The London School of Economics and Political Science

Is Consumption Growth only a Sideshow in Asset Pricing? Asset Pricing Implications of Demographic Change and Shocks to Time Preferences

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A thesis submitted to the Department of Finance of the London School of Economics for the degree of Doctor of Philosophy, London, April 2012

Declaration

I certify that the thesis I have presented for examination for the Ph.D. degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I declare that my thesis consists of approximately 30'000 words.

Abstract

I show that risk sources such as unexpected demographic changes or shocks to the agent's subjective time preferences may have stronger implications and be of greater importance for asset pricing than risk in the (aggregate) consumption growth process.

In the first chapter, I discuss stochastic changes to time preferences. Shocks to the agent's subjective time discounting of future utility cause stochastic changes in asset prices and the agent's value function. Independent of the consumption growth process, shocks to time discounting imply a covariation between asset returns and the marginal utility process, and the equity premium is non-zero. My model can generate both a reasonably low level and volatility in the risk-free real interest rate and a high stock price volatility and equity premium. If time discounting follows a process with mean-reversion, then the interest rate process is mean-reverting and stock returns are (at long horizons) negatively auto-correlated.

In the second chapter, I analyze the asset pricing implications of birth and death rate shocks in an overlapping generations model. The interest rate and the equity premium are time varying and under certain conditions the interest rate is lower and the equity premium is higher during periods characterized by a high birth rate and low mortality than in times of a low birth rate and high mortality. Demographic changes may explain substantial parts of the time variation in the real interest rate and the equity premium. Demographic uncertainty implies a large unconditional variation in asset returns and leads to stochastic changes in the conditional volatility of stock returns.

In the last chapter, I illustrate how shocks to the death rate may affect expected asset returns in the cross-section. An agent demands more of an asset with higher (lower) payoff in states of the world when he expects to live longer (shorter) and marginal utility is high (low) than an asset with the opposite payoff schedule. In equilibrium, the first asset pays a lower expected return than the latter. Empirical evidence supports the model. Out-of-sample evidence suggests that a strategy, which loads on uncertainty in the death rate, pays a positive unexplained return according to traditional market models.

Acknowledgements

I am grateful to my supervisors Dimitri Vayanos, Stavros Panageas and Jack Favilukis. Dimitri was my main supervisor since the first year of my PhD studies. He is truly an inspiring and enthusiastic academic advisor. I have benefited much from his challenging comments and constructive suggestions. I am particularly grateful to him for teaching me the necessary continuous time finance techniques to succeed as a scholar in the field of asset pricing. Moreover, he always had a ready answer to all my questions in academic and private matters.

Stavros started to supervise me at the beginning of the second year of my studies during his visit at the LSE. His keen support and steady encouragement was invaluable. His door was always open to me and he spent uncountable hours and spared no effort to guide me through the process of writing my thesis. I am particularly indebted to Stavros for his endeavor to teach me the many general equilibrium and overlapping generations modeling techniques and the numerical solution methods necessary to successfully write this thesis. In addition, Stavros shared much of his life experience with me and was undoubtedly more than an academic advisor. I am also extremely grateful for his enormous effort to make it possible for me to conduct one year of research at the University of Chicago Booth School of Business.

Jack became my supervisor during the second half of the second year of my PhD studies. He was always available to discuss my research ideas and his critical assessment and detail comments improved my work enormously. I am particularly thankful to him for helping me to gain a much deeper understanding of the economic dynamics in my models, which fundamentally enhanced the positioning and presentation of my thesis. Moreover, I am grateful to Jack for teaching me various empirical methods which are vital to test the empirical implications of my theoretical work as in the third chapter of this thesis.

I am also deeply indebted to Dimitri, Stavros and Jack for their outstanding effort to prepare me for the academic job market which was vital for me to succeed and receive an offer for an Assistant Professor of Finance position at Olin Business School, Washington University in St. Louis. I am further thankful to Kathy Yuan, Christian Julliard and Denis Gromb and many faculty members at the London School of Economics for their extraordinary advice and support during the job market.

Furthermore, I am extremely grateful to the three conference discussants Richard Roll, Francisco Gomes and Neil Mehrotra, whose critical comments and inspiring suggestions improved my work essentially. In addition, I give my special thanks to Bernard Dumas for many stimulating discussions and interesting questions. Finally, I am happy for many comments and suggestions by seminar participants at the London School of Economics, University of Chicago, City University of Hong Kong, INSEAD, EconCon 2011 Conference, HEC Lausanne, 24th Australasian Banking and Finance Conference, Stockholm School of Economics, University of Texas Austin, Indiana University, University of Washington, Arizona State University, University of Maryland, Duke University, University of New South Wales, Hong Kong University of Science and Technology, HEC Paris, University of Illinois Urbana-Champaign, Washington University in St Louis, and the Adam Smith Workshop for Asset Pricing.

I am also thankful to Mary Comben for her great administrative support throughout my PhD studies.

An important source to finance my PhD studies was the research scholarship by LSE for which I am grateful. Moreover, my one year visit at the University of Chicago would not have been possible without the very generous financial support by the Swiss National Science Foundation (SNF). In addition, I am extremely thankful for the generous financial support by my grandparents, Wilhelm and Elsa Maurer and Christian Menn.

I wish to thank my parents, Andreas and Claudia Maurer for their financial and moral support. They have undoubtedly provided me with the best prerequisites possible to pursue my PhD studies and had in every conceivable situation constructive advise ready. Their encouragement and constant care was invaluable.

Finally, I thank my girlfriend, Wing Yan Tam for her unlimited effort to support me. She has never left my side and shared many good times with me and comforted me at every necessary occasion. She ensured that I had the most comfortable and worry-free environment to work efficiently. Without her loving and caring support I would have been lost.

Contents

Chapter 1

Is Consumption Growth merely a Sideshow in Asset Pricing?

Abstract

Shocks to the agent's subjective time discounting of future utility cause stochastic changes in his consumption-to-wealth ratio. In general equilibrium, asset prices crucially depend on the current consumption-to-wealth ratio. Time discounting also affects the agent's value function and - given he has recursive preferences - his marginal utility. Independent of the consumption growth process, shocks to time discounting imply a covariation between asset returns and the marginal utility process, and the equity premium is non-zero. My model can generate both a reasonably low level and volatility in the risk-free real interest rate and a high stock price volatility and equity premium, even in absence of consumption growth shocks. If time discounting follows a process with mean-reversion, then the real interest rate follows a mean-reverting stochastic process and realized stock returns are negatively auto-correlated (at long horizons). The market price of risk, equity premium, and the conditional volatilities in the stock price and real interest rate follow stationary Markov diffusion processes. The price-earnings ratio has power to predict future stock market excess returns, and reveals contains about various unobservable key quantities in asset pricing.

1.1 Introduction

General equilibrium models in asset pricing literature build on the premise that uncertainty in (aggregate) consumption growth is the fundamental driving force for pricing, and other risk sources only matter if they covary with the consumption growth process. However, standard consumption-based asset pricing models as originally defined in Lucas (1978), Grossman and Shiller (1981), and Hansen and Singleton (1983) encounter many problems when trying to fit the data (for an overview of empirical stylized facts see Campbell (2003)).

Most prominent is Mehra and Prescott's (1985) equity premium puzzle and the closely related risk-free rate puzzle of Weil (1989).[1] Another important (and related) problem of standard consumption-based asset pricing models is that in equilibrium the stock price volatility essentially equals the variation in aggregate consumption growth. But in the data the aggregate consumption process is extremely smooth while the stock price is volatile. In addition, there are several other puzzles including the low correlation between stock returns and aggregate consumption growth, autocorrelation in realized stock returns and predictability of expected returns, while aggregate consumption growth is difficult to forecast.

Many extensions of the standard consumption-based models are introduced in the literature,[2] and while some empirical facts might be partially explained, we still do not have a fully satisfying answer to the asset pricing puzzles. I challenge the fundamental

[1] Moreover, Shiller (1982), Hansen and Jagannathan (1991), and Cochrane and Hansen (1992) show that the challenge to explain the equity premium puzzle is to find a way to introduce sufficient variation in the marginal utility process to match the large Sharpe ratio in the data.

[2] Extensions of standard consumption-based models include habit formation preferences (Abel (1990), Constantinides (1990), Campbell and Cochrane (1999)), information asymmetry (Wang (1993)), heterogeneous agents (Dumas (1989), Constantinides and Duffie (1996), Sandroni (1999); Chan and Kogan (2002), Brav, Constantinides and Geczy (2002), Constantinides, Donaldson and Mehra (2002), Storesletten, Telmer and Yaron (2007), Gomes and Michaelides (2008), Bhamra and Uppal (2010), Garleanu and Panageas (2010), Cvitanic, Jouini, Malamud and Napp (2011)), rare disasters (Rietz (1988), Barro (2006), Martin (2007), Wachter (2008), Gabaix (2011), Chen, Joslin and Tran (2012)), parameter uncertainty (Weitzmann (2007)), adjustment costs (Gabaix and Laibson (2001)), and taste shocks (Normandin and St-Amour (1998)), and 'long run risk' models (Barsky and De Long (1993), Bansal and Yaron (2004), Bansal, Khatchatrian and Yaron (2005), Bansal, Kiku and Yaron (2007a, 2007b), Bansal, Dittmar and Kiku (2008), Hanson, Heaton and Li (2008), Kaltenbrunner and Lochstoer (2010), Kung and Schmid (2011) - although 'long run risk' models are promising they seem inconsistent with many empirical facts documented by Le Roy and Porter (1981), Shiller (1981), Hall (1988), Campbell and Mankiw (1989), Yogo (2004), Pakos (2007), Vissing-Jorgensen (2002), Hasanov (2007), Bonaparte (2008), Campbell and Beeler (2009), Cochrane (2011)).

assumption that uncertainty in consumption growth is the main driving force in asset pricing. Pushing it even further, I question whether shocks to consumption growth have first order effects or are merely a sideshow in asset pricing.

I present a model with a risk source that is able to shed light on many of the asset pricing puzzles (empirical stylized facts) and yet can be completely independent of the consumption growth process. In general equilibrium the asset pricing implications of shocks to consumption growth are negligible in comparison to the effects of my new risk source.

I employ a standard Lucas (1978) economy with a representative agent who has recursive preferences, and introduce shocks to the representative agent's time preferences (subjective time discount rate of future utility) as a new source of uncertainty. Time preferences describe the agent's preferences over the trade-off between streams of (expected) utility received at diverse points in time (Uzawa (1968a, 1968b, 1969)), or simply the agent's impatience.

I illustrate that shocks to time preferences have substantial implications for asset pricing, and risk in the aggregate consumption growth is of secondary or essentially no importance. Uncertainty in time preferences generates both a large equity premium and substantial stock price volatility, and a reasonably low risk-free real interest rate which follows a 'smooth' process. In my model the market price of risk, equity premium, conditional stock price volatility, interest rate and its conditional volatility follow mean-reverting stochastic processes. The price-earnings ratio has power to predict future stock market excess returns, and realized stock returns are negatively auto-correlated at long horizons. The correlation between stock returns and the real interest rate (or its conditional volatility) is relatively low, and so is the correlation between stock returns and aggregate consumption growth. Because the consumption growth process has negligible asset pricing implications, I can simply choose a process to match aggregate consumption growth data.

In more detail, a shock to time discounting of future utility gives rise to an instant adjustment in the agent's consumption-to-wealth ratio and results in a stochastic change in the real interest rate and the stock price. Intuitively, if prices do not change with an increase in impatience, then the agent desires to liquidate assets and trade future consumption for current consumption. But because in equilibrium the representative agent cannot change his consumption path and asset holdings due to the feasibility constraint in the economy (or equivalently because market clearing has to be satisfied and the aggregate endowment and the supply in financial markets remain constant), asset prices have to adjust until the agent's desire to liquidate assets and increase current consumption vanishes. In particular, an increase in impatience results in a drop in the stock price and in financial wealth. Accordingly, because the absolute consumption level is unchanged, the consumption-to-wealth ratio increases.

If the agent has recursive preferences, then marginal utility depends on the current consumption level and the value function (time discounted future utility). The value function crucially depends on the agent's impatience. Equivalently, I show that the quadratic variation in the value function is well described by the quadratic variation in the consumption-to-wealth ratio. So, shocks to time preferences cause shocks to the consumption-to-wealth ratio and result in stochastic changes in the value function and marginal utility. It follows that the market price of risk is affected by uncertainty in the agent's subjective time discount rate. Intuitively, an increase in impatience is associated with a bad state of the world, and because the stock price is decreasing in impatience, the agent requires a positive compensation to hold stocks - that is, the equity premium is positive due to uncertainty in the agent's time preferences.

It is important to understand that uncertainty in the agent's time preferences is priced completely independent of the consumption growth process - that is, even if aggregate consumption is constant over time.

My model has several empirical implications. Most importantly, the price-earnings ratio - an observable variable in the data - is highly correlated with the representative agent's current time discount rate - the (unobservable) key state variable in the model. Therefore, the price-earnings ratio well captures the variation in the consumption-to-wealth ratio, the real interest rate level and conditional volatility, the conditional stock price volatility, the market price of risk and the equity premium - quantities which are not (directly) observable in the data. Furthermore, shocks to time preferences imply a low correlation between the real interest rate and the conditional expected consumption growth rate and therefore, the estimates of the elasticity of intertemporal substitution in literature are biased towards zero (even if household data is used for the estimation).

Though my model uses shocks to time preferences, the same modeling tools and asset pricing channel can be employed for various other risk sources. Crucial is that the agent has recursive preferences and the chosen risk source has a direct or (through market clearing) an indirect effect on the agent's consumption-to-wealth ratio. Indirect channels might be shocks in the bond market which result in shocks to the interest rate and make the agent instantly adjust his consumption-to-wealth ratio. In turn, shocks to the bond market might for instance be triggered by monetary or fiscal policy or changes in laws that lead firms to adjust their capital structure.

My paper relates to the literature on changes in time preferences and taste shocks. Nason (1991) explores how taste shocks affect an agents optimal consumption path, and Atkeson and Lucas (1992) and Farhi and Werning (2007) discuss the efficient consumption goods allocation under taste shocks. In international finance literature taste shocks have been employed to explain puzzles like the equity home bias, a low international consumption correlation, the exchange rate risk premium, and comovements between stock, bond and foreign exchange markets (Stockman and Tesar (1995), Bergin (2006), Pavlova and Rigobon (2007), Feng (2009), Jimenez-Martin and Cinca (2009)). Normandin and St-Amour (1998) attempt to explain the equity premium puzzle using taste shocks. They do not consider shocks to time discounting (see section 2.2 for a discussion on the difference between shocks to time discounting and instantaneous taste shocks) and work with a model in partial equilibrium. In contrast to uncertainty in time preferences, instantaneous taste shocks do not have implications for the stock price volatility and only affect the equity premium if they are correlated with the aggregate consumption growth process.

Time variation in the time discount rate is discussed in diverse contexts. Kreps (1979) discusses an agent's preference for flexibility in the context of time variations in preferences (see also Dekel, Lipman and Rustichini (2001), Dekel et al. (2007), Higashi, Hyogo and Takeoka (2009), Krishna and Sadowski (2010) and Higashi, Hyogo and Takeoka (2011) for extensions). Uzawa (1968a, 1968b, 1969) introduces state dependent time preferences (the discount rate is a function of the consumption path) from where a large body of literature emerged. Using extensions of Uzawa's preferences an agent's optimal consumption path is studied by Epstein (1983), Shi and Epstein (1993) and Acharya and Balvers (2004), an extension of the CCAPM is introduced by Bergman (1985), macroeconomic growth models are discussed by Epstein and Hynes (1983), Devereux (1991) and Sarkar (2007), and a real business cycle model is set up by Mendoza (1991). Becker and Mulligan (1997) introduce a model of endogenous time preferences in the sense that the agent can invest in a technology to reduce time discounting. Their approach is mainly applied in macroeconomic growth theory (Stern (2006), Le Van, Saglam and Erol (2009), Chen, Hsu and Lu (2010), Dioikitopoulos and Kalyvitis (2012)). Dutta and Michel (1998) and Karni and Zilcha (2000) study the optimal consumption path and the wealth distribution in an economy with stochastic changes in the time discount rate. Finally, there is empirical evidence that suggests discount rates to depend on state variables and to vary over time (Lawrence (1991), Samwick (1998), Bishai (2004), Meier and Sprenger (2010)).

Models of heterogeneous agents could produce a representative agent with a time varying time discount rate. However, the dynamics of the model and the pricing implications would differ from my paper. The individual agent would still faces a constant time discount rate in his individual optimization problem and the value function depended only through market clearing on a time varying "time discount rate" of the representative agent. Asset pricing literature has not considered a model with recursive preferences and heterogeneity in agents' time preferences.

The modeling tools and the pricing channel in this paper are closely related to my paper on demographic changes (Maurer (2012)), where shocks to birth and death rates are used to introduce risk in the consumption distribution across cohorts. In a similar spirit as uncertainty in time preferences, demographic shocks have pricing implications independent of the aggregate consumption growth process.

The chapter is organized as follows. In section 1.2, I introduce the setup of the model. In section 1.3, I derive the specification of the consumption-to-wealth ratio and discuss the qualitative implications for the stock price and the pricing kernel. In section 1.3.5, I illustrate the quantitative magnitude of the qualitative results. I conclude in section 1.4, and the proofs are in the appendix.

1.2 Model

I consider an endowment economy as in Lucas (1978) with a single consumption good and an infinitely lived representative agent. The consumption good is the numeraire in my analysis and follows an exogenously specified process. The novelty in my model is the specification of the agent's preferences. I let the agent's subjective time discount rate follow a stochastic process - that is, I introduce shocks to the agent's time preference structure, which determines his patience and willingness to defer consumption. In addition, the agent maximizes a recursive (non-time additive expected) utility function.

1.2.1 Aggregate Endowment, Financial Markets and Budget Constraint

The supply side in the consumption goods market is constituted by a representative firm which produces (or is endowed with) Y_t units of the consumption good at time t. As in Lucas (1978), 'production' is exogenous, in the sense that the firm cannot reinvested any of its output, and the evolution of Y is specified by the dynamics

$$\frac{dY_t}{Y_t} = \mu^{(Y)}dt + \sigma^{(Y)}d\widetilde{W}_t \tag{1.1}$$

with the constant drift term $\mu^{(Y)}$ and diffusion vector $\sigma^{(Y)}$, and $\widetilde{W}$ denoting a d-dimensional Wiener process. The firm pays earnings Y_t as dividends D_t to shareholders.

Financial markets are assumed to be dynamically complete.[3] π denotes the (unique) stochastic discount factor (SDF) in the economy and is determined in equilibrium. The agent's financial wealth at time t is denoted by W_t. He consumes c_t and invests the remaining part of his financial wealth in equities and bonds. Equities are claims on the stream of dividends D_t paid out by the representative firm. The price of equity

[3] This assumption is satisfied given equity and bond contracts as long as there is only one source of uncertainty, that is $\widetilde{W}$ is a one-dimensional Wiender process. In the general case where output, time discount factor and instantaneous taste shocks are not perfectly correlated, I implicitly suppose the existence of further contracts that dynamically complete financial markets - that is the existence of a complete set of Arrow-Debreu securities that allow to trade any of the d independend Brownian motions. My focus lies on equity and bond markets only.

denoted by P_t is equal to the present value of the stream of future dividends

$$P_t = E_t \left[\int_t^{\infty} \frac{\pi_s}{\pi_t} D_s ds \right] \tag{1.2}$$

The supply of equities is normalized to one. x_t denotes the number of equities purchased by the agent at time t. To do not permit arbitrage opportunities I restrict trading activities according to the standard technical assumption $\frac{x_t P_t}{W_t} \in \mathcal{L}^2$, with $\mathcal{L}^2 \equiv \left\{ x \in \mathcal{L} \mid \int_0^T x_t^2 dt < \infty \quad a.s. \right\}$ and $\mathcal{L}$ is the set of processes adapted to the filtration $\mathcal{F}^P$ generated by asset prices, $\mathcal{F}_t^P \equiv \sigma \{ P_u : u \leq t \}$. Bonds are instantaneously risk free and pay interest r_t with

$$r_t dt = E_t \left[-\frac{d\pi_t}{\pi_t} \right] \tag{1.3}$$

Bonds are in zero net supply. The part of the agent's financial wealth that is not used to buy stocks, $(W_t - x_t P_t)$ is invested in bonds.

An agent's financial wealth W_t evolves according to the dynamics

$$dW_t = \underbrace{W_t r_t dt}_{\text{risk free return}} + \underbrace{x_t \left(dP_t + D_t dt - P_t r_t dt \right)}_{\text{stock market excess return}} - \underbrace{c_t dt}_{\text{consumption}} \tag{1.4}$$

I define the set of feasible cash flows as $(M + W)_t \equiv \{ x : x - W_t \in M \}$, where M denotes the set of all marketable cash flows and a cash flow is marketable if it is financed by a trading strategy $\frac{x_t P_t}{W_t} \in \mathcal{L}^2$. The set of admissible cash flows is $\Im_t \equiv (M + W)_t \cap \mathcal{L}^+$, where $\mathcal{L}^+$ includes all non-negative processes adapted to $\mathcal{F}^P$. The agent's consumption c_t has to be an element of the set of admissible cash flows $\Im_t$.

1.2.2 Agent's Objective Function

The agent's preferences are an adaptation of the stochastic differential utility functions of the Kreps and Porteus (1978) type introduced by Duffie and Epstein (1992a, 1992b).[4] I extend their specification with shocks to the agent's taste and his subjective time preferences, which describes his preferences over the trade-off between streams of utility

[4] Stochastic differential utilities by Duffie and Epstein (1992a, 1992b) are the continuous time equivalent of the discrete time recursive preferences discussed by Epstein and Zin (1989, 1991).

received at different points in time. The recursive utility function V_t is characterized as

$$V_t = E_t \left[\int_t^{\infty} f\left(c_s, V_s, Z_s, \beta_s\right) ds \right] \tag{1.5}$$

with the aggregator function $f(.)$ given by

$$f\left(c_t, V_t, Z_t, \beta_t\right) = \frac{Z_t c_t^{\rho} - \beta_t \left[\left(1-\gamma\right) V_t\right]^{\frac{\rho}{1-\gamma}}}{\rho \left[\left(1-\gamma\right) V_t\right]^{\frac{\rho}{1-\gamma}-1}} \tag{1.6}$$

where Z_t captures instantaneous taste shocks, β_t describes discounting of future utility (time preferences), the term $\frac{1}{1-\rho}$ equals the elasticity of intertemporal substitution (EIS), and γ controls relative risk aversion. The specification allows to disentangle EIS and γ, and describes an agent's preferences over the timing of uncertainty resolution, as discussed in Kreps and Porteus (1978). The special case of time additive constant relative risk aversion preferences is recovered if $1 - \gamma = \rho$.

The subjective time discount rate β is defined on the space (β_L, β_H) and follows an Orenstein-Uhlenbeck process,

$$\begin{aligned} d\beta_t &= \mu_t^{(\beta)} dt + \sigma_t^{(\beta)} d\widetilde{W}_t \\ &= m^{(\beta)} \left(\overline{\beta} - \beta_t\right) dt + \sqrt{\beta_t - \beta_L}\sqrt{\beta_H - \beta_t}\sigma^{(\beta)} d\widetilde{W}_t \end{aligned} \tag{1.7}$$

where $m^{(\beta)}$ determines the speed of mean-reversion of β towards the long run (expected) level $\overline{\beta}$, and $\sigma^{(\beta)}$ is a constant diffusion vector. The sufficient condition $2m^{(\beta)}\overline{\beta} > (\beta_H - \beta_L)\,\sigma^{(\beta)} \left(\sigma^{(\beta)}\right)^T$ ensures that β is never absorbed at the boundaries β_L and β_H or crosses them (Feller (1951)).

I model the impatience parameter β by an Orenstein-Uhlenbeck process with well defined boundaries for several reasons. Most importantly, the specification allows to (roughly) match the dynamics of the model implied risk-free real interest rate with standard short rate models in the literature, which employ generalizations of the Orenstein-Uhlenbeck process (for instance Vasicek (1977), Cox, Ingersoll and Ross (1985), Hull and White (1990), Black and Karasinski (1991), Longstaff and Schwartz (1992), Chen (1996)). As I show below, the mean-reversion property of β results in a stationary real interest rate process which is expected to revert to its mean in the long run. It also implies that the (stock) price-earnings ratio is stationary and mean-reverting, and (at long horizons) stock returns are negatively auto-correlated. Finally, it seems natural to assume that the subjective time discount rate is bounded, and the assumption delivers well specified boundary conditions to solve the ordinary differential equation (1.9).

In contrast to shocks to time preferences (β), shocks to Z are interpreted as instantaneous taste shocks. I suppose that Z follows a martingale process with the dynamics

$$\begin{aligned} dZ_t &= \mu_t^{(Z)} dt + \sigma_t^{(Z)} d\widetilde{W}_t \\ &= Z_t \sigma^{(Z)} d\widetilde{W}_t \end{aligned} \tag{1.8}$$

A non-zero drift term $\mu_t^{(Z)}$ would affect time discounting of future utility - the effective subjective time discount rate equals $\beta + \mu^{(Z)}$ - and stochastic changes in $\mu^{(Z)}$ would be equivalent to shocks to β. Choosing $\mu_t^{(Z)} = 0$ allows to separate the instantaneous taste shocks Z from the dynamics in the time discount rate β. The focus in literature solely lies on instantaneous taste shocks and the interaction with risk in consumption growth, while the pricing implications of stochastic changes in the agent's time preferences are ignored (for instance Stockman and Tesar (1995), Normandin and St-Amour (1998), Pavlova and Rigobon (2007), Feng (2009)). The main purpose of my analysis is to study the asset pricing implications of shocks to time preferences and I introduce instantaneous taste shocks merely to point out the fundamental differences between the two risk sources and thus, the differences between my model and the literature on taste shocks.

The agent's objective is to maximize the value function subject to the dynamic or equivalently the static budget constraint,

$$\sup_{\{c,x\} \in (\Im \times \mathcal{L}^2)} \left\{ V_t(c) = E_t \left[\int_t^\infty f(c_s, V_s, Z_s, \beta_s)\, ds \right] \right\}, \qquad s.t.\ \ dY_s,\ d\beta_s,\ dZ_s \tag{\textbf{P1}}$$

1.3 Equilibrium

1.3.1 Definition of Equilibrium

An equilibrium is defined by a set of adapted processes $\{c, x, \pi\}$ such that at any time

(i) the agent's utility is maximized subject to the budget constraint (Problem (**P1**)),

(ii) the consumption goods market clears, $Y_t = c_t$,

(iii) the equity market clears, $1 = x_t$,

(iv) the bond market clears, $0 = W_t - x_t P_t$.

1.3.2 Consumption-to-Wealth Ratio

The key quantity in my analysis is the agent's consumption-to-wealth ratio $\psi_t = \frac{c_t}{W_t}$. Given the exogenous evolution of Y and the the consumption-to-wealth ratio ψ, it is straightforward to derive equilibrium asset prices.

Proposition 1.1 *There exists an equilibrium with the consumption-to-wealth ratio $\psi_t(\beta_t)$ described by the ordinary differential equation*

$$\begin{aligned} 0 \;=\; & -\psi_t + \beta_t - \rho\mu^{(Y)} + \frac{\mu_t^{(\psi)}}{\psi_t} + \frac{\gamma\rho}{2}\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T - (1-\gamma)\,\sigma^{(Y)}\left(\sigma^{(Z)}\right)^T \\ & + (1-\gamma)\,\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T - \frac{1-\gamma+\rho}{2\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T \\ & - \frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\left(\sigma^{(Z)}\right)^T + \frac{1-\gamma}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\sigma^{(Z)}\right)^T \end{aligned} \tag{1.9}$$

for $\beta_t \in (\beta_L, \beta_H)$, and $\psi_t(\beta_t)$ follows a Markov diffusion process

$$d\psi_t = \mu_t^{(\psi)}dt + \sigma_t^{(\psi)}d\widetilde{W}_t \tag{1.10}$$

with

$$\mu_t^{(\psi)} = \frac{\partial\psi_t}{\partial\beta_t}\mu_t^{(\beta)} + \frac{1}{2}\frac{\partial^2\psi_t}{\partial\beta_t^2}\sigma_t^{(\beta)}\left(\sigma_t^{(\beta)}\right)^T \tag{1.11}$$

$$\sigma_t^{(\psi)} = \frac{\partial\psi_t}{\partial\beta_t}\sigma_t^{(\beta)} \tag{1.12}$$

Proof. See Appendix. ■

It is important to notice that Proposition 1.1 does not qualitatively change for the special case where the agent has time additive preferences $(1-\gamma=\rho)$.

The consumption-to-wealth ratio ψ_t is a function of only the agent's current subjective time discount rate β_t. It is independent of the state variables Y and Z. The latter two state variables would matter in a setting where the aggregate endowment and the agent's taste followed more general diffusion processes rather than a geometric Brownian motion with constant drift and diffusion[5]. However, ψ_t depends on β_t independent of the specification of the stochastic process driving β - even in the case where β is specified by a geometric Brownian motion with constant drift and diffusion $\left(d\beta_t = \beta_t \mu^{(\beta)} + \beta_t \sigma^{(\beta)} d\widetilde{W}_t\right)$. The impact on the consumption-to-wealth ratio and the asset pricing implications of instantaneous taste shocks versus shocks to time preferences are very different.

The dependence of ψ_t on β_t arises from the direct relationship between the agent's (current) impatience and the desire to consume his wealth early in time. Intuitively, this relationship ought to be positive as an increase in the time discount of future utility suggests a desire to shift the optimal consumption plan from 'late' consumption and to consumption 'early' in time, such that marginal utility equals at every point in time. Mathematically, the sign of the dependence of ψ_t on β_t is not obvious because a change in β_t also affects the quantities $\frac{\partial \psi_t}{\partial \beta_t}$, $\frac{\partial^2 \psi_t}{\partial \beta_t^2}$, $\mu_t^{(\beta)}$ and $\sigma_t^{(\beta)}$. For instance, if an increase in β_t at some point $\beta_t = \widehat{\beta}$ leads to a large enough decrease in $-\frac{1-\gamma+\rho}{2\rho} \frac{\sigma_t^{(\psi)}}{\psi_t} \left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T$, then ψ_t might in fact be declining in β_t (at $\beta_t = \widehat{\beta}$). Economic intuition tells that a change in impatience may lead to a change in uncertainty in the agent's subjective time discount rate $\sigma_t^{(\beta)}$, and a change in the variation in the consumption-to-wealth ratio $\sigma_t^{(\psi)}$, which effectively captures the uncertainty perceived by the agent. In turn, a change in risk triggers a change in today's precautionary savings motive and in the current consumption-to-wealth ratio. An increase in impatience may increases the precautionary savings motive and decrease the consumption-to-wealth ratio.

Lemma 1.1 *If $\sigma^{(\beta)} \to 0$ and $(\beta_H - \beta_L) < \infty$, then $\psi_t(\beta_t)$ is continuous and monotonically increasing in β_t ($\forall \beta_t \in (\beta_L, \beta_H)$).*

Proof. See Appendix. ■

[5] See for instance the long run risk literature initiated by Bansal and Yaron (2004), who assume more flexible dynamics in the aggregate endowment process.

Intuitively, as $\sigma_t^{(\beta)}$ becomes very small, there is (almost) no uncertainty about changes in the agent's time preferences and his precautionary savings motive (almost) disappears. It follows the first intuition that an increase in impatience leads to the desire of more current consumption in expenses of future consumption. Given aggregate endowment is unaffected by the increase in impatience, the desire to liquidate assets and to increase current consumption implies that in equilibrium there has to be a downward adjustment in the stock price until the agent's desire to sell his assets and buy more consumption goods vanishes and all market clearing conditions are satisfied. A decrease in the stock price leads to a decrease in the agent's (financial) wealth and subsequently to an increase in the consumption-to-wealth ratio because the absolute consumption level remains constant.

Numerical solutions of the ODE (1.9) presented in section 3.5 suggest the same conclusion of a positive relation between the consumption-to-wealth ratio and the current impatience. The numerical results also suggest that $\frac{\partial \psi_t}{\partial \beta_t} < 1$ and mostly smaller than 0.65; that is, the variation in the consumption-to-wealth ratio is substantially lower than the variation in the subjective time discount rate. Intuitively, after a sudden increase in impatience the agent expects to revert to be more patient in the long run (because β_t follows a mean-reverting process) and thus, he does not increase his current consumption-to-wealth ratio as much as if the shock to time discounting was permanent. In equation (1.9) this mean-reversion (damper) effect is captured by the term $\frac{1}{\psi} \frac{\partial \psi_t}{\partial \beta_t} m^{(\beta)} \left(\overline{\beta} - \beta_t \right) < (>) \, 0$, for $\beta_t > (<) \, \overline{\beta}$. The mean-reversion property in the time discount rate process further implies that the consumption-to-wealth ratio follows a mean-reverting process.

1.3.3 Stock Price

Given the consumption-to-wealth ratio, it is straightforward to derive the stock price. Financial markets clearing (solving the equilibrium conditions (iii) and (iv) for P_t) tells us that in equilibrium the stock price has to equal the agent's (financial) wealth, $P_t = W_t$.[6] Rewriting the expression in terms of the consumption-to-wealth ratio ($\psi_t = \frac{c_t}{W_t}$)

[6] The same result is obtained when combining the definition of the stock price being equal to the present value of future dividends or equivalently the present value of aggregate endowment or consumption (consumption goods market clearing) with the agent's static budget constraint. That is,

and using the market clearing condition in the consumption goods market (equilibrium condition (ii), $Y_t = c_t$) yields

$$P_t = \frac{1}{\psi_t} Y_t \tag{1.13}$$

The (stock) price-earnings ratio (or equivalently the price-dividend ratio) is $\frac{P_t}{Y_t} = \frac{1}{\psi_t}$. From the discussion on the consumption-to-wealth ratio it follows immediately that the price-earnings ratio is continuous and monotonically decreasing in the agent's current impatience β_t, and $\frac{P_t}{Y_t}$ follows a stochastic process with mean-reversion. Given the mean-reversion property in the price-earnings ratio, I conjecture that the realized stock return is negatively correlated at relatively long horizons. Intuitively, at short horizons the mean-reversion in $\frac{P_t}{Y_t}$ is weak and noise dominates, while at long horizons the reverse is true. This is because the speed of mean-reversion is proportional to the time horizon while volatility is proportional to the square root of the time horizon.

The ex-dividend stock price is not a constant multiple of aggregate consumption as in Lucas (1978), but it is cointegrated with the aggregate endowment process because ψ_t is stationary. In other words, the ex-dividend stock price follows a 'noisy', mean-reverting process with the (less noisy) stochastic and non-stationary mean (trend) equal to the aggregate consumption process.

Applying Itô's Lemma to equation (1.13) shows that the stock price follows a Brownian diffusion process with the dynamics

$$dP_t = \mu_t^{(P)} dt + \sigma_t^{(P)} d\widetilde{W}_t \tag{1.14}$$

with

$$\frac{\mu_t^{(P)}}{P_t} = \mu^{(Y)} - \frac{\mu_t^{(\psi)}}{\psi_t} + \frac{\sigma_t^{(\psi)}}{\psi_t} \left(\frac{\sigma_t^{(\psi)}}{\psi_t} \right)^T - \sigma^{(Y)} \left(\frac{\sigma_t^{(\psi)}}{\psi_t} \right)^T \tag{1.15}$$

$$\frac{\sigma_t^{(P)}}{P_t} = \sigma^{(Y)} - \frac{\sigma_t^{(\psi)}}{\psi_t} \tag{1.16}$$

The (conditional) stock price volatility depends on shocks to aggregate endowment and to the agent's subjective time discount rate, even in the special case of time additive

$P_t = E_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} D_s ds \right] = E_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} Y_s ds \right] = E_t \left[\int_t^\infty \frac{\pi_s}{\pi_t} c_s ds \right] = W_t.$

preferences. Time preference shocks imply a quadratic variation in the consumption-to-wealth ratio (equation (1.9)) which causes an instantaneous volatility in the stock price (equation (1.13)). Instantaneous taste shocks do not matter, provided that Z follows a geometric Brownian motion with constant drift and diffusion terms.

For illustrative purposes I suppose for now that $\sigma^{(Y)}\left(\sigma^{(\beta)}\right)^T = 0$. It is important to understand that shocks to the agent's time preferences affect the stock price completely independent of aggregate endowment shocks; the volatility terms $\sigma^{(Y)}$ and $\sigma_t^{(\psi)}$ are additive in equation (1.16). Shocks to time preferences matter for asset pricing even in absence of consumption growth! The intuition is as mentioned earlier. Sudden changes in the agent's subjective time discount rate cause pure demand shocks while the supply is fixed. An increase in impatience means that at current prices the agent wants to liquidate assets and buy more consumption goods. Since the supply in all markets remains unchanged and in equilibrium market clearing must be satisfied, the stock price has to adjust (fall) until the agent revokes his plan to liquidate assets for more current consumption. The decline in the stock price causes the agent's (financial) wealth to drop and accordingly, the consumption-to-wealth ratio to increase, which explains the contemporaneous relation between P_t and ψ_t.

It follows that the stock price volatility is larger than the variation in aggregate consumption growth. Indeed, as illustrated in the Monte Carlo simulations in section 3.5 stochastic changes in time preferences are responsible for (almost) the entire variation in the stock price and the unconditional excess volatility in the stock market over the variation in aggregate consumption growth is substantial (the difference is more than one order of magnitude). Finally, because $\frac{\sigma_t^{(\psi)}}{\psi_t}$ is a function (or equivalently $\frac{\partial \psi_t}{\partial \beta_t}$, ψ_t and $\sigma_t^{(\beta)}$ are functions) of the current level β_t, the conditional stock price volatility follows a stationary stochastic process and is expected to revert in the long run to a constant mean.

The conditional expected stock return is $\frac{\mu_t^{(Y)}+D_t}{P_t}dt$. It is trivial that $\frac{\mu_t^{(Y)}+D_t}{P_t}dt$ crucially depends on the current level of the agent's impatience and for $\sigma_t^{(\beta)} \rightarrow 0$ it features the same qualitative properties as ψ_t. Most important, the conditional expected stock return follows a stationary stochastic process that reverts in the long run to a constant mean. Intuitively, if the agent is very impatient, then the price-earnings ratio lies far below its long run average and the stock is relatively cheap. Since the agent expects to become more patient in future (β_t reverts to $\overline{\beta}$) and the price-earnings ratio is expected to increase, the expected stock return is large. Moreover, the expected stock return is declining as β_t reverts to $\overline{\beta}$, the price-earnings ratio increases to its long run average and the stock becomes relatively more expensive. Given a persistence at short horizons in the conditional expected stock return process, I conjecture that realized stock returns are (slightly) positively correlated at short horizons.

The equity premium can be written as $\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}dt$. Alternatively, using the definition of the stock price being equal to the present value of future dividends (equation (1.2)) and noticing that $E_t\left[P_s\right]$ for $s<t$ is a local martingale yields

$$E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt=-\frac{dP_t}{P_t}\left(\frac{d\pi_t}{\pi_t}\right)^T=\frac{\sigma_t^{(P)}}{P_t}\left(\kappa_t\right)^T dt \tag{1.17}$$

where $\kappa_t d\widetilde{W}_t=-\frac{d\pi_t-E_t[d\pi_t]}{\pi_t}$ is the market price of risk. The interpretation of equation (1.17) is standard: if an asset pays off low (high) during times when the agent faces a high (low) marginal utility and desires much (does not need much) wealth (negative correlation between SDF and stock price), then the agent demands a premium to hold the asset (positive equity premium). Either expression of the equity premium requires me to solve for the risk-free real interest rate or the market price of risk to be able to give an interpretation.

1.3.4 Stochastic Discount Factor and Equity Premium

Proposition 1.2 *In an equilibrium with ψ_t as described in Proposition 1.1, the pricing kernel π follows a Markov diffusion process defined by*

$$\frac{d\pi_t}{\pi_t}=-r_tdt-\kappa_td\widetilde{W}_t \tag{1.18}$$

with the risk-free real interest rate r_t and the market price of risk κ_t characterized as

$$\begin{aligned} r_t &= \beta_t + (1-\rho)\,\mu^{(Y)} - \frac{\gamma\,(2-\rho)}{2}\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T \\ &+ \frac{1-\gamma-\rho}{2\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T - \frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\left(\sigma^{(Z)}\right)^T \\ &- \frac{1-\gamma-\rho}{\rho}\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T + \frac{(1-\gamma)\,(1-\rho)}{\rho}\sigma^{(Y)}\left(\sigma^{(Z)}\right)^T \end{aligned} \tag{1.19}$$

$$\kappa_t = \kappa_t^{(Y)} + \kappa_t^{(Z)} + \kappa_t^{(\beta)} = \gamma\sigma^{(Y)} - \frac{1-\gamma}{\rho}\sigma^{(Z)} + \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t} \tag{1.20}$$

The risk-free real interest rate and and market price of risk are functions of the current state β_t and follow a stationary Markov diffusion processes with the dynamics specified in equations (1.51), (1.52), (1.53) and (1.54). The equity premium can be written as

$$\begin{aligned} E_t\left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt &= \gamma\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T - \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T \\ &+ \frac{1-\gamma-\rho-\gamma\rho}{\rho}\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T \\ &- \frac{1-\gamma}{\rho}\left(\sigma^{(Y)} - \frac{\sigma_t^{(\psi)}}{\psi_t}\right)\left(\sigma^{(Z)}\right)^T \end{aligned} \tag{1.21}$$

The equity premium is a function of of the current state β_t and follows a stationary Markov diffusion process.

Proof. See Appendix. ■

The risk-free real interest rate depends of the three standard quantities: the representative agent's current time discount rate of future utility, his expected consumption growth over the next instant in time, and precautionary savings. Precautionary savings depend on aggregate endowment risk and uncertainty in the agent's taste and time preferences. In the special case of time additive preferences ($1-\gamma=\rho$), the precautionary savings motive depends only on aggregate consumption risk and the interaction (covariation) between aggregate consumption risk and taste shocks - in particular, shocks to time preferences are irrelevant.

In general, a change in the agent's time preferences has two effects on the interest rate. An increase in impatience has the direct consequence to generate a desire of the agent to increase current consumption and therefore liquidate his financial assets, which leads to a drop in the bond price and an increase in the interest rate. However, an increase in impatience may also lead to an increase in uncertainty about the (future) consumption-to-wealth ratio $\left(\psi_t, \frac{\partial \psi_t}{\partial \beta_t} \text{ and } \sigma_t^{(\beta)} \text{ are functions of } \beta_t\right)$. The additional risk may increase the agent's precautionary savings motive and cause a decline in the interest rate. It is hard to tell which of the two opposing effects dominates. The numerical results in section 3.5 show that the latter (precautionary savings) channel dominates if the agent is patient enough ($\frac{\partial r_t}{\partial \beta_t} < 0$), and the first (direct) effect dominates if the agent becomes sufficiently impatient ($\frac{\partial r_t}{\partial \beta_t} > 0$).

Garleanu and Panageas (2010) show that in an economy where agents are heterogeneous with respect to the curvatures in their objective functions, empirical estimates of the EIS are biased towards zero, if the econometrician uses aggregate consumption data in his estimation. In their paper the expected consumption growth of an individual agent ($dE_t\left[\ln\left(\frac{C_{t+1}^i}{C_t^i}\right)\right]$) and the interest rate ($dE_t\left[\ln\left(1+r_{t+1}\right)\right]$) are stochastic (due to stochastic changes in the distribution of aggregate consumption among agents), while expected aggregate consumption growth is constant ($dE_t\left[\ln\left(\frac{C_{t+1}^{agg}}{C_t^{agg}}\right)\right]$). Accordingly, if the true economy is populated by heterogeneous agents but the econometrician uses aggregate consumption data, then the estimation

$$EIS = \frac{dE_t\left[\ln\left(C_{t+1}^{agg}\right) - \ln\left(C_t^{agg}\right)\right]}{dE_t\left[\ln\left(1+r_{t+1}\right)\right]} \tag{1.22}$$

which is derived (approximated) from the Euler equation in a representative agent economy (Vissing-Jorgensen (2002)), will be biases towards zero. However, this problem can be circumvented and an unbiased estimate can be obtained if household consumption data is used to estimate the EIS ($= \frac{dE_t\left[\ln\left(C_{t+1}^i\right) - \ln\left(C_t^i\right)\right]}{dE_t\left[\ln\left(1+r_{t+1}\right)\right]}$) for each individual investor. Indeed empirical estimates of the EIS by Hall (1988), Campbell and Mankiw (1989), Yogo (2004) and Pakos (2007) - who use aggregate consumption data - are indistinguishable from zero. In contrast, estimates by Hasanov (2007) and Bonaparte (2008) - who use household specific data - yield a higher EIS of around 0.3. Garleanu and Panageas (2010) deliver a nice explanation for the difference in the estimates. However, in most economics literature $EIS = 0.3$ is still considered to be low.

The result of my analysis is even stronger than the conclusion in Garleanu and Panageas (2010). Since r_t is a function of β_t, the interest rate follows a stationary stochastic process which is expected to revert in the long to a constant mean.[7] It is important that there is variation in the interest rate which originates from shocks to time preferences and is unrelated to variation in either aggregate or the individual investor's (expected) consumption growth. Accordingly, due to shocks to time preferences there is a bias towards zero in the estimation of the EIS no matter whether the econometrician uses aggregate or household consumption data. This is in contrast to the case in Garleanu and Panageas (2010) where the use of household consumption data resolves the estimation problem. Therefore, shocks to time preferences explain why the estimates by Hasanov (2007) and Bonaparte (2008) are still lower than expected (and lower than the true EIS).

The market price of risk depends on risk in aggregate consumption growth, taste shocks, and uncertainty about time discounting. Although taste shock affect marginal utility, they only matter for the equity premium if they are correlated with either one of the two other risk sources. In contrast, aggregate consumption growth and time preferences matter for the equity premium independent of the correlation structure because they both affect marginal utility and the stock price volatility. Accordingly, the asset pricing implications of instantaneous taste shocks and stochastic changes in time preferences are very different.

For illustrative purposes, I suppose for now that aggregate consumption growth is constant ($\sigma^{(Y)} = 0$; no risk in the aggregate consumption process) and taste shocks are unrelated to risk in time discounting ($\sigma^{(Z)} \left(\sigma^{(\beta)}\right)^T = 0$). The equity premium simplifies to

$$E_t \left[\frac{dP_t + D_t dt}{P_t} \right] - r_t dt = -\frac{1-\gamma-\rho}{\rho} \frac{\sigma_t^{(\psi)}}{\psi_t} \left(\frac{\sigma_t^{(\psi)}}{\psi_t} \right)^T \tag{1.23}$$

It depends only on shocks to time preferences and it is positive if and only if $EIS \notin \left(\min\left(\frac{1}{\gamma}, 1\right), \max\left(1, \frac{1}{\gamma}\right)\right) \left(\text{or } -\frac{1-\gamma-\rho}{\rho} = \frac{1-\gamma EIS}{1-EIS} > 0\right)$. In the special case of time additive preferences the market price of risk and the equity premium are zero.[8]

[7] Remember that this result is one of the main reasons for me to model β as a mean-reverting process in the first place.

[8] In the case of CRRA utility, shocks to the agent's subjective discount factor only affect the equity premium if they are correlated with risk in the aggregate consumption growth process.

In the special case of time additive preferences ($1-\gamma=\rho$), the stream of utility the agent receives at some time t depends only on the current consumption level. Marginal utility also depends only on current consumption and the marginal utility process is a function of the agent's current consumption growth. Accordingly, the subjective time discount rate of future utility does not affect the marginal utility process and has no effect on the market price of risk (and the equity premium). In contrast, shocks to aggregate consumption and instantaneous taste shocks, which have a 'scaling' effect on the current consumption level, affect marginal utility and the market price of risk.

Under more general recursive preferences, the stream of utility the agent receives at some time t depends on a weighted average of past, current and future expected consumption. The agent's time preferences determine the 'weights' used in the 'weighted average' - that is the importance of consumption streams at different points in time. Marginal utility is again a function of a weighted average of past, current and future expected consumption and the marginal utility process depends on a weighted average of past, current and future expected consumption growth. Since the market price of risk is defined by stochastic changes in the marginal utility process and past consumption is realized (no uncertainty), the market price of risk is a function of only the quadratic variation in current consumption growth and future expected consumption growth. Equations (1.6) and (1.24) suggest that the variation in the value function is a sufficient statistic of the variation in future expected consumption growth. By definition of the value function, time preferences matter in a crucial way. A shock to the agent's subjective time discount rate means an unexpected change in the 'weighting' of future expected consumption and an unpredictable shock to the value function. This is equivalent to a stochastic change in marginal utility and the shock to time preferences affects the market price of risk. From section 3.3.3 I know that the stock price instantly reacts to sudden changes in the agent's time preferences. Accordingly, shocks to time preferences imply a covariation between the marginal utility process and the stock price, and the equity premium is non-zero. Finally, shocks to aggregate consumption and instantaneous taste shocks have the exact same effect on marginal utility and the market price of risk as in the case of time additive preferences.

In mathematical terms, from equations (1.6) and (1.24) it follows that marginal utility is a decreasing (increasing) function in $(1-\gamma) V_t$ if $\frac{1-\gamma-\rho}{1-\gamma} < (>) 0$. Equation (1.34) states that $(1-\gamma) V_t$ is decreasing (increasing) in ψ_t if $\frac{1-\gamma}{\rho} > (<) 0$. Accordingly, marginal utility is increasing in the consumption-to-wealth ratio if and only if $\frac{1-\gamma-\rho}{\rho} < 0$ (or $EIS \notin \left(\min\left(\frac{1}{\gamma}, 1\right), \max\left(1, \frac{1}{\gamma}\right)\right)$). If a shock to time preferences yields an increase (decrease) in the consumption-to-wealth ratio, then the agent is in a high (low) marginal utility state (if $\frac{1-\gamma-\rho}{\rho} < 0$). In equilibrium, an increase (decline) in the consumption-to-wealth ratio implies a drop (increase) in the stock price (equation (1.16)). Therefore, the stock return is negatively correlated with the marginal utility process and by definition (equation (1.17)) the equity premium is positive.

On a more intuitive level, an increase (decrease) in impatience means that the agent discounts his future utility is more (less) heavily. This is plausibly associated with a bad (good) state of the world and the agent desires much (does not need much) wealth. An increase (decrease) in time discounting also corresponds to a decline (rise) in the stock price (see section 3.3 for an intuition). Therefore, the stock pays off low (high) in a bad (good) state of the world, which is an undesirable payoff schedule, and the agent requires a positive premium to hold the stock (positive equity premium).

Finally, because $\frac{\sigma_t^{(\psi)}}{\psi_t}$ is a function of the current state β_t, the market price of risk and the equity premium both follow a stationary stochastic process which is expected to revert in the long run to a constant mean.

1.3.5 Numerical Results and Monte Carlo Simulations

I solve the model numerically to quantify the magnitudes of my qualitative results. Uncertainty in the agent's subjective time discount rate has quantitatively important implications for asset pricing (first order effect). In contrast, the pricing implications of aggregate consumption growth shocks are negligible (second order effect).

I suppose that in the long run the subjective time discount rate is always expected to revert to $\overline{\beta} = 0.045$. The speed of mean-reversion is assumed to be moderate, $m^{(\beta)} = 0.175$, and I chose the conditional volatility $\sigma^{(\beta)}\sqrt{\beta_H - \beta_t}\sqrt{\beta_t - \beta_L} = 0.125\sqrt{\beta_t - \beta_t^2}$, that is $\sigma^{(\beta)} = 0.125$, $\beta_L = 0$, and $\beta_H = 1$. Although the support of β_t is defined by the interval $(\beta_L, \beta_H) = (0, 1)$, the mean-reversion and the state dependent conditional volatility of β_t imply that β_t does not travel too far from the long run mean $\overline{\beta}$. Indeed

Monte Carlo simulations suggest that the time discount rate does not leave the interval $(0, 0.2775)$ with a confidence of 99.9% (see table 1.1 for an estimate of the effective unconditional distribution of the time discount rate). Simulations also show that the unconditional mean of β_t roughly equals $\overline{\beta}$, and the unconditional volatility of β_t is 4.26%, which matches the conditional volatility at $\beta_t = \overline{\beta}$ ($\sigma^{(\beta)}\sqrt{\overline{\beta} - \overline{\beta}^2} = 2.59\%$; the conditional volatility has to be adjusted for the speed of mean-reversion to compare it to the unconditional volatility).

Unconditional Distribution of Simulated Time Discount Rate

Percentile	0.1%	0.5%	1%	5%	10%	25%
β_t	0.0000	0.0002	0.0005	0.0025	0.0050	0.0136

Percentile	75%	90%	95%	99%	99.5%	99.9%
β_t	0.0634	0.1029	0.1313	0.1945	0.2202	0.2775

Table 1.1: Unconditional distribution of β_t from 100 simulations of 10'000 years of weekly data.

I choose $\mu^{(Y)} = 0.02$ and $\sigma^{(Y)} = 0.01$ to roughly match the first two unconditional moments of aggregate consumption growth in US data. Setting $\sigma^{(Y)} = 0$ hardly affects my results. It is hard to tell how large the correlation between shocks to time preferences and shocks to consumption growth is ought to be. Though, for instance theory papers by Uzawa (1968a, 1968b, 1969) and Becker and Mulligan (1997) suggest that $\sigma^{(Y)}\left(\sigma^{(\beta)}\right)^T < 0$.[9] For simplicity I set $\sigma^{(Y)}\left(\sigma^{(\beta)}\right)^T = 0$. Since I merely introduce instantaneous taste shocks in my analysis to point out the fundamental differences in pricing compared to uncertainty in time preferences (see sections 3.2 to 3.4 for a sufficient discussion), I suppose now that $Z_t = 1$. For the curvature in the agent's preferences I assume the two conservative values $\gamma = 0.5$ and $EIS = 0.9$.

Figure 1.1 and 1.2 plot the consumption-to-wealth ratio (ψ_t, $\frac{\sigma_t^{(\psi)}}{\psi_t}$), the interest rate ($r_t$, $\sigma_t^{(r)}$) and the stock price ($\kappa_t^{(\beta)}$, $\frac{\sigma_t^{(P)}}{P_t}$, $E_t\left[\frac{dP_t + D_t dt - r_t P_t dt}{P_t}\right]$, $\frac{P_t}{Y_t}$) against the agent's

[9] One could also estimate the correlation between β_t and Y_t from the expression $Corr\left(\frac{\sigma_t^{(P)}}{P_t}, \sigma^{(Y)}\right) = \frac{\frac{\sigma_t^{(P)}}{P_t}\left(\sigma^{(Y)}\right)^T}{\sqrt{\frac{\sigma_t^{(P)}}{P_t}\left(\frac{\sigma_t^{(P)}}{P_t}\right)^T \sigma^{(Y)}\left(\sigma^{(Y)}\right)^T}} = \frac{\left(\sigma^{(Y)} - \frac{\sigma_t^{(\psi)}}{\psi_t}\right)\left(\sigma^{(Y)}\right)^T}{\sqrt{\left(\sigma^{(Y)} - \frac{\sigma_t^{(\psi)}}{\psi_t}\right)\left(\sigma^{(Y)} - \frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T \sigma^{(Y)}\left(\sigma^{(Y)}\right)^T}}$, assuming one knows the correlation between the stock price and aggregate consumption growth. If I assume $Corr\left(\frac{\sigma_t^{(P)}}{P_t}, \sigma^{(Y)}\right) = 0.25$ (and the parameterization in section 3.5), then $Corr(\beta_t, Y_t) = -0.175$.

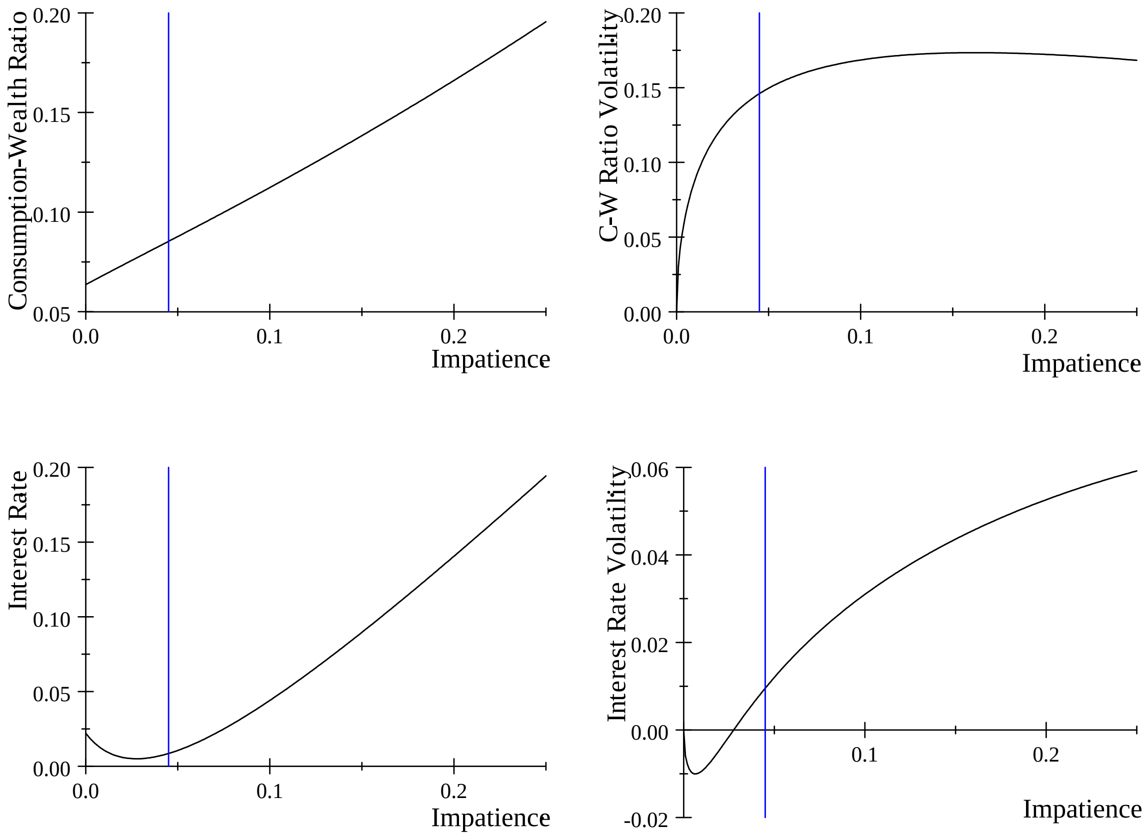

Figure 1.1: Black line: Dependence of the consumption-to-wealth ratio ψ_t (top-left panel), the conditional volatility in the consumption-to-wealth ratio $\frac{\sigma_t^{(\psi)}}{\psi_t}$ (top-right panel), the riskfree real interest rate r_t (bottom-left panel), and the conditional volatility in the interest rate (exposure of interest rate to changes in impatience) $\sigma_t^{(r)}$ (with $\sigma_t^{(r)}$ as defined in the appendix; a negative $\sigma_t^{(r)}$ means a that r_t is decreasing in β_t) (bottom-right panel) on the agent's impatience β_t (subjective time discount factor of future utility). The blue line indicates the unconditional expectation of the time discount factor ($\overline{\beta}$). The horizontal axis spans the interval between the bottom 0.1% and the top 99.9% percentiles of the stationary distribution of β_t.

subjective time discount rate (β_t). Although I solve the ODE (1.9) for the entire support (β_L, β_H) of β_t, the plots only display the results for $\beta_t \in (0, 0.2775)$, the interval between the bottom 0 and the top 99.9% percentiles of the unconditional distribution of β_t, and my discussion shall be limited to this smaller support.

In contrast to the numerical solutions (figure 1.1 and 1.2), table 1.2 presents the averages and standard deviations of estimated unconditional moments of the key variables from 100 Monte Carlo simulations of 10'000 years of weekly data; that is, for every simulation (10'000 years of weekly data) I estimate the 20 unconditional moments in table 1.2, and I report the average and standard deviation over the 100 estimates.

The consumption-to-wealth ratio is almost linear in the time discount rate and increases monotonically from 6.36% to 21.23% (top-left panel in figure 1.1). In the long run it is expected to be $\psi_t\left(\beta_t = \overline{\beta}\right) = 8.53\%$ which almost coincides with the estimated unconditional expected value $\hat{E}\left(\psi_t\right) = 8.55\%$ (table 1.2). The numerical results also show that $E_t\left[d\psi_t\right] < 0$ if and only if $\beta_t \in \left(\overline{\beta}, \beta_H\right)$, which confirms the mean-reversion property in process of ψ_t. Moreover, $\frac{\partial \psi_t}{\partial \beta_t} \in (0.479, 0.628)$, suggesting that the conditional variation in ψ_t is substantially lower than the conditional variation in β_t. Indeed the estimated unconditional volatility in the time discount rate (4.26%) is twice as large as the volatility in the consumption-to-wealth ratio (2.14%) (table 1.2). The top-right panel in figure 1.1 further presents the relation between the state variable β_t and the key quantity $\frac{\sigma_t^{(\psi)}}{\psi_t}$ (conditional volatility of percentage changes in ψ_t), which essentially describes the economic uncertainty introduced by shocks to time preferences and determines the precautionary savings motive, the stock price volatility and the equity premium. $\frac{\sigma_t^{(\psi)}}{\psi_t}$ is a hump-shaped function in β_t. The uncertainty is steeply sloping in the vicinity of the boundary β_L (and β_H) where $\sigma_t^{(\beta)}$ approaches zero as $\beta_t \to \beta_L$ (and $\beta_t \to \beta_H$) to ensure that $\beta_t \in (\beta_L, \beta_H)$ almost surely.

The risk-free real interest rate is a U-shaped function in the time discount rate (bottom-left panel in figure 1.1). It declines monotonically from 2.22% to 0.50% for $\beta_t \in (0, 0.028)$ ($\sigma_t^{(r)} < 0$), and is thereafter strictly increasing in β_t ($\sigma_t^{(r)} > 0$) and reaches its maximum of 22.40% at $\beta_t = 0.2775$. On the interval $(\beta_L, \beta_1) = (0, 0.028)$ there is a strong increase in uncertainty in the consumption-to-wealth ratio (top-right panel in figure 1.1) which gives rise to a rapid increase in precautionary savings and

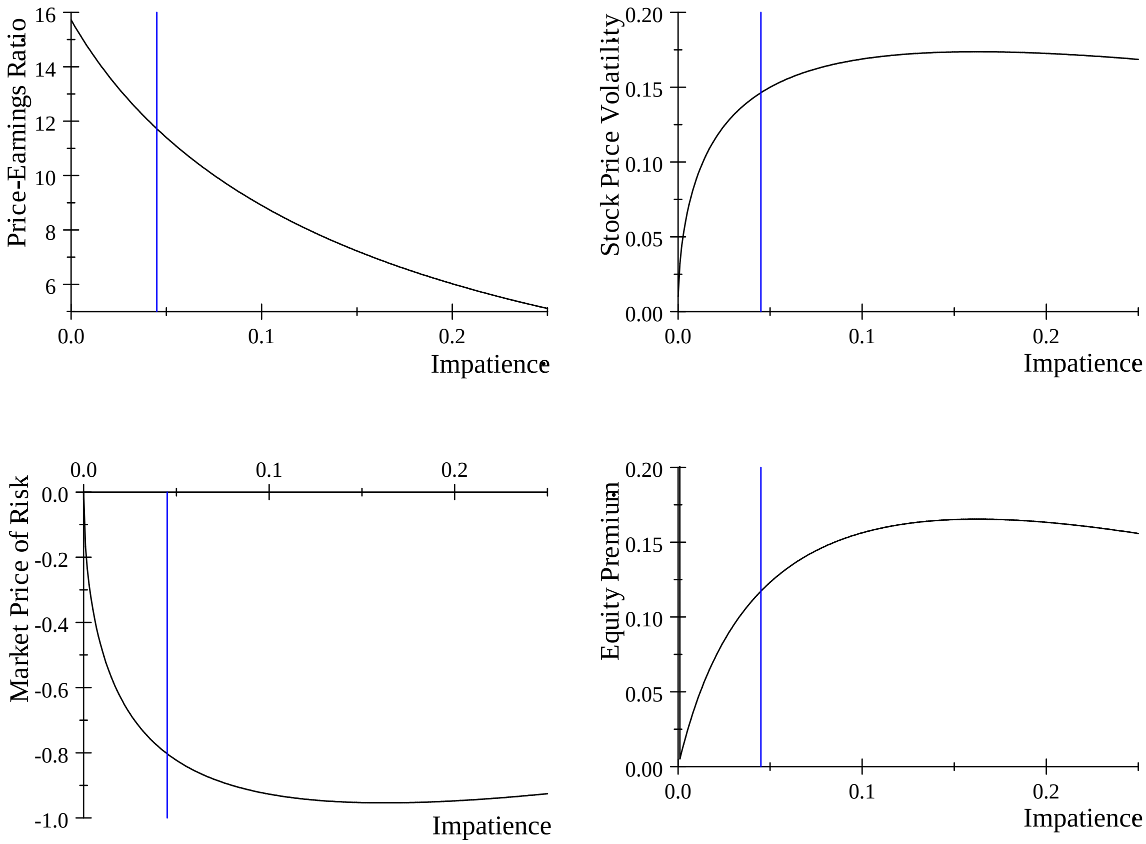

Figure 1.2: Dependence of the stock price-earnings ratio $\frac{P_t}{Y_t}$ (top-left panel), the conditional stock price volatility $\frac{\sigma_t^{(P)}}{P_t}$ (top-right panel), the market price of risk with respect to uncertainty in time preferences $\kappa_t^{(\beta)}$ (bottom-left panel), and the equity premium $E_t\left[\frac{dP_t+D_tdt-r_tP_tdt}{P_t}\right]$ (bottom-right panel) on the agent's impatience β_t (subjective time discount factor of future utility). The blue line indicates the unconditional expectation of the time discount factor $(\overline{\beta})$. The horizontal axis spans the interval between the bottom 0.1% and the top 99.9% percentiles of the stationary distribution of β_t.

results in a drop in the interest rate. The negative impact on the interest rate is large enough to dominate the positive effect due to the growing impatience of the agent. In turn, for $\beta_t > 0.028$ uncertainty does not increase by much and even starts to decrease for $\beta_t \geq 0.162$, and the positive effect on the interest rate (increasing impatience) dominates. Since r_t is not everywhere monotonically increasing in β_t, there exist some (few) circumstances where the interest rate process is temporarily expected to move away from its mean. However, in the long run the interest rate is always expected to revert to the level $r_t\left(\beta_t = \overline{\beta}\right) = 0.86\%$. Monte Carlo simulations show that risk-free bonds pay on average a real interest of 2.03% (table 1.2), which is higher than $r_t\left(\beta_t = \overline{\beta}\right)$ due to the convexity in the function $r_t\left(\beta_t\right)$.

The conditional volatility of the interest rate is hump-shaped on the interval $(\beta_L, \beta_1) = (0, 0.028)$ and strictly increasing in the agent's impatience (for $\beta_t \in (0.028, 0.2775)$). In the long run it is expected to revert to $\sigma_t^{(r)}\left(\beta_t = \overline{\beta}\right) = 0.96\%$ (bottom-right panel in figure 1.1). Simulations show that the unconditional volatility in the interest rate is much larger (2.56%), which is due to the substantial (unconditional) variation (1.57%) in the conditional interest rate volatility $\sigma_t^{(r)}$ (table 1.2).

The price-earnings ratio equals the inverse of the consumption-to-wealth ratio (equation (1.13)). It is monotonically decreasing in the agent's impatience and almost linear in $\frac{1}{\beta_t}$ (top-left panel in figure 1.2). Moreover, it follows a mean-reverting stochastic process with the long run level 11.72. The unconditional average is 12.27 (table 1.2).

The conditional stock price volatility $(\frac{\sigma_t^{(P)}}{P_t})$ strongly depends on the variation in time discounting. $\frac{\sigma_t^{(P)}}{P_t}$ is a linear function in $\frac{\sigma_t^{(\psi)}}{\psi_t}$, and the dependence on the state variable β_t is displayed in the top-right panel in figure 1.2. It follows a stationary stochastic process and is expected to revert to its the long run level of 14.63%. From simulations I estimate an unconditional mean and volatility in the conditional stock price volatility of 12.24% and 4.33%, while the estimated unconditional volatility in realized stock returns is 13.85% (table 1.2). Shocks to aggregate consumption growth merely generate a volatility of 1%, while risk in the time discount rate accounts for the remaining stock price volatility. Accordingly, if $\sigma^{(Y)}\left(\sigma^{(\beta)}\right)^T$ is small (I assume it is zero), the correlation between the stock market return and aggregate consumption growth is small (close to zero).

Unconditional Moments (Simulated Annual Data)

	Mean across Simulations	Std across Simulations
$\hat{E}\left[\beta_t\right]$	4.48%	0.13%
$\sqrt{\hat{E}\left[\left(\beta_t - \hat{E}\left[\beta_t\right]\right)^2\right]}$	4.26%	0.15%
$\hat{E}\left[\psi_t\right]$	8.55%	0.07%
$\sqrt{\hat{E}\left[\left(\psi_t - \hat{E}\left[\psi_t\right]\right)^2\right]}$	2.14%	0.08%
$\hat{E}\left[\sigma_t^{(\psi)}\right]$	1.11%	0.02%
$\sqrt{\hat{E}\left[\left(\sigma_t^{(\psi)} - \hat{E}\left[\sigma_t^{(\psi)}\right]\right)^2\right]}$	0.61%	0.01%
$\hat{E}\left[r_t\right]$	2.03%	0.07%
$\sqrt{\hat{E}\left[\left(r_t - \hat{E}\left[r_t\right]\right)^2\right]}$	2.56%	0.16%
$\hat{E}\left[\sigma_t^{(r)}\right]$	0.78%	0.05%
$\sqrt{\hat{E}\left[\left(\sigma_t^{(r)} - \hat{E}\left[\sigma_t^{(r)}\right]\right)^2\right]}$	1.57%	0.03%
$\hat{E}\left[\kappa_t^{(\beta)}\right]$	−66.56%	0.66%
$\sqrt{\hat{E}\left[\left(\kappa_t^{(\beta)} - \hat{E}\left[\kappa_t^{(\beta)}\right]\right)^2\right]}$	24.88%	0.31%
$\hat{E}\left[\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}\right]$	9.34%	0.14%
$\sqrt{\hat{E}\left[\left(\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t} - \hat{E}\left[\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}\right]\right)^2\right]}$	4.82%	0.04%
$\hat{E}\left[\frac{\sigma_t^{(P)}}{P_t}\right]$	12.24%	0.12%
$\sqrt{\hat{E}\left[\left(\frac{\sigma_t^{(P)}}{P_t} - \hat{E}\left[\frac{\sigma_t^{(P)}}{P_t}\right]\right)^2\right]}$	4.33%	0.05%
$\hat{E}\left[\frac{P_{t+1}-P_t+D_{t+1}}{P_t}\right]$	11.55%	0.08%
$\sqrt{\hat{E}\left[\left(\frac{P_{t+1}+D_{t+1}}{P_t} - \hat{E}\left[\frac{P_{t+1}+D_{t+1}}{P_t}\right]\right)^2\right]}$	13.85%	0.17%
$\hat{E}\left[\frac{P_t}{Y_t}\right]$	12.27	0.07
$\sqrt{\hat{E}\left[\left(\frac{P_t}{Y_t} - \hat{E}\left[\frac{P_t}{Y_t}\right]\right)^2\right]}$	2.40	0.04

Table 1.2: Estimation of unconditional moments of annual key quantities: average and standard deviation across 100 simulations of 10'000 years of weekly simulated data.

The market price of risk in the agent's time preferences ($\kappa_t^{(\beta)}$) is a linear function in $\frac{-\sigma_t^{(\psi)}}{\psi_t}$ and a U-shaped function in β_t (bottom-left panel in figure 1.2). It is sharply decreasing for small values in β_t as uncertainty in time discounting is rapidly increasing for β_t close to β_L. $\kappa_t^{(\beta)}$ is defined on the negative space - that is, marginal utility is increasing in the agent's impatience as discussed in section 3.4 (provided $EIS \notin \left(\min\left(\frac{1}{\gamma}, 1\right), \max\left(1, \frac{1}{\gamma}\right)\right)$). In the long run it is expected to revert to $\kappa_t^{(\beta)}\left(\beta_t = \overline{\beta}\right) = -80.28\%$. The unconditional mean and volatility in $\kappa_t^{(\beta)}$ are -66.56% and 24.88% (table 1.2). The market price of risk for uncertainty in time discounting is on average two orders of magnitude larger than the (constant) market price of risk for uncertainty in aggregate consumption growth ($\kappa_t^{(Y)} = \gamma\sigma^{(Y)} = 0.5\%$)!

The equity premium is a quadratic function in $\frac{\sigma_t^{(\psi)}}{\psi_t}$ and a hump-shaped function in β_t (bottom-right panel in figure 1.2). In the long run the equity premium is expected to revert to the level of 11.72%. The estimated unconditional mean and standard deviation in the equity premium are 9.34% and 4.82% (table 1.2). Risk in aggregate consumption growth is almost not compensated in financial markets; the equity premium due to aggregate consumption shocks is almost zero (0.005%). In contrast, uncertainty in the agent's time preferences generates essentially the entire equity premium. Moreover, all variation in the equity premium comes solely from changes in the agent's perceived uncertainty $\frac{\sigma_t^{(\psi)}}{\psi_t}$ due to changes in impatience.

In table 1.3 I report estimated unconditional correlations from 100 Monte Carlo simulations. For every simulation I generate a 10'000 year sample path for β and estimate the unconditional annual correlations between the quantities $\frac{P_t}{Y_t}$, ψ_t, r_t, $\sigma_t^{(r)}$, $\kappa_t^{(\beta)}$, $\frac{\sigma_t^{(P)}}{P_t}$, $\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}$, and $\frac{P_{t+1}-P_t+D_{t+1}}{P_t}$. I report the average over the 100 estimated correlation matrices in table 1.3.

I find a strong correlation between β_t and all the reported quantities since all the variables crucially depend on the agent's current impatience. The correlation between time discounting and the realized stock return over the subsequent year $(t, t+1)$ is weaker (than the other correlations) because $\frac{P_{t+1}-P_t+D_{t+1}}{P_t}$ depends only indirectly on β_t due to mean-reversion. Unfortunately, the representative agent's current subjective time discount rate is not observable in the data, and neither are the quantities ψ_t, r_t, $\sigma_t^{(r)}$, $\kappa_t^{(\beta)}$, $\frac{\sigma_t^{(P)}}{P_t}$, and $\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}$.

Unconditional Correlations (Simulated Annual Data)

	β_t	$\frac{P_t}{Y_t}$	ψ_t	r_t	$\sigma_t^{(r)}$	$\kappa_t^{(\beta)}$	$\frac{\sigma_t^{(P)}}{P_t}$	$\frac{\mu_t^{(P)}+D_{t+1}-r_tP_t}{P_t}$
$\frac{P_t}{Y_t}$	-0.96							
ψ_t	1.00	-0.96						
r_t	0.87	-0.71	0.88					
$\sigma_t^{(r)}$	0.97	-0.96	0.96	0.80				
$\kappa_t^{(\beta)}$	-0.75	0.88	-0.73	-0.33	-0.72			
$\frac{\sigma_t^{(P)}}{P_t}$	0.76	-0.89	0.74	0.35	0.74	-1.00		
$\frac{\mu_t^{(P)}+D_{t+1}-r_tP_t}{P_t}$	0.85	-0.96	0.83	0.47	0.86	-0.97	0.97	
$\frac{P_{t+1}-P_t+D_{t+1}}{P_t}$	0.45	-0.45	0.44	0.35	0.44	-0.38	0.38	0.41

Table 1.3: Estimation of unconditional correlations between annual key quantities: average across 100 simulations of 10'000 years of simulated data.

However, from the above discussion I know that there is a strong relation between the price-earnings ratio and the agent's subjective time discount rate ($\frac{P_t}{Y_t}$ depends almost linearly on $\frac{1}{\beta_t}$). Indeed the state variable β_t becomes (almost) observable through the price-earnings ratio. Therefore, particularly striking from an empirical perspective is the strong correlation between the price-earnings ratio - which is an observable variable in the data - and literally all the key quantities in my model. The price-earnings ratio reveals much information about the unobservable variables ψ_t, r_t, $\sigma_t^{(r)}$, $\kappa_t^{(\beta)}$, $\frac{\sigma_t^{(P)}}{P_t}$, and $\frac{\mu_t^{(P)}+D_t-r_tP_t}{P_t}$. Moreover, the current price-earnings ratio is able to predict future (realized) stock returns.

Most of the (postulated) relations in table 1.3 are almost impossible to test empirically because only noisy estimates are available of the true quantities. However, consistent with the results in table 1.3 empirical work by Campbell and Ammer (1993) suggests that there is a rather weak correlation between realized stock returns (and/or equity premia) and the interest rate. Keim and Stambaugh (1986), Campbell and Shiller (1988) and Fama and French (1988a) explore the power of the price-earnings ratio to forecast future stock returns and find evidence which is consistent with table 1.3.

Finally, table 1.3 makes clear that ex-post realized stock returns are not a good proxy for expected returns - the unconditional correlation is only 0.414. Empirical estimates of correlations between expected stock returns - using ex-post realized returns as a proxy - and quantities such as the price-earnings ratio, the conditional stock price volatility, the real interest rate or the conditional interest rate volatility will be severely biased towards zero (even if the econometrician was able to perfectly measure the real interest rate or the conditional volatilities in the stock price or interest rate).

Auto-Correlation in Realized Stock Returns $\frac{P_t - P_{t-1} + D_t}{P_{t-1}}$

	Holding Period					
	10 years	5 years	1 year	1 month	1 week	1 day
Mean across Simulations	-0.089	-0.152	-0.050	-0.005	-0.001	-0.000
Std across Simulations	0.036	0.022	0.011	0.004	0.002	0.001

Table 1.4: Estimation of auto-correlations in realized daily, weekly, monthly, annual, and 5-year stock returns: average across 100 simulations of 10'000 years of simulated data. Notice that the power of the test declines quickly if sample paths shorter than 10'000 years are simulated and used to estimate the auto-correlation.

In table 1.4 I present estimated auto-correlations in simulated stock returns for various holding periods. I find a negative auto-correlation in stock returns at long horizons; though for very long holding periods the auto-correlation starts to disappear again. For short holding periods the auto-correlation is small and does not significantly differ from zero. The negative auto-correlation is induced by the mean-reversion in the price-earnings ratio process as conjectured in the discussion on the price-earnings ratio in section 3.3. The pattern in table 1.4 is consistent with empirical evidence provided by Fama and French (1988b). My model illustrates that stock returns can be auto-correlated if investors are perfectly rational and informed, there are no frictions, and asset prices are efficient in the sense that they incorporate all available information in the economy (prices are forward looking).

1.4 Conclusion

Traditional consumption-based asset pricing models miss several empirical stylized facts. In contrast to the literature, I argue that the aggregate consumption growth process is merely of secondary importance while other (unrelated) risk sources are the main driving forces in asset pricing.

I show that shocks to the representative agent's subjective time discount rate of future utility have first order implications for asset pricing, while risk in the aggregate consumption growth process does not essentially matter. For illustrative purposes I suppose that shocks to time preferences are independent of the consumption growth process. This assumption helps to gain a better understanding of the newly introduced pricing channel and to demonstrate the fundamental differences to the pricing channel in related literature, which crucially depends on risk in the aggregate consumption growth process. Although I restrict my analysis to uncertainty in time preferences, the same modeling tools and asset pricing channel can be applied in the context of other risk sources. Important is that the agent has recursive preferences and the chosen risk source has implications for the representative agent's consumption-to-wealth ratio.

My model is able to match the data well and provides answers to various challenges in (empirical) asset pricing literature. Uncertainty in time discounting generates a large equity premium and stock price volatility. The risk-free real interest rate is low and has a moderate variation. The market price of risk, equity premium, conditional stock price volatility, interest rate and its conditional volatility follow mean-reverting stochastic processes. The price-earnings ratio has power to predict future stock market excess returns, and realized stock returns are negatively auto-correlated at long horizons. The correlation between stock returns and the real interest rate (or its conditional volatility) is low, and so is the correlation between stock returns and aggregate consumption growth. Because the consumption growth process has negligible asset pricing implications, I can simply choose a process to match aggregate consumption growth data.

The most important empirical implication of the model is that the price-earnings ratio, which is an observable variable in the data, is highly correlated with the unobservable current time discount rate of the representative agent. Accordingly, the price-earnings ratio reveals much information about other unobservable key quantities in finance such as the consumption-to-wealth ratio, the real interest rate level and its conditional volatility, the conditional stock price volatility, the market price of risk and the equity premium.

1.5 Appendix

Proof of Proposition 1.1 & 1.2. Following Duffie and Skiadas (1994, Theorem 2), the Gateau derivative (directional derivative) of the utility function at time s at $\bar{c}$ in the direction x is

$$\begin{aligned}\nabla V_s(\bar{c};x) &\equiv \lim_{\alpha\rightarrow 0}\frac{V_s(\bar{c}+\alpha x)-V_s(\bar{c})}{\alpha} \\ &= E_s\left[\int_s^{\infty} e^{\int_s^t \frac{\partial}{\partial V_u} f(\bar{c}_u,V_u,Z_u,\beta_u)du}\frac{\partial}{\partial c_t} f(\bar{c}_t,V_t,Z_t,\beta_t)\,xdt\right] \\ &= E_s\left[\int_s^{\infty} R_t x dt\right]\end{aligned} \tag{1.24}$$

The Riesz representation process R_t is defined as

$$R_t = e^{\int_s^t \frac{\partial}{\partial V_u} f(\bar{c}_u,V_u,Z_u,\beta_u)du}\frac{\partial}{\partial c_t} f(\bar{c}_t,V_t,Z_t,\beta_t) \tag{1.25}$$

Optimality implies (assuming that the optimal consumption plan c^* is in the interior)

$$\nabla V_s(c^*;[c-c^*]) = 0 \tag{1.26}$$

for all admissible consumption plans $c \in \Im$. Since $\left(c^s - c^{s^*}\right)$ spans the set of all marketable cash flows M,

$$\nabla V_s(c^*;x) = 0 \tag{1.27}$$

holds for all marketable cash flows $x \in M$. This implies that the Riesz representation process is a multiple of a SDF π,

$$R_t = \eta\pi_t \tag{1.28}$$

for some constant η. Since markets are dynamically complete, the found pricing kernel is unique. I can solve for the optimal consumption plan by plugging in the expression for the Riesz representation process (from now I drop the notation of indicating the optimum by a star)

$$\begin{aligned}
\eta\pi_t &= e^{\int_s^t \frac{\partial}{\partial V_u} f(c_u,V_u,Z_u,\beta_u)du} \frac{\partial}{\partial c_t} f\left(c_t, V_t, Z_t, \beta_t\right) && (1.29)\\
c_t &= \eta^{-\frac{1}{1-\rho}} Z_t^{\frac{1}{1-\rho}} e^{\frac{1}{1-\rho}\int_{-\infty}^t \frac{\partial}{\partial V_u} f(c_u,V_u,Z_u,\beta_u)du} \left[(1-\gamma) V_t\right]^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} \pi_t^{-\frac{1}{1-\rho}} && (1.30)\\
\frac{c_t}{c_s} &= \left(\frac{Z_t}{Z_s}\right)^{\frac{1}{1-\rho}} e^{\frac{1}{1-\rho}\int_s^t \frac{\partial}{\partial V_u} f(c_u,V_u,Z_u,\beta_u)du} \left(\frac{V_t}{V_s}\right)^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} \left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{1-\rho}} && (1.31)
\end{aligned}$$

Using dynamic programming to solve the utility maximization problem, I can state the Hamilton-Jacobi-Bellman equation as follows

$$0 = \sup_{\{c_t, X_t\}} \left\{ f\left(c_t, V\left(W, Z, \beta, t\right), , Z_t, \beta_t\right) dt + E_t\left[dV\left(W, Z, \beta, t\right)\right] \right\} \tag{1.32}$$

with W indicating the financial wealth. The first order condition with respect to optimal consumption is given by

$$\frac{\partial}{\partial c_t} f\left(c_t, V\left(W, Z, \beta, t\right), Z_t, \beta_t\right) = \frac{\partial}{\partial W} V\left(W, Z, \beta, t\right) \tag{1.33}$$

I make the following conjecture for the value function

$$V\left(W, Z, \beta, t\right) = \frac{\left(W_t\right)^{1-\gamma}}{1-\gamma} Z_t^{\frac{1-\gamma}{\rho}} \psi_t\left(Z_t, \beta_t\right)^{-\frac{(1-\gamma)(1-\rho)}{\rho}} \tag{1.34}$$

Plugging the conjectured value function into the FOC yields

$$c_t = W_t \psi_t \tag{1.35}$$

Plugging back into the conjectured value function and solving for c_t, allows us to rewrite the expression for optimal consumption as

$$c_t = \left[(1-\gamma) V_t\right]^{\frac{1}{(1-\gamma)}} \psi_t^{\frac{1}{\rho}} Z_t^{-\frac{1}{\rho}} \tag{1.36}$$

Combining this with the expression obtained from the martingale approach (equation (1.30)) and solving for the value function leaves us with

$$V_t = \frac{1}{1-\gamma}\eta^{-\frac{1-\gamma}{\gamma}} e^{\frac{1-\gamma}{\gamma}\int_{-\infty}^{t}\frac{\partial}{\partial V_u}f(c_u,V_u,Z_u,\beta_u)du} Z_t^{\frac{1-\gamma}{\gamma\rho}} \psi_t^{-\frac{(1-\gamma)(1-\rho)}{\gamma\rho}} \pi_t^{-\frac{1-\gamma}{\gamma}} \tag{1.37}$$

$\frac{\partial}{\partial V_t} f\left(c_t, V_t\right)$ is a function of only β, and it is a Markov process,

$$\begin{aligned} \frac{\partial}{\partial V_t} f\left(c_t, V_t, Z_t, \beta_t\right) &= \frac{1-\gamma-\rho}{\rho} Z_t c_t^{\rho}\left[\left(1-\gamma\right)V_t\right]^{\frac{-\rho}{1-\gamma}} - \frac{1-\gamma}{\rho}\beta_t \\ &= \frac{1-\gamma-\rho}{\rho}\psi_t - \frac{1-\gamma}{\rho}\beta_t \end{aligned} \tag{1.38}$$

Solving for optimal consumption yields

$$c_t = \eta^{-\frac{1}{\gamma}} e^{\frac{1}{\gamma}\int_{-\infty}^{t}\frac{1-\gamma-\rho}{\rho}\psi_u - \frac{1-\gamma}{\rho}\beta_u du} Z_t^{\frac{1-\gamma}{\gamma\rho}} \psi_t^{-\frac{1-\gamma-\rho}{\gamma\rho}} \pi_t^{-\frac{1}{\gamma}} \tag{1.39}$$

$$\frac{c_t}{c_s} = e^{\frac{1}{\gamma}\int_{s}^{t}\frac{1-\gamma-\rho}{\rho}\psi_u - \frac{1-\gamma}{\rho}\beta_u du} \left(\frac{Z_t}{Z_s}\right)^{\frac{1-\gamma}{\gamma\rho}} \left(\frac{\psi_t}{\psi_s}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{\gamma}} \tag{1.40}$$

The dynamics of the utility function are given by

$$\begin{aligned} \frac{dV_t}{V_t} = & \frac{1-\gamma}{\gamma}\frac{\partial}{\partial V_t} f\left(c_t, V_t, Z_t, \beta_t\right) dt + \frac{1-\gamma}{\gamma\rho}\frac{dZ_t}{Z_t} \\ & + \frac{\left(1-\gamma\right)\left(1-\gamma-\gamma\rho\right)}{2\gamma^2\rho^2}\left(\frac{dZ_t}{Z_t}\right)^2 - \frac{\left(1-\gamma\right)^2}{\gamma^2\rho}\frac{dZ_t}{Z_t}\frac{d\pi_t}{\pi_t} \\ & - \frac{\left(1-\gamma\right)^2\left(1-\rho\right)}{\gamma^2\rho^2}\frac{dZ_t}{Z_t}\frac{d\psi_t}{\psi_t} - \frac{\left(1-\gamma\right)\left(1-\rho\right)}{\gamma\rho}\frac{d\psi_t}{\psi_t} \\ & + \frac{\left(1-\gamma\right)^2\left(1-\rho\right)}{\gamma^2\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} - \frac{1-\gamma}{\gamma}\frac{d\pi_t}{\pi_t} + \frac{1-\gamma}{2\gamma^2}\left(\frac{d\pi_t}{\pi_t}\right)^2 \\ & + \frac{\left(1-\gamma-\rho+2\gamma\rho\right)\left(1-\gamma\right)\left(1-\rho\right)}{2\gamma^2\rho^2}\left(\frac{d\psi_t}{\psi_t}\right)^2 \end{aligned} \tag{1.41}$$

By definition the drift term of the value function equals $-f\left(c_t, V_t, Z_t, \beta_t\right)dt$, which boils down to a ODE determining the function $\psi_t\left(\beta\right)$ and at the same time verifies my conjecture about the value function (given a solution for the stated ODE exists)

$$
\begin{aligned}
0 \quad = \quad & \frac{1-\rho}{\rho}\psi_t - \frac{\beta_t}{\rho} + \frac{1}{\rho}\frac{1}{dt}E_t\left[\frac{dZ_t}{Z_t}\right] + \frac{1-\gamma-\gamma\rho}{2\gamma\rho^2}\frac{1}{dt}\left(\frac{dZ_t}{Z_t}\right)^2 \\
& -\frac{(1-\gamma)(1-\rho)}{\gamma\rho^2}\frac{1}{dt}\frac{dZ_t}{Z_t}\frac{d\psi_t}{\psi_t} - \frac{1-\gamma}{\gamma\rho}\frac{1}{dt}\frac{dZ_t}{Z_t}\frac{d\pi_t}{\pi_t} \\
& -\frac{1-\rho}{\rho}\frac{1}{dt}E_t\left[\frac{d\psi_t}{\psi_t}\right] - \frac{1}{dt}E_t\left[\frac{d\pi_t}{\pi_t}\right] \\
& +\frac{(1-\gamma)(1-\rho)}{\gamma\rho}\frac{1}{dt}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} + \frac{1}{2\gamma}\frac{1}{dt}\left(\frac{d\pi_t}{\pi_t}\right)^2 \\
& +\frac{(1-\gamma-\rho+2\gamma\rho)(1-\rho)}{2\gamma\rho^2}\frac{1}{dt}\left(\frac{d\psi_t}{\psi_t}\right)^2
\end{aligned}
\tag{1.42}
$$

The last step is equivalent to solving the HJB equation. I can use expression (1.39) to get the dynamics of the optimal consumption process

$$
\begin{aligned}
\frac{dc_t}{c_t} \quad = \quad & \frac{1-\gamma-\rho}{\gamma\rho}\psi_t dt - \frac{1-\gamma}{\gamma\rho}\beta_t dt + \frac{1-\gamma}{\gamma\rho}\frac{dZ_t}{Z_t} - \frac{1-\gamma-\rho}{\gamma\rho}\frac{d\psi_t}{\psi_t} \\
& -\frac{1}{\gamma}\frac{d\pi_t}{\pi_t} + \frac{(1-\gamma)(1-\gamma-\gamma\rho)}{2\gamma^2\rho^2}\left(\frac{dZ_t}{Z_t}\right)^2 + \frac{1+\gamma}{2\gamma^2}\left(\frac{d\pi_t}{\pi_t}\right)^2 \\
& +\frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma^2\rho^2}\left(\frac{d\psi_t}{\psi_t}\right)^2 + \frac{1-\gamma-\rho}{\gamma^2\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} \\
& -\frac{(1-\gamma-\rho)(1-\gamma)}{\gamma^2\rho^2}\frac{dZ_t}{Z_t}\frac{d\psi_t}{\psi_t} - \frac{1-\gamma}{\gamma^2\rho}\frac{dZ_t}{Z_t}\frac{d\pi_t}{\pi_t}
\end{aligned}
\tag{1.43}
$$

Setting equation (1.43) and using the market clearing condition in the consumption goods market $dY_t = dc_t$ and solving for the SDF

$$
\begin{aligned}
\frac{d\pi_t}{\pi_t} \quad = \quad & -\gamma\frac{dY_t}{Y_t} + \frac{1-\gamma-\rho}{\rho}\psi_t dt - \frac{1-\gamma}{\rho}\beta_t dt + \frac{1-\gamma}{\rho}\frac{dZ_t}{Z_t} \\
& -\frac{1-\gamma-\rho}{\rho}\frac{d\psi_t}{\psi_t} + \frac{(1-\gamma)(1-\gamma-\gamma\rho)}{2\gamma\rho^2}\left(\frac{dZ_t}{Z_t}\right)^2 \\
& +\frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma\rho^2}\left(\frac{d\psi_t}{\psi_t}\right)^2 + \frac{1+\gamma}{2\gamma}\left(\frac{d\pi_t}{\pi_t}\right)^2 \\
& -\frac{1-\gamma}{\gamma\rho}\frac{dZ_t}{Z_t}\frac{d\pi_t}{\pi_t} + \frac{1-\gamma-\rho}{\gamma\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} \\
& -\frac{(1-\gamma-\rho)(1-\gamma)}{\gamma\rho^2}\frac{dZ_t}{Z_t}\frac{d\psi_t}{\psi_t}
\end{aligned}
\tag{1.44}
$$

Plugging into equation (1.42) yields an ODE that determines $\psi_t(\beta)$

$$
\begin{aligned}
0 &= \frac{1-\rho}{\rho}\psi_t - \frac{\beta_t}{\rho} + r_t - \frac{1-\rho}{\rho}\frac{\mu_t^{(\psi)}}{\psi_t} + \frac{\mu_t^{(Z)}}{\rho Z_t} + \frac{\gamma}{2}\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T \qquad (1.45)\\
&\quad -\frac{(1-\gamma)(1-\rho)}{\rho^2}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(Z)}}{Z_t}\right)^T + \frac{1-\gamma-\rho}{2\rho^2}\frac{\sigma_t^{(Z)}}{Z_t}\left(\frac{\sigma_t^{(Z)}}{Z_t}\right)^T\\
&\quad -\gamma\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T + \frac{1-\gamma-\rho+2\gamma\rho}{2\rho^2}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T
\end{aligned}
$$

or

$$
\begin{aligned}
\psi_t &= \beta_t - \rho\frac{1}{dt}E_t\left[\frac{dY_t}{Y_t}\right] + \frac{\mu_t^{(\psi)}}{\psi_t} \qquad (1.46)\\
&\quad +\frac{\gamma\rho}{2}\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T - \frac{1-\gamma+\rho}{2\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T\\
&\quad -\frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\left(\sigma^{(Z)}\right)^T + (1-\gamma)\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T\\
&\quad -(1-\gamma)\sigma^{(Y)}\left(\sigma^{(Z)}\right)^T + \frac{1-\gamma}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\sigma^{(Z)}\right)^T
\end{aligned}
$$

with

$$
\begin{aligned}
\mu_t^{(\psi)} &= \frac{\partial\psi_t}{\partial\beta_t}\mu_t^{(\beta)} + \frac{1}{2}\frac{\partial^2\psi_t}{\partial\beta_t^2}\sigma_t^{(\beta)}\sigma_t^{(\beta)^T} \qquad (1.47)\\
\sigma_t^{(\psi)} &= \frac{\partial\psi_t}{\partial\beta_t}\sigma_t^{(\beta)} \qquad (1.48)
\end{aligned}
$$

which proves Proposition 1.1. Plugging back into equation (1.44), Proposition 1.2 follows from the definition of the SDF $\left(\frac{d\pi_t}{\pi_t} = -r_t dt - \kappa_t d\widetilde{W}_t\right)$

$$
\begin{aligned}
r_t &= \beta_t + (1-\rho)\mu^{(Y)} - \frac{\gamma(2-\rho)}{2}\sigma^{(Y)}\sigma^{(Y)^T} \qquad (1.49)\\
&\quad +\frac{1-\gamma-\rho}{2\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T - \frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\left(\sigma^{(Z)}\right)^T\\
&\quad -\frac{1-\gamma-\rho}{\rho}\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T + \frac{(1-\gamma)(1-\rho)}{\rho}\sigma^{(Y)}\left(\sigma^{(Z)}\right)^T
\end{aligned}
$$

$$\begin{aligned}
\kappa_t &= \kappa_t^{(Y)} + \kappa_t^{(Z)} + \kappa_t^{(\beta)} \qquad (1.50)\\
\kappa_t^{(Y)} &= \gamma\sigma^{(Y)}\\
\kappa_t^{(Z)} &= -\frac{1-\gamma}{\rho}\sigma^{(Z)}\\
\kappa_t^{(\beta)} &= \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}
\end{aligned}$$

with the dynamics

$$\begin{aligned}
dr_t &= \mu_t^{(r)}dt + \sigma_t^{(r)}d\widetilde{W}_t \qquad (1.51)\\
&= d\beta_t + \frac{1-\gamma-\rho}{2\rho}\sigma^{(\beta)}\left(\sigma^{(\beta)}\right)^T\frac{d\left(\varphi_t^2\right)}{\varphi_t^2} - \frac{1-\gamma-\rho}{\rho}\sigma^{(Y)}\left(\sigma^{(\beta)}\right)^T\frac{d\varphi_t}{\varphi_t}\\
d\kappa_t &= \mu_t^{(\kappa)}dt + \sigma_t^{(\kappa)}d\widetilde{W}_t \qquad (1.52)\\
&= \frac{1-\gamma-\rho}{\rho}\sigma^{(\beta)}\frac{d\varphi_t}{\varphi_t}
\end{aligned}$$

where

$$\begin{aligned}
\frac{d\varphi_t}{\varphi_t} &= \left(\frac{\psi_t\frac{\partial^2\psi}{\partial\beta^2}-\frac{\partial\psi}{\partial\beta}}{\psi_t\frac{\partial\psi}{\partial\beta}} + \frac{(\beta_H-\beta_t)-(\beta_t-\beta_L)}{2\left(\beta_H-\beta_t\right)\left(\beta_t-\beta_L\right)}\right)d\beta_t \qquad (1.53)\\
&\quad + \left(\begin{array}{c}\frac{\psi_t^2\frac{\partial^3\psi}{\partial\beta^3}-2\left(\psi_t\frac{\partial^2\psi}{\partial\beta^2}-\frac{\partial\psi}{\partial\beta}\right)}{\psi_t^2\frac{\partial\psi}{\partial\beta}} - \frac{(\beta_H-\beta_L)^2}{4(\beta_t-\beta_L)^2(\beta_H-\beta_t)^2}\\ +\frac{\left(\psi_t\frac{\partial^2\psi}{\partial\beta^2}-\frac{\partial\psi}{\partial\beta}\right)((\beta_H-\beta_t)-(\beta_t-\beta_L))}{\psi_t\frac{\partial\psi}{\partial\beta}(\beta_t-\beta_L)(\beta_H-\beta_t)}\end{array}\right)d\beta_t\left(d\beta_t\right)^T\\
\frac{d\left(\varphi_t^2\right)}{\varphi_t^2} &= 2\frac{d\varphi_t}{\varphi_t} + \frac{d\varphi_t}{\varphi_t}\left(\frac{d\varphi_t}{\varphi_t}\right)^T \qquad (1.54)
\end{aligned}$$

Finally, using equation (1.17) together with equation (1.50) the equity premium is given by

$$\begin{aligned}
E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt &= \gamma\sigma^{(Y)}\left(\sigma^{(Y)}\right)^T - \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^T \qquad (1.55)\\
&\quad +\frac{1-\gamma-\rho-\gamma\rho}{\rho}\sigma^{(Y)}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)\\
&\quad -\frac{1-\gamma}{\rho}\left(\sigma^{(Y)}-\frac{\sigma_t^{(\psi)}}{\psi_t}\right)\left(\sigma^{(Z)}\right)^T
\end{aligned}$$

■

Proof of Lemma 1.1. Suppose $\frac{\overline{\beta}-\rho\mu^{(Y)}+\frac{\gamma\rho}{2}\sigma^{(Y)}\sigma^{(Y)^T}-\frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\sigma^{(Z)^T}-(1-\gamma)\sigma^{(Y)}\sigma^{(Z)^T}}{m^{(\beta)}} > 0$. For $\sigma^{(\beta)} \to 0$ and $(\beta_H - \beta_L) < \infty$ the ODE (1.9) becomes

$$0 = -\psi_t + \beta_t + \frac{\frac{\partial \psi_t}{\partial \beta_t} m^{(\beta)} \left(\overline{\beta} - \beta_t\right)}{\psi_t} + C_1 \tag{1.56}$$

with $C_1 = -\rho\mu^{(Y)} + \frac{\gamma\rho}{2}\sigma^{(Y)}\sigma^{(Y)^T} - \frac{1-\gamma-\rho}{2\rho}\sigma^{(Z)}\sigma^{(Z)^T} - (1-\gamma)\,\sigma^{(Y)}\sigma^{(Z)^T}$. The solution of the ODE $\psi_t(\beta)$ is $\forall \beta_t \in \left(\beta_L, \overline{\beta}\right)$

$$\psi_t(\beta_t) = e^{\frac{\beta_t}{m^{(\beta)}}} \left(\overline{\beta} - \beta_t\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}} \frac{1}{C_2 + \int_{\beta_t}^{\overline{\beta}} e^{\frac{h}{m^{(\beta)}}} \frac{\left(\overline{\beta}-h\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}}}{m^{(\beta)}\left(\overline{\beta}-h\right)} dh} \tag{1.57}$$

and $\forall \beta_t \in \left(\overline{\beta}, \beta_H\right)$

$$\psi_t(\beta_t) = e^{\frac{\beta_t}{m^{(\beta)}}} \left(\beta_t - \overline{\beta}\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}} \frac{1}{C_2 + \int_{\overline{\beta}}^{\beta_t} e^{\frac{h}{m^{(\beta)}}} \frac{\left(h-\overline{\beta}\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}}}{m^{(\beta)}\left(h-\overline{\beta}\right)} dh} \tag{1.58}$$

with some constant C_2. Note that for $C_2 = 0$, $\psi_t\left(\beta_t = \overline{\beta}\right) = \overline{\beta} + C_1$. I show this using L'Hopital's rule as follows

$$\begin{aligned} \psi_t\left(\beta_t = \overline{\beta}\right) &= \lim_{\beta_t \downarrow \overline{\beta}} \psi_t(\beta_t) = \lim_{\beta_t \uparrow \overline{\beta}} \psi_t(\beta_t) \\ &= \lim_{\beta_t \uparrow \overline{\beta}} \frac{\frac{\partial\left(e^{\frac{\beta_t}{m^{(\beta)}}}\left(\overline{\beta}-\beta_t\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}}\right)}{\partial \beta_t}}{\frac{\partial\left(C_2 - \int_{\overline{\beta}}^{\beta_t} e^{\frac{h}{m^{(\beta)}}} \frac{\left(\overline{\beta}-h\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}}}{m^{(\beta)}\left(\overline{\beta}-h\right)} dh\right)}{\partial \beta_t}} \\ &= \lim_{\beta_t \uparrow \overline{\beta}} \frac{\frac{1}{m^{(\beta)}} e^{\frac{\beta_t}{m^{(\beta)}}} \left(\overline{\beta} - \beta_t\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}} - \frac{\overline{\beta}+C_1}{m^{(\beta)}} e^{\frac{\beta_t}{m^{(\beta)}}} \left(\overline{\beta} - \beta_t\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}-1}}{-e^{\frac{\beta_t}{m^{(\beta)}}} \frac{\left(\overline{\beta}-\beta_t\right)^{\frac{\overline{\beta}+C_1}{m^{(\beta)}}}}{m^{(\beta)}\left(\overline{\beta}-\beta_t\right)}} \\ &= \overline{\beta} + C_1 \end{aligned} \tag{1.59}$$

And for $C_2 \neq 0$, $\psi_t\left(\beta_t = \overline{\beta}\right) = 0$, which is not an economically meaningful quantity. Moreover, given the mean-reversion property of the process β, I know that $\lim_{t\to\infty} \beta_t = \overline{\beta}$, and β_t is absorbed at $\overline{\beta}$ as soon as β_t reaches $\overline{\beta}$ (since $\sigma^{(\beta)} \to 0$). At the point

$\beta_t = \overline{\beta}$, the model coincides with a standard Lucas (1978) economy with constant time preferences $\overline{\beta}$, for which the solution of the consumption-to-wealth ratio is well-known, that is $\psi = \overline{\beta} + C_1$. Note that for $C_2 = 0$, the function $\psi_t(\beta_t)$ is continuous everywhere $(\forall \beta_t \in (\beta_L, \beta_H))$. Next, I solve equation (1.56) for $\frac{\partial \psi_t}{\partial \beta_t}$

$$\frac{\partial \psi_t}{\partial \beta_t} = (\psi_t - (\beta_t + C_1)) \frac{\psi_t}{m^{(\beta)} \left(\overline{\beta} - \beta_t\right)} \tag{1.60}$$

$\frac{\partial \psi_t}{\partial \beta_t} > 0$ iff $\psi_t(\beta_t) > (<) \, \beta_1 + C_1$ for $\beta_t < (>) \, \overline{\beta}$, and $\frac{\partial \psi_t}{\partial \beta_t} < 0$ iff $\psi_t(\beta_t) < (>) \, \beta_t + C_1$ for $\beta_t < (>) \, \overline{\beta}$. Suppose now that $\psi_t\left(\widehat{\beta}_t\right) \leq \widehat{\beta}_t + C_1$ for some $\widehat{\beta}_t \in \left(\beta_L, \overline{\beta}\right)$. Then since $\frac{\partial \psi_t}{\partial \beta_t}|_{\beta_t = \widehat{\beta}_t} \leq 0$ and thus,

$$\begin{aligned} \psi_t\left(\widehat{\beta}_t + \varepsilon\right) &\leq \psi_t\left(\widehat{\beta}_t\right) \\ &\leq \widehat{\beta}_t + C_1 \\ &< \widehat{\beta}_t + \varepsilon + C_1 \end{aligned} \tag{1.61}$$

for $\varepsilon \downarrow 0$, it must be (by iteration) that $\psi_t\left(\widehat{\widehat{\beta}}_t\right) < \widehat{\widehat{\beta}}_t + C_1$, $\forall \widehat{\widehat{\beta}}_t \in \left(\widehat{\beta}_t, \overline{\beta}\right)$, and $\lim_{\beta_t \uparrow \overline{\beta}} \psi_t(\beta_t) < \overline{\beta} + C_1$, a contradiction to $\psi_t\left(\beta_t = \overline{\beta}\right) = \overline{\beta} + C_1$ (in combination with continuity of $\psi_t(\beta_t)$). Suppose next that $\psi_t\left(\widehat{\beta}_t\right) \geq \overline{\beta} + C_1$ for some $\widehat{\beta}_t \in \left(\beta_L, \overline{\beta}\right)$. Then since $\frac{\partial \psi_t}{\partial \beta_t}|_{\beta_t = \widehat{\beta}_t} > 0$ and thus,

$$\begin{aligned} \psi_t\left(\widehat{\beta}_t + \varepsilon\right) &> \psi_t\left(\widehat{\beta}_t\right) \\ &\geq \overline{\beta} + C_1 \end{aligned} \tag{1.62}$$

for $\varepsilon \downarrow 0$, it must be (by iteration) that $\psi_t\left(\widehat{\widehat{\beta}}_t\right) > \overline{\beta} + C_1$, $\forall \widehat{\widehat{\beta}}_t \in \left(\widehat{\beta}_t, \overline{\beta}\right)$, and $\lim_{\beta_t \uparrow \overline{\beta}} \psi_t(\beta_t) > \overline{\beta} + C_1$, a contradiction to $\psi_t\left(\beta_t = \overline{\beta}\right) = \overline{\beta} + C_1$ (in combination with continuity of $\psi_t(\beta_t)$). It follows that (in order to attain $\psi_t\left(\overline{\beta}\right) = \lim_{\beta_t \uparrow \overline{\beta}} \psi_t(\beta_t) = \overline{\beta} + C_1$) it must be that $\psi_t(\beta_t) \in \left(\beta_t + C_1, \overline{\beta} + C_1\right)$, $\forall \beta_t \in \left(\beta_L, \overline{\beta}\right)$, and thus, $\frac{\partial \psi_t}{\partial \beta_t} > 0$, $\forall \beta_t \in \left(\beta_L, \overline{\beta}\right)$. Accordingly, $\psi_t(\beta_t)$ is increasing in β_t, $\forall \beta_t \in \left(\beta_L, \overline{\beta}\right)$ and will approach $\overline{\beta} + C_1$ as $\beta_t \to \overline{\beta}$ $(\lim_{\beta_t \uparrow \overline{\beta}} \psi_t(\beta_t) = \overline{\beta} + C_1)$. Suppose further that $\psi_t\left(\widehat{\beta}_t\right) \geq \widehat{\beta}_t + C_1$

for some $\widehat{\beta}_t \in \left(\overline{\beta}, \beta_H\right)$. Then since $\frac{\partial \psi_t}{\partial \beta_t}|_{\beta_t=\widehat{\beta}_t} \leq 0$ and thus,

$$\begin{aligned} \psi_t\left(\widehat{\beta}_t+\varepsilon\right) &\geq \psi_t\left(\widehat{\beta}_t\right) \\ &\geq \widehat{\beta}_t+C_1 \\ &> \widehat{\beta}_t+\varepsilon+C_1 \end{aligned} \tag{1.63}$$

for $\varepsilon \uparrow 0$, it must be (by iteration) that $\psi_t\left(\widehat{\widehat{\beta}}_t\right) > \widehat{\widehat{\beta}}_t + C_1$, $\forall \widehat{\widehat{\beta}}_t \in \left(\overline{\beta}, \widehat{\beta}_t\right)$, and $\lim_{\beta_t \downarrow \overline{\beta}} \psi_t\left(\beta_t\right) > \overline{\beta} + C_1$, a contradiction to $\psi_t\left(\beta_t = \overline{\beta}\right) = \overline{\beta} + C_1$ (in combination with continuity of $\psi_t\left(\beta_t\right)$). Finally, suppose that $\psi_t\left(\widehat{\beta}_t\right) \leq \overline{\beta} + C_1$ for some $\widehat{\beta}_t \in \left(\overline{\beta}, \beta_H\right)$. Then since $\frac{\partial \psi_t}{\partial \beta_t}|_{\beta_t=\widehat{\beta}_t} > 0$ and thus,

$$\begin{aligned} \psi_t\left(\widehat{\beta}_t+\varepsilon\right) &< \psi_t\left(\widehat{\beta}_t\right) \\ &\leq \overline{\beta}+C_1 \end{aligned} \tag{1.64}$$

for $\varepsilon \uparrow 0$, it must be (by iteration) that $\psi_t\left(\widehat{\widehat{\beta}}_t\right) < \widehat{\widehat{\beta}}_t + C_1$, $\forall \widehat{\widehat{\beta}}_t \in \left(\overline{\beta}, \widehat{\beta}_t\right)$, and $\lim_{\beta_t \downarrow \overline{\beta}} \psi_t\left(\beta_t\right) < \overline{\beta} + C_1$, a contradiction to $\psi_t\left(\beta_t = \overline{\beta}\right) = \overline{\beta} + C_1$ (in combination with continuity of $\psi_t\left(\beta_t\right)$). Accordingly, (in order to attain $\psi_t\left(\overline{\beta}\right) = \lim_{\beta_t \downarrow \overline{\beta}} \psi_t\left(\beta_t\right) = \overline{\beta} + C_1$) it must be that $\psi_t\left(\beta_t\right) \in \left(\overline{\beta} + C_1, \beta_t + C_1\right)$, $\forall \beta_t \in \left(\overline{\beta}, \beta_H\right)$, and thus, $\frac{\partial \psi_t}{\partial \beta_t} > 0$, $\forall \beta_t \in \left(\overline{\beta}, \beta_H\right)$. Thus, $\psi_t\left(\beta_t\right)$ is increasing in β_t, $\forall \beta_t \in \left(\overline{\beta}, \beta_H\right)$ and will approach $\overline{\beta} + C_1$ as $\beta_t \to \overline{\beta}$ ($\lim_{\beta_t \downarrow \overline{\beta}} \psi_t\left(\beta_t\right) = \overline{\beta} + C_1$). Hence, there exists a $\psi_t\left(\beta_t\right)$ which is continuous and monotonically increasing in $\beta_t, \forall \beta_t \in \left(\beta_L, \beta_H\right)$. ■

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Chapter 2

Asset Pricing Implications of Demographic Change

Abstract

An overlapping generations (OLG) model featuring demographic uncertainty (stochastic changes in birth and death rates) is solved in general equilibrium. Given a moderate level of relative risk aversion (RRA) and a low enough elasticity of intertemporal substitution (EIS), the interest rate is decreasing in the birth rate and increasing in the death rate. If agents have recursive preferences, demographic uncertainty is priced in financial markets. The market price of risk and the equity premium are time varying and under certain conditions they are higher during periods characterized by a high birth rate (baby boom) and low mortality than in times of a low birth rate and a high death rate. Demographic changes appear to explain substantial parts of the time variation in the real interest rate, the market price of risk and the equity premium. Due to demographic uncertainty the conditional volatility of stock returns is stochastically changing over time and the unconditional volatility of asset returns is substantially larger than the unconditional variation in aggregate consumption growth.

2.1 Introduction

In the developed world there is a substantial demographic transition in progress (large increase in the retiree-to-worker ratio) caused by the post World War II baby boom in conjunction with declining mortality rates. The demographic change is likely to have a significant impact on the global economy, including GDP growth prospects, government policies, the solvency of social security systems, and financial markets.

I explore how optimal consumption decisions and asset prices are affected by demographic transitions and by uncertainty about the timing of future demographic changes. The focus lies on the time variation in the first two moments of financial asset returns.

I solve in general equilibrium an analytically tractable OLG model featuring stochastically changing birth and mortality rates. In contrast to the literature, I model births and deaths as Poisson events and in my model stochastic changes in birth and death rates have no effect on the instantaneous variation in the population size, labor supply and aggregate production output. Instead, ignoring total factor productivity (TFP) shocks for now, population and production output growth are locally deterministic processes (zero quadratic variation). More important, a shock to the birth or the death rate implies a redistribution of aggregate consumption within the population (across cohorts). The consumption distribution in the population is essential because pricing depends solely on consumption growth of existing agents in an OLG model.[1] But, because changes in the population size are perfectly predictable over a small instant in time, shifts in the consumption distribution are also perfectly forecast over a short horizon. Demographic shocks do not add any instantaneous risk, but only long run risk - shocks to the expected growth rate in consumption of existing agents - to the economy.

A smooth growth in the population size, as found in the model, is close to what we observe in the real world. In reality, birth and death rates are subject to unpredictable changes, but in the short run the population grows gradually and growth is highly predictable.

In my theoretical model the interest rate is decreasing in the birth rate and increas-

[1] Existing agents are agents which were already alive at time t and survive over the next dt time period.

ing in the death rate, given a moderate level of RRA and a small enough EIS. The key driving forces for the result are the following. A high birth rate implies an expectation for large new born cohorts to enter the economy in future. A large new born cohort claims a big share of aggregate consumption and growth in consumption of existing agents is expected to be moderate. A small growth in consumption of existing agents corresponds to a low interest rate in equilibrium. It is important to understand that the main driving force is not the change in expected aggregate consumption growth but the shift in the distribution of aggregate consumption within the population (from existing agents to the new born cohort), which results in a drop in the existing agents' expected consumption growth rate and a decline in the interest rate.

In contrast, a high death rate implies a short life expectancy, a high discount of future utility and few savings. Accordingly, the interest rate is high. In addition, aggregate consumption has to be split among only few survivors, if the death rate is high, and the consumption growth of existing agents is large. In equilibrium, this also corresponds to a high interest rate.

Because birth and death rates affect the interest rate through different channels, they have to be modelled separately and not as one general state variable that determines total population growth or the average age of the population.

The stock price volatility exceeds the variation in aggregate consumption growth because of demographic uncertainty. Stock prices respond to demographic changes through different channels. First, expected growth in labor supply, production output and dividends are sensitive to birth and death rate changes (Barsky and De Long (1993)). Second and more important, demographic changes have a similar effect on the discount rate of stocks as on the real interest rate. Because stock prices incorporate information about changes in future dividend growth and the discount rate, there is an instantaneous volatility in stock prices due to demographic uncertainty. In my model, uncertainty in the distribution of consumption within the population - which is the responsible channel for changes in the discount rate - is the major source for volatility in financial markets rather than stochastic changes in the expected production growth rate.

If agents maximize utility functions of the power utility family, then an immediate implication of a locally deterministic consumption process (no instantaneous shocks to consumption growth) is that the stochastic discount factor (SDF) has no quadratic variation. As demographic uncertainty adds (only) long run risk to the economy, it has an impact on the interest rate and the stock price volatility but the equity premium is not affected.

In the case of recursive utility, pricing depends on the covariation of asset returns and instantaneous and future consumption growth (Bansal and Yaron (2004)). The variation in the current consumption-to-wealth ratio is a sufficient statistic for the variation in future consumption growth. As the consumption-to-wealth ratio is a function of the interest rate and time discounting of future utility, it instantaneously responds to changes in birth and death rates. Accordingly, demographic uncertainty induces a covariation between stock returns and the consumption-to-wealth ratio, and is priced in financial markets. The equity premium is time varying and I provide conditions that suffice for it to be positive and increasing in the birth rate and decreasing in the death rate.

In contrast to other long run risk models, in my analysis shocks to future consumption growth of existing agents are not triggered by shocks to expected production (or aggregate consumption) growth but by shocks to the allocation of aggregate consumption to the new born generation versus existing agents. In other words, the redistribution risk of the aggregate endowment between the new born cohort and existing agents is the main channel of pricing rather than long run risk in production output. This also implies that pricing of demographic uncertainty works mostly through the channel of shocks to the discount rate rather than stochastic changes in future expected dividend growth.

My model suggest that demographic changes explain substantial parts of the time variation in the interest rate, the market price of risk and the equity premium. A large body of empirical literature explores stock return predictability. Returns are found to be more predictable at low frequencies than over the short run, and most of the predictable variation is due to variation in discount rates rather than changes in expected dividend growth (e.g. Keim and Stambaugh (1986), Fama and French (1988a, 1988b), Ammer and Campbell (1993), Goetzman and Jorion (1993), and Cochrane

(2011) as a summary). Moreover, Ferson and Harvey (1991) suggest that time variation in the market price of risk rather than time variation in the exposure of a stock to systemic risk is the driving force causing a time variation in discount rates. According to my qualitative and quantitative results, these facts may be explained (partly) by demographic changes.

My qualitative and quantitative results require preferences with a low EIS, which is consistent with a large body of empirical literature.[2] Hall (1988), Campbell and Mankiw (1989), Yogo (2004), and Pakos (2007) use aggregate consumption and financial data to estimate the EIS from the Euler equation in a representative agent model. They get estimates close to zero. Vissing-Jorgensen (2002) disentangles asset holders from non-asset holders and estimates an EIS coefficient of 0.3 for stockholders. Her estimates are noisy and not significantly different from zero. Hasanov (2007) and Bonaparte (2008) use household-specific consumption and portfolio choice data to take account for heterogeneity. They get an EIS of about 0.25, but the noise of the estimates is not negligible and much lower levels of the EIS are possible.

Interest Rate and Equity Premium (in %)

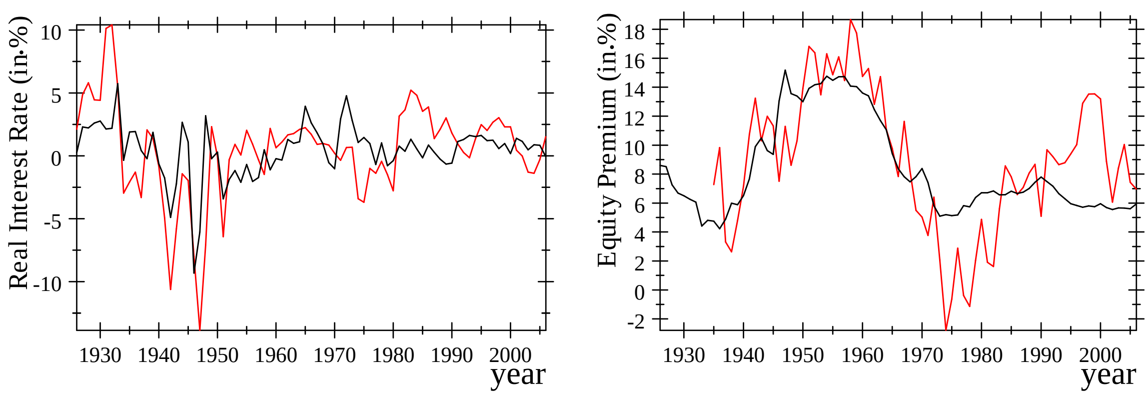

Figure 2.1: Left panel: Real interest rate in USA (red line), and implied interest rate from regression results and observed birth and death rates (black line). Right panel: 10 year averages of US stock market excess returns (red line), and implied equity premium from regression results and observed birth and death rates (black line).

According to the model changes in birth and death rates matter for asset pricing behind the effects on the interest rate. The interest rate is not a "sufficient statistic"

[2] Mostly I also require $RRA > 1$ but not too large, which is an unproblematic assumption.

that summarizes asset pricing implications of demographic changes. I illustrate this result empirically in figure 2.1 (details are in the appendix). Consistent with my qualitative results, a linear regression analysis suggests that the real interest rate depends negatively on the birth rate and positively on the death rate. Stock market excess returns are positively related to the birth rate and negatively related to the mortality rate. The picture suggests a strong relation between the demographic and financial quantities. Before the mid 1970's the relation appears stronger than in recent years. A weaker link in recent times might be explained by the increasing globalization in financial markets (Geanakoplos et al. (2004)).

Most related literature asks how retirement of the baby boomers affects asset prices. The focus lies on models with perfectly predictable baby boom and bust "cycles". In the long run there is a time variation in asset returns as baby boomers proceed through the life-cycle and the "average savings behavior" across the population slowly and predictably changes.

Empirical studies suggest that in the long run asset prices, price/dividend ratios, the interest rate and equity premia are linked to various demographic quantities (Mankiw and Weil (1989), Bakshi and Chen (1994), Erb, Harvey and Viskanta (1997), Porterba (2001), Geanakoplos et al. (2004), Goyal (2004), Ang and Maddaloni (2005), Huynh et al. (2006), Tamoni et al. (2007), Acemoglu and Johnson (2007), Hanewald (2010) and Takats (2010)). Calibrations by Brooks (2000, 2004) and Geanakoplos et al. (2004) suggest that predictable baby booms and busts cause the interest rate and equity prices to vary over time as the baby boomers live through the life-cycle. Most of the results are driven by the assumption that consumption-to-wealth ratios differ across cohorts because agents face a fixed lifetime horizon. Abel (2003) shows in an analytically tractable model that the price of a unit of capital is increasing in the birth rate and follows a mean reverting process. Unfortunately, the risk free rate and the equity premium cannot be disentangled in his model. Auerbach and Kotlikoff (1987), Kotlikoff, Smetters and Walliser (2001), Fehr et al. (2003), and Fehr, Jokisch and Kotlikoff (2004a, 2004b) use dynamic general equilibrium simulation models to explorer the possible impact of deterministic trends in birth and death rates on long run economic and fiscal conditions.

I take a different approach and ask how the uncertainty about future demographic shocks affect asset pricing. Recursive preferences are crucial for demographic uncertainty to be priced in financial markets and present a novel channel that introduces a time variation in the interest rate and the equity premium. Another key contribution is the tractability of the model which allows me to derive qualitative results about the implications of demographic changes on the level of and the time variation in the interest rate and the equity premium.[3] I am able to distinguish various (offsetting) channels through which asset returns are affected by birth and death rates, and for different preference parameter regions I can prove which channel is dominant. My calibration exercise replicates part of the (in US data) observed time variation in the interest rate and stock market excess returns, and the results are consistent with my empirical findings displayed in figure 2.1.

The paper is organized as follows. The next section describes the model. In the following I present my results in three steps. First, I discuss the simplest version of the model with constant birth and death rates and use comparative statics analysis to gain a first intuition about the dynamics in my model. Second, I proof that the intuition from the constant case carries over to a dynamic two state Markov switching model and derive further qualitative results. Third, I generalize the model to include TFP shocks and Brownian uncertainty and numerically illustrate the possible quantitative impact of my analytical results. Finally, I discuss limitations and extensions of my model and conclude.

2.2 The Economy

2.2.1 Demographics and Uncertainty

I consider an OLG model in continuous time that generalizes the Blanchard (1985) model. I disentangle birth and death rates and let them change stochastically over time.

[3] A key difference in my model to the previous literature is the assumption of age-independent mortality. This is essential to keep the model analytical tractable and has the advantage to isolate my results from the findings in previous literature. The results in Brooks (2000, 2004) and Geanakoplos et al. (2004) are mainly driven by the fact that an agent's "life expectancy" is decreasing in age. Thus, my results should be viewed as a complement to previous findings rather than as a substitute.

The economy is populated with a continuum of agents of measure N_t. The birth rate is denoted by n_t and the new born cohort at time t is of the size $n_t N_t dt$.[4] Each agent faces an instantaneous probability of death $\lambda_t dt$. Conditional on being alive at time t_1, an agent's survival probability until time $t_2 > t_1$ is $e^{-\int_{t_1}^{t_2} \lambda_v dv}$. To keep the model tractable, I do not allow for heterogeneity in the arrival rate of death. Imposing mortality rates to be age-independent is restrictive and counterfactual, but it is a small price to pay when one is interested in the common time variation in death rates. According to the much celebrated Lee and Carter (1992) approach, time variation in (age specific) death rates is mostly due to one (across cohorts) common stochastic time component.[5] The main general equilibrium implication of age-independent mortality is that the marginal propensity to consume is independent of age. Arguably the most interesting life-cycle effects on (age-dependent) consumption and savings behavior do not come from the time variation in the marginal propensity to consume but from the hump-shaped pattern of earnings over the life-cycle, and my model accounts for this feature.

Timing of death is uncertain to the individual, but on the aggregate the size of a cohort declines non-stochastically over the next instant of time because the economy is populated by a continuum rather than a finite number of agents. The size of cohort s (agents born at time s) shrinks to $n_s N_s e^{-\int_s^t \lambda_v dv} ds$ until time $t > s$. The population size is $N_t = \int_{-\infty}^{t} n_u N_u e^{-\int_u^t \lambda_v dv} du = N_s e^{\int_s^t n_v - \lambda_v dv}$.[6] $\frac{dN_t}{N_t} = (n_t - \lambda_t)\, dt$ is a term in only dt, and population size N_t follows a locally deterministic process (zero quadratic variation).

In the USA the crude birth rate (denoted by n_t in the model) declined from about 3% in 1910 to 1.5% in 2006. Annual changes are subject to a standard deviation of 3.8% (unconditionally), and shocks appear to be persistent (the current birth rate level is a good predictor of next year's level). In addition to "short term" uncertainty (annual volatility), there are major "long term" transitions. Statistics from other developed

[4]The instantaneous probability of an existing agent to give birth to a new agent at time t is $n_t dt$.

[5]The Lee and Carter (1992) model is widely used in demographic research and has also gained much attention in other fields of research. In asset pricing and household finance literature many papers employ it (Cox et al., 2006; Chen and Cox, 2009; Cocco and Gomes, 2009; DeNardi et al., 2009; Maurer, 2011; Hanewald and Post, 2010).

[6]I need the technical assumption that the economy already exists for an infinite amount of time.

Birth Rate and Stochastic Time Component in Death Rates (in %)

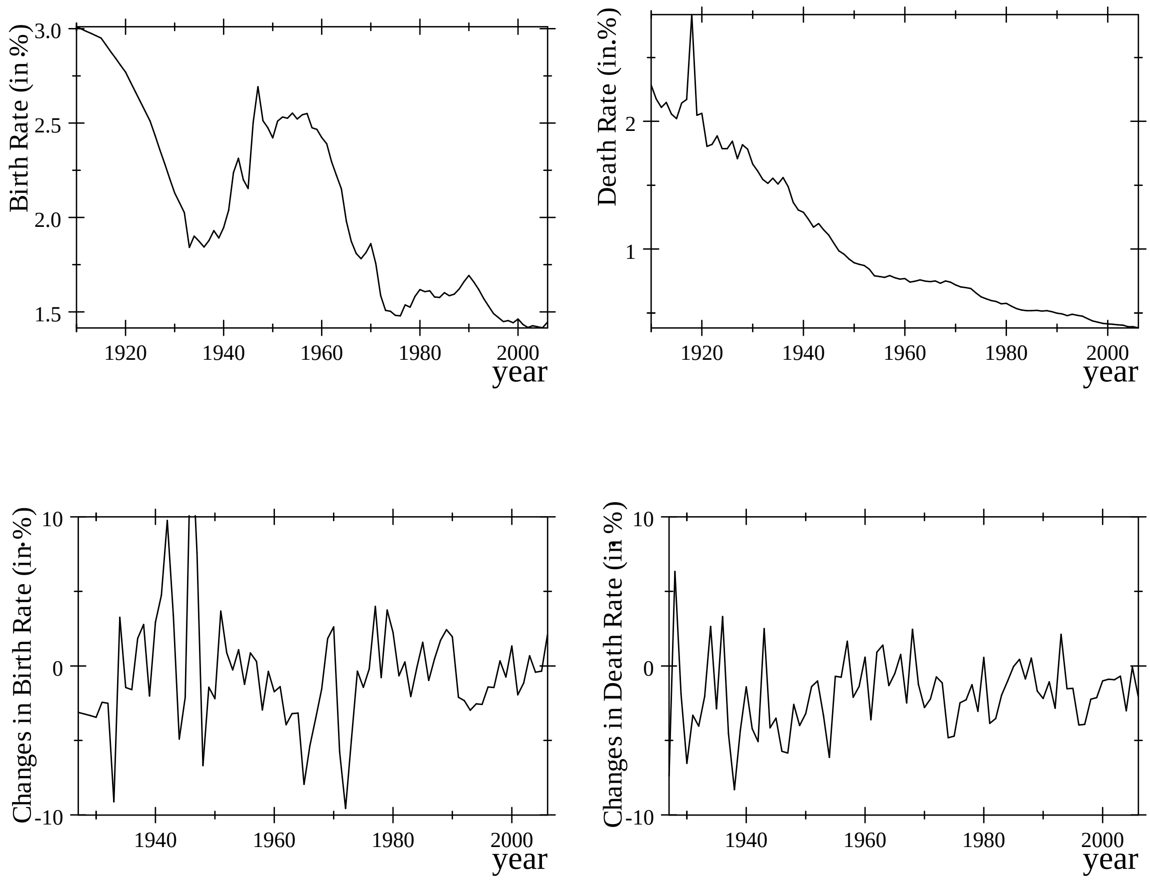

Figure 2.2: Top-left panel: Crude birth rate (in %) in the USA from 1910 until 2006. Source: Department of Health and Human Services, National Center for Health Statistics, USA; and The Human Mortality Database, University of California, Berkeley and Max Planck Institute for Demographic Research. Top-right panel: Lee and Carter (1992) model output (in %) for US mortality data from 1900 to 2006 provided by National Center for Health Statistics. Estimation of common stochastic time component across generations. Bottom-left panel: Percentage changes in crude birth rate. Bottom-right panel: Percentage changes in common stochastic time component of Lee and Carter (1992) model estimation.

countries reveal similar patterns. Mankiw and Weil (1988) also illustrate the uncertainty in the birth rate process and the difficulty the USA census bureau has to forecast the future evolution of the birth rate (figure 2 in their paper).

The US central death rate over all ages (denoted by λ_t in the model) is mostly decreasing and changes are subject to a yearly unconditional standard deviation of 5.1%. Shocks to the death rate are persistent (cf. Lee and carter (1992)). US population statistics are representative for the developed world with the exception that many European countries suffered more from the two world wars than the USA.

2.2.2 Production

The supply side in the consumption goods market is constituted by a representative firm which is endowed with capital stock K_t and has access to a technology described by a Cobb-Douglas production function $Y_t = A_t (G_t)^a (K_t)^{1-a}$. A_t denotes TFP, G_t the employed amount of labor, and Y_t determines the quantity of consumption goods produced by the firm. Except for the last section, I assume A_t to grow at an exogenously given (deterministic) rate $\frac{dA_t}{A_t} = \mu^{(A)} dt$, and there are no stochastic TFP shocks for illustrative purposes. The firm is assumed to not face any economic decision, and I presume for the capital stock K_t a deterministic growth path according to $\frac{dK_t}{K_t} = \mu^{(K)} dt$.[7]

I assume full employment in the economy. An agent born at time s supplies $G(s,t)$ labor efficiency units at time t. To match the hump-shaped profile of life-cycle earnings in Hubbard et al. (1993), I let $G(s,t) = \sum_{i=1}^{2} B_i e^{-\delta_i \int_s^t n_u du}$ with the technical assumption of $\delta_i > -1$.[8] $G(s,t)$ generates the desired hump-shape pattern if $B_1 > |B_2| > B_2$ and $\delta_1 < \delta_2$. For some derivations I use a simpler specification with $B_2 = 0$ (or $\delta_1 = \delta_2$). Aggregation yields the total amount of labor efficiency units employed by the firm, $G_t = \int_{-\infty}^{t} G(s,t)\, n_s N_s e^{-\int_s^t \lambda_u du} ds = N_t \sum_{i=1}^{2} \frac{B_i}{1+\delta_i}$. G_t is

[7] No economic decision in the sense that the firm does not invest, employs all supplied labor at a competitive wage equal to the marginal productivity of labor, and pays out all remaining earnings as dividends. Capital growth is understood as a byproduct of production (for free) as is technological progress. I may set $\mu^{(K)} = 0$ without altering any of my results.

[8] $\delta_i > -1$ must hold in order to ensure aggregate supply of labor to stay finite, $|G_t| = \left| \int_{-\infty}^{t} G(s,t)\, n_s N_s e^{-\int_s^t \lambda_u du} ds \right| < \infty$. See also Garleanu and Panageas (2010) for a similar specification.

Labor Efficiency Units over the Life-cycle, $G(s,t)$

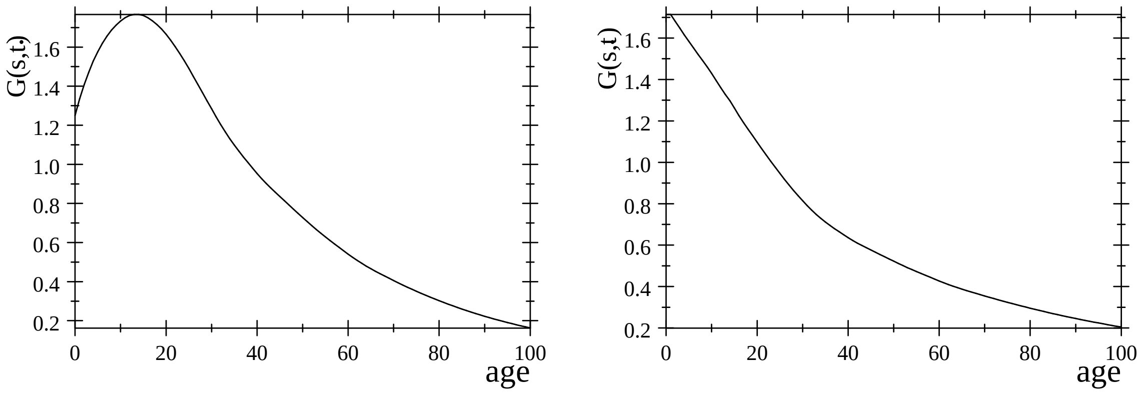

Figure 2.3: Double exponential function $G(s,t) = \sum_{i=1}^{2} B_i e^{-\delta_i \int_s^t n_u du}$ with the parameterisation $(B_1, B_2, \delta_1, \delta_2) = (31.25, -30, 2.65, 2.95)$ (left panel), and $(B_1, B_2, \delta_1, \delta_2) = (1.75, 0, 1.3, 0)$ (right panel).

locally deterministic, $\frac{dG_t}{G_t} = (n_t - \lambda_t)\, dt$, but has long run risk inherent. The supply of consumption goods follows a locally deterministic process with the growth rate $\frac{dY_t}{Y_t} = \mu_t^{(Y)} dt = \left[\mu^{(A)} + (1-a)\,\mu^{(K)} + a\,(n_t - \lambda_t)\right] dt$.

Labor is paid according to its marginal productivity, $y_t = a\frac{Y_t}{G_t}$. An agent of cohort s earns in exchange for his labor $y_t^s = aY_t\frac{G(s,t)}{G_t}$. The firm does not invest and pays the remaining fraction of output, $(1-a)\,Y_t$ as dividends D_t to the shareholders of the firm.

2.2.3 Financial Markets: Equity, Bond, and Insurance

Financial markets are assumed to be dynamically complete.[9] π denotes the (unique) SDF in the economy and is determined in equilibrium. Agents are born without any financial wealth but are endowed with labor. Financial wealth at time t of an agent of cohort s is denoted by W_t^s, and $\widehat{W}_t^s$ describes total wealth (financial and human wealth). An agent of cohort s consume c_t^s and allocates the remaining part of his financial wealth to equities and bonds. Equities are claims on the stream of dividends D_t paid out by the representative firm. The price of an equity is denoted by P_t. The supply of equities is normalized to one. X_t^s denotes the number of equities purchased

[9] This assumption is satisfied given equity, bond and annuity contracts as long as there is only one source of uncertainty, i.e. either the birth rate or the death rate is stochastic. In the general case where both rates are characterised as random processes I implicitly suppose the existence of further contracts that dynamically complete financial markets. My focus lies on equity, bond and annuity markets only.

by an agent of cohort s.[10] Bonds are instantaneously risk free and pay interest r_t. Bonds are in zero supply. The part of an agent's financial wealth that is not used to buy stocks, $(W_t^s - X_t^s P_t)$ is invested in bonds.

Agents have access to annuity contracts supplied by a large, competitive insurance company as in Blanchard (1985). A claim (long position) on an insurance contract pays off as follows: if the agent survives the next time period dt he receives the premium $\lambda_t dt$ from the insurer, and if he dies he pays 1. Agents have an incentive to fully annuitize because their objective functions are strictly increasing in consumption and they do not draw utility from bequest. The insurer breaks even almost surely (earnings and liabilities coincide).

2.2.4 Agents' Objective Functions and Budget Constraints

An agent's financial wealth W_t^s evolves according to the dynamics

$$dW_t^s = \underbrace{W_t^s \lambda_t dt}_{\text{insurance premium}} + \underbrace{W_t^s r_t dt}_{\text{risk free return}} + \underbrace{X_t^s \left(dP_t + D_t dt - P_t r_t dt\right)}_{\text{stock market excess return}} + \underbrace{y_t^s dt}_{\text{labor income}} - \underbrace{c_t^s dt}_{\text{consumption}} \tag{2.1}$$

with the initial condition $W_s^s = 0$. As in Blanchard (1985) I impose the transversality condition (given the agent is still alive at time u) $\lim_{u\to\infty} e^{-\int_s^u \lambda_v dv} \frac{\pi_u}{\pi_s} W_u^s = 0$. This ensures that agents do not borrow without limit, accumulate an infinite amount of debt, and protect themselves by buying annuity contracts.

The set of feasible cash flows is $(M + y^s + W_s^s) \equiv \left\{ x^s : F_s^{(\lambda)}\left(x^s - y^s\right) - W_s^s \in M \right\}$. $F_s^{(\lambda)}$ is a discount function such that $F_s^{(\lambda)}\left(x_t\right) = e^{-\int_s^t \lambda_u du} x_t$, and M denotes the set of all marketable cash flows.[11] The set of admissible cash flows is $\Im \equiv (M + y^s + W_s^s) \cap \mathcal{L}^+$. $\mathcal{L}^+$ includes all non-negative processes adapted to $\mathcal{F}^P$ (filtration generated by asset prices). An agent's consumption process c^s has to be an element of the set of admissible cash flows $\Im$.

Agents are assumed to feature homogeneous preferences and the only heterogeneity in the model is timing of death and wealth between agents across cohorts (but not within

[10] To do not permit arbitrage opportunities I restrict trading activities according to the standard technical assumption $\frac{X_t^s P_t}{W_t^s} \in \mathcal{L}^2$, where $\mathcal{L}^2 \equiv \left\{ x \in \mathcal{L} \mid \int_0^T x_t^2 dt < \infty \quad a.s. \right\}$ and $\mathcal{L}$ is the set of processes adapted to the filtration $\mathcal{F}^P$ generated by asset prices, $\mathcal{F}_t^P \equiv \sigma\left\{P_s : s \leq t\right\}$

[11] A cash flow is marketable if it is financed by a trading strategy $X^s \in \mathcal{L}^2$.

the same cohort). Preferences are described by a stochastic differential utility function of the Kreps and Porteus (1978) type introduced by Duffie and Epstein (1992a, 1992b). Following Duffie and Epstein (1992a) and adding the feature of lifetime uncertainty (for the formal derivation see the appendix), the utility specification is

$$V_t^s = E_t \left[\int_t^\infty f\left(c_u^s, V_u^s\right) du \right] \tag{2.2}$$

with the aggregator function $f(.)$ given by

$$f\left(c_u^s, V_u^s\right) = \frac{\beta \left(c_u^s\right)^\rho - \left(\beta + \frac{\rho}{1-\gamma}\lambda_u\right) \left[(1-\gamma) V_u^s\right]^{\frac{\rho}{1-\gamma}}}{\rho \left[(1-\gamma) V_u^s\right]^{\frac{\rho}{1-\gamma}-1}} \tag{2.3}$$

The term $\frac{1}{1-\rho}$ equals the IES, γ controls risk aversion, and β specifies time discounting.

The term $\frac{\rho}{1-\gamma}\lambda_u$ discounts future utility due to risk aversion towards uncertainty about the timing of death. Intuitively, the probability of dying early creates an incentive to save less than an infinitely-lived agent (or an agent with a fixed lifetime) because there is no bequest motive. In contrast, the possibility of surviving longer than life expectancy (state of high marginal utility) creates an incentive for precautionary savings. The former intuition corresponds to a positive discount of utility from future consumption, while the latter one implies that the agent cares relatively more about future consumption. It depends on the preference parameters whether the first or the second effect dominates and the discount is positive or negative. Under time additive utility only the first effect matters while the second effect is irrelevant because agents are risk neutral towards uncertainty about the timing of death.[12]

An agent's objective is to maximize the value function subject to the dynamic or equivalently the static budget constraint,

$$\sup_{\{c^s, X^s\} \in (\Im \times \mathcal{L}^2)} \left\{ V_s^s\left(c^s\right) = E_s \left[\int_s^\infty f\left(c_u^s, V_u^s\right) du \right] \right\}, \qquad s.t.\ \ d\lambda_s,\ dn_s \tag{\textbf{P1}}$$

[12] For a related discussion on risk aversion towards uncertainty about the timing of an agent's death I refer to Bommier (2003).

2.3 The Equilibrium

2.3.1 Definition of Equilibrium

An equilibrium is defined by a set of adapted processes $\{c, X, \pi\}$ such that

(i) for every agent utility is maximized subject to the dynamic budget constraint, problem (**P1**) is solved $\forall s$,

(ii) consumption markets clear, $Y_t = C_t = \int_{-\infty}^{t} c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds$, and

(iii) financial markets clear, $\int_{-\infty}^{t} X_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds = 1$ and $\int_{-\infty}^{t} \left(W_t^s - X_t^s P_t\right) n_s N_s e^{-\int_s^t \lambda_u du} ds = 0$.

2.3.2 General Remarks about the Equilibrium Analysis

The optimal consumption-to-wealth ratio is given by the function $\psi_t(\lambda, n) = \frac{c_t^s}{\widehat{W}_t^s}$, which is constant across cohorts. The functions $F^{y,t,(i)}(\lambda, n, t)$, $\forall i \in \{1, 2\}$ define the present value of labor income of a new born agent, $\widehat{W}_t^t = \frac{Y_t}{N_t} \sum_{i=1}^{2} F^{y,t,(i)}(\lambda, n, t)$. These quantities are essential to determine the aggregate consumption share of the new born cohort, $n_t \sum_{i=1}^{2} F^{y,t,(i)} \psi_t = \frac{c_t^t n_t N_t}{C_t}$. In equilibrium, the interest rate depends crucially on $\frac{c_t^t n_t N_t}{C_t}$. Moreover, the variation in $\psi_t(\lambda, n)$ is a sufficient statistic for the variation in future consumption growth, which is needed for pricing risky assets.

To understand what affects the SDF π, I provide a short sketch of its derivation. I employ the market clearing condition in the consumption goods market, which must hold almost surely at all times and implies $dY_t = dC_t$. Growth in aggregate consumption depends on three terms: aggregation of optimal consumption growth of individuals, dying agents who abruptly stop their stream of consumption, and consumption of the new born cohort,

$$dC_t = \int_{-\infty}^{t} \frac{dc_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds - \lambda_t C_t dt + c_t^t n_t N_t dt \tag{2.4}$$

Applying Itô's lemma to the first order condition of the optimal consumption choice problem (**P1**) implies that the dynamics of an individual's optimal consumption are independent of his cohort but dependent on birth and death rates and the dynamics of the SDF, $\frac{dc_t^s}{c_t^s} = \Xi\left(\frac{d\pi_t}{\pi_t}, n_t, \lambda_t, dn_t, d\lambda_t\right)$.[13] Plugging into (2.4), I can solve for the dynamics of the SDF,

$$\frac{d\pi_t}{\pi_t} = \Xi^{-1}\left(\underbrace{\frac{dY_t}{Y_t} + \lambda_t dt - \frac{c_t^t n_t N_t}{Y_t} dt}_{\text{consumption growth of existing agents}}, n_t, \lambda_t, dn_t, d\lambda_t\right) \tag{2.5}$$

According to (2.5) the dynamics of the SDF are a function of the consumption growth of existing agents, and it is the Euler equation of existing agents that matters for pricing. Therefore, because shocks to birth and death rates cause a redistribution of aggregate consumption within the population and particularly the distribution between the new born cohort and existing agents (which affects the consumption growth rate of existing agents), demographic shocks are crucial for pricing. It is important to understand that uncertainty in the distribution of consumption is a pricing channel which is independent of the pricing implications of (demographic) shocks to the expected production output growth rate. In the following I show that the pricing implications of uncertainty in the distribution of consumption within the population dominate the pricing implications of shocks to expected production output growth.

Lemma 2.1 *In equilibrium the stock price P_t is given by*

$$P_t = \underbrace{\frac{Y_t}{\psi_t(\lambda, n)}}_{\textit{aggregated total wealth}} - \underbrace{Y_t \sum_{i=1}^{2} \frac{F^{y,t,(i)}(\lambda, n, t)}{1+\delta_i}}_{\textit{aggregated present value of labor income}} \tag{2.6}$$

Proof. See appendix. ■

The stock price is determined in financial market clearing and is equal to aggregated financial wealth (total wealth minus present value of labor income).

Lemma 2.2 *The expected excess return of an asset paying the stream of dividends D_t is*

$$E_t\left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt = -\frac{dP_t}{P_t}\frac{d\pi_t}{\pi_t} \tag{2.7}$$

[13] Function $\Xi\left(\frac{d\pi_t}{\pi_t}, n_t, \lambda_t, dn_t, d\lambda_t\right)$ represents the left hand side of equation (2.102) combined with (2.86).

Proof. See appendix. ■

An asset is compensated with a positive (negative) risk premium if its instantaneous returns are negatively (positively) correlated with the marginal utility process.

2.3.3 Constant Birth and Mortality Rates

As a benchmark, I consider the case of no demographic changes. I explore the differences in the dependence of the interest rate on birth and death rates. Comparative statics help to get an intuition of how demographic changes affect the economy.

Proposition 2.1 *Consider an economy as described. Suppose that the birth rate and the mortality rate are constant over time and the two assumptions (i) $\mu^{(Y)} - (1+\delta_i)\, n < r$ and (ii) $\frac{\rho\gamma}{1-\gamma}\lambda + \beta > \rho r$ hold. There exists an equilibrium with a constant interest rate r which is a root to the equation*

$$r = \underbrace{\beta}_{\substack{time\\ discounting}} + \underbrace{(1-\rho)}_{=\frac{1}{EIS}} \underbrace{\left[\mu^{(Y)} + \lambda - \frac{nN_t c_t^t(r)}{C_t}\right]}_{\substack{consumption\ growth\\ of\ existing\ agents}} - \underbrace{\lambda}_{\substack{annuity\\ payoff}} + \underbrace{\frac{\rho}{1-\gamma}\lambda}_{\substack{time\ discounting\ due\\ to\ uncertain\ lifetime}} \tag{2.8}$$

with $c_t^t(r)$ specified in the appendix. The SDF is non-stochastic and the return on equities is constant and equal to the risk free interest rate r.

Proof. See appendix. ■

Conditions (i) and (ii) are required to ensure total wealth to stay finite and the consumption-to-wealth ratio to be non-negative. The equity premium and the volatility of asset prices are zero because there is no uncertainty on the aggregate. The focus lies on the interest rate.

The interest rate in an equivalent economy populated by an infinitely-lived representative agent is $r_* = \beta + (1-\rho)\, \mu_*^{(Y)}$ with $\mu_*^{(Y)} = \mu^{(A)} + (1-a)\, \mu^{(K)}$. It differs from the rate in the OLG economy by the term

$$r - r_* = \underbrace{(1-\rho)\left[\lambda - \frac{nN_t c_t^t(r)}{C_t}\right]}_{(I)} + \underbrace{(1-\rho)\, a\, (n-\lambda)}_{(II)} - \underbrace{\lambda}_{(III)} + \underbrace{\frac{\rho}{1-\gamma}\lambda}_{(IV)} \tag{2.9}$$

The difference arises for four reasons.

(I) Following equation (2.5), $r \neq r_*$ holds because aggregate consumption growth (consumption of the infinitely-lived agent) differs from consumption growth of existing agents in the OLG economy. Deaths of existing agents have a positive effect on consumption growth of surviving agents because survivors have to share total production output with less peers. Births of new agents mean a decline in the older cohorts' share of aggregate consumption and their consumption growth because new agents claim a fraction of aggregate consumption. The death rate increases and the birth rate decreases the interest rate compared to the rate found in the infinitely-lived agent economy.

(II) In an OLG economy the growth rate of total output depends on population growth, $(n_t - \lambda_t)\, dt$. A high birth rate causes total output to grow fast which positively affects the interest rate. A high death rate results the opposite.

(III) The insurance premium has the same impact on an agent's wealth dynamics (equation (2.1)) and optimal consumption path as the risk free interest rate. As the insurance premium works as a substitute to the interest rate, in equilibrium the interest rate is not required to be as high as in a world without insurance payments. Accordingly, the interest rate in an OLG economy is lower than the rate in an infinitely-lived agent economy due to annuity contracts.

(IV) In an OLG economy an agent faces risk aversion towards uncertainty about the length of his life. There is a trade off between how much savings an agent requires for consumption until death and how much he is willing to risk when facing the probability of an early death. The first reason tells that an agent saves more under lifetime uncertainty than if he knew the exact time of death because there is a chance that he will live an unexpectedly long life and thus, his marginal utility is high in future (precautionary savings). The latter reason says that an agent consumes bigger parts of his wealth early in time under lifetime uncertainty because he faces a probability that he will not be alive to consume his savings in future and draw utility from it. In an OLG economy agents save more (less) and the interest rate is smaller (larger) than in an infinitely-lived agent economy, if the discount of future utility due to agents' risk aversion towards uncertainty about the timing of death is negative (positive) and $\frac{\rho}{1-\gamma}\lambda < (>)\, 0$ holds (cf. discussion on time discounting due to lifetime uncertainty in section 2.4).

Lemma 2.3 *Suppose (i)* $\gamma > 1$*, (ii)* $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} > 1 - \frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$*, and the technical conditions in the appendix hold. There exists a cut-off value* $\overline{EIS}^{(r)}$ *such that the condition* $EIS < \overline{EIS}^{(r)}$ *suffices for the interest rate in an OLG economy to be smaller than the rate in an equivalent economy populated by an infinitely-lived representative agent* $(r < r_*)$*.*

Proof. See appendix.[14] ■

Condition (ii) requires life-cycle earnings to be sufficiently decreasing in age. For $B_2 = 0$, the condition is $\delta_1 > -\frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$. Agents have to save for retirement if life-cycle earnings are decreasing in age, and a big supply in savings implies a low interest rate (Blanchard, 1985).

Under a strong motive for consumption smoothing (small ρ, EIS), an agent seeks to flatten his consumption path over the life-cycle, which corresponds to a large consumption-to-wealth ratio and few savings (few financial wealth).[15] Given a large consumption-to-wealth ratio, the new born cohort claims a big fraction of aggregated consumption and $\frac{nN_t c_t^t}{C_t}$ is large enough to ensure expression (2.9) to be negative and $r < r_*$ to hold.

To get an intuition how a change in the birth rate affects the interest rate I take the first derivative of r with respect to n,

$$\frac{\partial r}{\partial n} = (1-\rho)\frac{\overbrace{a}^{\substack{\text{faster growth}\\ \text{in total output}}} - \overbrace{\frac{N_t c_t^t}{C_t}}^{\substack{\text{increase in new}\\ \text{born generation}}} + \overbrace{nN_t\frac{\partial}{\partial n}\left(-\frac{c_t^t}{C_t}\right)}^{\substack{\text{decrease in consumption share}\\ \text{of individual new born agent}}}}{1+(1-\rho)\,nN_t\frac{\partial}{\partial r}\left(\frac{c_t^t(r)}{C_t}\right)} \tag{2.10}$$

with $\frac{\partial}{\partial n}\left(-\frac{c_t^t}{C_t}\right) = \frac{1}{N_t}\sum_{i=1}^{2}\frac{-a+1+\delta_i}{r-\mu^{(Y)}+(1+\delta_i)n}F^{y,(i)}\psi$ ($F^{y,(i)}$ and ψ are specified in the appendix).

To ensure that the denominator in equation (2.10) is positive, I let $f^{(r)}(x) = \beta + (1-\rho)\left[\mu^{(Y)} + \lambda - \frac{nN_t c_t^t(x)}{C_t}\right] - \frac{1-\gamma-\rho}{1-\gamma}\lambda - x$ and suppose $f^{(r)}(x)$ to be decreasing at $x = r$.

[14] The conditions are sufficient but not necessary. The technical conditions in the appendix are easy to satisfy and I do not worry about them. The same is true for all Lemmas that follow.

[15] I suppose that an agent's consumption grows with age ($\frac{c_t^s}{c_s^s} > 1$, for $\forall\ s < t$). This is a natural assumption and is true for a large enough growth in GDP. $\frac{c_t^s}{c_s^s} > 1$ implies $\frac{\partial}{\partial(-\rho)}\left(\frac{c_t^s}{c_s^s}\right) < 0$, $\forall\ s < t$.

The requirement on the slope of $f^{(r)}(.)$ is not a strong assumption. For instance, under the conditions in Lemma 2.3 there exists $r \in \left(\mu^{(Y)}, r_*\right)$ that satisfies the requirement.

There are three offsetting effects of the birth rate on the interest rate. First, the workforce and production output grow faster as the birth rate increases, which has a positive impact on the consumption growth of existing agents ($r \nearrow$).

Second, holding c_t^t constant, an increase in the size of the new born generation causes the new born cohort's claim on aggregate consumption to rise. The interest rate is negatively affected by an increase in the aggregate consumption share of the new born cohort as it slows down consumption growth of existing agents ($r \searrow$).

Third, labor income is declining in the birth rate.[16] Intuitively, a boost in the workforce causes the labor market to become more competitive and wages to drop. A new born agent's total wealth is equal to the present value of his life-cycle earnings, which is sensitive to changes in labor income. In contrast, total wealth of an old agent is less prone to labor income shocks because a large fraction of his endowment consists of financial wealth. A negative shock to labor income implies a (relatively) stronger decline in total wealth of a new born agent than in total wealth of an old agent. Since the consumption-to-wealth ratio remains unchanged, the aggregate consumption share of a new born agent declines, and the consumption growth of existing agents increases as the birth rate rises ($r \nearrow$).

Lemma 2.4 *Suppose $\gamma > 1$ and the technical conditions in the appendix hold. There exists a cut-off value $\overline{EIS}^{(n)}$ such that the condition $EIS < \overline{EIS}^{(n)}$ suffices for $\frac{\partial r}{\partial n} < 0$.*

Proof. See appendix. ■

If I decrease the EIS, agents save less financial wealth. An old agent's total wealth becomes more sensitive to labor income shocks, and the relative difference in a drop of total wealth of old versus young agents due to an increase in the birth rate (and a decline in labor income) gets smaller. Accordingly the magnitude of $\frac{\partial}{\partial n}\left(-\frac{c_t^t}{C_t}\right)$ shrinks.

A strong motive for consumption smoothing implies a large consumption-to-wealth ratio and c_t^t (much consumption at birth). A large c_t^t ensures that (on the margin)

[16] I assume that $1 + \delta_1 > a$, so that the positive effect of an increase in output and aggregate labor income (due to an increase in n) is dominated by the negative effect of a decrease in marginal productivity of labor and productivity of agents. This is satisfied for a decreasing life-cycle earnings profile. For now I ignore feedback effects through the interest rate.

the additional new born agent consumes more than what he "produces" ($a\frac{Y_t}{N_t} - c_t^t = \left[a - \frac{N_t c_t^t}{C_t}\right]\frac{Y_t}{N_t} < 0$).

As a result, if the EIS is small enough, there is one key channel through which a change in the birth rate affects the interest rate. A rise in the birth rate causes more new born agents to enter the economy and to claim a bigger fraction of aggregate consumption. Accordingly, consumption growth of existing agents slows down and the interest rate declines.

Taking the first derivative of r with respect to λ yields,

$$\frac{\partial r}{\partial \lambda} = \frac{(1-\rho)\,[\overbrace{-a}^{\substack{\text{slower growth}\\ \text{in output}}} + \overbrace{1}^{\substack{\text{surviving agents share}\\ \text{aggregate consump-}\\ \text{tion with less peers}}} \overbrace{- nN_t \frac{\partial}{\partial \lambda}\left(\frac{c_t^t}{C_t}\right)}^{\substack{\text{change in con-}\\ \text{sumption share of}\\ \text{new born cohort}}}] - \overbrace{1}^{\substack{\text{higher pay-}\\ \text{ments from}\\ \text{annuities}}} + \overbrace{\frac{\rho}{1-\gamma}}^{\substack{\text{time discounting}\\ \text{due to lifetime}\\ \text{uncertainty}}}}{1 + (1-\rho)\, nN_t \frac{\partial}{\partial r}\left(\frac{c_t^t(r)}{C_t}\right)} \tag{2.11}$$

with $\frac{\partial}{\partial \lambda}\left(\frac{c_t^t}{C_t}\right) = -\frac{1}{N_t}\sum_{i=1}^{2} \frac{a}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)}\psi + \frac{1}{N_t}\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\sum_{i=1}^{2} F^{y,(i)}$.

The expression $(1-\rho)(-a+1) - 1 + \frac{\rho}{1-\gamma}$ summarizes the following four effects. An increase in mortality (i) decreases the workforce and production output ($r \searrow$), (ii) increases the consumption share and growth in consumption of survivors ($r \nearrow$), (iii) implies a high insurance premium ($r \searrow$), and (iv) increases the magnitude of time discounting of future utility due to risk aversion towards lifetime uncertainty (if $\frac{\rho}{1-\gamma} > (<)\, 0$, then $r \nearrow (\searrow)$).

Keeping the interest rate constant, the consumption share of the new born cohort changes with fluctuations in the death rate for two reasons. First, an increase in the death rate causes production output and the present value of labor income to decline. Following the argument in the discussion of a change in the birth rate, the new born cohort's aggregate consumption share decreases because of the negative labor income shock (term $-\sum_{i=1}^{2} \frac{an}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)}\psi$, $r \nearrow$). Second, as mortality increases agents discount future utility more positively (negatively) and increase (decrease) their consumption-to-wealth ratio, if $\frac{\rho}{1-\gamma} > (<)\, 0$. Accordingly, the consumption level at birth and the aggregate consumption share of the new born cohort increase (decline) (term $n\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\sum_{i=1}^{2} F^{y,(i)}$, $r \searrow (\nearrow)$).

Lemma 2.5 *Suppose $\gamma > 1$ and the technical conditions in the appendix hold. There exists a cut-off value $\overline{EIS}^{(\lambda)}$ such that the condition $EIS < \overline{EIS}^{(\lambda)}$ suffices for $\frac{\partial r}{\partial \lambda} > 0$.*

Proof. See appendix. ■

The term $(1-\rho)(-a+1)-1$ is positive if the EIS is small enough.

As I shrink the EIS, the aggregate consumption share of the new born cohort becomes less sensitive to changes in mortality. The intuition is similar to the discussion on changes in the birth rate.

For $\gamma > 1$ and $EIS < 1$, agents (positively) discount future utility because of risk aversion towards lifetime uncertainty. The discount and the (positive) effect on the interest rate become large if the EIS is small.

As a result, for a small enough EIS, I end up with the following key driving forces that causes the interest rate to be increasing in mortality. As the death rate increases, agents face a higher probability of dying early and discount future utility stronger. As a consequence they consume more of their wealth early in life and save less financial wealth, which causes the interest rate to increase in equilibrium. In addition, aggregate consumption has to be split among less survivors and the consumption growth of existing agents and the interest rate increase.

For the remaining discussion I assume $\frac{\partial r}{\partial n} < 0$ and $\frac{\partial r}{\partial \lambda} > 0$.

From Lemma 2.1 and 2.2 it is straightforward to derive the Gordon growth model

$$P_t = \frac{D_t}{r - \mu^{(Y)}} \tag{2.12}$$

The stock price is increasing in the birth rate. As more agents enter the workforce, growth in total output and future stock dividends increase and the discount rate declines.[17]

$$\frac{\partial P}{\partial n} = \frac{\overbrace{a}^{\text{increase in growth of future dividends}} - \overbrace{\frac{\partial r}{\partial n}}^{\text{decrease in discount rate}}}{r - \mu^{(Y)}} P_t \tag{2.13}$$

[17] Equivalently, the stock price increases because the demand for stocks hikes driven (i) by an increase in aggregate savings (boost in present value of aggregate labor income) and (ii) by a drop in interest paid by the risk free asset, which makes stocks more attractive as an alternative investment to the riskless bond market.

The stock price is decreasing in the death rate. An increase in mortality causes growth in output and future dividends to decline, and the discount rate to increase.

$$\frac{\partial P}{\partial \lambda} = \frac{\overbrace{-a}^{\substack{\text{decrease in growth}\\ \text{of future dividends}}} \quad - \quad \overbrace{\frac{\partial r}{\partial \lambda}}^{\substack{\text{increase in}\\ \text{discount rate}}}}{r - \mu^{(Y)}} P_t \tag{2.14}$$

The consumption-to-wealth ratio, ψ depends crucially on agents' preferences. Time discounting of future utility has a positive impact on ψ. Depending on the dominance of either income or substitution effect ($EIS < 1$ or $EIS > 1$), the rate of return on wealth (from bonds and annuities)[18] is positively or negatively related to ψ.

$$\psi = \overbrace{\frac{1}{1-\rho}}^{=EIS} \left[\overbrace{\beta}^{\substack{\text{time}\\ \text{discounting}}} + \overbrace{\frac{\rho}{1-\gamma}}^{\substack{\text{discounting due to}\\ \text{uncertain lifetime}}} \lambda - \overbrace{\rho(\underbrace{r}_{\substack{\text{interest}\\ \text{rate}}} + \underbrace{\lambda}_{\substack{\text{annuity}\\ \text{payoff}}})}^{\substack{\text{trade off between income}\\ \text{and substitution effect}}} \right] \tag{2.15}$$

The consumption-to-wealth ratio is decreasing in the birth rate if $EIS < 1$. As the interest rate declines the agent experiences a negative income shock; savings grow slower and the agent can afford less consumption in future. If the agent cares enough about consumption smoothing, he will save more and consume less today to compensate for the negative shock to future endowment/ consumption (income effect dominates substitution effect).

$$\frac{\partial \psi}{\partial n} = -\frac{\rho}{1-\rho}\frac{\partial r}{\partial n} \tag{2.16}$$

The consumption-to-wealth ratio is increasing in the death rate if $EIS < 1$ and $\gamma > 1$. Agents discount utility from future consumption stronger due to an increase in mortality ($\frac{\rho}{1-\gamma} > 0$) and prefer to consume a larger part of their wealth early in life. In addition, future consumption becomes cheaper as the interest rate and the insurance premium increase and agents instantly consume part of the "newly gained income" (income effect).

$$\frac{\partial \psi}{\partial \lambda} = \frac{1}{1-\rho}\left[\frac{\rho}{1-\gamma} - \rho\left(\frac{\partial r}{\partial \lambda} + 1\right)\right] \tag{2.17}$$

[18] Or equivalently the inverse of the price of future consumption.

2.3.4 Regime Shifts in the Birth Rate: Two State Markov Switching Model

I keep mortality constant and let the birth rate randomly jump between two levels. Random switches capture the long run profile of baby boom and bust transitions found in US birth rate data. Once the birth rate process is stochastic, long run risk is introduced in the economy and I can explore the impact of demographic uncertainty on pricing stocks.

I let the birth rate process be $dn_t = s^{(n)} dS_t^{(n)}$, with $s^{(n)} = n_H - n_L$. $S_t^{(n)} \in \{1, 0\}$ follows a two state, continuous time Markov switching process with transition probability matrix between time t and $t+\Delta$ given by $\Theta^{(S,n)}(\Delta) = \begin{pmatrix} 1-\overline{\theta}_H^{(n)}\Delta & \overline{\theta}_H^{(n)}\Delta \\ \overline{\theta}_L^{(n)}\Delta & 1-\overline{\theta}_L^{(n)}\Delta \end{pmatrix}$. The birth rate switches between the two values $n_t \in \{n_L, n_H\}$. Because the model has only two states, key variables, which depend on the birth rate, switch between two distinct values.

There are minor changes to the utility specification as described in the appendix. Agents' objectives stay the same.

Proposition 2.2 *Suppose an economy as described. In general, there exists an equilibrium with a SDF π that follows a stochastic process driven by the same two state Markov switching process $S^{(n)}$ as the birth rate. The equilibrium interest rate r_t switches between two distinct levels, $r_t \in \left\{ r_L^{(n)}, r_H^{(n)} \right\}$ defined by*

$$\begin{aligned} r_j^{(n)} &= \beta + (1-\rho)\left[\mu_j^{(Y,n)} + \lambda - n_j \sum_{i=1}^{2} F_j^{y,(i),(n)} \psi_j^{(n)}\right] - \lambda + \frac{\rho}{1-\gamma}\lambda \\ &\quad -\overline{\theta}_j^{(n)}\left(\left(\frac{\psi_k^{(n)}}{\psi_j^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) + \frac{1-\gamma-\rho}{1-\gamma}\overline{\theta}_j^{(n)}\left(\left(\frac{\psi_k^{(n)}}{\psi_j^{(n)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right) \end{aligned} \tag{2.18}$$

$\forall (j,k) \in \{(L,H), (H,L)\}$, with $r_t \mid [n_t = n_L] = r_L^{(n)}$ and $r_t \mid [n_t = n_H] = r_H^{(n)}$. The market price of risk jumps between two distinct values, $\kappa_t \in \left\{ \kappa_L^{(n)}, \kappa_H^{(n)} \right\}$ given by

$$\kappa_j^{(n)} = -\left(\left(\frac{\psi_k^{(n)}}{\psi_j^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) \tag{2.19}$$

$\forall (j,k) \in \{(L,H),(H,L)\}$, *with* $\kappa_t \mid [n_t = n_L] = \kappa_L^{(n)}$ *and* $\kappa_t \mid [n_t = n_H] = \kappa_H^{(n)}$. *Demographic uncertainty is priced in equilibrium and the equity premium is non-zero. In the special case of power utility, the SDF follows a locally deterministic process and the equity premium disappears. The functions* $F_L^{y,(1),(n)}$, $F_L^{y,(2),(n)}$, $F_H^{y,(1),(n)}$, $F_H^{y,(2),(n)}$, $\psi_L^{(n)}$, *and* $\psi_H^{(n)}$ *are determined in a system of 6 non-linear equations provided in the appendix.*

Proof. *See appendix.* ■

To understand why the market price of risk is non-zero in the general case of recursive utility and zero in the special case of CRRA preferences, it is best to look at the optimal consumption path for an individual agent

$$\underbrace{\frac{c_t^s}{c_s^s}}_{\substack{\text{variation in current}\\ \text{consumption growth}}} = e^{\frac{1}{1-\rho}\int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s)+\lambda_u du} \underbrace{\left(\frac{V_t^s(\psi_t,\pi_t)}{V_s^s(\psi_s,\pi_s)}\right)^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}}}_{\substack{\text{variation in future}\\ \text{consumption growth}}} \underbrace{\left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{1-\rho}}}_{\substack{\text{variation in marginal}\\ \text{utility process}}} \tag{2.20}$$

Suppose that the SDF had zero quadratic variation. Because the value function V^s features a discontinuity at the time of a regime shift, optimal consumption must jump as a regime shift occurs. As each agent is affected the same (dynamics of the value function are independent of the cohort), the aggregate consumption process features jumps. But, the aggregate supply of consumption goods has no instantaneous variation and markets could not possibly clear ($dY_t \neq dC_t$). To resolve the problem it must be that the SDF is driven by a jump process such that all discontinuities in V^s are exactly offset and optimal consumption of the individual follows a locally deterministic process (cf. equation (2.119)).

The SDF is defined as a (deterministic) multiple of the marginal utility process (Gateau derivative of the utility function), which depends on current and future consumption. As current consumption follows a locally deterministic process it does not introduce any stochastics in the marginal utility process and its dynamics are irrelevant for the derivation of the equity premium. The variation in the consumption-to-wealth ratio is a sufficient statistic of the variation in future consumption growth. As a result the market price of risk is a non-linear function of the ratio $\frac{\psi_H}{\psi_L}$.

In the case of CRRA preferences optimal consumption does not depend on the agent's value function ($c_t^s = c_s^s e^{-\frac{1}{\gamma}\beta(t-s)} \left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{\gamma}}$), and the consumption path of an individual agent is (locally) deterministic. The SDF must not be stochastic to ensure market clearing. The market price of risk is zero and pricing of risky assets is not affected by stochastic changes in the birth rate.[19]

Lemma 2.6 *Suppose $\gamma \in (1, 1-\rho)$ $(\rho < 0)$ and the technical conditions in the appendix hold. There exists a cut-off value $\overline{EIS}_1^{(n)}$ such that the condition $EIS < \overline{EIS}_1^{(n)}$ suffices for the interest rate during a period characterized by a high birth rate (baby boom) to be lower than the rate during times of a low birth rate (baby bust), $r_L^{(n)} > r_H^{(n)}$. The consumption-to-wealth ratio is decreasing and the magnitude of the market price of risk is increasing in the birth rate, $\psi_L^{(n)} > \psi_H^{(n)}$ and $|\kappa_L^{(n)}| < |\kappa_H^{(n)}|$.*

Proof. See appendix. ■

The intuition for $r_L^{(n)} > r_H^{(n)}$ and $\psi_L^{(n)} > \psi_H^{(n)}$ is equivalent to the argument provided in the static case. $|\kappa_L^{(n)}| < |\kappa_H^{(n)}|$ is a technical property of the model.

The stock price is a state dependent multiple of GDP (Lemma 2.1),

$$P_t^{(n_j)} = P_t \mid [n_t = n_j] = Y_t \left[\frac{1}{\psi_j^{(n)}} - \sum_{i=1}^{2} \frac{F_j^{y,(i),(n)}}{1+\delta_i} \right] \tag{2.21}$$

$\forall j \in \{L, H\}$. The growth rate is stochastic and conditional on the state of the world

$$\frac{dP_t}{P_t} \mid [n_t = n_j] = \mu_j^{(Y,n)} dt + \frac{Y_t}{P_t^{(n_j)}} \left[\frac{1}{\psi_k^{(n)}} - \frac{1}{\psi_j^{(n)}} - \sum_{i=1}^{2} \frac{F_k^{y,(i),(n)} - F_j^{y,(i),(n)}}{1+\delta_i} \right] |dS_t^{(n)}| \tag{2.22}$$

$\forall (j,k) \in \{(L,H), (H,L)\}$. GDP follows a locally deterministic process because demographic uncertainty introduces only long run risk in the economy. In contrast, the stock price has non-zero quadratic variation since stocks are forward looking and incorporate changes in growth prospects of the economy (information about future growth in dividends and future changes in the discount rate). Demographic uncertainty introduces in a natural way excess volatility of asset returns over the variation in aggregate consumption growth (Barsky and De Long (1993)).

[19] Another way to understand that the SDF is locally deterministic is by noticing that in case of time additive utilities, marginal utility depends solely on current consumption but not future consumption.

Following Lemma 2.2, the equity premium is

$$\frac{1}{dt}E_t\left[\frac{dP_t + D_t dt}{P_t}\right] - r_t \quad | \quad [n_t = n_j] = \overline{\theta}_j^{(n)} \frac{Y_t}{P_t^{(n_j)}} \left(\left(\frac{\psi_k^{(n)}}{\psi_j^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) \cdot \left[\frac{1}{\psi_j^{(n)}} - \frac{1}{\psi_k^{(n)}} + \sum_{i=1}^{2} \frac{F_k^{y,(i),(n)} - F_j^{y,(i),(n)}}{1+\delta_i}\right] \tag{2.23}$$

$\forall (j,k) \in \{(L,H),(H,L)\}$. Demographic uncertainty is priced in equilibrium and the equity premium switches between two distinct values. In the special case of power utility with $1-\gamma-\rho = 0$ the equity premium is zero.

Lemma 2.7 *Suppose* $\frac{1-\gamma-\rho}{\rho} < 0$ *and the technical conditions in the appendix hold. There exists a cut-off value* $\overline{EIS}_2^{(n)}$ *such that the condition* $EIS < \overline{EIS}_2^{(n)}$ *suffices for the equity premium to be positive in both states of the world.*

Proof. See appendix. ■

A key result is the inequality $|\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}}| - |\sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i}| > 0$. The stock price moves into the opposite (same) direction as a change in the consumption-to-wealth ratio (total wealth). This is consistent with the developed intuition from the comparative statics analyses in the previous section.

By equation (2.81) and (2.119), $(1-\gamma) V_t^s$ is decreasing (increasing) in the consumption-to-wealth ratio if $\frac{1-\gamma}{\rho} > (<) 0$. Combining equations (2.3) and (2.68), marginal utility is decreasing (increasing) in $(1-\gamma) V_t^s$ if $\frac{1-\gamma-\rho}{1-\gamma} < (>) 0$. Accordingly, condition $\frac{1-\gamma-\rho}{\rho} < 0$ is necessary for changes in the stock price (or total wealth) and the SDF to be negatively correlated. The payoff of stocks is low (high) in states of the world when marginal utility is high (low) and more (less) wealth is desired, and investors require a positive compensation for holding stocks.

Lemma 2.8 *Suppose* $\overline{\theta}_H^{(n)} > (<) \overline{\theta}_L^{(n)}$, *and the conditions in Lemma 2.6 and 2.7 hold. There exists a cut-off value* $\overline{EIS}_3^{(n)}$ *such that the condition* $EIS < \overline{EIS}_3^{(n)}$ *ensures that the equity premium is larger (lower) during a baby boom than the premium during times of a low birth rate.*

Proof. See appendix. ■

The parameters $\overline{\theta}_L^{(n)}$ and $\overline{\theta}_H^{(n)}$ determine the probability of a regime switch conditional on being in a low and high birth rate state. The ratio $\frac{\overline{\theta}_H^{(n)}}{\overline{\theta}_L^{(n)}}$ describes the ratio between the instantaneous risk in stock returns during a high and a low birth rate state. The equity premium is higher during a high birth rate state, if a baby boom lasts on average shorter (the risk for a regime switch is higher) compared to a low birth rate state. US population data over the last 100 years reveal that this seems true.

Consistent with the result in the previous section, Lemma 2.6 and 2.7 imply that the stock price is increasing in the birth rate,

$$P_t^{(n_H)} - P_t^{(n_L)} = Y_t \left[\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} - \sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i} \right] > 0 \tag{2.24}$$

A baby boom causes the stock market to boom and the growth rates of the stock price and dividends are high. There is an immediate stock market bust (negative jump) as soon as the baby boom stops (at the time of a regime shift from a high to a low birth rate). The model implies a slow growth in asset prices and in dividends when the birth rate is low, but it does not imply a major stock market bust as the baby boom generation "retires".[20] This follows because all key quantities are Markov processes and immediately adjust at the time of a regime shift, when agents are surprised by a change in the economic environment.

The result that the retirement of the baby boom generation does not have an impact on asset prices can be challenged on different grounds. First, capital accumulation (with convex adjustment costs) is likely to alter the result because a slow-down in population growth causes disinvestment. Because of convex adjustment costs there is not one immediate cut in the capital stock as the birth rate drops, but disinvestment continues over a long horizon and the desired capital stock is approached slowly (cf. Abel, 2003).

Second, the specification of the life-cycle earnings profile enforces by construction the Markov property of aggregate supply of labor efficiency units, which implies the consumption-to-wealth ratio and total wealth to be Markov processes. A choice of a more general path for life-cycle earnings (e.g. discontinuity at time of retirement) causes

[20] Retirement can be defined as the age when an agent is endowed with less labor efficiency units then some level x, for instance the data in Hubbard et al. (1993) suggest the people at age 65 earn about 35-40% of the maximal labor income over the lif-cycle.

the consumption-to-wealth ratio and total wealth to be history-dependent functions (in particular I have to keep track which cohort retires at which point in time). The introduction of age-dependent death rates also causes the variables to depend on the past. If the consumption-to-wealth ratio and total wealth are not Markov processes, then asset prices depend on past observations of the birth rate, and baby booms and busts have implications on asset prices for a long time after a regime shift occurs. As a result the model's answer to the question whether the retirement of the baby boomers causes a stock market bust has to be treated with caution. Brooks (2000, 2004) and Geanakoplos et al. (2004) complement my model with respect to these issues and deliver an answer to the question. My model is setup to explore how demographic uncertainty affects asset pricing in addition to the effects documented by Brooks (2000, 2004) and Geanakoplos et al. (2004).

A numerical exercise helps to illustrate the quantitative magnitude of my qualitative results. I set $n_L = 1.5\%$, $n_H = 2.5\%$, $\overline{\theta}_L^{(n)} = 1\%$, $\overline{\theta}_H^{(n)} = 6.7\%$ and $\lambda = 1\%$, which roughly captures the long run transitions in the birth rate and the average death rate in the USA in the 20th century. I chose $\beta = 0.005$, $a = 0.9$ and $\mu^{(A)} + (1-a)\,\mu^{(K)} = 1.55\%$, so that $\mu_L^{(Y,n)} = 2\%$ and $\mu_H^{(Y,n)} = 2.9\%$. I fix $\gamma = 2.5$ and numerically solve the model for $EIS \in [0.05, 0.25]$.

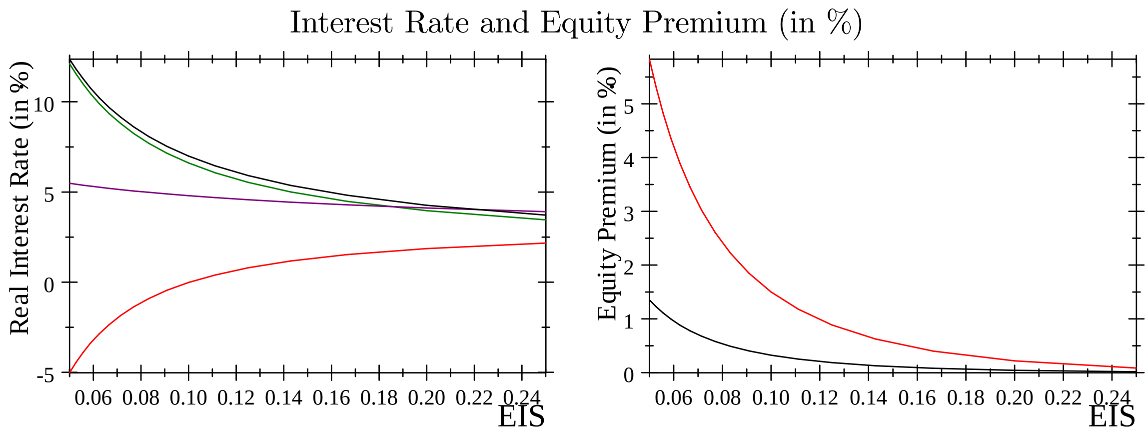

Figure 2.4: Left panel: Interest rate in the two state Markov switching model for $n_L = 1.5\%$ (black line), $n_H = 2.5\%$ (red line), and interest rate in a model with constant birth rate for $n_L = 1.5\%$ (green line), $n_H = 2.5\%$ (purple line). Rigth panel: Equity premium in the regime shifting model for $n_L = 1.5\%$ (black line), $n_H = 2.5\%$ (red line).

The black and the red line in the left panel in figure 2.4 report the equilibrium interest rates corresponding to n_L and n_H in the regime switching model. The green and the purple line indicate the interest rates in a static model with a constant birth rate equal to n_L respectively n_H (agents do not anticipate changes in the birth rate). The interest rate during a baby boom is lower than the rate in a state of a low birth rate. For a low enough EIS the interest rate during a baby boom is even negative. Because I exclude TFP shocks the interest rate is relatively high during times of a low birth rate. TFP shocks decrease the interest rate due to precautionary savings.

The right panel shows the risk premium paid by stocks due to random regime shifts in the birth rate. The black line indicates the premium paid in a state of a low birth rate and the red line describes the premium during a baby boom. Stocks pay a substantially higher excess return during a baby boom than in a state of a low birth rate. The introduction of levered equity would amplify the difference.

The model helps to explain some of the long run time variation in the interest rate and in stock market excess returns in the USA. Keeping the numerical results in mind I look at US data from 1926 to 2009. I define the period between the early 1940's and the early 1960's as a baby boom and the periods before the 1940's and after the early 1960's as times of a low birth rate. During the two periods of a low birth rate the real interest rate were on average between $1\% - 2\%$ and the stock market paid on average 5.5% in excess of the (nominal) interest rate. In contrast, during the baby boom the real interest rate was on average -2.5% and the stock market paid on average an excess return of 16%.

2.3.5 Regime Shifts in the Mortality Rate: Two State Markov Switching Model

To analyze the impact of random changes in the death rate on financial markets, I fix the birth rate while letting the mortality rate switch between a high and a low level. The results and the discussion are similar to the previous section.

I let the death rate process be $d\lambda_t = s^{(\lambda)} dS_t^{(\lambda)}$, with $s^{(\lambda)} = \lambda_H - \lambda_L$. $S_t^{(\lambda)} \in \{1, 0\}$ follows a two state, continuous time Markov switching process with transition probability matrix between time t and $t + \Delta$ given by $\Theta^{(S,\lambda)}(\Delta) = \begin{pmatrix} 1 - \overline{\theta}_H^{(\lambda)}\Delta & \overline{\theta}_H^{(\lambda)}\Delta \\ \overline{\theta}_L^{(\lambda)}\Delta & 1 - \overline{\theta}_L^{(\lambda)}\Delta \end{pmatrix}$.

The death rate switches between the two values $\lambda_t \in \{\lambda_L, \lambda_H\}$.

There are minor changes to the utility specification as described in the appendix. Agents' objectives stay the same.

Proposition 2.3 *Suppose an economy as described. In general, there exists an equilibrium with a SDF π that follows a stochastic process driven by the same two state Markov switching process $S^{(\lambda)}$ as the death rate. The equilibrium interest rate r_t switches between two distinct levels, $r_t \in \left\{r_L^{(\lambda)}, r_H^{(\lambda)}\right\}$ defined by*

$$\begin{aligned} r_j^{(\lambda)} &= \beta + (1-\rho)\left[\mu_j^{(Y,\lambda)} + \lambda_j - n\sum_{i=1}^{2} F_j^{y,(i),(\lambda)}\psi_j^{(\lambda)}\right] - \lambda_j + \frac{\rho}{1-\gamma}\lambda_j \\ &\quad -\overline{\theta}_j^{(\lambda)}\left(\left(\frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) + \frac{1-\gamma-\rho}{1-\gamma}\overline{\theta}_j^{(\lambda)}\left(\left(\frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right) \end{aligned} \tag{2.25}$$

$\forall\,(j,k) \in \{(L,H),(H,L)\}$, with $r_t \mid [\lambda_t = \lambda_L] = r_L^{(\lambda)}$ and $r_t \mid [\lambda_t = \lambda_H] = r_H^{(\lambda)}$. The market price of risk also jumps between two distinct values, $\kappa_t \in \left\{\kappa_L^{(\lambda)}, \kappa_H^{(\lambda)}\right\}$ given by

$$\kappa_j^{(\lambda)} = -\left(\left(\frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) \tag{2.26}$$

$\forall\,(j,k) \in \{(L,H),(H,L)\}$, with $\kappa_t \mid [\lambda_t = \lambda_L] = \kappa_L^{(\lambda)}$ and $\kappa_t \mid [\lambda_t = \lambda_H] = \kappa_H^{(\lambda)}$. Demographic uncertainty is priced in equilibrium and the equity premium is non-zero, except for the special case of power utility. The functions $F_L^{y,(1),(\lambda)}$, $F_L^{y,(2),(\lambda)}$, $F_H^{y,(1),(\lambda)}$, $F_H^{y,(2),(\lambda)}$, $\psi_L^{(\lambda)}$, and $\psi_H^{(\lambda)}$ are determined in a system of 6 non-liner equations provided in the appendix.

Proof. *See appendix.* ■

Lemma 2.9 *Suppose $\gamma \in (1, 1-\rho)$ ($\rho < 0$) and the technical conditions in the appendix hold. There exists a cut-off value $\overline{EIS}_1^{(\lambda)}$ such that the condition $EIS < \overline{EIS}_1^{(\lambda)}$ suffices for the interest rate during a period characterized by a high death rate to be higher than the rate during times of low mortality, $r_L^{(\lambda)} < r_H^{(\lambda)}$. The consumption-to-wealth ratio is increasing and the magnitude of the market price of risk is decreasing in the death rate, $\psi_L^{(\lambda)} < \psi_H^{(\lambda)}$ and $|\kappa_L^{(\lambda)}| > |\kappa_H^{(\lambda)}|$.*

Proof. See appendix. ■

The result is equivalent to the finding in the static case.

The stock price is a multiple of GDP (Lemma 2.1),

$$P_t^{(\lambda_j)} = P_t \mid [\lambda_t = \lambda_j] = Y_t \left[\frac{1}{\psi_j^{(\lambda)}} - \sum_{i=1}^{2} \frac{F_j^{y,(i),(\lambda)}}{1+\delta_i} \right] \tag{2.27}$$

$\forall j \in \{L, H\}$ and the growth rate is

$$\frac{dP_t}{P_t} \mid [\lambda_t = \lambda_j] = \mu_j^{(Y,\lambda)} dt + \frac{Y_t}{P_t^{(\lambda_j)}} \left[\frac{1}{\psi_k^{(\lambda)}} - \frac{1}{\psi_j^{(\lambda)}} - \sum_{i=1}^{2} \frac{F_k^{y,(i),(\lambda)} - F_j^{y,(i),(\lambda)}}{1+\delta_i} \right] |dS_t^{(\lambda)}| \tag{2.28}$$

$\forall (j,k) \in \{(L,H), (H,L)\}$. GDP is locally deterministic, while stock returns are subject to instantaneous volatility due to the forward looking property of the stock price.

According to Lemma 2.2, the equity premium is

$$\begin{aligned} \frac{1}{dt} E_t \left[\frac{dP_t + D_t dt}{P_t} \right] - r_t \quad | \quad [\lambda_t = \lambda_j] = \overline{\theta}_j^{(\lambda)} \frac{Y_t}{P_t^{(\lambda_j)}} \left(\left(\frac{\psi_k^{(\lambda)}}{\psi_j^{(\lambda)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \\ \cdot \left[\left(\frac{1}{\psi_j^{(\lambda)}} - \frac{1}{\psi_k^{(\lambda)}} \right) + \sum_{i=1}^{2} \frac{F_k^{y,(i),(\lambda)} - F_j^{y,(i),(\lambda)}}{1+\delta_i} \right] \end{aligned} \tag{2.29}$$

$\forall (j,k) \in \{(L,H), (H,L)\}$. In the special case of CRRA utility with $1 - \gamma - \rho = 0$, there is no equity premium.

Lemma 2.10 *Suppose* $\frac{1-\gamma-\rho}{\rho} < 0$ *and the technical conditions in the appendix hold. There exists a cut-off value* $\overline{EIS}_2^{(\lambda)}$ *such that the condition* $EIS < \overline{EIS}_2^{(\lambda)}$ *suffices for the equity premium to be positive in both states of the world.*

Proof. See appendix. ■

Lemma 2.10 is equivalent to Lemma 2.7.

Consistent with the comparative statics analysis in section 3.3, the stock price is decreasing in the death rate (under the conditions in Lemma 2.9 and 2.10),

$$P_t^{(\lambda_L)} - P_t^{(\lambda_H)} = Y_t \left[\frac{1}{\psi_L^{(\lambda)}} - \frac{1}{\psi_H^{(\lambda)}} - \sum_{i=1}^{2} \frac{F_L^{y,(i),(\lambda)} - F_H^{y,(i),(\lambda)}}{1+\delta_i} \right] > 0 \tag{2.30}$$

Lemma 2.11 *Suppose* $\overline{\theta}_L^{(\lambda)} > (<)\, \overline{\theta}_H^{(\lambda)}$ *and the conditions in Lemma 2.9 and 2.10 hold. There exists a cut-off value* $\overline{EIS}_3^{(\lambda)}$ *such that the condition* $EIS < \overline{EIS}_3^{(\lambda)}$ *ensures that the equity premium is larger (lower) during a period characterized by a low death rate than the premium in times of high mortality.*

Proof. See appendix. ■

2.3.6 General Model with Brownian Uncertainty: Numerical Illustration

I illustrate the possible quantitative magnitude of my results in a numerical exercise. I model the birth rate and the death rate as Brownian diffusion processes,

$$\frac{dn_t}{n_t} = \mu^{(n)}dt + \sigma^{(n)}d\widetilde{W}_t \tag{2.31}$$

$$\frac{d\lambda_t}{\lambda_t} = \mu^{(\lambda)}dt + \sigma^{(\lambda)}d\widetilde{W}_t \tag{2.32}$$

$\mu^{(i)}$ and $\sigma^{(i)}$ denote constant drift and diffusion terms of process i, and $\widetilde{W}_t$ is a d dimensional Brownian motion. I also introduce TFP shocks to the economy and let

$$\frac{dA_t}{A_t} = \mu^{(A)}dt + \sigma^{(A)}d\widetilde{W}_t \tag{2.33}$$

with $\sigma^{(A)}\left(\sigma^{(n)}\right)^T = \sigma^{(A)}\left(\sigma^{(\lambda)}\right)^T = 0_{(d\times d)}$.

Proposition 2.4 *Suppose an economy as described. In general, there exists an equilibrium with a SDF* π *with the dynamics*

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \kappa_t d\widetilde{W}_t \tag{2.34}$$

The interest rate r_t is

$$r_t = \beta + (1-\rho)\left[\mu_t^{(Y)} + \lambda_t - n_t \sum_{i=1}^{2} F^{y,t,(i)} \psi_t\right] - \lambda_t + \frac{\rho}{1-\gamma}\lambda_t \qquad (2.35)$$
$$\underbrace{+\frac{1-\gamma-\rho}{2\rho}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^2 - \frac{\gamma\,(2-\rho)}{2}\left(\sigma^{(A)}\right)^2}_{\textit{precautionary savings}}$$

and the market price of risk κ_t takes the form

$$\kappa_t = \underbrace{\frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}}_{\substack{\textit{pricing of long run risk/} \\ \textit{demographic uncertainty}}} + \underbrace{\gamma\sigma^{(A)}}_{\substack{\textit{pricing of ins-} \\ \textit{tantaneous risk}}} \qquad (2.36)$$

Demographic uncertainty is priced in equilibrium except in the special case of power utility. The functions $F^{y,t,(1)}(\lambda, n, t)$, $F^{y,t,(2)}(\lambda, n, t)$, and $\psi_t(\lambda, n)$ are determined in a system of 3 differential equations provided in the appendix.

Proof. See appendix. ■

Precautionary savings induced by TFP shocks and demographic uncertainty have a negative impact on the interest rate (if $\frac{1-\gamma-\rho}{\rho} < 0$).

The argument of the previous discussion carries over to explain why demographic uncertainty is priced under recursive preferences but not in case of power utility.

Following Lemma 2.1, the stock price is

$$P_t = Y_t\left[\frac{1}{\psi_t(\lambda, n)} - \sum_{i=1}^{2}\frac{F^{y,t,(i)}(\lambda, n, t)}{1+\delta_i}\right] \qquad (2.37)$$

with the dynamics

$$dP_t = \mu_{P,t}^{(P)} dt + \sigma_{P,t}^{(P)} d\widetilde{W}_t \qquad (2.38)$$
$$\mu_{P,t}^{(P)} = \mu_t^{(Y)} P_t dt + Y_t\left[-\frac{1}{\psi_t}\frac{\mu_t^{(\psi)}}{\psi_t} - \sum_{i=1}^{2}\frac{\mu_t^{\left(F^{y,(i)}\right)}}{1+\delta_i} + \frac{1}{\psi_t}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^2\right]$$
$$\sigma_{P,t}^{(P)} = \sigma^{(A)} P_t + Y_t\left[-\frac{1}{\psi_t}\frac{\sigma_t^{(\psi)}}{\psi_t} - \sum_{i=1}^{2}\frac{\sigma_t^{\left(F^{y,(i)}\right)}}{1+\delta_i}\right]$$

The expected return and volatility are stochastically changing over time. There is instantaneous excess volatility of financial assets over consumption growth.

The equity premium follows from Lemma 2.2,

$$E_t\left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt = \gamma\sigma^{(A)}\left(\sigma^{(A)}\right)^T - \frac{1-\gamma-\rho}{\rho}\frac{Y_t}{P_t}\frac{\sigma_t^{(\psi)}}{\psi_t}\left(\frac{\left(\sigma_t^{(\psi)}\right)^T}{\psi_t^2} + \sum_{i=1}^{2}\frac{\sigma_t^{\left(F^{y,(i)}\right)^T}}{1+\delta_i}\right) \tag{2.39}$$

Lemma 2.12 *Suppose* $\gamma \in (1, 1-\rho)$ $(\rho < 0)$*. There exists* $\overline{EIS}(n_t, \lambda_t)$ *such that* $EIS < \overline{EIS}(n_t, \lambda_t)$ *suffices for the interest rate to be decreasing in the birth rate and increasing in the mortality rate and the equity premium to be positive.*

Proof. See appendix. ■

The result is similar to the findings in the earlier discussion, but weaker. $\overline{EIS}(n_t, \lambda_t)$ depends on the current level of the birth rate and the death rate.

I illustrate numerically the quantitative importance of the model. To match the first two moments of US population statistics I set $\mu^{(n)} = -0.0076$, $\sigma^{(n)} = 0.0381$, $\mu^{(\lambda)} = -0.0071$, $\sigma^{(\lambda)} = 0.051$, and $correl\left(\frac{dn_t}{n_t}, \frac{d\lambda_t}{\lambda_t}\right) = -0.1$. I choose $\beta = 0.005$, $\mu^{(A)} + (1-a)\mu^{(K)} = 2.1\%$, $\sigma^{(A)} = 2.5\%$, $a = 0.9$, $\gamma = 5$, and $EIS = 0.067$.

Interest Rate and Equity Premium (in %): Calibration Output

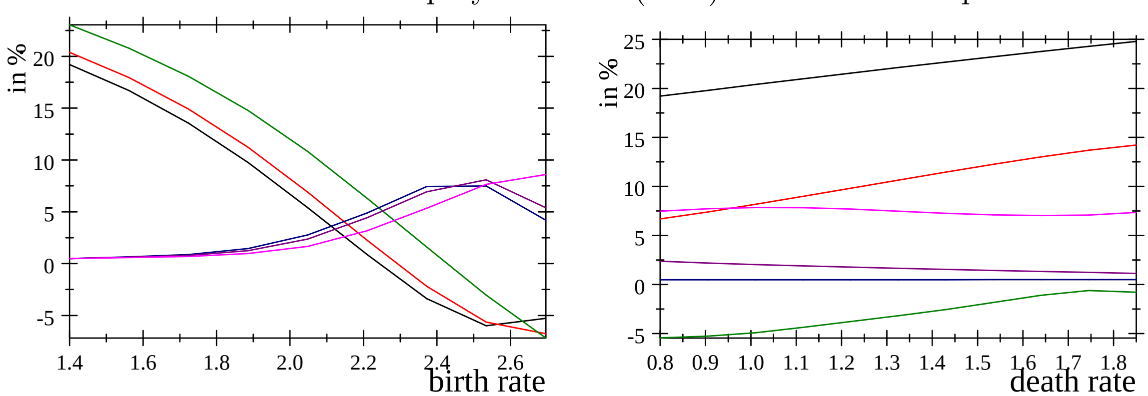

Figure 2.5: Left panel: Interest rate and equity premium for $\lambda = 0.8\%$ (black, blue), $\lambda = 1\%$ (red, purple), $\lambda = 1.5\%$ (green, mangenta). Right panel: Interest rate and equity premium for $n = 1.4\%$ (black, blue), $n = 2\%$ (red, purple), $n = 2.5\%$ (green, mangenta).

Figure 2.5 shows the equilibrium interest rate and equity premium dependent on the birth rate and the death rate. The interest rate is increasing in the death rate. It is mostly decreasing in the birth rate, but changes to be increasing if the birth rate is large. A high birth rate implies a low consumption-to-wealth ratio, and c_t^t is small. An additional increase in the birth rate causes the aggregate consumption share of the new born cohort to increase only little (as c_t^t is small), and puts moderate downward pressure on the interest rate. But, an increase in the birth rate also affects the interest rate positively due to the acceleration in production output growth, and the positive impact is independent of the current level in the birth rate. The latter (positive) effect is dominant if the birth rate is large, and the interest rate becomes increasing in the birth rate. Technically, Lemma 2.12 is more difficult to satisfy (a lower EIS is required) if the birth rate is high, i.e. $\frac{\partial \overline{EIS}(n_t,\lambda_t)}{\partial n} < 0$.

The equity premium is decreasing in the death rate and mostly increasing in the birth rate. Given a large current level in the birth rate, the equity premium starts to be decreasing in the birth rate because the consumption-to-wealth ratio (which is in a positive relation to the interest rate) becomes less sensitive to changes in the birth rate.

In the USA the birth rate was most of the time less than 2.5% in the 20th century, and (given $EIS = 0.067$) for $n < 2.5\%$ the interest rate and equity premium are decreasing respectively increasing in the birth rate.

The market price of risk compensating uncertainty in the birth rate follows a similar pattern as the equity premium. Given the birth rate is less than 2.5%, it is always increasing in the birth rate. The market price of risk compensating uncertainty in the death rate is slightly decreasing in mortality.

Similar, the exposure of the risky asset to uncertainty in the birth rate is increasing in the birth rate as long as the current level in the birth rate not too large, while the exposure to risk in the death rate is almost independent of the level in the death rate.

Changes in the birth rate cause a variation of considerable magnitude in the market price of (birth rate) risk, the exposure of the risky asset to uncertainty in the birth rate, and the equity premium. In contrast, changes in the death rate hardly cause any variation in neither the market price of (mortality) risk, the exposure of the stock to uncertainty in the death rate, or the equity premium.

Interest Rate and Equity Premium (in %): Calibration vs Data

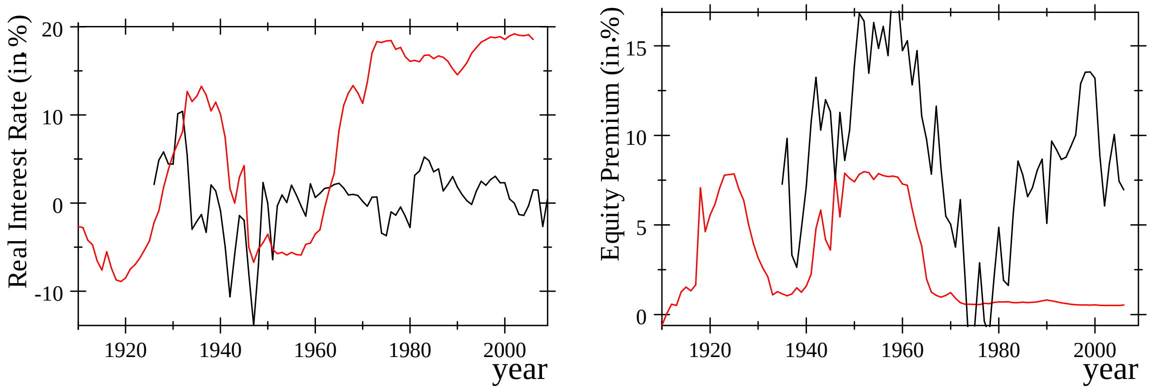

Figure 2.6: Left panel: Real interest rate in USA (black) and model implied interest rate using US birth and mortality (red). Right panel: 10 year averages of US stock market excess returns (black line) and model implied equity premium using US birth and mortality data (red).

Figure 2.6 compares the real interest rate and 10 year averages of stock market excess returns in the USA with the model implied interest rate and equity premium constructed from the calibration results and observed birth and death rates. The model is able to explain some of the time variation in the interest rate and the equity premium. Most of the variation in the equity premium is due to changes in the birth rate, while changes in the death rate have a small impact. The presented results do not consider leverage in equities. Leverage amplifies the time variation in the equity premium. In addition, features like financial constraints and limited asset market participation as discussed in Brooks (2004) and Geanakoplos et al. (2004) are likely to improve the results.

I simulate birth and death rate data and analyze the unconditional moments of aggregate consumption growth, the interest rate and stock returns. 100'000 simulations of 100 years of birth and death rate data show that on average the unconditional volatility of aggregate consumption growth is 3%, volatility of the interest rate is 8.5%, and volatility of stock returns is 24.8%. Demographic uncertainty introduces a substantial difference between the unconditional variation in consumption growth and asset returns.

Figure 2.7 illustrates the volatility clustering in stock returns due to demographic uncertainty. I simulate 1'000 years of birth and death rate data and calculate model implied stock returns. The left panel shows the stock returns and the right panel squared returns.

Simulated Stock Returns

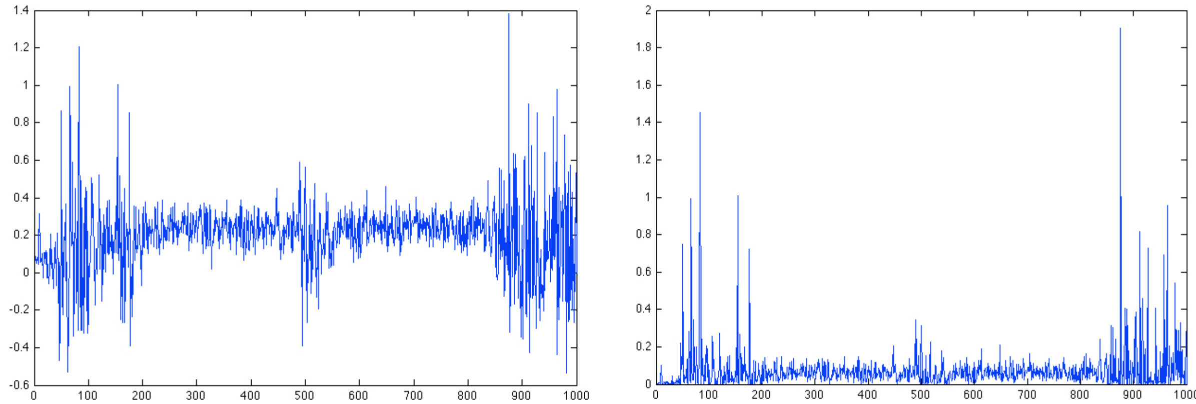

Figure 2.7: Simulation output using 1000 years of simulated birth and death rate data: stock return (left panel) and squared stock return (right panel).

2.4 Extension and Comments

I heuristically discuss three important extensions of the model.

1) Generalization of Birth and Death Rate Processes

For simplicity I have modelled birth and death rates as geometric Brownian motions with constant drifts and diffusions. Instead I may consider for instance autoregressive processes. An AR(1) process with a positive first autocorrelation term describes changes in the birth rate well. Changes in the death rate are not autocorrelated.

If changes in the birth rate (and the death rate) are described by an autoregressive process rather than white noise, I expect the consumption-to-wealth ratio and the interest rate to depend (in addition to the current level and volatility) on recent changes in the birth rate (and the death rate).

The static model in section 3.1 provides a good intuition. Assume that changes in the birth rate are positively autocorrelated (as found in the data). Consider a large past increase (decrease) in the birth rate. Accordingly, a further large increase (decrease) in the birth rate is expected in the near future. Because the consumption-to-wealth ratio is negatively related to the birth rate (for a small enough EIS), a large expected increase (decrease) in the birth rate creates an incentive for a forward-looking agent to choose a low (high) current consumption-to-wealth ratio. The property $\frac{\partial \psi}{\partial n} < 0$ and the positive relation between ψ_t and $\mu_t^{(\psi)} = \frac{1}{dt} E_t (d\psi_t)$ in equation (2.86) formalizes the intuition. In equilibrium a low (high) consumption-to-wealth ratio corresponds to a low (high) interest rate (if $EIS < 1$). As a result the consumption-to-wealth ratio and the interest rate are negatively related to recent changes in the birth rate. I show in the first section in the appendix that there is empirical evidence for a negative relationship between the level of the current interest rate and past changes in the birth rate. Geanakoplos et al. (2004) document a similar relation between changes in demographic quantities and the level in the interest rate.

2) Social Security and other Intergenerational Transfers

The simplest way to model a social security system is by letting agents pay a (possibly age-dependent) labor income tax which is redistributed to the entire population.[21] In the limit when all labor income is collected and agents receive/ consume per capita GDP, the consumption goods allocation is identical to the first best allocation in an Arrow economy with (intergenerational) market completeness.[22]

Other intergenerational transfers are modelled by assuming that agents care about other agents' utilities. For instance, a parent may care about how much utility his children obtain and vice versa. In the extreme case when agents care about other agents' utilities the same as about their own utility, the economy achieves the first best allocation.

I look at the extreme case when the intergenerational wealth redistribution leads to the first best allocation. Noticing that under first best $c_t^t = \frac{C_t}{N_t}$, equation (2.10) and equation (2.11) become $\frac{\partial r}{\partial n} = (1-\rho)(a-1)$ and $\frac{\partial r}{\partial \lambda} = (1-\rho)(1-a) + \frac{\rho}{1-\gamma} - 1$. The comparative statics analysis suggests that for $\min\left\{-\frac{a(1-\gamma)}{(1-a)(1-\gamma)-1}, 0\right\} > \rho$ the interest rate and the consumption-to-wealth ratio are decreasing in the birth rate and increasing in the death rate. The result is stronger than Lemma 2.4 and 2.5. The interpretation

[21] Alternatively, I may consider a set-up as in Gertler (1997) where agents randomly switch from a working state to retirement and social security is a transfer between workers and retirees. In that case, to keep my model tractable I must introduce a new set of contracts to let agents hedge the new retirement risk and to keep markets dynamically complete. However, in Gertler (1997) the results are driven by the market incompleteness due to retirment risk.

[22] See also Abel (2003) for a discussion on how a social security system can be employed to approach the Golden Rule in the eocnomy

and the key driving forces for the result remain the same. I expect the intuition to continue to hold in a dynamic model with stochastic changes in birth and death rates. Because the consumption-to-wealth ratio is sensitive to demographic changes, I expect demographic uncertainty to be priced and the equity premium to be time varying.

Since my results are not affected even if I impose the first best allocation, I do not expect that the introduction of a (reasonable) social security system or other intergenerational transfers alter the fundamental qualitative results of my model. Though the quantitative magnitude of the effects might change. For instance, Brooks (2004) argues that the introduction of a social security system has important quantitative asset pricing implications.

3) Capital Accumulation

For simplicity I have assumed that firms cannot invest. But demographic changes have a long term impact on the labor supply, and it is reasonable that a firm optimally adjusts its capital stock in response to highly predictable long run changes in the labor market.

I consider capital accumulation with convex adjustment costs as in Abel (2003). Suppose the birth rate increases (decreases) or the death rate decreases (increases). The stock price increases (declines). The firm starts to invest (disinvest) and less (more) units of production output will be available to consumers. Under time additive utilities, a drop (increase) in current aggregate consumption implies a high (low) marginal utility state. As a result I expect a positive correlation between the marginal utility process and stock returns which implies a negative equity premium.

In contrast, under recursive utilities it is not clear whether capital accumulation has a negative or a positive impact on the equity premium. It is still true that current aggregate consumption drops (increases) due to investment (disinvestment) by the representative firm, which has a positive (negative) effect on marginal utility. But future aggregate consumption will grow faster (slower) due to the initial investment (disinvestment) and under certain restrictions on the parameterization of the recursive preferences, this has a negative (positive) impact on marginal utility. The two effects are offsetting and it is not clear whether there is a positive or negative correlation between the marginal utility process and stock returns.

I expect capital accumulation to reduce the sensitivity of the interest rate and the consumption-to-wealth ratio towards changes in birth and death rates. Capital accumulation causes growth in production output to react stronger in response to changes in birth and death rates because an increase (decrease) in the birth rate or a decrease (increase) in the death rate comes with additional investment (disinvestment). Equation (2.10) and (2.11) suggest that $|\frac{\partial r}{\partial n}| < 0$, $|\frac{\partial \psi}{\partial n}| < 0$, $|\frac{\partial r}{\partial \lambda}| < 0$ and $|\frac{\partial \psi}{\partial \lambda}| < 0$. Accordingly, a decrease in the sensitivity of the consumption-to-wealth ratio to changes in birth and death rates causes the market price of risk and the volatility of asset prices to decline.

2.5 Conclusion

I answer the questions how demographic transitions affect the value of financial assets and whether demographic uncertainty is priced in financial markets. I solve an analytically tractable general equilibrium model with stochastically changing birth and death rates. The interest rate is time varying due to demographic changes. For a small enough EIS and a moderate RRA the interest rate is decreasing in the birth rate and increasing in the death rate. I limit my focus to a discussion on the short rate and leave term structure implications for future research. The equity premium is stochastically changing over time and I provide conditions that suffice for the equity premium to be increasing in the birth rate and decreasing in the death rate.

An important result for future empirical research is that the identified asset pricing implications of changes in death and birth rates work through different channels and it is essential to model birth and death rates separately and not as one general state variable that determines total population growth or the average age of the population.

Numerical calibrations suggest that stochastic changes in the birth rate have stronger implications on asset pricing than changes in the death rate. Demographic uncertainty explains part of the equity premium puzzle and the excess volatility of asset returns over volatility in aggregate consumption growth. Moreover, the model is able to explain some of the time variation in the interest rate and stock market excess returns in the USA in the 20th century. In particular, the model helps to explain why the average interest rate was only -2.5% and the average stock market excess return was

16% during the baby boom period (early 1940's to 1960's), and the interest rate was roughly $1\% - 2\%$ and the stock market excess return was only 5.5% in 1926-1940 and after the early 1960's.

2.6 Appendix

2.6.1 Empirical Motivation

I use data from 1926 to 2006 provided by CRSP. I approximate the annual real interest rate by the difference between the annualized nominal interest rate (multiplication of gross returns on 30 days Treasury Bills) and the realized inflation (CPI). I assume that in expectations realized inflation equals expected inflation. Stock market excess returns are the returns on the value weighted stock index provided by CRSP minus the nominal interest rate. I use a Hodrick-Prescott filter, a Baxter-King band pass filter and an average over 10 years rolling windows of the stock market excess returns to get estimates of the equity premium; all methods yield similar results.

Birth rate statistics are provided by the Department of Health and Human Services, National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute. The immigration rate is found in the Annual Statistical Yearbook by the U.S. Bureau of Citizenship and Immigration Services. Data on the central death rate are provided by the National Center for Health Statistics, USA, and The Human Mortality Database, University of California, Berkeley and Max Planck Institute.

It is questionable whether only birth rate data or a combination of birth and immigration rates should be used to analyze my model. In the USA immigrants are rather young and hence, in the model immigration could be considered as an equivalent to births of agents. Over the past two centuries in the USA roughly 15%-20% of all immigrants were children (younger than 14 years old), about 65%-70% were young adults (between 14 and 44 years old) and only 10%-15% were older than 44 years old (Source: Economic History Association, EH.net).

I run the regression $r_t = \alpha^{(r)} + \beta_n^{(r)} n_t + \beta_i^{(r)} i_t + \beta_\lambda^{(r)} \lambda_t + \beta_{\Delta n}^{(r)} \Delta n_t + \beta_{\Delta i}^{(r)} \Delta i_t + \beta_{\Delta\lambda}^{(r)} \Delta\lambda_t +$

Interest Rate and Equity Premium (in %)

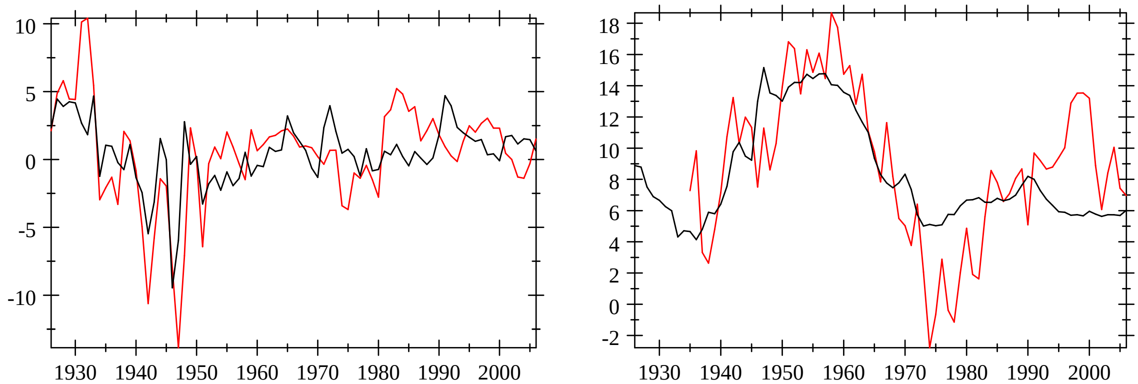

Figure 2.8: Left panel: Real interest rate in USA (red line), and implied interest rate from regression results and demographic statistics (black line). Right panel: 10 year averages of US stock market excess returns (red line), and implied equity premium from regression results and observed birth, immigration and death rates (black line).

$\epsilon_t^{(r)}$.[23] r_t represents the real interest rate, $\alpha^{(r)}$ is a constant term, $\beta_n^{(r)}$, $\beta_i^{(r)}$, $\beta_\lambda^{(r)}$, $\beta_{\Delta n}^{(r)}$, $\beta_{\Delta i}^{(r)}$, and $\beta_{\Delta\lambda}^{(r)}$ are regression coefficients, n_t, i_t and λ_t are the birth rate, the immigration rate and the death rate, Δ is a lag operator, and $\epsilon_t^{(r)}$ is an error term. The regression estimates are $\alpha^{(r)} = -0.007$, $\beta_n^{(r)} = -1.66$, $\beta_i^{(r)} = 9.19$, $\beta_\lambda^{(r)} = 2.54$, $\beta_{\Delta n}^{(r)} = -22.88$, $\beta_{\Delta i}^{(r)} = -5$, and $\beta_{\Delta\lambda}^{(r)} = -3.78$. I construct an implied real interest rate using the regression estimates and the observed birth, immigration and death rates. The result is displayed in the left panel in figure 2.8. I also run the regression $r_t^{(x)} = \alpha^{\left(r^{(x)}\right)} + \beta_n^{\left(r^{(x)}\right)} n_t + \beta_i^{\left(r^{(x)}\right)} i_t + \beta_\lambda^{\left(r^{(x)}\right)} \lambda_t + \epsilon_t^{\left(r^{(x)}\right)}$. $r_t^{(x)}$ represents the expected excess return on the value weighted stock market index provided by CRSP, $\alpha^{\left(r^{(x)}\right)}$ is a constant term, $\beta_n^{\left(r^{(x)}\right)}$, $\beta_i^{\left(r^{(x)}\right)}$, and $\beta_\lambda^{\left(r^{(x)}\right)}$ are regression coefficients, n_t is the birth rate, i_t is the immigration rate, λ_t is the death rate, and $\epsilon_t^{\left(r^{(x)}\right)}$ is an error term. The regression output is $\alpha^{\left(r^{(x)}\right)} = -0.068$, $\beta_n^{\left(r^{(x)}\right)} = 9.8$, $\beta_i^{\left(r^{(x)}\right)} = 1.17$, and $\beta_\lambda^{\left(r^{(x)}\right)} = -4.56$. I construct an implied equity premium using the regression estimates and the observed birth and death rates. The result is presented in the right panel in figure 2.8.

Next, I run the two regression but exclude the immigration rate as an explanatory variable. For the interest rate I have the regression $r_t = \alpha^{(r)} + \beta_n^{(r)} n_t + \beta_\lambda^{(r)} \lambda_t +$

[23] It is not obvious whether the interest rate, the equity premium and the birth and death rates are stationary or non-stationary processes. In the case of non-stationarity I interpret the estimation as a Engle-Granger regression.

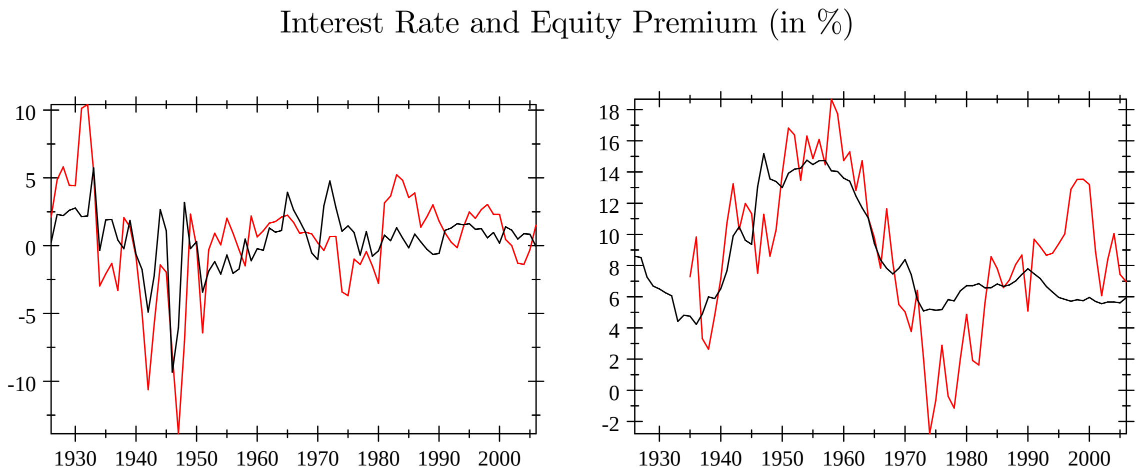

Figure 2.9: Left panel: Real interest rate in USA (red line), and implied interest rate from regression results and demographic statistics (black line). Right panel: 10 year averages of US stock market excess returns (red line), and implied equity premium from regression results and observed birth and death rates (black line).

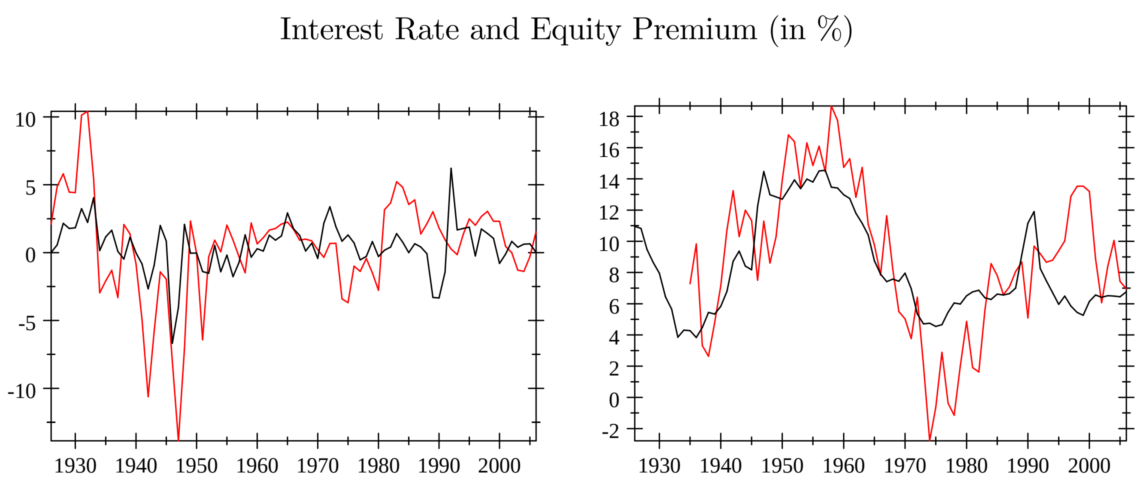

Figure 2.10: Left panel: Real interest rate in USA (red line), and implied interest rate from regression results and demographic statistics (black line). Right panel: 10 year averages of US stock market excess returns (red line), and implied equity premium from regression results and observed birth plus immigration rate and death rate (black line).

$\beta^{(r)}_{\Delta n}\Delta n_t + \beta^{(r)}_{\Delta\lambda}\Delta\lambda_t + \epsilon^{(r)}_t$, and I get the output $\alpha^{(r)} = 0.037$, $\beta^{(r)}_n = -2.47$, $\beta^{(r)}_\lambda = 1.4$, $\beta^{(r)}_{\Delta n} = -24.05$, and $\beta^{(r)}_{\Delta\lambda} = -1.71$. For the equity premium the regression takes the form $r^{(x)}_t = \alpha^{\left(r^{(x)}\right)} + \beta^{\left(r^{(x)}\right)}_n n_t + \beta^{\left(r^{(x)}\right)}_\lambda \lambda_t + \epsilon^{\left(r^{(x)}\right)}_t$, and the estimation gives $\alpha^{\left(r^{(x)}\right)} = -0.063$, $\beta^{\left(r^{(x)}\right)}_n = 9.7$, and $\beta^{\left(r^{(x)}\right)}_\lambda = -4.72$. I reconstruct again the model implied interest rate and equity premium and present the results in figure 2.9.

Finally, I run the two regressions but add up birth and immigration rates and use them as one explanatory variable ($\widehat{n}_t = n_t + i_t$). For the interest rate I have the specification $r_t = \alpha^{(r)} + \beta^{(r)}_{\widehat{n}}\widehat{n}_t + \beta^{(r)}_\lambda\lambda_t + \beta^{(r)}_{\Delta\widehat{n}}\Delta\widehat{n}_t + \beta^{(r)}_{\Delta\lambda}\Delta\lambda_t + \epsilon^{(r)}_t$, and the estimation results $\alpha^{(r)} = 0.03$, $\beta^{(r)}_{\widehat{n}} = -1.47$, $\beta^{(r)}_\lambda = 0.53$, $\beta^{(r)}_{\Delta\widehat{n}} = -15.71$, and $\beta^{(r)}_{\Delta\lambda} = 2.41$. In the case of the equity premium I run the regression $r^{(x)}_t = \alpha^{\left(r^{(x)}\right)} + \beta^{\left(r^{(x)}\right)}_{\widehat{n}}\widehat{n}_t + \beta^{\left(r^{(x)}\right)}_\lambda\lambda_t + \epsilon^{\left(r^{(x)}\right)}_t$, and estimate $\alpha^{\left(r^{(x)}\right)} = -0.098$, $\beta^{\left(r^{(x)}\right)}_{\widehat{n}} = 9.72$, and $\beta^{\left(r^{(x)}\right)}_\lambda = -2.96$. I compare the observed and the model implied financial quantities of the last specification in figure 2.10.

2.6.2 Additional Calibrations of the Model

First, I present some further results obtained for the calibration exercise in the paper. The left panel in figure 2.11 illustrates the model implied time variation in the market price of risk for uncertainty in the birth respectively the death rate. Note that a negative market price of risk for uncertainty in the death rate implies a positive equity premium because the stock price reacts negatively to shocks in the death rate. The right panel in figure 2.11 displays the (conditional) model implied stock price volatility due to birth respectively death rate uncertainty. Clearly, uncertainty in the birth rate have a much stronger impact on pricing of financial assets than mortality risk. Moreover, most of the time variation in the stock price volatility and the equity premium are due to changes in the birth rate while changes in the death rate have a rather moderate impact.

I present two more calibration outputs of the general model with Brownian uncertainty. First, I numerically solve the model with the inputs $\mu^{(n)} = -0.0049$, $\sigma^{(n)} = 0.0472$, $\mu^{(\lambda)} = -0.0071$, $\sigma^{(\lambda)} = 0.051$, $correl\left(\frac{dn_t}{n_t}, \frac{d\lambda_t}{\lambda_t}\right) = -0.1$, $\beta = 0.005$, $\mu^{(A)} + (1-a)\,\mu^{(K)} = 2.1\%$, $\sigma^{(A)} = 2.5\%$, $a = 0.9$, $\gamma = 5$, and $EIS = 0.067$. The difference to the calibration in the text is that I use the moments of the sum of US birth and immigration rates instead of only the birth rate. I plot the model implied interest rate

Model Implied Market Price of Risk and Stock Price Volatility (in %)

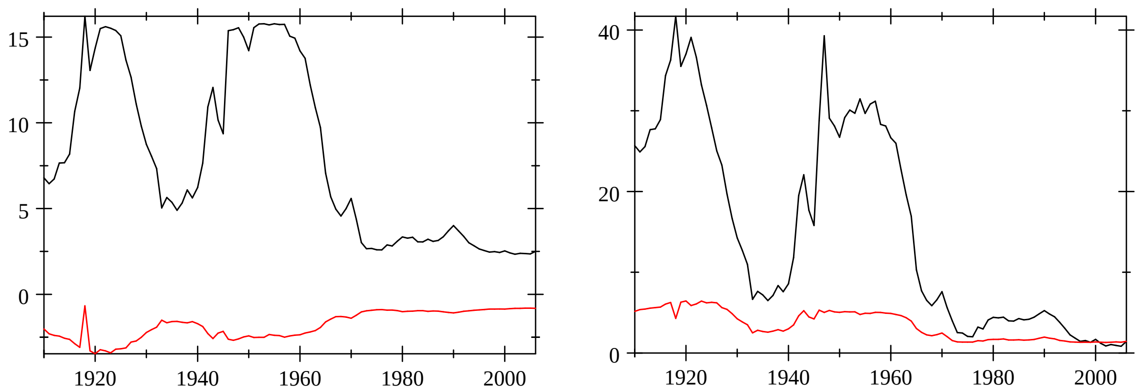

Figure 2.11: Left panel: Model implied market price of birth rate risk (black) and model implied market price of mortality risk (red). Right panel: Model implied stock market volatility due to birth rate uncertainty (black) and model implied stock price volatility due to death rate uncertainty (red).

Interest Rate and Equity Premium (in %): Calibration vs Data

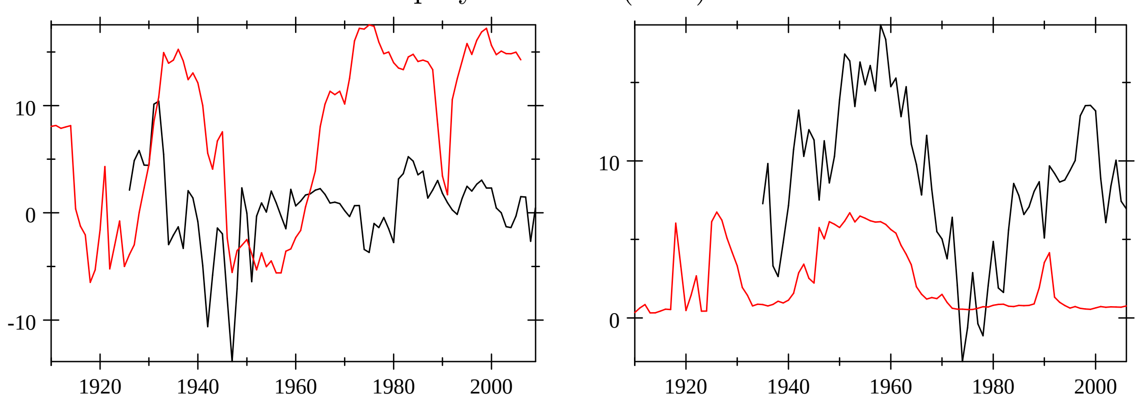

Figure 2.12: Left panel: Real interest rate in USA (black) and model implied interest rate using US birth and mortality (red). Right panel: US stock market excess returns using Hodrick-Prescott filter (black) and model implied equity premium using US birth and mortality data (red).

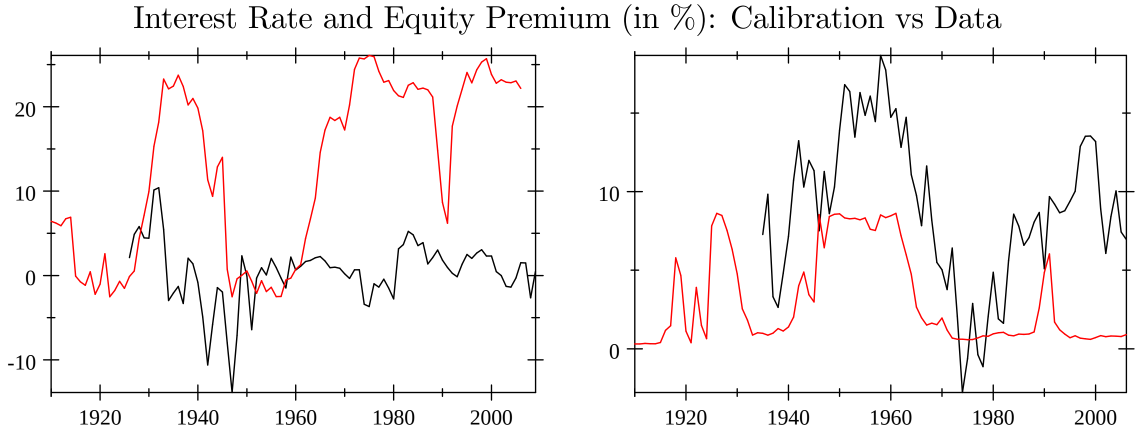

Figure 2.13: Left panel: Real interest rate in USA (black) and model implied interest rate using US birth and mortality (red). Right panel: US stock market excess returns using Hodrick-Prescott filter (black) and model implied equity premium using US birth and mortality data (red).

and equity premium using the observed birth plus immigration rates and the death rate.

The major difference to the calibration output in the text (without immigration) is that the short-lived immigration wave in the USA at around 1990 generates a substantial but short-lived hike in the interest rate and the equity premium. In figure 2.12 I plot observed/ estimated and model implied (real) interest rate and equity premium.

Second, I repeat the calibration exercise (case of birth plus immigration) but change the EIS from 0.067 to 0.05. As expected the interest rate increases and the variation in the equity premium becomes stronger and closer to what I observe in financial data. The results are provided in figure 2.13.

2.6.3 Derivation of Kreps and Porteus (1978) Type Stochastic Differential Utilities given Uncertain Lifetimes

Stochastic differential utilities are a continuous time counterpart to the recursive utilities discussed by Epstein and Zin (1989, 1991). Duffie and Epstein (1992a) restrict their derivation to the case of Brownian information. In a model with uncertain lifetimes the dynamics of the value function include a Poisson jump term that sets the value function to zero when the agent passes away. If information is generated by a Brownian motion and a Poisson jump process (due to lifetime uncertainty), then I have to make some adaptations to the utility specification introduced in Duffie and Epstein (1992a). Following the notation in Duffie and Epstein (1992a), the dynamics of the utility (given the agent is still alive at time t and will die at time τ) have to be rewritten as (given $t \leq \tau$)

$$dV_t = \mu_t dt + \sigma_t dB_t - V_t dQ_t \tag{2.40}$$

B is a Brownian motion, Q is a compensated Poisson jump process with hazard rate λ_t. The agent dies if Q jumps the first time since the agent is born and I denote the time of the first jump by τ. The arrival rate of death is time varying and stochastic, i.e. Q is a doubly stochastic process (Cox process). Following the lines in Duffie and Epstein (1992a) this implies

$$\begin{aligned}
\mu_t &= -f(c_t, V_t) - \frac{1}{2}A(V_t)\sigma_t^2 - \lambda_t [M(V_t, V_{t-}) - M(V_{t-}, V_{t-})] \qquad (2.41)\\
&= -f(c_t, V_t) - \frac{1}{2}A(V_t)\sigma_t^2 - \lambda_t [M(V_{t-} - V_{t-}, V_{t-}) - M(V_{t-}, V_{t-})]\\
&= -f(c_t, V_t) - \frac{1}{2}A(V_t)\sigma_t^2 + \lambda_t M(V_{t-}, V_{t-})
\end{aligned}$$

$M(y, x) = \frac{h(y)}{h'(x)}$ is the local gradient representation of the certainty equivalent m, i.e. $\nabla m(\delta_x; p) = \int M(y, x)\, dp(y)$, and $h(.)$ is defined as $h(m(V)) = E(h(V))$.

Since

$$V_t = E_t[V_T \mid \tau > T] + E_t\left[\int_t^T -\mu_s ds\right] \tag{2.42}$$

it follows that as T goes to infinity

$$V_t = E_t\left[\int_t^T f(c_s, V_s) + \frac{1}{2}A(V_s)\sigma_s^2 - \lambda_s M(V_s, V_s)\, ds\right] \tag{2.43}$$

I can show as in Duffie and Epstein (1992a) that the following transformation leads to an equivalent utility function $\overline{V}_t = \phi(V_t)$ with

$$\begin{aligned}
f(c_t, z) &= \frac{\overline{f}(c_t, \phi(z))}{\phi'(z)} && (2.44)\\
m(z) &= \phi^{-1}(\overline{m}[\phi(z)]) && (2.45)\\
\phi'(z) M(y, z) &= \overline{M}(\phi(y), \phi(z)) && (2.46)\\
A(x) &= \phi'(x)\overline{A}(\phi(x)) + \frac{\phi''(x)}{\phi'(x)} && (2.47)
\end{aligned}$$

This follows from

$$\begin{aligned}
d\overline{V}_t &= \left(\phi'(V_t)\left[-f(c_t,V_t)-\frac{1}{2}A(V_t)\sigma_t^2+\lambda_t M(V_t,V_t)\right]+\frac{1}{2}\phi''(V_t)\sigma_t^2\right)dt \\
&\quad +\phi'(V_t)\sigma_t dB_t-\phi(V_t)dQ_t \qquad (2.48)\\
&= \left[-\overline{f}\left(c_t,\overline{V_t}\right)-\frac{1}{2}\overline{A}\left(\overline{V}_t\right)\overline{\sigma}_t^2+\lambda_t\overline{M}\left(\overline{V}_t,\overline{V}_t\right)\right]dt+\overline{\sigma}_t dB_t-\overline{V}_t dQ_t
\end{aligned}$$

with

$$\overline{\sigma}_t=\phi'(V_t)\sigma_t \qquad (2.49)$$

and

$$\begin{aligned}
& -f(c_t,V_t)-\frac{1}{2}A(V_t)\sigma_t^2+\lambda_t M(V_t,V_t) \qquad (2.50)\\
&= -\frac{\overline{f}\left(c_t,\overline{V_t}\right)}{\phi'(V_t)}-\frac{1}{2}\frac{\overline{A}\left(\overline{V}_t\right)}{\phi'(V_t)}\overline{\sigma}_t^2-\frac{1}{2}\frac{\phi''(V_t)}{\phi'(V_t)}\sigma_t^2+\lambda_t\frac{\overline{M}\left(\overline{V}_t,\overline{V}_t\right)}{\phi'(V_t)}\\
&= -\frac{\overline{f}\left(c_t,\overline{V_t}\right)}{\phi'(V_t)}-\frac{1}{2}\left[\overline{A}\left(\overline{V}_t\right)\phi'(V_t)+\frac{\phi''(V_t)}{\phi'(V_t)}\right]\sigma_t^2+\lambda_t\frac{\overline{M}\left(\overline{V}_t,\overline{V}_t\right)}{\phi'(V_t)}
\end{aligned}$$

Choosing $\phi''(x)=\phi'(x)A(x)$ implies $\overline{A}(x)=0$, and thus, $\overline{m}[x]=E[x]$.

For the specification introduced in Duffie and Epstein (1992a), featuring the Kreps and Porteus (1978) property of preferences over the timing of risk resolution,

$$\begin{aligned}
f(c_s,V_s) &= \frac{\beta}{\rho}\frac{c_s^\rho-V_s^\rho}{V_s^{\rho-1}} \qquad (2.51)\\
m(x) &= \left(E\left[x^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}} \qquad (2.52)\\
V_t &= E_t\left[\int_t^\infty f(c_s,V_s)+\frac{1}{2}A(V_s)\sigma_s^2-\lambda_s M(V_s,V_s)\,ds\right] \qquad (2.53)\\
&= E_t\left[\int_t^\infty \frac{\beta}{\rho}\frac{c_s^\rho-V_s^\rho}{V_s^{\rho-1}}-\frac{1}{2}\frac{\gamma}{V_s}\sigma_s^2-\lambda_s\frac{V_s}{1-\gamma}ds\right]
\end{aligned}$$

Letting $\phi(x)=\frac{1}{1-\gamma}x^{1-\gamma}$ to get an equivalent utility function $\overline{V}_t=\phi(V_t)$, I end up with

$$\begin{aligned}
\overline{f}\left(c_s,\overline{V}_s\right) &= \phi'\left(\phi^{-1}\left(\overline{V}_s\right)\right)f\left(c_s,\phi^{-1}\left(\overline{V}_s\right)\right) \qquad (2.54)\\
&= \frac{\beta}{\rho}\frac{c_s^\rho-\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}}}{\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}-1}}\\
\overline{m}(x) &= E[x] \qquad (2.55)
\end{aligned}$$

I get the utility specification

$$\begin{aligned}
\overline{V}_t &= E_t\left[\int_t^\infty \overline{f}\left(c_s,\overline{V}_s\right)+\frac{1}{2}\overline{A}\left(\overline{V}_s\right)\sigma_s^2-\lambda_s\overline{M}\left(\overline{V}_s,\overline{V}_s\right)ds\right] \qquad (2.56)\\
&= E_t\left[\int_t^\infty \frac{\beta}{\rho}\frac{c_s^\rho-\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}}}{\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}-1}}-\lambda_s\overline{V}_s ds\right]\\
&= E_t\left[\int_t^\infty \frac{\beta c_s^\rho-\left(\beta+\frac{\rho}{(1-\gamma)}\lambda_s\right)\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}}}{\rho\left[(1-\gamma)\overline{V}_s\right]^{\frac{\rho}{1-\gamma}-1}}ds\right]\\
&= E_t\left[\int_t^\infty \widehat{f}\left(c_s,\overline{V}_s\right)ds\right]
\end{aligned}$$

As shown in the online appendix of Garleanu and Panageas (2010) the same specification can be obtained as a continuous time limit of the discrete time recursive utility function

$$\begin{aligned}
V_t &= \left\{c_t^\rho+(1-\beta)E_t\left[\left(1_sV_{t+1}\right)^{1-\gamma}\right]^{\frac{\rho}{1-\gamma}}\right\}^{\frac{1}{\rho}} \qquad (2.57)\\
&= \left\{c_t^\rho+(1-\beta)E_t\left[(1-\lambda_t)V_{t+1}^{1-\gamma}\right]^{\frac{\rho}{1-\gamma}}\right\}^{\frac{1}{\rho}}\\
&= \left\{c_t^\rho+(1-\beta)(1-\lambda_t)^{\frac{\rho}{1-\gamma}}E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{\rho}{1-\gamma}}\right\}^{\frac{1}{\rho}}
\end{aligned}$$

1_s is an indicator function determining whether the agent survives ($1_s = 1$) or passes away ($1_s = 0$). The non-linear "discounting" term $(1-\lambda_t)^{\frac{\rho}{1-\gamma}}$ captures risk aversion towards the timing of death. This relates to the discussion by Bommier (2003). Depending on the preference parameters, $\frac{\rho}{1-\gamma} > (<)\, 0$, an agent is less (more) concerned about future consumption (utility) and wants to save less (more) than under a certain length of life. The utility specification in the paper of Garleanu and Panageas (2010) differs from my specification because (opposed to my approach) they exclude risk aversion towards the timing of death.

Because the utility function is a continuous time version of the recursive utility function introduced by Epstein and Zin (1989, 1991), in order for the agent to have a preference for early (late) resolution of risk (in the sense of Kreps and Porteus (1978)), I need

$$1-\gamma < (>)\, \rho \qquad (2.58)$$

This insight becomes clear when considering the discrete time recursive utility function

$$V_t = \left[c_t^\rho + (1-\beta) E_t \left[1_s V_{t+1}^{1-\gamma} \right]^{\frac{\rho}{1-\gamma}} \right]^{\frac{1}{\rho}} \tag{2.59}$$

I define $V_t = \overline{V}_t^{\frac{1}{\rho}}$ and rewrite the discrete time utility specification as

$$\overline{V}_t = \left[c_t^\rho + (1-\beta) E_t \left[1_s \overline{V}_{t+1}^{\frac{1-\gamma}{\rho}} \right]^{\frac{\rho}{1-\gamma}} \right] \tag{2.60}$$

For $\rho > 0$, $\arg\sup_{\{c^s, X^s\} \in (\Im \times \mathcal{L}^2)} \{V_t\} = \arg\sup_{\{c^s, X^s\} \in (\Im \times \mathcal{L}^2)} \{\overline{V}_t\}$ and by Jensen's inequality early (late) resolution of risk is preferred if $\frac{1-\gamma}{\rho} < (>) \, 1$ or $1 - \gamma < (>) \, \rho$. For $\rho < 0$, $\arg\sup_{\{c^s, X^s\} \in (\Im \times \mathcal{L}^2)} \{V_t\} = \arg\sup_{\{c^s, X^s\} \in (\Im \times \mathcal{L}^2)} \max \{-\overline{V}_t\}$ and by Jensen's inequality early (late) resolution is preferred if $\frac{1-\gamma}{\rho} > (<) \, 1$ or $1 - \gamma < (>) \, \rho$.

The specification nests the special case of a time additive expected utility function featuring a CRRA profile with $\gamma = \frac{1}{IES}$. Indeed setting $\gamma = 1 - \rho$ reduces to the familiar specification for power utilities

$$\begin{aligned} \overline{V}_t &= E_t \left[\int_t^\infty \frac{\beta c_s^{1-\gamma}}{1-\gamma} - (\beta + \lambda_s) \overline{V}_s ds \right] \\ &= E_t \left[\int_t^\infty \frac{\beta c_s^{1-\gamma}}{1-\gamma} e^{-\int_t^s \beta + \lambda_u du} ds \right] \end{aligned} \tag{2.61}$$

The condition $\gamma = 1 - \rho$ implies indifference with respect to timing of risk resolution; neither early nor late resolution of risk is preferred. In the case of time additive utility agents also become risk neutral towards uncertainty about the timing of death (cf. also Bommier (2003)).

There are a few comments on the specification. As the utility function may be defined on the negative space, it might seem that being dead is desirable. I can rule out this problem by not giving the agent the option to commit suicide. One may also circumvent the problem of suicidal agents by adding a large enough constant term to the aggregator function $f(.)$, so that the agent draws utility from simply being alive. Such a constant term does not matter in the utility maximization problem. Further, the specification here excludes bequest motives. This is restrictive, but in turn a too large bequest motive may give rise to suicidal behavior of an agent and counter-intuitive results (for more details see Maurer, 2010).

The derivation of the utility specification in the economy with regime shifts (Markov switching model) follows the same steps. Let the state of the world be indicated by the state variable $S_t \in \{0,1\}$, which jumps when a regime shift occurs. Adjustments have to be done with respect to the dynamics in the value function,

$$dV_t = \mu_t dt + 1_{\{S_t=1\}} s_1^{(V)} |d\widehat{S}_t| + 1_{\{S_t=0\}} s_0^{(V)} |d\widehat{S}_t| - V_t dQ_t \tag{2.62}$$

$\widehat{S}_t$ is a compensated Poisson jump process corresponding to the Markov switching process S_t, and $s_i^{(V)}$ denotes the change in the value function due to a jump from state $i \in \{1,0\}$ to the other state. The drift term is given by

$$\begin{aligned} \mu_t &= -f\left(c_t, V_t\right) + 1_{\{S_t=1\}} \overline{\theta}_H \left[M\left(V_{t-} + s_1^{(V)} V_{t-}\right) - M\left(V_{t-}, V_{t-}\right) \right] \\ &\quad +1_{\{S_t=0\}} \overline{\theta}_L \left[M\left(V_{t-} + s_0^{(V)}, V_{t-}\right) - M\left(V_{t-}, V_{t-}\right) \right] + \lambda_t M\left(V_{t-}, V_{t-}\right) \end{aligned} \tag{2.63}$$

The remaining of the derivation follows by applying the same lines of argument as above. The specification of the SDU in case of regime shifts in the birth rate becomes

$$V_s^s = E_s \left[\int_s^\infty f\left(c_u^s, V_u^s\right) du \right] \tag{2.64}$$

with

$$f\left(c_u^s, V_u^s\right) = \frac{\beta \left(c_u^s\right)^\rho - \left(\beta + \frac{\rho}{1-\gamma} \lambda\right) \left[\left(1-\gamma\right) V_u^s\right]^{\frac{\rho}{1-\gamma}}}{\rho \left[\left(1-\gamma\right) V_u^s\right]^{\frac{\rho}{1-\gamma}-1}} - \left[\begin{array}{c} 1_{\left\{S_u^{(n)}=1\right\}} \overline{\theta}_H^{(n)} s_1^{(V^s,n)} \\ +1_{\left\{S_u^{(n)}=0\right\}} \overline{\theta}_L^{(n)} s_0^{(V^s,n)} \end{array} \right] \tag{2.65}$$

The specification in case of regime shifts in the death rate is written as

$$V_s^s = E_s \left[\int_s^\infty f\left(c_u^s, V_u^s\right) du \right] \tag{2.66}$$

with

$$f\left(c_u^s, V_u^s\right) = \frac{\beta \left(c_u^s\right)^\rho - \left(\beta + \frac{\rho}{1-\gamma} \lambda_u\right) \left[\left(1-\gamma\right) V_u^s\right]^{\frac{\rho}{1-\gamma}}}{\rho \left[\left(1-\gamma\right) V_u^s\right]^{\frac{\rho}{1-\gamma}-1}} - \left[\begin{array}{c} 1_{\left\{S_u^{(\lambda)}=1\right\}} \overline{\theta}_H^{(\lambda)} s_1^{(V^s,\lambda)} \\ +1_{\left\{S_u^{(\lambda)}=0\right\}} \overline{\theta}_L^{(\lambda)} s_0^{(V^s,\lambda)} \end{array} \right] \tag{2.67}$$

2.6.4 Proofs of Propositions

Proposition 2.1 is a special cases of Proposition 2.4. Proposition 2.2 and 2.3 are closely related to Proposition 2.4. I provide a proof for the general case and show afterwards how to get from there the other Propositions.

Proof of Proposition 2.4. Following Duffie and Skiadas (1994, Theorem 2), the Gateau derivative (directional derivative) of the utility function in equation (2.2) at $\overline{c}^s$ in the direction x is

$$\begin{aligned}\nabla V_s^s\left(\overline{c}^s;x\right) &\equiv \lim_{\alpha\to 0}\frac{V_s^s\left(\overline{c}^s+\alpha x\right)-V_s^s\left(\overline{c}^s\right)}{\alpha} \\ &= E_s\left[\int_s^\infty e^{\int_s^t \frac{\partial}{\partial V_u^s} f(\overline{c}_u^s, V_u^s)du}\frac{\partial}{\partial c_t} f\left(\overline{c}_t^s, V_t^s\right)xdt\right] \\ &= E_s\left[\int_s^\infty R_t x dt\right]\end{aligned} \tag{2.68}$$

The Riesz representation process R_t is defined as

$$R_t = e^{\int_s^t \frac{\partial}{\partial V_u^s} f(\overline{c}_u^s, V_u^s)du}\frac{\partial}{\partial c_t} f\left(\overline{c}_t^s, V_t^s\right) \tag{2.69}$$

Optimality implies for any agent born at time s (assuming that the optimal consumption plan c^{s^*} is in the interior)

$$\nabla V_s^s\left(c^{s^*};\left[c^s-c^{s^*}\right]\right)=0 \tag{2.70}$$

for all admissible consumption plans $c^s\in\Im$. Since $F_s^{(\lambda)}\left(c^s-c^{s^*}\right)$ spans the set of all marketable cash flows M,

$$\nabla V_s^s\left(c^{s^*};F_s^{(\lambda)^{-1}}\left(x\right)\right)=0 \tag{2.71}$$

holds for all marketable cash flows $x\in M$. This implies that the Riesz representation process is a multiple of a SDF π,

$$R_t e^{\int_s^t \lambda_u du} = \eta^s \pi_t \tag{2.72}$$

for some constant η^s. Since markets are dynamically complete, the found pricing kernel is unique. I can solve for the optimal consumption plan for any agent born at time s

by plugging in the expression for the Riesz representation process (from now I drop the notation indicating the optimum by a star)

$$
\begin{aligned}
\eta^s \pi_t &= e^{\int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \frac{\partial}{\partial c_t} f\left(c_t^s, V_t^s\right) && (2.73) \\
c_t^s &= \left(\eta^s\right)^{-\frac{1}{1-\rho}} \beta^{\frac{1}{1-\rho}} e^{\frac{1}{1-\rho} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \left[(1-\gamma) V_t^s\right]^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} \left(\pi_t\right)^{-\frac{1}{1-\rho}} && (2.74) \\
c_t^s &= c_s^s e^{\frac{1}{1-\rho} \int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \left(\frac{V_t^s}{V_s^s}\right)^{\frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}} \left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{1-\rho}} && (2.75)
\end{aligned}
$$

Using dynamic programming to solve the utility maximization problem of an agent born at time s, I can state the Hamilton-Jacobi-Bellman equation as follows

$$
0 = \sup_{\{c_t^s, X_t^s\}} \left\{ f\left(c_t^s, V^s\left(\widehat{W}^s, \lambda, n, t\right)\right) dt + E_t\left[dV^s\left(\widehat{W}^s, \lambda, n, t\right)\right]\right\} \tag{2.76}
$$

with $\widehat{W}_t^s = W_t^s + E_t\left[\int_t^\infty e^{-\int_t^u \lambda_v dv} \frac{\pi_u}{\pi_t} y_u^s du\right]$ representing the agent's total wealth while W^s indicates his financial wealth. The first order condition with respect to optimal consumption is given by

$$
\frac{\partial}{\partial c_t^s} f\left(c_t^s, V^s\left(\widehat{W}^s, \lambda, n, t\right)\right) = \frac{\partial}{\partial \widehat{W}^s} V^s\left(\widehat{W}^s, \lambda, n, t\right) \tag{2.77}
$$

This holds conditional on survival. In the following I also condition on survival and although it is not written explicitly, I keep in mind that the variables c^s, W^s, and V^s jump to zero when the agent dies. I make the following conjecture for the value function

$$
V^s\left(\widehat{W}^s, \lambda, n, t\right) = \frac{\left(\widehat{W}_t^s\right)^{1-\gamma}}{1-\gamma} \beta^{\frac{1-\gamma}{\rho}} \psi_t\left(\lambda, n\right)^{-\frac{(1-\gamma)(1-\rho)}{\rho}} \tag{2.78}
$$

Plugging the conjectured value function into the FOC yields

$$
c_t^s = \widehat{W}_t^s \psi_t \tag{2.79}
$$

Plugging back into the conjectured value function and solving for c_t^s, allows us to rewrite the expression for optimal consumption as

$$
c_t^s = \left[(1-\gamma) V_t^s\right]^{\frac{1}{1-\gamma}} \psi_t^{\frac{1}{\rho}} \beta^{-\frac{1}{\rho}} \tag{2.80}
$$

Combining this with the expression obtained from the martingale approach (equation (2.74)) and solving for the value function leaves us with

$$V_t^s = \frac{1}{1-\gamma} (\eta^s)^{-\frac{1-\gamma}{\gamma}} e^{\frac{1-\gamma}{\gamma}\int_s^t \frac{\partial}{\partial V_u^s} f(c_u^s, V_u^s) + \lambda_u du} \beta^{\frac{1-\gamma}{\gamma\rho}} \psi_t^{-\frac{(1-\gamma)(1-\rho)}{\gamma\rho}} \pi_t^{-\frac{1-\gamma}{\gamma}} \tag{2.81}$$

$\frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s)$ is independent of the agent's time of birth, but is a function of only λ and n, and it is a Markov process,

$$\frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s) = \frac{1-\gamma-\rho}{\rho}\psi_t - \frac{1-\gamma}{\rho}\beta - \lambda_t \tag{2.82}$$

Solving for optimal consumption yields

$$\begin{aligned} c_t^s &= (\eta^s)^{-\frac{1}{\gamma}} \beta^{\frac{1-\gamma}{\gamma\rho}} e^{\frac{1}{\gamma}\int_s^t \frac{1-\gamma-\rho}{\rho}\psi_u - \frac{1-\gamma}{\rho}\beta du} \psi_t^{-\frac{1-\gamma-\rho}{\gamma\rho}} \pi_t^{-\frac{1}{\gamma}} && (2.83)\\ c_t^s &= c_s^s e^{\frac{1}{\gamma}\int_s^t \frac{1-\gamma-\rho}{\rho}\psi_u - \frac{1-\gamma}{\rho}\beta du} \left(\frac{\psi_t}{\psi_s}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}} \left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1}{\gamma}} && (2.84) \end{aligned}$$

The dynamics of the utility function are given by (assuming the agent survives over the next instant of time)

$$\begin{aligned} \frac{dV_t^s}{V_t^s} = \; & \frac{1-\gamma}{\gamma}\left(\frac{\partial}{\partial V_t^s} f(c_t^s, V_t^s) + \lambda_t\right) dt - \frac{1-\gamma}{\gamma}\frac{d\pi_t}{\pi_t} && (2.85)\\ & + \frac{(1-\gamma)^2(1-\rho)}{\gamma^2\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} - \frac{(1-\gamma)(1-\rho)}{\gamma\rho}\frac{d\psi_t}{\psi_t} \\ & + \frac{1}{2}\left(\frac{(1-\gamma)^2}{\gamma^2} + \frac{1-\gamma}{\gamma}\right)\left(\frac{d\pi_t}{\pi_t}\right)^2 \\ & + \frac{1}{2}\left(\frac{(1-\gamma)^2(1-\rho)^2}{\gamma^2\rho^2} + \frac{(1-\gamma)(1-\rho)}{\gamma\rho}\right)\left(\frac{d\psi_t}{\psi_t}\right)^2 \end{aligned}$$

According to the definition of the value function, the drift term equals $-f(c_t^s, V_t^s)\,dt$, which boils down to a PDE determining the function $\psi_t(\lambda, n)$ and at the same time verifies my conjecture about the value function (given a solution for the stated PDE exists)

$$
\begin{aligned}
0 \quad = \quad & \frac{1-\rho}{\rho}\psi_t\left(\lambda,n\right)-\frac{\beta}{\rho}-\frac{\gamma}{1-\gamma}\lambda_t-\frac{1-\rho}{\rho}\frac{1}{dt}E_t\left[\frac{d\psi_t}{\psi_t}\right] \\
& +\frac{1}{2\gamma}\frac{1}{dt}\left(\frac{d\pi_t}{\pi_t}\right)^2+\frac{\left(1-\gamma\right)\left(1-\rho\right)}{\gamma\rho}\frac{1}{dt}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} \\
& +\frac{1}{dt}E_t\left[-\frac{d\pi_t}{\pi_t}\right]+\frac{1}{2}\left(\frac{\left(1-\gamma\right)\left(1-\rho\right)^2}{\gamma\rho^2}+\frac{1-\rho}{\rho}\right)\frac{1}{dt}\left(\frac{d\psi_t}{\psi_t}\right)^2
\end{aligned}
\tag{2.86}
$$

The last step is equivalent to solving the HJB equation.

Next, I make use of the static budget constraint,

$$
0=W_s^s=E_s\left[\int_s^\infty e^{-\int_s^t\lambda_v dv}\frac{\pi_t}{\pi_s}\left(c_t^s-y_t^s\right)dt\right]
\tag{2.87}
$$

to solve for the optimal consumption level of new born agents, c_s^s. Plugging in expression (2.83) for optimal consumption yields

$$
c_s^s=\frac{E_s\left[\int_s^\infty e^{-\int_s^t\lambda_v dv}\frac{\pi_t}{\pi_s}y_t^s dt\right]}{E_s\left[\int_s^\infty e^{-\int_s^t\lambda_v dv}e^{\frac{1}{\gamma}\int_s^t\frac{1-\gamma-\rho}{\rho}\psi_u-\frac{1-\gamma}{\rho}\beta du}\left(\frac{\psi_t}{\psi_s}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1-\gamma}{\gamma}}dt\right]}
\tag{2.88}
$$

I define the following functions

$$
F^{c,s}\left(\lambda,n,s\right)=E_s\left[\int_s^\infty e^{-\int_s^t\lambda_v dv}e^{\frac{1}{\gamma}\int_s^t\frac{1-\gamma-\rho}{\rho}\psi_u-\frac{1-\gamma}{\rho}\beta du}\left(\frac{\psi_t}{\psi_s}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_t}{\pi_s}\right)^{-\frac{1-\gamma}{\gamma}}dt\right]
\tag{2.89}
$$

and

$$
F^{y,s,(i)}\left(\lambda,n,s\right)=E_s\left[\int_s^\infty\frac{\pi_t}{\pi_s}a\frac{Y_t}{Y_s}\frac{B_i e^{-(1+\delta_i)\int_s^t n_v dv}}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}dt\right]
\tag{2.90}
$$

with

$$
\frac{Y_s}{N_s}\sum_{i=1}^{2}F^{y,s,(i)}\left(\lambda,n,s\right)=E_s\left[\int_s^\infty e^{-\int_s^t\lambda_v dv}\frac{\pi_t}{\pi_s}y_t^s dt\right]
\tag{2.91}
$$

Thus, I have

$$
c_s^s=\frac{Y_s}{N_s}\frac{\sum_{i-1}^{2}F^{y,s,(i)}\left(\lambda,n,s\right)}{F^{c,s}\left(\lambda,n,s\right)}
\tag{2.92}
$$

I define the variables

$$
\begin{aligned}
Z_s^c &= E_s\left[\int_{-\infty}^{\infty} e^{-\int_{-\infty}^{t}\lambda_v dv} e^{\frac{1}{\gamma}\int_{-\infty}^{t}\frac{1-\gamma-\rho}{\rho}\psi_u - \beta\frac{1-\gamma}{\rho}du}\left(\frac{\psi_t}{\psi_{-\infty}}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_t}{\pi_{-\infty}}\right)^{-\frac{1-\gamma}{\gamma}} dt\right] \\
&= e^{-\int_{-\infty}^{s}\lambda_v dv} e^{\frac{1}{\gamma}\int_{-\infty}^{s}\frac{1-\gamma-\rho}{\rho}\psi_u - \beta\frac{1-\gamma}{\rho}du}\left(\frac{\psi_s}{\psi_{-\infty}}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_s}{\pi_{-\infty}}\right)^{-\frac{1-\gamma}{\gamma}} F_s^{c,s} \qquad (2.93) \\
&+ \int_{-\infty}^{s} e^{-\int_{-\infty}^{t}\lambda_v dv} e^{\frac{1}{\gamma}\int_{-\infty}^{t}\frac{1-\gamma-\rho}{\rho}\psi_u - \beta\frac{1-\gamma}{\rho}du}\left(\frac{\psi_t}{\psi_{-\infty}}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_t}{\pi_{-\infty}}\right)^{-\frac{1-\gamma}{\gamma}} dt
\end{aligned}
$$

and

$$
\begin{aligned}
Z_s^{y,(i)} &= E_s\left[\int_{-\infty}^{\infty}\frac{\pi_t}{\pi_{-\infty}} a\frac{Y_t}{Y_{-\infty}}\frac{B_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} e^{-(1+\delta_i)\int_{-\infty}^{t} n_v dv} dt\right] \qquad (2.94) \\
&= \frac{\pi_s}{\pi_{-\infty}}\frac{Y_s}{Y_{-\infty}} e^{-(1+\delta_i)\int_{-\infty}^{s} n_v dv} F_s^{y,s} \\
&+ \int_{-\infty}^{s}\frac{\pi_t}{\pi_{-\infty}} a\frac{Y_t}{Y_{-\infty}}\frac{B_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} e^{-(1+\delta_i)\int_{-\infty}^{t} n_v dv} dt
\end{aligned}
$$

Noticing that the newly defined quantities, Z_s^c, $Z_s^{y,(1)}$ and $Z_s^{y,(2)}$ are (local) martingales and (by the tower property of conditional expectations) their drift terms equal zero, I get PDE's that determine the functions $F^{c,s}(\lambda, n, s)$, $F^{y,s,(1)}(\lambda, n, s)$ and $F^{y,s,(2)}(\lambda, n, s)$

$$
\begin{aligned}
0 &= \left[-\lambda_s + \frac{1-\gamma-\rho}{\gamma\rho}\psi_s - \frac{1}{\gamma}\beta\frac{1-\gamma}{\rho}\right] ds F_s^{c,s} + ds - \frac{1-\gamma}{\gamma}\frac{d\pi_s}{\pi_s} dF_s^{c,s} \qquad (2.95) \\
&+ E_s\left[dF_s^{c,s}\right] - \frac{1-\gamma}{\gamma} E_s\left[\frac{d\pi_s}{\pi_s}\right] F_s^{c,s} - \frac{1-\gamma-\rho}{\gamma\rho} E_s\left[\frac{d\psi_s}{\psi_s}\right] F_s^{c,s} \\
&+ \frac{1}{2}\frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{\gamma^2\rho^2}\left(\frac{d\psi_s}{\psi_s}\right)^2 F_s^{c,s} + \frac{1}{2}\frac{1-\gamma}{\gamma^2}\left(\frac{d\pi_s}{\pi_s}\right)^2 F_s^{c,s} \\
&+ \frac{(1-\gamma-\rho)(1-\gamma)}{\gamma^2\rho}\frac{d\psi_s}{\psi_s}\frac{d\pi_s}{\pi_s} F_s^{c,s} - \frac{1-\gamma-\rho}{\gamma\rho}\frac{d\psi_s}{\psi_s} dF_s^{c,s}
\end{aligned}
$$

and $\forall i \in \{1, 2\}$

$$
\begin{aligned}
0 &= \left[\frac{1}{ds} E_s\left[\frac{d\pi_s}{\pi_s}\right] + \mu_s^{(Y)} + \frac{1}{ds}\frac{d\pi_s}{\pi_s}\frac{dY_s}{Y_s} - (1+\delta_i) n_s\right] ds F_s^{y,s,(i)} \qquad (2.96) \\
&+ E_s\left[dF_s^{y,s,(i)}\right] + dF_s^{y,s(i)}\frac{d\pi_s}{\pi_s} + dF_s^{y,s(i)}\frac{dY_s}{Y_s} + a\frac{B_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} ds
\end{aligned}
$$

Alternatively, using equation (2.79) I can derive $F^{c,s}(\lambda, n, s)$ as follows

$$\begin{aligned} c_s^s &= \widehat{W}_s^s \psi_s = \left(W_s^s + E_s \left[\int_s^\infty e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} y^s dt \right] \right) \psi_s && (2.97) \\ &= E_s \left[\int_s^\infty e^{-\int_s^t \lambda_v dv} \frac{\pi_t}{\pi_s} c_t^s dt \right] \psi_s = c_s^s F_s^{c,s} \psi_s \\ F_s^{c,s} &= \frac{1}{\psi_s} && (2.98) \end{aligned}$$

Combining with the PDE determining $F^{c,s}(\lambda, n, s)$ (equation (2.95)) I end up with

$$\begin{aligned} 0 = {} & \frac{1-\rho}{\rho} \psi_s ds - \frac{\beta}{\rho} ds - \frac{\gamma}{1-\gamma} \lambda_s ds - \frac{1-\rho}{\rho} E_s \left[\frac{d\psi_s}{\psi_s} \right] && (2.99) \\ & + E_s \left[-\frac{d\pi_s}{\pi_s} \right] + \frac{1}{2} \left(\frac{(1-\gamma)(1-\rho)^2}{\gamma \rho^2} + \frac{1-\rho}{\rho} \right) \left(\frac{d\psi_s}{\psi_s} \right)^2 \\ & + \frac{1}{2\gamma} \left(\frac{d\pi_s}{\pi_s} \right)^2 + \frac{(1-\gamma)(1-\rho)}{\gamma\rho} \frac{d\psi_s}{\psi_s} \frac{d\pi_s}{\pi_s} \end{aligned}$$

which is the same as the PDE (2.86) that determines $\psi_t(\lambda, n)$. This verifies the conjecture about the value function (2.78). Equation (2.79) also tells us that $\psi_t(\lambda, n)$ describes the consumption to wealth ratio.

Market clearing in the consumption market implies

$$dY_t = dC_t \tag{2.100}$$

Growth in aggregate output is exogenously given and for aggregate consumption I have

$$\begin{aligned} dC_t &= d\left(\int_{-\infty}^t c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds \right) && (2.101) \\ &= c_t^t n_t N_t dt + \int_{-\infty}^t \frac{dc_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_u du} ds - \lambda_t C_t dt \end{aligned}$$

I can use expression (2.83) to get the dynamics of the optimal consumption process

$$\begin{aligned}\frac{dc_t^s}{c_t^s} &= \frac{1-\gamma-\rho}{\rho\gamma}\psi_t dt - \beta\frac{1-\gamma}{\gamma\rho}dt - \frac{1-\gamma-\rho}{\gamma\rho}\frac{d\psi_t}{\psi_t} \\ &\quad -\frac{1}{\gamma}\frac{d\pi_t}{\pi_t} + \frac{1+\gamma}{2\gamma^2}\left(\frac{d\pi_t}{\pi_t}\right)^2 + \frac{1-\gamma-\rho}{\gamma^2\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} \\ &\quad +\frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma^2\rho^2}\left(\frac{d\psi_t}{\psi_t}\right)^2\end{aligned} \tag{2.102}$$

Plugging back into the market clearing condition and solving for growth in the SDF gives

$$\begin{aligned}\frac{d\pi_t}{\pi_t} &= -\gamma\frac{dY_t}{Y_t} - \gamma\lambda_t dt + \gamma n_t \sum_{i-1}^{2} F^{y,t,(i)}\psi_t dt + \frac{1-\gamma-\rho}{\rho}\psi_t dt \\ &\quad -\beta\frac{1-\gamma}{\rho}dt - \frac{1-\gamma-\rho}{\rho}\frac{d\psi_t}{\psi_t} + \frac{1-\gamma-\rho}{\gamma\rho}\frac{d\psi_t}{\psi_t}\frac{d\pi_t}{\pi_t} \\ &\quad +\frac{1+\gamma}{2\gamma}\left(\frac{d\pi_t}{\pi_t}\right)^2 + \frac{(1-\gamma-\rho)(1-\gamma)(1-\rho)}{2\gamma\rho^2}\left(\frac{d\psi_t}{\psi_t}\right)^2\end{aligned} \tag{2.103}$$

Using (2.86) and (2.96), and plugging in the expression for the SDF, I can derive a system of 3 differential equations that determines the quantities ψ_t and $F^{y,t,(i)}$, $\forall i \in \{1,2\}$

$$\begin{aligned}0 &= -\left[r_t - \mu_t^{(Y)} + \gamma\left(\sigma^{(A)}\right)^2 + \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t}\sigma^{(A)} + (1+\delta_i)\,n_t\right]F_t^{y,t,(i)} \\ &\quad +\mu_t^{(F^{y,(i)})} - \frac{1-\gamma-\rho}{\rho}\sigma_t^{(F^{y,(i)})}\frac{\sigma_t^{(\psi)}}{\psi_t} + (1-\gamma)\,\sigma^{(A)}\sigma_t^{(F^{y,(i)})} + \frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}\end{aligned} \tag{2.104}$$

$$\begin{aligned}0 &= -\psi_t + \frac{1}{1-\rho}\beta + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda_t - \frac{\rho}{1-\rho}r_t + \frac{\mu_t^{(\psi)}}{\psi_t} \\ &\quad -\frac{1}{2}\left(\frac{1-\gamma}{\rho}+\frac{\gamma}{1-\rho}\right)\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^2 - \frac{1}{2}\frac{\gamma\rho}{1-\rho}\left(\sigma^{(A)}\right)^2 + \frac{\gamma\rho}{1-\rho}\sigma^{(A)}\frac{\sigma_t^{(\psi)}}{\psi_t}\end{aligned} \tag{2.105}$$

with the dynamics of $F^{y,t,(i)}$ and ψ_t defined as

$$\begin{aligned}
\mu_t^{(F^{y,(i)})} &= F_\lambda^{y,(i)}\lambda_t\mu^{(\lambda)} + F_n^{y,(i)}n_t\mu^{(n)} + \frac{1}{2}F_{\lambda\lambda}^{y,(i)}\left(\lambda_t\sigma^{(\lambda)}\right)^2 && (2.106)\\
&\quad +\frac{1}{2}F_{nn}^{y,(i)}\left(n_t\sigma^{(n)}\right)^2 + F_{\lambda n}^{y,(i)}\lambda_t n_t\sigma^{(\lambda)}\left(\sigma^{(n)}\right)^T \\
\sigma_t^{(F^{y,(i)})} &= F_\lambda^{y,(i)}\lambda_t\sigma^{(\lambda)} + F_n^{y,(i)}n_t\sigma^{(n)} && (2.107)\\
\mu_t^{(\psi)} &= \psi_\lambda\lambda_t\mu^{(\lambda)} + \psi_n n_t\mu^{(n)} + \frac{1}{2}\psi_{\lambda\lambda}\left(\lambda_t\sigma^{(\lambda)}\right)^2 && (2.108)\\
&\quad +\frac{1}{2}\psi_{nn}\left(n_t\sigma^{(n)}\right)^2 + \psi_{\lambda n}\lambda_t n_t\sigma^{(\lambda)}\left(\sigma^{(n)}\right)^T \\
\sigma_t^{(\psi)} &= \psi_\lambda\lambda_t\sigma^{(\lambda)} + \psi_n n_t\sigma^{(n)} && (2.109)
\end{aligned}$$

By definition of $r_t = E_t\left[-\frac{d\pi_t}{\pi_t}\right]$ and $\kappa_t = -\frac{d\pi_t}{\pi_t} - E_t\left[-\frac{d\pi_t}{\pi_t}\right]$, Proposition 2.4 follows,

$$\begin{aligned}
r_t &= \beta + (1-\rho)\left[\mu_t^{(Y)} + \lambda_t - n_t\sum_{i-1}^{2}F^{y,t,(i)}\psi_t\right] - \frac{1-\gamma-\rho}{1-\gamma}\lambda_t && (2.110)\\
&\quad +\frac{1-\gamma-\rho}{2\rho}\left(\frac{\sigma_t^{(\psi)}}{\psi_t}\right)^2 - \frac{\gamma(2-\rho)}{2}\left(\sigma^{(A)}\right)^2 - \frac{1-\gamma-\rho}{\rho}\sigma^{(A)}\frac{\sigma_t^{(\psi)}}{\psi_t}\\
\kappa_t &= \frac{1-\gamma-\rho}{\rho}\frac{\sigma_t^{(\psi)}}{\psi_t} + \gamma\sigma^{(A)} && (2.111)
\end{aligned}$$

■

Proof of Proposition 2.1. Proposition 2.1 is a special case of Proposition 2.4 and follows immediately when using $\sigma^{(A)} = 0$, $d\lambda_t = 0$ and $dn_t = 0$. Moreover, I rewrite

$$\begin{aligned}
c_t^t(r) &= \frac{C_t}{N_t}\sum_{i=1}^{2}F^{y,t,(i)}\psi_t && (2.112)\\
&= \frac{C_t}{N_t}a\frac{\int_0^\infty e^{\left(\mu^{(Y)}-n-r\right)t}G(0,t)\,dt}{\int_0^\infty G(0,s)\,ne^{-ns}ds}\left(\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda + \frac{1}{1-\rho}\beta - \frac{\rho}{1-\rho}r\right)
\end{aligned}$$

■

Proof of Proposition 2.2 and 2.3. The proof of Proposition 2.2 and 2.3 are basically the same. The main difference to Proposition 2.4 is that the function $\psi_t(\lambda, n)$ does not follow a continuous diffusion process, but jumps between two distinct values. The argument follows basically the same lines. The derivation from equation (2.68) to equation (2.83) is carried over without any change. The further derivation differs slightly and is described now. To derive optimal consumption of new born agents I make use of the static budget constraint and define the functions $F^{c,s}(\lambda, n, s)$, $F^{y,s,(i)}(\lambda, n, s)$, Z_s^c and $Z_s^{y,(i)}$ as before. Using the martingale property of the Z functions and setting the drift zero yields

$$\begin{aligned} 0 \quad = \quad & \left[-\lambda_{s-} + \frac{1-\gamma-\rho}{\gamma\rho}\psi_{s-} - \frac{1-\gamma}{\gamma\rho}\beta\right] dsF^{c,s-} + ds \\ & -\frac{1-\gamma}{\gamma}E_s\left[\frac{d\pi_{s-}^{(cp)}}{\pi_{s-}}\right]F^{c,s-} + \frac{1}{2}\frac{1-\gamma}{\gamma^2}\left(\frac{d\pi_{s-}^{(cp)}}{\pi_{s-}}\right)^2 F^{c,s-} \\ & +E_s\left[\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_s}{\pi_{s-}}\right)^{-\frac{1-\gamma}{\gamma}}\frac{F^{c,s}}{F^{c,s-}} - 1\right)F^{c,s-}|dS_s|\right] \end{aligned} \tag{2.113}$$

and

$$\begin{aligned} 0 \quad = \quad & \left[E_s\left[\frac{d\pi_{s-}^{(cp)}}{\pi_{s-}}\right] + \mu_{s-}^{(Y)}ds - (1+\delta_i)\, n_{s-}ds\right]F^{y,s-,(i)} \\ & +E_s\left[\left(\frac{\pi_s}{\pi_{s-}}\frac{F^{y,s,(i)}}{F^{y,s-,(i)}} - 1\right)F^{y,s-,(i)}|dS_s|\right] + \frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}}ds \end{aligned} \tag{2.114}$$

where the superscript (cp) denotes the continuous (smooth) part of the process (for notational details see Shreve, 2004). Using the relation $F^{c,s} = \frac{1}{\psi_s}$ gives the equation determining ψ_t

$$\begin{aligned} 0 \quad = \quad & \frac{1-\rho}{\rho}\psi_{s-}(\lambda, n) - \frac{\beta}{\rho} - \frac{\gamma}{1-\gamma}\lambda_{s-} - \frac{1}{ds}E_s\left[\frac{d\pi_{s-}^{(cp)}}{\pi_{s-}}\right] \\ & +\frac{\gamma}{1-\gamma}E_s\left[\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{(1-\gamma)(1-\rho)}{\gamma\rho}}\left(\frac{\pi_s}{\pi_{s-}}\right)^{-\frac{1-\gamma}{\gamma}} - 1\right)|dS_s|\right] \end{aligned} \tag{2.115}$$

Given these functions, it holds

$$c_s^s = \frac{Y_s}{N_s}\sum_{i=1}^{2} F^{y,s,(i)}\psi_s \tag{2.116}$$

The dynamics of the optimal consumption process for the individual agent are (using equation (2.83))

$$\frac{dc_t^s}{c_{t-}^s} = \left(\frac{1-\gamma-\rho}{\rho\gamma}\psi_{t-}(\lambda,n) - \frac{1-\gamma}{\rho\gamma}\beta\right)dt$$
$$-\frac{1}{\gamma}\frac{d\pi_{t-}^{(cp)}}{\pi_{t-}} + \frac{1+\gamma}{\gamma^2}\left(\frac{d\pi_{t-}^{(cp)}}{\pi_{t-}}\right)^2 + \frac{c_t^s - c_{t-}^s}{c_{t-}^s}|dS_t| \tag{2.117}$$

and $\frac{c_t^s - c_{t-}^s}{c_{t-}^s}$ is given by

$$\frac{c_t^s - c_{t-}^s}{c_{t-}^s} = \left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma-\rho}{\gamma\rho}}\left(\frac{\pi_t}{\pi_{t-}}\right)^{-\frac{1}{\gamma}} - 1 \tag{2.118}$$

From equation (2.118) and the fact that on the aggregate consumption and output are smooth, $C_t = C_{t-}$ and $Y_t = Y_{t-}$ (no discontinuities), it follows that the pricing kernel process must have a jump component inherent, and in particular, it must hold

$$\frac{\pi_t}{\pi_{t-}} = \left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} \tag{2.119}$$

$$\frac{d\pi_{t-}}{\pi_{t-}} - \frac{d\pi_{t-}^{(cp)}}{\pi_{t-}} = \left(\left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)|dS_t| \tag{2.120}$$

Finally, imposing market clearing in the consumption good market as before ($dY_t = dC_t$) and solving for the SDF yields

$$\frac{d\pi_t^{(cp)}}{\pi_t} = \left[-\gamma\mu_t^{(Y)} - \frac{1-\gamma}{\rho}\beta - \gamma\lambda_t + \gamma n_t \sum_{i-1}^{2} F^{y,t,(i)}\psi_t + \frac{1-\gamma-\rho}{\rho}\psi_t\right]dt$$
$$+\frac{1+\gamma}{2\gamma}\left(\frac{d\pi_t^{(cp)}}{\pi_t}\right)^2 \tag{2.121}$$

Adding the jump component leaves us with the quantities

$$r_{t-} = \beta + (1-\rho)\left[\mu_{t-}^{(Y)} + \lambda_{t-} - n_{t-}\sum_{i-1}^{2} F^{y,t-,(i)}\psi_{t-}\right] - \frac{1-\gamma-\rho}{1-\gamma}\lambda_{t-} \tag{2.122}$$
$$-\overline{\theta}_{t-}\left(\left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) + \frac{1-\gamma-\rho}{1-\gamma}\overline{\theta}_{t-}\left(\left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)$$

and

$$\kappa_{t-} = -\left(\left(\frac{\psi_t}{\psi_{t-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) \tag{2.123}$$

with

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \kappa_t \left|d\widehat{S}_t\right| \tag{2.124}$$

and

$$\begin{aligned}
\psi_{s-} &= \frac{1}{1-\rho}\beta + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda_{s-} - \frac{\rho}{1-\rho}r_{s-} \qquad (2.125)\\
&\quad -\frac{\rho}{1-\rho}\overline{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) - \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\overline{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)\\
&= \beta - \rho\mu_{s-}^{(Y)} + \rho n_{s-}\sum_{i-1}^{2} F^{y,s-,(i)}\psi_{s-} + \frac{\gamma\rho}{1-\gamma}\lambda_{s-} - \frac{\rho}{1-\gamma}\overline{\theta}_{s-}\left(\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)
\end{aligned}$$

$$F^{y,s-,(i)} = \frac{1}{r_{s-} - \mu_{s-}^{(Y)} + (1+\delta_i)\, n_{s-} + \overline{\theta}_{s-}\left(\frac{\psi_s}{\psi_{s-}}\right)^{-\frac{1-\gamma-\rho}{\rho}}\left(1 - \frac{F^{y,s,(i)}}{F^{y,s-,(i)}}\right)}\frac{aB_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \tag{2.126}$$

■

2.6.5 Proofs of Lemmas

Proof of Lemma 2.1. From market clearing in financial markets it follows immediately that

$$P_t = \int_{-\infty}^{t} W_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds \tag{2.127}$$

From the static budget constraint it follows the expression for financial wealth

$$W_t^s = E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t}\left(c_u^s - y_u^s\right) du\right] \tag{2.128}$$

The constraint is binding at optimum because of local non-satiation (utility is increasing in consumption). Plugging in yields

$$\begin{aligned}
P_t &= \int_{-\infty}^{t} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t}c_u^s du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \qquad (2.129)\\
&\quad -\int_{-\infty}^{t} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t}y_u^s du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds
\end{aligned}$$

For the first term I have

$$\begin{aligned}
&\int_{-\infty}^{t} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t}c_u^s du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \qquad (2.130)\\
&= \int_{-\infty}^{t} \widehat{W}_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds = \int_{-\infty}^{t} \frac{\widehat{W}_t^s}{c_t^s} c_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds\\
&= \int_{-\infty}^{t} \frac{1}{\psi_t(\lambda, n)} c_t^s n_s N_s e^{-\int_s^t \lambda_v dv} ds = \frac{1}{\psi_t(\lambda, n)} C_t
\end{aligned}$$

For the second term it holds

$$\begin{aligned}
&\int_{-\infty}^{t} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t}y_u^s du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds \qquad (2.131)\\
&= \int_{-\infty}^{t} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t} a\frac{Y_u}{G_u}\left(\sum_{i=1}^{2} B_i e^{-\delta_i \int_s^t n_v dv} e^{-\delta_i \int_t^u n_v dv}\right) du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds\\
&= \int_{-\infty}^{t} \sum_{i=1}^{2} e^{-\delta_i \int_s^t n_v dv} E_t\left[\int_t^{\infty} e^{-\int_t^u \lambda_v dv}\frac{\pi_u}{\pi_t} a\frac{Y_u}{G_u} B_i e^{-\delta_i \int_t^u n_v dv} du\right] n_s N_s e^{-\int_s^t \lambda_v dv} ds\\
&= \int_{-\infty}^{t} \sum_{i=1}^{2} \frac{Y_t}{N_t} e^{-\delta_i \int_s^t n_v dv} F^{y,t,(i)}(\lambda, n, t)\, n_s N_s e^{-\int_s^t \lambda_v dv} ds\\
&= Y_t \sum_{i=1}^{2} F^{y,t,(i)}(\lambda, n, t) \int_{-\infty}^{t} n_s e^{-(1+\delta_i)\int_s^t n_v dv} ds = Y_t \sum_{i=1}^{2} \frac{F^{y,t,(i)}(\lambda, n, t)}{1+\delta_i}
\end{aligned}$$

Combining and imposing market clearing in the consumption goods market ($Y_t = C_t$) gives

$$P_t = Y_t \left[\frac{1}{\psi_t(\lambda, n)} - \sum_{i=1}^{2} \frac{F^{y,t,(i)}(\lambda, n, t)}{1+\delta_i}\right] \qquad (2.132)$$

■

Proof of Lemma 2.2. Following the definition of the price of an asset that pays dividends D_t, I can write

$$P_t = E_t\left[\int_t^T \frac{\pi_s}{\pi_t} D_s ds + \frac{\pi_T}{\pi_t} P_T\right] \qquad (2.133)$$

$$E_t[P_0] = E_t\left[\int_0^T \frac{\pi_s}{\pi_0} D_s ds + \frac{\pi_T}{\pi_0} P_T\right] = P_t \frac{\pi_t}{\pi_0} + \int_0^t \frac{\pi_s}{\pi_0} D_s ds \qquad (2.134)$$

Noticing that the $E_t[P_0]$ is a martingale (according to the tower property of conditional expectations), it follows immediately

$$0 = E_t\left[d\left(E_t\left[P_0\right]\right)\right] = E_t\left[dP_t\frac{\pi_t}{\pi_0} + P_t\frac{d\pi_t}{\pi_0} + dP_t\frac{d\pi_t}{\pi_0} + \frac{\pi_t}{\pi_0}D_t dt\right] \tag{2.135}$$

$$E_t\left[\frac{dP_t + D_t}{P_t}\right] - r_t = -\frac{dP_t}{P_t}\frac{d\pi_t}{\pi_t} \tag{2.136}$$

■

Proof of Lemma 2.3. First note that I often use the notation $\Phi_t\left(r\right) = \frac{N_t c_t^t(x)}{C_t} = \sum_{i=1}^{2} F_t^{y,(i)}\psi_t$ to describe the ratio between consumption of a new born agent and per capita GDP. Let $\overline{\rho}^{(r)} = \min_{\rho\in\left\{\Upsilon^{(r)}\cup 0\right\}}\left\{\rho\right\}$ with $\Upsilon^{(r)} = \left\{\rho : \upsilon^{(r)}\left(\rho\right) = 0, \rho < 0\right\}$ and $\upsilon^{(r)}\left(\rho\right) = a\left(n-\lambda\right) - \frac{\gamma}{1-\gamma}\lambda - n\Phi\left(r\left(\rho\right)\right)$. I show that the condition $\rho < \overline{\rho}^{(r)}$ (or $EIS < \overline{EIS}^{(r)} = \frac{1}{1-\overline{\rho}^{(r)}}$) suffices for the interest rate in an OLG economy to be smaller than the rate in an equivalent economy populated by a representative infinitely-lived agent ($r < r_*$). Moreover, I show that for $B_2 = 0$ of $\delta_1 = \delta_2$, the function $\upsilon^{(r)}\left(\rho\right)$ is monotonically decreasing in $-\rho$ (for $\rho < 0$), and if $\lim_{\rho\nearrow 0}\left[\upsilon^{(r)}\left(\rho\right)\right] < 0$, then the set $\Upsilon^{(r)}$ is single valued, and otherwise empty. It follows that for $B_2 = 0$ or $\delta_1 = \delta_2$ there exists no $\rho > \overline{\rho}^{(r)}$ that satisfies $\upsilon^{(r)}\left(\rho\right) < 0$. In the general case ($B_2 \neq 0$ and $\delta_1 \neq \delta_2$) there might exist $\rho > \overline{\rho}^{(r)}$ that satisfies $\upsilon^{(r)}\left(\rho\right) < 0$. I need the technical conditions $r - \frac{\gamma}{1-\gamma}\lambda - \beta$, $r - \mu^{(Y)} > 0$, and $\mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$. Given $\gamma > 1$, I have

$$\left(1-\rho\right)\left[a\left(n-\lambda\right)+\lambda-n\Phi\left(r\right)\right] - \frac{1-\gamma-\rho}{1-\gamma}\lambda < \left(1-\rho\right)\left[a\left(n-\lambda\right) - \frac{\gamma}{1-\gamma}\lambda - n\Phi\left(r\right)\right] \tag{2.137}$$

The expression in equation (2.9) is negative and it holds $r < r_*$, if the sufficient condition

$$n\Phi\left(r\right) > a\left(n-\lambda\right) - \frac{\gamma}{1-\gamma}\lambda \tag{2.138}$$

is satisfied. Because ψ is constant across cohorts, it holds

$$\frac{c_t^t}{c_t^s} = \frac{\widehat{W}_t^t}{\widehat{W}_t^s} = \frac{Y_t}{Y_s}\frac{N_s}{N_t}\frac{\widehat{W}_s^s}{\widehat{W}_t^s} = e^{\int_s^t \mu^{(Y)}-(n-\lambda)-(r+\lambda-\psi)du} = e^{\int_s^t \mu^{(Y)}-n-r+\psi du} \tag{2.139}$$

and

$$\Phi(r) = \frac{c_t^t}{\frac{C_t}{N_t}} = \frac{c_t^t}{\int_{-\infty}^t c_t^s n_s \frac{N_s}{N_t} e^{-\int_s^t \lambda du} ds} = \frac{1}{\int_{-\infty}^t \frac{c_t^s}{c_t^t} n e^{-\int_s^t n du} ds} \tag{2.140}$$

$$= \frac{1}{\int_{-\infty}^t e^{\int_s^t r-\mu^{(Y)}-\psi du} n ds} = \frac{\psi+\mu^{(Y)}-r}{n}$$

with $\psi+\mu^{(Y)}-r>0$ since $\psi+\mu^{(Y)}-r=n\Phi(r)$. I look at how condition (2.138) behaves in the limit when the EIS approaches zero. Taking the limit of ρ approaching $-\infty$, I get for the key variables

$$\lim_{\rho\to-\infty} \frac{r}{1-\rho} = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - n \lim_{\rho\to-\infty} \Phi(r) \tag{2.141}$$

and

$$\lim_{\rho\to-\infty} \frac{\psi}{1-\rho} = \lim_{\rho\to-\infty} \frac{r}{1-\rho} \tag{2.142}$$

and $\forall i \in \{1,2\}$

$$\lim_{\rho\to-\infty} (1-\rho) F^{y,(i)} = \frac{1}{\lim_{\rho\to-\infty} \frac{r}{1-\rho}} \frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.143}$$

Suppose $|\lim_{\rho\to-\infty} \frac{r}{1-\rho}| < \infty$ and $\lim_{\rho\to-\infty} \frac{r}{1-\rho} \neq 0$, I get

$$\lim_{\rho\to-\infty} \Phi(r) = a\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.144}$$

Indeed $|\lim_{\rho\to-\infty} \frac{r}{1-\rho}| = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} < \infty$ and in general $\lim_{\rho\to-\infty} \frac{r}{1-\rho} = \mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$. In the case of $B_2=0$ or $\delta_1=\delta_2$, $\lim_{\rho\to-\infty}\Phi(r)$ simplifies to

$$\lim_{\rho\to-\infty} \Phi(r) = a(1+\delta_1) \tag{2.145}$$

In the limit as the EIS approaches zero condition (2.138) is satisfied if $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} > 1 - \frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$. For $B_2=0$ or $\delta_1=\delta_2$, the condition boils down to $\delta_1 > -\frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$. Using $\gamma>1$, $r-\frac{\gamma}{1-\gamma}\lambda-\beta$ and the conditions of Lemma 2.3 ($r-\mu^{(Y)}>0$), and taking the derivative of r with respect to $-\rho$ gives

$$\begin{aligned}
\frac{\partial r}{\partial(-\rho)} &= \frac{\mu^{(Y)}+\lambda-n\Phi(r)-n\frac{1}{1-\rho}\left(r-\beta-\frac{\gamma}{1-\gamma}\lambda\right)\sum_{i=1}^{2}F^{y,(i)}-\frac{1}{1-\gamma}\lambda}{1+(1-\rho)\,n\Phi'(r)} \qquad (2.146)\\
&= \frac{\mu^{(Y)}-\frac{\gamma}{1-\gamma}\lambda-n\Phi(r)-\left(\frac{1}{1-\rho}r-\frac{1}{1-\rho}\beta-\frac{1}{1-\rho}\frac{\gamma}{1-\gamma}\lambda\right)\frac{1}{\psi}n\Phi(r)}{1+(1-\rho)\,n\Phi'(r)}\\
&= \frac{\frac{1}{\psi}\left(\psi-n\Phi(r)\right)\left(\mu^{(Y)}-\frac{\gamma}{1-\gamma}\lambda-n\Phi(r)\right)}{1+(1-\rho)\,n\Phi'(r)}\\
&= \frac{\frac{1}{\psi}\left(r-\mu^{(Y)}\right)\left(r-\frac{\gamma}{1-\gamma}\lambda-\psi\right)}{1+(1-\rho)\,n\Phi'(r)} = \frac{\frac{1}{\psi}\left(r-\mu^{(Y)}\right)\frac{1}{1-\rho}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)}{1+(1-\rho)\,n\Phi'(r)} > 0
\end{aligned}$$

Assuming $\rho<0$ and taking the derivative of ψ with respect to $-\rho$ yields

$$\begin{aligned}
\frac{\partial\psi}{\partial(-\rho)} &= \frac{1}{(1-\rho)^2}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)-\frac{\rho}{1-\rho}\left(-\frac{\partial r}{\partial\rho}\right) \qquad (2.147)\\
&= \frac{1}{(1-\rho)^2}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\left(1-\rho\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)\,n\Phi'(r)}\right)>0
\end{aligned}$$

Analyzing the function $\Phi(r)$, I get

$$\frac{\partial\Phi(r)}{\partial r} = -\sum_{i=1}^{2}\left(\frac{1}{r-\mu^{(Y)}+(1+\delta_i)\,n}F^{y,(i)}\psi+F^{y,(i)}\frac{\rho}{1-\rho}\right) \qquad (2.148)$$

and

$$\begin{aligned}
\frac{\partial\left(\Phi\left(r\right)\right)}{\partial\left(-\rho\right)} &= \frac{1}{n}\frac{\partial\left(\psi+\mu^{(Y)}-r\right)}{\partial\left(-\rho\right)}=\frac{1}{n}\left(\frac{\partial\psi}{\partial\left(-\rho\right)}-\frac{\partial r}{\partial\left(-\rho\right)}\right) \qquad (2.149)\\
&= \frac{1}{n}\frac{1}{\left(1-\rho\right)^{2}}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\left(1-\frac{\frac{1}{\psi}\left(r-\mu^{(Y)}\right)}{1+\left(1-\rho\right)n\Phi'\left(r\right)}\right)\\
&= \frac{1}{\left(1-\rho\right)^{2}}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}\left(\frac{\psi+\mu^{(Y)}-r}{n}+\left(1-\rho\right)\Phi'\left(r\right)\psi\right)\\
&= \frac{1}{\left(1-\rho\right)^{2}}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}\\
&\quad\cdot\left(\Phi\left(r\right)-\left(1-\rho\right)\psi\sum_{i=1}^{2}\left(\frac{1}{r-\mu^{(Y)}+\left(1+\delta_{i}\right)n}F^{y,(i)}\psi+F^{y,(i)}\frac{\rho}{1-\rho}\right)\right)\\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}\left(\Phi\left(r\right)-\psi\sum_{i=1}^{2}\frac{1}{r-\mu^{(Y)}+\left(1+\delta_{i}\right)n}F^{y,(i)}\psi\right)\\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}\sum_{i=1}^{2}F^{y,(i)}\psi\left(1-\psi\frac{1}{r-\mu^{(Y)}+\left(1+\delta_{i}\right)n}\right)\\
&= \frac{1}{1-\rho}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}n\sum_{i=1}^{2}\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+\left(1+\delta_{i}\right)n}\left(1+\delta_{i}-\Phi\left(r\right)\right)
\end{aligned}$$

$\frac{\partial(\Phi(r))}{\partial(-\rho)}>0$ holds and $\Phi\left(r\right)$ is strictly increasing in $-\rho$ if $\sum_{i=1}^{2}\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+(1+\delta_i)n}\left(1+\delta_{i}-\Phi\left(r\right)\right)$ >0. In general it is hard to tell whether this condition is satisfied. However, if $B_2=0$ or $\delta_1=\delta_2$, then

$$\frac{\partial\left(\Phi\left(r\right)\right)}{\partial\left(-\rho\right)}=\frac{1}{1-\rho}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+\left(1-\rho\right)n\Phi'\left(r\right)}n\frac{\Phi\left(r\right)}{r-\mu^{(Y)}+\left(1+\delta_{1}\right)n}\left(1+\delta_{1}-\Phi\left(r\right)\right) \quad (2.150)$$

and $a\left(1+\delta_1\right)>\Phi\left(r\right)$ and $\frac{\partial(\Phi(r))}{\partial(-\rho)}>0$ must hold for $\rho<0$. I proof this statement as follows. Suppose $\Phi\left(r\right)=\Phi^{*}>1+\delta_1$ was true for some $\widehat{\rho}<0$. Then, since $\Phi\left(r\right)$ is a continuous function and $1+\delta_1>(<)\,\Phi\left(r\right)$ implies $\frac{\partial(\Phi(r))}{\partial(-\rho)}>(<)\,0$, $1+\delta_1>\Phi\left(r\right)$ can never occur for $\rho<\widehat{\rho}$ and $\Phi\left(r\right)$ will converge (from above) to $\widehat{\Phi}\in\left[1+\delta_1,\Phi^{*}\right)$ as ρ approaches $-\infty$. But this contradicts $\lim_{\rho\to-\infty}\Phi\left(r\right)=a\left(1+\delta_1\right)$. Similar, suppose $\Phi\left(r\right)=\Phi^{*}\in\left(a\left(1+\delta_1\right),1+\delta_1\right)$ for some $\widehat{\rho}<0$. Then, since $\Phi\left(r\right)$ is a continuous function and $1+\delta_1>(<)\,\Phi\left(r\right)$ implies $\frac{\partial(\Phi(r))}{\partial(-\rho)}>(<)\,0$, $\Phi\left(r\right)$ is approaching the limit $\widehat{\Phi}\in\left(\Phi^{*},1+\delta_1\right]$ (from below) as ρ approaches $-\infty$, which contradicts $\lim_{\rho\to-\infty}\Phi\left(r\right)=a\left(1+\delta_1\right)$. The only possibility is that $\Phi\left(r\right)$ is strictly increasing and approaches the limit $a(1+\delta_1)$ (from below) as ρ approaches $-\infty$. Finally, since $\Phi(r)$ is a continuous function in ρ (for $\rho < 0$), $\rho < \overline{\rho}^{(r)}$ for $\overline{\rho}^{(r)} = \min_{\rho \in \left(\Upsilon^{(r)} \cup \{0\}\right)} \{\rho\}$ with $\Upsilon^{(r)} \equiv \left\{\rho : \upsilon^{(r)} = 0, \rho \leq 0\right\}$ and $\upsilon^{(r)} = a(n-\lambda) - \frac{\gamma}{1-\gamma}\lambda - n\Phi(r(\rho))$ satisfies condition (2.138), if $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} > 1 - \frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$. Moreover, in the case of $B_2 = 0$ or $\delta_1 = \delta_2$ and $\delta_1 > -\frac{1+\frac{1-a}{a}\gamma}{1-\gamma}\frac{\lambda}{n}$, if condition (2.138) is not satisfied for $\rho \nearrow 0$, then $\Upsilon^{(r)}$ is single valued (as $\upsilon^{(r)}$ is monotonic), and if condition (2.138) is satisfied for $\rho \nearrow 0$, then $\Upsilon^{(r)}$ is empty.

A similar result can be achieved following the argument in Garleanu and Panageas (2010). Provided the (sufficient) conditions

$$n\Phi(r_*) > a(n-\lambda) + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda \tag{2.151}$$

and

$$\frac{\gamma\rho}{1-\gamma}\lambda + \beta - \rho\mu^{(Y)} > 0 \tag{2.152}$$

and

$$\beta - a(n-\lambda) - \rho\mu_*^{(Y)} > 0 \tag{2.153}$$

the equilibrium interest rate in the OLG economy is lower than the respective interest rate in an economy populated by an infinitely-lived representative agent ($r < r_*$). I define $r_* = (1-\rho)\mu_*^{(Y)} + \beta$ (interest rate in infinitely-lived representative agent economy). Let the function $f^{(r)}(x) = (1-\rho)\mu^{(Y)} + \frac{\gamma\rho}{1-\gamma}\lambda + \beta - (1-\rho)n\Phi(x) - x$. First, I note that condition (2.153) implies that $r_* > \mu^{(Y)}$. Next, condition (2.151) implies $0 > f^{(r)}(r_*)$, and condition (2.152) implies $0 < f^{(r)}\left(\mu^{(Y)}\right)$. Since the function $f^{(r)}(.)$ is continuous, then by the intermediate value theorem, there exists $\left\{r : r \in \left(\mu^{(Y)}, r_*\right),\ f^{(r)}(r) = 0\right\}$. This means that there exists an equilibrium interest rate in the OLG economy that is larger than the growth rate in GDP but smaller than the rate in an equivalent economy populated by an infinitely-lived agent. As pointed out by Garleanu and Panageas (2010), condition (2.151) can be interpreted as a requirement on life-cycle earnings to be sufficiently strong declining in age. Assuming the special parameterization of $G(0,t) = B_1 e^{-\delta_1 nt}$ ($B_2 = 0$), it becomes

$$\delta_1 > \frac{\left(a(n-\lambda) + \frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda\right)\left(\beta - \rho\mu_*^{(Y)} - a(n-\lambda)\right)}{n\left(a\beta - a\rho\mu_*^{(Y)} - a(n-\lambda) - (1-a)\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\lambda\right)} - 1 \tag{2.154}$$

It requires δ_1 to be large enough and life-cycle earnings to decrease fast enough as an agent age. Condition (2.152) is implied by $\gamma > 1$. This is because condition (2.152) it is implied by condition (ii) of Proposition 2.1 if $\rho > 0$ and $r > \mu^{(Y)}$ (which is an implication of the just discussed intermediate value theorem), or it is satisfied for $\rho < 0$ and $\gamma \notin \left(\frac{\mu^{(Y)}}{\mu^{(Y)}+\lambda}, 1\right)$. The condition (2.153) requires the IES to be small enough, $\rho < \frac{\beta - a(n-\lambda)}{\mu^{(Y)} - a(n-\lambda)}$ (given $\mu_*^{(Y)} > 0$).

The argument of both discussed proofs are interdependent and complement each other. I need a sufficiently decreasing life-cycle earnings profile and a strong enough consumption smoothing motive. The difference is that once I explore the magnitude of the EIS and once I focus on the labor income path. ■

Proof of Lemma 2.4. Let $\overline{\rho}^{(n)} = \min_{\rho \in \left(\Upsilon^{(n)} \cup \{0\}\right)} \{\rho\}$ with $\Upsilon^{(n)} = \{\rho : \upsilon^{(n)}(\rho) = 0, \rho < 0\}$ and $\upsilon^{(n)}(\rho) = \Phi(r(\rho)) - a\frac{r(\rho) - \mu^{(Y)} + (1+\delta_1)n}{r(\rho) - \mu^{(Y)} + an}$. I show that the condition $\rho < \overline{\rho}^{(n)}$ (or $EIS < \overline{EIS}^{(n)} \equiv \frac{1}{1-\overline{\rho}^{(n)}}$) suffices for $\frac{\partial r}{\partial n} < 0$ to hold. Moreover, I show that for $B_2 = 0$ or $\delta_1 = \delta_2$, the function $\upsilon^{(n)}(\rho)$ is monotonically increasing in $-\rho$ (for $\rho < 0$), and if $\lim_{\rho \nearrow 0} \left[\upsilon^{(n)}(\rho)\right] < 0$, then the set $\Upsilon^{(n)}$ is single valued, and otherwise $\Upsilon^{(n)}$ is empty. It follows that for $B_2 = 0$ or $\delta_1 = \delta_2$ there exists no $\rho > \overline{\rho}^{(n)}$ that satisfies $\upsilon^{(n)}(\rho) > 0$. In the general case ($B_2 \neq 0$ and $\delta_1 \neq \delta_2$) there might exist $\rho > \overline{\rho}^{(n)}$ that satisfies $\upsilon^{(n)}(\rho) > 0$. I need the technical conditions $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$, $\delta_1 > 0$, $r - \frac{\gamma}{1-\gamma}\lambda - \beta > 0$, $\mu^{(Y)} - \frac{\gamma}{1-\gamma}\lambda - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$ and the conditions of Proposition 2.1 and Lemma 2.3 to hold. For $\frac{\partial r}{\partial n} < 0$ to hold, I need $a - \Phi(r) + n\sum_{i=1}^{2} \frac{-a+1+\delta_i}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)}\psi < 0$. Using the assumptions $\delta_1 > 0$ and Lemma 2.3 ($r - \mu^{(Y)} > 0$), I get

$$\begin{aligned} & a - \Phi(r) + n\sum_{i=1}^{2} \frac{-a+1+\delta_i}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)}\psi && (2.155) \\ &= a - \Phi(r) + \sum_{i=1}^{2} \left(1 - \frac{r-\mu^{(Y)}+an}{r-\mu^{(Y)}+(1+\delta_i)n}\right) F^{y,(i)}\psi \\ &= a - \sum_{i=1}^{2} \frac{r-\mu^{(Y)}+an}{r-\mu^{(Y)}+(1+\delta_i)n} F^{y,(i)}\psi < a - \frac{r-\mu^{(Y)}+an}{r-\mu^{(Y)}+(1+\delta_1)n}\Phi(r) \end{aligned}$$

which implies that

$$\Phi(r) > a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an} \tag{2.156}$$

suffices for $\frac{\partial r}{\partial n} < 0$ to hold. First, I look at how condition (2.156) behaves in the limit when the EIS approaches zero. Following the result in Lemma 2.3, I get

$$\lim_{\rho\to-\infty}\Phi(r)=a\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.157}$$

and for $B_2=0$ or $\delta_1=\delta_2$, $\lim_{\rho\to-\infty}\Phi(r)$ simplifies to

$$\lim_{\rho\to-\infty}\Phi(r)=a(1+\delta_1) \tag{2.158}$$

Moreover,

$$\lim_{\rho\to-\infty}a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}=\lim_{\rho\to-\infty}a\frac{\frac{r}{1-\rho}+\frac{-\mu^{(Y)}+(1+\delta_1)n}{1-\rho}}{\frac{r}{1-\rho}+\frac{-\mu^{(Y)}+an}{1-\rho}}=a \tag{2.159}$$

In the limit as the EIS approaches zero condition (2.156) is satisfied ($\lim_{\rho\to-\infty}\Phi(r)>\lim_{\rho\to-\infty}a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}$) if $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}>1$ or equivalently $B_1>\frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$. In the case of $B_2=0$ or $\delta_1=\delta_2$ condition $\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}>1$ boils down to $\delta_1>0$. Next, I note that $a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}\in(a,1+\delta_1)$. Using $\gamma>1$, $r-\frac{\gamma}{1-\gamma}\lambda-\beta$, $1+\delta_1>a$ and the conditions of Lemma 2.3 ($r-\mu^{(Y)}>0$), I see that the term $a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}$ is strictly decreasing in $-\rho$ until it approaches a in the limit where ρ approaches $-\infty$, because

$$\begin{aligned}\frac{\partial\left(a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}\right)}{\partial(-\rho)} &= a\frac{\partial\left(1+\frac{(1+\delta_1)n-an}{r-\mu^{(Y)}+an}\right)}{\partial r}\frac{\partial r}{\partial(-\rho)} \\ &= -an\frac{1+\delta_1-a}{\left(r-\mu^{(Y)}+an\right)^2}\frac{\partial r}{\partial(-\rho)}<0\end{aligned} \tag{2.160}$$

Following Lemma 2.3, I also know that $\frac{\partial(\Phi(r))}{\partial(-\rho)}>0$ holds and $\Phi(r)$ is strictly increasing in $-\rho$ if

$$\sum_{i=1}^{2}\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+(1+\delta_i)n}(1+\delta_i-\Phi(r))>0 \tag{2.161}$$

It is hard to tell whether this condition is satisfied in general. However, if $B_2=0$ or $\delta_1=\delta_2$, then

$$\frac{\partial(\Phi(r))}{\partial(-\rho)}=\frac{1}{1-\rho}\frac{1}{\psi}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+(1-\rho)n\Phi'(r)}n\frac{\Phi(r)}{r-\mu^{(Y)}+(1+\delta_1)n}(1+\delta_1-\Phi(r)) \tag{2.162}$$

and $a(1+\delta_1) > \Phi(r)$ and $\frac{\partial(\Phi(r))}{\partial(-\rho)} > 0$ must hold for $\rho < 0$ (as shown in Lemma 2.3). In conclusion, since $\Phi(r)$ and $a\frac{r-\mu^{(Y)}+(1+\delta_1)n}{r-\mu^{(Y)}+an}$ are continuous functions in ρ (for $\rho < 0$), $\rho < \overline{\rho}^{(n)}$ satisfies condition (2.156) and $\frac{\partial r}{\partial n} < 0$ (if $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$). Moreover, in the case of $B_2 = 0$ or $\delta_1 = \delta_2$, if condition (2.156) is not satisfied for $\rho \nearrow 0$, then $\Upsilon^{(n)}$ is single valued (as $\upsilon^{(n)}$ is monotonic), and if condition (2.156) is satisfied for $\rho \nearrow 0$, then $\Upsilon^{(n)}$ is empty. ■

Proof of Lemma 2.5. Let $\overline{\rho}^{(\lambda)} = \min_{\rho\in\left(\Upsilon^{(\lambda)}\cup\{0\}\right)}\{\rho\}$ with $\Upsilon^{(\lambda)} = \{\rho : \upsilon^{(\lambda)}(\rho) = 0, \rho < 0\}$ and $\upsilon^{(\lambda)}(\rho) = 1 - a - \left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi(\rho)} - \frac{a}{r(\rho)-\mu^{(Y)}+(1+\delta_1)n}\right) na(1+\delta_1) - \frac{1-\gamma-\rho}{(1-\gamma)(1-\rho)}$. I show that the condition $\rho < \overline{\rho}^{(\lambda)}$ (or $EIS < \overline{EIS}^{(\lambda)} \equiv \frac{1}{1-\overline{\rho}^{(\lambda)}}$) suffices for $\frac{\partial r}{\partial\lambda} > 0$ to hold. Moreover, I show that for $B_2 = 0$ or $\delta_1 = \delta_2$, the function $\upsilon^{(\lambda)}(\rho)$ is monotonically increasing in $-\rho$ (for $\rho < 0$), and if $\lim_{\rho\nearrow 0}\left[\upsilon^{(\lambda)}(\rho)\right] < 0$, then the set $\Upsilon^{(\lambda)}$ is single valued, and otherwise $\Upsilon^{(\lambda)}$ is empty. It follows that for $B_2 = 0$ or $\delta_1 = \delta_2$ there exists no $\rho > \overline{\rho}^{(\lambda)}$ that satisfies $\upsilon^{(\lambda)}(\rho) > 0$. In the general case ($B_2 \neq 0$ and $\delta_1 \neq \delta_2$) there might exist $\rho > \overline{\rho}^{(\lambda)}$ that satisfies $\upsilon^{(\lambda)}(\rho) > 0$. The technical conditions needed are the same as in Lemma 2.4. For $\frac{\partial r}{\partial\lambda} > 0$ to hold, I need $(1-\rho)\left(1 - a + \sum_{i=1}^{2}\frac{an}{r-\mu^{(Y)}+(1+\delta_i)n}F^{y,(i)}\psi - n\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\sum_{i=1}^{2}F^{y,(i)}\right) - \frac{1-\gamma-\rho}{1-\gamma} > 0$. Suppose that $\rho < 0$. Using $\gamma > 1$, and $\Phi(r) < a(1+\delta_1)$ (result of Lemma 2.3), I note that

$$
\begin{aligned}
&(1-\rho)\left(\begin{array}{c} 1 - a + \sum_{i=1}^{2}\frac{an}{r-\mu^{(Y)}+(1+\delta_i)n}F^{y,(i)}\psi \\ -n\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\sum_{i=1}^{2}F^{y,(i)} \end{array}\right) - \frac{1-\gamma-\rho}{1-\gamma} \qquad (2.163)\\
&> (1-\rho)\left(1 - a - \left(\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\frac{1}{\psi} - \frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}\right)n\Phi(r)\right) - \frac{1-\gamma-\rho}{1-\gamma}\\
&> (1-\rho)\left(1 - a - \left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi} - \frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}\right)n\Phi(r)\right) - \frac{1-\gamma-\rho}{1-\gamma}\\
&> (1-\rho)\left(1 - a - \left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi} - \frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}\right)na(1+\delta_1)\right) - \frac{1-\gamma-\rho}{1-\gamma}
\end{aligned}
$$

Condition

$$
(1-\rho)\left(1 - a - \left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi} - \frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}\right)na(1+\delta_1)\right) - \frac{1-\gamma-\rho}{1-\gamma} > 0 \qquad (2.164)
$$

suffices for $\frac{\partial r}{\partial\lambda} > 0$. For the case when the EIS approaches zero condition (2.164) is satisfied since

$$\lim_{\rho\to-\infty}\left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi}-\frac{a}{r-\mu^{(Y)}+(1+\delta_1)\,n}\right)=0 \tag{2.165}$$

and

$$\lim_{\rho\to-\infty}(1-\rho)\left(1-a-\left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi}-\frac{a}{r-\mu^{(Y)}+(1+\delta_1)\,n}\right)na\,(1+\delta_1)\right)-\frac{1-\gamma-\rho}{1-\gamma}>0 \tag{2.166}$$

The term $-\frac{1-\gamma-\rho}{1-\gamma}$ is increasing in $-\rho$ (since $\gamma>1$). The expression $-\frac{\gamma}{1-\gamma}\frac{1}{\psi}-\frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}$ is decreasing in $-\rho$ if $\sum\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+(1+\delta_i)n}\left(1+\delta_i-\Phi\left(r\right)\right)>0$, because

$$\begin{aligned}
&\frac{\partial\left(-\frac{\gamma}{1-\gamma}\frac{1}{\psi}-\frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}\right)}{\partial\left(-\rho\right)} \qquad (2.167)\\
&=\frac{\gamma}{1-\gamma}\frac{1}{\psi^2}\frac{\partial\psi}{\partial\left(-\rho\right)}+\frac{a}{\left(r-\mu^{(Y)}+(1+\delta_1)\,n\right)^2}\frac{\partial r}{\partial\left(-\rho\right)}\\
&=\frac{\gamma}{1-\gamma}\frac{1}{\psi^2}\frac{1}{\left(1-\rho\right)^2}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\left(1-\rho\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)\,n\Phi'\left(r\right)}\right)\\
&+\frac{a}{\left(r-\mu^{(Y)}+(1+\delta_1)\,n\right)^2}\frac{\frac{1}{\psi}\left(r-\mu^{(Y)}\right)\frac{1}{1-\rho}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)}{1+(1-\rho)\,n\Phi'\left(r\right)}\\
&=\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\frac{1}{\left(1-\rho\right)^2}\left[\begin{array}{c}\frac{\gamma}{1-\gamma}\frac{1}{\psi^2}\left(1-\rho\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\right)\\+\frac{a}{\left(r-\mu^{(Y)}+(1+\delta_1)n\right)^2}\left(1-\rho\right)\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\end{array}\right]\\
&<\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\frac{1}{\left(1-\rho\right)^2}\frac{1}{\psi^2}\left[\begin{array}{c}\frac{\gamma}{1-\gamma}\left(1-\rho\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\right)\\+a\left(1-\rho\right)\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\end{array}\right]\\
&<\frac{\gamma}{1-\gamma}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\frac{1}{\left(1-\rho\right)^2}\frac{1}{\psi^2}\left[\begin{array}{c}\left(1-\rho\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\right)\\-\left(1-\rho\right)\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)n\Phi'(r)}\end{array}\right]\\
&=\frac{\gamma}{1-\gamma}\left(r-\frac{\gamma}{1-\gamma}\lambda-\beta\right)\frac{1}{\left(1-\rho\right)^2}\frac{1}{\psi^2}\left[1-\frac{1}{\psi}\frac{r-\mu^{(Y)}}{1+(1-\rho)\,n\Phi'\left(r\right)}\right]\\
&=\frac{\gamma}{1-\gamma}\frac{1}{1-\rho}\frac{1}{\psi^3}\frac{r-\frac{\gamma}{1-\gamma}\lambda-\beta}{1+(1-\rho)\,n\Phi'\left(r\right)}n^2\sum_{i=1}^{2}\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+(1+\delta_i)\,n}\left(1+\delta_i-\Phi\left(r\right)\right)
\end{aligned}$$

where the first inequality follows from $\psi<r-\mu^{(Y)}+(1+\delta_1)\,n$ and the second one from $\frac{-\gamma}{1-\gamma}>a$. As in the discussion in Lemma 2.4, it is hard to tell whether $\sum\frac{F^{y,(i)}\psi}{r-\mu^{(Y)}+(1+\delta_i)n}\left(1+\delta_i-\Phi\left(r\right)\right)>0$ is satisfied in general. However, if $B_2=0$ or $\delta_1=\delta_2$, then $a\,(1+\delta_1)>\Phi\left(r\right)$ and $\frac{\partial(\Phi(r))}{\partial(-\rho)}>0$ must hold for $\rho<0$. Since $-\frac{\gamma}{1-\gamma}\frac{1}{\psi}-$

$\frac{a}{r-\mu^{(Y)}+(1+\delta_1)n}$ and $-\frac{1-\gamma-\rho}{1-\gamma}$ are continuous function in ρ (for $\rho < 0$), it follows that $\rho < \overline{\rho}$ satisfies condition (2.164) and $\frac{\partial r}{\partial \lambda} > 0$ (if $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$). Moreover, in the case of $B_2 = 0$ or $\delta_1 = \delta_2$, if condition (2.164) is not satisfied for $\rho \nearrow 0$, then $\Upsilon^{(\lambda)}$ is single valued (as $\upsilon^{(\lambda)}$ is monotonic), and if condition (2.164) is satisfied for $\rho \nearrow 0$, then $\Upsilon^{(\lambda)}$ is empty. ■

Proof of Lemma 2.6. Let $\overline{\rho}_1^{(n)} = \min_{\rho \in \left\{\Upsilon_1^{(n)}, 0\right\}} \{\rho\}$ with $\Upsilon_1^{(n)} = \{\rho : \upsilon_1^{(n)}(\rho) = 0, \rho < 0\}$ and $\upsilon_1^{(n)}(\rho) = r_L^{(n)}(\rho) - r_H^{(n)}(\rho)$. I show that for $\rho < \overline{\rho}_1^{(n)}$ (or $EIS < \overline{EIS}_1^{(n)} = \frac{1}{1-\overline{\rho}_1^{(n)}}$), the interest rate during a period characterized by a high birth rate (baby boom) is lower than the rate during times of a low birth rate (baby bust), $r_H^{(n)} < r_L^{(n)}$. This is a sufficient condition and there might exist some $\rho > \overline{\rho}_1^{(n)}$ that satisfies $r_H^{(n)} < r_L^{(n)}$. I need the technical conditions $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$, $\delta_1 > 0$, $\mu_L^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_L a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$ and $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_H a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$. The conditions $\rho < 0$ and $\gamma \in (1, 1-\rho)$ imply $\frac{1-\gamma-\rho}{\rho} < 0$, and $\psi_H^{(n)} < (>)\, \psi_L^{(n)}$ is true if $r_H^{(n)} < (>)\, r_L^{(n)}$ holds. I can show this using a proof by contradiction. Suppose $\psi_H^{(n)} > \psi_L^{(n)}$ and $r_H^{(n)} < r_L^{(n)}$. I have

$$
\begin{aligned}
\psi_H^{(n)} - \psi_L^{(n)} = {} & \underbrace{-\frac{\rho}{1-\rho}\left(r_H^{(n)} - r_L^{(n)}\right)}_{<0} \underbrace{-\frac{\rho}{1-\rho}\overline{\theta}_H^{(n)}\left(\left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)}_{<0} \\
& \underbrace{-\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\overline{\theta}_H^{(n)}\left(\left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)}_{<0} \\
& \underbrace{+\frac{\rho}{1-\rho}\overline{\theta}_L^{(n)}\left(\left(\frac{\psi_H^{(n)}}{\psi_L^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)}_{<0} \\
& \underbrace{+\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\overline{\theta}_L^{(n)}\left(\left(\frac{\psi_H^{(n)}}{\psi_L^{(n)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)}_{<0}
\end{aligned}
\tag{2.168}
$$

which contradicts the assumption $\psi_H^{(n)} > \psi_L^{(n)}$. Hence, if there exists a solution, then $\psi_H^{(n)} < \psi_L^{(n)}$ and $r_H^{(n)} < r_L^{(n)}$ must hold. The same line of argument holds for $r_H^{(n)} > r_L^{(n)}$ and $\psi_H^{(n)} > \psi_L^{(n)}$.

Next, I look at the difference between the interest rate during a baby bust and the rate during a baby boom. To proof the Lemma I have to find conditions such that the $r_L^{(n)} - r_H^{(n)} > 0$ holds. I explore the behavior of $r_L^{(n)} - r_H^{(n)}$ under the limit when ρ approaches $-\infty$. I first suppose that $\forall j \in \{L, H\}$, $i \in \{1, 2\}$, $|\lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho}| < \infty$, $\lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho} \neq 0$, $|\lim_{\rho \to -\infty} \frac{\psi_j^{(n)}}{1-\rho}| < \infty$, $\lim_{\rho \to -\infty} \frac{\psi_j^{(n)}}{1-\rho} \neq 0$, $|\lim_{\rho \to -\infty} (1-\rho) F_j^{y,(i),(n)}| < \infty$, and $\lim_{\rho \to -\infty} (1-\rho) F_j^{y,(i),(n)} \neq 0$ hold, and verify these assumptions in the end. For the interest rate I have $\forall (j, h) \in \{(L, H), (H, L)\}$

$$\lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho} = \mu_j^{(Y,n)} - \frac{\gamma}{1-\gamma} \lambda - n_j \lim_{\rho \to -\infty} \left(\sum_{i=1}^{2} F_j^{y,(i),(n)} \psi_j^{(n)} \right) \tag{2.169}$$

For the consumption to wealth ratio I get $\forall (j, h) \in \{(L, H), (H, L)\}$

$$\lim_{\rho \to -\infty} \frac{\psi_j^{(n)}}{1-\rho} = \lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho} \tag{2.170}$$

For the function $F_j^{y,i,(n)}$, $\forall i \in \{1, 2\}$, $(j, h) \in \{(L, H), (H, L)\}$ it holds

$$\lim_{\rho \to -\infty} (1-\rho) F_j^{y,(i),(n)} = \frac{1}{\lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho}} \frac{a B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \tag{2.171}$$

The consumption share of the new born cohort is $\forall j \in \{L, H\}$

$$n_j \lim_{\rho \to -\infty} \left(\sum_{i=1}^{2} F_j^{y,(i),(n)} \psi_j^{(n)} \right) = n_j a \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \tag{2.172}$$

Plugging the last expression into the equation of $\lim_{\rho \to -\infty} \frac{r_j^{(n)}}{1-\rho}$, it follows that my assumptions are indeed true. I can now compare how $r_H^{(n)}$ and $r_L^{(n)}$ behave in the limit,

$$\lim_{\rho \to -\infty} \left(\frac{r_L^{(n)}}{1-\rho} - \frac{r_H^{(n)}}{1-\rho} \right) = a (n_H - n_L) \left(\frac{\sum_{i=1}^{2} B_i}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} - 1 \right) \tag{2.173}$$

As a result, in the limit as ρ approaches $-\infty$, $\lim_{\rho \to -\infty} \left(r_L^{(n)} - r_H^{(n)} \right) > 0$ is satisfied, if $\frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} > 1$ or equivalently $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}} |B_2|$ holds. This is the same condition as in the static case (Lemma 2.4), and requires that $\frac{B_1}{|B_2|}$ is large enough and $(\delta_2 - \delta_1)$ is small enough. In the case of $B_2 = 0$ or $\delta_1 = \delta_2$, the condition becomes $\delta_1 > 0$. In conclusion, since the functions $r_L^{(n)}(\rho)$ and $r_H^{(n)}(\rho)$ are continuous in ρ, the condition

$\rho < \overline{\rho}_1^{(n)}$ ensures that $r_L^{(n)} > r_H^{(n)}$ holds.

It is straightforward that $|\kappa_L^{(n)}| < |\kappa_H^{(n)}|$ must hold given $r_H^{(n)} < r_L^{(n)}$, $\psi_H^{(n)} < \psi_L^{(n)}$ and $\frac{1-\gamma-\rho}{\rho} < 0$. It is true that

$$0 < \left[\left(\psi_H^{(n)}\right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left(\psi_L^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\right]^2 \tag{2.174}$$

and rearranging yields

$$\begin{aligned} & \left(\psi_L^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}} \left(\psi_H^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}} - \left(\psi_H^{(n)}\right)^{-2\frac{1-\gamma-\rho}{\rho}} \\ < \; & \left(\psi_L^{(n)}\right)^{-2\frac{1-\gamma-\rho}{\rho}} - \left(\psi_L^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}} \left(\psi_H^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}} \end{aligned} \tag{2.175}$$

and dividing both sides by $\left(\psi_L^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}} \left(\psi_H^{(n)}\right)^{-\frac{1-\gamma-\rho}{\rho}}$ gives

$$\underbrace{-\left(\left(\frac{\psi_H^{(n)}}{\psi_L^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)}_{|\kappa_L^{(n)}|} < \underbrace{\left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1}_{|\kappa_H^{(n)}|} \tag{2.176}$$

■

Proof of Lemma 2.7. Let $\overline{\rho}_2^{(n)} = \min_{\rho \in \left\{\Upsilon_2^{(n)}, 0\right\}} \{\rho\}$ with $\Upsilon_2^{(n)} = \{\rho : \upsilon_2^{(n)}(\rho) = 0,$ $\rho < 0\}$ and $\upsilon_2^{(n)}(\rho) = \frac{1}{\psi_H^{(n)}(\rho)} - \frac{1}{\psi_L^{(n)}(\rho)} - \sum_{i=1}^{2} \frac{F_H^{y,(i),(n)}(\rho) - F_L^{y,(i),(n)}(\rho)}{1+\delta_i}$. I show that for $\rho < \overline{\rho}_2^{(n)}$ (or $EIS < \overline{EIS}_2^{(n)} = \frac{1}{1-\overline{\rho}_2^{(n)}}$), the equity premium is positive in both states of the world. This is a sufficient condition and there might exist some $\rho > \overline{\rho}_2^{(n)}$ which is consistent with a positive equity premium in both states of the world. I need the technical conditions $\mu_L^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_L a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ and $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda - n_H a \frac{B_1+B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$. Condition $\frac{1-\gamma-\rho}{\rho} < 0$ implies (independent of $\psi_L^{(n)} > \psi_H^{(n)}$ or $\psi_L^{(n)} < \psi_H^{(n)}$) $\forall\, (i,j) \in \{(L,H), (H,L)\}$

$$-\overline{\theta}_i^{(n)} \frac{Y_t}{P_t^{(n_i)}} \frac{1}{\psi_i^{(n)}} \left(\left(\frac{\psi_j^{(n)}}{\psi_i^{(n)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right) \left(\left(\frac{\psi_j^{(n)}}{\psi_i^{(n)}}\right)^{-1} - 1\right) > 0 \tag{2.177}$$

To ensure that $E_t\left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt > 0$, $\forall\; n_t \in \{n_L, n_H\}$, it is sufficient to show that the two inequalities

$$
\begin{aligned}
& -\overline{\theta}_i^{(n)} \frac{Y_t}{P_t^{(n_i)}} \frac{1}{\psi_i^{(n)}} \left(\left(\frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \left(\left(\frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-1} - 1 \right) \\
& > |\overline{\theta}_i^{(n)} \frac{Y_t}{P_t^{(n_i)}} \left(\left(\frac{\psi_j^{(n)}}{\psi_i^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \sum_{k=1}^{2} \frac{F_j^{y,(k),(n)} - F_i^{y,(k),(n)}}{1+\delta_k} |
\end{aligned}
\tag{2.178}
$$

$\forall\ (i,j) \in \{(L,H)\,,),(H,L)\}$, hold. Note that $(1-\rho)\, P_t^{(n_L)}$ and $(1-\rho)\, P_t^{(n_H)}$ are positive and finite for $\rho < 0$. Rewriting both inequalities yields the single condition

$$
(1-\rho)\, |\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}}| \geq (1-\rho)\, |\sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i}| \tag{2.179}
$$

From Lemma 2.6 it follows that

$$
\begin{aligned}
& \lim_{\rho\to-\infty} (1-\rho)\, |\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}}| - \lim_{\rho\to-\infty} (1-\rho)\, |\sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i}| \\
& = (1-a)\, |\frac{1}{\lim_{\rho\to-\infty} \frac{r_H^{(n)}}{1-\rho}} - \frac{1}{\lim_{\rho\to-\infty} \frac{r_L^{(n)}}{1-\rho}}| > 0
\end{aligned}
\tag{2.180}
$$

In the limit as ρ approaches $-\infty$, condition (2.179) is satisfied. Since the function $\upsilon_2^{(n)}(\rho)$ is continuous, the condition $\rho < \overline{\rho}_2^{(n)}$ ensures that condition (2.179) is satisfied and $E_t \left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt > 0,\ \forall\ n_t \in \{n_L, n_H\}$. ■

Proof of Lemma 2.8. Let $\overline{\rho}_3^{(n)} = \min_{\rho \in \left\{\Upsilon_1^{(n)}, \Upsilon_2^{(n)}, \Upsilon_3^{(n)}, 0\right\}} \{\rho\}$ with $\Upsilon_3^{(n)} = \{\rho : \upsilon_3^{(n)}(\rho) = 0,\ \rho < 0\}$, $\upsilon_3^{(n)}(\rho) = \frac{\overline{\theta}_H^{(n)}}{\overline{\theta}_L^{(n)}} - \frac{1-\sum_{i=1}^{2} \frac{F_H^{y,(i),(n)}(\rho)\psi_H^{(n)}(\rho)}{1+\delta_i}}{1-\sum_{i=1}^{2} \frac{F_L^{y,(i),(n)}(\rho)\psi_L^{(n)}(\rho)}{1+\delta_i}} \left(\frac{\psi_L^{(n)}(\rho)}{\psi_H^{(n)}(\rho)}\right)^{\frac{1-\gamma}{\rho}}$ and $\Upsilon_1^{(n)}$ and $\Upsilon_2^{(n)}$ as defined in Lemma 2.6 and 2.7. I show that the condition $\rho < \overline{\rho}_3^{(n)}$ (or $EIS < \overline{EIS}_3^{(n)} \equiv \frac{1}{1-\overline{\rho}_3^{(n)}}$) ensures that the equity premium is larger (lower) during a period characterized by a high birth rate (baby boom) than the premium during times of a low birth rate (baby bust). This is a sufficient condition and there might exist some $\rho > \overline{\rho}_2^{(n)}$ which is consistent with the result of the Lemma. To give proof I have to show that for $\overline{\theta}_H^{(n)} > (<)\, \overline{\theta}_L^{(n)}$, it holds

$$
E_t \left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt \mid [n_t = n_H] - E_t \left[\frac{dP_t + D_t dt}{P_t}\right] - r_t dt \mid [n_t = n_L] > (<)\, 0 \tag{2.181}
$$

or plugging in the expression for the equity premium

$$0 \;<\; (>)\, \overline{\theta}_H^{(n)} \frac{Y_t}{P_t^{(n_H)}} \left(\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}} - \sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i} \right) \cdot \left[\left(\left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) + \frac{\overline{\theta}_L^{(n)}}{\overline{\theta}_H^{(n)}} \frac{P_t^{(n_H)}}{P_t^{(n_L)}} \left(\left(\frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} - 1 \right) \right] \tag{2.182}$$

Since $\psi_H^{(n)} < \psi_L^{(n)}$ (by Lemma 2.6), $|\frac{1}{\psi_H^{(n)}} - \frac{1}{\psi_L^{(n)}}| > |\sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} - F_L^{y,(i),(n)}}{1+\delta_i}|$ (by Lemma 2.7), and $\rho < 0$, $\gamma \in (1, 1-\rho)$ (by Lemma 2.6 and 2.7), it suffices to show that the last term in square brackets is positive (negative),

$$\frac{\overline{\theta}_H^{(n)}}{\overline{\theta}_L^{(n)}} > (<) \frac{P_t^{(n_H)}}{P_t^{(n_L)}} \left(\frac{\psi_H^{(n)}}{\psi_L^{(n)}} \right)^{-\frac{1-\gamma-\rho}{\rho}} = \frac{1 - \sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} \psi_H^{(n)}}{1+\delta_i}}{1 - \sum_{i=1}^{2} \frac{F_L^{y,(i),(n)} \psi_L^{(n)}}{1+\delta_i}} \left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{\frac{1-\gamma}{\rho}} \tag{2.183}$$

In the limit as the EIS approaches zero condition (2.183) is satisfied if

$$\frac{\overline{\theta}_H^{(n)}}{\overline{\theta}_L^{(n)}} > (<) \lim_{\rho \to -\infty} \left[\frac{1 - \sum_{i=1}^{2} \frac{F_H^{y,(i),(n)} \psi_H^{(n)}}{1+\delta_i}}{1 - \sum_{i=1}^{2} \frac{F_L^{y,(i),(n)} \psi_L^{(n)}}{1+\delta_i}} \left(\frac{\psi_L^{(n)}}{\psi_H^{(n)}} \right)^{\frac{1-\gamma}{\rho}} \right] = 1 \tag{2.184}$$

Since the function $v_3^{(n)}(\rho)$ is continuous in $\rho < 0$, the condition $\rho < \overline{\rho}_3^{(n)}$ ensures that condition (2.183) is satisfied and $E_t \left[\frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_H] > (<)\, E_t \left[\frac{dP_t + D_t dt}{P_t} \right] - r_t dt \mid [n_t = n_L]$. ■

Proof of Lemma 2.9. Let $\overline{\rho}_1^{(\lambda)} = \min_{\rho \in \left\{ \Upsilon_1^{(\lambda)}, 0 \right\}} \{\rho\}$ with $\Upsilon_1^{(\lambda)} = \{\rho : v_1^{(\lambda)}(\rho) = 0,\ \rho < 0\}$ and $v_1^{(\lambda)}(\rho) = r_H^{(\lambda)}(\rho) - r_L^{(\lambda)}(\rho)$. I show that for $\rho < \overline{\rho}_1^{(\lambda)}$ (or $EIS < \overline{EIS}_1^{(\lambda)} \equiv \frac{1}{1-\overline{\rho}_1^{(\lambda)}}$), the interest rate during a period characterized by a high death rate is higher than the rate during times of a low mortality, $r_H^{(\lambda)} > r_L^{(\lambda)}$. This is a sufficient condition and there might exist some $\rho > \overline{\rho}_1^{(\lambda)}$ which is consistent with the result of the Lemma. The proof follows the same line of argument as the proof of Lemma 2.6. I need the technical conditions $\mu_L^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_L - na \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$ and $\mu_H^{(Y,\lambda)} - \frac{\gamma}{1-\gamma} \lambda_H - na \frac{B_1 + B_2}{\frac{B_1}{1+\delta_1} + \frac{B_2}{1+\delta_2}} \neq 0$. The conditions $\rho < 0$ and $\gamma \in (1, 1-\rho)$ imply $\frac{1-\gamma-\rho}{\rho} < 0$, and $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$ is implied by $r_H^{(\lambda)} > r_L^{(\lambda)}$ holds. I can show this using a proof by contradiction. Suppose $\psi_H^{(\lambda)} < \psi_L^{(\lambda)}$

and $r_H^{(\lambda)} > r_L^{(\lambda)}$. I have

$$
\begin{aligned}
\psi_H^{(\lambda)} - \psi_L^{(\lambda)} &= \underbrace{\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\left(\lambda_H - \lambda_L\right)}_{>0} \underbrace{-\frac{\rho}{1-\rho}\left(r_H^{(\lambda)} - r_L^{(\lambda)}\right)}_{>0} \\
&\quad \underbrace{-\frac{\rho}{1-\rho}\overline{\theta}_H^{(\lambda)}\left(\left(\frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)}_{>0} \\
&\quad \underbrace{+\frac{\rho}{1-\rho}\overline{\theta}_L^{(\lambda)}\left(\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} - 1\right)}_{>0} \\
&\quad \underbrace{-\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\overline{\theta}_H^{(\lambda)}\left(\left(\frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)}_{>0} \\
&\quad \underbrace{+\frac{\rho}{1-\rho}\frac{\gamma}{1-\gamma}\overline{\theta}_L^{(\lambda)}\left(\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{-\frac{1-\gamma}{\rho}} - 1\right)}_{>0}
\end{aligned} \tag{2.185}
$$

which contradicts the assumption $\psi_H^{(\lambda)} < \psi_L^{(\lambda)}$. Hence, if there exists a solution, then $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$ and $r_H^{(\lambda)} > r_L^{(\lambda)}$ holds.

Next, I look at the difference between the interest rate in a high death rate state and the rate in a low death rate state. To proof the Lemma I have to find conditions such that the $r_H^{(\lambda)} - r_L^{(\lambda)} > 0$ holds. In the limit as EIS goes to zero my key quantities are essentially the same as derived in Lemma 2.6, for $j \in \{L, H\}$, $k \in \{1, 2\}$

$$
\lim_{\rho\to-\infty}\frac{r_j^{(\lambda)}}{1-\rho} = \mu_j^{(Y,\lambda)} - \frac{\gamma}{1-\gamma}\lambda_j - n\lim_{\rho\to-\infty}\left(\sum_{i=1}^{2} F_j^{y,(i),(\lambda)}\psi_j^{(\lambda)}\right) \tag{2.186}
$$

$$
\lim_{\rho\to-\infty}\frac{\psi_j^{(\lambda)}}{1-\rho} = \lim_{\rho\to-\infty}\frac{r_j^{(\lambda)}}{1-\rho} \tag{2.187}
$$

$$
\lim_{\rho\to-\infty}\left(1-\rho\right)F_j^{y,(k),(\lambda)} = \frac{1}{\lim_{\rho\to-\infty}\frac{r_j^{(\lambda)}}{1-\rho}}\frac{aB_k}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.188}
$$

$$
n\lim_{\rho\to-\infty}\left(\sum_{i=1}^{2} F_j^{y,(k),(\lambda)}\psi_j^{(\lambda)}\right) = na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.189}
$$

and

$$\lim_{\rho\to-\infty}\left(\frac{r_H^{(\lambda)}}{1-\rho}-\frac{r_L^{(\lambda)}}{1-\rho}\right)=\left(-\frac{\gamma}{1-\gamma}-a\right)(\lambda_H-\lambda_L) \tag{2.190}$$

Thus, for $\gamma > 1$, $\lim_{\rho\to-\infty}\left(r_H^{(\lambda)}-r_L^{(\lambda)}\right) > 0$ is satisfied. In conclusion, since the functions $r_L^{(\lambda)}(\rho)$ and $r_H^{(\lambda)}(\rho)$ are continuous in $\rho < 0$, the condition $\rho < \overline{\rho}_1^{(\lambda)}$ ensures that $r_H^{(\lambda)} > r_L^{(\lambda)}$ holds.

It is straightforward that $|\kappa_L^{(\lambda)}| > |\kappa_H^{(\lambda)}|$ must hold given $r_H^{(\lambda)} > r_L^{(\lambda)}$, $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$ and $\frac{1-\gamma-\rho}{\rho} < 0$. It is true that

$$0 < \left[\left(\psi_H^{(\lambda)}\right)^{-2\frac{1-\gamma-\rho}{\rho}}-\left(\psi_L^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\right]^2 \tag{2.191}$$

and rearranging yields

$$\begin{aligned}&\left(\psi_H^{(\lambda)}\right)^{-2\frac{1-\gamma-\rho}{\rho}}-\left(\psi_L^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\left(\psi_H^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\\ > \; &\left(\psi_L^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\left(\psi_H^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}-\left(\psi_L^{(\lambda)}\right)^{-2\frac{1-\gamma-\rho}{\rho}}\end{aligned} \tag{2.192}$$

and dividing both sides by $\left(\psi_L^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}\left(\psi_H^{(\lambda)}\right)^{-\frac{1-\gamma-\rho}{\rho}}$ gives

$$\underbrace{\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}}-1}_{|\kappa_L^{(\lambda)}|} > \underbrace{-\left(\left(\frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}}-1\right)}_{|\kappa_H^{(\lambda)}|} \tag{2.193}$$

■

Proof of Lemma 2.10. Let $\overline{\rho}_2^{(\lambda)} = \min_{\rho\in\left\{\Upsilon_2^{(\lambda)},0\right\}}\{\rho\}$ with $\Upsilon_2^{(\lambda)} = \{\rho : \upsilon_2^{(\lambda)}(\rho) = 0,$ $\rho < 0\}$ and $\upsilon_2^{(\lambda)}(\rho) = \frac{1}{\psi_L^{(\lambda)}(\rho)}-\frac{1}{\psi_H^{(\lambda)}(\rho)}-\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}(\rho)-F_H^{y,(i),(\lambda)}(\rho)}{1+\delta_i}$. I show that for $\rho < \overline{\rho}_2^{(\lambda)}$ (of $EIS < \overline{EIS}_2^{(\lambda)} \equiv \frac{1}{1-\overline{\rho}_2^{(\lambda)}}$), the equity premium is positive in both states of the world. This is a sufficient condition and there might exist $\rho > \overline{\rho}_2^{(\lambda)}$ which is consistent with a positive equity premium in both states of the world. The proof is the same as the proof of Lemma 2.7. I need the technical conditions $\mu_L^{(Y,\lambda)} - \frac{\gamma}{1-\gamma}\lambda_L - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$ and $\mu_H^{(Y,n)} - \frac{\gamma}{1-\gamma}\lambda_H - na\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$. As in Lemma 2.7, under the condition $\frac{1-\gamma-\rho}{\rho} < 0$ I only have to show that $\upsilon_2^{(\lambda)}(\rho) > 0$. Using the results of Lemma 2.9, I have

$$\lim_{\rho\to-\infty}(1-\rho)\left(|\frac{1}{\psi_L^{(\lambda)}}-\frac{1}{\psi_H^{(\lambda)}}|-|\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}-F_H^{y,(i),(\lambda)}}{1+\delta_i}|\right) \tag{2.194}$$
$$= (1-a)\,|\frac{1}{\lim_{\rho\to-\infty}\frac{r_L^{(\lambda)}}{1-\rho}}-\frac{1}{\lim_{\rho\to-\infty}\frac{r_H^{(\lambda)}}{1-\rho}}| > 0$$

It follows that in the limit as ρ approaches $-\infty$, the equity premium is positive in both states of the world. Since the function $\upsilon_2^{(\lambda)}(\rho)$ is continuous in $\rho < 0$, the condition $\rho < \overline{\rho}_2^{(\lambda)}$ ensures $E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt > 0,\ \forall\ \lambda_t\in\{\lambda_L,\lambda_H\}$. ■

Proof of Lemma 2.11. Let $\overline{\rho}_3^{(\lambda)} = \min_{\rho\in\left\{\Upsilon_1^{(\lambda)},\Upsilon_2^{(\lambda)},\Upsilon_3^{(\lambda)},0\right\}}\{\rho\}$ with $\Upsilon_3^{(\lambda)} = \{\rho : \upsilon_3^{(\lambda)}(\rho) = 0,\ \rho < 0\}$, $\upsilon_3^{(\lambda)}(\rho) = \frac{\overline{\theta}_L^{(\lambda)}}{\overline{\theta}_H^{(\lambda)}} - \frac{1-\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}\psi_L^{(\lambda)}}{1+\delta_i}}{1-\sum_{i=1}^{2}\frac{F_H^{y,(i),(\lambda)}\psi_H^{(\lambda)}}{1+\delta_i}}\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{\frac{1-\gamma}{\rho}}$ and $\Upsilon_1^{(\lambda)}$ and $\Upsilon_2^{(\lambda)}$ as defined in Lemma 2.9 and 2.10. I show that the condition $\rho < \overline{\rho}_3^{(\lambda)}$ (or $EIS < \overline{EIS}_3^{(\lambda)} \equiv \frac{1}{1-\overline{\rho}_3^{(\lambda)}}$) ensures that the equity premium is larger (lower) during a period characterized by a low death rate than the premium in times of high mortality. This is a sufficient condition and there might exist some $\rho > \overline{\rho}_3^{(\lambda)}$ which is consistent with the result of the Lemma. The proof follows the same argument as the proof of Lemma 2.8. I have to show that for $\overline{\theta}_L^{(n)} > (<)\,\overline{\theta}_H^{(n)}$ the condition

$$E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt \mid [\lambda_t=\lambda_L]-E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt \mid [\lambda_t=\lambda_H] > (<)\,0 \tag{2.195}$$

holds or plugging in the expression for the equity premium

$$0 \;<\; (>)\,\overline{\theta}_L^{(\lambda)}\frac{Y_t}{P_t^{(\lambda_L)}}\left(\frac{1}{\psi_L^{(\lambda)}}-\frac{1}{\psi_H^{(\lambda)}}-\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}-F_H^{y,(i),(\lambda)}}{1+\delta_i}\right) \tag{2.196}$$
$$\cdot\left[\left(\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}}-1\right)+\frac{\overline{\theta}_H^{(\lambda)}}{\overline{\theta}_L^{(\lambda)}}\frac{P_t^{(\lambda_L)}}{P_t^{(\lambda_H)}}\left(\left(\frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}}-1\right)\right]$$

Since $\psi_H^{(\lambda)} > \psi_L^{(\lambda)}$ (by Lemma 2.9), $|\frac{1}{\psi_L^{(\lambda)}}-\frac{1}{\psi_H^{(\lambda)}}|-|\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}-F_H^{y,(i),(\lambda)}}{1+\delta_i}|$ (by Lemma 2.12), and $\rho<0$, $\gamma\in(1,1-\rho)$ (by Lemma 2.9 and 2.10), it suffices to show that

$$\frac{\overline{\theta}_L^{(\lambda)}}{\overline{\theta}_H^{(\lambda)}} > (<)\,\frac{P_t^{(\lambda_L)}}{P_t^{(\lambda_H)}}\left(\frac{\psi_L^{(\lambda)}}{\psi_H^{(\lambda)}}\right)^{-\frac{1-\gamma-\rho}{\rho}} = \frac{1-\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}\psi_L^{(\lambda)}}{1+\delta_i}}{1-\sum_{i=1}^{2}\frac{F_H^{y,(i),(\lambda)}\psi_H^{(\lambda)}}{1+\delta_i}}\left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{\frac{1-\gamma}{\rho}} \tag{2.197}$$

In the limit as the EIS approaches zero condition (2.197) is satisfied,

$$\frac{\overline{\theta}_L^{(\lambda)}}{\overline{\theta}_H^{(\lambda)}} > (<) \lim_{\rho\to-\infty} \frac{1-\sum_{i=1}^{2}\frac{F_L^{y,(i),(\lambda)}\psi_L^{(\lambda)}}{1+\delta_i}}{1-\sum_{i=1}^{2}\frac{F_H^{y,(i),(\lambda)}\psi_H^{(\lambda)}}{1+\delta_i}} \left(\frac{\psi_H^{(\lambda)}}{\psi_L^{(\lambda)}}\right)^{\frac{1-\gamma}{\rho}} = 1 \tag{2.198}$$

Since the function $\upsilon_3^{(\lambda)}(\rho)$ is continuous in $\rho<0$, the condition $\rho<\overline{\rho}_3^{(\lambda)}$ ensures that condition (2.197) is satisfied and $E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt \mid [\lambda_t=\lambda_L] > (<) E_t\left[\frac{dP_t+D_tdt}{P_t}\right]-r_tdt \mid [\lambda_t=\lambda_H]$. ■

Proof of Lemma 2.12. I need the technical conditions $B_1 > \frac{1+\frac{1}{\delta_1}}{1+\frac{1}{\delta_2}}|B_2|$ and $\mu_t^{(Y)} - \frac{\gamma}{1-\gamma}\lambda_t - n_t a\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \neq 0$. Suppose for now that $\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\neq 0$, $|\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}|<\infty$, $|\lim_{\rho\to-\infty}\frac{\mu_t^{(\psi)}}{\psi_t}|<\infty$, $|\lim_{\rho\to-\infty}\frac{\sigma_t^{(\psi)}}{\psi_t}|<\infty$, $|\lim_{\rho\to-\infty}(1-\rho)\,\mu_t^{(F^{y,(i)})}|<\infty$, and $|\lim_{\rho\to-\infty}(1-\rho)\,\sigma_t^{(F^{y,(i)})}|<\infty$. I will later see that these assumptions are indeed true. In the limit as the EIS approaches zero I have

$$\lim_{\rho\to-\infty}\frac{r_t}{1-\rho} = \mu_t^{(Y)} - \frac{\gamma}{1-\gamma}\lambda_t - n_t n_t \lim_{\rho\to-\infty}\left[\sum_{i=1}^{2}F^{y,t,(i)}(\lambda,n,t)\,\psi_t(\lambda,n)\right] \tag{2.199}$$

$$\lim_{\rho\to-\infty}\frac{\partial}{\partial\lambda}\left(\frac{r_t}{1-\rho}\right) = -\frac{\gamma}{1-\gamma}-a>0 \tag{2.200}$$

$$\lim_{\rho\to-\infty}\frac{\partial}{\partial n}\left(\frac{r_t}{1-\rho}\right) = a\left(1-\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}\right)<0 \tag{2.201}$$

$$\lim_{\rho\to-\infty}\frac{\psi_t(\lambda,n)}{1-\rho} = \lim_{\rho\to-\infty}\frac{r_t}{1-\rho} \tag{2.202}$$

$$\lim_{\rho\to-\infty}\frac{\sigma_t^{(\psi)}}{1-\rho} = \left(-\frac{\gamma}{1-\gamma}-a\right)\lambda_t\sigma^{(\lambda)} + a\left(1-\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}\right)n_t\sigma^{(n)} \tag{2.203}$$

$$\lim_{\rho\to-\infty}\frac{\mu_t^{(\psi)}}{1-\rho} = \left(-\frac{\gamma}{1-\gamma}-a\right)\lambda_t\mu^{(\lambda)} + a\left(1-\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}\right)n_t\mu^{(n)} \tag{2.204}$$

$$\lim_{\rho\to-\infty}(1-\rho)\,F^{y,t,(i)}(\lambda,n,t) = \frac{1}{\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}}\frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.205}$$

$$\lim_{\rho \to -\infty} (1-\rho)\, \sigma_t^{(F^{y,(i)})} = -\frac{\frac{\partial}{\partial \lambda}\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2}\frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}\lambda_t\sigma^{(\lambda)} - \frac{\frac{\partial}{\partial n}\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2}\frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}n_t\sigma^{(n)} \tag{2.206}$$

$$\begin{aligned}\lim_{\rho \to -\infty} (1-\rho)\, \mu_t^{(F^{y,(i)})} = \frac{aB_i}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}}[&\frac{\lim_{\rho\to-\infty}\frac{\partial}{\partial\lambda}\frac{r_t}{1-\rho}\lambda_t\mu^{(\lambda)}}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2}+\frac{\lim_{\rho\to-\infty}\frac{\partial}{\partial n}\frac{r_t}{1-\rho}n_t\mu^{(n)}}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2} \\ &-\frac{\left(\lim_{\rho\to-\infty}\frac{\partial}{\partial\lambda}\frac{r_t}{1-\rho}\right)^2\left(\lambda_t\sigma^{(\lambda)}\right)^2}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^3}-\frac{\left(\lim_{\rho\to-\infty}\frac{\partial}{\partial n}\frac{r_t}{1-\rho}\right)^2\left(n_t\sigma^{(n)}\right)^2}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^3} \\ &-\frac{\left(\lim_{\rho\to-\infty}\frac{\partial}{\partial\lambda}\frac{r_t}{1-\rho}\right)\left(\lim_{\rho\to-\infty}\frac{\partial}{\partial n}\frac{r_t}{1-\rho}\right)\lambda_t n_t\sigma^{(\lambda)}\left(\sigma^{(n)}\right)^T}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^3}]\end{aligned} \tag{2.207}$$

$$\lim_{\rho\to-\infty}\sum_{i=1}^{2}F^{y,t,(i)}(\lambda,n,t)\,\psi_t(\lambda,n) = a\frac{B_1+B_2}{\frac{B_1}{1+\delta_1}+\frac{B_2}{1+\delta_2}} \tag{2.208}$$

Obviously, all assumptions are satisfied. The equity premium is positive in the limit because

$$\lim_{\rho\to-\infty}E_t\left[\frac{dP_t+D_tdt}{P_t}-r_t\right] = -\frac{1-\gamma-\rho}{\rho}\left[\begin{array}{c}\frac{\frac{\partial}{\partial\lambda}\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)\lambda_t\sigma^{(\lambda)}}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2} \\ +\frac{\frac{\partial}{\partial\lambda}\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)n_t\sigma^{(n)2}}{\left(\lim_{\rho\to-\infty}\frac{r_t}{1-\rho}\right)^2}\end{array}\right] > 0 \tag{2.209}$$

Thus, $\lim_{\rho\to-\infty}\frac{\partial r_t}{\partial n} < 0$, $\lim_{\rho\to-\infty}\frac{\partial r_t}{\partial\lambda} > 0$, and $\lim_{\rho\to-\infty}E_t\left[\frac{dP_t+D_t}{P_t}-r_t\right] > 0$. Since r_t, ψ_t , $F^{y,t,(i)}$, $\mu_t^{(\psi)}$, $\sigma_t^{(\psi)}$, $\mu_t^{(F^{y,(i)})}$, and $\sigma_t^{(F^{y,(i)})}$ are continuous in ρ (for $\rho<0$), there exists $\overline{\rho}(n_t,\lambda_t)$ such that $\rho<\overline{\rho}(n_t,\lambda_t)$ (or $EIS<\overline{EIS}(n_t,\lambda_t)\equiv\frac{1}{1-\overline{\rho}(n_t,\lambda_t)}$) ensures $\frac{\partial r_t}{\partial n}<0$, $\frac{\partial r_t}{\partial\lambda}>0$, and $E_t\left[\frac{dP_t+D_t}{P_t}-r_t\right]>0$. Note that $\overline{\rho}(n_t,\lambda_t)$ ($\overline{EIS}(n_t,\lambda_t)$) depends crucially on the current level of n_t and λ_t. ■

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Chapter 3

Uncertain Life Expectancy, Optimal Portfolio Choice and the Cross-Section of Asset Returns

Abstract

A model with stochastic changes in an agent's death rate and a dependence between asset prices and his life expectancy is solved. An agent demands more of an asset that pays off high (low) in states of the world when he expects to live longer (shorter) than an asset with the opposite payoff schedule. In equilibrium, an asset with a positive correlation between its returns and (unexpected) changes in the death rate pays a higher expected return than an (equivalent) asset with a negative correlation between its returns and changes in the death rate. Empirical evidence supports the model. A trading strategy is constructed which exploits the (theoretical) relationship between assets' expected returns and their correlations to changes in the death rate. Out-of-sample evidence suggests that the strategy pays a positive unexplained return according to traditional market models.

3.1 Introduction

The financial economics literature usually assumes that investors face a constant or deterministically changing probability of death. But empirical work by Lee and Carter (1992) and a growing body of demographic literature suggest that mortality rates are stochastically changing over time.

I explore the implications of stochastic changes in the life expectancy (or equivalently the death rate) on an investor's optimal portfolio choice and intertemporal consumption decision. In equilibrium, I analyze how the cross-sectional relationship between expected asset returns (yields) is affected by the behavior of investors facing stochastic changes in their life expectancies.

The literature consistently suggests that there is essentially no impact of lifetime uncertainty on an agent's optimal portfolio composition (Yaari (1965), Hakansson (1969), Merton (1971, 1973), Richard (1975)). However, all models do assume a constant probability distribution of death.

Given the agent's utility function is of the time additive form, stochastic changes in the life expectancy are not sufficient on its own to have any interesting impact on the optimal portfolio composition. Once the assumption is added that changes in the mortality rate are correlated with asset returns, it turns out that lifetime uncertainty affects an agent's optimal investment strategy and equilibrium asset pricing.

I introduce a continuous time finance model featuring a dependency between asset prices and agents' arrival rates of death. Agents maximize expected lifetime utility over intermediate consumption. Ceteris paribus, an agent invests more respectively less in an asset that pays off high (low) respectively low (high) in states of the world when he expects to live longer (shorter).

Under certain homogeneity conditions, I am able to state a "Three Fund Separation Theorem" in a similar spirit as introduced by Merton (1973). I derive an equilibrium asset pricing equation which states that the expected excess return of any asset depends in a linear fashion on the expected excess return of the market portfolio and the expected excess return of a fund, which features a positive correlation between its returns and changes in mortality rates. The derived asset pricing equation predicts that assets with a relatively strong positive correlation between their returns and changes in death rates earn a relatively high equity premium.

The theoretical results hold for any possible specifications of time additive utilities, given the restriction that utility at any time t is strictly increasing and concave in consumption at time t.

The intuition for my result is simple. Consider an asset that is likely to pay off high (low) in a state of the world where an agent expects to live longer (shorter). If an agent suddenly expects to live longer (state of high marginal utility of consumption), he would like to have more wealth to support his living standard. He is expected to buy more of the asset (than predicted in a standard CAPM) since its payoff is high (low) in states when the agent requires more (less) wealth and has a relatively high (low) marginal utility. In equilibrium it is expected that the asset will have a lower equity premium (than predicted in a standard CAPM) because of its hedging property.

The presence of a market for annuities (in the sense of Blanchard (1985)) does not alter my results.

Empirically, I test the asset pricing implications of my theoretical model. Consistent with my theory, empirical evidence suggests that assets with a relatively strong positive correlation between their returns and changes in the (aggregate) death rate outperform other assets on average. I construct a dynamic trading strategy which buys (sells) assets with a strong (weak) positive correlation to changes in the (aggregate) death rate. Out-of-sample evidence suggests that the constructed trading strategy earns a positive unexplained return according to traditional market models. I also find that a factor based on the mentioned trading strategy helps to explain the cross-sectional relationship in expected asset returns in addition to traditional factors.

The financial economics literature has (implicitly) assumed that there is no dependency between asset prices and an economic agent's time of death. I claim that this assumption is wrong.

A relation between asset returns and death rates is supported by empirical studies in the literature and my own empirical findings. Empirical research suggests that mortality depends on the level of development and availability of human capital, medical technology, technological conditions, and the economic environment (Mokyr, 1993; Schultz, 1993; Easterlin, 1999; Smith, 1999; Lichtenberg, 1998, 2002, 2003; Cutler et al., 2006; Soares, 2005, 2007), key quantities which are also linked to asset prices.

Life expectancy is naturally related to GDP growth (and asset prices) because growth in the population size and the workforce (which is an input factor in production and aggregate output) are affected by changes in death rates. Empirical studies by Acemoglu and Johnson (2007) and Hanwald (2010) confirm a link between changes in death rates and GDP.

Intuitively, it is reasonable to believe that an individual's life expectancy changes as technology advances, and it is natural to assume that asset prices depend on technological progress. Consider a firm in the health industry. If the company develops new pharmaceuticals, say a cure for cancer, an agent's life expectancy increases and the company is expecting high future earnings and its stock price increases. In contrast, if the new medicine turns out to be faulty, the firm's stock price and the life expectancy drop. This example induces a positive correlation between changes in the life expectancy and the company's asset returns. The impact for an individual who knows that he has cancer is much bigger than for someone who is healthy at the time but might possibly get cancer in future. The correlation between asset returns and changes in the life expectancy may differ considerably across agents.

There are many more examples for other industries. For instance, I also suspect firms in the insurance industry to be exposed to changes in the life expectancy as insurers offer many types of contracts related to the death of either an individual or events related to the aggregate death rate.

Another way how a dependency is induced is by (rare) events like a pandemic or a war. If suddenly a pandemic flu of a comparable size to the "Spanish Flu" in 1918 was circling the globe, the life expectancy would drop drastically as would most asset prices (since business activity would be severely constrained in such times). Within the last decade, three major influenzas were spreading: the severe acute respiratory syndrome (SARS), the avian flu, and the swine flu. Fortunately, none of them turned out to be as severe as the Spanish flu, and yet they affected the life expectancy and asset prices in local areas of outbreak (in particular Hong Kong and Mexico). For wars a similar argument applies as for the event of a pandemic.

Another channel is given through the institution of defined benefit pension plans. In the USA, on retirement employees oftentimes receive a pension from their former employer. A defined benefit plan promises an employee on retirement for the rest of his life a specific monthly payment depending on factors such as his time of employment, salary history, etc, but not on investment returns. The employee receives a deterministic regular payment on retirement while the employer bears all investment risk.[1] Accordingly, it is bad (good) news for the employer if there is an increases (decreases) in the life expectancy of retired employees because the employer is obliged to pay more (less) to fund defined benefit plans. Through this channel a negative correlation evolves.

According to the U.S. Bureau of Labor Statistics (2009) about 20% of private-industry workers in the USA have a defined benefit plan, and Coronado et al. (2008) find that about two-thirds of large companies in the USA offer defined benefit pension plans to employees. A substantial share of US stocks are affected through this channel and are expected to show a negative correlation between their asset returns and changes in the aggregate life expectancy of US citizens.

There is few and limited research on stochastically changing mortality rates in the theoretical financial economics literature (for instance Cox et al. (2006), Yogo (2009), Chen and Cox (2009), Cocco and Gomes (2009), DeNardi et al. (2009), Maurer (2011)), and none of the papers has explored the cross-sectional yield relationship between assets with a focus on stochastic shocks in death rates. Karatzas and Wang (2001) solve in a complete market environment a utility maximization problem subject to a random stopping time which is adapted to the filtration generated by asset prices.

To my research most closely related papers are by Martellini and Urosevic (2005) and Blanchet-Scalliet et al. (2008). Both papers consider a portfolio choice problem with the feature that the agent's exit time is uncertain and not independent of asset returns. In the special case of only one risky and one riskless asset and power utility it is shown that if the probability of exiting the market is positively correlated with the return on the risky asset, then the investment in the risky asset is higher than in a case of no uncertainty about exiting the market.

My contribution in addition to latter two papers is that my results hold for any

[1] For an extensive treatment on defined benefit plans I refer to Barnow and Ehrenberg (1979), Bodie, Marcus and Merton (1988), Harrison and Sharpe (1983), Bulow and Scholes (1983), Haberman (1997), Exley, Mehta and Smith (1997) and theirs references.

specification of time additive utilities and for an arbitrarily large universe of assets. Moreover, under fairly general conditions I can state a yield relationship between assets and show that assets which are positively correlated with changes in death rates are paying a higher expected return than assets which are negatively correlated with changes in death rates. I also provide empirical evidence emphasizing the quantitative relevance of my model.

The paper is organized as follows. Section 3.2 introduces the theoretical model, and it is solved in section 3.3. Section 3.4 states a Tree Fund Separation Theorem and discusses a yield relationship among assets in partial equilibrium. In section 3.5, I show that the introduction of annuity markets does not change the results. Section 3.6 illustrates the empirical importance of the model. Section 3.7 concludes.

3.2 The Model

Financial markets are modelled in partial equilibrium as introduced by Merton (1971, 1973). I choose the simplest case of a constant investment opportunity set. I omit a discussion of the specific assumptions needed and refer to Merton (1971, 1973) for details.

Let $(\Omega, \mathcal{F}_\infty, \mathbb{P})$ be a (complete) probability space endowed with a filtration $F = (\mathcal{F}_t)\,,\ t \in [0,\infty)$ - a right continuous, non-decreasing collection of (augmented) σ-algebras, $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}_\infty$, $0 \leq s \leq t \leq \infty$. Let there be a d-dimensional Wiener process $W = (W_t, \mathcal{F}_t)\,,\ t \in [0,\infty)$ on $(\Omega, \mathcal{F}_\infty, P)$[2].

My model assumes the existence of one risk free asset with a deterministic price process $P^{[0]} = \left(P_t^{[0]}, \mathcal{F}_t\right)\,,\ t \in [0,\infty)$ which increases with a constant rate of return r.[3] The dynamics of $P_t^{[0]}$ are

$$dP_t^{[0]} = rP_t^{[0]}dt \tag{3.1}$$

There are N risky assets characterized by an N-dimensional (random) price process $P^{[1N]} = \left(P_t^{[1N]}, \mathcal{F}_t\right)\,,\ t \in [0,\infty)$ which follows an N-dimensional geometric Brownian

[2] I employ the notation $W = (W_t, \mathcal{F}_t)\,,\ t \in [0,\infty)$ as found in Liptser and Shiryaev (2001) to denote a random process $W = (W_t)\,,\ t \in [0,\infty)$ to be adapted to the filtration F.

[3] If I write constant, it is in the meaning of non-stochastic, i.e. in the sense that a constant variable might vary over time but is deterministic, as in Merton (1973).

motion,

$$dP_t^{[1N]} = I_{P_t}\mu_P^{[1N]}dt + I_{P_t}\sigma_P^{[1N]}dW_t \tag{3.2}$$

where I_{P_t} is a $(N \times N)$ diagonal matrix with diagonal element i equal to the price of risky asset i $\left(P_t^{[i]}\right)$, $\mu_P^{[1N]}$ is a $(N \times 1)$ vector denoting the constant expected rate of returns, and $\sigma_P^{[1N]}$ is a $(N \times d)$ dimensional constant diffusion matrix. Let us write the $(N+1)$ price vector of all asset prices as $P = \begin{pmatrix} P^{[0]} & P^{[1N]^T} \end{pmatrix}^T$.

Next, I specify population growth. Let there be a one-dimensional, right continuous, doubly stochastic Poisson process $K = (K_t, \mathcal{F}_t)$, $t \in [0, \infty)$ with intensity κ_t and initial value K_0 on $(\Omega, \mathcal{F}_\infty, P)$. $K \mid \kappa$ is a non-homogeneous Poisson process with (time varying) intensity κ. $K \mid \kappa$ is independent of W. For a general treatment about doubly stochastic or Cox processes I refer to Cox (1955, 2001) and his references. I let the number of agents born up to time t be equal to K_t, and denote agent k's date of birth by $b^{(k)}$, where $b^{(k)}$ is equal to the time of the $(k - K_0)$th jump of K if $k > K_0$, and 0 otherwise. For the dynamics of the stochastic intensity parameter process $\kappa = (\kappa_t, \mathcal{F}_t)$, $t \in [0, \infty)$ I assume

$$d\kappa_t = \mu_{\kappa,t}dt + \kappa_t\sigma_{\kappa,t}dW_t \tag{3.3}$$

where $\mu_{\kappa,t}$ is a drift term which might depend on κ and t (and must be such that $\kappa > 0$ $a.s.$), $\sigma_{\kappa,t}$ is a $(1 \times d)$ dimensional diffusion vector depending on t, and κ_0 is the process' initial value at time 0. Notice that the specification of arrivals of new agents is general and flexible. It is easy to match real birth and population growth data by adjusting $\mu_{\kappa,t}$ and $\sigma_{\kappa,t}$.

To define uncertainty of death, I let $J = (J_t, \mathcal{F}_t)$, $t \in [0, \infty)$ be a K_∞-dimensional, right continuous, doubly stochastic Poisson process with intensity λ_t and $J_0 = 0$ on $(\Omega, \mathcal{F}_\infty, P)$. $J \mid \lambda$ is a non-homogeneous Poisson process and independent of both W and K. I define the time of agent k's death as the point in time when the first jump of $J^{(k)}$ (the kth element of J) since time $b^{(k)}$ occurs and denote it by $\tau_{b^{(k)}}^{(k)}$. For each k, the intensity parameter of process $J^{(k)}$, $\lambda^{(k)} = \left(\lambda_t^{(k)}, \mathcal{F}_t\right)$, $t \in [0, \infty)$ itself is an adapted random process and follows a geometric Brownian motion,

$$d\lambda_t^{(k)} = \left\{ \begin{array}{c} \lambda_t^{(k)} \mu_{\lambda, b^{(k)}, t}^{(k)} dt + \lambda_t^{(k)} \sigma_\lambda^{(k)} dW_t, \text{ if } t > b^{(k)} \\ \lambda_t^{(k)} \mu_{\lambda, t}^{(k)} dt + \lambda_t^{(k)} \sigma_\lambda^{(k)} dW_t, \text{ otherwise} \end{array} \right\} \tag{3.4}$$

where $\mu_{\lambda,t}^{(k)}$ is a drift term which depends on only t, $\mu_{\lambda,b^{(k)},t}^{(k)}$ is a drift term which depends on $b^{(k)}$ and t (i.e. on time of birth and age), and $\sigma_\lambda^{(k)}$ is a $(1 \times d)$ dimensional constant diffusion vector.

This specification ensures positivity of $\lambda^{(k)}$, the arrival rate of death of agent k. There is great flexibility in this general specification of the drift term. Oftentimes in financial economics literature, the arrival time of death is modelled by a Poisson process with a constant intensity parameter. Such a specification fits real mortality data badly. For instance, it implies that many people pass away in a young age and fewer old people. But, in the data we observe that young people have a low probability of dying while most of the people die at an age of about 80 years (see for example Edwards, 2003). This is no problem in my specification because the drift term is a function of an agent's age, and I can start off with a small λ (arrival rate of death) and let it grow as the agent gets older, which matches real mortality data well. Moreover, the process of λ at any time $t < b^{(k)}$ represents the evolution of the aggregate death rate for newborn agents. My specification also matches the well-known model by Lee and Carter (1992).

The Lee and Carter (1992) approach is arguably the most popular two factor stochastic mortality modeling method in (demographic) literature (Hanewald, 2010). Although there are more sophisticated models from the epidemiology literature (Booth et al., 2006), it is the simplicity and yet its power to forecast and model much of the variation in (age-dependent) death rates that makes the Lee and Carter (1992) model attractive to a variety of disciplines. In the asset pricing and household finance literature there are many papers which employ it (Cox et al. (2006), Chen and Cox (2009), Cocco and Gomes (2009), DeNardi et al. (2009), Maurer (2011), Hanewald and Post (2010)). Moreover, it has become a benchmark for the forecasts of the US Census bureau, and the US Social Security Technical Advisory Panel recently suggested Trustees to employ the method (Chen and Cox, 2009).

From the dynamics of the asset prices and the death rates it becomes evident that depending on how I choose the diffusion terms I can construct either a case with a (non-zero) correlation between changes in the death rates and asset returns or a case with death rates which are independent of asset prices. The birth rate may also be chosen to be dependent or independent of the other variables.

From the fact that $J^{(k)}$ given the realizations of $\lambda^{(k)}$ is a non-homogeneous Poisson process, it is trivial to derive the probability of dying before time t given the realizations of $\lambda^{(k)}$

$$\begin{aligned} H^{(k)}(t) &\equiv cdf_{\tau^{(k)}_{b^{(k)}}|\Lambda^{(k)}_t}(t) = \Pr\left\{\tau^{(k)}_{b^{(k)}} \in [b^{(k)}, t] \mid \Lambda^{(k)}_t\right\} \\ &= \Pr\left\{\overline{J}^{(k)}_t \geq 1 \mid \Lambda^{(k)}_t\right\} = 1 - \exp\left[-\int_{b^{(k)}}^{t} \lambda^{(k)}_s ds\right] \mid \Lambda^{(k)}_t \end{aligned} \tag{3.5}$$

and the probability of agent k surviving until time t

$$\overline{H}^{(k)}(t) \equiv 1 - H^{(k)}(t) \tag{3.6}$$

for $0 \leq b^{(k)} \leq t < s < \infty$, where $\Lambda^{(k)}_t = \left\{\lambda^{(k)}_s\right\}_{s\in[0,t]}$, $\overline{J}^{(k)}_t = J^{(k)}_t - J^{(k)}_{b^{(k)}}$. The *pdf* of $\tau^{(k)}_{b^{(k)}}$ given the realizations of $\lambda^{(k)}$ is

$$h^{(k)}(t) \equiv pdf_{\tau^{(k)}_{b^{(k)}}|\Lambda^{(k)}_t}(t) = \frac{\partial H^{(k)}(t)}{\partial t} = \lambda^{(k)}_t \exp\left[-\int_{b^{(k)}}^{t} \lambda^{(k)}_s ds\right] \mid \Lambda^{(k)}_t. \tag{3.7}$$

The conditional probability of surviving until time s given the agent has survived up to time t is

$$\overline{H}^{(k)}(t,s) \equiv \frac{\overline{H}^{(k)}(s)}{\overline{H}^{(k)}(t)} \tag{3.8}$$

and the conditional *pdf* of $\tau^{(k)}_{b^{(k)}}$ is

$$h^{(k)}(t,s) \equiv \frac{h^{(k)}(s)}{\overline{H}^{(k)}(t)} \tag{3.9}$$

Lemma 3.1 *Agent k's life expectancy $LE^{(k)}_t$ at time t is a function of his present arrival rate of death $\lambda^{(k)}_t$ (and further deterministic parameters), and is inversely related to it, i.e.* $\frac{\partial LE^{(k)}_t}{\partial \lambda^{(k)}_t} < 0$.

Proof. See Appendix. ■

Intuitively, if the arrival rate of death increases, then (for any $s > t$) the probability of still being alive at some future point in time s decreases, and life expectancy decreases. Therefore, there is a negative relationship between an agent k's life expectancy $LE^{(k)}$ and his arrival rate of death $\lambda^{(k)}$.

At every point in time t there are $K_t - \sum_{k=1}^{K_t} \mathbf{1}_{\left\{\tau^{(k)}_{b^{(k)}} < t\right\}}$ agents in the market, where $\mathbf{1}_{\left\{\tau^{(k)}_{b^{(k)}} < t\right\}}$ is an indicator function. Agent k is born at time $b^{(k)}$ and dies at time $\tau^{(k)}_{b^{(k)}}$. His preferences are described by a time additive expected utility function over consumption $c^{(k)}$. Following the specification first introduced by Yaari (1965), expected lifetime utility of agent k at time $t \in [b^{(k)}, \tau^{(k)}_{b^{(k)}})$ is given by

$$E_t\left[U^{(k)}(c^{(k)}) \mid \tau^{(k)}_{b^{(k)}} > t\right] = E_t\left[\int_t^{\infty} \overline{H}^{(k)}(t,s) u^{(k)}_s\left(c^{(k)}_s\right) + h^{(k)}(t,s) B^{(k)}\left(\overline{W}^{(k)}, s\right) ds\right] \tag{3.10}$$

where the functions $u^{(k)}_s(x)$ and $B^{(k)}(x,s)$ are strictly increasing and concave in x, and there is some time discounting. $u^{(k)}_s(x)$ is the stream of utility obtained from current consumption at time s, and $B^{(k)}(x,s)$ is the utility from bequest given the agent dies at time s.

I restrict my analysis to the class of time additive utility functions and do not discuss stochastic differential utilities as introduced in Duffie and Epstein (1992a, 1992b). Latter specifications, which include utilities of the Kreps and Porteus (1978) type or functions incorporating habit formation, may be able to generate additional interesting asset pricing implications of lifetime uncertainty and stochastically changing mortality rates (Maurer (2011)). In addition, Bommier (2003) and Maurer (2011) point out that the formulation of time additive utility functions implies agents to be risk neutral towards lifetime uncertainty. Nevertheless, I do not want to worry about this in my analysis. Focusing on time additive utilities ensures that the model is analytically tractable and I can easily relate my innovations to the previous literature.

At time of birth, agent k's initial wealth is given by $\overline{W}^{(k)}_{b^{(k)}}$. Agent k allocates his wealth at every point in time t to consumption $c^{(k)}_t$, a risky asset portfolio $X^{[1N](k)}_t$ (measured in numbers of shares) or alternatively $\phi^{[1N](k)}_t = \frac{1}{\overline{W}^{(k)}_t} I_{P_t} X^{[1N](k)}_t$ (relative portfolio holdings), and the risk free asset. His wealth is a random process and evolves according to the dynamics

$$d\overline{W}_t^{(k)} = \overline{W}_t^{(k)} r dt + \overline{W}_t^{(k)} \phi_t^{[1N](k)^T} \left(\mu_P^{[1N]} dt + \sigma_P^{[1N]} dW_t - r 1_{(1\times N)} dt\right) - c_t^{(k)} dt \quad (3.11)$$

To prohibit arbitrage opportunities I make the technical assumption of $\phi_t^{[1N](k)} \in \left(\mathcal{L}^2\right)^N$, where $\mathcal{L}^2 = \left\{x \in \mathcal{L} \mid \int_0^T x_t^2 dt < \infty \quad a.s.\right\}$ and $\mathcal{L}$ is the set of processes adapted to the filtration $\mathcal{F}^P$ generated by asset prices, $\mathcal{F}_t^P = \sigma\{P_s : s \leq t\}$. I set the further restriction that $\overline{W}_t^{(k)}$ at $t = \tau_{b^{(k)}}^{(k)}$ has to be non-negative almost surely.

Problem 3.1 *Agent k tackles the following expected lifetime utility maximization problem (given $\tau_{b^{(k)}}^{(k)} > t$):*

$$\begin{aligned}
V^{(k)} \quad & \left(\overline{W}^{(k)}, \lambda^{(k)}, \overline{J}^{(k)}, t\right) = \sup_{\left\{c^{(k)}, \phi^{[1N](k)}\right\} \in \Re_+^1 \times (\mathcal{L}^2)^N} \left\{E_t\left[U^{(k)}(c^{(k)}) \mid \tau_{b^{(k)}}^{(k)} > t\right]\right\} \\
& s.t. \\
& d\overline{W}_t^{(k)} = \overline{W}_t^{(k)} r dt + \overline{W}_t^{(k)} \phi_t^{[1N](k)^T} \left(\mu_P^{[1N]} - r 1_{(1\times N)}\right) dt \\
& \qquad\qquad + \overline{W}_t^{(k)} \phi_t^{[1N](k)^T} \sigma_P^{[1N]} dW_t - c_t^{(k)} dt \\
& d\lambda_t^{(k)} = \lambda_t^{(k)} \mu_{\lambda, b^{(k)}, t}^{(k)} dt + \lambda_t^{(k)} \sigma_\lambda^{(k)} dW_t \\
& \overline{W}_{\tau_{b^{(k)}}^{(k)}}^{(k)} \geq 0
\end{aligned} \qquad (P1)$$

where $V^{(k)}\left(\overline{W}^{(k)}, \lambda^{(k)}, \overline{J}^{(k)}, t\right)$ is the value function.

$V^{(k)}\left(\overline{W}^{(k)}, \lambda^{(k)}, \overline{J}^{(k)}, t\right)$ depends on the agent's wealth, his arrival rate of death, time, and on whether he is still alive $\overline{J}^{(k)} = 0$ or dead $\overline{J}^{(k)} \neq 0$. The dynamics of κ and $\lambda^{(l)}$, for $l \neq k$, do not matter.

3.3 Solution to the Model

I employ dynamic programming to solve the agent's maximization problem.

Lemma 3.2 *The optimal investment strategy* $\phi_t^{*[1N](k)}$ *of agent* k *is given by*

$$\phi_t^{*[1N](k)} = \frac{1}{-V_{\overline{WW}}^{(k)}\overline{W}_t^{(k)}}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left[V_{\overline{W}}^{(k)}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) + V_{\overline{W}\lambda}^{(k)}\sigma_P^{[1N]}\sigma_\lambda^{(k)^T}\lambda_t^{(k)}\right] \tag{3.12}$$

The optimal investment strategy is affected by stochastic changes in the agent k's instantaneous probability of death, but independent of changes in the birth rate or changes in other agents' death rates.

Proof. See Appendix. ∎

Notice that the optimal investment strategy is not affected if labor income is introduced, as long as it is independent of asset prices.

It becomes evident that a correlation between asset returns and changes in the agent's arrival rate of death affects the optimal investment strategy $\phi_t^{*[1N](k)}$. To examine the effect in more detail, it is crucial to explorer the properties of the value function.

Lemma 3.3 *Assume agent* k *does not derive any utility from bequest,* $B^{(k)}\left(\overline{W}^{(k)}, s\right) = 0$*. For any specification of the utility function satisfying the general conditions described in the model, the value function* $V^{(k)}\left(\overline{W}^{(k)}, \lambda^{(k)}, \overline{J}^{(k)}, t\right)$ *defined on* $t \in [b^{(k)}, \tau_{b^{(k)}}^{(k)})$*, has the following properties:*

1) the value function is strictly increasing in wealth, $V_{\overline{W}}^{(k)} > 0$*,*

2) the value function is strictly concave, $V_{\overline{WW}}^{(k)} < 0$*, and*

3) the value function satisfies $V_{\overline{W}\lambda}^{(k)} < 0$*.*

Proof. See Appendix. ∎

The first two properties - the value function is strictly increasing and concave in wealth - are standard.

The intuition for the third property is straightforward. Consider a decrease (increase) in the instantaneous probability of death. The agent faces a longer (shorter) expected lifetime horizon, and he expects that his wealth has to be allocated to more (less) periods of time than before the drop (increase) in the death rate. The stream of consumption in each period decreases (increases) and marginal utility increases (decreases) because the utility function is concave in consumption. Accordingly, an additional dollar is more (less) valuable, and $V_{\overline{W}\lambda}^{(k)} < 0$ follows.

Alternatively, when the value function is defined only on the positive space, the result can be understood as follows. Ceteris paribus, an increase in the mortality rate leads to a lower expected utility for any feasible consumption plan. It follows that the optimal consumption plan $c^*(\lambda_l)$ given a low death rate λ_l yields a higher expected utility then the optimal consumption $c^*(\lambda_h)$ given the probability of death is high (λ_h), $E_t\left[U^{(k)}(c^*(\lambda_l)) \mid \lambda_l\right] > E_t\left[U^{(k)}(c^*(\lambda_h)) \mid \lambda_h\right]$. If this was not the case, then given λ_l plan $c^*(\lambda_l)$ would be dominated by $c^*(\lambda_h)$ because $c^*(\lambda_h)$ yields a higher expected utility under λ_l than under λ_h, $E_t\left[U^{(k)}(c^*(\lambda_h)) \mid \lambda_l\right] > E_t\left[U^{(k)}(c^*(\lambda_h)) \mid \lambda_h\right] > E_t\left[U^{(k)}(c^*(\lambda_l)) \mid \lambda_l\right]$. This is a contradiction to $c^*(\lambda_l)$ being optimal given λ_l. Given $V_\lambda^{(k)} < 0$, one expects that one dollar is used 'more efficiently' if λ is low, and $V_{\overline{W}\lambda}^{(k)} < 0$ holds.

In the formal proof the result follows from the concavity of the utility function and the fact that optimal consumption is increasing in the mortality rate. There is a simple intuition for the claim that optimal consumption is increasing in the mortality rate. If the probability of dying goes up at any time in future, the agent rather eats up more of his wealth today than having the chance to die with a lot of wealth from which he cannot derive any utility (no utility from bequest).

From the last argument it becomes evident why the assumption of no bequest utility is important. If the agent derived utility from bequest, equation (3.45) (see Appendix) had to be adjusted by the term $\int_t^\infty h^{(k)}(t,s)\,\frac{\partial B^{(k)}\left(\overline{W}^{(k)},s\right)}{\partial \overline{W}_s^{(k)}}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}ds$. By equation (3.38) (see Appendix) $h^{(k)}(t,s)$ is not always decreasing in $\lambda_t^{(k)}$, and for s small enough it is increasing. This causes an ambiguity about the sign of $\frac{\partial c_t^{*(k)}}{\partial \lambda_t^{(k)}}$. If (for s small enough) marginal utility from bequest is large (small) enough, then $\frac{\partial c_t^{*(k)}}{\partial \lambda_t^{(k)}} < (>) 0$ holds. Intuitively, an agent prefers to take the chance to die with a lot of wealth and does not want to consume too much today if the utility from bequest is large enough. The sign of $V_{\overline{W}\lambda}^{(k)}$ depends crucially on the size of bequest utility relative to consumption utility.

The question arises whether it is reasonable to assume a bequest utility that dominates consumption utility enough such that $\frac{\partial c_t^{*(k)}}{\partial \lambda_t^{(k)}} < 0$ holds. Indeed, such a constellation may imply a suicidal agent, in the sense that he desires to face a high probability of dying, if his bequest utility dominates his consumption utility by too much.

Having established the properties of the value function I can explorer the impact of a dependency between asset prices and the arrival rate of death on optimal portfolio choice.

Proposition 3.1 *The optimal investment strategy of agent k is given by*

$$\phi_t^{*[1N](k)} = \left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left[A_t^{(k)}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) - B_t^{(k)}\sigma_P^{[1N]}\sigma_\lambda^{(k)^T}\right] \tag{3.13}$$

for some positive parameters $A_t^{(k)}$ and $B_t^{(k)}$. The optimal portfolio composition consists of a standard myopic and a hedging demand. Ceteris paribus, agent k's demand is higher (lower) for assets with returns that are stronger negatively (positively) correlated with changes in his arrival rate of death and positively (negatively) correlated with changes in his life expectancy. In general, $\sigma_P^{[1N]}\sigma_\lambda^{(k)^T} \neq \sigma_P^{[1N]}\sigma_\lambda^{(l)^T}$, $\forall k \neq l$ is true, which implies that every agent chooses a uniquely tailored optimal asset allocation depending on his personal health condition and lifestyle.

Proof. See Appendix. ■

In the general case of many risky assets, I cannot unambiguously tell whether the hedging demand is positive or negative (which crucially depends on the structure of the covariance matrix). In the case of only one risky asset, the hedging demand is positive (negative) if the asset's returns are negatively (positively) correlated with changes in agent k's arrival rate of death. One might speculate that this should lead to a lower (higher) equity premium then in a model without lifetime uncertainty if the market was negatively (positively) correlated with changes in the aggregate death rate. Indeed, empirical evidence show both a 'too high' equity premium in stock markets and an overall positive relation between changes in the aggregate death rate and stock returns (see below). I elaborate more on asset pricing implications of lifetime uncertainty later.

The last statement of the proposition says that agents' exposures to diverse risks of death differ depending on many personal characteristics. I illustrate this by two examples. Consider agent A who is HIV (human immunodeficiency virus) positive and the healthy agent B. A discovery of a cure for AIDS means a huge decrease (increase) in agent A's arrival rate of death (life expectancy) while the decrease (increase) in agent

B's arrival rate of death (life expectancy) is going to be moderate. The dependency between asset returns of a company that is likely to ever produce pharmaceuticals related to HIV and changes in the arrival rate of death for agent A is stronger than for agent B, implying differences in their optimal portfolio choice decisions.

In the same spirit, consider agent A who lives in a global city and agent B who lives in the countryside. A global pandemic outbreak threatens agent A much more than agent B, and changes in agent A's life expectancy are more likely to be stronger positively correlated with a large set of asset returns than changes in agent B's life expectancy, which implies a difference in their optimal investment strategies.

My result implies that the effect on optimal portfolio choice is exactly the opposite as found in the special case of CRRA utility by Blanchet-Scalliet et al. (2008). The difference arises because in Blanchet-Scalliet et al. (2008) agents maximize utility over terminal wealth (or in some sense there is only utility form bequest), rather than time additive expected utility over a stream of intermediate consumption. My analysis confirms the finding of Blanchet-Scalliet et al. (2008) and even generalizes it to a more general class of time additive utility functions (not only for CRRA utility $u(x) = x^{1-\gamma}/(1-\gamma)$, with $\gamma > 1$)[4]. If agent k is maximizing expected utility over terminal wealth (bequest), rather than expected lifetime utility, then from the (adjusted) HJB equation I get the same FOC with respect to $\phi_t^{[1N](k)}$. $V_{\overline{W}}^{(k)} > 0$ and $V_{\overline{WW}}^{(k)} < 0$ still holds, which can be seen when taking the first and second derivatives with respect to $\overline{W}_t^{(k)}$ from the value function and noticing that $B^{(k)}(x,s)$ is a strictly increasing and concave function in x,

$$V_{\overline{W}}^{(k)} = E_t\left[\int_t^{\infty} h^{(k)}(t,s)\frac{\partial B^{(k)}\left(\overline{W}^{(k)},s\right)}{\partial \overline{W}_s^{(k)}}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}ds\right] > 0 \tag{3.14}$$

$$V_{\overline{WW}}^{(k)} = E_t\left[\int_t^{\infty} h^{(k)}(t,s)\frac{\partial^2 B^{(k)}\left(\overline{W}^{(k)},s\right)}{\partial \overline{W}_s^{(k)}\partial \overline{W}_s^{(k)}}\left(\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}\right)^2 ds\right] < 0. \tag{3.15}$$

[4]Notice, however, that I do not get a closed form solution, while Blanchet-Scalliet et al. (2008) were able to get in their special case a solution in closed form.

In contrast to Lemma 3.3, now $V^{(k)}_{\overline{W}\lambda} > 0$ holds, which follows directly from equation (3.38) (see Appendix) in combination with

$$V^{(k)}_{\overline{W}\lambda} = E_t \left[\int_t^\infty \frac{\partial h^{(k)}(t,s)}{\partial \lambda_t^{(k)}} \frac{\partial B^{(k)}\left(\overline{W}^{(k)}, s\right)}{\partial \overline{W}_s^{(k)}} \frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}} ds \right] > 0 \tag{3.16}$$

and the assumption of a large enough risk aversion coefficient such that

$$E_t \left[\frac{\partial B^{(k)}\left(\overline{W}^{(k)}, r\right)}{\partial \overline{W}_r^{(k)}} \frac{\partial \overline{W}_r^{(k)}}{\partial \overline{W}_t^{(k)}} \right] > E_t \left[\frac{\partial B^{(k)}\left(\overline{W}^{(k)}, s\right)}{\partial \overline{W}_s^{(k)}} \frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}} \right], \forall r < s \tag{3.17}$$

which is similar to the condition of $\gamma > 1$ in the paper by Blanchet-Scalliet et al. (2008). In the case of only one risky asset, the hedging demand is positive (negative) if the asset's returns are positively (negatively) correlated with changes in agent k's arrival rate of death and negatively (positively) correlated with changes in agent k's life expectancy, which is analogous to the results from Blanchet-Scalliet et al. (2008).

3.4 Special Case of some Homogeneity

In general agents choose individualized investment strategies depending on their specific health conditions and lifestyles. Now, I make the simplifying assumption of some homogeneity among agents in form of $\sigma_P^{[1N]} \sigma_\lambda^{(k)^T} = \sigma_P^{[1N]} \sigma_\lambda^T$, $\forall k$. This does not imply that every agent's arrival rate of death is the same. It only says that all unexpected (percentage) changes in arrival rates of death which are adapted to the filtration generated by asset prices are equal among agents. $\lambda^{(k)}$ and $\lambda^{(l)}$ (for $k \neq l$) may still load differently on some Brownian motions which are not driving asset prices, i.e. $\sigma_\lambda^{(k)} \neq \sigma_\lambda^{(l)}$ and yet $\sigma_P^{[1N]} \sigma_\lambda^{(k)^T} = \sigma_P^{[1N]} \sigma_\lambda^{(l)^T}$, $\forall k \neq l$ holds.

Agent k's absolute demand for risky assets is

$$\begin{aligned} d_t^{(k)} &\equiv \overline{W}_t^{(k)} \phi_t^{*[1N](k)} \\ &= \left(\sigma_P^{[1N]} \sigma_P^{[1N]^T} \right)^{-1} \left[\widetilde{A}_t^{(k)} \left(\mu_P^{[1N]} - r 1_{(1\times N)} \right) - \widetilde{B}_t^{(k)} \sigma_P^{[1N]} \sigma_\lambda^T \right] \end{aligned} \tag{3.18}$$

where $\widetilde{A}_t^{(k)} = A_t^{(k)} \overline{W}_t^{(k)}$ and $\widetilde{B}_t^{(k)} = B_t^{(k)} \overline{W}_t^{(k)}$. The demand differs among agents only in terms of $\widetilde{A}_t^{(k)}$ and $\widetilde{B}_t^{(k)}$. It is possible to derive a Three Fund Separation Theorem in the same spirit as introduced by Merton (1973).

Theorem 3.1 ("Three Fund Separation") *Given the structure of the model described above and further assuming $\sigma_P^{[1N]}\sigma_\lambda^{(k)^T} = \sigma_P^{[1N]}\sigma_\lambda^T$, $\forall k$, then there exist three funds, such that (i) all K_t economic agents described above will be indifferent between allocating their wealth to the original (N+1) assets or to the three funds; (ii) the construction of the funds is based on an asset allocation depending solely on "technological" criteria, but not on any characteristics of an individual agent (only dependent on variables in the investment opportunity set and the covariances of asset returns to common changes in arrival rates of death); and (iii) an agent does neither require knowledge of the investment opportunity set for individual assets nor of the composition of the three funds.*

Proof. See Appendix. ■

For the following (partial) equilibrium analysis, I assume that at death of agent k, at $\tau_{b(k)}^{(k)}$, his entire portfolio is liquidated. The remaining wealth (which by assumption is non-negative almost surely) must either be stored (invested in the risk free asset) and distributed to newborn agents, or buried with the dead agent. It does not matter for the analysis how I handle the remaining wealth as long as I do not distribute it to agents who were born before $\tau_{b(k)}^{(k)}$ because otherwise these agents' maximization problems must depend on the probability of other agents passing away and the chance of receiving part of their legacies[5].

Having discussed the demand functions for every agent k, I derive aggregate demand which, not surprisingly, is a linear combination of the three funds derived in the Three Fund Separation Theorem,

$$\begin{aligned} D_t^{[1N]} &= \sum_{k=1}^{K_t}\left[d_t^{(k)}\mathbf{1}_{\left\{\tau_{b(k)}^{(k)}<t\right\}}\right] \qquad (3.19)\\ &= \left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left[A_t\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) - B_t\sigma_P^{[1N]}\sigma_\lambda^T\right] \end{aligned}$$

[5] To illustrate this point, think of the following example. Intuitively, an agent would have much incentive to invest in a company which is doing a lot of research to find a cure for cancer, if the agent's wealthy father is suffering from cancer. This is because the payoff of receiving the father's legacy and asset returns of the firm are negatively related.

where $A_t = \sum_{k=1}^{K_t} \left[\widetilde{A}_t^{(k)} \mathbf{1}_{\left\{ \tau_{b^{(k)}}^{(k)} < t \right\}} \right]$ and $B_t = \sum_{k=1}^{K_t} \left[\widetilde{B}_t^{(k)} \mathbf{1}_{\left\{ \tau_{b^{(k)}}^{(k)} < t \right\}} \right]$.

I consider an exogenous supply of assets as in Merton (1973) and refer to his discussion about modelling details. Aggregate supply in numbers of risky assets is denoted by $X_t^{(M)}$, and the total value of risky assets supplied is $M_t = X_t^{(M)^T} P_t^{[1N]}$. The market portfolio denoted by $\phi_t^{[1N](M)}$ is

$$\phi_t^{[1N](M)} = \frac{1}{M_t} I_{P_t} X_t^{(M)} \tag{3.20}$$

I can now state the implications of uncertain life expectancy on the yield relationship among assets in partial equilibrium.

Theorem 3.2 *Given the structure of the model described above and assuming* $\sigma_P^{[1N]} \sigma_\lambda^{(k)^T} = \sigma_P^{[1N]} \sigma_\lambda^T$, $\forall k$, *then in a financial markets equilibrium every asset* i *satisfies an expected (excess) return relationship given by*

$$\begin{aligned} \mu_i - r = \quad & \left(\mu_M - r\right) \beta_i^{(M)} + \left(\mu_\delta - r\right) \beta_i^{(\delta)} \\ & \textit{with} \\ & \beta_i^{(M)} = \frac{\sigma_i}{\sigma_M} \frac{\rho_{i,M} - \rho_{M,\delta}\rho_{i,\delta}}{1 - \left(\rho_{M,\delta}\right)^2} \\ & \beta_i^{(\delta)} = \frac{\sigma_i}{\sigma_\delta} \frac{\rho_{i,\delta} - \rho_{M,\delta}\rho_{i,M}}{1 - \left(\rho_{M,\delta}\right)^2} \end{aligned} \tag{3.21}$$

where μ_j *is the expected return of asset* j, σ_j *is the standard deviation of asset* j, $\rho_{j,p}$ *is the correlation between returns of asset* j *and* p, $\forall \left(j,p\right) \in \left[i, M, \delta\right] \times \left[i, M, \delta\right]$, *and* δ *is the portfolio strategy* $\delta^{[1N]} = \left(\sigma_P^{[1N]} \sigma_P^{[1N]^T}\right)^{-1} \sigma_P^{[1N]} \sigma_\lambda^T$, $\delta^{[0]} = 1 - \left(1_{(1 \times N)}\right)^T \delta^{[1N]}$. *Assuming* $\mu_\delta > r$, *an asset with a positive (negative) correlation between its returns and common unexpected changes in arrival rates of death, pays a higher (lower) equity premium, than what is expected in a model without uncertain life expectancy.*

Proof. See Appendix. ■

The introduced equilibrium yield relationship is an extension of the ICAPM. The second statement in the proposition confirms the intuition given at the beginning of the paper.

The asset pricing equation offers a suitable groundwork for performing empirical tests. $\beta_i^{(M)}$ and $\beta_i^{(\delta)}$ are regression coefficients of running a linear regression of asset i's excess returns on excess returns of the market portfolio and an uncertain life expectancy hedging portfolio which is positively related to changes in mortality rates. $(\mu_M - r)$ and $(\mu_\delta - r)$ are risk premia for the two factors represented by the market portfolio $\phi_t^{(M)}$ and the uncertain life expectancy hedging portfolio δ, on which asset i loads with $\beta_i^{(M)}$ and $\beta_i^{(\delta)}$, respectively.

3.5 Market for Annuities

Within the above stated model I was able to show that uncertainty in the life expectancy has an impact on an agent's intertemporal consumption choice, optimal portfolio composition and on capital asset pricing. Although, I have considered a rather general universe of assets, I did exclude the possibility to trade annuities. The payoff of an annuity depends on the specific buyer's mortality rate and date of death, but is not an available asset for other agents. This characteristic is not incorporated in the asset universe considered. In a discussion about lifetime uncertainty the question naturally arises in how far the analysis alters in presence of a market for annuities and whether the results continue to hold.

I consider annuities of a similar type as in Blanchard (1985). Let there be a large competitive insurance company that offers at any time t individualized contracts to each agent k with claims depending on the instantaneous survival respectively death of the agent. A contract bought by agent k at time t has a payoff as follows: conditional on survival over the next time period dt agent k receives an amount of money equal to his instantaneous probability of death $\lambda_t^{(k)} dt$, and in case of death the insurance company collects 1 from the agent. Because agents do not derive any utility from bequest in my model and utility is strictly increasing in consumption, agents desire to buy as many annuities as possible subject to the non-negativity constraint of wealth at time of death. At any time t it is optimal for agent k to buy exactly $\overline{W}_t^{(k)}$ contracts such that the insurer collects the agent's entire wealth in case of death and the agent maximizes the stream of payments received from the insurer as long as he stays alive.

In expectation the insurance company breaks even because she gets $\overline{W}_t^{(k)}$ from every agent k with probability $\lambda_t^{(k)}dt$ and has to pay $\overline{W}_t^{(k)}\lambda_t^{(k)}dt$ to every agent k with probability $\left(1-\lambda_t^{(k)}dt\right)$; expected earnings and liabilities equal. In Blanchard (1985) the insurance company breaks even almost surely because there is an infinite number of agents who face identical mortality rates and wealth is equally distributed within a cohort. In my model there is a finite number of agents who have individualized mortality rates, and the break even condition holds only in expectation. However, by the law of large numbers in the limit as the number of agents in the market approaches infinity (and assuming total wealth in the economy is not concentrated to only few agents) the profit of the insurance company converges (in probability) to zero.

Given a market for annuities, the wealth process of agent k has to be adjusted by the additional income stream from the annuity ($\overline{W}_t^{(k)}\lambda_t^{(k)}dt$), and becomes

$$d\overline{W}_t^{(k)} = \overline{W}_t^{(k)}rdt+\overline{W}_t^{(k)}\lambda_t^{(k)}dt+\overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\left(\mu_P^{[1N]}dt+\sigma_P^{[1N]}dW_t-r1_{(1\times N)}dt\right)-c_t^{(k)}dt. \tag{3.22}$$

The maximization problem does not essentially change except for the new wealth dynamics. The value function $V^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},\overline{J}^{(k)},t\right)$ is again a function of the agent's wealth, his mortality rate, a state variable determining whether he is still alive, and time. The impact of the death rate on the value function differs compared to the earlier discussion without annuities. The death rate is not only an important state variable because it determines the life expectancy and the expected lifetime horizon of the agent, but also because it matters for the dynamics of the wealth process. This becomes evident when writing down the HJB equation. The HJB equation looks similar to the one in the case without annuities expect that the Itô-Doeblin formula has to be adjusted by the new drift term of the wealth process, i.e. the term $V_{\overline{W}}^{(k)}\overline{W}_t^{(k)}\lambda_t^{(k)}$ has to be added. The adjustment of the HJB equation does effectively change the value function and particularly the dependency of the value function on $\lambda_t^{(k)}$.

The first order conditions with respect to the two controls $c_t^{(k)}$ and $\phi_t^{[1N](k)}$ are up to the change in the value function identical to the first order conditions derived in absence of an annuity market. The proofs of Lemma 3.2 and 3.3 continue to hold. Even though there is a new value function, the earlier derived qualitative properties still hold. $V^{(k)}$ is strictly increasing and concave in $\overline{W}^{(k)}$ and $V^{(k)}_{\overline{W}\lambda} < 0$. Given the FOC is essentially unchanged and Lemma 3.2 and 3.3 continue to hold, all further derived results about the impact of uncertain life expectancy on optimal portfolio choice and capital asset pricing remain valid.

Accordingly, a market for annuities does not essentially matter for the analysis and the achieved results. Even though the qualitative results are the same, the quantitative impact may change. The quantitative impact depends crucially on (the magnitude of) the value function and its dependency on $\lambda^{(k)}$, which is essential to determine the magnitude of the hedging demand term in the specification of the optimal investment strategy.

To get an intuition why annuity markets do not change my (qualitative) results, I first observe that the expected lifetime income paid by an annuity is equal to 1 independent of time and the level of the death rate

$$E_t\left[\int_t^{\infty} \lambda_s^{(k)} e^{-\int_t^s \lambda_u^{(k)} du} ds\right] = E_t\left[\left[-e^{-\int_t^s \lambda_u^{(k)} du}\right]_t^{\infty}\right] = 1. \qquad (3.23)$$

It follows that the present value of an annuity is negatively (positively) related to the death rate (life expectancy). An annuity bought by agent k pays a low (high) stream of income if k's mortality rate is low (high) and his life expectancy and marginal utility both are high (low). Accordingly, annuities exacerbate the uncertain life expectancy problem. Assets with a negative correlation between returns and changes in the death rate are expected to be even more desirable for their hedging property than in a world without an annuity market. In equilibrium the hedging property should be priced with an even higher premium than in a world without annuities.

3.6 Empirical Evidence

3.6.1 Data

I try to quantify and test the implications of the proposed model using data from the USA ranging from 1927 to 2005.[6] Mortality data is provided by the National Center

[6] Data on mortality is available since 1900.

for Health Statistics (NCHS). I consider the 25 Fama-French portfolios (formed on size and book-to-market characteristics) as my universe of assets (Fama and French, 1993). As a robustness check, I repeat all empirical tests using the 100 Fama-French portfolios (formed on size and book-to-market); the results are weaker in the sense that the magnitude of the effects is smaller, but the same conclusions follow.

Unfortunately, the estimation of the model is difficult. An agent's instantaneous probability of death is not observable. The best I can do is to use a noisy estimate based on death counts and population size data. Second, the NCHS only provides death rate data on an annual frequency. Accordingly, my sample size in terms of the number of data points is small and it is hard to accurately estimate correlations between asset returns and changes in death rates. I also expect the true correlation to be moderate which makes it difficult to recognize it at all (because the estimates are noisy). Finally, I do not expect that the correlation between an asset's returns and changes in death rates is constant over time, but the only way to get an estimation is by assuming a constant relation.

Following Theorem 3.2, I am interested in the correlation between asset returns (in excess of the risk free rate) and common changes in death rates across all agents. Indeed, Lee and Carter (1992) suggest that the stochastic time variation in age specific death rates is mostly driven by one across cohorts common stochastic time component which is well described by a Brownian diffusion process. This result is similar to my theoretical homogeneity condition $\sigma_P^{[1N]}\sigma_\lambda^{(k)^T} = \sigma_P^{[1N]}\sigma_\lambda^T$, $\forall k$, which is needed in the model, and is vital for my analysis. Following Lee and Carter (1992), I estimate the 'aggregate death rate' (the common stochastic time component in age-dependent death rates) λ_t as follows. Let $m_{x,t}$ denote the mortality rate of cohort x at time t,[7] a_x and b_x two time-invariant constants which differ across age x, λ_t a stochastic time component (which is common to all cohorts), and $u_{x,t}$ a white noise term for cohort x and time t. The Lee and Carter (1992) model states the relationship

$$\ln(m_{x,t}) = a_x + b_x \ln(\lambda_t) + u_{x,t} \tag{3.24}$$

[7] I estimate $m_{x,t}$ from the data as the ratio between the number of agents aged x passing away in year $[t, t+1)$ and the number of agents aged x in year t.

with $\sum_x a_x = 0$ and $\sum_x b_x = \sum_x 1$ (so that $\sum_t \ln(\lambda_t) = \sum_t \frac{1}{\sum_x 1} \sum_x \ln(m_{x,t})$). For estimation details I refer to Lee and Carter (1992).

Estimation: Age Factors a_x and b_x

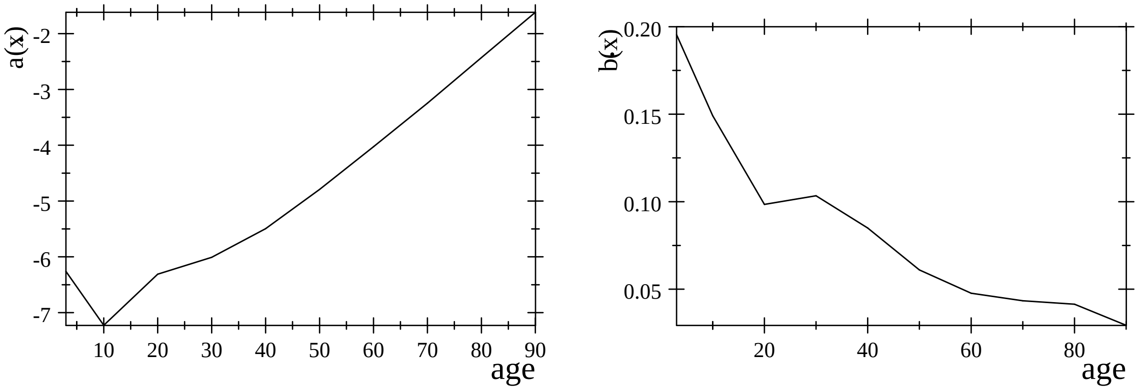

Figure 3.1: Left panel: Estimation of age factor a_x. Right panel: Estimation of age factor b_x.

Estimation: Stochastic Time Component $\lambda(t)$ and $\ln\left(\frac{\lambda(t)}{\lambda(t-1)}\right)$

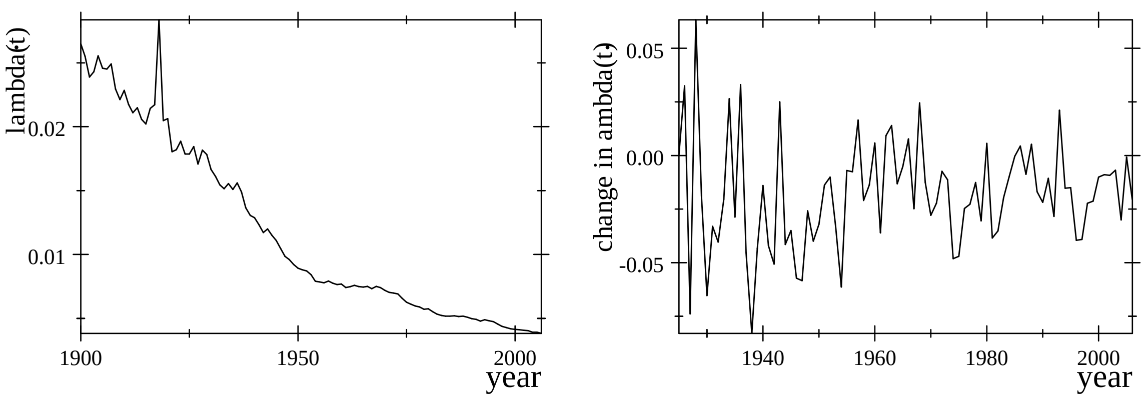

Figure 3.2: Left panel: Estimation of stochastic time component λ_t. Right panel: Percentage changes in λ_t.

Figure 3.1 and 3.2 display the estimation output for the age specific factors a_x and b_x and the stochastic time component λ_t. Not surprisingly, a_x is an increasing function in age. More interestingly, b_x is found to be declining in age; old people are relatively less exposed to the stochastic time component than young people.[8] For illustrative

[8] This might lead to interesting portfolio choice differences between young and old investors; I do not discuss this issue here.

purposes, I exclude the Spanish flu in the late 1910's from the plot of $\ln\left(\frac{\lambda(t)}{\lambda(t-1)}\right)$. Excluding the event of the Spanish flu from the sample, the time series $\ln\left(\frac{\lambda(t)}{\lambda(t-1)}\right)$ has a mean of -1.7% and a standard deviation of 3%.

3.6.2 In-sample Evidence

For the sample from 1927 to 2005, I estimate (using annual data) for each asset i (25 Fama-French portfolios) the unconditional correlation between the asset's excess returns $r_t^{(i)} - r_t$ and the percentage changes in λ_t.[9] I denote the estimated correlation coefficient for asset i by $\widehat{\rho}_{i,\lambda} = Corr\left(r_t^{(i)} - r_t, \ln\left(\frac{\lambda(t)}{\lambda(t-1)}\right)\right)$. For the same sample period, I estimate for each asset i the average excess return $\widehat{E}\left(r_t^{(i)} - r_t\right)$, and the unexplained expected excess return and the factor loadings of asset i according to the CAPM, the 3 factor Fama-French model and a 4 factor model (which includes a momentum factor). Under the assumption of the CAPM, in order to estimate the unexplained expected excess return of asset i, $\alpha_i^{(CAPM)}$ and the factor loading $\beta_i^{(CAPM)}$, I run for each asset i the time series regression

$$r_t^{(i)} - r_t = \alpha_i^{(CAPM)} + \beta_{m,i}^{(CAPM)}\left(r_t^{(m)} - r_t\right) + \epsilon_t^{(CAPM,i)} \tag{3.25}$$

where $r_t^{(m)}$ denotes the return on the market portfolio, and $\epsilon_t^{(CAPM,i)}$ is white noise.

In case of the 3 factor Fama-French model I estimate the equation

$$r_t^{(i)} - r_t = \alpha_i^{(3F)} + \beta_{m,i}^{(3F)}\left(r_t^{(m)} - r_t\right) + \beta_{SMB,i}^{(3F)} SMB_t + \beta_{HML,i}^{(3F)} HML_t + \epsilon_t^{(3F,i)} \tag{3.26}$$

where SMB_t and HML_t describe the size and book-to-market factors,[10] and $\epsilon_t^{(3F,i)}$ is white noise.

Accordingly, for the 4 factor model I estimate

$$r_t^{(i)} - r_t = \alpha_i^{(4F)} + \beta_{m,i}^{(4F)}\left(r_t^{(m)} - r_t\right) + \beta_{SMB,i}^{(4F)} SMB_t + \beta_{HML,i}^{(4F)} HML_t + \beta_{MOM,i}^{(4F)} MOM_t + \epsilon_t^{(4F,i)} \tag{3.27}$$

[9] $r_t^{(i)}$ and r_t denote the annual return of asset i and the risk free return in year t.
[10] Data for SMB_t and HML_t are provided in K. French's data library.

where MOM_t denotes the momentum factor,[11] and $\epsilon_t^{(4F,i)}$ is white noise.

I explorer the cross-sectional relation between the estimates $\widehat{\rho}_{i,\lambda}$ and the diverse financial quantities. I find a highly significant positive relation between $\widehat{\rho}_{i,\lambda}$ and $\widehat{E}\left(r_t^{(i)} - r_t\right)$ (with a confidence higher then 99%). The correlation is $Corr\left(\widehat{\rho}_{i,\lambda}, \widehat{E}\left(r_t^{(i)} - r_t\right)\right) = 0.52$.

In table 3.1 I show the various cross-sectional correlations between $\widehat{\rho}_{i,\lambda}$ and the unexplained expected excess returns and the factor loadings according to the three considered market models.

Correlations between $\widehat{\rho}_{i,\lambda}$ and financial quantities

Market Model (j)	Correlations $\left(\widehat{\rho}_{i,\lambda}, \alpha_i^{(j)}\right)$	$\left(\widehat{\rho}_{i,\lambda}, \beta_{m,i}^{(j)}\right)$	$\left(\widehat{\rho}_{i,\lambda}, \beta_{SMB,i}^{(j)}\right)$	$\left(\widehat{\rho}_{i,\lambda}, \beta_{HML,i}^{(j)}\right)$	$\left(\widehat{\rho}_{i,\lambda}, \beta_{MOM,i}^{(j)}\right)$
CAPM	0.55****	0.01			
3 Factor	0.47****	−0.36**	0.28*	0.04	
4 Factor	0.43***	−0.33**	0.28*	0.06	0.41***

Table 3.1: Cross-sectional estimation of correlations between $\widehat{\rho}_{i,\lambda}$ and the unexplained expected excess return α_i and the factor loadings $\beta_{f,i}$ according to the CAPM, the Fama-French 3 factor and 4 factor models. Significance of estimation: 4 'stars' denote an estimation of a coefficient which is statistically different from 0 on a 1% significance level, 3 'stars' denote a significance on a 5% level, 2 'stars' denote a significance on a 10% level, and 1 'star' denotes a significance on a 15% level.

I find a strong relation between an asset's correlation to changes in the death rate and its unexplained expected excess return according to the three tested market models. There is also some evidence for a link between $\widehat{\rho}_{i,\lambda}$ and firm size and the momentum factor. $\widehat{\rho}_{i,\lambda}$ appears to be independent of a firm's book-to-market ratio.

The scatter plots in figure 3.3 visualize the correlation between $\widehat{\rho}_{i,\lambda}$ and $\alpha_i^{(j)}$, $j \in \{CAPM, 4F\}$. The positive relationship is evident for both displayed market models, the CAPM and the 4 factor model.

Next, I sort the assets according to their correlations with changes in the death rate ($\widehat{\rho}_{i,\lambda}$), and divide them into $Z = \{2, 3, 4, 5, 10\}$ percentiles. I compare the average (annualized) $\widehat{E}\left(r_t^{(i)} - r_t\right)$ and (annualized) $\alpha_i^{(j)}$, $j \in \{CAPM, 3F, 4F\}$ of the assets in the top percentile (assets with large $\widehat{\rho}_{i,\lambda}$) to the assets in the bottom percentile (assets

[11] Data for SMB_t and HML_t are provided in K. French's data library.

Scatter Plots

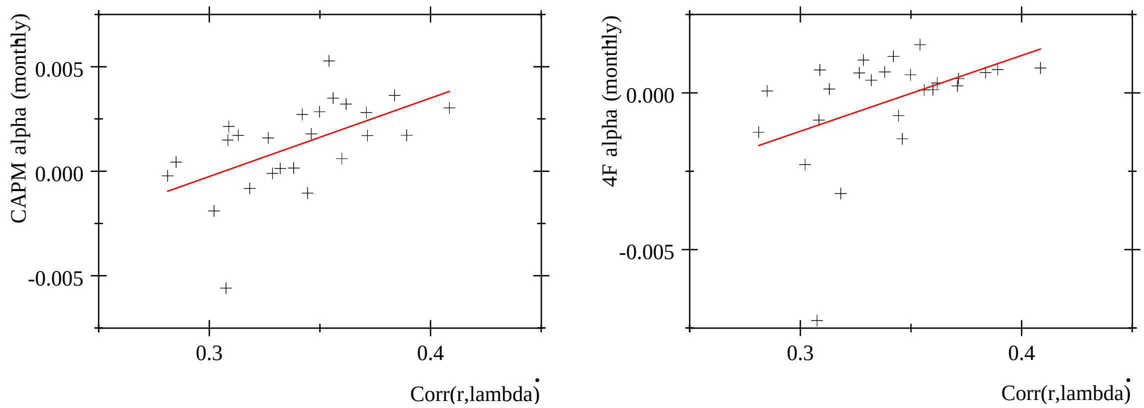

Figure 3.3: Left panel: Scatter plot and fitted linear line showing positive relation between $\widehat{\rho}_{i,\lambda}$ and $\alpha_i^{(CAPM)}$ (monthly data). Right panel: Scatter plot and fitted line showing positive relation between $\widehat{\rho}_{i,\lambda}$ and $\alpha_i^{(4F)}$ (monthly data).

with low $\widehat{\rho}_{i,\lambda}$). Table 3.2 presents the results. Consistent with the results in table 3.1 and the scatter plot, the assets with a large $\widehat{\rho}_{i,\lambda}$ outperform the assets with a low $\widehat{\rho}_{i,\lambda}$.

Top vs Bottom Percentiles of Assets sorted according to $\widehat{\rho}_{i,\lambda}$

Z	$\widehat{E}\left(r_t^{(top)} - r_t^{(bottom)}\right)$	$\alpha_{top}^{(CAPM)} - \alpha_{bottom}^{(CAPM)}$	$\alpha_{top}^{(3F)} - \alpha_{bottom}^{(3F)}$	$\alpha_{top}^{(4F)} - \alpha_{bottom}^{(4F)}$
2	4.1%	3.1%	2.0%	1.5%
3	4.4%	3.9%	3.4%	2.8%
4	4.1%	3.7%	3.3%	2.8%
5	5.4%	4.7%	3.9%	3.4%
10	5.2%	3.3%	2.2%	1.6%

Table 3.2: Assets are sorted according to $\widehat{\rho}_{i,\lambda}$ and grouped into Z percentiles. I estimate the average performance of the assets in each percentile and compare the top versus the bottom percentile according to the measures $\widehat{E}\left(r_t^{(top)} - r_t^{(bottom)}\right)$, $\alpha_{top}^{(CAPM)} - \alpha_{bottom}^{(CAPM)}$, $\alpha_{top}^{(3F)} - \alpha_{bottom}^{(3F)}$, and $\alpha_{top}^{(4F)} - \alpha_{bottom}^{(4F)}$. The comparison is done 5 times for different numbers of percentiles, and each row of the table shows into how many percentiles I have divided the sample ($Z = \{2, 3, 4, 5, 10\}$).

The empirical results are robust in the sense that they do not essentially change if I perform the same tests within a subsample of data (e.g. data from 1927-1980 or 1947-2005) or if I use the 100 Fama-French portfolios (formed on size and book-to-market) as the asset universe.

The empirical (in-sample) tests confirm the implications of my theoretical model that assets whose returns are relatively strong positively correlated to changes in the (aggregate) death rate (large $\widehat{\rho}_{i,\lambda}$) outperform other assets according to traditional market models.

3.6.3 Out-of-sample Evidence

In the previous section, I have presented evidence for a general relation between an asset's performance and its correlation to changes in the death rate. In this section, I test the out-of-sample performance of a dynamic (zero-cost) trading strategy that buys (sells) assets which are relatively strong (weak) positively correlated to changes in the death rate. I argue that this trading strategy is related to the Fama and French (1992, 1993, 1996) factors, and my model on uncertainty in life expectancy provides a theoretical rational for these factors.

At the end of each year $t-1$, I estimate for every asset i (25 Fama-French portfolios) the correlation between its excess returns and the changes in the death rate over the past 15 years (including the realization in year $t-1$) and denote the estimated correlation coefficient by $\widehat{\rho}_t^{(i,\lambda)}$.[12] $\widehat{\rho}_t^{(i,\lambda)}$ is an estimate of the (unobservable) conditional correlation coefficient $Corr_t\left(r_t^{(i)}-r_t, \ln\left(\frac{\lambda(t)}{\lambda(t-1)}\right)\right)$ in year t. I sort the assets according to the estimates $\widehat{\rho}_t^{(i,\lambda)}$ and divide them into 5 percentiles. I form an equally weighted portfolio based on the assets in the top (bottom) quintile (20% assets associated with the highest (lowest) $\widehat{\rho}_t^{(i,\lambda)}$) and denote the (monthly) returns of the 'high correlation' ('low correlation') portfolio in year t by $r_{t,m}^{(Top)}$ $\left(r_{t,m}^{(Bottom)}\right)$.[13] My trading strategy takes a long position in the 'high correlation' portfolio and a short position in the 'low correlation' portfolio. I denote the (monthly) returns of the trading strategy in year t by $r_{t,m}^{(TMB)}=r_{t,m}^{(Top)}-r_{t,m}^{(Bottom)}$.

The trading strategy's turn over is by construction moderate. Over the sample period from 1927 to 2005, the 'high' and the 'low' correlation portfolios pay (on average) an annual excess return of 14.1% respectively 9.7%, while facing almost the same risk. The unconditional volatilities of the 'high' and 'low' correlation portfolios are 24.1%

[12] The results hardly change and the conclusions are the same, if I drop the last observation, i.e. if I estimate $\widehat{\rho}_t^{(i,\lambda)}$ based on the data over the past 15 years excluding the realization in year $t-1$. This might be preferred to ensure that (without any doubt) all data is available to get an estimate of $\widehat{\rho}_t^{(i,\lambda)}$ at the beginning of year t.

[13] Subscribt $(t,m)=\{(t,1),(t,2),...,(t,12)\}$ denotes a vector of monthly data in year t.

Performance of Trading Strategy and 'High' and 'Low' Correlation Portfolios (Monthly Returns)

CAPM:

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \epsilon_{t,m}^{(i)}$$

	α_i	$\beta_{m,i}$
High Corr. Portfolio	$0.25\%^{***}$	1.11^{***}
Low Corr. Portfolio	-0.02%	1.06^{***}
Trading Strategy	$0.27\%^{***}$	0.05^{**}

3 Factor Model:

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \beta_{SMB,i}SMB_{t,m} + \beta_{HML,i}HML_{t,m} + \epsilon_{t,m}^{(i)}$$

	α_i	$\beta_{m,i}$	$\beta_{SMB,i}$	$\beta_{HML,i}$
High Corr. Portfolio	-0.01%	1.06^{***}	0.55^{***}	0.38^{***}
Low Corr. Portfolio	$-0.22\%^{***}$	1.04^{***}	0.34^{***}	0.29^{***}
Trading Strategy	$0.21\%^{**}$	0.02	0.21^{***}	0.09^{**}

4 Factor Model:

$$\begin{aligned} r_{t,m}^{(i)} - r_{t,m} = \alpha_i &+ \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \beta_{SMB,i}SMB_{t,m} + \beta_{HML,i}HML_{t,m} \\ &+ \beta_{MOM,i}MOM_{t,m} + \epsilon_{t,m}^{(i)} \end{aligned}$$

	α_i	$\beta_{m,i}$	$\beta_{SMB,i}$	$\beta_{HML,i}$	$\beta_{MOM,i}$
High Corr. Portfolio	0.00%	1.06^{***}	0.55^{***}	0.37^{***}	-0.01
Low Corr. Portfolio	$-0.17\%^{***}$	1.04^{***}	0.34^{***}	0.28^{***}	-0.05^{***}
Trading Strategy	$0.17\%^{*}$	0.02	0.21^{***}	0.09^{***}	0.04

Table 3.3: At the end of each year $t-1$, I sort assets according to their correlations with changes in the death rate over the past 15 years $\left(\widehat{\rho}_t^{(i,\lambda)}\right)$. I split the asset universe into 5 percentiles (based on the $\widehat{\rho}_t^{(i,\lambda)}$ sort), and form an equally weighted portfolio consisting of the assets in the highest (lowest) quintile and call it the 'high' ('low') correlation portfolio with monthly returns in year t denoted by $r_{t,m}^{(TOP)}$ $\left(r_{t,m}^{(Bottom)}\right)$. The trading strategy takes a long position in the 'high' correlation portfolio and a short position in the 'low' correlation portfolio, and pays in year t the monthly returns $r_{t,m}^{(TMB)} = r_{t,m}^{(Top)} - r_{t,m}^{(Bottom)}$. I estimate for the two correlation portfolios and the trading strategy the unexplained expected excess return and the factor loadings according to the CAPM, the Fama-French 3 factor and 4 factor models. The regression coefficients are reported in the table. Significance of estimation: 3 'stars' denote a significance on a 1% level that the estimated coefficient is different from 0, 2 'stars' denote a significance on a 5% level, and 1 'stars' denotes a significance on a 10% level.

and 22.5%, and the CAPM-β's are 1.11 respectively 1.06.

Table 3.3 compares the performance between the trading strategy and the 'high' and 'low' correlation portfolios. According to the CAPM the trading strategy pays an annual unexplained expected excess return of 3.29% and has almost no exposure to systemic risk. Within the 3 factor and 4 factor models the trading strategy pays an annual unexplained expected excess return of 2.55% respectively 2.06%. While the trading strategy is (mostly) market neutral (no exposure to changes in market portfolio excess returns), it loads on the Fama-French factors.

In table 3.4, 3.5 and 3.6 I illustrate that the constructed trading strategy is a suitable risk factor (I call it $TMB_{t,m}$) which helps to explain the cross-sectional relation between expected asset returns. The factor $TMB_{t,m} = r_{t,m}^{(TMB)}$ is simply equal to the (excess) returns of the constructed trading strategy. Within the considered asset universe (25 Fama-French portfolios), the market portfolio on its own (CAPM) fails to explain a substantial part of the cross-sectional yield relationship. It is well-known that adding the Fama-French size and book-to-market factors reduces the unexplained expected returns a lot (within the sample). However, a new 3 factor market model (I call it the 'Uncertain Life Expectancy 3 Factor Model') which replaces the Fama-French size factor ($SMB_{t,m}$) by the new factor $TMB_{t,m}$ explains the cross-sectional variation in expected asset returns even better. I conclude that the factor $TMB_{t,m}$ is able to explain substantial parts of the size (and some parts of the value) premium documented by Fama-French (1992).

Finally, I test whether assets load on the factor $TMB_{t,m}$ if I add it to the CAPM, the 3 factor or 4 factor models. I run the following time series regression for each asset

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{TMB,i} TMB_{t,m} + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \epsilon_{t,m}^{(i)} \tag{3.28}$$

I find that 17 out of the 25 estimated $\beta_{TMB,i}$ coefficients are statistically different from zero on a 1% significance level. I further run the regression

$$\begin{aligned} r_{t,m}^{(i)} - r_{t,m} \quad = \quad & \alpha_i + \beta_{TMB,i} TMB_{t,m} + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) \\ & + \beta_{SMB,i} SMB_{t,m} + \beta_{HML,i} HML_{t,m} + \epsilon_{t,m}^{(i)} \end{aligned} \tag{3.29}$$

Estimation of Unexplained Expected Excess Returns of 25 Fama French Portfolios according to CAPM (Monthly Returns)

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \epsilon_{t,m}^{(i)}$$

		Book-to-Market				
Size		Low	2	3	4	High
Small		$-0.45\%^{**}$	0.13%	$0.28\%^{**}$	$0.53\%^{***}$	$0.67\%^{***}$
	(t-stat)	(-2.22)	(0.82)	(2.19)	(4.29)	(4.60)
2		$-0.30\%^{**}$	0.09%	$0.36\%^{***}$	$0.42\%^{***}$	$0.55\%^{***}$
	(t-stat)	(-2.27)	(0.91)	(3.80)	(4.28)	(4.47)
3		$-0.23\%^{**}$	$0.16\%^{**}$	$0.24\%^{***}$	$0.41\%^{***}$	$0.47\%^{***}$
	(t-stat)	(-2.16)	(1.99)	(3.09)	(4.67)	(4.06)
4		$-0.13\%^{*}$	0.03%	$0.29\%^{***}$	$0.33\%^{***}$	$0.41\%^{***}$
	(t-stat)	(-1.75)	(0.55)	(4.11)	(3.97)	(3.58)
Big		-0.07%	0.00%	$0.19\%^{***}$	$0.22\%^{***}$	$0.26\%^{**}$
	(t-stat)	(-1.32)	(0.06)	(2.69)	(2.61)	(2.23)

coeff significant on 1%, 5%, 10% level: 13,19, 20

Table 3.4: The table reports for each Fama-French portfolio the estimated α_i coefficient according to the CAPM. The t-statistics of the estimates are provided in brackets. 3 'stars' indicate a coefficient different from 0 on the 1% significance level, 2 'stars' indicate a significance on the 5% level, and 1'star' indicates a significance on the 10% level.

Estimation of Unexplained Expected Excess Returns of 25 FF Portfolios according to Fama-French 3 Factor Model (Monthly Returns)

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \beta_{SMB,i} SMB_{t,m} + \beta_{HML,i} HML_{t,m} + \epsilon_{t,m}^{(i)}$$

		Book-to-Market				
Size		Low	2	3	4	High
Small		$-0.55\%^{***}$	-0.05%	-0.01%	$0.16\%^{***}$	$0.12\%^{**}$
	(t-stat)	(-4.12)	-0.57	-0.17	3.01	2.01
2		$-0.22\%^{***}$	-0.04%	$0.10\%^{**}$	0.05%	0.01%
	(t-stat)	(-3.17)	-0.65	1.99	1.00	0.29
3		-0.05%	0.04%	-0.02%	0.03%	-0.08%
	(t-stat)	(-0.92)	0.63	-0.31	0.49	-1.21
4		0.06%	-0.08%	0.05%	-0.02%	-0.11%
	(t-stat)	(1.16)	-1.35	0.85	-0.35	-1.43
Big		$0.14\%^{***}$	-0.02%	0.06%	$-0.11\%^{**}$	$-0.24\%^{***}$
	(t-stat)	(3.28)	-0.39	0.94	-2.04	-3.04

coeff significant on 1%, 5%, 10% level: 5, 8, 8

Table 3.5: The table reports for each Fama-French portfolio the estimated α_i coefficient according to the Fama-French 3 factor model. The t-statistics of the estimates are provided in brackets. 3 'stars' indicate a coefficient different from 0 on the 1% significance level, 2 'stars' indicate a significance on the 5% level, and 1'star' indicates a significance on the 10% level.

Estimation of Unexplained Expected Excess Returns of 25 FF Portfolios according to Uncertain Life Expectancy 3 Factor Model (Monthly Returns)

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{m,i}\left(r_{t,m}^{(m)} - r_{t,m}\right) + \beta_{HML,i}HML_{t,m} + \beta_{m,i}TMB_{t,m} + \epsilon_{t,m}^{(i)}$$

		Book-to-Market				
Size		Low	2	3	4	High
Small		-0.34%	0.08%	0.09%	$0.25\%^{**}$	$0.23\%^{*}$
	(t-stat)	(-1.63)	(0.52)	(0.73)	(2.12)	(1.73)
2		-0.11%	0.03%	$0.15\%^{*}$	0.11%	0.09%
	(t-stat)	(-0.85)	(0.29)	(1.67)	(1.23)	(0.93)
3		0.03%	0.07%	0.07%	0.05%	-0.03%
	(t-stat)	(0.30)	(0.87)	(0.10)	(0.78)	(-0.40)
4		0.10%	-0.08%	0.06%	-0.01%	-0.07%
	(t-stat)	(1.51)	(-1.26)	(1.01)	(-0.11)	(-0.87)
Big		$0.11\%^{**}$	-0.06%	0.03%	$-0.13\%^{**}$	$-0.23\%^{***}$
	(t-stat)	(2.33)	(-1.03)	(0.50)	(-2.14)	(-2.93)

coeff significant on 1%, 5%, 10% level: 1, 4, 6

Table 3.6: The table reports for each Fama-French portfolio the estimated α_i coefficient according to the Uncertain Life Expectancy 3 factor model. The t-statistics of the estimates are provided in brackets. 3 'stars' indicate a coefficient different from 0 on the 1% significance level, 2 'stars' indicate a significance on the 5% level, and 1'star' indicates a significance on the 10% level.

and find that 8 out of 25 estimated $\beta_{TMB,i}$ coefficients are statistically different from zero on a 1% significance level. For the regression

$$r_{t,m}^{(i)} - r_{t,m} = \alpha_i + \beta_{TMB,i} TMB_{t,m} + \beta_{m,i} \left(r_{t,m}^{(m)} - r_{t,m} \right) + \beta_{SMB,i} SMB_{t,m} + \beta_{HML,i} HML_{t,m} + \beta_{MOM,i} MOM_{t,m} + \epsilon_{t,m}^{(i)} \tag{3.30}$$

I also find that 8 out of 25 estimated $\beta_{TMB,i}$ coefficients are statistically different from zero on a 1% significance level. Moreover, for all three regressions I test whether the loadings on $TMB_{t,m}$ are jointly different from zero. A χ^2 test rejects the null hypothesis of $\beta_{TMB,i} = 0, \forall i$ on a 1% significance level.

My tests suggest that the factor $TMB_{t,m}$ adds important information to the CAPM, the 3 factor model and the 4 factor model and is not absorbed by the other explanatory variables.

As a robustness check I repeat the tests for the data set of 100 Fama-French portfolios and/or the subsample periods 1927-1985 and 1947-2005. The conclusions are the same.

3.7 Conclusion

I explore implications of a time variation in the life expectancy on an agent's optimal portfolio choice. Imposing a financial markets equilibrium, I analyze (in a partial equilibrium model) how the relation between assets' expected returns are affected by the behavior of uncertain lived agents.

I introduced a continuous time finance model featuring a dependency between asset prices and agents' arrival rates of death. Agents maximize expected lifetime utility over intermediate consumption. The model is solved for any possible specification of time additive utilities subject to the (weak) restriction that the utility is strictly increasing and concave in consumption. The optimal asset allocation of an agent is affected by a time variation in the arrival rate of death. Ceteris paribus, an agent invests relatively more respectively less in an asset that pays off high (low) respectively low (high) in states of the world when he expects to live longer (shorter), than in an asset which behaves independently of changes in his death rate. In general the portfolio composition of an agent depends on some individual characteristics such as his health condition and lifestyle, and every agent will choose a tailored investment strategy which differs from the portfolio choice of other agents.

If certain homogeneity is assumed, I state a "Three Fund Separation Theorem" which works in a similar spirit as introduced by Merton (1973). I derive an equilibrium asset pricing equation which states that the expected excess return of any asset depends in a linear fashion on the expected excess returns of the market portfolio and the expected excess return of a fund which is positively related to changes in the (aggregate) death rate. An asset with a positive (negative) correlation between its returns and unexpected changes in the (aggregate) arrival rate of death, has a higher (lower) equity premium, than what was expected in a model without lifetime uncertainty. I show that my results continue to hold if a market for annuities is introduced.

My theoretical asset pricing model is supported by empirical evidence. Indeed, assets with a relatively strong positive correlation between their returns and changes in the (aggregate) death rate outperform other assets on average. I construct a dynamic trading strategy which buys (sells) assets with a strong (weak) positive correlation to changes in the (aggregate) death rate. Out-of-sample evidence suggests that the constructed trading strategy earns a positive unexplained return according to traditional market models. Finally, I construct a factor based on the mentioned trading strategy and provide evidence that it helps to explain the cross-sectional relationship in expected asset returns in addition to traditional factors.

3.8 Appendix

3.8.1 Specification of Expected Lifetime Utility and Wealth dynamics

The expected lifetime utility of agent k at time $t \in [b^{(k)}, \tau^{(k)}_{b^{(k)}})$ is given by

$$\begin{aligned} E_t & \left[U^{(k)}(c^{(k)}) \mid \tau^{(k)}_{b^{(k)}} > t\right] \\ &= E_t\left[\int_t^{\tau^{(k)}_{b^{(k)}}} u^{(k)}_s\left(c^{(k)}_s\right) ds + B^{(k)}\left(\overline{W}^{(k)}, \tau^{(k)}_{b^{(k)}}\right) \mid \tau^{(k)}_{b^{(k)}} > t\right] \\ &= E_t\left[\int_t^{\infty} 1_{\left\{\overline{J}^{(k)}_s=0\right\}} u^{(k)}_s\left(c^{(k)}_s\right) + 1_{\left\{\overline{J}^{(k)}_s=1 \cap \overline{J}^{(k)}_{s-}=0\right\}} B^{(k)}\left(\overline{W}^{(k)}, s\right) ds \mid \overline{J}^{(k)}_t = 0\right] \\ &= E_t\left[\int_t^{\infty} h^{(k)}(t,m)\left(\int_t^m u^{(k)}_s\left(c^{(k)}_s\right) ds + B^{(k)}\left(\overline{W}^{(k)}, m\right)\right) dm\right] \\ &= E_t\left[\int_t^{\infty} \overline{H}^{(k)}(t,s) u^{(k)}_s\left(c^{(k)}_s\right) + h^{(k)}(t,s) B^{(k)}\left(\overline{W}^{(k)}, s\right) ds\right]. \end{aligned} \tag{3.31}$$

Agent k's wealth is a random process and evolves according to the dynamics

$$\begin{aligned} d\overline{W}^{(k)}_t &= r\left(\overline{W}^{(k)}_t - X^{[1N](k)T}_t P^{[1N]}_t\right) dt + X^{[1N](k)^T}_t dP^{[1N]}_t - c^{(k)}_t dt \\ &= \overline{W}^{(k)}_t r dt + \overline{W}^{(k)}_t \phi^{[1N](k)^T}_t \left(\left(I_{P_t}\right)^{-1} dP^{[1N]}_t - r1_{(1\times N)} dt\right) - c^{(k)}_t dt \\ &= \overline{W}^{(k)}_t r dt + \overline{W}^{(k)}_t \phi^{[1N](k)^T}_t \left(\mu^{[1N]}_P dt + \sigma^{[1N]}_P dW_t - r1_{(1\times N)} dt\right) - c^{(k)}_t dt, \end{aligned} \tag{3.32}$$

3.8.2 Proofs of Lemmas

Proof of Lemma 3.1. It is straightforward to show that

$$\begin{aligned} LE^{(k)}_t &= E_t\left[\tau^{(k)}_{b^{(k)}} \mid \tau^{(k)}_{b^{(k)}} > t\right] \\ &= \frac{1}{\overline{H}^{(k)}(t)} E_t\left[\int_t^{\infty} s h^{(k)}(u)\, ds\right], \end{aligned} \tag{3.33}$$

and after plugging in the formulas for $\overline{H}^{(k)}(t)$ and $h^{(k)}(t)$, and further rewriting for $s > t$, $\lambda_s^{(k)}$ in terms of $\lambda_t^{(k)}$, I end up with a function in $\lambda_t^{(k)}$,

$$\begin{aligned} LE_t^{(k)} &= \int_{-\infty}^{\infty}\int_t^{\infty} s\lambda_t^{(k)} \exp\left\{\int_t^s \mu_{\lambda,b^{(k)},r}^{(k)} dr - \frac{1}{2}\sigma_\lambda^{(k)}\sigma_\lambda^{(k)^T}(s-t) + \sigma_\lambda^{(k)}\sqrt{s-t}z\right\} \\ &\quad \exp\left\{-\int_t^s \lambda_t^{(k)} \exp\left\{\int_t^u \mu_{\lambda,b^{(k)},r}^{(k)} dr - \frac{1}{2}\sigma_\lambda^{(k)}\sigma_\lambda^{(k)^T}(u-t) + \sigma_\lambda^{(k)}\sqrt{u-t}z\right\} du\right\} ds. \end{aligned} \tag{3.34}$$

From this expression I can now determine whether $LE_t^{(k)}$ is increasing or decreasing in $\lambda_t^{(k)}$. Note that

$$\begin{aligned} \overline{H}^{(k)}(t,s) &= \exp\left\{-\int_t^s \lambda_u^{(k)} du\right\} \\ &= \exp\left\{-\int_t^s \lambda_t^{(k)} \exp\left\{\begin{array}{c} \int_t^u \mu_{\lambda,b^{(k)},r}^{(k)} dr - \frac{1}{2}\sigma_\lambda^{(k)}\sigma_\lambda^{(k)^T}(u-t) \\ +\sigma_\lambda^{(k)}(W_u - W_t) \end{array}\right\} du\right\}, \end{aligned} \tag{3.35}$$

and thus,

$$\frac{\partial \overline{H}^{(k)}(t,s)}{\partial \lambda_t^{(k)}} = \frac{\ln\left(\overline{H}^{(k)}(t,s)\right)}{\lambda_t^{(k)}}\overline{H}^{(k)}(t,s) < 0 \quad a.s., \tag{3.36}$$

since $\overline{H}^{(k)}(t,s) \in (0,1)$ $a.s.$ Then, since

$$\begin{aligned} h^{(k)}(t,s) &\equiv \frac{h^{(k)}(s)}{\overline{H}^{(k)}(t)} \\ &= \lambda_s^{(k)}\overline{H}^{(k)}(t,s) \\ &= \lambda_t^{(k)} \exp\left\{\int_t^s \mu_{\lambda,b^{(k)},r}^{(k)} dr - \frac{1}{2}\sigma_\lambda^{(k)}\sigma_\lambda^{(k)^T}(s-t) + \sigma_\lambda^{(k)}(W_s - W_t)\right\}\overline{H}^{(k)}(t,s), \end{aligned} \tag{3.37}$$

I get

$$\frac{\partial h^{(k)}(t,s)}{\partial \lambda_t^{(k)}} = \frac{\lambda_s^{(k)}}{\lambda_t^{(k)}}\overline{H}^{(k)}(t,s)\left[1 + \ln\left(\overline{H}^{(k)}(t,s)\right)\right] \begin{array}{l} > 0, \text{ for s small} \\ < 0, \text{ for s large} \end{array} \quad a.s., \tag{3.38}$$

and hence, for $\lambda_t^{(k)\prime} < \lambda_t^{(k)\prime\prime}$

$$\begin{aligned} LE_t^{(k)\prime} &= E_t\left[\int_t^{\infty} sh^{(k)\prime}(t,s)\, ds\right] \\ &> E_t\left[\int_t^{\infty} sh^{(k)\prime\prime}(t,s)\, ds\right] = LE_t^{(k)\prime\prime}. \end{aligned} \tag{3.39}$$

I also can think of it in another way. For $\lambda_t^{(k)\prime} < \lambda_t^{(k)\prime\prime}$ by (3.36)

$$\overline{H}^{(k)\prime}(t,s) > \overline{H}^{(k)\prime\prime}(t,s) \qquad a.s., for\, s \in (t,\infty) \tag{3.40}$$

holds, and hence,

$$LE_t^{(k)\prime} > LE_t^{(k)\prime\prime} \tag{3.41}$$

must be true. ■

Proof of Lemma 3.2. Following Merton (1971), the Hamilton-Jacobi-Bellman optimality principle implies that the maximization problem $(P1)$ has the same solution as

$$\begin{aligned} 0 \quad = \quad & \sup_{\left\{c_t^{(k)},\phi_t^{[1N](k)}\right\}\in\Re_+^1\times(\mathcal{L}^2)^N} \left\{u_t^{(k)}\left(c_t^{(k)}\right)dt + E_t\left[dV^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},\overline{J}^{(k)},t\right)\right]\right\} \\ & s.t. \\ & d\overline{W}_t^{(k)} = \overline{W}_t^{(k)} r dt + \overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right)dt \\ & \qquad\quad + \overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\sigma_P^{[1N]}dW_t - c_t^{(k)}dt \\ & d\lambda_t^{(k)} = \lambda_t^{(k)}\mu_{\lambda,b^{(k)},t}^{(k)}dt + \lambda_t^{(k)}\sigma_\lambda^{(k)}dW_t \end{aligned} \tag{HJB}$$

with the Transversality condition $\underset{t\to\infty}{lim} E_t\left[V^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},\overline{J}^{(k)},t\right)\right] = 0$. Applying the Itô-Doeblin theorem for jump processes to the term $dV^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},\overline{J}^{(k)},t\right)$, I can rewrite the HJB equation as

$$0 = \sup_{\left\{c_t^{(k)},\phi_t^{[1N](k)}\right\}\in\Re_+^1\times(\mathcal{L}^2)^N} \left\{ \begin{array}{c} u_t^{(k)}\left(c_t^{(k)}\right) + V_\lambda^{(k)}\lambda_t^{(k)}\mu_{\lambda,b^{(k)},t}^{(k)} + V_t^{(k)} \\ +V_{\overline{W}}^{(k)}\left[\overline{W}_t^{(k)}r + \overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) - c_t^{(k)}\right] \\ +\frac{1}{2}V_{\overline{W}\overline{W}}^{(k)}\overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\sigma_P^{[1N]}\sigma_P^{[1N]^T}\phi_t^{[1N](k)}\overline{W}_t^{(k)} \\ +V_{\overline{W}\lambda}^{(k)}\overline{W}_t^{(k)}\phi_t^{[1N](k)^T}\sigma_P^{[1N]}\sigma_\lambda^{(k)^T}\lambda_t^{(k)} + \frac{1}{2}V_{\lambda\lambda}^{(k)}\lambda_t^{(k)}\sigma_\lambda^{(k)}\sigma_\lambda^{(k)^T}\lambda_t^{(k)} \\ +\lambda_t^{(k)}[\underbrace{B^{(k)}\left(\overline{W}^{(k)},t\right)}_{V^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},J^{(k)}=1,t\right)} - V^{(k)}\left(\overline{W}^{(k)},\lambda^{(k)},J^{(k)}=0,t^-\right)] \end{array} \right\} \tag{3.42}$$

The first order conditions with respect to the controls $c_t^{(k)}$ and $\phi_t^{[1N](k)}$ imply

$$\frac{\partial}{\partial c_t^{*(k)}}\left[u_t^{(k)}\left(c_t^{*(k)}\right)\right] - V_{\overline{W}}^{(k)} = 0 \tag{3.43}$$

and

$$0 = V_{\overline{W}\overline{W}}^{(k)}\overline{W}_t^{(k)}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\phi_t^{*[1N](k)} + V_{\overline{W}}^{(k)}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) + V_{\overline{W}\lambda}^{(k)}\sigma_P^{[1N]}\sigma_\lambda^{(k)^T}\lambda_t^{(k)} \tag{3.44}$$

The result follows immediately. ∎

Proof of Lemma 3.3. Statement 1) follows immediately from the first FOC and the definition of $u_t^{(k)}(.)$ being strictly increasing, hence, $V_{\overline{W}}^{(k)} = \frac{\partial}{\partial c}\left[u_t^{(k)}\left(c_t^{*(k)}\right)\right] > 0$. In order to show proof for the latter two claims, I first note that at optimum

$$\frac{\partial u_t^{(k)}\left(c_t^{*(k)}\right)}{\partial c_t} = \overline{H}^{(k)}(t,s))\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s}\frac{\partial\overline{W}_s^{(k)}}{\partial\overline{W}_t^{(k)}}, \forall s\in[t,\infty) \qquad a.s. \tag{3.45}$$

must hold. Intuitively, this equation says that the utility agent k gets from consuming one additional dollar at time t must in expectation equal the utility that agent k derives from investing one dollar at time t and consuming the proceeds at time $s \in [t,\infty)$ in addition to $c_s^{*(k)}$. This becomes evident by taking the first derivative with respect to

$\overline{W}_t^{(k)}$ from the formal definition of the value function $V^{(k)}\left(\overline{W}^{(k)}, \lambda^{(k)}, \overline{J}^{(k)}, t\right)$

$$V_{\overline{W}}^{(k)} = E_t\left[\int_t^\infty \overline{H}^{(k)}(t,s)\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s}\frac{\partial c_s^{*(k)}}{\partial \overline{W}_s^{(k)}}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}ds\right], \tag{3.46}$$

noting that $\int_t^\infty \frac{\partial c_s^{*(k)}}{\partial \overline{W}_s^{(k)}}ds$ must equal 1 almost surely,[14] and combining it with the FOC,

$$\frac{\partial u_t^{(k)}\left(c_t^{*(k)}\right)}{\partial c_t} = E_t\left[\int_t^\infty \overline{H}^{(k)}(t,s)\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}\frac{\partial c_s^{*(k)}}{\partial \overline{W}_s^{(k)}}ds\right], \tag{3.47}$$

what has (3.45) as a solution. An increase in $\overline{W}_t^{(k)}$ leads agent k to adjust his optimal consumption plan and thus, to increase expected $c_s^{*(k)}$, for some (at least one) $s \in [t,\infty)$. But then, by (3.45) at optimum expected consumption at every point in time $s \in [t,\infty)$ must increase, and hence, $\frac{\partial c_t^{*(k)}}{\partial \overline{W}_t^{(k)}} > 0$. Then, remembering that $u_t^{(k)}(.)$ is strictly concave by definition and taking the derivative of the first FOC with respect to $\overline{W}_t^{(k)}$ gives

$$V_{\overline{WW}}^{(k)} = \underbrace{\frac{\partial^2}{\partial c_t^2}\left[u_t^{(k)}\left(c_t^{*(k)}\right)\right]}_{<0}\underbrace{\frac{\partial c_t^{*(k)}}{\partial \overline{W}_t^{(k)}}}_{>0} < 0 \tag{3.48}$$

what proves claim 2). To show statement 3), note that an increase in $\lambda_t^{(k)}$ means by (3.36) a decrease in survival probabilities $\overline{H}^{(k)}(t,s)$, $\forall s > t$, and ceteris paribus, a decrease in $\overline{H}^{(k)}(t,s))\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}$, $\forall s > t$. But then, besides some other (here uninteresting) adjustments to the optimal consumption plan, $\frac{\partial u_t^{(k)}\left(c_t^{*(k)}\right)}{\partial c_t}$ necessarily has to decrease as well such that (3.45) will be satisfied again, what means that $c_t^{*(k)}$ increases. Hence, I have shown that $\frac{\partial c_t^{*(k)}}{\partial \lambda_t^{(k)}} > 0$, and thus, using again the definition that $u_t^{(k)}(.)$ is strictly concave and taking the first derivative with respect to $\lambda_t^{(k)}$ of the

[14] This is true since $\int_t^\infty \frac{\partial c_s^{*(k)}}{\partial \overline{W}_s^{(k)}}ds = \int_t^\infty \frac{\partial c_s^{*(k)}}{\partial \overline{W}_s^{(k)}}\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}\left(\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}\right)^{-1}ds = \int_t^\infty \frac{\partial c_s^{*(k)}}{\partial \overline{W}_t^{(k)}}\left(\frac{\partial \overline{W}_s^{(k)}}{\partial \overline{W}_t^{(k)}}\right)^{-1}ds = 1$ *a.s.*, where the last equation holds because of a simple accounting rule that a change in wealth at time t by one dollar must equal a change of sumed up recent and future consumption by exactly one dollar plus invested proceeds.

FOC, I get

$$V^{(k)}_{\overline{W}\lambda} = \underbrace{\frac{\partial^2}{\partial c_t^2}\left[u_t^{(k)}\left(c_t^{*(k)}\right)\right]}_{<0}\underbrace{\frac{\partial c_t^{*(k)}}{\partial \lambda_t^{(k)}}}_{>0} < 0, \tag{3.49}$$

what completes the proof.

Yet another way to proof statement 2) and 3), is by employing the martingale approach introduced by Cox and Huang (1989) to tackle agent k's optimization problem. First, I see that the dynamic budget constraint implies the static constraint

$$\overline{W}_t^{(k)} \geq E_t\left[\int_t^\infty \overline{H}^{(k)}(t,s)\frac{\pi_s}{\pi_t}c_s^{(k)}ds\right] \tag{3.50}$$

where π is a stochastic discount factor. It is important to notice that the dynamic and the static constraint are equivalent in the case of complete markets where there is a unique SDF. However, in the case of incomplete markets with multiple SDFs the dynamic constraint implies the static one but it does not hold the other way. Moreover, assuming that the optimal consumption plan $c^{*(k)}$ is in the interior, the Gateaux derivative of expected utility at point of optimal consumption $c^{*(k)}$ in direction $\left(c^{(k)} - c^{*(k)}\right)$ equals zero

$$\nabla E_t\left[U^{(k)}(c^{*(k)})\right]\left[c^{(k)} - c^{*(k)}\right] = 0 \tag{3.51}$$

for all admissible consumption plans $c^{(k)}$. Defining $Q_t^{(k)}(.)$ as a function such that $Q_t^{(k)}(x_s) = \overline{H}^{(k)}(t,s)x_s$ and noticing that $Q_t^{(k)}\left(c^{(k)} - c^{*(k)}\right)$ spans the set of all marketable cash flows I get

$$\nabla E_t\left[U^{(k)}(c^{*(k)})\right]\left[Q_t^{(k)^{-1}}(x)\right] = 0 \tag{3.52}$$

for all marketable cash flows x. But this implies that the Riesz representation process R_s times $\frac{1}{\overline{H}^{(k)}(t,s)}$ is equal to a SDF times a constant $\eta^{(k)}$. Since for time additive utilities as defined here the Riesz representation process R is given by

$$R_s = \overline{H}^{(k)}(t,s)\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s}, \tag{3.53}$$

I can derive for optimal consumption $c_t^{*(k)}$ the condition

$$\frac{\partial u_s^{(k)}\left(c_s^{*(k)}\right)}{\partial c_s} = \eta^{(k)}\pi_s. \tag{3.54}$$

If markets are complete, the SDF is unique and there is no problem to determine (3.54). In incomplete markets there are multiple SDFs and I have to choose the one SDF that minimizes expected utility

$$\pi^{(k)} = \arg\inf_{\pi\in\Pi}\left\{E_t\left[U^{(k)}(I\left(\eta^{(k)}\pi\right))\right]\right\} \tag{3.55}$$

where the function $I\,(.)$ is the inverse function of the marginal utility process

$$I\left(\frac{\partial u_s^{(k)}\,(x_s)}{\partial x_s}\right) = x_s. \tag{3.56}$$

Note that since $u_s^{(k)}\,(.)$ is by assumption strictly increasing and concave, $I\,(.)$ is positive and strictly decreasing. The constant $\eta^{(k)}$ is determined through the static budget constraint using the SDF $\pi^{(k)}$

$$\overline{W}_t^{(k)} = E_t\left[\int_t^\infty \overline{H}^{(k)}(t,s)\frac{\pi_s^{(k)}}{\pi_t^{(k)}}I\left(\eta^{(k)}\pi_s^{(k)}\right)ds\right]. \tag{3.57}$$

Finally, combining condition (3.54) with the first FOC from dynamic programming, I have

$$V_{\overline{W}}^{(k)} = \eta^{(k)}\pi_t^{(k)} \tag{3.58}$$

Now, I can take the first derivative from both sides in equation (3.57) with respect to $\overline{W}_t^{(k)}$

$$1 = E_t\left[\int_t^\infty \overline{H}^{(k)}(t,s)\frac{\pi_s^{(k)}}{\pi_t^{(k)}}\underbrace{\frac{\partial I\left(\eta^{(k)}\pi_s^{(k)}\right)}{\partial\left(\eta^{(k)}\pi_s^{(k)}\right)}}_{<0}\frac{\partial\eta^{(k)}}{\partial\overline{W}_t^{(k)}}\pi_s^{(k)}ds\right] \tag{3.59}$$

what implies $\frac{\partial\eta^{(k)}}{\partial\overline{W}_t^{(k)}} < 0$. In turn when taking the first derivative from (3.58) with respect to $\overline{W}_t^{(k)}$, I get

$$V_{\overline{W}\overline{W}}^{(k)} = \frac{\partial \eta^{(k)}}{\partial \overline{W}_t^{(k)}} \pi_t^{(k)} < 0 \tag{3.60}$$

what proves statement 2). Moreover, taking the first derivative from (3.57) with respect to $\lambda_t^{(k)}$,

$$0 = E_t \left[\int_t^\infty \underbrace{\frac{\partial \overline{H}^{(k)}(t,s)}{\partial \lambda_t^{(k)}}}_{<0} \frac{\pi_s^{(k)}}{\pi_t^{(k)}} I\left(\eta^{(k)}\pi_s^{(k)}\right) + \overline{H}^{(k)}(t,s) \frac{\pi_s^{(k)}}{\pi_t^{(k)}} \underbrace{\frac{\partial I\left(\eta^{(k)}\pi_s^{(k)}\right)}{\partial\left(\eta^{(k)}\pi_s^{(k)}\right)}}_{<0} \frac{\partial \eta^{(k)}}{\partial \lambda_t^{(k)}} \pi_s^{(k)} ds \right] \tag{3.61}$$

implies $\frac{\partial \eta^{(k)}}{\partial \lambda_t^{(k)}} < 0$. Finally, taking the first derivative with respect to $\lambda_t^{(k)}$ from (3.58) yields

$$V_{\overline{W}\lambda}^{(k)} = \frac{\partial \eta^{(k)}}{\partial \lambda_t^{(k)}} \pi_t^{(k)} < 0 \tag{3.62}$$

what completes the proof of Lemma 3.2. ■

3.8.3 Proofs of Propositions and Theorems

Proof of Proposition 3.1. From the FOC of the HJB, the expressions of $\phi_t^{*[1N](k)}$, $A_t^{(k)}$ and $B_t^{(k)}$ follow immediately. Moreover, using Lemma 3.2 it is trivial to see that $A_t^{(k)} = \frac{V_{\overline{W}}^{(k)}}{-V_{\overline{W}\overline{W}}^{(k)} \overline{W}_t^{(k)}} > 0$ and $B_t^{(k)} = \frac{-V_{\overline{W}\lambda}^{(k)}}{-V_{\overline{W}\overline{W}}^{(k)} \overline{W}_t^{(k)}} \lambda_t^{(k)} > 0$. The remainder of the proposition follows straightforward. ■

Proof of Theorem 3.1 ("Three Fund Separation"). Denote the three funds by δ_1, δ_2 and δ_3, and use the superscript $[1N]$ and $[0]$ to indicate investments in risky assets and the riskless asset, respectively. Let there be one fund consisting of the risk free asset only

$$\delta_1^{[1N]} = 0, \ \delta_1^{[0]} = 1, \tag{3.63}$$

another fund being equal to the well known tangency portfolio in a standard CAPM

$$\delta_2^{[1N]} = \frac{\left(\sigma_P^{[1N]} \sigma_P^{[1N]^T}\right)^{-1} \left(\mu_P^{[1N]} - r1_{(1\times N)}\right)}{\left(1_{(1\times N)}\right)^T \left(\sigma_P^{[1N]} \sigma_P^{[1N]^T}\right)^{-1} \left(\mu_P^{[1N]} - r1_{(1\times N)}\right)}, \ \delta_2^{[0]} = 0, \tag{3.64}$$

and finally a third fund which invests in a combination of the risk free and risky assets

$$\delta_3^{[1N]} = -\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\sigma_P^{[1N]}\sigma_\lambda^T,\ \delta_3^{[0]} = 1 - \left(1_{(1\times N)}\right)^T\delta_3^{[1N]}. \tag{3.65}$$

Let agent k's investments in the three funds, denoted by $\left(\theta_{1,t}^{(k)}, \theta_{2,t}^{(k)}, \theta_{3,t}^{(k)}\right)$, be

$$\theta_{1,t}^{(k)} = \overline{W}_t^{(k)} - \theta_{2,t}^{(k)} - \theta_{3,t}^{(k)}, \tag{3.66}$$

$$\theta_{2,t}^{(k)} = \widetilde{A}_t^{(k)}\left(1_{(1\times N)}\right)^T\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right), \tag{3.67}$$

$$\theta_{3,t}^{(k)} = \widetilde{B}_t^{(k)}. \tag{3.68}$$

Noting that the strategy $\left(\theta_{1,t}^{(k)}, \theta_{2,t}^{(k)}, \theta_{3,t}^{(k)}\right)$ replicates exactly the demand $d_t^{(k)}$, proves (i). Moreover, the three funds clearly satisfy (ii). Finally, noting that

$$\left(1_{(1\times N)}\right)^T\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) = \frac{\mu_{\delta_2} - r}{\sigma_{\delta_2}^2}, \tag{3.69}$$

where $\mu_{\delta_2} - r$ and $\sigma_{\delta_2}^2$ are the risk premium respectively the variance of fund δ_2, proves (iii). ■

Proof of Theorem 3.2. Imposing equilibrium in financial markets, i.e. equating aggregate demand and supply of risky assets

$$\begin{aligned} M_t\phi_t^{[1N](M)} &= D_t^{[1N]} \\ &= A_t\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) \\ &\quad -B_t\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)^{-1}\sigma_P^{[1N]}\sigma_\lambda^T, \end{aligned} \tag{3.70}$$

and solving for the vector of expected excess returns

$$\mu_P^{[1N]} - r1_{(1\times N)} = \frac{M_t}{A_t}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\phi_t^{[1N](M)} + \frac{B_t}{A_t}\sigma_P^{[1N]}\sigma_\lambda^T, \tag{3.71}$$

allows us to get the following expression for any asset i's (portfolio $\phi_t^{(i)}$'s) expected excess return by pre-multiplying (3.71) by $\phi_t^{[1N](i)}$

$$\begin{aligned}
\mu_i - r &= \phi_t^{[1N](i)^T}\left(\mu_P^{[1N]} - r1_{(1\times N)}\right) \qquad (3.72)\\
&= \frac{M_t}{A_t}\phi_t^{[1N](i)^T}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\phi_t^{[1N](M)} + \frac{B_t}{A_t}\phi_t^{[1N](i)^T}\sigma_P^{[1N]}\sigma_\lambda^T\\
&= \frac{M_t}{A_t}\phi_t^{[1N](i)^T}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\phi_t^{[1N](M)} + \frac{B_t}{A_t}\phi_t^{[1N](i)^T}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\delta^{[1N]}\\
&= \frac{M_t}{A_t}\rho_{i,M}\sigma_i\sigma_M + \frac{B_t}{A_t}\rho_{i,\delta}\sigma_i\sigma_\delta.
\end{aligned}$$

Moreover, (3.72) holds for $i = M$

$$\mu_M - r = \frac{M_t}{A_t}\sigma_M^2 + \frac{B_t}{A_t}\rho_{M,\delta}\sigma_M\sigma_\delta, \qquad (3.73)$$

and $i = \delta$

$$\mu_\delta - r = \frac{M_t}{A_t}\rho_{\delta,M}\sigma_\delta\sigma_M + \frac{B_t}{A_t}\sigma_\delta^2. \qquad (3.74)$$

Thus, I can solve the system of equations (3.73) and (3.74) for $\frac{M_t}{A_t}$ and $\frac{B_t}{A_t}$, and plug the solutions back into (3.72) to get

$$\mu_i - r = (\mu_M - r)\frac{\sigma_i}{\sigma_M}\frac{\rho_{i,M} - \rho_{M,\delta}\rho_{i,\delta}}{1 - \left(\rho_{M,\delta}\right)^2} + (\mu_\delta - r)\frac{\sigma_i}{\sigma_\delta}\frac{\rho_{i,\delta} - \rho_{M,\delta}\rho_{i,M}}{1 - \left(\rho_{M,\delta}\right)^2}. \qquad (3.75)$$

Finally, to prove the second claim in the proposition, I first note that

$$\begin{aligned}
\rho_{i,\delta}\sigma_i\sigma_\delta &= \phi_t^{[1N](i)^T}\left(\sigma_P^{[1N]}\sigma_P^{[1N]^T}\right)\delta^{[1N]} \qquad (3.76)\\
&= \phi_t^{[1N](i)^T}\sigma_P^{[1N]}\sigma_\lambda^T = \rho_{i,\lambda}\sigma_i\sigma_\lambda,
\end{aligned}$$

and hence, the sign of the correlation between asset i and λ is the same as the one of the correlation between asset i and δ, i.e.

$$sign\left(\rho_{i,\delta}\right) = sign\left(\rho_{i,\lambda}\right). \qquad (3.77)$$

Further, in absence of lifetime uncertainty the second term in the equilibrium yield relationship vanishes. Thus, if for asset i $\rho_{i,\lambda} > (<)0$, then $(\mu_\delta - r)\,\beta_i^{(\delta)} > (<)0$, which completes the proof. ■

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