0$, we use (1.30) to show

$$\begin{aligned}\mathbb{E}\left[\int_0^T (Z_t^2)^2 dt\right] &= \int_0^T \mathbb{E}\left[(Z_t^2)^2\right] dt \\ &\leq C\int_0^T \left[\frac{1-e^{-\delta t}}{\delta} - \frac{e^{-2\gamma t}-e^{-\delta t}}{\delta-2\gamma}1_{\{\delta\neq 2\gamma\}} - e^{-\delta t}t1_{\{\delta=2\gamma\}}\right] dt < \infty,\end{aligned}$$

for every $T > 0$. Since we have the same property applying for Z_t^1, we can conclude that (1.29) holds. Hence, for our control $u(Z_t)$, we obtain

$$\limsup_{T\to\infty}\frac{1}{T}\mathbb{E}\left[\int_0^T \left\{q(Z_t^1 - Z_t^2)^2 + ru(Z_t)^2\right\} dt\right] = \eta.$$

This completes the proof. □

Proof of Proposition 1.2.2. The proof follows the exact same lines as Proposition 1.2.1.

This requires defining the process $Z_t \in \mathbb{R}^{N+1}$ as

$$dZ_t = \begin{pmatrix} -\gamma_1 Z_t^1 \\ \vdots \\ -\gamma_N Z_t^N \\ u_t \end{pmatrix} dt + \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_N \\ 0 \end{pmatrix} dW_t,$$

$$Z_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ x_0^N \end{pmatrix}.$$

The HJB equation is then modified accordingly and the optimal feedback control is of the form, for $z \in \mathbb{R}^{N+1}$,

$$u(z) = \sum_{i=1}^N \alpha_i \frac{\delta^2}{\delta + \gamma_i} z_i - \delta z_{N+1}.$$

The verification argument can then easily be arranged to accomodate for the target Y^N as it suffices to extract each subcomponent Y^{γ_i} in the estimates and follow the one dimensional case. □

1.3.2 Proof of Lemma 1.2.4

Proof of Lemma 1.2.4. For $Y_t^\gamma = \int_0^t e^{-\gamma(t-s)} dW_s$, we have

$$\mathbb{E}\left[Y_t^\gamma Y_t^{\bar\gamma}\right] = \frac{1 - e^{-t(\gamma+\bar\gamma)}}{\gamma + \bar\gamma} \to \frac{1}{\gamma+\bar\gamma}, \tag{1.32}$$

as $t \to \infty$.
Next, we recall that $\hat\xi_t^\gamma = \frac{\delta}{\delta+\gamma} Y_t^\gamma$. Hence, we have, as $t \to \infty$,

$$\mathbb{E}\left[\hat\xi_t^\gamma Y_t^{\bar\gamma}\right] = \frac{\delta}{\delta+\gamma} \frac{1 - e^{-t(\gamma+\bar\gamma)}}{\gamma + \bar\gamma} \to \frac{\delta}{\delta+\gamma}\frac{1}{\gamma+\bar\gamma}.$$

In a similar way, we obtain

$$\mathbb{E}\left[\hat\xi_t^\gamma \hat\xi_t^{\bar\gamma}\right] \to \frac{\delta}{\delta+\gamma}\frac{\delta}{\delta+\bar\gamma}\frac{1}{\gamma+\bar\gamma}.$$

Next, using $X_t^\gamma = e^{-\delta t} x_0^\gamma + \int_0^t \delta e^{-\delta(t-s)} \hat\xi_s^\gamma ds$, we consider,

$$\mathbb{E}\left[X_t^\gamma Y_t^{\bar\gamma}\right] = \int_0^t \delta e^{-\delta(t-s)} \mathbb{E}\left[\hat\xi_s^\gamma Y_t^{\bar\gamma}\right] ds = \int_0^t \delta e^{-\delta(t-s)} \frac{\delta}{\delta+\gamma} \mathbb{E}\left[Y_s^\gamma Y_t^{\bar\gamma}\right] ds,$$

where the term involving x_0^γ vanishes since Y^γ is a centered Gaussian process. We need to compute, for $0 \le s \le t$,

$$\begin{aligned}\mathbb{E}\left[Y_s^\gamma Y_t^{\bar\gamma}\right] &= \mathbb{E}\left[\int_0^s e^{-\gamma(s-v)}dW_v \int_0^s e^{-\bar\gamma(t-v)}dW_v\right] = e^{-\bar\gamma(t-s)}\mathbb{E}\left[Y_s^\gamma Y_s^{\bar\gamma}\right] \\ &= e^{-\bar\gamma(t-s)}\left(\frac{1-e^{-s(\gamma+\bar\gamma)}}{\gamma+\bar\gamma}\right). \end{aligned} \tag{1.33}$$

Hence, we have,

$$\begin{aligned}\mathbb{E}\left[X_t^\gamma Y_t^{\bar\gamma}\right] &= \frac{\delta^2}{\delta+\gamma}\frac{1}{\gamma+\bar\gamma}\int_0^t e^{-(\delta+\bar\gamma)(t-s)}\left\{1-e^{-s(\gamma+\bar\gamma)}\right\}ds \\ &= \frac{\delta^2}{\delta+\gamma}\frac{1}{\gamma+\bar\gamma}\left[\frac{1-e^{-(\delta+\bar\gamma)t}}{\delta+\bar\gamma} - e^{-(\delta+\bar\gamma)t}\left\{\frac{e^{(\delta-\gamma)t}-1}{\delta-\gamma}\right\}\right], \end{aligned} \tag{1.34}$$

where the term in brackets needs to be understood in the following sense,

$$\left\{\frac{e^{(\delta-\gamma)t}-1}{\delta-\gamma}\right\} = \left\{\frac{e^{(\delta-\gamma)t}-1}{\delta-\gamma}\right\}1_{\{\delta\neq\gamma\}} + t1_{\{\delta=\gamma\}}.$$

As $t\to\infty$, since $\gamma,\bar\gamma>0$, the limit is

$$\lim_{t\to\infty}\mathbb{E}\left[X_t^\gamma Y_t^{\bar\gamma}\right] = \frac{\delta}{\delta+\gamma}\frac{\delta}{\delta+\bar\gamma}\frac{1}{\gamma+\bar\gamma}.$$

Since $\hat\xi_t^{\bar\gamma} = \delta/(\delta+\bar\gamma)Y_t^{\bar\gamma}$, we have

$$\mathbb{E}\left[X_t^\gamma\hat\xi_t^{\bar\gamma}\right] = \frac{\delta}{\delta+\bar\gamma}\mathbb{E}\left[X_t^\gamma Y_t^{\bar\gamma}\right],$$

and the limit follows from the previous result.

For the last limit, we start by writing

$$\begin{aligned}\mathbb{E}\left[X_t^\gamma X_t^{\bar\gamma}\right] &= e^{-2\delta t}(x_0^\gamma)^2 + \mathbb{E}\left[\int_0^t \delta e^{-\delta(t-s)}\hat\xi_s^\gamma ds\int_0^t \delta e^{-\delta(t-r)}\hat\xi_r^{\bar\gamma}dr\right] \\ &= e^{-2\delta t}(x_0^\gamma)^2 + \int_0^t\int_0^t \delta e^{-\delta(t-s)}\delta e^{-\delta(t-r)}\mathbb{E}\left[\hat\xi_s^\gamma\hat\xi_r^{\bar\gamma}\right]dsdr.\end{aligned}$$

In what follows, we assume $x_0^\gamma = 0$ for simplicity. Notice that, using (1.33), we have,

$$\mathbb{E}\left[\hat\xi_s^\gamma\hat\xi_r^{\bar\gamma}\right] = \frac{\delta^2}{(\delta+\gamma)(\delta+\bar\gamma)}\mathbb{E}\left[Y_s^\gamma Y_r^{\bar\gamma}\right] = \begin{cases}\frac{\delta^2}{(\delta+\gamma)(\delta+\bar\gamma)}e^{-\bar\gamma(r-s)}\left(\frac{1-e^{-s(\gamma+\bar\gamma)}}{\gamma+\bar\gamma}\right) & : \quad r\ge s \\ \frac{\delta^2}{(\delta+\gamma)(\delta+\bar\gamma)}e^{-\gamma(s-r)}\left(\frac{1-e^{-r(\gamma+\bar\gamma)}}{\gamma+\bar\gamma}\right) & : \quad r\le s\end{cases}.$$

Hence, we separate the expectation computation as follows,

$$
\begin{aligned}
&\mathbb{E}\left[X_t^{\gamma}X_t^{\bar{\gamma}}\right] \\
&= \int_0^t\int_0^s \delta e^{-\delta(t-s)}\delta e^{-\delta(t-r)}\mathbb{E}\left[\hat{\xi}_s^{\gamma}\hat{\xi}_r^{\bar{\gamma}}\right] drds + \int_0^t\int_s^t \delta e^{-\delta(t-s)}\delta e^{-\delta(t-r)}\mathbb{E}\left[\hat{\xi}_s^{\gamma}\hat{\xi}_r^{\bar{\gamma}}\right] drds \\
&=: P_1(t) + P_2(t),
\end{aligned} \tag{1.35}
$$

and we start by computing $P_1(t)$. Denoting

$$
d = \frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})},
$$

we have

$$
\begin{aligned}
\frac{P_1(t)}{d} &= \int_0^t \delta e^{-\delta(t-s)}\int_0^s \delta e^{-\delta(t-r)}e^{-\gamma(s-r)}\left(\frac{1-e^{-r(\gamma+\bar{\gamma})}}{\gamma+\bar{\gamma}}\right) drds \\
&= \frac{\delta}{\gamma+\bar{\gamma}}\int_0^t \delta e^{-\delta(t-s)}\left(e^{-\delta t}e^{-\gamma s}\left[\frac{e^{(\delta+\gamma)s}-1}{\delta+\gamma}\right] - e^{-\delta t}e^{-\gamma s}\left[\frac{e^{(\delta-\bar{\gamma})s}-1}{\delta-\bar{\gamma}}\right]\right) ds,
\end{aligned}
$$

where as in (1.34), the second term in square brackets equals s when $\delta = \bar{\gamma}$. The first part of $P_1(t)/d$ satifies,

$$
\begin{aligned}
&\frac{\delta}{\gamma+\bar{\gamma}}\int_0^t \delta e^{-\delta(t-s)}e^{-\delta t}e^{-\gamma s}\left[\frac{e^{(\delta+\gamma)s}-1}{\delta+\gamma}\right] ds \\
&= \frac{\delta^2}{\gamma+\bar{\gamma}}\frac{1}{\delta+\gamma}e^{-2\delta t}\int_0^t\left\{e^{2\delta s} - e^{(\delta-\gamma)s}\right\} ds \\
&= \frac{\delta^2}{\gamma+\bar{\gamma}}\frac{1}{\delta+\gamma}\left(\frac{1-e^{-2\delta t}}{2\delta} - e^{-2\delta t}\left[\frac{e^{(\delta-\gamma)t}-1}{\delta-\gamma}\right]\right) \\
&\to \frac{\delta^2}{\gamma+\bar{\gamma}}\frac{1}{\delta+\gamma}\frac{1}{2\delta},
\end{aligned}
$$

as $t \to \infty$ and where the fraction in bracket again is to be understood as in (1.34). The second part of $P_1(t)/d$ satisfies,

$$
\begin{aligned}
&\frac{\delta}{\gamma+\bar{\gamma}}\int_0^t \delta e^{-\delta(t-s)}e^{-\delta t}e^{-\gamma s}\left[\frac{e^{(\delta-\bar{\gamma})s}-1}{\delta-\bar{\gamma}}\right] ds \\
&= \frac{\delta^2}{\gamma+\bar{\gamma}}e^{-2\delta t}\int_0^t e^{\delta s}e^{-\gamma s}\left[\frac{e^{(\delta-\bar{\gamma})s}-1}{\delta-\bar{\gamma}}\right] ds \\
&= \frac{\delta^2}{\gamma+\bar{\gamma}}e^{-2\delta t}\left(\frac{1}{\delta-\bar{\gamma}}\left[\int_0^t\left\{e^{(2\delta-\gamma-\bar{\gamma})s} - e^{(\delta-\gamma)s}\right\} ds\right]1_{\{\delta\neq\bar{\gamma}\}} + \left[\int_0^t e^{(\delta-\gamma)s}sds\right]1_{\{\delta=\bar{\gamma}\}}\right) \\
&= \frac{\delta^2}{\gamma+\bar{\gamma}}e^{-2\delta t}\frac{1}{\delta-\bar{\gamma}}\left[\left\{\frac{e^{(2\delta-\gamma-\bar{\gamma})t}-1}{2\delta-\gamma-\bar{\gamma}}\right\} - \frac{e^{(\delta-\gamma)t}-1}{\delta-\gamma}\right]1_{\{\delta\neq\bar{\gamma}\}} \\
&\quad + \frac{\delta^2}{\gamma+\bar{\gamma}}e^{-2\delta t}\left[\frac{e^{(\delta-\gamma)t}\{(\delta-\gamma)t-1\}}{(\delta-\gamma)^2} + \frac{1}{(\delta-\gamma)^2}\right]1_{\{\delta=\bar{\gamma}\}} \\
&\to 0,
\end{aligned}
$$

as $t \to \infty$, and where we used the fact that $\int e^{ax} x dx = (1/a^2)e^{ax}(ax-1)$, for $a \in \mathbb{R}\backslash\{0\}$, and where the fraction in between the curved brackets in the last equality is again to be understood in the sense of (1.34). Hence, we have

$$\frac{P_1(t)}{d} \to \frac{1}{2}\frac{1}{\gamma+\bar{\gamma}}\frac{\delta}{\delta+\gamma}.$$

Using Fubini and by symmetry, we know that the limit for $P_2(t)/d$ can be obtained from the limit of $P_1(t)/d$ where we replace γ by $\bar{\gamma}$. Indeed, we have

$$\begin{aligned}\frac{P_2(t)}{d} &= \int_0^t \int_s^t \delta e^{-\delta(t-s)} \delta e^{-\delta(t-r)} e^{-\bar{\gamma}(r-s)} \left(\frac{1-e^{-s(\gamma+\bar{\gamma})}}{\gamma+\bar{\gamma}}\right) drds \\ &= \int_0^t \delta e^{-\delta(t-r)} \int_0^r \delta e^{-\delta(t-s)} e^{-\bar{\gamma}(r-s)} \left(\frac{1-e^{-s(\gamma+\bar{\gamma})}}{\gamma+\bar{\gamma}}\right) dsdr,\end{aligned}$$

which is $P_1(t)/d$ with $\bar{\gamma}$ instead of γ. It follows that

$$\frac{P_1(t)}{d} \to \frac{1}{2}\frac{1}{\gamma+\bar{\gamma}}\frac{\delta}{\delta+\bar{\gamma}},$$

and therefore,

$$\lim_{t\to\infty} \mathbb{E}\left[X_t^\gamma X_t^{\bar{\gamma}}\right] = \lim_{t\to\infty} P_1(t) + P_2(t) = \frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})}\frac{1}{\gamma+\bar{\gamma}}\frac{\delta}{2}\left(\frac{1}{\delta+\bar{\gamma}}+\frac{1}{\delta+\gamma}\right).$$

□

1.3.3 Proof of Proposition 1.2.7

Proof of Proposition 1.2.7. We start the proof by recalling that by Remark 1.2.6, we can write the optimal trading rate with the signal process as,

$$u_t = \delta(\hat{\xi}_t - X_t).$$

Solving for X_t, given $X_0 = x_0$, we obtain

$$X_t = e^{-\delta t}x_0 + \int_0^t \delta e^{-\delta(t-s)}\hat{\xi}_s ds = e^{-\delta t}x_0 + \int_0^\infty \int_0^t \delta e^{-\delta(t-s)}\hat{\xi}_s^\gamma ds\, \mu(d\gamma),$$

Notice that X_t is a Gaussian process with vanishing mean and $\hat{\xi}_t$ does not depend on x_0. Hence, we can assume for the objective computation, without loss of generality and for simplicity, $x_0 = 0$. We can then write

$$\begin{aligned}X_t &= \int_0^\infty X_t^\gamma \mu(d\gamma), \\ u_t &= \int_0^\infty u_t^\gamma \mu(d\gamma).\end{aligned}$$

with $x_0^\gamma = 0$.

For simplicity, in what follows, we write u instead of $\hat{u}$ and we call deviation the process $X_t - W_t^H$. Computing the minimised objective amounts to computing the stationary variance of both Gaussian processes $X_t - W_t^H$ and $\hat{\xi} - X_t$. Hence, we first compute separately

$$\lim_{t\to\infty} \mathbb{E}\left[(X_t - W_t^H)^2\right], \quad \lim_{t\to\infty} \mathbb{E}\left[(\hat{\xi}_t - X_t)^2\right].$$

Starting with the deviation, we use the integral representations to rewrite

$$\mathbb{E}\left[(X_t - W_t^H)^2\right] = \int_0^\infty \int_0^\infty \mathbb{E}\left[(X_t^\gamma - Y_t^\gamma)(X_t^{\bar{\gamma}} - Y_t^{\bar{\gamma}})\right] \mu(d\gamma)\mu(d\bar{\gamma}).$$

Hence, using the limits as $t \to \infty$ of $\mathbb{E}\left[X_t^\gamma X_t^{\bar{\gamma}}\right], \mathbb{E}\left[X_t^\gamma Y_t^{\bar{\gamma}}\right], \mathbb{E}\left[X_t^{\bar{\gamma}} Y_t^\gamma\right], \mathbb{E}\left[Y_t^{\bar{\gamma}} Y_t^\gamma\right]$, from Lemma 1.2.4 together with a monotone convergence argument, we obtain,

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[(X_t - W_t^H)^2\right] \\
&= \int_0^\infty \int_0^\infty \left[\frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})} \frac{1}{\gamma+\bar{\gamma}} \frac{\delta}{2} \left(\frac{1}{\delta+\gamma} + \frac{1}{\delta+\bar{\gamma}}\right)\right. \\
&\qquad\qquad\qquad \left. - \quad \frac{2}{\gamma+\bar{\gamma}} \frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})} + \frac{1}{\gamma+\bar{\gamma}}\right] \mu(d\gamma)\mu(d\bar{\gamma}).
\end{aligned}$$

Denoting

$$d = \frac{1}{\gamma+\bar{\gamma}} \frac{1}{2} \frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2},$$

the expression in brackets, can be rewritten as

$$d\left\{\delta^3(\gamma+\bar{\gamma}) + 2\delta^2(\bar{\gamma}+\gamma)^2 + 4\delta(\gamma\bar{\gamma}^2 + \gamma^2\bar{\gamma}) + 2\gamma^2\bar{\gamma}^2\right\} =: d\left\{P_1 + P_2 + P_3 + P_4\right\}.$$

Computing now each integrals separately, we have for the first part,

$$\begin{aligned}
&\int_0^\infty \int_0^\infty d(P_1)\mu(d\gamma)\mu(d\bar{\gamma}) = \frac{\delta^3}{2} \int_0^\infty \int_0^\infty \frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2} \mu(d\gamma)\mu(d\bar{\gamma}) \\
&= \frac{\delta^{-2H}}{2\Gamma(H+1/2)^2\Gamma(1/2-H)^2} \int_0^\infty \int_0^\infty \frac{1}{(1+\gamma)^2(1+\bar{\gamma})^2} \gamma^{-H-1/2}\bar{\gamma}^{-H-1/2} d\gamma d\bar{\gamma} \\
&= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2} \frac{1}{8}(1+2H)^2\pi^2\sec(\pi H)^2,
\end{aligned}$$

where $\sec(x) = 1/\cos(x)$. The details for the last equality are described in the section 1.3.4. The formulas obtained for the upcoming double integrals involving P_2, P_3, P_4 and later Q_1, Q_2 are obtained through similar computations as for P_1 and are therefore omitted.

For the second part, we have

$$\int_0^\infty \int_0^\infty d(P_2)\mu(d\gamma)\mu(d\bar{\gamma}) = \delta^2 \int_0^\infty \int_0^\infty \frac{\gamma+\bar{\gamma}}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2}\mu(d\gamma)\mu(d\bar{\gamma})$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\int_0^\infty \int_0^\infty \frac{\gamma+\bar{\gamma}}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{2}(1-4H^2)\pi^2\sec(\pi H)^2.$$

For the third part, we compute,

$$\int_0^\infty \int_0^\infty d(P_3)\mu(d\gamma)\mu(d\bar{\gamma}) = 2\delta \int_0^\infty \int_0^\infty \frac{(\gamma\bar{\gamma}^2+\gamma^2\bar{\gamma})}{\gamma+\bar{\gamma}}\frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2}\mu(d\gamma)\mu(d\bar{\gamma})$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\int_0^\infty \int_0^\infty \frac{(\gamma\bar{\gamma}^2+\gamma^2\bar{\gamma})}{\gamma+\bar{\gamma}}\frac{2}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{2}(1-2H)^2\pi^2\sec(\pi H)^2.$$

The fourth part is computed as,

$$\int_0^\infty \int_0^\infty d(P_4)\mu(d\gamma)\mu(d\bar{\gamma}) = \int_0^\infty \int_0^\infty \frac{\gamma^2\bar{\gamma}^2}{\gamma+\bar{\gamma}}\frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2}\mu(d\gamma)\mu(d\bar{\gamma})$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\int_0^\infty \int_0^\infty \frac{\gamma^2\bar{\gamma}^2}{\gamma+\bar{\gamma}}\frac{1}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{8}\left\{-5-4(H-2)H+4(1-H)\csc(\pi H)\right\}\pi^2\sec(\pi H)^2,$$

where $\csc(x) = 1/\sin(x)$. Summing up the four parts and simplfying, we obtain,

$$\lim_{t\to\infty}\mathbb{E}\left[(X_t - W_t^H)^2\right]$$
$$= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{2}(1-H)\left\{1+\csc(\pi H)\right\}\pi^2\sec(\pi H)^2.$$

We follow a similar approach for the optimal rate. We have,

$$\mathbb{E}\left[(u_t)^2\right] = \delta^2\int_0^\infty \int_0^\infty \mathbb{E}\left[(\hat{\xi}_t^\gamma - X_t^\gamma)(\hat{\xi}_t^{\bar{\gamma}} - X_t^{\bar{\gamma}})\right]\mu(d\gamma)\mu(d\bar{\gamma}),$$

and using Lemma 1.2.4 results together with the monotone convergence argument allows us to write

$$\lim_{t\to\infty}\mathbb{E}\left[(u_t)^2\right]$$
$$= \delta^2\int_0^\infty \int_0^\infty \Bigg[\frac{\delta}{\delta+\gamma}\frac{\delta}{\delta+\bar{\gamma}}\frac{1}{\gamma+\bar{\gamma}} - \frac{\delta}{\delta+\gamma}\left(\frac{\delta}{\delta+\bar{\gamma}}\right)^2\frac{1}{\gamma+\bar{\gamma}} - \frac{\delta}{\delta+\bar{\gamma}}\left(\frac{\delta}{\delta+\gamma}\right)^2\frac{1}{\gamma+\bar{\gamma}}$$
$$+ \frac{\delta^2}{(\delta+\gamma)(\delta+\bar{\gamma})}\frac{1}{\gamma+\bar{\gamma}}\frac{\delta}{2}\left(\frac{1}{\delta+\gamma}+\frac{1}{\delta+\bar{\gamma}}\right)\Bigg]\mu(d\gamma)\mu(d\bar{\gamma}).$$

The expression in bracket can be rewritten as,

$$d\left[2\delta^2\gamma\bar{\gamma}+\delta^3(\gamma+\bar{\gamma})\right] =: d\left[Q_1+Q_2\right],$$

which again, allows us to compute each integrals separately. For the first part, we have,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty d(Q_1)\mu(d\gamma)\mu(d\bar{\gamma}) = \delta^2\int_0^\infty\int_0^\infty \frac{\gamma\bar{\gamma}}{\gamma+\bar{\gamma}}\frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2}\mu(d\gamma)\mu(d\bar{\gamma})\\
&= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\int_0^\infty\int_0^\infty \frac{\gamma\bar{\gamma}}{\gamma+\bar{\gamma}}\frac{1}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}\\
&= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{8}\left(4H\csc(\pi H)-4H^2-1\right)\pi^2\sec(\pi H)^2.
\end{aligned}$$

For the second part, we have,

$$\begin{aligned}
&\int_0^\infty\int_0^\infty d(Q_2)\mu(d\gamma)\mu(d\bar{\gamma}) = \frac{\delta^3}{2}\int_0^\infty\int_0^\infty \frac{1}{(\delta+\gamma)^2(\delta+\bar{\gamma})^2}\mu(d\gamma)\mu(d\bar{\gamma})\\
&= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{2}\int_0^\infty\int_0^\infty \frac{1}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}\\
&= \frac{\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{8}(1+2H)^2\pi^2\sec(\pi H)^2.
\end{aligned}$$

Summing up the two parts, we obtain,

$$\lim_{t\to\infty}\mathbb{E}\left[(u_t)^2\right] = \frac{\delta^{2-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\frac{1}{2}H\left\{1+\csc(\pi H)\right\}\pi^2\sec(\pi H)^2.$$

Therefore, using $\delta=\sqrt{q/r}$, the minimised objective value is given by

$$\begin{aligned}
J(u;1) &= q\lim_{t\to\infty}\mathbb{E}\left[(X_t-W_t^H)^2\right] + r\lim_{t\to\infty}\mathbb{E}\left[(u_t)^2\right]\\
&= \frac{q\delta^{-2H}}{\Gamma(H+1/2)^2\Gamma(1/2-H)^2}\left\{\frac{1+\csc(\pi H)}{2}\right\}\pi^2\sec(\pi H)^2\\
&= q\delta^{-2H}\frac{1}{2}\left\{1+\csc(\pi H)\right\}
\end{aligned}$$

where the last equality follows from the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},\quad z\notin\mathbb{Z},$$

and the identity $\sin(x+\pi/2)=\cos(x)$. □

1.3.4 Additional details for the Proof of Proposition 1.2.7

We now provide the details for the following equality found in the proof of Proposition 1.2.7,

$$\frac{1}{2}\int_0^\infty\int_0^\infty \frac{1}{(1+\gamma)^2(1+\bar{\gamma})^2}\gamma^{-H-1/2}\bar{\gamma}^{-H-1/2}d\gamma d\bar{\gamma}$$
$$=\frac{1}{8}(1+2H)^2\pi^2\sec(\pi H)^2.$$

Since this double integral is simply the product of two integrals, we need to show

$$\int_0^\infty \frac{1}{(1+\gamma)^2}\gamma^{-H-1/2}d\gamma = \frac{1}{4}(1+2H)^2\pi^2\sec(\pi H)^2.$$

In the other cases where a double integral involves a non-separable integrand, we obtain similar formulas by integrating with respect to one variable after another while keeping the same methodology presented for this case. We therefore omit to present these computations.

We denote by ${}_2F_1(a,b,c,x)$ the hypergeometric function satisfying for $a\in\mathbb{R}$ and $c>b>0$,

$${}_2F_1(a,b,c,x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}dt, \quad |x|<1,$$

and

$$\frac{\partial_2F_1(a,b,c,x)}{\partial x}(a,b,c,x) = \frac{ab}{c}{}_2F_1(a+1,b+1,c+1,x).$$

In that case, we have the following expression for the indefinite integral, up to an additive constant,

$$\int \frac{1}{(1+\gamma)^2}\gamma^{-H-1/2}d\gamma = -\frac{2}{3+2H}\left(\frac{1}{1+\gamma}\right)^{\frac{3}{2}+H}{}_2F_1\left(\frac{1}{2}+H,\frac{3}{2}+H,\frac{5}{2}+H,\frac{1}{1+\gamma}\right)$$
$$=: g(\gamma).$$

We have

$${}_2F_1\left(\frac{1}{2}+H,\frac{3}{2}+H,\frac{5}{2}+H,\frac{1}{1+\gamma}\right) = \frac{\Gamma\left(\frac{5}{2}+H\right)}{\Gamma\left(\frac{3}{2}+H\right)\Gamma(1)}\int_0^1 t^{H+1/2}\left(1-\frac{t}{1+\gamma}\right)^{-1/2-H}dt$$

so clearly $g(\gamma) \to 0$ as $\gamma \to \infty$. It follows,

$$
\begin{aligned}
\int_0^\infty \frac{1}{(1+\gamma)^2}\gamma^{-H-1/2}d\gamma &= -g(0) \\
&= \frac{2}{3+2H}\,{}_2F_1\left(\frac{1}{2}+H, \frac{3}{2}+H, \frac{5}{2}+H, 1\right) \\
&= \frac{2}{3+2H}\frac{\Gamma\left(\frac{5}{2}+H\right)}{\Gamma\left(\frac{3}{2}+H\right)}\int_0^1 t^{H+1/2}(1-t)^{-1/2-H}dt \\
&= \frac{2}{3+2H}\frac{\Gamma\left(\frac{5}{2}+H\right)}{\Gamma\left(\frac{3}{2}+H\right)}\beta(H+3/2, 1/2-H),
\end{aligned}
$$

where $\beta(a,b)$ denotes the Beta function, i.e., for $a, b > 0$,

$$
\beta(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt = \int_0^\infty t^{a-1}(1+t)^{-a-b}dt
$$

and it satisfies in particular,

$$
\begin{aligned}
\beta(a+1,b) &= \beta(a,b)\frac{a}{a+b} \\
\beta(a,1-a) &= \frac{\pi}{\sin(\pi a)}.
\end{aligned}
\tag{1.36}
$$

Using the above identities for the Beta function, we obtain

$$
\begin{aligned}
\beta(H+3/2, 1/2-H) &= \beta(H+1/2, 1/2-H)(H+1/2) \\
&= \frac{\pi}{\sin(\pi H + \pi/2)}(H+1/2) \\
&= \frac{\pi}{\cos(\pi H)}\frac{1}{2}(1+2H) \\
&= \frac{1}{2}(1+2H)\pi\sec(\pi H),
\end{aligned}
$$

where we recall $\sec(x) = 1/\cos(x)$. Hence, we have

$$
\begin{aligned}
\int_0^\infty \frac{1}{(1+\gamma)^2}\gamma^{-H-1/2}d\gamma &= \frac{2}{3+2H}\frac{\Gamma\left(\frac{5}{2}+H\right)}{\Gamma\left(\frac{3}{2}+H\right)}\frac{1}{2}(1+2H)\pi\sec(\pi H) \\
&= \frac{1}{2}(1+2H)\pi\sec(\pi H),
\end{aligned}
$$

where we used the property $\Gamma(z+1) = z\Gamma(z)$.

1.3.5 Additional details for the Proof of Corollary 1.2.10

Next, to complete the proof of Corollary 1.2.10, we want to show that

$$
\left\{\frac{\Gamma(H+1/2)}{c_H}\right\}^2 \frac{q}{\delta^{2H}}\frac{1}{2}\{1+\csc(\pi H)\} = \frac{q}{\delta^{2H}}\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\},
$$

or equivalently

$$\left\{\frac{\Gamma(H+1/2)}{c_H}\right\}^2 \frac{1}{\sin(\pi H)} = \Gamma(2H+1), \tag{1.37}$$

where we recall

$$c_H^2 = \frac{1}{2H} + \int_0^\infty \left\{(1+u)^{H-1/2} - u^{H-1/2}\right\}^2 du.$$

We start by rewriting

$$\begin{aligned}
&\int_0^\infty \left\{(1+u)^{H-1/2} - u^{H-1/2}\right\}^2 du \\
&= \int_0^\infty \left\{(1+u)^{2H-1} - 2(1+u)^{H-1/2}u^{H-1/2} + u^{2H-1}\right\} du \\
&= \int_0^\infty (1+u)^{2H-1} du - \int_0^\infty u^{2H-1} du + \int_0^\infty \left\{2u^{2H-1} - 2(1+u)^{H-1/2}u^{H-1/2}\right\} du \\
&= -\frac{1}{2H} + 2\int_0^\infty u^{H-1/2}\left\{u^{H-1/2} - (1+u)^{H-1/2}\right\} du.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
c_H^2 &= 2\int_0^\infty u^{H-1/2}\left\{u^{H-1/2} - (1+u)^{H-1/2}\right\} du \\
&= 2^{1-4H}\int_0^\infty t^{-H}(1+t)^{-1-H} dt \\
&= 2^{1-4H}\beta(1-H, 2H) \\
&= 2^{1-4H}\frac{\Gamma(1-H)\Gamma(2H)}{\Gamma(H+1)},
\end{aligned}$$

where the second equality is obtained by a change of variable and the last equality follows from

$$\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

Using the identity

$$\Gamma(z)\Gamma(z+1/2) = 2^{1-2z}\sqrt{\pi}\Gamma(2z),$$

first with $z = H + 1/2$ and then with $z = H$, we obtain

$$\begin{aligned}
c_H^2 &= \frac{2^{1-2H}\Gamma(H+1/2)\Gamma(1-H)\Gamma(2H)}{\Gamma(2H+1)\sqrt{\pi}} \\
&= \frac{\Gamma(H+1/2)^2\Gamma(1-H)\Gamma(H)}{\Gamma(2H+1)\pi}.
\end{aligned}$$

Next we use identity (1.36),

$$\frac{\pi}{\sin(\pi H)} = \Gamma(H)\Gamma(1-H),$$

to obtain

$$c_H^2 = \frac{\Gamma(H+1/2)^2}{\Gamma(2H+1)}\frac{1}{\sin(\pi H)},$$

which is exactly (1.37).

Chapter 2

Small-cost Expansion for Tracking Problems of Rough Targets

This chapter is based on joint work with Dr. Christoph Czichowsky and Prof. Johannes Muhle-Karbe.

2.1 Framework and statement

In this section, we consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a fractional Brownian motion (fBM) $(B_t^H)_{0\leq t\leq T}$ and we denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the completed filtration it generates. We then consider the finite horizon linear-quadratic tracking problem of a rough target ξ that is given by $\xi_t = f(B_t^H)$ for $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^2(\mathbb{R})$. The objective is given by

$$J_T(u) := \mathbb{E}\left[\int_0^T \left\{\nu_s(X_s^u - \xi_s)^2 + \lambda\kappa_s(u_s)^2\right\} ds\right], \tag{2.1}$$

for the regular control

$$X_t^u = x + \int_0^t u_s ds,$$

starting at $x > 0$ and cost coefficients $\nu = (\nu_t)_{0\leq t\leq T}$ and $\kappa = (\kappa_t)_{0\leq t\leq T}$ given by positive adapted processes that will be specified later. The tracking problem then consists of minimising

$$\min_{u\in\mathcal{A}} J_T(u), \tag{2.2}$$

over the set $\mathcal{A}$ of admissible regular controls, that is, $\mathbb{F}$-adapted processes $u = (u_t)_{0\leq t\leq T}$ such that $X_t^u = x + \int_0^t u_s ds$ is well defined for $0 \leq t \leq T$ and $J_T(u) < \infty$.

In general, finding an explicit solution to (2.2) is not possible for random costs coefficients. Indeed, if a solution exists, it is generally given in terms of solutions to BSDEs as in Kohlmann and Tang (2002) in a semimartingale setup.

Motivated by the portfolio optimisation problem with small transactions costs introduced in the Introduction, we consider the small cost limit of the tracking problem (2.2), as $\lambda \to 0$. Our aim is then to derive the leading order expansion of the objective (2.1) as well as asymptotically optimal trading rates $\hat{u}^a$ that attain the leading order in the asymptotic expansion, but are more explicit than the true solution.

Heuristic Argument

Let us assume for this discussion that the optimal trading rate for Problem (2.2) exists and denote it by $\hat{u}^\lambda$. Then, it is clear that the optimised objective $J_T(\hat{u}^\lambda) \to 0$ as $\lambda \to 0$.

Since we here consider a rough target depending on a fBM, we expect the leading order of the minimised objective to be linked via a rescaling of space and time to the self-similarity property of fBM. This together with the structure of our problem in terms of cost and control suggests that the leading order should be λ^H.

It is known that leading order coefficients for such tracking problems are linked to ergodic control problem of the underlying driving noise, see for example ergodic control problems of BM in Cai *et al.* (2017a).

In order to derive the leading order coefficient, the first difficulty that arises comes from the fact that fBM is not a semimartingale and tools from stochastic calculus, such as Itô's formula, are not available. However, because we do a rescaling in space and time, it will turn out that an application of the classical Taylor expansion of our rough target in place of applying Itô formula will be sufficient. More precisely, for $0 < t < s$, with $|t - s| < \delta$, we have

$$f(B_s^H) = f(B_t^H) + f'(B_{u(t,s)}^H)(B_s^H - B_t^H), \tag{2.3}$$

for $u(t,s) \in [t,s]$ by continuity of the paths of fBM. Moreover independence of increments in the classical case is here replaced by the mixing property of increments of fBM. This together with the heuristic approach to derive the leading order coefficient exposed in Cai *et al.* (2017a) can be made precise to provide an expression for the leading order and its full proof in the case where the cost coefficients are constant $\nu_t \equiv \nu$ and $\kappa_t \equiv \kappa$.

In particular, the constant cost coefficient case has an explicit solution that allows straightforward computations, see Bank *et al.* (2017). In the case of random coefficients, the solutions are expressed in terms of BSDEs. We therefore first establish the convergence of the rescaled key BSDE solution. This key BSDE solution describes the urgency for reverting towards some signal process. Then, we express the minimised objective as follows,

$$J_T(\hat{u}^\lambda) = \lambda^H \mathbb{E}\left[\int_0^T \left(\nu_t \left\{\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)\right\}^2 + \kappa_t \left\{\lambda^{(1-H)/2}\hat{u}_t^\lambda\right\}^2\right) dt\right].$$

and show that the rescaled processes, $\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)$ and $\lambda^{(1-H)/2}\hat{u}_t^\lambda$ converge to stationary processes linked to the ergodic control problem of fBM. This derivation relies on the above Taylor expansion and also requires us to first characterise the behavior of the solutions $\hat{X}^\lambda, \hat{u}^\lambda$ with our characterisation of the key BSDE as $\lambda \to 0$.

Assumptions

For completeness, we recall important facts from Nourdin (2012) that will justify our assumptions.

Theorem 2.1.1 (Theorem 4.2 in Nourdin (2012))**.** *Let $X = (X_t)_{t\in[0,T]}$ be a centered and continuous Gaussian process. Set $\sigma^2 = \sup_{t\in[0,T]} Var(X_t)$. Then, $m := \mathbb{E}\left[\sup_{u\in[0,T]} X_u\right]$ is finite and we have, for all $x > m$,*

$$\mathbb{P}\left(\sup_{u\in[0,T]} X_u \geq x\right) \leq e^{-\frac{(x-m)^2}{2\sigma^2}}.$$

For a Gaussian process satisfying $X \overset{d}{=} -X$, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{u\in[0,T]} |X_u| \geq x\right) &\leq \mathbb{P}\left(\sup_{u\in[0,T]} X_u \geq x\right) + \mathbb{P}\left(\sup_{u\in[0,T]} -X_u \geq x\right)\\ &= 2\mathbb{P}\left(\sup_{u\in[0,T]} X_u \geq x\right).\end{aligned}$$

Since for a nonnegative random variable Z, we have

$$\mathbb{E}\left[|Z|^p\right] = \int_0^\infty pt^{p-1}\mathbb{P}\left(Z > t\right)dt, \tag{2.4}$$

it follows that $\sup_{t\in[0,T]} |X_u|$ has all moments for $p \geq 1$. Moreover, (2.4) also implies that $\sup_{u\in[0,T]} |X_u|$ has all exponential moments, i.e, for all $\theta \in \mathbb{R}$,

$$\mathbb{E}\left[e^{\theta \sup_{u\in[0,T]}|X_u|}\right] < \infty.$$

In our application framework in Chapter 3, the targets we consider are of the form $\xi_t = \eta e^{bX_t}$, $\eta > 0, b \in \mathbb{R}$, for X either a fBM, a RLfBM or a fOU. These processes all satisfy the assumption of Theorem 2.1.1. Moreover, the positive cost coefficients ν_t, κ_t for the control problem and their inverses $1/\nu_t, 1/\kappa_t$, are all of the same form as ξ_t. Hence, for a process of the form of ξ_t, we have

$$\mathbb{E}\left[\left|\sup_{t\in[0,T]} \xi_t\right|^p\right] \leq \eta^p\mathbb{E}\left[e^{bp\sup_{t\in[0,T]}|X_t|}\right] < \infty, \quad \forall p \geq 1.$$

This justifies the following assumptions we will work with for our main result. These assumptions are satisfied in our application.

Assumption 2.1.2. *We have $f \in \mathcal{C}^2(\mathbb{R})$, $\nu_t, \kappa_t > 0$ for all $t \in [0,T]$ and for all $p \geq 1$,*

$$\sup_{t\in[0,T]} f(B_t^H) \in L^p(\mathbb{P}),\ \sup_{t\in[0,T]} f'(B_t^H) \in L^p(\mathbb{P})$$
$$\sup_{t\in[0,T]} \nu_t \in L^p(\mathbb{P}),\ \sup_{t\in[0,T]} \frac{1}{\nu_t} \in L^p(\mathbb{P})$$
$$\sup_{t\in[0,T]} \kappa_t \in L^p(\mathbb{P}),\ \sup_{t\in[0,T]} \frac{1}{\kappa_t} \in L^p(\mathbb{P}).$$

Main Result

Theorem 2.1.3. *Under the Assumptions 2.1.2, we have the following small cost leading order expansion of $J_T(\hat{u}^\lambda)$ for the target $\xi_t = f(B_t^H)$:*

$$\begin{aligned} J_T(\hat{u}^\lambda) &= \mathbb{E}\left[\int_0^T \nu_t(\hat{X}_t^\lambda - \xi_t)^2 dt + \lambda \int_0^T \kappa_t(\hat{u}_t^\lambda)^2 dt\right] \\ &= \lambda^H \int_0^T \mathbb{E}\left[I(f'(B_t), \nu_t, \kappa_t)\right] dt + o(\lambda^H), \end{aligned} \tag{2.5}$$

where for $\alpha \in \mathbb{R}$ and $\nu, \kappa > 0$,

$$\begin{aligned} I(\alpha, \nu, \kappa) &= \inf_{u\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \nu(X_t - \alpha B_t^H)^2 dt + \kappa(u_t)^2 dt\right] \\ &=: \inf_{u\in\mathcal{A}} J(u,\alpha) \\ &= \alpha^2 \frac{q}{\delta^{2H}}\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\}, \quad \delta = \sqrt{\frac{\nu}{\kappa}}, \end{aligned}$$

corresponds to the minimised objective of the linear-quadratic ergodic control of fBM over the set of adapted controls $(u_t)_{t\geq 0}$ such that $X_t = x + \int_0^t u_s ds$, and $J(u,\alpha) < \infty$ as studied in Chapter 1.

Moreover, an asymptotically optimal strategy, denoted $\hat{u}_t^a$, which attains the leading order in (2.5) is given by

$$\begin{aligned} \hat{\xi}_t^a &:= f(B_t^H) + f'(B_t^H)\mathbb{E}\left[\int_t^T \sqrt{\frac{\nu_t}{\kappa_t}}\frac{1}{\sqrt{\lambda}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}\frac{1}{\sqrt{\lambda}}(u-t)} \left(B_u^H - B_t^H\right) du \,\middle|\, \mathcal{F}_t\right], \\ \hat{u}_t^a &:= \sqrt{\frac{\nu_t}{\kappa_t}}\frac{1}{\sqrt{\lambda}}\left(\hat{\xi}_t^a - \hat{X}_t^a\right), \\ \hat{X}_t^a &:= x + \int_0^t \hat{u}_s^a ds. \end{aligned}$$

In Chapter 3, we provide interpretation for the formulas of the leading order coefficient obtained in Theorem 2.1.3 and we also simulate realisations of the asymptotically optimal signal process $\hat{\xi}^a$.

2.2 Proof of Theorem 2.1.3

Throughout this proof, we separate for λ small enough, the interval $[0,T]$ into intervals $[0,\lambda^a] \cup [\lambda^a, T-\lambda^a] \cup [T-\lambda^a, T]$, where $a > 0$ satisfies,

$$0 < \max\left\{H, \frac{1-H}{2}\right\} < a < \frac{1}{2}. \tag{2.6}$$

In particular, we have $\lambda^{a-1/2} \to \infty$ as $\lambda \to 0$. We sometimes also denote $\eta(\lambda) = T - \lambda^a$. We essentially show that the main statement holds on the interval $[\lambda^a, T-\lambda^a]$ and that the remaining intervals have effect of order $o(\lambda^H)$.

2.2.1 BSDE solution

Under our Assumptions 2.1.2, we have by Corollary 1 in Kruse and Popier (2016) that a solution to the BSDE (2.8) below exists. Then, by the verification argument of Theorem 3.4 in Bank and Voß (2018), we have the existence of a solution to the control problem (2.2) and its optimal solution is described by

$$\begin{aligned} d\hat{X}_t^\lambda &= \hat{u}_t^\lambda dt \\ \hat{u}_t^\lambda &= \frac{c_t^\lambda}{\lambda\kappa_t}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right), \end{aligned} \tag{2.7}$$

where c_t^λ is given by the BSDE

$$dc_t^\lambda = \left\{\frac{(c_t^\lambda)^2}{\lambda\kappa_t} - \nu_t\right\} dt - dN_t^\lambda, \quad 0 \le t < T, \quad c_T^\lambda = 0, \tag{2.8}$$

for N^λ a martingale, or in its conditional form,

$$c_t^\lambda = \mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{c_r^\lambda}{\lambda\kappa_r} dr} \nu_s \, ds \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t < T. \tag{2.9}$$

The signal process $\hat{\xi}^\lambda = (\hat{\xi}_t^\lambda)_{0\le t\le T}$ is given by the form

$$\hat{\xi}_t^\lambda = \frac{1}{c_t^\lambda}\mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{c_r^\lambda}{\lambda\kappa_r} dr} \nu_s \xi_s \, ds \,\middle|\, \mathcal{F}_t\right], \quad 0 \le t < T.$$

We notice from (2.7) that the optimal rate of trading is described by a reversion towards a process $\hat{\xi}^\lambda$. The *signal process* $\hat{\xi}^\lambda$ weights in future values of the target. This permits to take advantage of any autocorrelation of the target as in the case of the linear-quadratic ergodic control of fBM problem in Chapter 1.

Moreover, we see that the process c^λ provides the urgency at which the reversion towards the signal needs to happen. In particular, we notice it appears on both side of (2.9) making it harder to study. Therefore, characterising the behavior of c^λ as $\lambda \to 0$ will be of crucial importance for our work.

We can derive the above BSDE by considering the value function of the problem (2.2)

$$\operatorname*{ess\,inf}_{\tilde{u}\in C(u,t)} J_t(\tilde{u}) := \operatorname*{ess\,inf}_{\tilde{u}\in C(u,t)} \mathbb{E}\left[\int_t^T \left\{(X_s^{\tilde{u}}-\xi_s)^2\nu_s + \lambda\kappa_s(\tilde{u}_s)^2\right\} ds \,\middle|\, \mathcal{F}_t\right],$$

for $C(u,t)$ being the set of admissible strategies starting at time t for the initial the position X_t^u. In particular, it can be shown that the value function is of the form

$$J_t(u) = \alpha_t(X_t^u)^2 - 2\beta_t X_t^u + \gamma_t.$$

This together with the martingale optimality principle imply that α, β, γ satisfy the following BSDEs,

$$\begin{aligned} d\alpha_t &= \left\{\frac{(\alpha_t^\lambda)^2}{\lambda\kappa_t} - \nu_t\right\} dt - dN_t^\alpha, \quad \alpha_T^\lambda = 0, \\ d\beta_t &= \left\{\frac{\alpha_t}{\lambda k_t}\beta_t - \xi_t\nu_t\right\} dt - dN_t^\beta, \quad \beta_T = 0, \\ d\gamma_t &= \left\{\frac{\beta_t^2}{\lambda\kappa_t} - \xi_t^2\nu_t\right\} dt - dN_t^\gamma, \quad \gamma_T = 0, \end{aligned} \tag{2.10}$$

where $N^\alpha, N^\beta, N^\eta$ are martingales and the terminal conditions encode the terminal constraint. We realise that $\alpha \equiv c^\lambda$ as above and that

$$\beta_t = \mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{\alpha_r}{\lambda\kappa_r}dr}\nu_s\xi_s \, ds \,\middle|\, \mathcal{F}_t\right] = \hat{\xi}_t^\lambda c_t^\lambda.$$

We can therefore also rewrite the value function as

$$\begin{aligned} J_t(u) &= c_t^\lambda\left(X_t^u - \hat{\xi}_t^\lambda\right)^2 + r_t, \\ r_t &= \gamma_t - c_t^\lambda(\hat{\xi}_t^\lambda)^2. \end{aligned} \tag{2.11}$$

We now study the behavior of the BSDE solution c_t^λ as $\lambda \to 0$. Some of the results provided below are given in terms of assumptions that are always implied by our Assumptions 2.1.2. In what follows, we sometimes write $\|\cdot\|_p$ short for $\|\cdot\|_{L^p(\mathbb{P})}$.

Proposition 2.2.1. *Suppose that $\nu = (\nu_t)_{t\in[0,T]}$ and $\kappa = (\kappa_t)_{t\in[0,T]}$ are continuous, $\nu_t, \kappa_t > 0$ for $t \in [0,T]$ and $\sup_{0\le t\le T}\sqrt{\nu_t\kappa_t}$ is in $L^p(\mathbb{P})$ for some $p > 1$. Let $(c_t^\lambda, N_t^\lambda)_{t\in[0,T]}$ be, as in Bank and Voß (2018), the solution to the BSDE*

$$dc_t^\lambda = \left\{\frac{(c_t^\lambda)^2}{\lambda\kappa_t} - \nu_t\right\} dt - dN_t^\lambda, \quad c_T^\lambda = 0, \tag{2.12}$$

given by

$$c_t^\lambda = \mathbb{E}\left[\int_t^T e^{-\int_t^s \frac{c_r^\lambda}{\lambda\kappa_r}dr}\nu_s \, ds \,\middle|\, \mathcal{F}_t\right]. \tag{2.13}$$

Then,

$$\sup_{0\le t\le\eta(\lambda)}\left|\frac{c_t^\lambda}{\sqrt{\lambda}}-\sqrt{\nu_t\kappa_t}\right|\to 0,\ \ as\ \lambda\to 0,$$

$\mathbb{P}$*-a.s. and in* $L^p(\mathbb{P})$.

For the proof of Proposition 2.2.1, we consider first the following intermediary result. We know from our previous discussion that, for $C(u,t)$ being the set of admissible strategies starting at time t for the initial position X_t^u,

$$\operatorname*{ess\,inf}_{\tilde u\in C(u,t)} J_t(\tilde u):=\operatorname*{ess\,inf}_{\tilde u\in C(u,t)}\mathbb{E}\left[\int_t^T\left\{(X_s^{\tilde u}-\xi_s)^2\nu_s+\lambda\kappa_s(\tilde u_s)^2\right\}ds\,\middle|\,\mathcal{F}_t\right]=c_t^\lambda(X_t^u-\hat\xi_t^\lambda)^2+r_t,$$

where c^λ satisfies the BSDE (2.12) and $\hat\xi^\lambda$ and r are as in (2.11). In particular, when $(\xi_v)_{v\in[t,T]}\equiv 1$, we have $\hat\xi^\lambda\equiv 1$. Moreover in the notation from equations (2.10) we have, when $\xi\equiv 1$, $\beta_t\equiv\alpha_t\equiv\gamma_t$ and therefore,

$$r_t=\gamma_t-c_t^\lambda=0.$$

It follows that we can write

$$(X_t^u-1)^2c_t^\lambda=\operatorname*{ess\,inf}_{\tilde u\in C(u,t)} J_t(\tilde u).$$

Choosing the initial position $X_t^u=0$, we obtain the following representation of c^λ as

$$c_t^\lambda=\operatorname*{ess\,inf}_{\tilde u\in C(u,t),X_t^u=0}\mathbb{E}\left[\int_t^T\left\{(X_s^{\tilde u}-1)^2\nu_s+\lambda\kappa_s(\tilde u_s)^2\right\}ds\,\middle|\,\mathcal{F}_t\right].\tag{2.14}$$

Note that representation (2.14) is independent of the target ξ_t and has been proved in similar contexts, see for example Theorem 1.3 in Ankirchner *et al.* (2014). In what follows, we use $J_t(\tilde u)$ to denote the conditional expectation in (2.14). We prove Proposition 2.2.1 in several steps involving the following lemmas.

Lemma 2.2.2. *Let* $(c_t^\lambda)_{0\le t\le T}$ *be as in* (2.14) *and consider*

$$\tilde u_s^\lambda=\frac{1}{\sqrt{\lambda}}\sqrt{\frac{\nu_s}{\kappa_s}}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_t^s\sqrt{\frac{\nu_r}{\kappa_r}}dr\right),$$

for $s\ge t$*. Then,* $X_s^{\tilde u}=\int_t^s\tilde u_v^\lambda dv=1-\exp\left\{-(1/\sqrt{\lambda})\int_t^s\sqrt{\nu_r/\kappa_r}dr\right\}$,

$$\frac{c_t^\lambda}{\sqrt{\lambda}}\le\frac{J_t(\tilde u^\lambda)}{\sqrt{\lambda}}:=\mathbb{E}\left[\int_t^T\exp\left(-\frac{2}{\sqrt{\lambda}}\int_t^s\sqrt{\frac{\nu_r}{\kappa_r}}dr\right)\frac{2}{\sqrt{\lambda}}\sqrt{\frac{\nu_s}{\kappa_s}}\sqrt{\nu_s\kappa_s}ds\,\middle|\,\mathcal{F}_t\right].\tag{2.15}$$

Proof. By the fact that $(\tilde u_s^\lambda)_{t\le s\le T}$ is an admissible strategy in (2.14), we immediately obtain (2.15). □

Lemma 2.2.3. *Under the assumptions of Proposition 2.2.1, we have, for $p > 1$, for all $\eta(\lambda) \in (0, T)$ such that $\eta(\lambda) \to T$ and $\frac{T-\eta(\lambda)}{\sqrt{\lambda}} \to \infty$, as $\lambda \to 0$, that*

$$\sup_{s\in[t,\eta(\lambda)]} \left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du - \sqrt{\nu_s \kappa_s} \right| \to 0,$$

$\mathbb{P}$-a.s. and in $L^p(\mathbb{P})$, as $\lambda \to 0$, and for $\tilde{u}^\lambda$ from Lemma 2.2.2,

$$\tilde{w}(\eta(\lambda), \lambda) := \sup_{r\in[t,\eta(\lambda)]} \left| \frac{J_r(\tilde{u}^\lambda)}{\sqrt{\lambda}} - \sqrt{\nu_r \kappa_r} \right| \to 0,$$

$\mathbb{P}$-a.s. and in $L^p(\mathbb{P})$, as $\lambda \to 0$.

Proof. Fix $\varepsilon > 0$ and choose $\delta \in (0, T - t)$ such that $|\sqrt{\nu_u \kappa_u} - \sqrt{\nu_s \kappa_s}| < \varepsilon$, for all $u, s \in [t, T]$ with $|u - s| < \delta$. Then, writing $\sqrt{\nu_u \kappa_u} = \sqrt{\nu_u \kappa_u} - \sqrt{\nu_s \kappa_s} + \sqrt{\nu_s \kappa_s}$, we obtain, for $s \in [t, \eta(\lambda)]$,

$$\begin{aligned}
&\left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du - \sqrt{\nu_s \kappa_s} \right| \\
&\leq \int_s^{(s+\delta)\wedge T} \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} |\sqrt{\nu_u \kappa_u} - \sqrt{\nu_s \kappa_s}| \, du \\
&\quad + 2 \sup_{u\in[t,T]} |\sqrt{\nu_u \kappa_u}| \int_{(s+\delta)\wedge T}^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} du \\
&\quad + \sqrt{\nu_s \kappa_s} \left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} du - 1 \right| \\
&\leq \varepsilon + 2 \sup_{u\in[t,T]} |\sqrt{\nu_u \kappa_u}| \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^{(s+\delta)\wedge T} \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \left\{1 - \exp\left(-\frac{2}{\sqrt{\lambda}} \int_{(s+\delta)\wedge T}^T \sqrt{\frac{\nu_r}{\kappa_r}} dr\right)\right\} \\
&\quad + \sqrt{\nu_s \kappa_s} \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^T \sqrt{\frac{\nu_r}{\kappa_r}} dr\right),
\end{aligned} \tag{2.16}$$

where the last inequality follows by simply integrating each du integral. Because $t \mapsto \sqrt{\nu_t / \kappa_t} > 0$ and is continuous, we have that

$$\exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^{(s+\delta)\wedge T} \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \leq \exp\left(-\frac{2}{\sqrt{\lambda}} \left\{(T - \eta(\lambda)) \wedge \delta\right\} \inf_{u\in[t,T]} \sqrt{\frac{\nu_u}{\kappa_u}}\right),$$

and

$$\exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^T \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \leq \exp\left(-\frac{2}{\sqrt{\lambda}} \{T - \eta(\lambda)\} \inf_{u\in[t,T]} \sqrt{\frac{\nu_u}{\kappa_u}}\right),$$

for all $s \in [t, \eta(\lambda)]$. Therefore, the right hand side of (2.16) is smaller than 2ε for all $s \in [t, \eta(\lambda)]$, when λ is sufficiently small. This implies that

$$\sup_{s\in[t,\eta(\lambda)]} \left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr\right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du - \sqrt{\nu_s \kappa_s} \right| \overset{\mathbb{P}-a.s.}{\to} 0, \tag{2.17}$$

as $\lambda \to 0$.

Because

$$\left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr \right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du \right| \leq \sup_{s \in [t,T]} \sqrt{\nu_s \kappa_s} \in L^p(\mathbb{P}),$$

we obtain by Lebesgue's dominated convergence theorem that the convergence in (2.17) as $\lambda \to 0$ also holds in $L^p(\mathbb{P})$.

Defining for $v \in [t, T]$,

$$M_v^\lambda = \mathbb{E}\left[\sup_{s \in [t,\eta(\lambda)]} \left| \int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr \right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du - \sqrt{\nu_s \kappa_s} \right| \,\middle|\, \mathcal{F}_v \right],$$

we can apply Doob's maximal inequality as in Proposition 7.16 in Kallenberg (2006) and use the convergence (2.17) in $L^p(\mathbb{P})$ to obtain

$$\sup_{v \in [t,T]} \left| M_v^\lambda \right| \to 0,$$

in $L^p(\mathbb{P})$, as $\lambda \to 0$. Because for $\tilde{u}^\lambda$ from Lemma 2.2.2 we have,

$$\frac{J_s(\tilde{u}^\lambda)}{\sqrt{\lambda}} = \mathbb{E}\left[\int_s^T \exp\left(-\frac{2}{\sqrt{\lambda}} \int_s^u \sqrt{\frac{\nu_r}{\kappa_r}} dr \right) \frac{2}{\sqrt{\lambda}} \sqrt{\frac{\nu_u}{\kappa_u}} \sqrt{\nu_u \kappa_u} du \,\middle|\, \mathcal{F}_s \right],$$

this implies

$$\tilde{w}(\eta(\lambda), \lambda) = \sup_{s \in [t,\eta(\lambda)]} \left| \frac{J_s(\tilde{u}^\lambda)}{\sqrt{\lambda}} - \sqrt{\nu_s \kappa_s} \right| \to 0$$

$\mathbb{P}$-a.s. and in $L^p(\mathbb{P})$, as $\lambda \to 0$. □

Combining Lemmas 2.2.2 and 2.2.3, we already have

$$\limsup_{\lambda \to 0} \frac{c_t^\lambda}{\sqrt{\lambda}} \leq \limsup_{\lambda \to 0} \frac{J_t(\tilde{u}^\lambda)}{\sqrt{\lambda}} = \sqrt{\nu_t \kappa_t},$$

in $L^p(\mathbb{P})$ and $\mathbb{P}$-a.s. We next consider $\liminf_{\lambda \to 0} c_t^\lambda / \sqrt{\lambda}$.

Lemma 2.2.4. *Under the assumptions of Proposition 2.2.1, fix any $\eta(\lambda) \in (0, T)$ such that $\eta(\lambda) \to T$ and $\frac{T - \eta(\lambda)}{\sqrt{\lambda}} \to \infty$, as $\lambda \to 0$. Define $\tilde{\eta}(\lambda) = T - \frac{T - \eta(\lambda)}{2} \in (0, T)$ and*

$$\tilde{J}_t(\eta(\lambda), \lambda) := \mathbb{E}\left[\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s \frac{1}{\sqrt{\lambda} \kappa_r} \left\{ \sqrt{\nu_r \kappa_r} + \tilde{w}(\tilde{\eta}(\lambda), \lambda) \right\} dr \right) \nu_s ds \,\middle|\, \mathcal{F}_t \right]. \quad (2.18)$$

Then, we have that $\tilde{\eta}(\lambda) \to T$ and $\frac{\tilde{\eta}(\lambda) - \eta(\lambda)}{\sqrt{\lambda}} \to \infty$, as $\lambda \to 0$, and

$$\frac{c_t^\lambda}{\sqrt{\lambda}} \geq \frac{\tilde{J}_t(\eta(\lambda), \lambda)}{\sqrt{\lambda}}$$

and

$$\sup_{r\in[t,\eta(\lambda)]}\left|\frac{\tilde{J}_r(\eta(\lambda),\lambda)}{\sqrt{\lambda}}-\sqrt{\nu_r\kappa_r}\right|\to 0,$$

$\mathbb{P}$*-a.s. and in* $L^p(\mathbb{P})$*, for* $p>1$*, as* $\lambda\to 0$*.*

Proof. Using the expression (2.13) of c^λ, we have

$$\begin{aligned}
\frac{c_t^\lambda}{\sqrt{\lambda}} &= \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s \frac{c_r^\lambda}{\lambda\kappa_r}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\,\middle|\,\mathcal{F}_t\right]\\
&\geq \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s \frac{J_r(\tilde{u}^\lambda)}{\lambda\kappa_r}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\,\middle|\,\mathcal{F}_t\right]\\
&\geq \mathbb{E}\left[\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s \frac{1}{\sqrt{\lambda}\kappa_r}\left\{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)\right\}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\,\middle|\,\mathcal{F}_t\right],\\
&= \frac{\tilde{J}_t(\eta(\lambda),\lambda)}{\sqrt{\lambda}},
\end{aligned}$$

where the first inequality follows from Lemma 2.2.2, the second inequality follows from Lemma 2.2.3 and $\tilde{\eta}(\lambda)<T$.

Proceeding now as in the proof of Lemma 2.2.3, we first show that

$$\sup_{0\leq t\leq\eta(\lambda)}\left|\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s \frac{1}{\sqrt{\lambda}\kappa_r}\left\{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)\right\}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds-\sqrt{\nu_t\kappa_t}\right|\to 0, \quad (2.19)$$

$\mathbb{P}$-a.s. as $\lambda\to 0$. Indeed, by writing

$$\begin{aligned}
&\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s \frac{1}{\sqrt{\lambda}\kappa_r}\left\{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)\right\}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\\
&=\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s \frac{1}{\sqrt{\lambda}\kappa_r}\left\{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)\right\}dr\right)\\
&\qquad\qquad\cdot\left(\frac{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\sqrt{\lambda}\kappa_s}\right)\frac{\nu_s\kappa_s}{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}ds,
\end{aligned}$$

we observe that the mollifier

$$\exp\left(-\frac{2}{\sqrt{\lambda}}\int_t^s\sqrt{\frac{\nu_r}{\kappa_r}}dr\right)\frac{2}{\sqrt{\lambda}}\sqrt{\frac{\nu_s}{\kappa_s}}$$

simply needs to be replaced by the mollifier

$$\exp\left(-\frac{1}{\sqrt{\lambda}}\int_t^s\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\frac{1}{\sqrt{\lambda}}\frac{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_s},$$

and the same type of estimates as in (2.16) carry over. We still have

$$\begin{aligned}&\exp\left(-\frac{1}{\sqrt{\lambda}}\int_s^{(s+\delta)\wedge\tilde{\eta}(\lambda)}\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\\&\leq\exp\left(-\frac{\{\tilde{\eta}(\lambda)-\eta(\lambda)\}\wedge\delta}{\sqrt{\lambda}}\inf_{u\in[0,T]}\sqrt{\frac{\nu_u}{\kappa_u}}\right),\end{aligned}$$

and

$$\exp\left(-\frac{1}{\sqrt{\lambda}}\int_s^{\tilde{\eta}(\lambda)}\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\leq\exp\left(-\frac{\tilde{\eta}(\lambda)-\eta(\lambda)}{\sqrt{\lambda}}\inf_{u\in[0,T]}\sqrt{\frac{\nu_u}{\kappa_u}}\right),$$

holds for all $s\in[0,\eta(\lambda)]$. Therefore, we obtain (2.19) as in (2.17). Moreover, by combining

$$\begin{aligned}&\sup_{0\leq t\leq\eta(\lambda)}\left|\int_t^{\tilde{\eta}(\lambda)}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_t^s\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\right|\\&\leq\sup_{0\leq t\leq\eta(\lambda)}\left|\int_t^{\tilde{\eta}(\lambda)}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_t^s\frac{\sqrt{\nu_r\kappa_r}}{\kappa_r}dr\right)\frac{1}{\sqrt{\lambda}}\frac{\kappa_s}{\sqrt{\nu_s\kappa_s}}\sqrt{\nu_s\kappa_s}ds\right|\\&\leq\sup_{0\leq t\leq T}|\sqrt{\nu_t\kappa_t}|\in L^p(\mathbb{P})\end{aligned}$$

together with (2.19), we have by Lebesgue's dominated convergence theorem that

$$\sup_{s\in[0,\eta(\lambda)]}\left|\int_s^{\tilde{\eta}(\lambda)}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_s^u\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds-\sqrt{\nu_s\kappa_s}\right|\to 0,$$

in $L^p(\mathbb{P})$, as $\lambda\to 0$. By Doob's maximal inequality, this also implies that for

$$\begin{aligned}&Y_v:=\\&\mathbb{E}\left[\sup_{s\in[0,\eta(\lambda)]}\left|\int_s^{\tilde{\eta}(\lambda)}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_s^u\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\frac{\nu_u}{\sqrt{\lambda}}ds-\sqrt{\nu_s\kappa_s}\right|\,\middle|\,\mathcal{F}_v\right],\end{aligned}$$

we have

$$\sup_{v\in[0,T]}|Y_v|\to 0,\tag{2.20}$$

in $L^p(\mathbb{P})$, as $\lambda\to 0$. This allows to conclude as in the Proof of Lemma 2.2.3. □

Proof of Proposition 2.2.1. Combining that

$$\begin{aligned}\frac{\tilde{J}_t(\eta(\lambda),\lambda)}{\sqrt{\lambda}}&=\mathbb{E}\left[\int_t^{\tilde{\eta}(\lambda)}\exp\left(-\frac{1}{\sqrt{\lambda}}\int_t^s\frac{\sqrt{\nu_r\kappa_r}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\kappa_r}dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\,\middle|\,\mathcal{F}_t\right]\\&\leq\frac{c_t^\lambda}{\sqrt{\lambda}}\leq\frac{J_t(\tilde{u}^\lambda)}{\sqrt{\lambda}},\end{aligned}$$

for all $t\in[0,\eta(\lambda)]$, by Lemmas 2.2.2, 2.2.3 and 2.2.4, with the convergence of the upper and lower bound to $\sqrt{\nu_t \kappa_t}$, we obtain that

$$\sup_{t \in [0,\eta(\lambda)]} \left| \frac{c_t^\lambda}{\sqrt{\lambda}} - \sqrt{\nu_t \kappa_t} \right| \to 0,$$

in $L^p(\mathbb{P})$ as $\lambda \to 0$. □

For later use we establish a variation of Proposition 2.2.1.

Proposition 2.2.5. *Under the assumptions of Proposition 2.2.1, suppose that* $\sup_{0 \le t \le T} \left(\sqrt{\kappa_t / \nu_t} \right) \in L^{q_1}(\mathbb{P})$ *and* $\sup_{0 \le t \le T} (1/\sqrt{\nu_t \kappa_t}) \in L^{q_2}(\mathbb{P})$*, for some* $1 < q_1, q_2 < \infty$*. Then, we have, for any* $\eta(\lambda) \in (0,T)$ *such that* $\eta(\lambda) \to T$ *and* $(T - \eta(\lambda))/\sqrt{\lambda} \to \infty$ *as* $\lambda \to 0$*, that*

$$\mathbb{E}\left[\sup_{0 \le t \le \eta(\lambda)} \left| \frac{1}{c_t^\lambda / \sqrt{\lambda}} - \frac{1}{\sqrt{\nu_t \kappa_t}} \right|^p \right] \to 0, \tag{2.21}$$

as $\lambda \to 0$ *for* $p > 1$ *such that* $1/p = 1/q_1 + 1/q_2$*.*

Proof. By Proposition 2.2.1, we have that

$$\sup_{0 \le t \le \eta(\lambda)} \left| c_t^\lambda / \sqrt{\lambda} - \sqrt{\nu_t \kappa_t} \right| \overset{\mathbb{P}-a.s.}{\to} 0,$$

as $\lambda \to 0$. Because $\sup_{0 \le t \le T} 1/\sqrt{\nu_t \kappa_t} < \infty$ implies $\inf_{0 \le t \le T} \sqrt{\nu_t \kappa_t} > 0$, we have that

$$\sup_{0 \le t \le \eta(\lambda)} \left| \frac{1}{c_t^\lambda / \sqrt{\lambda}} - \frac{1}{\sqrt{\nu_t \kappa_t}} \right| \overset{\mathbb{P}-a.s.}{\to} 0,$$

as $\lambda \to 0$, since the function $f(x) = 1/x$ is Lipschitz continuous on $[\varepsilon, \infty)$ for any $\varepsilon > 0$. Therefore, we only need to establish an integrable majorant for $\sup_{0 \le t \le \eta(\lambda)} \left| \frac{1}{c_t^\lambda / \sqrt{\lambda}} \right|$ in order to conlcude (2.21) by Lebesgue's dominated convergence theorem. For this, we recall that

$$\begin{aligned} \frac{\tilde{J}_t(\eta(\lambda), \lambda)}{\sqrt{\lambda}} &= \mathbb{E}\left[\int_t^{\tilde{\eta}(\lambda)} \exp\left(- \int_t^s \frac{1}{\sqrt{\lambda} \kappa_r} \left\{ \sqrt{\nu_r \kappa_r} + \tilde{w}(\tilde{\eta}(\lambda), \lambda) \right\} dr \right) \frac{\nu_s}{\sqrt{\lambda}} ds \,\middle|\, \mathcal{F}_t \right] \\ &\le \frac{c_t^\lambda}{\sqrt{\lambda}}, \end{aligned}$$

for all $t \in [0,T)$ by Lemma 2.2.4 and hence,

$$\frac{\sqrt{\lambda}}{\tilde{J}_t(\eta(\lambda), \lambda)} \ge \frac{1}{c_t^\lambda / \sqrt{\lambda}},$$

for all $t \in [0, \eta(\lambda)]$. We denote for $r \in [t, \tilde{\eta}(\lambda)]$,

$$g(r) = \frac{1}{\sqrt{\lambda} \kappa_r} \left\{ \sqrt{\nu_r \kappa_r} + \tilde{w}(\tilde{\eta}(\lambda), \lambda) \right\},$$

so that

$$\frac{\tilde{J}_t(\eta(\lambda),\lambda)}{\sqrt{\lambda}} = \mathbb{E}\left[\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s g(r)dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds\,\middle|\,\mathcal{F}_t\right].$$

We then estimate the integral as,

$$\begin{aligned}
&\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s g(r)dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds \\
&= \int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s g(r)dr\right) g(s)\frac{1}{g(s)}\frac{\nu_s}{\sqrt{\lambda}}ds \\
&\geq \int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s g(r)dr\right) g(s)ds \left\{\inf_{0\leq s\leq T}\frac{\nu_s\kappa_s}{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}\right\} \\
&\geq \left\{1-\exp\left(-\int_t^{\tilde{\eta}(\lambda)} g(r)dr\right)\right\}\left\{\inf_{0\leq s\leq T}\frac{\nu_s\kappa_s}{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}\right\} \\
&\geq \left(1-\frac{1}{e}\right)\left\{\int_t^{\tilde{\eta}(\lambda)} g(r)dr\wedge 1\right\}\left\{\inf_{0\leq s\leq T}\frac{\nu_s\kappa_s}{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}\right\},
\end{aligned}$$

where we use the fact that $1-e^{-x}\geq(1-1/e)x$, for $x\in[0,1]$. Since, $\tilde{w}(\tilde{\eta}(\lambda),\lambda)>0$, we obtain

$$\begin{aligned}
&\int_t^{\tilde{\eta}(\lambda)} \exp\left(-\int_t^s g(r)dr\right)\frac{\nu_s}{\sqrt{\lambda}}ds \\
&\geq \left(1-\frac{1}{e}\right)\left\{\left(\frac{\tilde{\eta}(\lambda)-\eta(\lambda)}{\sqrt{\lambda}}\right)\left(\inf_{0\leq s\leq T}\sqrt{\frac{\nu_s}{\kappa_s}}\right)\wedge 1\right\}\left\{\inf_{0\leq s\leq T}\frac{\nu_s\kappa_s}{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}\right\}.
\end{aligned}$$

Using Jensen's inequality for conditional expectations with the convex function $h(x)=1/x$, we obtain

$$\frac{1}{c_t^\lambda/\sqrt{\lambda}} \leq \frac{\sqrt{\lambda}}{\tilde{J}_t(\eta(\lambda),\lambda)} \leq M_t,$$

where

$$M_t := \mathbb{E}\left[\frac{1}{\left(1-\frac{1}{e}\right)}\left\{\frac{\sqrt{\lambda}}{\eta(\lambda)}\left(\sup_{0\leq s\leq T}\sqrt{\frac{\kappa_s}{\nu_s}}\right)\vee 1\right\}\left\{\sup_{0\leq s\leq T}\frac{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\nu_s\kappa_s}\right\}\,\middle|\,\mathcal{F}_t\right].$$

Taking the supremum and then using Doob's maximal inequality for martingale, we obtain

$$\mathbb{E}\left[\left(\sup_{0\leq t\leq\eta(\lambda)}\left|\frac{1}{c_t^\lambda/\sqrt{\lambda}}\right|\right)^p\right] \leq \mathbb{E}\left[\left(\sup_{0\leq t\leq T} M_t\right)^p\right] \leq C\mathbb{E}\left[|M_T|^p\right],$$

for some constant $C > 0$. Hence, we obtain by Jensen's and by Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\left[\left(\sup_{0\le t\le \eta(\lambda)}\left|\frac{1}{c_t^\lambda/\sqrt{\lambda}}\right|\right)^p\right]\\
&\le C\mathbb{E}\left[\left|\frac{1}{\left(1-\frac{1}{e}\right)}\left\{\frac{\sqrt{\lambda}}{\eta(\lambda)}\left(\sup_{0\le s\le T}\sqrt{\frac{\kappa_s}{\nu_s}}\right)\vee 1\right\}\left\{\sup_{0\le s\le T}\frac{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\nu_s\kappa_s}\right\}\right|^p\right]\\
&\le C\left\|\left(\sup_{0\le s\le T}\sqrt{\frac{\kappa_s}{\nu_s}}\right)\vee 1\right\|_{q_1}^p\left\|\sup_{0\le s\le T}\frac{\sqrt{\nu_s\kappa_s}+\tilde{w}(\tilde{\eta}(\lambda),\lambda)}{\nu_s\kappa_s}\right\|_{q_2}^p<\infty,
\end{aligned}$$

for λ small enough where $q_1, q_2 > 0$ satisfy $1/q_1 + 1/q_2 = 1/p$, and C was used as an absorbing constant. ☐

This last lemma about c^λ will be necessary when establishing convergences for the processes $\hat{\xi}_t^\lambda$ and $\hat{X}^\lambda$.

Lemma 2.2.6. *Under the Assumption of Proposition 2.2.5, we have*

$$\lambda^{-H/2}\sup_{0\le t\le T-2\lambda^a} e^{-\int_t^{t+\lambda^a}\frac{c_r^\lambda}{\lambda\kappa_r}dr}\to 0, \tag{2.22}$$

as $\lambda \to 0$, $\mathbb{P}$*-a.s. and in* $L^p(\mathbb{P})$ *for* $p > 1$.

Proof. We have

$$\lambda^{-H/2}e^{-\int_t^{t+\lambda^a}\frac{c_r^\lambda}{\lambda\kappa_r}dr}\le \lambda^{-H/2}e^{-\inf_{0\le r\le T-\lambda^a}\left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r}\right)\lambda^{a-1/2}}$$

and want to show the right hand side is bounded in $L^p(\mathbb{P})$. For this, we define the function

$$f(x,y)=x^{\frac{-H}{2a-1}}e^{-xy},$$

for $x \in (1,\infty)$ and $y \in (0,\infty)$. Notice that

$$f\left(\lambda^{a-1/2},\inf_{0\le r\le T-\lambda^a}\left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r}\right)\right)=\lambda^{-H/2}e^{-\inf_{0\le r\le T-\lambda^a}\left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r}\right)\lambda^{a-1/2}}.$$

We then have

$$\max_{x\in(1,\infty)}f(x,y)=\left(\frac{H}{1-2a}\right)^{\frac{H}{1-2a}}y^{-\frac{H}{1-2a}}e^{-\frac{H}{1-2a}}>0,$$

which is positive since we have $a > 1/2$. Therefore, the collection of random variables

$$C_1=\left\{\left(\lambda^{-H/2}e^{-\inf_{0\le r\le T-\lambda^a}\left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r}\right)\lambda^{a-1/2}}\right)^p\,\middle|\,\lambda\in(0,1)\right\}$$

is dominated by the collection of random variables,

$$C_2 = \left\{ \left(\frac{H}{1-2a} \right)^{\frac{pH}{1-2a}} e^{-\frac{pH}{1-2a}} \left\{ \inf_{0 \le r \le T-\lambda^a} \left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r} \right) \right\}^{-\frac{pH}{1-2a}} \middle| \lambda \in (0,1) \right\}$$

in the sense that for each random variable $X_1 \in C_1$, we can find $X_2 \in C_2$, such that $0 \le X_1 \le X_2$.

Since

$$\left\{ \inf_{0 \le r \le T-\lambda^a} \left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r} \right) \right\}^{\frac{pH}{1-2a}} \ge \left(\inf_{t \in [0, T-\lambda^a]} \frac{c_t^\lambda}{\sqrt{\lambda}} \right)^{\frac{pH}{1-2a}} \left(\inf_{t \in [0,T]} \frac{1}{\kappa_t} \right)^{\frac{pH}{1-2a}},$$

we have

$$\left\{ \inf_{0 \le r \le T-\lambda^a} \left(\frac{c_r^\lambda}{\sqrt{\lambda}\kappa_r} \right) \right\}^{-\frac{pH}{1-2a}} \le \left(\sup_{t \in [0, T-\lambda^a]} \frac{1}{c_t/\sqrt{\lambda}} \right)^{\frac{pH}{1-2a}} \left(\sup_{t \in [0,T]} \kappa_t \right)^{\frac{pH}{1-2a}}$$

and the right hand side is integrable by Corollary 2.2.5. It follows that the set C_2 is uniformly integrable and so is the set C_1. The convergence in (2.22) follows by applying Lebesgue's dominated convergence theorem, as Theorem 1.21 in Kallenberg (2006), and the fact that we have the convergence

$$\lambda^{-H/2} \sup_{0 \le t \le T-2\lambda^a} e^{-\int_t^{t+\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r} dr} \to 0,$$

$\mathbb{P}$-a.s., as $\lambda \to 0$. This convergence follows directly from Proposition 2.2.1 and the fact that $\lambda^{a-1/2} \to \infty$ as $\lambda \to 0$. □

2.2.2 Convergence of rescaled processes

We now show several results related to the convergence of the rescaled processes we will encounter. Before moving to the rescaled processes involved in the solution to Problem (2.2), we start this section by studying the behavior of some processes that will emerge from our proof together with the behavior of rescaled fBM. In particular, the next lemma deals with the stationarity behavior of the processes involved in the solution to the ergodic linear-quadratic control of fBM.

Lemma 2.2.7. *Let $(\tilde{B}_t^H)_{t\in\mathbb{R}}$ a fractional Brownian motion on $\mathbb{R}$. For $t \in \mathbb{R}$, set*

$$\tilde{\xi}_t^0 = \mathbb{E}\left[\int_t^\infty \sqrt{\frac{\nu}{\kappa}} e^{-\sqrt{\frac{\nu}{\kappa}}(s-t)} \tilde{B}_s^H ds \,\middle|\, \tilde{\mathcal{F}}_t \right],$$
$$\tilde{X}_t^0 = e^{-\sqrt{\frac{\nu}{\kappa}}t} x + \int_0^t \sqrt{\frac{\nu}{\kappa}} e^{-\sqrt{\frac{\nu}{\kappa}}(t-s)} \tilde{\xi}_s^0 ds,$$

for constants $\nu, \kappa > 0$. Then, $(\tilde{\xi}_t^0 - \tilde{B}_t^H)_{t\in\mathbb{R}}$, $(\tilde{\xi}_t^0 - \tilde{X}_t^0)_{t\in\mathbb{R}}$ and $(\tilde{B}_t^H - \tilde{X}_t^0)_{t\in\mathbb{R}}$ are Gaussian *processes and*

$$\tilde{\xi}_t^0 - \tilde{B}_t^H \overset{L^2(\mathbb{P})}{\to} Z_1 \sim \mathcal{N}(0,\sigma_1^2),$$

$$\tilde{\xi}_t^0 - \tilde{X}_t^0 \overset{L^2(\mathbb{P})}{\to} Z_2 \sim \mathcal{N}(0,\sigma_2^2), \tag{2.23}$$

$$\tilde{B}_t^H - \tilde{X}_t^0 \overset{L^2(\mathbb{P})}{\to} Z_3 \sim \mathcal{N}(0,\sigma_3^2), \tag{2.24}$$

as $t \to \infty$, with

$$\sigma_2^2 = \delta^{-2H} H\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\}$$
$$\sigma_3^2 = \delta^{-2H}(1-H)\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\},$$

where $\delta = \sqrt{\nu/\kappa}$.

Proof. We start the proof by using the Mandelbrot-Van Ness representation of fBM to show

$$\begin{aligned}\mathbb{E}\left[\tilde{B}_{t+h}^H - \tilde{B}_t^H \,\middle|\, \tilde{\mathcal{F}}_t\right] &= \frac{1}{c_H}\int_{-\infty}^t \left\{(t+h-s)^{H-1/2} - (t-s)^{H-1/2}\right\} d\tilde{W}_s \\ &\overset{d}{=} \frac{1}{c_H}\int_{-\infty}^0 \left\{(h-s)^{H-1/2} - (-s)^{H-1/2}\right\} d\tilde{W}_s \\ &= \mathbb{E}\left[\tilde{B}_h^H \,\middle|\, \tilde{\mathcal{F}}_0\right],\end{aligned}$$

for a constant $c_H > 0$, where we used the fact that increments of the Brownian motion $(\tilde{W}_t)_{t\in\mathbb{R}}$ are stationary. Hence, the process

$$\left(\mathbb{E}\left[\tilde{B}_{t+h}^H - \tilde{B}_t^H \,\middle|\, \tilde{\mathcal{F}}_t\right]\right)_{t\geq 0}$$

is for each $h > 0$, a stationary Gaussian process with respect to the filtration $(\tilde{\mathcal{F}}_t)_{t\in\mathbb{R}}$. Denoting $\delta = \sqrt{\nu/\kappa}$, this implies that

$$\begin{aligned}\tilde{\xi}_t^0 - \tilde{B}_t^H &= \int_0^\infty \delta e^{-\delta h}\mathbb{E}\left[\left(\tilde{B}_{t+h}^H - \tilde{B}_t^H\right) \,\middle|\, \tilde{\mathcal{F}}_t\right] dh \\ &\overset{d}{=} \int_0^\infty \delta e^{-\delta h}\frac{1}{c_H}\int_{-\infty}^0 \left\{(h-s)^{H-1/2} - (-s)^{H-1/2}\right\} d\tilde{W}_s dh \\ &= \int_{-\infty}^0 \left(\frac{1}{c_H}\int_0^\infty \delta e^{-\delta h}\left\{(h-s)^{H-1/2} - (-s)^{H-1/2}\right\} dh\right) d\tilde{W}_s \end{aligned} \tag{2.25}$$

is a stationary Gaussian process as well.

The process $\tilde{\xi}_t^0 - \tilde{X}_t^0$ can be decomposed as follows

$$\begin{aligned}\tilde{\xi}_t^0 - \tilde{X}_t^0 &= \tilde{\xi}_t^0 - e^{-\delta t}\left(x + \int_0^t \delta e^{\delta s}\tilde{\xi}_s^0 ds\right) \\ &= \tilde{\xi}_t^0 - e^{-\delta t}\left(x + \int_0^t \delta e^{\delta s}\tilde{\xi}_s^0 ds + \int_{-\infty}^0 \delta e^{\delta s}\tilde{\xi}_s^0 ds - \int_{-\infty}^0 \delta e^{\delta s}\tilde{\xi}_s^0 ds\right) \\ &= -\int_{-\infty}^t \delta e^{-\delta(t-s)}\tilde{\xi}_s^0 ds + \tilde{\xi}_t^0 - e^{-\delta t}\left(-\int_{-\infty}^0 \delta e^{\delta s}\tilde{\xi}_s^0 ds + x\right). \end{aligned} \tag{2.26}$$

The process

$$Y_t := -\int_{-\infty}^t \delta e^{-\delta(t-s)}\tilde{\xi}_s^0 ds + \tilde{\xi}_t^0,$$

is a stationary Gaussian process since $\tilde{\xi}^0$, as the sum of two processes with stationary increments in (2.25), has stationary increments.

The remainder term in (2.26),

$$Y_t^R := -e^{-\delta t}\left(-\int_{-\infty}^0 \delta e^{\delta s}\tilde{\xi}_s^0 ds + x\right)$$

converges to 0 in $L^2(\mathbb{P})$, as $t \to 0$. Finally, since we have

$$\tilde{B}_t^H - \tilde{X}_t^0 = \left(\tilde{B}_t^H - \tilde{\xi}_t^0\right) + \left(\tilde{\xi}_t^0 - \tilde{X}_t^0\right),$$

the result as $t \to \infty$, for the process $(\tilde{B}_t^H - \tilde{X}_t^0)_{t\in\mathbb{R}}$ follows from the two previous parts of the proof.

The explicit formulas for the limiting variances in the statement follow from rescaling the formulas in Proposition 1.2.7 in Chapter 1 to adapt from RLfBM to fBM. □

Lemma 2.2.8. *For $H < 1/2$, let $(B_u^H)_{u\geq 0}$ and $(\tilde{B}_u^H)_{u\in(-\infty,\infty)}$ be fractional Brownian motions with respect to the filtrations $(\mathcal{F}_u)_{u\geq 0}$ and $(\tilde{\mathcal{F}}_u)_{u\in(-\infty,\infty)}$, respectively. Then, for each $t > 0$, $s \in \mathbb{R}$, we have*

$$\mathbb{E}\left[\lambda^{-H}(B_{t+\lambda s}^H - B_t^H)\,\middle|\,\mathcal{F}_t\right] \xrightarrow{d} \mathbb{E}\left[\tilde{B}_s^H\,\middle|\,\tilde{\mathcal{F}}_0\right],$$

as $\lambda \to 0$.

Proof. First, recall from the representation in Nuzman and Poor (2000) that there are Brownian motions $(W_t)_{t\geq 0}$ and $(\tilde{W}_t)_{t\in(-\infty,\infty)}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t\in(-\infty,\infty)}$ respectively, such that the following representations hold, for $t \geq 0$,

$$\begin{aligned} B_t^H &= \frac{1}{\Gamma(H+1/2)}\int_0^t \left\{\frac{u}{t}\right\}^{1/2-H}(t-u)^{H-1/2}dW_u \\ &\quad + \frac{1/2-H}{\Gamma(H+1/2)}\int_0^t u^{H-1/2}\beta_{1-u/t}(H+1/2, 1-2H)dW_u =: B_t^{H,1} + B_t^{H,2}, \end{aligned}$$

where $B_t^{H,1}$ and $B_t^{H,2}$ correspond to each integral and

$$\beta_v(a,b) = \int_0^v r^{a-1}(1-r)^{b-1}dr,$$

and, for $t \in (-\infty, \infty)$,

$$\tilde{B}_t^H = e_H \int_{-\infty}^t \left[(t-u)^{H-1/2} 1_{(0,\infty)}(t-u) - (-u)^{H-1/2} 1_{(0,\infty)}(-u) \right] d\tilde{W}_u,$$

where $e_H = 1/\Gamma(H+1/2) > 0$, is a normalising constant. Then, we compute,

$$\begin{aligned}
&\mathbb{E}\left[\lambda^{-H}(B_{t+\lambda s}^{H,1} - B_t^{H,1}) \,\middle|\, \mathcal{F}_t \right] \\
&= e_H \lambda^{-H} \left(\int_0^t \left\{ \frac{u}{t+\lambda s} \right\}^{H-1/2} (t+\lambda s - u)^{H-1/2} dW_u - \int_0^t \left\{ \frac{u}{t} \right\}^{1/2-H} (t-u)^{H-1/2} dW_u \right) \\
&\overset{d}{=} e_H \lambda^{-H} \left(\int_{-t}^0 \left[\left\{ \frac{\tilde{u}+t}{t+\lambda s} \right\}^{1/2-H} (\lambda s - \tilde{u})^{H-1/2} - \left\{ \frac{\tilde{u}+t}{t} \right\}^{1/2-H} (-\tilde{u})^{H-1/2} \right] d\tilde{W}_{\tilde{u}} \right) \\
&= e_H \left(\int_{-t/\lambda}^0 \left[\left\{ \frac{t+\lambda\tilde{u}}{t+\lambda s} \right\}^{1/2-H} (s-\tilde{u})^{H-1/2} - \left\{ \frac{t+\lambda\tilde{u}}{t} \right\}^{1/2-H} (-\tilde{u})^{H-1/2} \right] d\frac{1}{\lambda^{1/2}} \tilde{W}_{\lambda\tilde{u}} \right) \\
&\overset{d}{=} e_H \left(\int_{-t/\lambda}^0 \left[\left\{ \frac{t+\lambda v}{t+\lambda s} \right\}^{1/2-H} (s-v)^{H-1/2} - \left\{ \frac{t+\lambda v}{t} \right\}^{1/2-H} (-v)^{H-1/2} \right] d\tilde{W}_v \right) =: X_\lambda,
\end{aligned}$$

where we use the self-similarity property of Brownian motion in the last equality.

The collection $(X_\lambda)_{\lambda \in (0,1)}$ consists of Gaussian centered random variables. In order to show the convergence in distribution, we only need to show the convergence of the variances $\mathrm{Var}(X_\lambda) = \mathbb{E}\left[X_\lambda^2\right]$, as $\lambda \to 0$. To this end, we compute,

$$\begin{aligned}
\mathbb{E}\left[X_\lambda^2\right] &= e_H^2 \int_{-t/\lambda}^0 \left[\left\{ \frac{t+\lambda v}{t+\lambda s} \right\}^{1/2-H} (s-v)^{H-1/2} - \left\{ \frac{t+\lambda v}{t} \right\}^{1/2-H} (-v)^{H-1/2} \right]^2 dv \\
&= e_H^2 \int_{-t/\lambda}^0 \left[\left\{ \frac{t+\lambda v}{t} \right\}^{1/2-H} (-v)^{H-1/2} \left\{ \left(\frac{(t+\lambda s)(s-v)}{t(-v)} \right)^{H-1/2} - 1 \right\} \right]^2 dv \\
&= e_H^2 \int_0^{t/\lambda} \left\{ \frac{t-\lambda v}{t} \right\}^{1-H} (v)^{2H-1} \left\{ \left(\frac{(t+\lambda s)(s+v)}{tv} \right)^{H-1/2} - 1 \right\}^2 dv.
\end{aligned}$$

Next, we estimate the expression

$$\left\{ \frac{t-\lambda v}{t} \right\}^{1-H} \leq 1.$$

Moreover, by convexity of the function $x \mapsto (1+x)^{H-1/2}$, we have $\left|(1+x)^{H-1/2} - 1\right| \leq x$ for $x \geq 0$. We then obtain for λ small enough, and some constant $C > 0$,

$$\left\{ \left(\frac{(t+\lambda s)(s+v)}{tv} \right)^{H-1/2} - 1 \right\}^2 \leq C \frac{s^2}{v^2}.$$

Therefore,

$$\left\{\frac{t-\lambda v}{t}\right\}^{1-H}(v)^{2H-1}\left\{\left(\frac{(t+\lambda s)(s+v)}{tv}\right)^{H-1/2}-1\right\}^2 \\ \leq v^{2H-1}1_{(0,1)}(v)+Cs^2v^{2H-3}1_{(1,\infty)}(v),$$

for $v \in (0,t/\lambda)$. Since $\int_0^1 v^{2H-1}dv + C\int_1^\infty s^2v^{2H-3}dv < \infty$, we can apply Lebesgue's dominated convergence theorem and obtain

$$\mathbb{E}\left[X_\lambda^2\right] \to \int_0^\infty \left\{(s+v)^{H-1/2}-v^{H-1/2}\right\}^2 dv,$$

as $\lambda \to 0$. Finally, observe that

$$X_0 = \mathbb{E}\left[\tilde{B}_s^H \,\middle|\, \tilde{\mathcal{F}}_0\right] = \int_{-\infty}^0 \left\{(s-v)^{H-1/2}-(-v)^{H-1/2}\right\} d\tilde{W}_v,$$

is a centered Gaussian random variable with $\mathrm{Var}(X_0)=\int_0^\infty \left\{(s+v)^{H-1/2}-v^{H-1/2}\right\}^2 dv$. This allows to conclude that $X_\lambda \xrightarrow{d} X_0$, as $\lambda \to 0$.

For the second term, denoting $e'_H = (1/2-H)e_H$, we have,

$$\mathbb{E}\left[\lambda^{-H}(B_{t+\lambda s}^{H,2}-B_t^{H,2})\,\middle|\,\mathcal{F}_t\right] \\ = \lambda^{-H}e'_H\int_0^t u^{H-1/2}\left[\beta_{1-u/(t+\lambda s)}(H+1/2,1-2H)-\beta_{1-u/t}(H+1/2,1-2H)\right]dW_u.$$

Writing

$$g(t;u) = \beta_{1-u/(t+\lambda s)}(H+1/2,1-2H)-\beta_{1-u/t}(H+1/2,1-2H) \\ = \int_{1-u/t}^{1-u/(t+\lambda s)} r^{H-1/2}(1-r)^{-2H}dr,$$

we have the following estimate,

$$g(t;u) \leq (1-u/t)^{H-1/2}\int_{1-u/t}^{1-u/(t+\lambda s)}(1-r)^{-2H}dr \\ \leq (1-u/t)^{H-1/2}\frac{1}{1-2H}u^{1-2H}\left[\left(\frac{1}{t}\right)^{1-2H}-\left(\frac{1}{t+\lambda s}\right)^{1-2H}\right] \\ \leq (1-u/t)^{H-1/2}\frac{1}{1-2H}u^{1-2H}(1-2H)t^{2H-2}(\lambda s),$$

where we use the convexity of the function $h(x)=(1/x)^{1-2H}$ in the last inequality.

Since $\mathbb{E}\left[\lambda^{-H}(B_{t+\lambda s}^{H,2}-B_t^{H,2})\,\middle|\,\mathcal{F}_t\right]$ is again a Gaussian centered random variable, we estimate its second moment using the estimate for $g(t;u)$, and some constant $C > 0$,

$$\begin{aligned}
\mathbb{E}\left[\left(\lambda^{-H}\int_0^t u^{H-1/2}g(t;u)dW_u\right)^2\right] &= \lambda^{-2H}\int_0^t u^{2H-1}g(t;u)^2du \\
&\leq \lambda^{-2H}C\int_0^t (1-u/t)^{2H-1}u^{1-2H}t^{4H-4}(\lambda s)^2du \\
&\leq \lambda^{2(1-H)}s^2t^{4H-4}C\int_0^t u^{1-2H}(1-u/t)^{2H-1}du \\
&\leq \lambda^{2(1-H)}s^2t^{2H-2}C\int_0^1 v^{1-2H}(1-v)^{2H-1}dv \\
&\leq \lambda^{2(1-H)}s^2t^{2H-2}C\beta_1(2-2H,2H).
\end{aligned}$$

It follows that as $\lambda \to 0$, the second moment of the random variable vanishes. □

Lemma 2.2.9. *Fix $H \in (0,1/2)$ and let $(B_t^H)_{t\in\mathbb{R}}$ be a fractional Brownian motion. For $t > 0$, define the process*

$$X_u^{t,\lambda} := \lambda^{-H}\left(B_{t+\lambda u}^H - B_t^H\right)$$

for $u \in \mathbb{R}$. Then, the process $(B_u^H, X_u^{t,\lambda})_{u\in\mathbb{R}}$ converges in distribution to $(B_u^H, W_u^{H,t})_{u\in\mathbb{R}}$, where $(W_u^{H,t})_{u\in\mathbb{R}}$ is a fractional Brownian motion on $\mathbb{R}$ that is independent of $(B_u^H)_{u\in\mathbb{R}}$.

Moreover, there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and random variables $(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})_{u\in\mathbb{R}}$ and $(\tilde{B}_u^H, \tilde{W}_u^{H,t})_{u\in\mathbb{R}}$ on that probability space such that

$$\begin{aligned}
(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})_{u\in\mathbb{R}} &\overset{d}{=} (B_u^H, X_u^{t,\lambda})_{u\in\mathbb{R}}, \\
(\tilde{B}_u^H, \tilde{W}_u^{H,t})_{u\in\mathbb{R}} &\overset{d}{=} (B_u^H, W_u^{H,t})_{u\in\mathbb{R}}
\end{aligned}$$

and

$$(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})_{u\in\mathbb{R}} \overset{\tilde{\mathbb{P}}\text{-a.s.}}{\to} (\tilde{B}_u^H, \tilde{W}_u^{H,t})_{u\in\mathbb{R}},$$

on $C(\mathbb{R},\mathbb{R}^2)$, as $\lambda \to 0$, where we equip $C(\mathbb{R},\mathbb{R}^2)$ with the metric of uniform convergence on compacts and thus have a Polish space.

For $\alpha > 1$, define the measure μ^α on $\big(\mathbb{R},\mathcal{B}(\mathbb{R})\big)$ by

$$\mu^\alpha(dw) = C_\alpha(1+|w|)^{-\alpha}, \quad C_\alpha = \frac{2}{\alpha-1},$$

so that the product measure $\mathbb{P}(d\omega)\times\mu^\alpha(du)$ is a probability measure on $\Big(\Omega\times\mathbb{R}, \mathcal{F}\otimes\mathcal{B}(\mathbb{R})\Big)$. Then, $\big((|(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})|^\beta)_{u\in\mathbb{R}}\big)_{\lambda\in(0,1)}$ is uniformly integrable with respect to $\mathbb{P}\times\mu^\alpha$ for $\beta \in (0,(\alpha-1)/H)$ and, for $u \in \mathbb{R}$,

$$(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda}) \overset{L^\beta(\tilde{\mathbb{P}}\times\mu^\alpha)}{\to} (\tilde{B}_u^H, \tilde{W}_u^{H,t}), \quad \text{as } \lambda \to 0.$$

Proof. By the self-similarity and stationarity of fractional Brownian motion, we clearly have,

$$(X_u^{t,\lambda})_{u\in\mathbb{R}} \stackrel{d}{=} (W_u^{H,t})_{u\in\mathbb{R}}.$$

The tightness follows from Corollary 16.9 of Kallenberg (2006), since

$$\mathbb{E}\left[|(B_u^H, X_u^{t,\lambda}) - (B_s^H, X_s^{t,\lambda})|_{\mathbb{R}^2}\right] = \mathrm{Var}[B_u^H - B_s^H] + \mathrm{Var}[X_u^{t,\lambda} - X_s^{t,\lambda}] \leq 2|u-s|^{2H}$$

We next verify the claim on independence. For this, let us fix $t > 0$ and $u, v \in \mathbb{R}$. Using the covariance formula of fBM, we compute

$$\begin{aligned}\mathrm{Cov}(X_u^{t,\lambda}, B_v^H) &= \mathbb{E}\left[\lambda^{-H}(B_{t+\lambda u}^H - B_t^H)B_v^H\right] \\ &= \lambda^{-H}\frac{1}{2}\left\{\left(|t+\lambda u|^{2H} - |t|^{2H}\right) + \left(|t-v|^{2H} - |t-v+\lambda u|^{2H}\right)\right\}.\end{aligned}$$

By convexity of the function $g(x) = x^{2H}$, we have for $0 < x < y$,

$$y^{2H} - x^{2H} \leq 2Hx^{2H-1}(y-x).$$

Hence, we obtain

$$\begin{aligned}\left|\mathrm{Cov}(X_u^{t,\lambda}, B_v^H)\right| &\leq \lambda^{1-H}H\left\{(|t+\lambda u| \wedge |t|)^{2H-1} + (|t-v| \wedge |t-v+\lambda u|)^{2H-1}\right\}|u| \\ &\to 0, \qquad (2.27)\end{aligned}$$

as $\lambda \to 0$. Therefore, the correlation between the random variables goes to 0 and since B^H is a Gaussian process, we get the independence between the processes.

This implies that the tight family of random variables has a unique limit law and, hence, converges in distribution.

Since $C(\mathbb{R}, \mathbb{R}^2)$ with the metric of uniform convergence on compacts is a Polish space, we have, by Skorokhod representation as in Theorem 4.30 in Kallenberg (2006), that there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ such that

$$(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})_{u\in\mathbb{R}} \stackrel{\tilde{\mathbb{P}}-a.s.}{\to} (\tilde{B}_u^H, \tilde{W}_u^{H,t})_{u\in\mathbb{R}}$$

in $C(\mathbb{R}, \mathbb{R}^2)$, as $\lambda \to 0$.

Because $(\tilde{B}_u^H)_{u\in\mathbb{R}}$ and $(\tilde{X}_u^{t,\lambda})_{u\in\mathbb{R}}$ are both fractional Brownian motions, we have

$$E_{\tilde{\mathbb{P}}}[|\tilde{B}_u^H|^\beta] = E_{\tilde{\mathbb{P}}}[|\tilde{X}_u^{t,\lambda}|^\beta] = c_\beta u^{\beta H}$$

for all $u \in \mathbb{R}$ and for $c_\beta > 0$. Therefore, $E_{\tilde{\mathbb{P}}\times\mu^\alpha}[|(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})|^\beta] = \int_{\mathbb{R}} C_\beta u^{\beta H}\mu^\alpha(du) = \frac{C_\beta}{\alpha-1}\frac{1}{H\beta-\alpha-1}$, for some $C_\beta > 0$, for all $\beta \in \big(0, (\alpha-1)/H\big)$. Hence, $\big((|(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda})|^\beta)_{u\in\mathbb{R}}\big)_{\lambda\in(0,1)}$ is uniformly integrable with respect to $\tilde{\mathbb{P}} \times \mu^\alpha$ for $\beta \in \big(0, (\alpha-1)/H\big)$. By Vitali's convergence theorem, as in Proposition 4.12 of Kallenberg (2006), we have, for $u \in \mathbb{R}$, that

$$(\tilde{B}_u^H, \tilde{X}_u^{t,\lambda}) \stackrel{L^\beta(\tilde{\mathbb{P}}\times\mu^\alpha)}{\to} (\tilde{B}_u^H, \tilde{W}_u^{H,t}), \quad \text{as } \lambda \to 0.$$

□

For the next proofs, we also introduce the two auxiliary measures μ_-^α and μ_+^α defined on $((-\infty,0],\mathcal{B}((-\infty,0]))$ and $([0,\infty),\mathcal{B}([0,\infty)))$, respectively by,

$$\mu_+^\alpha(dw) = \frac{C_\alpha}{2}(1+|w|)^{-\alpha},$$
$$\mu_-^\alpha(dw) = \frac{C_\alpha}{2}(1+|w|)^{-\alpha}.$$

To simplify notation, we also write μ_+,μ_-. We now turn our attention to the rescaled process involved in the solution of Problem (2.2).

Lemma 2.2.10. *Under the Assumptions 2.1.2, for $U_\lambda = (-\lambda^{a-1/2},0)$, we have that*

$$\begin{aligned}\sup_{\lambda^a\le t\le T-2\lambda^a}&\Bigg\|\left\{\frac{\sqrt{\lambda}}{c^\lambda_{t+\sqrt{\lambda}w}}\int_0^{\lambda^{a-1/2}} e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\left(X^{t,\lambda}_{v+w}\right)\nu_{t+\sqrt{\lambda}(w+v)}dv\right\}\mathbb{1}_{U_\lambda}(w)\\ &\qquad -\int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left(\tilde{W}^{H,t}_{v+w}\right)dv\Bigg\|_{L^p(\mu_-\times\mathbb{P})}\\ =:&\sup_{\lambda^a\le t\le T-2\lambda^a}\left\|N^\lambda_t\right\|_{L^p(\mu_-\times\mathbb{P})}\to 0,\end{aligned} \tag{2.28}$$

for $p>1$, as $\lambda\to 0$.

Proof. We start by denoting $V_\lambda = (0,\lambda^{a-1/2})$ and define

$$Z^{t,\lambda}_{w,v} := \left|\frac{\sqrt{\lambda}\nu_{t+\sqrt{\lambda}(w+v)}}{c^\lambda_{t+\sqrt{\lambda}w}}e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w) - \sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\right| \tag{2.29}$$

for $t\in[\lambda^a, T-2\lambda^a]$. Then, we rewrite

$$\begin{aligned}&\left\{\frac{\sqrt{\lambda}}{c^\lambda_{t+\sqrt{\lambda}w}}\int_0^{\lambda^{a-1/2}} e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\left(X^{t,\lambda}_{v+w}\right)\nu_{t+\sqrt{\lambda}(w+v)}dv\right\}\mathbb{1}_{U_\lambda}(w)\\ &=\int_0^\infty\frac{\sqrt{\lambda}}{c^\lambda_{t+\sqrt{\lambda}w}}e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\left(X^{t,\lambda}_{v+w}\right)\nu_{t+\sqrt{\lambda}(w+v)}\mathbb{1}_{U_\lambda}(w)\mathbb{1}_{V_\lambda}(v)dv\end{aligned}$$

and we need to estimate first the $\|\cdot\|_{L^p(\mu_-)}$ norm of

$$\begin{aligned}N^\lambda_t := \int_0^\infty&\left\{\frac{\sqrt{\lambda}}{c^\lambda_{t+\sqrt{\lambda}w}}e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\left(X^{t,\lambda}_{v+w}\right)\nu_{t+\sqrt{\lambda}(w+v)}\mathbb{1}_{U_\lambda}(w)\mathbb{1}_{V_\lambda}(v)\right.\\ &\left.\qquad -\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left(\tilde{W}^{H,t}_{v+w}\right)\right\}dv.\end{aligned}$$

We then use the fact that $\mu_+(dv) = (C_\alpha/2)(1+|v|)^{-\alpha}dv$ and consider the product probability measure $d\mathbb{P}(\omega)\times\mu_+(dv)\times\mu_-(dw)$ to apply Hölder inequality. It follows, by applying first Jensen's inequality, the triangular inequality and Hölder's inequality twice,

$$\begin{aligned}
&\left\|N_t^\lambda\right\|_{L^p(\mathbb{P}\times\mu_-)}\\
&\leq C\left\|Z_{w,v}^{t,\lambda}(1+|v|)^\alpha \mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\right\|_{L^{q_1}(\mathbb{P}\times\mu_+\times\mu_-)}\left\|\left|X_{v+w}^{t,\lambda}\right|\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\right\|_{L^{q_2}(\mathbb{P}\times\mu_+\times\mu_-)}\\
&\quad + C\left\|\left(X_{v+w}^{t,\lambda}\right)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w) - \left(\tilde{W}_{v+w}^{H,t}\right)\right\|_{L^{q_2}(\mathbb{P}\times\mu_+\times\mu_-)}\\
&\qquad\qquad\qquad\qquad\cdot\left\|\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}(1+|v|)^\alpha\right\|_{L^{q_1}(\mathbb{P}\times\mu_+\times\mu_-)}
\end{aligned} \tag{2.30}$$

for some $C>0$ and $1<q_1,q_2<\infty$ such that $1/q_1+1/q_2=1/p$.

We next show that for $\varepsilon>0$,

$$\sup_{\lambda\in(0,1)}\sup_{\lambda^a\leq t\leq T-2\lambda^a}\mathbb{E}\left[\int_{-\infty}^0\int_0^\infty\left\{Z_{w,v}^{t,\lambda}(1+|v|)^\alpha\right\}^{q_1+\varepsilon}\mathbb{1}_{U_\lambda}(w)\mathbb{1}_{V_\lambda}(v)\mu_+(dv)\mu(dw)\right] \tag{2.31}$$

is bounded and hence, $\left\{Z_{w,v}^{t,\lambda}(1+|v|)^\alpha\right\}^{q_1}$ is uniformly integrable with respect to $d\mathbb{P}(\omega)\times\mu_+(dv)\times\mu_-(dw)$. For this, we consider both terms in (2.29) separately. Using the inequality, $(1+|v|)^p\leq C(1+|v|^p)$, for $C>0$, we estimate first, for some absorbing constants $C_1,C_2>0$, $w\in U_\lambda$,

$$\begin{aligned}
&\int_0^{\lambda^{a-1/2}}\left|\frac{\nu_{t+\sqrt{\lambda}(w+v)}}{c^\lambda_{t+\sqrt{\lambda}w}/\sqrt{\lambda}}e^{-\int_0^v\frac{c^\lambda_{t+\sqrt{\lambda}(w+u)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(w+u)}}du}\right|^{q_1+\varepsilon}(1+|v|)^{\alpha(q_1+\varepsilon)}\mu_+(dv)\\
&\leq C_1\left\{\sup_{t\in[0,T-\lambda^a]}\frac{\nu_t}{c_t^\lambda/\sqrt{\lambda}}\right\}^{q_1+\varepsilon-1}\\
&\quad + C_2\left\{\sup_{t\in[0,T-\lambda^a]}\frac{\nu_t}{c_t^\lambda/\sqrt{\lambda}}\right\}^{q_1+\varepsilon}\frac{\Gamma(\alpha(q_1+\varepsilon-1)+1)}{\left\{\inf_{0\leq t\leq T-\lambda^a}c_t/(\sqrt{\lambda}\kappa_t)\right\}^{\alpha(q_1+\varepsilon-1)+1}}.
\end{aligned}$$

This estimate which we call Z_1 is a random variable in $L^1(\mathbb{P})$ by Assumptions 2.1.2. Similarly, we can estimate

$$\begin{aligned}
&\int_0^{\lambda^{a-1/2}}\left\{\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}(1+|v|)^\alpha\right\}^{q_1+\varepsilon}\mu_+(dv)\\
&\leq C_1\left\{\sqrt{\frac{\nu_t}{\kappa_t}}\right\}^{q_1+\varepsilon-1}+C_2\left\{\sqrt{\frac{\nu_t}{\kappa_t}}\right\}^{(1-\alpha)(q_1+\varepsilon-1)}\Gamma(\alpha(q_1+\varepsilon-1)+1)=:Z_2
\end{aligned}$$

which is again a random variable in $L^1(\mathbb{P})$ by Assumption 2.1.2. Hence, since $Z_1+Z_2\in L^1(\mathbb{P})$, this implies the L^{q_1}-boundedness in (2.31) with respect to the product measure. Because $Z_{v,w}^{t,\lambda}(1+|v|)^\alpha \overset{\mathbb{P}-a.s.}{\to} 0$, as $\lambda\to 0$, for fixed v,w,t, we obtain by the generalised

Lebesgue convergence theorem that

$$\left\| Z^{t,\lambda}_{v,w}(1+|v|)^{\alpha}\mathbb{1}_{V_\alpha}(v)\mathbb{1}_{U_\alpha}(w)\right\|_{L^{q_1}(\mu_+\times\mu_-\times\mathbb{P})} \to 0, \tag{2.32}$$

as $\lambda \to 0$.

Next, we show that

$$\begin{aligned}
&\sup_{\lambda^a\le t\le T-2\lambda^a}\left\|\left(X^{t,\lambda}_{v+w}\right)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)-\left(\tilde W^{H,t}_{v+w}\right)\right\|_{L^{q_2}(\mathbb{P}\times\mu_+\times\mu_-)} \qquad (2.33)\\
&=\sup_{\lambda^a\le t\le T-2\lambda^a}\int_{-\infty}^{0}\int_0^{\infty}\mathbb{E}\left[\left|\left(X^{t,\lambda}_{v+w}\right)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)-\left(\tilde W^{H,t}_{v+w}\right)\right|^{q_2}\right]\mu_+(dv)\mu_-(dw)
\end{aligned}$$

converges to 0. Denoting

$$Y^{t,\lambda}_{v,w} := \left(X^{t,\lambda}_{v+w}\right)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)-\left(\tilde W^{H,t}_{v+w}\right),$$

we observe that $(X^{t,\lambda}_s)_{s\in\mathbb{R}}$ and $(\tilde W^{H,t}_s)_{s\in\mathbb{R}}$ are both fractional Brownian motions. Therefore, we have that

$$\begin{aligned}
\mathbb{E}\left[\left|X^{t,\lambda}_{v+w}\right|^{q_2+\varepsilon}\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\right] &\le C\,|v+w|^{(q_2+\varepsilon)H}\,\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\\
&\le C_1\left(|v|^{(q_2+\varepsilon)H}+|w|^{(q_2+\varepsilon)H}\right)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w),
\end{aligned}$$

for some $C_1>0$ and for $C_2>0$,

$$\mathbb{E}\left[\left|\tilde W^{H,t}_{v+w}\right|^{q_2+\varepsilon}\right]\le C_2\left(|v|^{(q_2+\varepsilon)H}+|w|^{(q_2+\varepsilon)H}\right).$$

This implies that $\left|Y^{t,\lambda}_{v,w}\right|^{q_2}$ is uniformly integrable.

Since by Lemma 2.2.9 we have $(X^{t,\lambda}_v)_{v\in\mathbb{R}} \to (\tilde W^{H,t}_v)_{v\in\mathbb{R}}$, $\mathbb{P}$-.a.s, as $\lambda\to 0$, we obtain that

$$\sup_{\lambda^a\le t\le T-2\lambda^a}\int_{-\infty}^{0}\int_0^{\infty}\mathbb{E}\left[\left|Y^{t,\lambda}_{v,w}\right|^{q_2}\right]\mu_+(dv)\mu_-(dw)\to 0, \tag{2.34}$$

as $\lambda \to 0$.

Finally, combining both convergences (2.32) and (2.34) with the estimates in (2.30) yields the claimed convergence in (2.28). □

Lemma 2.2.11. *Under the Assumptions 2.1.2, set*

$$\tilde\xi^{0,t}_w = \mathbb{E}\left[\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}r}\tilde W^{H,t}_{r+w}dr\,\middle|\,\tilde{\mathcal{F}}_w\right],$$

for $w\in\mathbb{R}$. Then, we have

$$\sup_{\lambda^a\le t\le T-2\lambda^a}\left\|\lambda^{-H/2}\left(\hat\xi^\lambda_{t+\sqrt{\lambda}w}-\xi_t\right)\mathbb{1}_{(-\lambda^{a-1/2},0)}(w)-f'(B^H_t)\tilde\xi^{0,t}_w\right\|_{L^p(\mathbb{P}\times\mu_-)}\to 0,$$

for $p > 1$, as $\lambda \to 0$.

Proof. For $r \in (t-\lambda^a, t+\lambda^a)$, we use Taylor's formula to obtain

$$\xi_r = f(B_r^H) = f(B_t^H) + f'(B_{u(r,t)}^H)(B_r^H - B_t^H)$$

for some $u(r,t) \in (\min\{r,t\}, \max\{r,t\})$. This implies that

$$\begin{aligned}
&\lambda^{-H/2}\left(\hat{\xi}_s^\lambda - \xi_t\right) \\
&= \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_s^{s+\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left(\xi_r - \xi_t\right)\nu_r dr \,\middle|\, \mathcal{F}_s\right] + R_s^{1,\lambda} \\
&= \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_s^{s+\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left\{f'(B_{u(r,t)}^H)(B_r^H - B_t^H)\right\}\nu_r dr \,\middle|\, \mathcal{F}_s\right] + R_s^{1,\lambda} \\
&= \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_s^{s+\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left\{f'(B_s^H)(B_r^H - B_t^H)\right\}\nu_r dr \,\middle|\, \mathcal{F}_s\right] + R_s^{1,\lambda} + R_s^{2,\lambda},
\end{aligned}$$

where the remainders are given by

$$\begin{aligned}
R_s^{1,\lambda} &:= \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_{s+\lambda^a}^T e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left(\xi_r - \xi_t\right)\nu_r dr \,\middle|\, \mathcal{F}_s\right], \\
R_s^{2,\lambda} &:= \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_s^{s+\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left\{(f'(B_{u(r,t)}^H) - f'(B_s^H))(B_r^H - B_t^H)\right\}\nu_r dr \,\middle|\, \mathcal{F}_s\right]
\end{aligned}$$

and we use, for $s \in [0,T)$, that

$$c_s^\lambda = \mathbb{E}\left[\int_s^T e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\nu_r dr \,\middle|\, \mathcal{F}_s\right].$$

After the change of variable $s = t + \sqrt{\lambda}w$, for $s \in (t-\lambda^a, t)$, we can write

$$\begin{aligned}
&\frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[\int_s^{s+\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\left\{f'(B_s^H)(B_r^H - B_t^H)\right\}\nu_r dr \,\middle|\, \mathcal{F}_s\right] \qquad (2.35)\\
&= \frac{\sqrt{\lambda}}{c_s^\lambda}f'(B_s^H)\mathbb{E}\left[\int_0^{\lambda^{a-1/2}} e^{-\int_0^v \frac{c^\lambda_{s+\sqrt{\lambda}x}}{\sqrt{\lambda}\kappa_{s+\sqrt{\lambda}x}}dx}\left\{\lambda^{-H/2}(B_{s+\sqrt{\lambda}v}^H - B_t^H)\right\}\nu_{s+\sqrt{\lambda}v}dv \,\middle|\, \mathcal{F}_s\right] \\
&= \frac{\sqrt{\lambda}}{c^\lambda_{t+\sqrt{\lambda}w}}f'(B_{t+\sqrt{\lambda}w}^H)\mathbb{E}\left[\int_0^{\lambda^{a-1/2}} e^{-\int_0^v \frac{c^\lambda_{t+\sqrt{\lambda}(x+w)}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}(x+w)}}dx}\left\{X_{v+w}^{t,\lambda}\right\}\nu_{t+\sqrt{\lambda}(v+w)}dv \,\middle|\, \mathcal{F}_{t+\sqrt{\lambda}w}\right].
\end{aligned}$$

Therefore,

$$\lambda^{-H/2}\left(\hat{\xi}^\lambda_{t+\sqrt{\lambda}w} - \xi_t\right) = f'(B_t^H)\tilde{\xi}_w^{0,t} + R^{1,\lambda}_{t+\sqrt{\lambda}w} + R^{2,\lambda}_{t+\sqrt{\lambda}w} + R^{3,\lambda}_{t+\sqrt{\lambda}w} + R^{4,\lambda}_{t+\sqrt{\lambda}w} + R^{5,\lambda}_{t+\sqrt{\lambda}w}$$

where the remainders $R^{3,\lambda}$, $R^{4,\lambda}$, $R^{5,\lambda}$ are given by, for $s = t + \sqrt{\lambda}w$,

$$R_s^{3,\lambda}$$
$$= f'(B_s^H)\mathbb{E}\left[\int_0^\infty \left(\frac{\sqrt{\lambda}}{c_s^\lambda} e^{-\int_0^v \frac{c^\lambda_{s+\sqrt{\lambda}x}}{\sqrt{\lambda}\kappa_{s+\sqrt{\lambda}x}}dx} \nu_{s+\sqrt{\lambda}v}\mathbb{1}_{V_\lambda}(v) - \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\right)\left\{X^{t,\lambda}_{v+w}\right\} dv \,\middle|\, \mathcal{F}_s\right],$$
$$R_s^{4,\lambda} = (f'(B_s^H) - f'(B_t^H))\mathbb{E}\left[\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left\{X^{t,\lambda}_{v+w}\right\} dv \,\middle|\, \mathcal{F}_s\right],$$
$$R_s^{5,\lambda} = f'(B_t^H)\mathbb{E}\left[\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left\{X^{t,\lambda}_{v+w} - \tilde{W}^{H,t}_{v+w}\right\} dv \,\middle|\, \mathcal{F}_s\right],$$

for $V^\lambda = (0, \lambda^{a-1/2})$.

Now, it only remains to show that the remainder terms $R^{1,\lambda}$, $R^{2,\lambda}$, $R^{3,\lambda}$ and $R^{4,\lambda}$ converge to zero in the norm $\|\cdot\|_{L^p(\mathbb{P}\times\mu_-)}$ for all $t \in [\lambda^a, T - 2\lambda^a]$.

By the tower property of conditional expectation, we can further rewrite $R_s^{1,\lambda}$ as

$$R_s^{1,\lambda} = \frac{\lambda^{-H/2}}{c_s^\lambda}\mathbb{E}\left[e^{-\int_s^{s+\lambda^a} \frac{c_v^\lambda}{\lambda\kappa_v}dv} c^\lambda_{s+\lambda^a}\left(\hat{\xi}^\lambda_{s+\lambda^a} - \xi_t\right) \,\middle|\, \mathcal{F}_s\right].$$

By Jensen's and Hölder's inequality for $1 < q_1, q_2 < \infty$ such that $1/p = 1/q_1 + 1/q_2$, we have that

$$\left\|R_s^{1,\lambda}\right\|_{L^p(\mathbb{P})} \le \left\|\lambda^{-H/2} e^{-\int_s^{s+\lambda^a} \frac{c_v^\lambda}{\lambda\kappa_v}dv}\right\|_{L^{q_1}(\mathbb{P})} \left\|\frac{c^\lambda_{s+\lambda^a}}{c^\lambda_s}\left(\hat{\xi}^\lambda_{s+\lambda^a} - \xi_t\right)\right\|_{L^{q_2}(\mathbb{P})},$$

By Lemma 2.2.6, the first term on the right-hand side converges to 0 uniformly for $s \in [0, T - 2\lambda^a]$. The second term can be decomposed as estimated by

$$\left\|\frac{c^\lambda_{s+\lambda^a}}{c^\lambda_s}\left(\hat{\xi}^\lambda_{s+\lambda^a} - \xi_t\right)\right\|_{L^{q_2}(\mathbb{P})} \le \left\|\frac{c^\lambda_{s+\lambda^a}}{\sqrt{\lambda}}\right\|_{L^{q_3}(\mathbb{P})} \left\|\frac{\sqrt{\lambda}}{c^\lambda_s}\right\|_{L^{q_4}(\mathbb{P})} \left\|\left(\hat{\xi}^\lambda_{s+\lambda^a} - \xi_t\right)\right\|_{L^{q_5}(\mathbb{P})},$$

with $1/q_2 = 1/q_3 + 1/q_4 + 1/q_5$ for $1 < q_3, q_4, q_5 < \infty$. By Proposition 2.2.1 and Proposition 2.2.5, we have the first two norm being bounded uniformly in s. For the last norm, we have,

$$\left\|\left(\hat{\xi}^\lambda_{s+\lambda^a} - \xi_t\right)\right\|_{L^{q_5}(\mathbb{P})} \le \left\|\hat{\xi}^\lambda_{s+\lambda^a}\right\|_{L^{q_5}(\mathbb{P})} + \|\xi_t\|_{L^{q_5}(\mathbb{P})} < \infty,$$

and this shows $R^{1,\lambda}$ converges to 0. Here, for the finiteness of the first norm, we have the following decomposition

$$|\hat{\xi}^\lambda_s| \le \left|\mathbb{E}\left[\frac{1}{c_s^\lambda}\int_s^{T-\lambda^a} e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\xi_r\nu_r dr \,\middle|\, \mathcal{F}_s\right] + \mathbb{E}\left[\frac{1}{c_s^\lambda}\int_{T-\lambda^a}^T e^{-\int_s^r \frac{c_v^\lambda}{\lambda\kappa_v}dv}\xi_r\nu_r dr \,\middle|\, \mathcal{F}_s\right]\right|. \tag{2.36}$$

Then, by a change of variable, we have for the first conditional expectation above the following upper bound,

$$\mathbb{E}\left[\frac{\sqrt{\lambda}}{c_s^\lambda}\int_0^{\lambda^{-1/2}(T-\lambda^a)} e^{-\int_0^u \frac{c^\lambda_{s+\sqrt{\lambda}r}}{\sqrt{\lambda}\kappa_{s+\sqrt{\lambda}r}}dv}\left|\xi_{s+\sqrt{\lambda}u}\right|\nu_{s+\sqrt{\lambda}u}du\,\middle|\,\mathcal{F}_s\right]$$
$$\leq \mathbb{E}\left[\left(\sup_{t\in[0,T-\lambda^a]}\frac{\sqrt{\lambda}}{c_t^\lambda}\right)\left(\sup_{t\in[0,T]}|\xi_t\nu_t|\right)\left(\sup_{t\in[0,T-\lambda^a]}\frac{\sqrt{\lambda}}{c_t^\lambda\kappa_t}\right)\middle|\,\mathcal{F}_s\right]. \tag{2.37}$$

For the second conditional expectation in (2.36), we have the following estimate,

$$\mathbb{E}\left[\frac{\sqrt{\lambda}}{c_s^\lambda}\sup_{t\in[0,T]}|\xi_t\nu_t|\,e^{-\int_s^{T-\lambda^a}\frac{c_v^\lambda}{\lambda\kappa_v}dv}\int_0^{\lambda^{a-1/2}} e^{-\int_{T-\lambda^a}^{T-\lambda^a+\sqrt{\lambda}v}\frac{c_r^\lambda}{\lambda\kappa_r}dr}dv\,\middle|\,\mathcal{F}_s\right]$$
$$\leq \mathbb{E}\left[\left(\sup_{t\in[0,T-\lambda^a]}\frac{\sqrt{\lambda}}{c_t^\lambda}\right)\left(\sup_{t\in[0,T]}|\xi_t\nu_t|\right)\left(\sup_{t\in[0,T-\lambda^a]}\lambda^{a-1/2}e^{-\int_s^{T-\lambda^a}\frac{c_r^\lambda}{\lambda\kappa_r}dr}\right)\middle|\,\mathcal{F}_s\right]. \tag{2.38}$$

Since we have $a > (1-H)/2$, we have $H/2 > 1/2-a$. We can therefore apply Lemma 2.2.6 for the last part of the integrand of (2.38). Therefore, we can estimate the expressions inside the conditional expectations of (2.37) and (2.38). In particular, with our Assumptions 2.1.2, we have that the $\|\cdot\|_{q_5}$ of these expression is finite (up to adding further Hölder inequality steps). It follows by Doob's maximal inequality that $\left\|\sup_{t\in[0,T-\lambda^a]}\left|\hat{\xi}_t^\lambda\right|\right\|_{q_5} < \infty$.

Note that $\left\|\sup_{0\leq s\leq T-2\lambda^a}|\hat{\xi}_s^\lambda|\right\|_{L^{q_5}(\mathbb{P})} < \infty$ also implies that

$$\left\|\sup_{0\leq t\leq T-2\lambda^a}\left|\hat{X}_t^\lambda\right|\right\|_{L^{q_5}(\mathbb{P})} \leq \left\|\sup_{0\leq t\leq T-2\lambda^a}\left|e^{-\int_0^t\frac{c_s^\lambda}{\lambda\kappa_s}ds}\left(x+\int_0^t\frac{c_s^\lambda}{\lambda\kappa_s}e^{\int_0^s\frac{c_r^\lambda}{\lambda\kappa_r}dr}\hat{\xi}_s^\lambda ds\right)\right|\right\|_{L^{q_5}(\mathbb{P})}$$
$$\leq |x| + \left\|\sup_{0\leq s\leq T-2\lambda^a}\left|\hat{\xi}_s^\lambda\right|\right\|_{L^{q_5}(\mathbb{P})} < \infty. \tag{2.39}$$

Because $t\mapsto f'(B_t^H)$ is uniformly continuous on $[0,T]$ and $\sup_{0\leq t\leq T}\left|f'(B_t^H)\right| \in L^q(\mathbb{P})$, we have that

$$R^{f,\lambda} = \sup_{\lambda^a\leq t\leq T-2\lambda^a}\ \sup_{s\in[t-\lambda^a,t+\lambda^a]}\left|f'(B_t^H)-f'(B_s^H)\right| \to 0,$$

in $L^q(\mathbb{P})$ as $\lambda\to 0$ for $q>1$. Hence, we have

$$\left\|R_s^{2,\lambda}\right\|_{L^p(\mathbb{P})} \leq \left\|R^{f,\lambda}\right\|_{L^{q_1}(\mathbb{P})}\left\|\int_s^{s+\lambda^a} e^{-\int_s^r\frac{c_v^\lambda}{\lambda\kappa_v}dv}\frac{\nu_r}{c_s^\lambda}\left|\lambda^{-H/2}(B_r^H-B_t^H)\right|dr\right\|_{L^{q_2}(\mathbb{P})}$$

Therefore we have,

$$\left\|R_{t+\sqrt{\lambda}w}^{2,\lambda}\right\|_{L^p(\mathbb{P}\times\mu_-)} \leq \left\|R^{f,\lambda}\right\|_{L^{q_1}(\mathbb{P})}\left\|\int_s^{s+\lambda^a} e^{-\int_s^r\frac{c_v^\lambda}{\lambda\kappa_v}dv}\frac{\nu_r}{c_s^\lambda}\left|\lambda^{-H/2}(B_r^H-B_t^H)\right|dr\right\|_{L^{q_2}(\mathbb{P}\times\mu_-)}.$$

After a change of variable $v=\lambda^{-1/2}(r-s)$, and $s=t+\sqrt{\lambda}w$, the second norm is exactly the norm of the left expression in the statement of Lemma 2.2.10. Hence, since we have

$$\sup_{t\in[\lambda^a,T-2\lambda^a]}\left\|\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left(\tilde{W}^{H,t}_{v+w}\right)dv\right\|_{L^p(\mu_-\times\mathbb{P})}<\infty,$$

by Assumption 2.1.2, the remainder $R^{2,\lambda}$ converges to 0.

Similarly, we have for $R^{3,\lambda}_s$,

$$\left\|R^{3,\lambda}_s\right\|_{L^p(\mathbb{P})}\le\sup_{s\in[0,T]}\left\|f'(B^H_s)\right\|_{L^{q_1}(\mathbb{P})}$$
$$\cdot\left\|\int_0^\infty\left(\frac{\sqrt{\lambda}}{c^\lambda_s}e^{-\int_0^v\frac{c^\lambda_{s+\sqrt{\lambda}x}}{\sqrt{\lambda}\kappa_{s+\sqrt{\lambda}x}}dx}\nu_{s+\sqrt{\lambda}v}\mathbb{1}_{V_\lambda}(v)-\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\right)\left\{X^{t,\lambda}_{v+w}\right\}dv\right\|_{L^{q_2}(\mathbb{P})},$$

and therefore, to have $\left\|R^{3,\lambda}_s\right\|_{L^p(\mathbb{P}\times\mu_-)}$ converging to 0 for all $t\in[\lambda^a,T-2\lambda^a]$, we need to check that

$$\left\|\int_0^\infty\left(\frac{\sqrt{\lambda}}{c^\lambda_s}e^{-\int_0^v\frac{c^\lambda_{s+\sqrt{\lambda}x}}{\sqrt{\lambda}\kappa_{s+\sqrt{\lambda}x}}dx}\nu_{s+\sqrt{\lambda}v}\mathbb{1}_{V_\lambda}(v)-\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\right)\left\{X^{t,\lambda}_{v+w}\right\}dv\right\|_{L^{q_2}(\mathbb{P}\times\mu_-)}$$

converges to 0 for all $t\in[\lambda^a,T-2\lambda^a]$. But this follows directly from the proof of Lemma 2.2.10 as this can be estimated by the first term in (2.30).

Finally, for $R^{4,\lambda}_s$ we have,

$$\left\|R^{4,\lambda}_s\right\|_{L^p(\mathbb{P})}\le\left\|R^{f,\lambda}\right\|_{L^{q_1}(\mathbb{P})}\left\|\mathbb{E}\left[\int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left\{X^{t,\lambda}_{v+w}\right\}dv\,\middle|\,\mathcal{F}_s\right]\right\|_{L^{q_2}(\mathbb{P})}.$$

By the proof of Lemma 2.2.10, it is clear that

$$\left\|\int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left\{X^{t,\lambda}_{v+w}\right\}dv\right\|_{L^{q_2}(\mathbb{P}\times\mu_-)}$$

is finite. Hence, we have $\left\|R^{4,\lambda}_s\right\|_{L^p(\mathbb{P}\times\mu_-)}$ converging to 0 for all $t\in[\lambda^a,T-\lambda^a]$.

For $R^{5,\lambda}_s$, we have for $s=t+\sqrt{\lambda}w$,

$$\left\|R^{5,\lambda}_s\right\|_{L^p(\mathbb{P})}\le\left\|f'(B^H_t)\right\|_{L^{q_1}(\mathbb{P})}\left\|\int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v}\left|X^{t,\lambda}_{v+w}-\tilde{W}^{H,t}_{v+w}\right|dv\right\|_{L^{q_2}(\mathbb{P})},$$

and we write

$$\begin{aligned}
&\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v} \left| X^{t,\lambda}_{v+w} - \tilde{W}^{H,t}_{v+w} \right|^{q_2} dv \\
&= \frac{2}{C_\alpha} \int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v} \left| X^{t,\lambda}_{v+w} - \tilde{W}^{H,t}_{v+w} \right|^{q_2} (1+v)^\alpha \mu_+(dv) \\
&\leq C \left\| \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v} (1+v)^{-\alpha} \right\|_{L^{q_6}(\mu_+)} \left\| \left| X^{t,\lambda}_{v+w} - \tilde{W}^{H,t}_{v+w} \right|^{q_2} \right\|_{L^{q_7}(\mu_+)}
\end{aligned}$$

for $C > 0$ and $1 < q_6, q_7 < \infty$ such that $1/q_6 + 1/q_7 = 1$. It follows that

$$\left\| R^{5,\lambda}_s \right\|^p_{L^p(\mathbb{P}\times\mu_-)} \leq \left\| \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v} (1+v)^{-\alpha} \right\|_{L^{q_6}(\mathbb{P}\times\mu_+)} \left\| \left| X^{t,\lambda}_{v+w} - \tilde{W}^{H,t}_{v+w} \right|^{q_2} \right\|_{L^{q_7}(\mathbb{P}\times\mu_+\times\mu_-)}.$$

By Lemma 2.2.9, we get that the second norm vanishes to 0. Using this and Lemma 2.2.8, we obtain the convergence of the conditional expectation

$$\mathbb{E}\left[\int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}v} \left\{ X^{t,\lambda}_{v+w} \right\} dv \,\middle|\, \mathcal{F}_s \right]$$

towards $\tilde{\xi}^{0,t}_w$ in $L^p(\mathbb{P} \times \mu_-)$ for all $t \in [\lambda^a, T - \lambda^a]$. □

Proposition 2.2.12. *Under the Assumptions 2.1.2, we have that*

$$\sup_{\lambda^a \leq t \leq T - 2\lambda^a} \left\| \lambda^{-H/2}(\hat{X}^\lambda_t - \xi_t) - f'(B^H_t)(\tilde{X}^{0,t}_0) \right\|_{L^p(\mathbb{P})} \to 0,$$

for $p > 1$, as $\lambda \to 0$, where

$$\begin{aligned}
\tilde{X}^{0,t}_0 &= \int_{-\infty}^0 \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}}s} \tilde{\xi}^{0,t}_s ds, \\
\tilde{\xi}^{0,t}_s &= \int_0^\infty \sqrt{\frac{\nu_t}{\kappa_t}} e^{-\sqrt{\frac{\nu_t}{\kappa_t}}r} \mathbb{E}\left[\tilde{W}^{H,t}_{r+s} \,\middle|\, \tilde{\mathcal{F}}_s \right] dr
\end{aligned}$$

for $s \in \mathbb{R}$.

Proof. The solution to the ODE

$$\begin{aligned}
d\hat{X}^\lambda_t &= \frac{c^\lambda_t}{\lambda\kappa_t} \left(\hat{\xi}^\lambda_t - \hat{X}^\lambda_t \right) dt, \quad 0 \leq t \leq T, \\
\hat{X}^\lambda_0 &= x
\end{aligned}$$

is given by

$$\hat{X}^\lambda_t = e^{-\int_0^t \frac{c^\lambda_s}{\lambda\kappa_s} ds} \left(x + \int_0^t \frac{c^\lambda_s}{\lambda\kappa_s} e^{\int_0^s \frac{c^\lambda_r}{\lambda\kappa_r} dr} \hat{\xi}^\lambda_s ds \right).$$

Moreover, denoting $g_\lambda(r) := c_r^\lambda/(\lambda\kappa_r)$, we have

$$
\begin{aligned}
1 &= h_\lambda(t)\int_0^t g_\lambda(s)e^{-\int_s^t g_\lambda(r)dr}ds,\\
h_\lambda(t) &= \frac{1}{1-e^{-\int_0^t g_\lambda(r)dr}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lambda^{-H/2}\left(\hat{X}_t^\lambda - \xi_t\right) \qquad (2.40)\\
&= \lambda^{-H/2}\left\{e^{-\int_0^t \frac{c_s^\lambda}{\lambda\kappa_s}ds}x + e^{-\int_{t-\lambda^a}^t \frac{c_s^\lambda}{\lambda\kappa_s}ds}\int_0^{t-\lambda^a}\frac{c_s^\lambda}{\lambda\kappa_s}e^{-\int_s^{t-\lambda^a}\frac{c_r^\lambda}{\lambda\kappa_r}dr}\left(\hat{\xi}_s^\lambda - \xi_t\right)ds\right\}\\
&\quad + \int_{t-\lambda^a}^t \frac{c_s^\lambda}{\lambda\kappa_s}e^{-\int_s^t \frac{c_r^\lambda}{\lambda\kappa_r}dr}\lambda^{-H/2}\left(\hat{\xi}_s^\lambda - \xi_t\right)ds + \lambda^{-H/2}\xi_t\frac{1-h_\lambda(t)}{h_\lambda(t)},
\end{aligned}
$$

and we have

$$
\lambda^{-H/2}\xi_t\frac{1-h_\lambda(t)}{h_\lambda(t)} = -\lambda^{-H/2}e^{-\int_0^t g_\lambda(r)dr}\xi_t.
$$

Because we have by previous results,

$$
\sup_{0\le t\le T-\lambda^a}\left(\lambda^{-H/2}e^{-\int_{t-\lambda}^t \frac{c_r^\lambda}{\lambda\kappa_r}dr}\right) \to 0
$$

in $L^q(\mathbb{P})$, as $\lambda \to 0$, and

$$
\sup_{0\le t\le T-\lambda^a}\left\|\int_0^{t-\lambda^a}\frac{c_s^\lambda}{\lambda\kappa_s}e^{-\int_s^{t-\lambda^a}\frac{c_r^\lambda}{\lambda\kappa_r}dr}\hat{\xi}_s^\lambda ds\right\|_{L^r(\mathbb{P})}
$$

is bounded for $\lambda \in (0,1)$ by (2.39) for some $q, r > 1$, we can estimate three out of four expressions in (2.40). More precisely, we have for $1 < q_1, q_2 < \infty$ such that $1/p = 1/q_1 + 1/q_2$,

$$
\left\|\lambda^{-H/2}e^{-\int_0^t g_\lambda(r)dr}\xi_t\right\|_{L^p(\mathbb{P})} \le \|\xi_t\|_{L^{q_1}(\mathbb{P})}\left\|\lambda^{-H/2}e^{-\int_0^t g_\lambda(r)dr}\right\|_{L^{q_2}(\mathbb{P})}.
$$

But then, we have

$$
e^{-\int_0^t g_\lambda(r)dr} \le e^{-\frac{t}{\sqrt{\lambda}}\inf_{v\in[0,T-\lambda^a]}\left(\frac{c_v^\lambda}{\sqrt{\lambda}\kappa_v}\right)} \le e^{-\lambda^{a-1/2}\inf_{v\in[0,T-\lambda^a]}\left(\frac{c_v^\lambda}{\sqrt{\lambda}\kappa_v}\right)},
$$

for every $t \in [\lambda^a, T-\lambda^a]$. As seen in the proof of Lemma 2.2.6, the random variables

$$
\lambda^{-H/2}e^{-\lambda^{a-1/2}\inf_{v\in[0,T-\lambda^a]}\left(\frac{c_v^\lambda}{\sqrt{\lambda}\kappa_v}\right)},
$$

indexed by $\lambda \in (0,1)$ are uniformly integrable in $L^{q_2}(\mathbb{P})$. Hence, we obtain

$$\sup_{t\in[\lambda^a,T-\lambda^a]} \left\| \lambda^{-H/2} e^{-\int_0^t g_\lambda(r)dr} \right\|_{L^{q_2}(\mathbb{P})} \to 0.$$

By Assumption 2.1.2 on ξ_t, we get the convergence towards 0. The term involving the initial condition x is dealt with the same way without the need for Hölder's inequality.

Next we estimate

$$\begin{aligned}
&\left\| \lambda^{-H/2} e^{-\int_{t-\lambda^a}^t \frac{c_s^\lambda}{\lambda\kappa_s}ds} \int_0^{t-\lambda^a} \frac{c_s^\lambda}{\lambda\kappa_s} e^{-\int_s^{t-\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r}dr} \left(\hat{\xi}_s^\lambda - \xi_t\right) ds \right\|_{L^p(\mathbb{P})} \\
&\leq \left\| \lambda^{-H/2} e^{-\int_{t-\lambda^a}^t \frac{c_s^\lambda}{\lambda\kappa_s}ds} \right\|_{L^{q_1}(\mathbb{P})} \left\| \int_0^{t-\lambda^a} \frac{c_s^\lambda}{\lambda\kappa_s} e^{-\int_s^{t-\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r}dr} \left(\hat{\xi}_s^\lambda - \xi_t\right) ds \right\|_{L^{q_2}(\mathbb{P})}
\end{aligned}$$

and we know the first norm is vanishing in the limit. For the second one, we have the upper bound

$$\sup_{t\in[\lambda^a,T-\lambda^a]} \left\{ \left\| \int_0^{t-\lambda^a} \frac{c_s^\lambda}{\lambda\kappa_s} e^{-\int_s^{t-\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r}dr} \hat{\xi}_s^\lambda ds \right\|_{L^{q_2}(\mathbb{P})} + \left\| \int_0^{t-\lambda^a} \frac{c_s^\lambda}{\lambda\kappa_s} e^{-\int_s^{t-\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r}dr} \xi_t ds \right\|_{L^{q_2}(\mathbb{P})} \right\}$$

and for the second term, since $1/h_\lambda(t) \leq 1$ for any $t > 0$, we have

$$\sup_{t\in[\lambda^a,T-\lambda^a]} \left\| \int_0^{t-\lambda^a} \frac{c_s^\lambda}{\lambda\kappa_s} e^{-\int_s^{t-\lambda^a} \frac{c_r^\lambda}{\lambda\kappa_r}dr} \xi_t ds \right\|_{L^{q_2}(\mathbb{P})} \leq \sup_{t\in[\lambda^a,T-\lambda^a]} \|\xi_t\|_{L^{q_2}(\mathbb{P})} < \infty.$$

For the first term, we have the finiteness following from (2.39).

For the rest of the proof, we do a change of variable, $s = t + \sqrt{\lambda}w$. We denote by $\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)$ the main term in (2.40). We also denote $V_\lambda = (-\lambda^{a-1/2}, 0)$, and introduce the measure $\mu_-(dw) = (C_\alpha/2)(1+|w|)^{-\alpha}dw$ and we have by Hölder's inequality that

$$\begin{aligned}
&\sup_{\lambda^a\leq t\leq T-2\lambda^a} \left\| \lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t) - f'(B_t^H)\tilde{X}_0^{0,t} \right\|_{L^p(\mathbb{P})} \\
&= \sup_{\lambda^a\leq t\leq T-2\lambda^a} \left\| \int_{-\infty}^0 \left\{ \frac{c_{t+\sqrt{\lambda}w}^\lambda}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}} e^{-\int_w^0 \frac{c_{t+\sqrt{\lambda}r}^\lambda}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}}dr} \lambda^{-H/2}(\hat{\xi}_{t+\sqrt{\lambda}w}^\lambda - \xi_t)\mathbb{1}_{V_\lambda}(w) \right.\right. \\
&\qquad\qquad \left.\left. - f'(B_t^H)\sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}}w} \tilde{\xi}_w^{0,t} \right\} dw \right\|_{L^p(\mathbb{P})} \\
&\leq \sup_{\lambda^a\leq t\leq T-2\lambda^a} \left\| \left\| \left| \frac{c_{t+\sqrt{\lambda}w}^\lambda}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}} e^{-\int_w^0 \frac{c_{t+\sqrt{\lambda}r}^\lambda}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}}dr} \mathbb{1}_{V_\lambda}(w) - \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}}w} \right| (1+|w|)^\alpha \right\|_{L^{q_1}(\mu_-)} \right. \\
&\qquad\qquad \left. \cdot \left\| \lambda^{-H/2}(\hat{\xi}_{t+\sqrt{\lambda}w}^\lambda - \xi_t) \right\|_{L^{q_2}(\mu_-)} \right\|_{L^p(\mathbb{P})} \\
&\quad + \sup_{\lambda^a\leq t\leq T-2\lambda^a} \left\| \int_{-\infty}^0 \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}}w} \left| \lambda^{-H/2}(\hat{\xi}_{t+\sqrt{\lambda}w}^\lambda - \xi_t)\mathbb{1}_{V_\lambda}(w) - f'(B_t^H)\tilde{\xi}_w^{0,t} \right| dw \right\|_{L^p(\mathbb{P})},
\end{aligned}$$

again for $1 < q_1, q_2 < \infty$ such that $1/p = 1/q_1 + 1/q_2$. Applying Hölder's inequality again and introducing the product probability measure $\mathbb{P}(d\omega) \times \mu_-(dw)$ this can further be estimated by

$$
\begin{aligned}
&\sup_{\lambda^a \le t \le T-2\lambda^a} \left\| \lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t) - f'(B_t^H)\tilde{X}_t^0 \right\|_{L^p(\mathbb{P})} \\
&\le \\
&\sup_{\lambda^a \le t \le T-2\lambda^a} \left\{ \left\| \left(\frac{c^\lambda_{t+\sqrt{\lambda}w}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}} e^{-\int_w^0 \frac{c^\lambda_{t+\sqrt{\lambda}r}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}} dr} \mathbb{1}_{V_\lambda}(w) - \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}} w} \right) (1+|w|)^\alpha \right\|_{L^{q_1}(\mathbb{P}\times\mu_-)} \right. \\
&\qquad \left. \cdot \left\| \lambda^{-H/2}(\hat{\xi}^\lambda_{t+\sqrt{\lambda}w} - \xi_t)\mathbb{1}_{V_\lambda}(w) \right\|_{L^{q_2}(\mathbb{P}\times\mu_-)} \right\} \\
&\quad + \sup_{\lambda^a \le t \le T-2\lambda^a} \left\| \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}} w} (1+|w|)^\alpha \right\|_{L^{q_1}(\mathbb{P}\times\mu_-)} \\
&\qquad \cdot \left\| \lambda^{-H/2}(\hat{\xi}^\lambda_{t+\sqrt{\lambda}w} - \xi_t)\mathbb{1}_{V_\lambda}(w) - f'(B_t^H)\tilde{\xi}^{0,t}_w \right\|_{L^{q_2}(\mathbb{P}\times\mu_-)}.
\end{aligned} \tag{2.41}
$$

By Lemma 2.2.11, we have that

$$
\left\| \lambda^{-H/2}(\hat{\xi}^\lambda_{t+\sqrt{\lambda}w} - \xi_t)\mathbb{1}_{V_\lambda}(w) - f'(B_t^H)\tilde{\xi}^{0,t}_w \right\|_{L^{q_2}(\mathbb{P}\times\mu_-)} \to 0,
$$

as $\lambda \to 0$ and therefore, the second term in (2.41) converges to 0. For the first term in (2.41), we show that the random variables,

$$
\begin{aligned}
Y^{1,\lambda} &= \left\{ \frac{c^\lambda_{t+\sqrt{\lambda}w}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}} e^{-\int_w^0 \frac{c^\lambda_{t+\sqrt{\lambda}r}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}} dr} (1+|w|)^\alpha \mathbb{1}_{V_\lambda}(w) \right\}^{q_1}, \\
Y^{2,\lambda} &= \left\{ \sqrt{\frac{\nu_t}{\kappa_t}} e^{\sqrt{\frac{\nu_t}{\kappa_t}} w} (1+|w|)^\alpha \mathbb{1}_{(-\infty,0)}(w) \right\}^{q_1}
\end{aligned}
$$

are bounded in $L^{1+\varepsilon}(\mathbb{P}\times\mu_-)$ and hence uniformly integrable. We estimate

$$
\begin{aligned}
&\mathbb{E}_{\mathbb{P}\times\mu_-}\left[(Y^{1,\lambda})^{1+\varepsilon} \right] \\
&\le C\mathbb{E}\left[\int_{-\lambda^{a-1/2}}^0 \left(\frac{c^\lambda_{t+\sqrt{\lambda}w}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}} \right)^{q_1(1+\varepsilon)} e^{-q_1(1+\varepsilon)\int_w^0 \frac{c^\lambda_{t+\sqrt{\lambda}r}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}} dr} (1+|w|)^{\alpha q_1(1+\varepsilon)-\alpha} dw \right] \\
&\le C_1\mathbb{E}\left[\left\{ \sup_{t\in[0,T-\lambda^a]} \frac{c_t^\lambda}{\sqrt{\lambda}\kappa_t} \right\}^{q_1(1+\varepsilon)} \frac{1}{\inf_{t\in[0,T-\lambda^a]} c_t^\lambda/(\sqrt{\lambda}\kappa_t)} \right] \\
&\quad + C_2\mathbb{E}\left[\left\{ \sup_{t\in[0,T-\lambda^a]} \frac{c_t^\lambda}{\sqrt{\lambda}\kappa_t} \right\}^{q_1(1+\varepsilon)} \frac{\Gamma(\alpha q_1(1+\varepsilon)-\alpha+1)}{\left\{\inf_{t\in[0,T-\lambda^a]} c_t^\lambda/(\sqrt{\lambda}\kappa_t)\right\}^{\alpha q_1(1+\varepsilon)-\alpha+1}} \right].
\end{aligned}
$$

for $C_1, C_2 > 0$ and

$$\mathbb{E}_{\mathbb{P}\times\mu_-}\left[(Y^{2,\lambda})^{1+\varepsilon}\right] \leq C_1\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sqrt{\frac{\nu_t}{\kappa_t}}\right)^{q_1(1+\varepsilon)-1}\right] + C_2\Gamma(\alpha(q_1(1+\varepsilon)-1)+1)\mathbb{E}\left[\left(\sup_{t\in[0,T]}\sqrt{\frac{\nu_t}{\kappa_t}}\right)^{(1-\alpha)q_1(1+\varepsilon)-1+\alpha}\right].$$

The uniform integrability of $Y^{1,\lambda}$ and $Y^{2,\lambda}$ then allows us to interchange the limit with taking expectation with respect to $\mathbb{P}\times\mu$. Because

$$\sup_{0\leq t\leq T-\lambda^a}\left|\frac{c_t^\lambda}{\sqrt{\lambda}} - \sqrt{\nu_t\kappa_t}\right| \to 0,$$

$\mathbb{P}$-a.s., as $\lambda \to 0$, we have

$$\sup_{\lambda^a\leq t\leq T-2\lambda^a}\left\|\left(\frac{c^\lambda_{t+\sqrt{\lambda}w}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}w}}e^{-\int_w^0\frac{c^\lambda_{t+\sqrt{\lambda}r}}{\sqrt{\lambda}\kappa_{t+\sqrt{\lambda}r}}dr}\mathbb{1}_{V_\lambda}(w) - \sqrt{\frac{\nu_t}{\kappa_t}}e^{\sqrt{\frac{\nu_t}{\kappa_t}}w}\right)(1+|w|)^\alpha\right\|_{L^{q_1}(\mathbb{P}\times\mu_-)} \to 0,$$

as $\lambda \to 0$. This shows the convergence of the right-hand side of (2.41) towards 0. □

Lemma 2.2.13. *Under the Assumptions 2.1.2, we have*

$$\sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \xi_t\right)\right\} - f'(B_t^H)\tilde{\xi}_0^{0,t}\right\|_{L^p(\mathbb{P})} \to 0,$$

for $p > 1$, as $\lambda \to 0$, where

$$\tilde{\xi}_s^{0,t} = \int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}r}\mathbb{E}\left[\tilde{W}^{H,t}_{r+s}\middle|\tilde{\mathcal{F}}_s\right]dr,$$

for $s \in \mathbb{R}$.

Proof. The proof follows the proofs of Lemma 2.2.11 and Lemma 2.2.10, with $w = 0$. □

Proposition 2.2.14. *Under the Assumptions 2.1.2, we have*

$$\sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\} - f'(B_t^H)\left(\tilde{\xi}_0^{0,t} - \tilde{X}_t^{0,t}\right)\right\|_{L^p(\mathbb{P})} \to 0,$$

for $p > 1$, as $\lambda \to 0$, where

$$\tilde{X}_0^{0,t} = \int_{-\infty}^0\sqrt{\frac{\nu_t}{\kappa_t}}e^{\sqrt{\frac{\nu_t}{\kappa_t}}s}\tilde{\xi}_s^{0,t}ds,$$

$$\tilde{\xi}_s^{0,t} = \int_0^\infty\sqrt{\frac{\nu_t}{\kappa_t}}e^{-\sqrt{\frac{\nu_t}{\kappa_t}}r}\mathbb{E}\left[\tilde{W}^{H,t}_{r+s}\middle|\tilde{\mathcal{F}}_s\right]dr$$

for $s \in \mathbb{R}$.

Proof. We write

$$\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right) = \lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \xi_t\right) + \lambda^{-H/2}\left(\xi_t - \hat{X}_t^\lambda\right)$$

and the result follows directly from Proposition 2.2.12 and Lemma 2.2.13. □

2.2.3 Main proof

Finally, before moving to the main proof of Theorem 2.1.3, we must show that the first and last interval, $[0, \lambda^a]$ and $[T - \lambda^a, T]$ have effect of order $o(\lambda^H)$.

Lemma 2.2.15. *In the setup of Theorem 2.1.3, we have*

$$J_{0,\lambda^a}(\hat{u}^\lambda) := \mathbb{E}\left[\int_0^{\lambda^a} \nu_t(\hat{X}_t^\lambda - \xi_t)^2 dt + \lambda \int_0^{\lambda^a} \kappa_t(\hat{u}_t^\lambda)^2 dt\right] = o(\lambda^H).$$

Proof. Without any loss of generality, we assume that $\hat{X}_0^\lambda = 0$ and we choose the suboptimal strategy on $[0, \lambda^a]$ given by

$$\bar{u}_t = 0, \quad t \in [0, \lambda^a].$$

It follows that

$$J_{0,\lambda^a}(\hat{u}^\lambda) \le J_{0,\lambda^a}(\bar{u}) = \mathbb{E}\left[\int_0^{\lambda^a} \nu_t(\xi_t)^2 dt\right].$$

Then, on the interval $[0, \lambda^a]$ we have the following Taylor expansion of the target,

$$\xi_t = f(B_t^H) = f(B_0^H) + f'(B_{u(0,t)}^H)(B_t^H - B_0^H),$$

for some $u(0,t) \in [0, t]$. Then, we have

$$\begin{aligned}
&\mathbb{E}\left[\nu_t(\xi_t)^2\right] \\
&\le 2f(B_0^H)^2 \sup_{0\le t\le\lambda^a} \mathbb{E}[\nu_t] + 2\mathbb{E}\left[\left\{\sup_{0\le t\le T} \nu_t\right\}\left\{\sup_{0\le t\le T} f'(B_t^H)^2\right\}(B_t^H - B_0^H)^2\right] \\
&\le 2f(B_0^H)^2 \left\|\sup_{0\le t\le T} \nu_t\right\|_{L^1(\mathbb{P})} \\
&\quad + 2\left\|\sup_{0\le t\le T} \nu_t\right\|_{L^{q_1}(\mathbb{P})} \left\|\sup_{0\le t\le T} f'(B_t^H)^2\right\|_{L^{q_2}(\mathbb{P})} \left\|(B_t^H - B_0^H)^2\right\|_{L^{q_3}(\mathbb{P})},
\end{aligned}$$

where $1 < q_1, q_2, q_3 < \infty$ and $1 = 1/q_1 + 1/q_2 + 1/q_3$. Hence using Assumptions 2.1.2, we obtain, for some constants $C_1, C_2, C_3 > 0$,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\lambda^a} \nu_t(\xi_t)^2 dt\right] &\le C_1\lambda^a + C_2 \int_0^{\lambda^a} t^{2H} dt \\
&\le C_1\lambda^a + C_3\lambda^{a(2H+1)}.
\end{aligned}$$

Since $a > H$, we have $a > H/(2H+1)$, and we conclude the higher order than λ^H. $\square$

The next lemma shows that including liquidation constraint for our problem will not affect our leading order expansion as liquidation has an effect of order $o(\lambda^H)$.

Lemma 2.2.16. *In the setup of Theorem 2.1.3, define for $X_t^u = x + \int_0^t u_s ds$,*

$$J_{T-\lambda^a,T}(u) = \mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t (X_t^u - \xi_t)^2 dt + \lambda \int_{T-\lambda^a}^T \kappa_t (u_t)^2 dt\right],$$

and let us denote by $(\hat{u}_t^{L,\lambda})_{t\in[0,T]}$ the optimal solution to minimising $J_T(u)$ as in (2.1) over the set of admissible stratetgies u satifying the terminal liquidation constraint $X_T^u = 0$. Then, we have

$$J_{T-\lambda^a,T}(\hat{u}^{L,\lambda}) = \mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t (\hat{X}_t^{L,\lambda} - \xi_t)^2 dt + \lambda \int_{T-\lambda^a}^T \kappa_t (\hat{u}_t^{L,\lambda})^2 dt\right] = o(\lambda^H).$$

Proof. We start by considering $\hat{u}^\lambda$ the unconstrained optimal solution to (2.1) and denote $X_{T-\lambda^a} = \hat{X}_{T-\lambda^a}^\lambda$. We then define the following suboptimal liquidation strategy (where we drop the λ exponent to alleviate notation) starting at $T - \lambda^a$,

$$\bar{u}_t = -\frac{X_{T-\lambda^a}}{\lambda^a}, \quad t \in [T-\lambda^a, T], \tag{2.42}$$

such that

$$\bar{X}_t = \frac{X_{T-\lambda^a}}{\lambda^a}(T-t), \quad t \in [T-\lambda^a, T].$$

We then have the following inequalities

$$J_{T-\lambda^a,T}(\hat{u}^\lambda) \le J_{T-\lambda^a,T}(\hat{u}^{L,\lambda}) \le J_{T-\lambda^a,T}(\bar{u}^\lambda).$$

and we then prove that

$$J_{T-\lambda^a,T}(\bar{u}) = \mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t (\bar{X}_t - \xi_t)^2 dt + \lambda \int_{T-\lambda^a}^T \kappa_t (\bar{u}_t)^2 dt\right] = o(\lambda^H).$$

to get the result.

To that effect, we start by writing

$$\begin{aligned}
\mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t (\bar{X}_t - \xi_t)^2 dt\right] &\le 3\mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t \left(\frac{X_{T-\lambda^a}}{\lambda^a}(T-t) - X_{T-\lambda^a}\right)^2 dt\right] \\
&\quad + 3\mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t \left(X_{T-\lambda^a} - \xi_{T-\lambda^a}\right)^2 dt\right] \\
&\quad + 3\mathbb{E}\left[\int_{T-\lambda^a}^T \nu_t \left(\xi_{T-\lambda^a} - \xi_t\right)^2 dt\right],
\end{aligned}$$

and we estimate each part separately. We have

$$\mathbb{E}\left[\int_{T-\lambda^a}^{T}\nu_t\left(\frac{X_{T-\lambda^a}}{\lambda^a}(T-t)-X_{T-\lambda^a}\right)^2 dt\right] \leq \frac{\lambda^a}{3}\mathbb{E}\left[\left(\sup_{t\in[0,T]}\nu_t\right)(X_{T-\lambda^a})^2\right]$$
$$\leq \frac{\lambda^a}{3}\left\|\sup_{t\in[0,T]}\nu_t\right\|_{q_1}\left\|(X_{T-\lambda^a})^2\right\|_{q_2},$$

with $1 < q_1, q_2 < \infty$, $1/q_1 + 1/q_2 = 1$. With (2.39) and Assumptions 2.1.2, the estimate is of higher order than λ^H since we have $H < a < 1/2$. Next, we consider

$$\lambda^{-H}\mathbb{E}\left[\int_{T-\lambda^a}^{T}\nu_t\left(X_{T-\lambda^a}-\xi_{T-\lambda^a}\right)^2 dt\right] \leq \lambda^a\left\|\sup_{t\in[0,T]}\nu_t\right\|_{q_1}\left\|\left\{\lambda^{-H/2}(X_{T-\lambda^a}-\xi_{T-\lambda^a})\right\}^2\right\|_{q_2},$$

which converges to 0 as $\lambda \to 0$ by Proposition 2.2.12. Next, we use Taylor expansion for $t \in [T-\lambda^a, T]$,

$$f(B_t^H) = f(B_{T-\lambda^a}) + f'(B_{T-\lambda^a}^H)(B_t^H - B_{T-\lambda^a}^H).$$

and we then have

$$\lambda^{-H}\mathbb{E}\left[\int_{T-\lambda^a}^{T}\nu_t\left(\xi_{T-\lambda^a}-\xi_t\right)^2 dt\right]$$
$$\leq \left\|\left\{\sup_{t\in[0,T]}\nu_t\right\}\left\{\sup_{t\in[0,T]}f'(B_t^H)^2\right\}\right\|_{q_3}\left\|\int_{T-\lambda^a}^{T}\left\{\lambda^{-H/2}(B_t^H-B_{T-\lambda^a}^H)\right\}^2 dt\right\|_{q_4},$$

for $1 < q_3, q_4 < \infty$, $1/q_4 + 1/q_3 = 1$. We further estimate

$$\left\|\left\{\sup_{t\in[0,T]}\nu_t\right\}\left\{\sup_{t\in[0,T]}f'(B_t^H)^2\right\}\right\|_{q_3} \leq \left\|\sup_{t\in[0,T]}\nu_t\right\|_{q_5}\left\|\sup_{t\in[0,T]}f'(B_t^H)^2\right\|_{q_6} < \infty,$$

by Assumption 2.1.2, with $1 < q_5, q_6 < \infty$, $1/q_3 = 1/q_5 + 1/q_6$. Furthermore, by Jensen's inequality, and the fact that the variance of increments of fBM is increasing with the size of the increments, we have

$$\mathbb{E}\left[\left(\int_{T-\lambda^a}^{T}\left\{\lambda^{-H/2}(B_t^H-B_{T-\lambda^a}^H)\right\}^2 dt\right)^{q_4}\right] \leq \lambda^{aq_4}\mathbb{E}\left[\left\{\lambda^{-H/2}(B_T^H-B_{T-\lambda^a}^H)\right\}^{2q_4}\right]$$
$$\leq C\lambda^{q_4\{a+2H(a-1/2)\}}$$

for some $C > 0$, where we use the fact that $\lambda^{-H/2}(B_T^H - B_{T-\lambda^a}^H) \sim \mathcal{N}\left(0, \lambda^{H(2a-1)}\right)$. We then have,

$$\left\|\int_{T-\lambda^a}^{T}\left\{\lambda^{-H/2}(B_t^H-B_{T-\lambda^a}^H)\right\}^2 dt\right\|_{q_4} \leq C^{1/q_4}\lambda^{\{a+2H(a-1/2)\}} \to 0,$$

as we have $a > H$ and therefore, $a > H/(1+2H)$.

Finally, we consider the term

$$\begin{aligned}\mathbb{E}\left[\int_{T-\lambda^a}^{T} \lambda\kappa_t(\bar{u}_t)^2 dt\right] &\leq \lambda^{1-a}\mathbb{E}\left[\left(\sup_{t\in[0,T]} \kappa_t\right)(X_{T-\lambda^a})^2\right] \\ &\leq \lambda^{1-a}\left\|\sup_{t\in[0,T]} \kappa_t\right\|_{q_1} \left\|(X_{T-\lambda^a})^2\right\|_{q_2}\end{aligned}$$

and where we use (2.42). Using (2.39) and Assumption 2.1.2, it then follows,

$$\lambda^{-H}\mathbb{E}\left[\int_{T-\lambda^a}^{T} \lambda\kappa_t(\bar{u}_t)^2 dt\right] \leq \lambda^{1-a-H}\left\|\sup_{t\in[0,T]} \kappa_t\right\|_{q_1} \left\|(X_{T-\lambda^a})^2\right\|_{q_2} \to 0,$$

since $H < a < 1/2$ and therefore, $H + a < 1$. □

We are now ready to provide the proof of Theorem 2.1.3.

Proof of Theorem 2.1.3. By Lemma 2.2.15 and 2.2.16, we can ignore the first interval $[0, \lambda^a]$ and the last interval $[T - \lambda^a, T]$ as both their respective effects are of order $o(\lambda^H)$. We then rewrite

$$\begin{aligned}&J_{\lambda^a,T-\lambda^a}(\hat{u}^\lambda) \\ &= \lambda^H\left(\int_{\lambda^a}^{T-\lambda^a} \mathbb{E}\left[\nu_t\left\{\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)\right\}^2\right]dt + \int_{\lambda^a}^{T-\lambda^a} \mathbb{E}\left[\kappa_t\left\{\lambda^{(1-H)/2}(\hat{u}_t^\lambda)\right\}^2\right]dt\right).\end{aligned}$$

For the deviation, we show

$$\sup_{t\in[\lambda^a,T-\lambda^a]}\left|\mathbb{E}\left[\nu_t\left\{\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)\right\}^2\right] - \mathbb{E}\left[\nu_t f'(B_t^H)^2 Z_D(\delta_t)^2\right]\right| \to 0,$$

as $\lambda \to 0$, where for $\delta > 0$, the random variable $Z_D(\delta)$ has the same law as (2.24) in Lemma 2.2.7. For this, we have

$$\begin{aligned}&\sup_{t\in[\lambda^a,T-\lambda^a]}\left|\mathbb{E}\left[\nu_t\left\{\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)\right\}^2\right] - \mathbb{E}\left[\nu_t f'(B_t^H)^2 Z_D(\delta_t)^2\right]\right| \\ &\leq \left\|\sup_{t\in[0,T]} \nu_t\right\|_{L^{q_1}(\mathbb{P})} \sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\left\{\lambda^{-H/2}(\hat{X}_t^\lambda - \xi_t)\right\}^2 - f'(B_t^H)^2 Z_D(\delta_t)^2\right\|_{L^{q_2}(\mathbb{P})},\end{aligned}$$

for $1 < q_1, q_2 < \infty$ such that $1/q_1 + 1/q_2 = 1$. The convergence towards 0 follows immediatly by Proposition 2.2.12 and Lemma 2.2.7.

For the rate, we show

$$\sup_{t\in[0,T-\lambda^a]}\left|\mathbb{E}\left[\kappa_t\left\{\lambda^{(1-H)/2}(\hat{u}_t^\lambda)\right\}^2\right] - \mathbb{E}\left[\nu_t f'(B_t^H)^2 Z_R(\delta_t)^2\right]\right| \to 0,$$

as $\lambda \to 0$. where for $\delta > 0$, the random variable $Z_R(\delta)$ has the same law as (2.23) in

Lemma 2.2.7. We recall that

$$\hat{u}_t^\lambda = \frac{c_t^\lambda}{\lambda\kappa_t}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right),$$

and therefore,

$$\kappa_t\left\{\lambda^{(1-H)/2}(\hat{u}_t^\lambda)\right\}^2 = \frac{1}{\kappa_t}\left(\frac{c_t^\lambda}{\sqrt{\lambda}}\right)^2\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2.$$

We then have,

$$\begin{aligned}
&\sup_{t\in[\lambda^a,T-\lambda^a]}\left|\mathbb{E}\left[\kappa_t\left\{\lambda^{(1-H)/2}(\hat{u}_t^\lambda)\right\}^2\right] - \mathbb{E}\left[\nu_t f'(B_t^H)^2 Z_R(\delta_t)^2\right]\right| \\
&\leq \sup_{t\in[\lambda^a,T-\lambda^a]}\mathbb{E}\left[\left|\left\{\frac{1}{\kappa_t}\left(\frac{c_t^\lambda}{\sqrt{\lambda}}\right)^2 - \nu_t\right\}\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2\right|\right] \qquad (2.43) \\
&\quad + \sup_{t\in[\lambda^a,T-\lambda^a]}\mathbb{E}\left[\nu_t\left|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2 - f'(B_t^H)^2 Z_R(\delta_t)^2\right|\right]. \qquad (2.44)
\end{aligned}$$

We start with the term (2.44). For this we have

$$\begin{aligned}
&\sup_{t\in[\lambda^a,T-\lambda^a]}\mathbb{E}\left[\nu_t\left|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2 - f'(B_t^H)^2 Z_R(\delta_t)^2\right|\right] \\
&\leq \left\|\sup_{t\in[0,T]}\nu_t\right\|_{L^{q_1}(\mathbb{P})}\sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\left|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2 - f'(B_t^H)^2 Z_R(\delta_t)^2\right|\right\|_{L^{q_2}(\mathbb{P})}.
\end{aligned}$$

By Proposition 2.2.14 we obtain the convergence towards 0. For the term (2.43), we have the upper bound

$$\left\|\sup_{t\in[0,T]}\frac{1}{\kappa_t}\right\|_{L^{q_3}(\mathbb{P})}\sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\frac{c_t^\lambda}{\sqrt{\lambda}} - \nu_t\kappa_t\right\|_{L^{q_4}(\mathbb{P})}\sup_{t\in[\lambda^a,T-\lambda^a]}\left\|\left\{\lambda^{-H/2}\left(\hat{\xi}_t^\lambda - \hat{X}_t^\lambda\right)\right\}^2\right\|_{L^{q_5}(\mathbb{P})},$$

with $1 < q_3, q_4, q_5 < \infty$ with $1 = 1/q_3 + 1/q_4 + 1/q_5$. The convergence towards 0 then follows from Assumptions 2.1.2, Proposition 2.2.1, Proposition 2.2.14

Combining the results obtained for the deviation and the rate, we can write

$$J_{\lambda^a,T-\lambda^a}(\hat{u}^\lambda) = \lambda^H\int_{\lambda^a}^{T-\lambda^a}\mathbb{E}\left[\nu_t f'(B_t^H)^2\left\{Z_D(\delta_t)^2 + Z_R(\delta_t)^2\right\}\right]dt + o(\lambda^H).$$

From Lemma 2.2.7, we have

$$\begin{aligned}
\mathbb{E}\left[Z_D(\delta_t)^2\,\middle|\,\mathcal{F}_t\right] &= \delta_t^2(1-H)\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\}, \\
\mathbb{E}\left[Z_R(\delta_t)^2\,\middle|\,\mathcal{F}_t\right] &= \delta_t^2 H\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\},
\end{aligned}$$

and it follows

$$\mathbb{E}\left[\nu_t f'(B_t^H)^2 \left\{Z_D(\delta_t)^2 + Z_R(\delta_t)^2\right\}\right] = \mathbb{E}\left[I(f'(B_t), \nu_t, \kappa_t)\right].$$

Finally, we add the effects from the first and last intervals to obtain, as $\lambda \to 0$,

$$\begin{aligned} J_T(\hat{u}^\lambda) &= J_{\lambda^a, T-\lambda^a}(\hat{u}^\lambda) + o(\lambda^H) \\ &= \lambda^H \int_0^T \mathbb{E}\left[I(f'(B_t), \nu_t, \kappa_t)\right] dt + o(\lambda^H). \end{aligned}$$

For an asymptotically optimal strategy, we realise that deriving the corresponding expression comes as a biproduct from our proof. Indeed, if we plug the expressions for $\hat{\xi}^a, \hat{X}^a, \hat{u}^a$ from the statement of Theorem 2.1.3 into the objective value (2.1) our aim is to show that $\lambda^{-H} J_T(\hat{u}^a)$ attains the leading order coefficient we derived. In particular rescaling the processes as they appear in $J_T(\hat{u}^a)$ will result in similar expression encountered before[1]. This becomes particularly clear when looking at the proofs Lemma 2.2.10, 2.2.11 and Proposition 2.2.12. □

2.3 Extension to other rough targets

We further extend the result of Theorem 2.1.3 to other types of rough target we consider in our introductory framework.

Fractional Ornstein-Uhlenbeck

Consider a fractional Ornstein-Uhlenbeck proces (fOU) given by, for $t > 0$,

$$Y_t^H = e^{-\alpha_Y t}\left(y + \nu \int_0^t e^{\alpha_Y u} dB_u^H\right),$$

for $\alpha_Y, \nu > 0$, $y \in \mathbb{R}$, and solving the equation

$$Y_t^H = y - \alpha_Y \int_0^t Y_u^H du + \nu B_t^H.$$

In particular, for $t > s$, we obtain that the increment of fOU is

$$Y_t^H - Y_s^H = -\alpha_Y \int_s^t Y_u^H du + \nu(B_t^H - B_s^H). \tag{2.45}$$

This suggest that once we perform the rescaling, the dt-integral will be of higher order and only the rescaled fBM, which is again an fBM, will remain.

In the proof of Theorem 2.1.3, the key part that requires additional results for other types of rough target is the one involving Taylor formula in Lemma 2.2.11. Indeed, we need to rescale increments like (2.45) and show that the rescaled dt-integral vanishes. The corresponding term in Lemma 2.2.11 is given in (2.35). In order to show that this term

[1]With in fact less remainder terms to control.

vanishes, we need to verify (2.33) in the proof of Lemma 2.2.10, which is a consequence of the next Lemma.

Lemma 2.3.1. *Provided* (2.6) *holds, we have, for all* $p \geq 1$,

$$\sup_{t\in[\lambda^a, T-\lambda^a]} \sup_{s\in[t-\lambda^a, t+\lambda^a]} \left\| \lambda^{-H/2}(Y_t^H - Y_s^H) - \lambda^{-H/2}\nu(B_t^H - B_s^H) \right\|_{L^p(\mathbb{P})} \to 0,$$

as $\lambda \to 0$.

Proof. We have for $s > t$,

$$\lambda^{-H/2}(Y_t^H - Y_s^H) - \lambda^{-H/2}\nu(B_t^H - B_s^H) = -\lambda^{-H/2}\alpha_Y \int_s^t Y_u^H du \tag{2.46}$$

and we have the following estimate (which also holds for $s < t$),

$$\begin{aligned}\sigma_\lambda^2 := \mathbb{E}\left[\left|\lambda^{-H/2}\alpha_Y \int_s^t Y_u du\right|^2\right] &\leq \lambda^{-H}\alpha_Y^2 \max_{u\in[0,T]} \mathbb{E}\left[\left|Y_u^H\right|^2\right](s-t)^2 \\ &\leq \lambda^{2a-H}\alpha_Y^2 \max_{u\in[0,T]} \mathbb{E}\left[\left|Y_u^H\right|^2\right] \to 0,\end{aligned}$$

as $\lambda \to 0$, if we choose $a > H > H/2$, which is the case for the proof of Theorem 2.1.3. Therefore, since Y^H is a Gaussian process, the random variable on the right-hand side of (2.46) is Gaussian. We obtain the result since $\sigma_\lambda \to 0$ as $\lambda \to 0$. □

Hence, denoting,

$$A_t - A_s = (Y_t^H - Y_s^H) - \nu(B_t^H - B_s^H),$$

we also clearly have for $U_\lambda = (-\lambda^{a-1/2}, 0)$, $V_\lambda = (0, \lambda^{a-1/2})$,

$$\sup_{t\in[\lambda^a, T-\lambda^a]} \left\| \left|A_{t+\sqrt{\lambda}(v+w)} - A_t\right| \mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w) \right\|_{L^p(\mathbb{P}\times\mu_-\times\mu_+)} \to 0.$$

From the proof of Lemma 2.3.1, we also deduce that, for $t > s$,

$$\mathbb{E}\left[(Y_t^H - Y_s^H)^2\right] \leq C_T(t-s)^2 + C(t-s)^{2H}, \tag{2.47}$$

for some constants $C_T, C > 0$. This last equation is necessary to adapt the proof for the higher order of the first and last intervals.

Therefore, under Assumptions 2.1.2, we can adapt the proof of Theorem 2.1.3 to obtain the following variation of (2.5).

$$\begin{aligned}J_T(\hat{u}^\lambda) &= \mathbb{E}\left[\int_0^T \nu_t(\hat{X}_t^\lambda - \xi_t)^2 dt + \lambda \int_0^T \kappa_t(\hat{u}_t^\lambda)^2 dt\right] \\ &= \lambda^H \int_0^T \mathbb{E}\left[I(\nu f'(Y_t^H), \nu_t, \kappa_t)\right] dt + o(\lambda^H),\end{aligned} \tag{2.48}$$

Equation (2.48) tells us that whenever the target is of the form $\xi_t = f(Y_t^H)$, we can essentially replace Y_t^H by νB_t^H in the ergodic control formula since the dt-part of increments of fOU is of higher order.

Riemann-Liouville fractional Brownian Motion

In a similar manner as for fOU, we extend the result of Theorem 2.1.3 to functions of Riemann-Liouville fBM W^H. We recall that we have the following representation,

$$B_t^H = \frac{1}{a_H}\left(I_t + W_t^H\right),$$

with

$$\begin{aligned} I_t &:= \frac{1}{\Gamma(H+1/2)}\int_{-\infty}^{0}\left\{(t-u)^{H-1/2} - (-u)^{H-1/2}\right\} dW_u, \\ W_t^H &:= \frac{1}{\Gamma(H+1/2)}\int_0^t (t-u)^{H-1/2} dW_u, \end{aligned}$$

where $a_H = c_H/\Gamma(H+1/2)$ and

$$c_H = \sqrt{\frac{1}{2H} + \int_0^\infty \left\{(1+u)^{H-1/2} - u^{H-1/2}\right\}^2 du}.$$

Hence, we have for $t > s$,

$$W_t^H - W_s^H = a_H(B_t^H - B_s^H) + a_H(I_t - I_s),$$

and as for fOU, we focus our attention on the increment $(I_t - I_s)$.

Lemma 2.3.2. *Given I_t defined above, we have the following result for $t > s > 0$,*

$$\mathbb{E}\left[(I_t - I_s)^2\right] \le C'_H(t-s)(s^{2H-1} - t^{2H-1}),, \tag{2.49}$$

$$\mathbb{E}\left[(I_t - I_s)^2\right] \le C_H(t-s)^{2H} \tag{2.50}$$

where C_H, C'_H are positive constants that depend on $H < 1/2$.

Proof. We first notice that the process I_t is differentiable at any $t > 0$, and its derivative is given by

$$i_t = \frac{(H-1/2)}{\Gamma(H+1/2)}\int_{-\infty}^{0}(t-s)^{H-3/2} dW_s,$$

and it satisfies

$$\mathbb{E}\left[(i_t)^2\right] = \left(\frac{H-1/2}{\Gamma(H+1/2)}\right)^2 \int_0^\infty (t+s)^{2H-3} ds = D_H t^{2H-2},$$

with $D_H > 0$. By Jensen's inequality, it follows that,

$$\mathbb{E}\left[(I_t - I_s)^2\right] = \mathbb{E}\left[\left(\int_s^t i_u du\right)^2\right] \leq (t-s)\int_s^t E[(i_u)^2]du$$
$$\leq (t-s)C'_H(s^{2H-1} - t^{2H-1}),$$

with $C'_H > 0$.

For the second estimate, we write

$$\begin{aligned}\mathbb{E}\left[(I_t - I_s)^2\right] &= \frac{1}{\Gamma(H+1/2)^2}\int_0^\infty \left\{(t+v)^{H-1/2} - (s+v)^{H-1/2}\right\}^2 dv \\ &= \frac{1}{\Gamma(H+1/2)^2}(t-s)^{2H}\int_s^\infty \left\{(1+u)^{H-1/2} - (u)^{H-1/2}\right\}^2 \\ &\leq (t-s)^{2H}\frac{1}{\Gamma(H+1/2)^2}\int_0^\infty \left\{(1+u)^{H-1/2} - (u)^{H-1/2}\right\}^2 = C_H(t-s)^{2H},\end{aligned}$$

with $C_H > 0$ and where we changed the variable $u = (s+v)/(t-s)$. □

In particular, we see from the previous lemma that the term $(I_t - I_s)$ scales like fBM. This ensures that in the proof of Theorem 2.1.3, the first and last interval are of higher order. Again, to adapt the proof of Theorem 2.1.3, we need to verify the key property (2.33) found in Lemma 2.2.10.

Lemma 2.3.3. *Provided* (2.6) *holds, we have, for* $U_\lambda = (-\lambda^{a-1/2}, 0)$*,* $V_\lambda = (0, \lambda^{a-1/2})$ *and for any* $q > 1$*,*

$$\sup_{t\in[2\lambda^a, T-2\lambda^a]} \left\| \lambda^{-H/2}(I_{t+\sqrt{\lambda}(v+w)} - I_t)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\right\|_{L^q(\mathbb{P}\times\mu_-\times\mu_+)} \to 0.$$

Proof. By Lemma 2.3.2, we have for any $t \in [2\lambda^a, T-2\lambda^a]$, for some $C > 0$

$$\begin{aligned}&\left\|\lambda^{-H/2}(I_{t+\sqrt{\lambda}(v+w)} - I_t)\mathbb{1}_{V_\lambda}(v)1_{U_\lambda}(w)\right\|_{L^q(\mathbb{P})} \\ &\leq C\lambda^{-H/2}\left|\sqrt{\lambda}(w+v)\right|^{1/2}\left\{(t+\sqrt{\lambda}(w+v))^{2H-1} - t^{2H-1}\right\}^{1/2}\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w) \\ &\leq C\lambda^{-H/2}\left|\sqrt{\lambda}(w+v)\right|^{1/2}(\lambda^a)^{(2H-1)/2}\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left\|\lambda^{-H/2}(I_{t+\sqrt{\lambda}(v+w)} - I_t)\mathbb{1}_{V_\lambda}(v)\mathbb{1}_{U_\lambda}(w)\right\|_{L^q(\mathbb{P}\times\mu_-\times\mu_+)} \\ &= \left\|\left\|\lambda^{-H/2}(I_{t+\sqrt{\lambda}(v+w)} - I_t)\mathbb{1}_{V_\lambda}(v)1_{U_\lambda}(w)\right\|_{L^q(\mathbb{P})}\right\|_{L^q(\mu_-\times\mu_+)} \\ &\leq C\lambda^{1/4-H/2+a(2H-1)/2}\left\{\int_0^\infty\int_{-\infty}^0 |w+v|^q(1+|w|)^{-\alpha}(1+|v|)^{-\alpha}dwdv\right\}^{1/q},\end{aligned}$$

which converges to 0 if we have α large enough and if, $1/2 - H - (1-2H)a > 0$, which is the same as requiring $a < 1/2$. Hence, we have the result. □

Similarly to the fOU case, under Assumptions 2.1.2, we can adapt the proof of Theorem 2.1.3 to obtain the following variation of (2.5).

$$\begin{aligned} J_T(\hat{u}^\lambda) &= \mathbb{E}\left[\int_0^T \nu_t(\hat{X}_t^\lambda - \xi_t)^2 dt + \lambda \int_0^T \kappa_t(\hat{u}_t^\lambda)^2 dt\right] \\ &= \lambda^H \int_0^T \mathbb{E}\left[I(a_H f'(W_t^H), \nu_t, \kappa_t)\right] dt + o(\lambda^H). \end{aligned} \tag{2.51}$$

In Chapter 1, we derived the formula for the ergodic control problem of RLfBM. If we denote this formula $I_W(f'(W_t^H), \nu_t, \kappa_t)$, then we obtain

$$I_W(f'(W_t^H), \nu_t, \kappa_t) = I(a_H f'(W_t^H), \nu_t, \kappa_t),$$

as proved in Corollary 1.2.10.

Chapter 3

Rough and Classical Stochastic Volatility Model Comparison

This chapter is based on joint work with Dr. Christoph Czichowsky and Prof. Johannes Muhle-Karbe[1].

We now compare the quantitative impact of small transaction costs in rough and classical stochastic volatility models. To make the models comparable, the respective parameters need to be chosen appropriately; we address this by matching the first and second moments of volatility so that, in particular, the expected volatility and frictionless positions coincide in both models.

Throughout, the time horizon is

$$T = 5 \text{ years.}$$

For the risk premium in the dynamics of the risky asset,

$$dS_t = \mu dt + \sigma_t dW_t, \tag{3.1}$$

we use the estimate

$$\mu = \$131.7 \text{ per year.}$$

This is obtained from the S&P500 time series from December 31, 2013 (when the price was \$1848.36) until December 31, 2018 (when the price was \$2506.85). (The average S&P500 price $\bar{S}$ during this period was 2255.61\$, so that the average risk premium was about 5.84%.) For the absolute risk-aversion parameter, we follow Gârleanu and Pedersen

[1]We would like to thank Dr. Mikko Pakkanen, Dr. Eduardo Abi Jaber and Aitor Muguruza for useful discussions and shared resources.

(2013) and set

$$\gamma = 10^{-9}.$$

3.1 Volatility models

As a prototype for a *rough volatility model*, we use the model of Gatheral *et al.* (2018) where the volatility process is the exponential of a fractional Ornstein-Uhlenbeck, that is an Ornstein-Uhlenbeck process driven by fractional Brownian motion with Hurst index $H \in (0, 1/2)$:

$$\sigma_t^H = \eta_H e^{\nu_H Y_t^H}, \quad \eta_H, \nu_H > 0,$$

where

$$dY_t^H = -\kappa_H Y_t^H dt + dW_t^H, \quad Y_0^H = 0, \quad \kappa_H > 0.$$

Here, we recall that $(W_t^H)_{t\in[0,T]}$ is a Riemann-Liouville fBM[1] (RLfBM) driven by the Brownian motion $(W_t)_{t\in[0,T]}$ generating the underlying filtration:

$$W_t^H = \frac{1}{\Gamma(H + 1/2)} \int_0^t (t - s)^{H-1/2} dW_s.$$

We will compare this model to a *classical volatility model*, where the Ornstein-Uhlenbeck process is instead driven by a standard Brownian motion $(W_t)_{t\in[0,T]}$ with Hurst index $H = 1/2$:

$$\sigma_t = \eta e^{Y_t}, \quad \eta > 0,$$

where

$$dY_t = -\kappa Y_t dt + dW_t, \quad Y_0 = 0, \quad \kappa > 0.$$

Rough volatility model parameters

The following estimates are obtained in Gatheral *et al.* (2018) from a time series of S&P 500 prices:[2]

$$H = 0.14, \quad \nu_H = 0.3 \times 0.74 = 0.22, \quad \eta_H = 2255.61 \times e^{-5} = 15.2, \quad \kappa_H = 5 \times 10^{-4}.$$

For time horizons that are not too long, Gatheral *et al.* (2018) observe that the mean-reversion speed κ_H in the rough model can be set to zero without materially affecting the properties of the model. The intuition for this is that fBM already has negative autocorrelation for $H < 1/2$, and thereby displays similar features as a mean-reverting process.

[1] In Gatheral *et al.* (2018) a fBM rather than a RLfBM is used.

[2] More precisely, Gatheral *et al.* (2018) work with the geometric version of the model, where σ_t^H is the volatility of the log price. In order to obtain broadly consistent values in our arithmetic model, we multiply their log volatility with the average price $2255.61 of the S&P 500 index over the sample period from December 31, 2013 until December 31, 2018. Moreover, since Gatheral *et al.* (2018) use fBM instead of RLfBM; to match the variances of the two processes, we multiply their value of ν by $\sqrt{2H}\Gamma(H+1/2) = 0.74$.

To simplify the moment matching below, we follow this approach and henceforth focus on

$$\sigma_t^H = \eta_H e^{\nu_H W_t^H}. \tag{3.2}$$

Classical volatility model parameters

To identify the two parameters η, κ in the classical volatility model, we need two conditions. To this end, we require that the time-averages of the volatility and the inverse of its square coincide in both models:

$$\frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t^H\right] dt = \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t\right] dt, \tag{3.3}$$

$$\frac{1}{T}\int_0^T \mathbb{E}\left[(\sigma_t^H)^{-2}\right] dt = \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t^{-2}\right] dt. \tag{3.4}$$

Matching the moments of the inverse squared volatility ensures that the time-averages of the frictionless positions $\mu/(\gamma\sigma_t^2), \mu/(\gamma(\sigma_t^H)^2)$ and the corresponding frictionless performances are the same in both models. Moreover, we match the time-averaged level of volatility for both models.

For the rough model, recall that the RLfBM W_t^H is Gaussian with mean 0 and variance $v_H t^{2H}$, where $v_H = (2H\Gamma(H+1/2)^2)^{-1}$. As a result, the moment-generating function of normal distribution allows us to compute the time-average moments of the volatility process $\sigma_t^H = \eta_H \exp(\nu_H W_t^H)$:

$$M_1 = \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t^H\right] dt = \frac{\eta_H}{T}\int_0^T e^{\frac{1}{2}\nu_H^2 v_H t^{2H}} dt,$$
$$M_{-2} = \frac{1}{T}\int_0^T \mathbb{E}\left[(\sigma_t^H)^{-2}\right] dt = \frac{\eta_H^{-2}}{T}\int_0^T \left(e^{2\nu_H^2 v_H t^{2H}}\right) dt.$$

For the classical model, observe that $Y_t \sim \mathcal{N}(0, \frac{1}{2\kappa}(1-e^{-2\kappa t}))$. As a result the corresponding time average moments of the volatility process $\sigma_t = \eta \exp(Y_t)$ are given by

$$M_1(\nu,\kappa) = \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t\right] dt = \frac{\eta}{T}\int_0^T e^{\frac{1}{4\kappa}(1-e^{-2\kappa t})} dt =: \eta I_1(\kappa),$$
$$M_{-2}(\nu,\kappa) = \frac{1}{T}\int_0^T \mathbb{E}\left[\sigma_t^{-2}\right] dt = \frac{\eta^{-2}}{T}\int_0^T e^{\frac{1}{\kappa}(1-e^{-2\kappa t})} dt =: \eta^{-2} I_{-2}(\kappa),$$

Hence, to achieve the moment matchings (3.3) and (3.4), η and κ need to satisfy

$$M_1 = \eta I_1(\kappa),$$
$$M_{-2} = \eta^{-2} I_{-2}(\kappa).$$

From the first equation, we have

$$\eta = I_1(\kappa)^{-1} M_1. \tag{3.5}$$

Inserting this into the second equation gives

$$M_{-2} = I_1(\kappa)^2 M_1^{-2} I_{-2}(\kappa),$$

or, equivalently,

$$I_1(\kappa)^2 I_{-2}(\kappa) = M_1^2 M_{-2}.$$

Since the right-hand side is given in terms of the parameters H, η_H, ν_H of the rough volatility model, it remains to solve this scalar equation numerically for κ, and in turn compute the corresponding value of η using (3.5). For the rough volatility parameters from Gatheral *et al.* (2018), this yields

$$\kappa = 4.40, \quad \eta = 0.0067.$$

Figure 3.1 shows simulated paths for both models with the parameter matching we have done. To facilitate comparison, the Brownian motion used to simulate the Ornstein-Uhlenbeck is also used to simulate the RLfBM. We see that the processes display broadly similar behavior, but the fractional volatility process fluctuates much more rapidly.

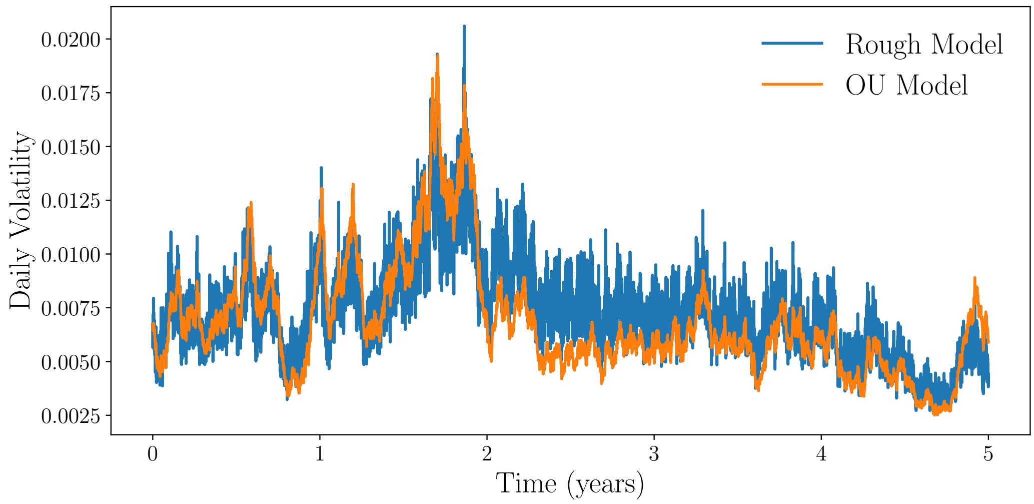

Figure 3.1: Simulated paths of daily volatility for the rough and the classical volatility models with the matched model parameters.

3.2 Asymptotic expansion of utility losses

For the rough and classical volatility models with parameters matched as above, we now compare the effect of small transaction costs. Recall that in the classical volatility model, the agent maximises

$$\mathbb{E}\left[\int_0^T \left(\mu\varphi_t - \frac{\gamma\sigma_t^2}{2}\varphi_t^2 - \lambda\dot{\varphi}_t^2\right) dt\right]; \tag{3.6}$$

in the rough version of the model, σ_t is replaced by σ_t^H. As seen in the Introduction, maximising this goal functional is equivalent to solving the following linear-quadratic tracking problem for the frictionless optimal strategy $\hat{\varphi}_t = \mu/(\gamma\sigma_t^2)$ (resp., $\hat{\varphi}_t = \mu/(\gamma(\sigma_t^H)^2)$ for the rough version of the model):

$$\min_{\dot{\varphi}} \mathbb{E}\left[\int_0^T \left\{\bar{\gamma}_t(\varphi_t - \hat{\varphi}_t)^2 + \lambda\dot{\varphi}_t^2\right\} dt\right], \quad \text{where } \bar{\gamma}_t = \tfrac{\gamma\sigma_t^2}{2}. \tag{3.7}$$

For small transaction costs λ , we have studied in the previous Chapter 2 the leading-order expansion of the minimised objective (3.7). In the case where the target is a rough process, namely a function of fractional Brownian motion $\hat{\varphi}_t = f(B_t^H)$ with $f \in \mathcal{C}^2(\mathbb{R})$, we found in Theorem 2.1.3 that

$$\min_{\dot{\varphi}} \mathbb{E}\left[\int_0^T \left\{\bar{\gamma}_t(\varphi_t - \hat{\varphi}_t)^2 + \lambda\dot{\varphi}_t^2\right\} dt\right] = \lambda^H \int_0^T \mathbb{E}\left[I(f'(B_t), \bar{\gamma}_t, 1)\right] dt + o(\lambda^H), \tag{3.8}$$

where for $\alpha, q, r > 0$,

$$\begin{aligned} I(\alpha, q, r) &= \inf_{u\in\mathcal{A}} \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T q(X_t - \alpha B_t^H)^2 dt + r(u_t)^2 dt\right] \\ &=: \inf_{u\in\mathcal{A}} J(u, \alpha) \\ &= \alpha^2 \frac{q}{\delta^{2H}}\Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\}, \quad \delta = \sqrt{\frac{q}{r}}, \end{aligned} \tag{3.9}$$

corresponds to the minimised objective of the linear-quadratic ergodic control of fBM[1]. An asymptotically optimal strategy $\hat{u}_t^a$ which attains the leading order in (3.8) is given by

$$\begin{aligned} \hat{\xi}_t^a &= f(B_t^H) + f'(B_t^H)\mathbb{E}\left[\int_t^T \sqrt{\frac{\bar{\gamma}_t}{\lambda}} e^{-\sqrt{\frac{\bar{\gamma}_t}{\lambda}}(u-t)} \left(B_u^H - B_t^H\right) du \,\middle|\, \mathcal{F}_t\right] \\ \hat{u}_t^a &= \sqrt{\frac{\bar{\gamma}_t}{\lambda}} \left(\hat{\xi}_t^a - \hat{X}_t^a\right), \\ \hat{X}_t^a &= x + \int_0^t \hat{u}_s^a ds, \end{aligned} \tag{3.10}$$

and we call $\hat{\xi}_t^a$ the asymptotically optimal signal process.

If we do not take terminal liquidation (which has order of $O(\lambda^{1/2})$) into account, the same results holds for the semimartingale stochastic volatility case so we can formally set $H = 1/2$ in the above equations[2]. In that case, the signal process is given by the target $\hat{\varphi}_t$ itself by the martingale property of Brownian motion. Accordingly, the asymptotically optimal rate always trades towards the current frictionless target and the corresponding controlled deviation between the target $\hat{\varphi}_t$ and the controlled process $\hat{X}^a$, as well as the optimal rate itself, have Ornstein-Uhlenbeck dynamics.

In the rough volatility case, a first naive guess would be to similarly consider a re-

[1] We also derived in Chapter 2 the corresponding formula for RLfBM but the formula for fBM is easier to interpret.

[2] Compare also with the formula for the lower bounds obtained in Cai *et al.* (2017a).

version towards the target. By looking at the underlying ergodic problem, this would correspond to reverting towards the fBM rather than some signal process. Therefore, the controlled deviation and optimal rate of the underlying ergodic control problem would be of fractional Ornstein-Uhlenbeck (fOU) type with unit mean-reversion speed and unit volatility parameter[1]. In particular, by Remark 2.4 in Cheridito *et al.* (2003), we know that the stationary variance of such a process is given by $\Gamma(2H+1)/2$. Since both the optimal rate and the controlled deviation are of fOU type, we obtain an ergodic objective value of $\Gamma(2H+1)$. Let us now compare this to the formula (3.9) for the leading-order optimum:

$$g(H) = \Gamma(2H+1)\left\{\frac{1+\sin(\pi H)}{2}\right\}.$$

The graphs of $\Gamma(2H+1)$ and $g(H)$ are shown in Figure 3.2. As $\{1+\sin(\pi H)\}/2 \leq 1$, for $H \neq 1/2$ the asymptotically optimal deviation leads to a better performance than its (fractional) Ornstein-Uhlenbeck counterpart, by taking into account future values of the frictionless target in the signal process $\hat{\xi}^a_t$. Thereby, the optimal trading rate exploits the autocorrelation of the target position, similarly as for mean-reverting target strategies with non-asymptotically small transaction costs, compare Gârleanu and Pedersen (2013).

Finally, the leading-order term in the expansion (3.8) for the rough volatility model is of order $O(\lambda^H)$, whereas it is of order $O(\lambda^{1/2})$ for Itô process targets. Whence, for sufficiently small cost parameters, the performance losses are always bigger in the rough model, due to the faster oscillations of the frictionless target strategy.

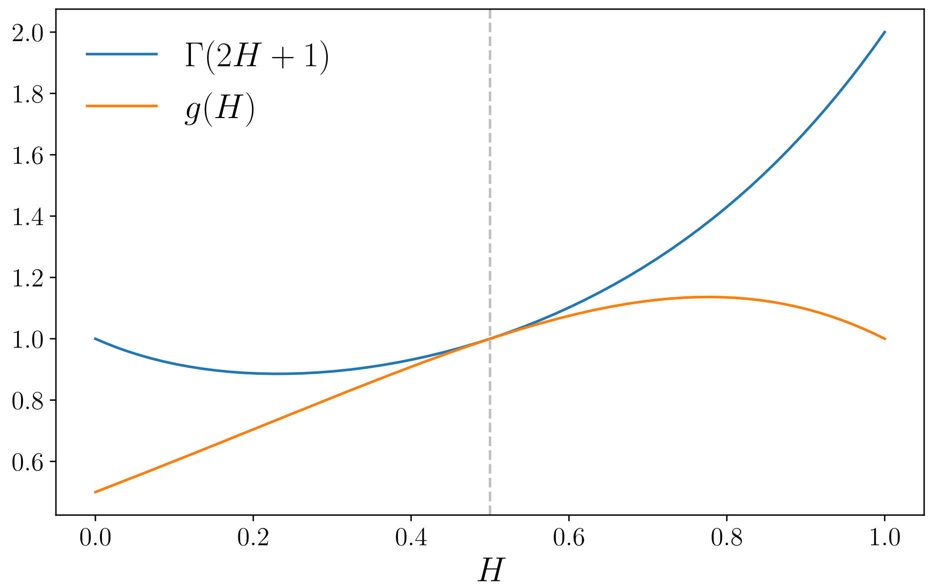

Figure 3.2: Plot of the function $g(H)$ and twice the stationary variance of a fractional Ornstein-Uhlenbeck process $\Gamma(2H+1)$.

[1]We assume that our discussion takes place in the setting $\alpha, q, r = 1$ in (3.9)

For the rough volatility model parameter chosen in our calibration, a typical path of the asymptotically optimal signal process $\hat{\xi}^a$ and position $\hat{X}^a$ from (3.10) is depicted in Figure 3.3. Clearly, the signal smooths out the fluctuations of the target and thereby prevents excessive portfolio adjustments. We provide more details on the algorithm used for our simulations in section 3.2.2.

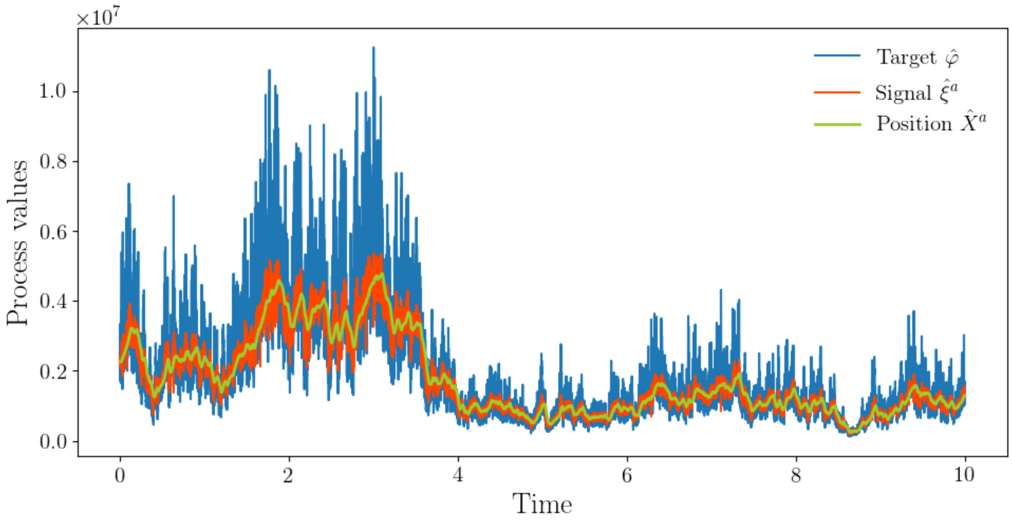

Figure 3.3: Target process $\hat{\varphi}$ with asymptotically optimal signal and controlled process $\hat{\xi}^a_t$ and $\hat{X}^a$ for cost parameter value of $\lambda = 3 \cdot 10^{-8}$.

Leading-order utility loss for the rough volatility model

We now compare the leading-order losses (3.8) due to small transaction costs for the rough and classical volatility models with parameters matched as described in Section 3.1. For the rough volatility model from Section 3.1, the frictionless optimiser $\hat{\varphi}$ is of the form

$$\hat{\varphi}_t = f(W_t^H),$$

where

$$f(x) = \frac{\mu}{\gamma \eta_H^2} e^{-2\nu_H x}, \quad f'(x) = -2\nu_H f(x).$$

In view of (3.8), the leading-order utility loss due to small transaction costs is

$$J_H(\lambda) := \lambda^H \mathbb{E}\left[\int_0^T \left(\frac{\gamma \sigma_t^2}{2}\right)^{1-H} f'(W_t^H)^2 dt\right] \Gamma(2H+1)\left(\frac{1+\sin(\pi H)}{2}\right)(a_H)^2,$$

where a_H is the rescaling constant that allows us to pass from the ergodic linear-quadratic objective value for tracking fBM to the ergodic linear-quadratic objective value for tracking RLfBM as seen in the proof of Corollary 1.2.10 in Chapter 1. In particular, we have that,

$$(a_H)^2 \Gamma(2H+1)(1+\sin(\pi H))/2 = (1/2)\{1+\csc(\pi H)\} =: E_H,$$

which corresponds to the ergodic linear-quadratic objective value of tracking a RLfBM as derived directly in Chapter 1[1].

After inserting the value of f' in the leading-order coefficient, we obtain

$$\mathbb{E}\left[\int_0^T \left\{\frac{\gamma(\sigma_t^H)^2}{2}\right\}^{1-H} f'(W_t^H)^2 dt\right] = \mathbb{E}\left[\int_0^T \left\{\frac{\gamma(\sigma_t^H)^2}{2}\right\}^{1-H} (2\nu_H)^2 \left\{\frac{\mu}{\gamma(\sigma_t^H)^2}\right\}^2 dt\right]$$
$$= \left(\frac{\gamma}{2}\right)^{1-H} \left(\frac{2\mu\nu_H}{\gamma}\right)^2 \int_0^T \mathbb{E}\left[\left\{(\sigma_t^H)^2\right\}^{-H-1}\right] dt.$$

Moreover, recalling $\sigma_t^H = \eta_H \exp(\nu_H W_t^H)$ and $(1+H)2\nu_H W_t^H \sim \mathcal{N}(0, 4(1+H)^2\nu_H^2 v_H t^{2H})$, we have

$$\mathbb{E}\left[\left\{(\sigma_t^H)^2\right\}^{-H-1}\right] = \eta_H^{-2(1+H)}\mathbb{E}\left[e^{-(1+H)2\nu_H W_t^H}\right] = \eta_H^{-2(1+H)} e^{2(1+H)^2\nu_H^2 v_H t^{2H}}.$$

As a consequence,

$$\mathbb{E}\left[\int_0^T \left\{\frac{\gamma(\sigma_t^H)^2}{2}\right\}^{1-H} f'(W_t^H)^2 dt\right] = \left(\frac{\gamma}{2}\right)^{1-H} \left(\frac{2\mu\nu_H}{\gamma}\right)^2 \eta_H^{-2(1+H)} \int_0^T e^{2(1+H)^2\nu_H^2 v_H t^{2H}} dt.$$

In summary, we obtain

$$J_H(\lambda) = \lambda^H E_H \left(\frac{\gamma}{2}\right)^{1-H} \left(\frac{2\mu\nu_H}{\gamma}\right)^2 \eta_H^{-2(1+H)} \int_0^T e^{2(1+H)^2\nu_H^2 v_H t^{2H}} dt. \tag{3.11}$$

Note that we can also rewrite the last integral as

$$\int_0^T e^{2(1+H)^2\nu_H^2 v_H t^{2H}} dt$$
$$= \frac{T}{2H} \int_0^1 e^{2(1+H)^2\nu_H^2 v_H T^{2H} v} v^{\frac{1}{2H}-1} dv$$
$$= \frac{T}{2H} \left\{-2(1+H)^2\nu_H^2 v_H T^{2H}\right\}^{\frac{1}{2H}-2} \Gamma\left(\frac{1}{2H}, -2(1+H)^2\nu_H^2 v_H T^{2H}\right),$$

where $\Gamma(\alpha, x)$ is the lower incomplete Gamma function:

$$\Gamma(\alpha, x) := \int_0^x e^{-t} t^{\alpha-1} dt.$$

Leading-order utility loss for the classical volatility model

For the classical volatility model, the frictionless optimiser is of the form

$$\hat{\varphi}_t = \frac{\mu}{\gamma\sigma_t^2} = f(Y_t),$$

[1]With parameters $\alpha, q, r = 1$ in (3.9)

where

$$f(x) = \frac{\mu}{\gamma\eta^2} e^{-2x}, \quad f'(x) = -2f(x).$$

The corresponding leading-order utility loss is

$$J(\lambda) := \lambda^{1/2} \mathbb{E}\left[\int_0^T \left(\frac{\gamma\sigma_t^2}{2}\right)^{1/2} f'(Y_t)^2 dt\right].$$

After inserting the expression for $f'(Y_t)$, this can be simplified to

$$\begin{aligned} J(\lambda) &= \lambda^{1/2} \mathbb{E}\left[\int_0^T \left(\frac{\gamma\sigma_t^2}{2}\right)^{1/2} \left(\frac{-2\mu}{\gamma\sigma_t^2}\right)^2 dt\right] \\ &= \lambda^{1/2} \left(\frac{\gamma}{2}\right)^{1/2} \left(\frac{2\mu}{\gamma}\right)^2 \int_0^T \mathbb{E}\left[\sigma_t^{-3}\right] dt. \end{aligned}$$

Moreover, since $\sigma_t = \eta \exp(Y_t)$ and $Y_t \sim \mathcal{N}(0, \frac{1}{2\kappa}(1 - e^{2\kappa t}))$, we have

$$\mathbb{E}\left[\sigma_t^{-3}\right] = \eta^{-3} e^{\frac{9}{4\kappa}(1-e^{-2\kappa t})}.$$

In summary, the leading-order utility loss due to small transaction costs in the classical volatility model therefore can be written as

$$J_C(\lambda) = \lambda^{1/2} \left(\frac{\gamma}{2}\right)^{1/2} \left(\frac{2\mu}{\gamma}\right)^2 \eta^{-3} \int_0^T e^{\frac{9}{4\kappa}(1-e^{-2\kappa t})} dt. \tag{3.12}$$

Note that we can also rewrite the last integral as

$$\begin{aligned} \int_0^T e^{\frac{9}{4\kappa}(1-e^{-2\kappa t})} dt &= e^{\frac{9}{4\kappa}} \int_{e^{-2\kappa T}}^1 \frac{1}{2\kappa} \frac{1}{u} e^{-\frac{9}{4\kappa}u} du \\ &= e^{\frac{9}{4\kappa}} \times \frac{Ei(9/4\kappa) - Ei(e^{2\kappa T} 9/4\kappa)}{2\kappa}, \end{aligned}$$

where

$$Ei(x) = -\int_x^\infty \frac{e^{-t}}{t} dt$$

is the exponential integral.

Comparison of the leading order utility losses

We now have closed-form expressions in terms of special functions at hand for the leading-order utility losses in the rough and classical volatility models. This allows to compare the corresponding values for the matched parameters from Section 3.1 and various choices of the transaction cost λ in a straightforward manner:

$$J_H(\lambda) = 1.4280 \times 10^9 \times \lambda^{0.14},$$
$$J_C(\lambda) = 9.1106 \times 10^{11} \times \lambda^{0.5}.$$

These functions are plotted for various values of λ in Figure 3.4. The leading-order utility losses intersect for $\lambda_i = 1.6153 \times 10^{-8}$; the effect of smaller transaction costs is more pronounced in the rough volatility model. In contrast, larger transaction costs have a smaller effect in the rough model. Intuitively, larger transaction costs imply smaller trading activity in which case any form of predictability in the target becomes an advantage. Gârleanu and Pedersen (2013) and Cartea and Jaimungal (2016) estimate transaction costs of the order of 10^{-7} for commodities and individual stocks, respectively. Accordingly, for even more liquid indices like the S&P considered here, the effect of the transaction costs are likely of a similar order of magnitude in both models.

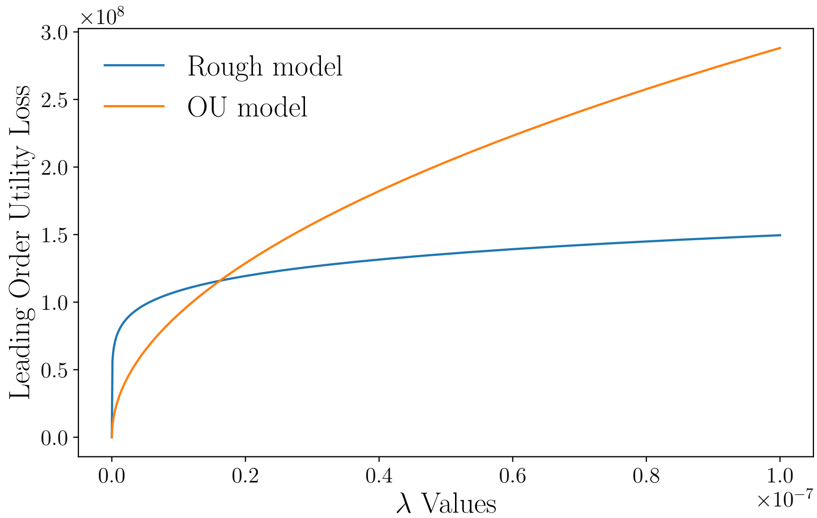

Figure 3.4: Leading-order utility losses $J_H(\lambda)$ (rough model) and $J_C(\lambda)$ (classical model).

A disadvantage of the above formulas is that they depend on the risk premium μ, which is notoriously difficult to estimate. To avoid this, one can consider the relative utility loss compared to the frictionless optimal utility J, which is the same in both models due to our moment matching:

$$J = 9.3002 \times 10^8.$$

We then compute the fractions of utility lost due to transaction costs in the rough and classical models. To wit, using (3.11), first ratio can be written as

$$\frac{J_H(\lambda)}{J} = \lambda^H E_H \gamma^{-H} \left(\frac{1}{2}\right)^{-H} (2\nu_H)^2 \eta^{-2H} \frac{\int_0^T e^{2(1+H)^2 \nu_H^2 v_H t^{2H}} dt}{\int_0^T e^{2\nu_H^2 v_H t^{2H}} dt}.$$

By (3.12), the second one is given by

$$\frac{J_C(\lambda)}{J} = \lambda^{1/2}\gamma^{-1/2}\left(\frac{1}{2}\right)^{-1/2} 2^2\eta^{-1}\frac{\int_0^T e^{\frac{9}{4\kappa}(1-e^{-2\kappa t})}dt}{\int_0^T e^{\frac{9}{4\kappa}(1-e^{-2\kappa t})}dt}.$$

These representations no longer depend on the risk premium μ, since it appears quadratically in all expressions and therefore cancels. Some values of the relative utility losses due to small transaction costs are shown in the following table.

λ	10^{-9}	$1.6153 \cdot 10^{-8}$	10^{-7}
Ratio $J_H(\lambda)/J$	8.4%	12.5%	16.1%
Ratio $J_C(\lambda)/J$	3.1%	12.5%	31%

With this table, we notice that differences in utility loss can be quite substantial depending on the size of the cost. Hence, for a cost parameter λ in the order of the estimated range of 10^{-7}, one would prefer to work with a rough volatility model.

3.2.1 Validity of asymptotic framework

A key question in any asymptotic analysis is whether the approximate formulas it provides provide good approximations to the original problem for realistic parameter values. In the present context, it is difficult to compare the optimal trading rates and their performances to the exact optimisers, because these are very difficult to compute.

However, we can test the accuracy of the formulas for the leading-order minimal utility losses by comparing them to the actual performance of the asymptotically optimal trading strategies in a simulation study. For the rough volatility model, a typical pathwise realisation of the frictionless target $\hat{\varphi}$ and the corresponding asymptotically optimal signal process $\hat{\xi}^a$ and position $\hat{X}^a$ have already been shown on Figure 3.3. We iterate this simulation $M = 500$ times for 6 values of λ between $\lambda = 9 \times 10^{-9}$ and $\lambda = 10^{-7}$. We then compute the realised utility loss on the left-hand side of (3.8) and compare them with the leading order term for the same utility loss on the right-hand side of (3.8). The utility loss estimates obtained through simulation are the points on Figure 3.5 and the asymptotic formula for the leading-order utility loss is represented by the blue curve.

The results depicted in Figure 3.5 clearly indicate that the asymptotical expansion provides an excellent approximation of the utility loss for realistic values of the transaction cost λ.

3.2.2 Simulation method

We have considered two ways to obtain a simulation of the signal process $\hat{\xi}^a$ of our rough volatility model, given by

$$\hat{\xi}^a_t = f(W^H_t) + f'(W^H_t)\int_t^T \sqrt{\frac{\bar{\gamma}_t}{\lambda}}e^{-\sqrt{\frac{\bar{\gamma}_t}{\lambda}}(u-t)}\mathbb{E}\left[\left(W^H_u - W^H_t\right)\middle|\mathcal{F}_t\right]du. \tag{3.13}$$

Obtaining simulations of the path of Riemann-Liouville fBM is already well documented, see for instance the rDonsker approach in Horvath *et al.* (2017) or the Hybrid scheme in Bennedsen *et al.* (2017). The main challenge in simulating (3.13) comes from the integral of weighted conditional expectations.

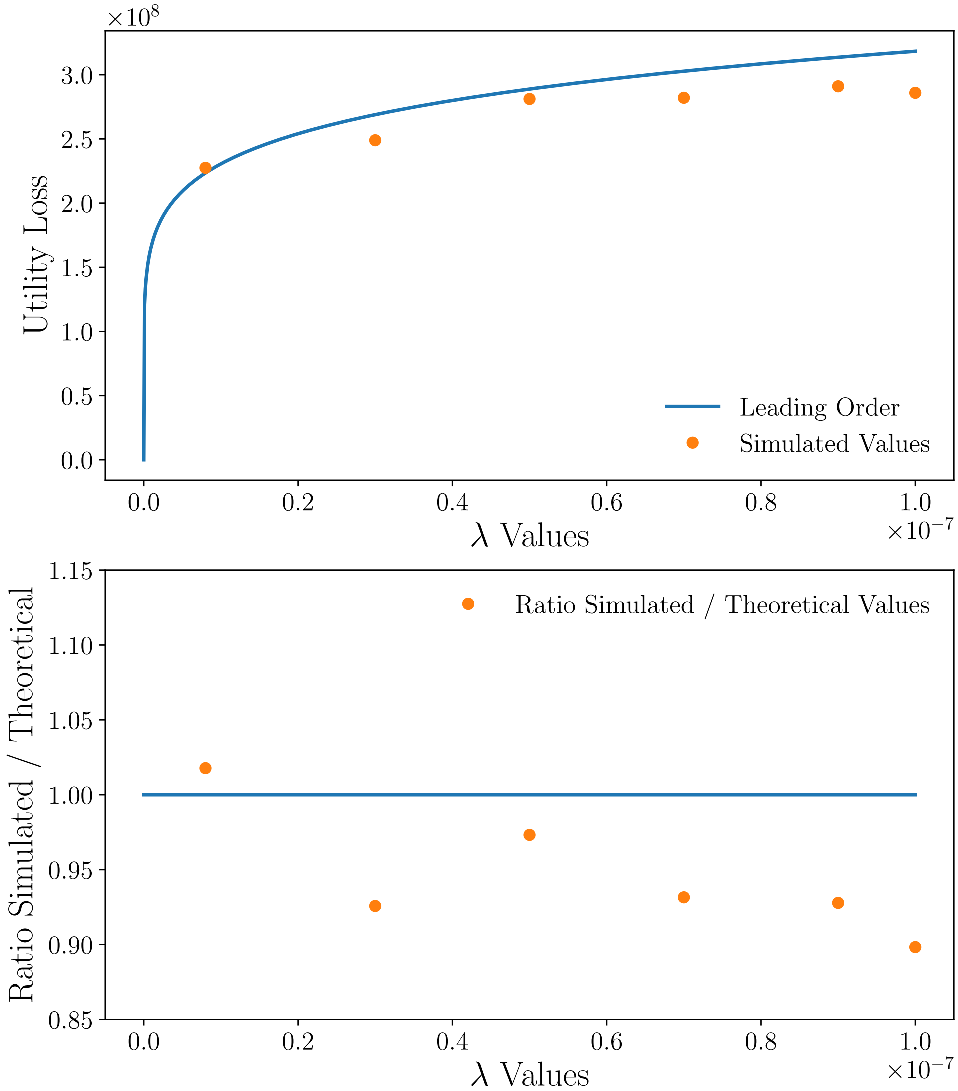

Figure 3.5: Top: Estimated utility losses (points) and the asymptotic leading-order utility loss graph $J_H(\lambda)$ (line) for different values of the cost parameter λ. Bottom: Ratio between estimated utility losses and the values of $J_H(\lambda)$.

More precisely, let us consider a partition $\{t_k = k\Delta t\}_{k=0,\ldots,N}$ of $[0,T]$ with step size Δt. A prediction formula for RLfBM as in Nuzman and Poor (2000) exists. However, in order to compute the integral expression in (3.13) this requires to compute at each time t_k, conditional expectations $\mathbb{E}[W_s^H \,|\, \mathcal{F}_{t_k}]$ for a number N_f^λ of $s > t_k$ values. This approach appears at a first glance computationally slow. In particular, our goal is to compute pathwise utility loss and perform averaging over a large number M of simulations for several values of λ.

Infinite dimensional Markovian approach

As an alternative, motivated by recent work in Abi Jaber and El Euch (2019) and Abi Jaber (2019), we can use the infinite dimensional Markovian representation of RLfBM given by

$$W_t^H = \int_0^\infty Y_t^\gamma \mu(d\gamma), \tag{3.14}$$

for

$$\mu(d\gamma) = \frac{1}{\Gamma(H+1/2)\Gamma(1/2-H)} \frac{1}{\gamma^{1/2+H}} d\gamma,$$

with Ornstein-Uhlenbeck processes Y_t^γ, $\gamma > 0$, satisfying the dynamics

$$\begin{aligned} dY_t^\gamma &= -\gamma Y_t^\gamma dt + dW_t \\ Y_0^\gamma &= 0 \end{aligned}$$

or equivalently

$$Y_t^\gamma = \int_0^t e^{-\gamma(t-s)} dW_s.$$

Denoting $\delta_t^\lambda = \sqrt{\bar{\gamma}_t/\lambda}$ and using (3.14) and Fubini's theorem allows to compute[1]

$$\int_t^\infty \delta_t^\lambda e^{-\delta_t^\lambda (u-t)} \int_0^\infty \mathbb{E}\left[Y_u^\gamma \mid \mathcal{F}_t\right] \mu(d\gamma) du = \int_0^\infty \frac{\delta_t^\lambda}{\delta_t^\lambda + \gamma} Y_t^\gamma \mu(d\gamma).$$

Hence, following the methodology exposed in Abi Jaber (2019), we can approximate the $\mu(d\gamma)-$integral by

$$\int_0^\infty \frac{\delta_t^\lambda}{\delta_t^\lambda + \gamma} Y_t^\gamma \mu(d\gamma) \approx \sum_{i=1}^n c_i^n \frac{\delta_t^\lambda}{\delta_t^\lambda + \gamma_i} Y_t^{\gamma_i^n},$$

where the weights c_i^n are computed in a similar way as in Abi Jaber (2019). Due to its Markovian nature, this last method is theoretically faster than the first approach, as we only need to store n values (typically, $n = 500$) of OU processes at any time t_k.

The main difficulty we encountered in implementing this approach comes from the fact that mimicking the precise roughness of RLfBM with a finite number of OU processes is not always ensured. The methodology described in Abi Jaber (2019) involves an additional tuning parameter r necessary in the discretisation of the measure μ. The choice of (n, r) is then responsible for the roughness of our simulations. In particular, as mentioned in Abi Jaber (2019) when the step size Δt becomes sufficiently small, statistical tests of roughness will return an estimated Hurst parameter of 0.5. Since our problem relies on ergodic properties of the path, it is crucial that the roughness of RLfBM is always matched adequately.

[1] Since λ is small, we do the approximation $T = \infty$ in (3.13).

An additional challenge we faced is that implementing stable schemes for $\hat{u}^a$[1] requires a small step size Δt due to the small values of λ. On the one hand, too little OU processes in relation to Δt lead to a semimartingale process rather than a rough process. On the other hand, a too large value of Δt leads to inaccurate approximations of the $dt-$integral. Unfortunately, we could not find a good pair (n, r) that balanced this tradeoff. In particular, taking a very large amount n (such as $n = 10^6$) of OU processes to balance our small values of Δt becomes too heavy in terms of memory allocation. Therefore, with the capped value of n we considered, the simulated path obtained were not able to mimick the roughness of RLfBM with $H = 0.14$.

rDonsker

Since simulations using the Markovian approach where not precise enough for our purpose, we considered the slower approach introduced at the beginning of this section. This approach requires to compute several, N_f, conditional expectations of RLfBM at each time t_k and weight them according to the exponential kernel in (3.13).

For this, we chose a simulation method that specifically aims at providing accurate roughness. This is the enhanced performance rDonsker scheme described in (Horvath *et al.*, 2017, Section 3.3.1). This approach is in essence a modified version of the left-point approximation of the integral

$$W_t^H = \frac{1}{\Gamma(H+1/2)} \int_0^t (t-s)^{H-1/2} dW_u,$$

such that

$$W_{t_i}^H \approx \sum_{k=1}^{i} (t_k^{*,i})^{H-1/2} \Delta W_k$$

for appropriately scaled independent increments of BM, ΔW_k and

$$t_k^{*,i} = \left(\frac{n}{2H} \left[(t_i - t_{k-1})^{2H} - (t_i - t_k)^{2H} \right] \right)^{1/(2H-1)}, k = 1, \dots, i,$$

is such that the first two moments of the approximation match those of RLfBM. This is implemented with a convolution. In particular, we modify the weights in the convolution to compute conditional expectations of future value of RLfBM since for $t > s$,

$$\mathbb{E}[W_t^H \,|\, \mathcal{F}_s] = \frac{1}{\Gamma(H+1/2)} \int_0^s (t-u)^{H-1/2} dW_u. \tag{3.15}$$

Alternatively, we can also use the prediction formula as in Nuzman and Poor (2000)[2],

$$\mathbb{E}[W_t^H \,|\, \mathcal{F}_s] \propto (t-s)^{H+1/2} \int_0^s \frac{1}{(s-u)^{H+1/2}} \frac{1}{(t-u)} W_u^H du$$

[1] We recall that $\hat{u}^a = \lambda^{-1/2} \bar{\gamma}_t^\lambda (\hat{\xi}_t^a - \hat{X}_t^a)$.

[2] See also the prediction formula for RLfBM in the unpublished work of Forde, Smith, Viitasaari (2019).

to obtain an approximation for (3.15). Implementing both approaches, we noticed some differences between the predictions obtained. For our simulations, we preferred to use the result obtained through our modified rDonsker approach as the prediction formula involves an exploding kernel that can reduce the precision of our predictions.

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