Two Problems in Graph Theory

Stanislav Kučera

A thesis submitted for the degree of
Master of Philosophy

Department of Mathematics
London School of Economics
and Political Science

September 2019

Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is my own work, created in collaboration with my supervisor Jan van den Heuvel.

Abstract

In the thesis we study two topics in graph theory.

The first one is concerned with the famous conjecture of Hadwiger that every graph G without a minor of a complete graph on $t+1$ vertices can be coloured with t colours. We investigate how large an induced subgraph of G can be, so that the subgraph can be coloured with t colours. We show that G admits a t-colourable induced subgraph on more than half of its vertices. Moreover, if such graph G on n vertices does not contain any triangle, we show it admits a t-colourable induced subgraph on at least $4n/5$ vertices and show even better bounds for graphs with larger odd girth.

The second topic is a variant of a well-known two player Maker-Breaker connectivity game in which players take turns choosing an edge in each step in order to achieve their respective goals. While a complete characterisation is known for the connectivity game in which both players choose a single edge, much less is known in all other cases. We study the variant in which both players choose two edges, or more generally, the variant in which the first player decides whether both players choose one or two edges in the next round.

Acknowledgements

I would like to thank the whole Department of Mathematics at LSE, and in particular to my supervisor, Jan van den Heuvel, for giving me an opportunity to join the academic family. I also thank Jan for introducing me to many interesting problems, guiding me through our research journeys, and for showing me what it means to be a member of academia.

I would like to thank all PhD students sharing the office, Enfale and Sarah, for enriching my stay in London.

The last but not least, I thank the London School of Economics and Political Science for providing me with financial support.

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Introduction

1.1 Chapters 2 and 3: around Hadwiger's conjecture

In the next two chapters we discuss results related to a well-known conjecture in graph theory that connects proper vertex colourings with existence of minors of complete graphs:

Conjecture 1.1 (Hadwiger [1943]). *Every graph G without a minor of K_{t+1} can be coloured with t colours.*

Hadwiger [1943] showed the conjecture to be true for $t \leq 3$. Wagner [1937] showed that the case $t = 4$ is equivalent to the Four Colour Theorem later proved by Appel and Haken [1976]. Robertson, Seymour, and Thomas [1993] proved the conjecture for $t = 5$. Despite serious effort, the conjecture remains open for $t \geq 6$.

Consider a graph G and a (proper) vertex colouring of G that uses $\chi(G)$ colours. Each subset of vertices coloured by the same colour forms an independent set; consequently, its size is at most $\alpha(G)$. Hence we have $|V(G)| \leq \alpha(G)\chi(G)$. Combining this inequality with Hadwiger's conjecture we get the following.

Conjecture 1.2. *Every K_{t+1}-minor-free graph G satisfies $|V(G)| \leq \alpha(G) \cdot t$, or equivalently $\alpha(G) \geq \dfrac{|V(G)|}{t}$.*

This conjecture is also wide open. However, there are classes of graphs for which Conjecture 1.2 is true while Hadwiger's conjecture remains undecided. For example, Fradkin [2012] showed that the family of claw-free graphs of independence number at least 3 is one such class.

Consider the following result of Duchet and Meyniel [1982], which is only a constant factor away from Conjecture 1.2.

Theorem 1.3. *Every K_{t+1}-minor-free graph G satisfies* $\alpha(G) \geq \frac{|V(G)|}{2t}$.

There have been several improvements of this bound. Most notably, Fox [2010] used the theory of claw-free graphs and decreased the constant factor 2, for the first time, to $2-c$ for $c = \frac{29-\sqrt{813}}{28} \approx 0.017$. Later, Balogh and Kostochka [2011] used a similar approach and optimized the constants to get the currently best bound.

Theorem 1.4. *Every K_{t+1}-minor-free graph G satisfies*

$$\alpha(G) \geq \frac{|V(G)|}{(2-c)t} \quad \textit{for} \quad c = \frac{80-\sqrt{5392}}{126} \approx 0.05214.$$

We use these results in pursuit of yet another question:

Question 1.5. *Let G be a K_{t+1}-minor-free graph on n vertices. What is the maximum fraction of vertices of G that spans a t-colourable induced subgraph?*

Consider the following greedy algorithm on our graph G. Find an independent set I of maximum size in G and colour vertices in I with a single colour. Then find an independent set of maximum size in $G-I$ and colour it with a different colour. Repeat t times, and colour some vertices of G with t colours. Intuitively, it may seem that using Theorem 1.4 to bound the size of each independent set yields a colouring of at least $t\frac{n}{(2-c)t} = \frac{n}{2-c}$ vertices of G. But this assumes the sizes of independent sets we choose in our algorithm do not decrease, or that we gain at the beginning more than we lose at the end. We do not attempt to analyse this greedy approach.

Seymour [2016] pointed out that the same proof as the one used for Theorem 1.3 also shows there always exists a t-colourable induced subgraph on at least $n/2$ vertices in a K_{t+1}-minor-free graph G on n vertices. In Chapter 2 we show this proof, improve the bound, and show the following.

Theorem 2.5. *For every $t \geq 4$ and a K_{t+1}-minor-free graph G on n vertices, there is a t-colourable induced subgraph H of G such that*

$$|V(H)| \geq \left(\frac{1}{2} + \frac{1}{(2-c)t} - \frac{1}{2(2-c)^2t^2}\right)n \quad \text{where} \quad c = \frac{80-\sqrt{5392}}{126} \approx 0.05214.$$

In Chapter 3 we further restrict the K_{t+1}-minor-free graph G by forbidding all short odd cycles, or all short cycles.

Theorem 3.6. *For every $c \geq 0$ and every K_{t+1}-minor-free graph G with odd girth $g = 2q+1 > 3$, there is a $(t+c)$-colourable induced subgraph H of G such that*

$$|V(H)| > \left(1 - \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor\frac{g+1}{8}\right\rfloor + \left\lfloor\frac{g+3}{8}\right\rfloor\right)\right)^{-1}\right)|V(G)|.$$

Moreover, if G has girth g, there is a $(t+c)$-colourable induced subgraph H of G such that

$$|V(H)| > \frac{(c+2)^{(g-3)/2}-1}{(c+2)^{(g-3)/2}}|V(G)|.$$

We conclude the chapter by showing:

Theorem 3.9. *For every $\varepsilon, \delta > 0$ there exists t_0 such that for every $t > t_0$, every triangle-free and K_{t+1}-minor-free graph G admits a $(1+\delta)t$-colourable induced subgraph on at least $(1-\varepsilon)|V(G)|$ vertices.*

1.2 Chapter 4: Maker-Breaker connectivity game

An (a,b) Maker-Breaker connectivity game is a game of two players, Maker and Breaker, in which both players take turns claiming previously unclaimed edges of a given (multi)graph G. Breaker plays first, in each turn he claims b edges, and then Maker claims a edges. The game ends when Maker wins by claiming all edge of some spanning tree of G, or when Breaker wins by claiming all edges of some edge cut. Note that these conditions are mutually exclusive and we can say that Breaker wins if he can prevent Maker from winning.

There are other versions of the game, for example the Hamiltonicity game in which Maker needs to claim a Hamiltonian subgraph to win, or versions played on hypergraphs. In this thesis we only consider the above defined connectivity game. While the game is sometimes studied for simple graphs (no loops or parallel edges), the results we mention generalise naturally for multigraphs and we therefore use them.

One possible line of work studies the following questions. What is the smallest n such that Maker has a winning strategy for K_n in the (a,b) game? For fixed a and n, what is the smallest threshold b_0 such that Breaker has a winning strategy for K_n in the (a,b_0) game? The initial results of this kind can be found in Chvátal and Erdős [1978]. The threshold was asymptotically determined for the $(1,b)$ game by Gebauer and Szabó [2009], and further studied for the general (a,b) game by Hefetz, Mikalački, and Stojaković [2012].

We study the following question.

Question 1.6. *Given a multigraph G, which player wins the game if they both play optimally?*

This turns out to be tightly connected with the number of edge-disjoint spanning trees in G. A complete characterisation for the $(1,1)$ game on matroids is given by Lehman [1964] and Edmonds [1965]. The version for multigraphs builds on work by Nash-Williams [1961], Tutte [1961].

Theorem 1.7. *Maker has a winning strategy for a graph G in the* $(1,1)$ *game if and only if G contains* 2 *edge-disjoint spanning trees.*

It is not difficult to generalise the strategy-stealing argument, and show that if Maker has a winning strategy for a multigraph G in the $(1,b)$ game, then G contains $b+1$ edge-disjoint spanning trees. The converse implication does not hold. As remarked by Chvátal and Erdős [1978], Bondy observed that there are multigraphs G with an arbitrary large number of edge-disjoint spanning trees and yet such that Breaker can win even the $(1,2)$ game. One such example is a long enough path on n vertices with each edge of multiplicity k.

Even less is known for the (a,b) game with $a > 1$. This has been studied for other games by Beck [2008], Gebauer [2012]. While the natural next step is to solve the $(2,2)$ game, for convenience, we study an intermediary version we call $(1/2,1/2)$ game. The game is played on a multigraph G, and Breaker plays first. In each round Breaker first chooses a constant $c \in \{1,2\}$. Then Breaker claims c previously unclaimed edges, and Maker claims c unclaimed edges. Maker wins if the edges she claimed throughout the game form a connected spanning subgraph of G; otherwise, Breaker wins.

In Chapter 4 we investigate for which multigraphs Maker has a winning strategy, but provide no characterisation. Rather, we discuss some directions in which this problem can be considered.

1.3 Notation and preliminaries

A *graph* is defined by a finite set of *vertices* and a set of pairs of vertices called *edges*. We do not allow edges of the form $\{u,v\}$ with $u = v$ (usually called loops). Only in Chapter 4 we allow multiple edges between the same pair of vertices (*parallel edges*), in which case we consider edges to form a multiset, and we then speak of a *multigraph*.

For a given (multi)graph G, we denote by $V(G)$ its vertex set, and by $E(G)$ its edge (multi)set. We say that vertices $v,u \in V(G)$ are *adjacent* or that v is a *neighbour of* u if $\{u,v\} \in E(G)$, and abbreviate this by $uv \in E(G)$. The vertex v is *incident* with $e \in E(G)$ if $v \in e$. The *degree of a vertex* in a (multi)graph is the number of edges incident with the vertex.

In the following, let $G = (V,E)$ and H be graphs, $X,Y \subseteq V$ be subsets of vertices of G, $F \subseteq E$ be a subset of its edges, $u,v \in V$ be two vertices of G and $e \in E$ be an edge of G. For a set S we denote by 2^S the set of all subsets of S, and denote by $\binom{S}{2}$ the set of all subsets of S of size 2.

We denote by $G-e$ the graph created from G by deleting the edge e from its edge set, and refer to this operation as *edge deletion*. We denote by $G-v$ the graph created from G by deleting the vertex v from its vertex set together with all edges $e \in E(G)$ such that $v \in e$. This operation is called a *vertex deletion*. Iteratively, we define $G \setminus X$ to be the graph created from G by deleting each vertex in X, and $G \setminus F$ to be the graph created from G by deleting each edge in F.

The graph H is a *subgraph* of the graph G if H can be obtained from G by a sequence of vertex and edge deletions, and it is an *induced subgraph* of G if H can be obtained form G by a sequence of vertex deletions. The graph G is *H-free* if H is not an induced subgraph of G. We say *there is a (copy of) graph H* in the graph

G if H is a subgraph of G.

The *complete graph* on n vertices, denoted by K_n, is the graph on n vertices where each pair of distinct vertices forms an edge. The *cycle* on $n > 2$ vertices, denoted by C_n, is the graph in which we have a linear order on vertices $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(C_n) = \{v_{i-1}v_i \mid 2 \leq i \leq n\} \cup \{v_n v_1\}$. The *path* on n vertices, denoted by P_n, is a graph in which we have a linear order on its vertices $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $E(P_n) = \{v_{i-1}v_i \mid 2 \leq i \leq n\}$. We call vertices v_1, v_n the *endpoints* of P_n. The *length* of a path and cycle is the number of edges it contains.

We say there is a *path from u to v* in G if there is a path P as a subgraph in G such that u and v are the endpoints of P. The distance of u and v is the length of the shortest path from u to v, 0 for $u = v$, and infinity if there is no such path. The distance of u and X is the minimum length of a path from u to any $v \in X$, and the distance between X and Y is the minimum length of a path from any $u \in X$ to any $v \in Y$. A cycle or path is *odd* if it has odd length, and it is *even* if it has even length. The *girth* of a graph G is the length of the shortest cycle in G, and the *odd girth* of G is the length of the shortest odd cycle in G.

We say that G is *connected* if there is a path from u to v for every $u, v \in V$. We denote by $G[X]$ the graph with vertex set X and edge set $\{e \in E(G) \mid e \in \binom{X}{2}\}$, and say that $G[X]$ is the *graph induced by X*. The set X is *connected* if $G[X]$ is a connected graph, and it is a *(connected) component* of G if it is a maximal (in inclusion) connected set. We say that X is an *independent set* if $G[X]$ has an empty edge set. The set X is a *dominating set* if each vertex either belongs to X or has a neighbour in X. The set X is a *vertex cut set* if $G \setminus X$ is not connected, and F is an *edge cut set* if $G \setminus F$ is not connected. Notice that if G is not connected then every subset of its edges is an edge cut set.

A *contraction of an edge $uv \in E$* is an operation in which we add to G an edge uw for every vertex w adjacent to v where $w \neq u$, and delete the vertex v from the graph. Unless we allow multigraphs in our context we also delete all created parallel edges (multiple edges between the same pair of vertices). The graph H is a *minor* of G if it can be obtained from G by a sequence of vertex and edge deletions and edge contractions. The graph G is *H-minor-free* if H is not a minor of G.

We denote by G/X the graph obtained from G by contracting all edges in $E(G[X])$.

If X is connected, then we call this operation *contracting X into a single vertex*. We can alternatively say that H is a *minor* of G if there is a *model* of H in G, that is, an assignment $m : V(H) \to 2^{V(G)}$ satisfying that for all $u, v \in V(H)$ such that $u \neq v$: (1) $m(v)$ is connected, (2) sets $m(u)$ and $m(v)$ are disjoint and (3) if $uv \in E(H)$, then there are $u' \in m(u), v' \in m(v)$ such that $u'v' \in E(G)$. In other words H is a subgraph of a graph created from G by contracting each $m(v)$ to a single vertex.

We denote by $\alpha(G)$ the *independence number* of G, that is, the maximum size of an independent set in G. We denote by $\omega(G)$ the *clique number* of G, that is, the maximum size of a set X such that $G[X]$ is a complete graph.

A *partial (proper vertex) colouring* of G is an assignment of *colours* to some vertices of G such that each pair of vertices u, v that are assigned the same colour satisfies $uv \notin E$. A *(proper vertex) colouring* of G is a partial colouring that colours all vertices of G. We say that G is k-colourable if there is a colouring of G that uses at most k distinct colours. We denote by $\chi(G)$ the *chromatic number* of G, that is, the minimum k such that G is k-colourable.

A *tree* is a connected graph without any cycle. A *spanning tree* of G is a subgraph of G that is a tree and contains all vertices of G. If G is not connected, it has no spanning tree. The graph G has k edge-disjoint spanning trees if there are k spanning trees $T_1, \ldots, T_k$ in G such that each $e \in E(G)$ belongs to at most one T_i for $1 \leq i \leq k$.

2

Colouring graphs without large clique minors

Let G be a K_{t+1}-minor-free graph and H be an induced subgraph of G which can be properly vertex coloured with t colours. We are interested in the maximum number of vertices of such graph H; or, in other words, the maximum fraction of vertices of G that induce a t-colourable graph.

We first investigate a result of Duchet and Meyniel [1982] that establishes a relationship between t and the size of the maximum independent set in G. We then show two different lower bounds for the size of H, both based on induction, and combine them together. Lastly, we use that Hadwiger's conjecture holds for $t < 6$, and use yet another inductive argument to get the best bound of this chapter.

2.1 Introducing β_t

Definition 2.1. *For $t \geq 1$ let β_t be the maximum positive real number such that for every K_{t+1}-minor-free graph G there exists a t-colourable induced subgraph H of G satisfying $|V(H)| \geq \beta_t |V(G)|$.*

For any fixed $t \geq 1$ and $n \geq 1$ the candidate for β_t comes from a graph on n vertices that allows a t-colourable subgraph on the minimum number of vertices. The minimum over these candidates over all values of n is either attained for a fixed n_0, or in the limit. In both cases β_t is well defined.

While Hadwiger [1943] conjectured that the answer is $\beta_t = 1$ for all $t \geq 1$, the currently best known bound for general t comes from the proof technique used in the proof of Theorem 1.3. This was pointed out in the survey by Seymour [2016] and we include the proof for completeness.

Lemma 2.2. *Every connected graph $G = (V,E)$ contains a connected dominating set A such that $\alpha(G[A]) > \frac{|A|}{2}$.*

Proof. We describe how to find sets $A_0, \ldots, A_l = A$ such that all sets are con-

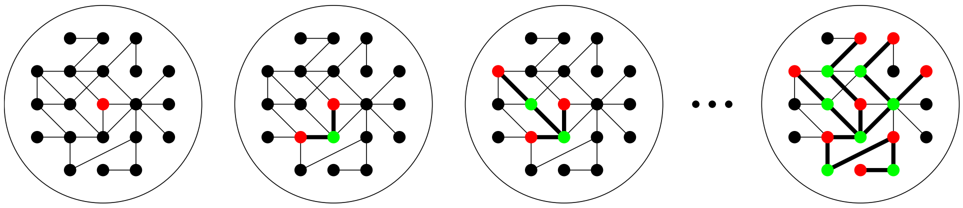

Figure 2.1: An example of step-by-step construction of set A. Red vertices form an independent set and bold edges maintain connectivity.

nected, satisfy $\alpha(G[A_i]) > |A_i|/2$ and A is dominating in G. We start by setting $A_0 := \{w\}$ for any $w \in V$. This set A_0 is connected and $\alpha(G[A_0]) = |A_0|$.

Consider a set A_i for $i \geq 0$. If there exists a pair of vertices $u, v \in V$ such that $uv \in E$, u is adjacent to a vertex in A_i and v is not adjacent to any vertex in A_i, we let $A_{i+1} := A_i \cup \{u,v\}$. Set A_{i+1} is connected. Because v is not adjacent to any vertex in A_i, we have

$$\alpha(G[A_{i+1}]) = 1 + \alpha(G[A_i]) > 1 + \frac{|A_i|}{2} = \frac{|A_{i+1}|}{2}.$$

See Figure 2.1 for a step-by-step construction. If there is no such pair u and v, then A_i is a dominating set we were looking for. □

In the paper by Duchet and Meyniel [1982] a lower bound for the size of A is also given, but we do not need it in our proofs, and so we omit it. Using the lemma it is now easy to show the following bound:

Theorem 2.3. *Every K_{t+1}-minor-free graph G admits a t-colourable induced subgraph H such that $|V(H)| > \frac{|V(G)|}{2}$.*

Proof. Let us first assume G is connected. We proceed by induction on t. For $t = 1$ the graph G has no edges and it can be coloured with 1 colour.

For $t > 1$, Lemma 2.2 gives us a connected dominating set A in G such that $\alpha(G[A]) > \frac{|A|}{2}$. Assume $G \setminus A$ contains a minor of K_t. Because A dominates G, every vertex in $G \setminus A$ is adjacent to some vertex in A. We can contract A to a single vertex and get a K_{t+1} minor in G, which is a contradiction. Therefore, $G \setminus A$ is K_t-minor-free. By the induction hypothesis it contains a $(t-1)$-colourable induced subgraph H' satisfying $|V(H')| > \frac{|V(G-A)|}{2} = \frac{|V(G)|}{2} - \frac{|A|}{2}$. Let H be the subgraph of G induced by the vertices in H' and vertices in the independent set of maximum size in $G[A]$. Since H' is $(t-1)$-colourable, H is t-colourable. We have

$$|V(H)| > |V(H')| + \frac{|A|}{2} > \frac{|V(G)|}{2} - \frac{|A|}{2} + \frac{|A|}{2} = \frac{|V(G)|}{2}.$$

If G is not connected, we find a t-colourable induced subgraph for each connected component, and let H be their union. □

The same bound can also be achieved using the decomposition we introduce in Chapter 3.

Since Hadwiger's conjecture is known for small t, we have $\beta_t = 1$ for $t \le 5$. Theorem 2.3 then implies

$$\beta_t \ge \begin{cases} 1 & 1 \le t \le 5 \\ 1/2 & 5 < t. \end{cases}$$

2.2 Improving bounds for β_t

Consider a K_{t+1}-minor-free graph $G = (V, E)$. By Lemma 2.2 there is a connected dominating set A of G such that $\alpha(G[A]) > \frac{|A|}{2}$. We denote any fixed independent set of maximum size in $G[A]$ by A^I, and consider 2 methods of partially colouring G with t colours.

2.2.1 Induction on t

The first method completely follows the induction in the proof of Theorem 2.3. Since A is a connected dominating set, $G \setminus A$ is K_t-minor-free and we can colour

β_{t-1} fraction of its vertices using $t-1$ colours. Putting one extra colour on A^I, the number of vertices we colour with t colours is at least $\beta_{t-1}(|V|-|A|)+|A^I|$. With Lemma 2.2 we have

$$\beta_t|V| \geq \beta_{t-1}(|V|-|A|)+|A^I| > \beta_{t-1}|V|+|A^I|(1-2\beta_{t-1}). \tag{2.1}$$

2.2.2 Induction on $|V|$

The graph $G\setminus A^I$ remains K_{t+1}-minor-free and its vertex set is smaller so we can use the induction hypothesis to obtain a t-colouring of a β_t fraction of its vertices. By the pigeon hole principle one of the colours is used on at most $\frac{\beta_t(|V|-|A^I|)}{t}$ vertices. We alter the colouring by using the least used colour on A^I instead. This gives a partial t-colouring of at least $\frac{t-1}{t}\beta_t(|V|-|A^I|)+|A^I|$ vertices of G. So we have

$$\beta_t|V| \geq \frac{t-1}{t}\beta_t(|V|-|A^I|)+|A^I|. \tag{2.2}$$

2.2.3 The best of both worlds

It holds that $\beta_t \geq 0.5$ and $\beta_t \geq \beta_{t+1}$ for all $t \geq 1$, so following the induction on t gives a good bound when A^I has a small size while the induction on $|V|$ works well when A^I is big.

Let $1 \geq \gamma_t \geq 0.5$ for $t \geq 1$ be defined as the minimum solution of recurrences 2.1, 2.2 and $\gamma_1 = 1$, that is

$$\gamma_t|V| = \max\left(\beta_{t-1}|V|+|A^I|(1-2\beta_{t-1}), \frac{t-1}{t}\beta_t(|V|-|A^I|)+|A^I|\right) \text{ for } t>1,$$

$$\gamma_1 = 1.$$

Clearly, we have $\beta_t \geq \gamma_t$ for all values of t. Let us find the size of A^I for which both methods intersect:

$$\gamma_{t-1}|V|+|A^I|(1-2\gamma_{t-1}) = \frac{t-1}{t}\gamma_t(|V|-|A^I|)+|A^I|$$

$$|A^I| = |V|\frac{t\gamma_{t-1}-t\gamma_t+\gamma_t}{2t\gamma_{t-1}-t\gamma_t+\gamma_t}.$$

We can substitute for $|A^I|$ to Inequality 2.1:

$$\gamma_t|V| \geq |V| \left(\gamma_{t-1} + \frac{t\gamma_{t-1} - t\gamma_t + \gamma_t}{2t\gamma_{t-1} - t\gamma_t + \gamma_t}(1 - 2\gamma_{t-1}) \right)$$

$$\gamma_t^2(1-t) + \gamma_t(t\gamma_{t-1} + \gamma_{t-1} + t - 1) - t\gamma_{t-1} \geq 0$$

We find the discriminant $D = \gamma_{t-1}^2(t^2+2t+1) + \gamma_{t-1}(-2t^2+4t-2) + t^2 - 2t + 1$ and have

$$\gamma_t = \frac{t\gamma_{t-1} + \gamma_{t-1} + t - 1 \pm \sqrt{D}}{2(t-1)} = \frac{1}{2} + \gamma_{t-1}\frac{t+1}{2(t-1)} \pm \frac{\sqrt{D}}{2(t-1)}. \tag{2.3}$$

Using induction on t, we show $\gamma_t > 0.5$ for all $t \geq 1$. For $t = 1$ we have $\gamma_1 = 1$. Let $1 \geq \gamma_{t-1} > 0.5$ and consider the middle term of the discriminant above.

$$\gamma_{t-1}(-2t^2+4t-2) < \frac{1}{2}(-2t^2+4t-2)$$

$$D < \gamma_{t-1}^2(t^2+2t+1) - t^2 + 2t - 1 + t^2 - 2t + 1 = \gamma_{t-1}^2(t+1)^2$$

$$\gamma_{t-1}\frac{t+1}{2(t-1)} \geq \frac{\sqrt{\gamma_{t-1}^2(t+1)^2}}{2(t-1)} > \frac{\sqrt{D}}{2(t-1)}$$

Finally, we substitute in 2.3 and get

$$\gamma_t = \frac{1}{2} + \gamma_{t-1}\frac{t+1}{2(t-1)} \pm \frac{\sqrt{D}}{2(t-1)} > \frac{1}{2}.$$

Some numeric values of γ_t can be found in Table 2.1. Even though the lower bounds on $\beta_t \geq \gamma_t$ produced by this approach can be further analysed, we omit this as the following approach gives a better bound for all $t > 10$.

2.3 Improving bounds for β_t differently, mostly better

We first prove a stronger version of Theorem 2.3

Theorem 2.4. *For every $t \geq 4$ and a K_{t+1}-minor-free graph G, there is a $(t-2)$-colourable induced subgraph H such that $|V(H)| \geq \frac{|V(G)|}{2}$.*

Proof. We proceed by induction on t. For $t = 4$ Hadwiger's conjecture is true and so there is a 4-colouring of every K_5-minor-free graph. If we only consider the 2 most frequent colours, we have a partial 2-colouring of at least half of $V(G)$. For $t > 4$, Lemma 2.2 gives us a connected dominating set A of G such that $\alpha(G[A]) > \frac{|A|}{2}$. We use the induction hypothesis on a K_t-minor-free $G \setminus A$ to obtain a $(t-3)$-colouring of $\frac{|V(G)|-|A|}{2}$ of its vertices. We extend this partial colouring in G by using a single extra colour on the maximum independent set in A to obtain a partial $(t-2)$-colouring of at least $\frac{|V(G)|-|A|}{2} + \frac{|A|}{2} = \frac{|V(G)|}{2}$ vertices of G and let H be induced by coloured vertices. □

Now that we only need $t-2$ colours to cover half of the graph, we can use two extra colours on the two largest independent sets to get a good bound.

Theorem 2.5. *For every $t \geq 4$ and a K_{t+1}-minor-free graph G on n vertices, there is a t-colourable induced subgraph H such that*

$$|V(H)| \geq \left(\frac{1}{2} + \frac{1}{(2-c)t} - \frac{1}{2(2-c)^2t^2}\right)n \quad \textit{where} \quad c = \frac{80-\sqrt{5392}}{126}.$$

Proof. Let I be an independent set of maximum size in G and let J be an independent set of maximum size in $G - I$. By Theorem 1.4 we have $|I| \geq \frac{n}{(2-c)t}$ and $|J| \geq \frac{n-|I|}{(2-c)t}$ where $c = \frac{80-\sqrt{5392}}{126}$. From Theorem 2.4 there is a $(t-2)$-colourable induced subgraph H' of $(G-I)-J$ on at least half of its vertices. We let H be a subgraph of G induced by vertices in H' and vertices in sets I and J. Clearly H is a t-colourable induced subgraph of G. We have

$$|V(H)| = |V(H')| + |I| + |J| \geq \frac{n-|I|-|J|}{2} + |I| + |J| = \frac{n+|I|+|J|}{2} \geq$$

$$\geq \frac{n+|I|}{2} + \frac{n-|I|}{2(2-c)t} = n\left(\frac{1}{2} + \frac{1}{2(2-c)t}\right) + |I|\left(1 - \left(\frac{1}{2} + \frac{1}{2(2-c)t}\right)\right) \geq$$

$$\geq n\left(\frac{1}{2} + \frac{1}{2(2-c)t}\right) + \frac{n}{(2-c)t}\left(\frac{1}{2} - \frac{1}{2(2-c)t}\right).$$

Graph H has at least $n\left(\frac{1}{2} + \frac{1}{(2-c)t} - \frac{1}{2(2-c)^2t^2}\right)$ vertices and we are done. □

See Table 2.1 for some approximate values of lower bounds for β_t produced by both discussed approaches.

t	First approach	Second approach
6	0.710102051443	0.581903427145
7	0.628353792429	0.570651174471
8	0.589311750665	0.562113933139
9	0.566651761211	0.555415751763
10	0.552027207871	0.550020600646
11	0.541922961806	0.545582183773
100	0.500471669149	0.505120663553
1000	0.500004750282	0.500513252390
10000	0.500000047545	0.500051337099
100000	0.500000000476	0.500005133829
1000000	0.500000000012	0.500000513384

Table 2.1: A table with lower bounds on β_t for some values of t produced by both approaches

2.4 Conclusion

We showed several lower bounds for the maximum number of vertices in a t-colourable induced subgraph of a K_{t+1}-minor-free graph on n vertices. In particular we showed β_t satisfies

$$\beta_t \geq \begin{cases} 1 & 0 < t \leq 5 \\ \dfrac{1}{2} + \beta_{t-1}\dfrac{t+1}{2(t-1)} - \dfrac{\sqrt{D}}{2(t-1)} & 5 < t \leq 10 \\ \dfrac{1}{2} + \dfrac{1}{(2-c)t} - \dfrac{1}{2(2-c)^2t^2} & 10 < t, \end{cases}$$

for $D = \beta_{t-1}^2(t^2+2t+1) + \beta_{t-1}(-2t^2+4t-2) + t^2 - 2t + 1$ and $c = \dfrac{80-\sqrt{5392}}{126}$.
As t grows, unfortunately the bound decreases to 0.5. While it may be too ambitious to proof $\beta_t = 1$ for all $t \geq 1$, could the following open questions be answered?

Question 2.6. *Is there $\delta > 0$ such that for all $t \geq 1$ it holds $\beta_t \geq 0.5 + \delta$?*

Question 2.7. *Can we prove better bounds for β_t for small $t > 5$?*

3

Colouring graphs without large clique minors and large (odd) girth

Let G be a K_{t+1}-minor-free graph with large odd girth g, and let H be an induced subgraph of G that is colourable with t colours. Like in Chapter 2 we want to maximize the number of vertices of H.

The additional condition that G has high odd girth g means that there is no induced odd cycle in G on less than g vertices. To avoid rounding errors, we always consider g to be an odd number. We will see that forbidding short odd cycles already helps a lot. If we also forbid short even cycles, we get even stronger results, but as this is quite a strong condition, we leave it as a separate case in our statements.

We state the results in this chapter in a very general way. Given constants c, g, t and a K_{t+1}-minor-free graph G having (odd) girth g, we search for a maximum sized induced subgraph that can be coloured with $(t+c)$ colours. We show:

Theorem 3.6. *For every $c \geq 0$ and every K_{t+1}-minor-free graph G with odd girth $g = 2q+1 > 3$, there is a $(t+c)$-colourable induced subgraph H of G such that*

$$|V(H)| > \left(1 - \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor \frac{g+1}{8} \right\rfloor + \left\lfloor \frac{g+3}{8} \right\rfloor\right)\right)^{-1}\right)|V(G)|.$$

Moreover, if G has girth g, there is a $(t+c)$-colourable induced subgraph H of G

such that

$$|V(H)| > \frac{(c+2)^{(g-3)/2}-1}{(c+2)^{(g-3)/2}}|V(G)|.$$

Even though we want to only use t colours, we use a parameter $c \geq 0$ throughout the chapter to quantify the number of extra colours we are allowed to use in addition to t. For simplicity, we can imagine $c = 0$, in which case we really only colour vertices with t colours. At the end of the chapter, we use this parameter c to show that with big enough t, we can colour all but $\delta|V(G)|$ vertices of G with $(1+\varepsilon)t$ colours, for any $\delta > 0$ and $\varepsilon > 0$.

We follow a graph decomposition approach used by Reed and Seymour [1998] to show that every K_{t+1}-minor-free graph has fractional chromatic number at most $2t$. Later, van den Heuvel and Wood [2018] also used it to show that every K_{t+1}-minor-free graph has clustered chromatic number $2t$ and defective chromatic number t.

Our aim is to find a decomposition of our graph into disjoint sets we call *blobs*. Our decomposition has many useful properties both for the structure of each blob, and for the way blobs *touch* each other.

Definition 3.1. *For a graph $G = (V,E)$ let $U,W \subseteq V$ be disjoint subsets of its vertices and $u \in V$ a vertex such that $u \notin W$. We say that u* touches *W if there is a vertex $v \in W$ such that $uv \in E$. More generally, U* touches *W if there is a vertex $u \in U$ that touches W.*

Each blob consists of three parts, where two of them form an independent set and can be coloured by a single colour each. The third part does not have this property, but we keep its size small.

The decomposition orders the blobs is such a way that every blob touches at most $t-1$ blobs that precede it. Since we can use two colours to colour almost everything in each blob, a greedy algorithm (use lowest unused colour in the given order of blobs) gives us immediately a $2t$-colouring of most of the graph. We instead use a bit more complicated way to colour the vertices, which yields a colouring of all these vertices with only t colours.

3.1 Definitions and notation

Definition 3.2. *A* blob *B in a graph G is a connected subset of vertices of G which consists of three disjoint sets B^C (*covered*), B^I (*independent*), B^W (*wasted*) such that B^C and B^I are independent sets.*

Note that every connected subset of vertices of a graph is a blob, as we can always place all vertices in B^W. While a path on 4 vertices is certainly a blob, there are many ways to split its vertices among B^C, B^I and B^W. In the rest of the chapter we will always try to keep B^W small.

As said, our final goal is to colour G with $t+c$ colours. Clearly, if there is any vertex v in G which is adjacent to less than $t+c$ other vertices, we can ignore it, colour all other vertices first, and then greedily colour v. In our decomposition we need to keep these low-degree vertices aside, which we do in a set S (vertices Safe for final colouring). Moreover, we also place in S vertices that do not necessarily have low degree, as long as they can also be ignored and coloured when all of their neighbours are already coloured.

Definition 3.3. *Let $B_1, B_2, \dots, B_l$ be disjoint blobs in a graph G and $v \in V(G)$ be a vertex such that $v \notin B_i$ for $1 \le i \le l$. Let I_v be the number of blobs B_i such that v touches B_i^I, and let N_v be the number of neighbours of v in $G \setminus \bigcup_{i=1}^{l}(B_i^I \cup B_i^W)$. We say that v is $(t+c)$-*safe *for G and blobs $B_1, \dots, B_l$ if $I_v + N_v < t+c$.*

In other words, if we consider a partial $(t+c)$-colouring of G, in which all vertices in B_i^I are coloured by a single colour for each $1 \le i \le l$ then a $(t+c)$-safe vertex neighbours at most $t+c-1$ vertices of distinct colours and so it can also be coloured with one of the $t+c$ colours.

3.2 The decomposition

We first state a lemma that we use when building a decomposition into blobs.

Lemma 3.4. *For every connected graph G with odd girth $g = 2q+1 \ge 5$, every vertex $v \in V(G)$, every $c \ge 0$ and every $0 < d < \dfrac{g-1}{2}$, there are disjoint sets $A, B \subseteq V(G)$ such that*

1. *A and B are independent sets,*
2. *$G[A \cup B]$ is connected,*
3. *every vertex $u \in V(G)$ at distance at most d from v belongs to A or B, and*
4. *B does not touch $V(G \setminus A)$.*

Moreover, if there exists a vertex at distance d from v, and all vertices at distance less than d from v have degree at least $c+3$, then

$$|A \cup B| \geq d+2+(c+1)\left(\left\lfloor \frac{d+2}{4} \right\rfloor + \left\lfloor \frac{d+3}{4} \right\rfloor\right).$$

If in addition to all above G has girth g, then $|A \cup B| \geq (c+2)^d$.

Proof. We say that a vertex u is at *odd distance* from v if the distance between the vertices is an odd number, and it is at *even distance* from v if the distance between them is an even number. Every vertex is at even distance 0 from itself.

If d is odd, let A be the set of all vertices in G at odd distance at most d from v, and let B be the set of all vertices in G at even distance at most d from v. Otherwise, if d is even, we let A be the set of all vertices in G at even distance at most d from v, and let B be the set of all vertices in G at odd distance at most d from v.

For contradiction with Property 1 assume there is an edge $uu' \in E(G[A])$. Clearly, u and u' are at the same distance $0 < d' \leq d$ from v. Consider any fixed paths of minimum length from u to v and from u' to v. Figure 3.1 shows that together with uu' they give us an odd cycle of length at most $2d'+1 \leq 2d+1 < g$, which is a contradiction with G having odd girth g. Hence, A is independent, and by the same argument B is also an independent set.

The graph $G[A \cup B]$ consists exactly of vertices at distance at most d from v, and so it is connected. Only vertices at distance d from v can have neighbours outside $A \cup B$, and because all these belong to A, B has no neighbours in $V(G \setminus A)$.

Assume there is a vertex u at distance d from v and that every vertex at distance less than d from v has degree at least $c+3$. The graph $G[A \cup B]$ contains a path P of length $d+1$ between u and v. Because v has $c+3$ neighbours, there are $c+2$ more vertices in $G[A \cup B]$ that do not belong to P. Each vertex on P at distance

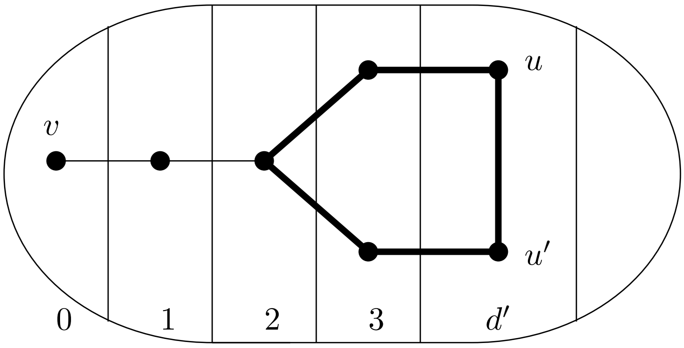

Figure 3.1: An odd cycle of length less than g when A is not independent.

$1 \leq 4k < d$ or $1 \leq 4k+1 < d$ from v has $c+1$ more neighbours outside P. In total

$$|V(G[A \cup B])| \geq d+1+(c+1)\left\lfloor \frac{d+2}{4} \right\rfloor + (c+1)\left\lfloor \frac{d+3}{4} \right\rfloor + 1 =$$

$$= d+2+(c+1)\left(\left\lfloor \frac{d+2}{4} \right\rfloor + \left\lfloor \frac{d+3}{4} \right\rfloor\right).$$

See Figure 3.2 for the case $c = 0$.

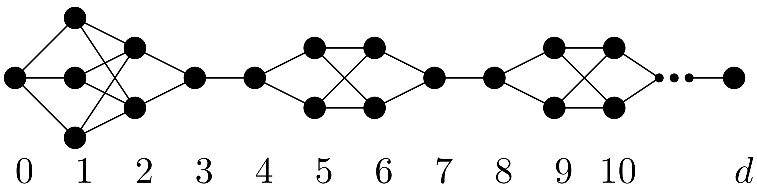

Figure 3.2: An example of $G[A \cup B]$ with the minimum number of vertices.

Assume additionally that G has girth g. Because $G[A \cup B]$ contains no cycles, it forms a tree. Vertex v has $c+3$ neighbours, each of them has $c+2$ more neighbours and so on. Therefore,

$$|V(G[A \cup B])| \geq 1 + \sum_{i=0}^{d-1} (c+3)(c+2)^i = 1 + (c+3)\frac{(c+2)^d - 1}{c+1} \geq (c+2)^d.$$

□

Next, we construct a decomposition into blobs.

Theorem 3.5. *For every $c \geq 0$ and every K_{t+1}-minor-free graph G with odd girth $g = 2q+1 > 3$, there exists a decomposition into a sequence of disjoint blobs $B_1, \ldots, B_b$ and an ordered set S with the following properties:*

1. *For every* $1 \leq i \leq |S|$ *and* $s_i \in S$*, there exists* $b' \leq b$ *such that* s_i *is* $(t+c)$*-safe for* $G \setminus \bigcup_{j=1}^{i-1} \{s_j\}$ *and* $B_1, \ldots, B_{b'}$*.*

2. *For every* $1 \leq i < j < k \leq b$*, if* B_k *touches both* B_i *and* B_j *then* B_i *touches* B_j*.*

3. *For every* $1 \leq i, j \leq b$*,* B_i^C *does not touch* B_j^C*.*

4. *For every* $1 \leq i \leq b$*, every vertex in* B_i^C *touches less than* t *blobs from* $B_1, \ldots, B_b$*.*

5. *For every* $1 \leq i \leq b$*,* $\dfrac{|B_i^W|}{|B_i|} < \left(\dfrac{g+3}{2} + (c+1) \left(\left\lfloor \dfrac{g+1}{8} \right\rfloor + \left\lfloor \dfrac{g+3}{8} \right\rfloor \right) \right)^{-1}$*.*

Moreover, if G *has girth* g*, then for every* $1 \leq i \leq b$*,* $|B_i^W|/|B_i| < (c+2)^{-(g-3)/2}$*.*

Proof. We iteratively extend a partial decomposition $B_1, \ldots, B_l, S$ by one of four operations. We apply the operations in order, that is, we attempt to apply the second operation only if the first one cannot be used. At the end we show that if none of the operations can be used, then our decomposition covers the whole graph.

We begin by setting

$$l := 1,\ B_1^I := \{w\},\ B_1^C := \emptyset,\ B_1^W := \emptyset,\ S := \emptyset,$$

for some vertex $w \in V(G)$.

Consider any partial decomposition $B_1, \ldots, B_l, S$. We maintain *the Invariant* that every maximal connected component C of $G \setminus \left(S \cup \bigcup_{i=1}^{l} B_i \right)$ satisfies the following. If C touches blobs B_i, B_j for $1 \leq i < j \leq l$ then B_i touches B_j. It follows directly that C touches at most $t-1$ blobs; otherwise, C and the blobs touched by C induce a model of K_{t+1}, which is a contradiction with G being K_{t+1}-minor-free. Let us refer to the blobs among $B_1, \ldots, B_l$ touched by C simply by *C-touched blobs*. Clearly, *the Invariant* is satisfied for $l = 1$ because C touches the only blob B_1.

For the rest of the proof, let C be any fixed maximal connected component of $G \setminus \left(S \cup \bigcup_{i=1}^{l} B_i \right)$.

I. Put safe vertices in S

Firstly, if there exists a vertex in C that is $(t+c)$-safe for $G \setminus S$ and $B_1, \ldots, B_l$, we add it in S. Notice that by placing some vertex v_1 in S, and creating $S' = S \cup \{v_1\}$, it may happen that another vertex v_2 becomes $(t+c)$-safe for $G \setminus S'$ and $B_1, \dots, B_l$, even if v_2 is not $(t+c)$-safe for $G \setminus S$ and $B_1, \dots, B_l$.

Because we place vertices to S only when they are $(t+c)$-safe for the current partial decomposition, Property 1 follows trivially. None of the remaining conditions or *the Invariant* are concerned with S.

II. Create a trivial blob

If there is a vertex $v \in C$ that touches all C-touched blobs, we create a new blob

$$B^I_{l+1} := \{v\},\ B^C_{l+1} := \emptyset,\ B^W_{l+1} := \emptyset.$$

Consider a maximal connected component C' of $G' := G \setminus \left(S \cup \bigcup_{i=1}^{l+1} B_i\right)$. If C' is disjoint from C, *the Invariant* for C' remains unchanged. Otherwise, we have $C' \subseteq C$. From $v \in C \setminus C'$ it follows that C' is a proper subset of C. We need to check that for each $1 \leq i \leq l$, if C' touches both B_i and B_{l+1}, then B_i touches B_{l+1}. This is true because C' only touches C-touched blobs and the new blob B_{l+1}, while $B_{l+1} = \{v\}$ touches all C-touched blobs. Property 2 follows directly from *the Invariant* for C. Properties 3,4 and 5 are trivial because both B^C_{l+1} and B^W_{l+1} are empty.

III. Create a non-trivial blob

For a vertex $v \in C$ and a blob B_i touched by C let the C-distance from v to B_i be the minimum length of a path from v to any vertex in B_i which only uses vertices in C as their inner points (vertices that are not end-points of the path). Since C is connected, C-distance from any vertex in C to any C-touched blob is a finite number.

If there is a vertex $v \in C$ at C-distance at most $d := \frac{g-1}{2}$ from each C-touched blob, we again create a new blob B_{l+1}. Let A, B be the sets from Lemma 3.4 used with $G[C], g, v, c, d-1$. We set

$$B^I_{l+1} := A,\ B^C_{l+1} := B,\ B^W_{l+1} := \emptyset.$$

Because B_{l+1} touches each C-touched blob, *the Invariant* and Property 2 follow by the same arguments as in the previous case. Because of properties of B, no vertex $v \in B^C_{l+1}$ can touch any later created or extended blob, and Property 3 holds. If some vertex of B^C_{l+1} touches all C-touched blobs, we create a trivial blob instead; otherwise, it touches at most $t-2$ C-touched blobs and B_{l+1}, so Property 4 is satisfied. Properties 5 is still trivial, as B^W_{l+1} is empty.

IV. Extend a blob

We can assume no vertex of C is at C-distance at most d from each C-touched blob. Let u be any vertex of C at C-distance at least 2 and less than g from B_i for some $1 \leq i \leq l$, and let P be the shortest path in $G[C \cup B_i]$ between u and any vertex in B_i. If u touches $t-2$ of the remaining C-touched blobs besides B_i, then the middle vertex on P is at distance at most d from all C-touched blobs, which is a contradiction. So any such vertex u touches at most $t-3$ C-touched blobs, and because it is not $(t+c)$-safe, it has at least $c+3$ neighbours in C.

Let v be a vertex at C-distance exactly $d+1$ from B_i for some $1 \leq i \leq l$, and let A, B be the sets from Lemma 3.4 used with $G[C], g, v, c, d-1$. We replace blob B_i with $\bar{B}_i$

$$\bar{B}^I_i := B^I_i \cup A,\ \bar{B}^C_i := B^C_i \cup B,\ \bar{B}^W_i := B^W_i \cup \{w\},$$

where w is any vertex of C at C-distance d from v that touches B_i.

Consider a vertex $s_k \in S$ that is $(t+c)$-safe for $G \setminus \bigcup_{j=1}^{k-1} s_j$ and $B_1, \ldots, B_i, \ldots, B_l$. It is possible that some of its neighbours in C are moved to $\bar{B}$, but the number of its neighbours cannot increase, so s_k remains $(t+c)$-safe for $G \setminus \bigcup_{j=1}^{k-1} s_j$ and $B_1, \ldots, \bar{B}_i, \ldots, B_l$, and Propery 1 holds.

Notice that $\bar{B}_i$ and B_i touch the same blobs in $B_1, \ldots, B_l$, because of *the Invariant* for C. This implies that both *the Invariant* and Property 2 are satisfied. Because of properties of B, no vertex $v \in \bar{B}^C_i$ can touch any later created or extended blob, and Property 3 holds. Every vertex in $\bar{B}^C_i \setminus B^C_i$ only touches the C-touched blobs, and we have at most $t-1$ of these, so Property 4 is satisfied.

Consider Property 5. Every time we extend blob B_i, we have $|\bar{B}^W_i| = |B^W_i| + 1$ and from Lemma 3.4

$$|\bar{B}_i| = |B_i| + |A \cup B| + 1 \geq |B_i| + d + 2 + (c+1)\left(\left\lfloor \frac{d+1}{4} \right\rfloor + \left\lfloor \frac{d+2}{4} \right\rfloor\right) =$$

$$= |B_i| + \frac{g+3}{2} + (c+1)\left(\left\lfloor \frac{g+1}{8} \right\rfloor + \left\lfloor \frac{g+3}{8} \right\rfloor\right).$$

Let us consider how $\bar{B}_i$ comes to existence. It is first created, at which point we have no vertices in ${\bar{B}_i}^W$. Then it is extended x times. Over these x extensions, we add x vertices to ${\bar{B}_i}^W$ and at least $x(|A \cup B|+1)$ vertices to the whole $\bar{B}_i$. So the ratio between the wasted part and total number of vertices in the blob always stays below $(|A \cup B|+1)^{-1}$. As a result:

$$\frac{|\bar{B}_i^W|}{|\bar{B}_i|} < \frac{1}{|A \cup B|+1} \leq \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor\frac{g+1}{8}\right\rfloor + \left\lfloor\frac{g+3}{8}\right\rfloor\right)\right)^{-1}$$

Now assume G has girth g. According to Lemma 3.4 we have

$$|A \cup B|+1 \geq (c+2)^{d-1} \text{ and } \frac{|\bar{B}_i^W|}{|\bar{B}_i|} < \frac{1}{(c+2)^{(g-3)/2}}.$$

It remains to show that if none of the four opearations can be performed then we have a full decomposition. But this is straightforward, because if there still exists any maximal connected component C of $G \setminus \left(S \cup \bigcup_{i=1}^{l} B_i\right)$ then there is a vertex $v \in C$ at C-distance at most d from all C-touched blobs, or all vertices are at C-distance at least $d+1$ from some C-touched blob. In the former case we create a non-trivial blob. In the latter case we consider any vertex v and blob B_i that is at C-distance at least $d+1$ from v. Let p be a shortest path in $C \cup B_i$ connecting v with B_i and let w be the vertex on p at C-distance $t+1$ from B_i. Then we can extend blob B_i using w. □

For $g=3$, the odd girth imposes no condition, but we also cannot use the quantitative part of Lemma 3.4, and we only get $\frac{|B_i^W|}{|B_I|} \leq \frac{1}{2}$. This leads to the same result as in Theorem 2.3.

The bound we get on the ratio between the sizes of B_i^W and B_i is much better if G has large girth. In that case Lemma 3.4 gives sets A, B of size exponential in g, which is a huge improvement over the linear bound we get for graph with large odd girth but low girth. We point out here that an even stronger result holds once C_4 is forbidden as an induced subgraph. Kühn and Osthus [2003] showed that all C_4-free graphs with large enough chromatic number satisfy Hadwiger's conjecture.

3.3 Colouring

We use the decomposition from Theorem 3.5 to properly vertex colour a big induced subgraph of our graph G with $t+c$ colours.

Theorem 3.6. *For every $c \geq 0$ and every K_{t+1}-minor-free graph G with odd girth $g = 2q+1 > 3$, there is a $(t+c)$-colourable induced subgraph H of G such that*

$$|V(H)| > \left(1 - \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor \frac{g+1}{8} \right\rfloor + \left\lfloor \frac{g+3}{8} \right\rfloor\right)\right)^{-1}\right)|V(G)|.$$

Moreover, if G has girth g, there is a $(t+c)$-colourable induced subgraph H of G such that

$$|V(H)| > \frac{(c+2)^{(g-3)/2} - 1}{(c+2)^{(g-3)/2}}|V(G)|.$$

Proof. From Theorem 3.5, we have a decomposition of G into a sequence of blobs $B_1, \ldots, B_b$ and a set S.

In order of increasing $1 \leq i \leq b$ we first colour the independent sets B_i^I, using one colour for each independent set. Assume that for some $1 \leq l \leq b$, the sets B_i^I for $1 \leq i < l$ are coloured and we want to colour B_l^I. By Property 2 of the decomposition, all preceding blobs touched by B_l touch each other, and because G is K_{t+1}-minor-free, B_l^I touches at most $t-1$ of the blobs. Each of them is coloured by a single colour, so there is a remaining colour for B_{l+1}^I. We only need t colours in this step.

Next, we colour the independent sets B_i^C, which can be coloured independently, thanks to Property 3. For all $1 \leq i \leq b$, each vertex $v \in B_i^C$ touches at most $t-1$ sets from $B_1^I, \ldots, B_b^I$ by Property 4, each of those is coloured by a single colour, so we can use a remaining colour to also colour v. We still only need t colours.

Last, we colour all vertices in $S = \{s_1, \ldots, s_{|S|}\}$ in the reverse order of the one in which we added them. From Property 1, for every $1 \leq i \leq |S|$ vertex $s_i \in S$ is $(t+c)$-safe for $G \setminus S'$ and $B_1, \ldots, B_l$ for some $0 < l \leq b$ and $S' = \{s_j \mid j < i\}$. Assume s_i touches I_{s_i} of sets $B_1^I, \ldots, B_l^I$. Then it also touches less than $t+c-I_{s_i}$ vertices in $G \setminus \bigcup_{j=1}^{l}(B_j^I \cup B_j^W)$. Therefore, it sees less than $t+c$ colours on its neighbours and there is at least 1 more colour that we can use.

We use $t+c$ colours to colour all vertices of G except those that belong to B_i^W for some $1 \leq i \leq b$. It remains to show that the graph H induced by coloured vertices has many vertices. We have

$$|V(H)| = |V(G)| - \sum_{i=1}^{b} |B_i^W| >$$

$$> |V(G)| - \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor\frac{g+1}{8}\right\rfloor + \left\lfloor\frac{g+3}{8}\right\rfloor\right)\right)^{-1} \sum_{i=1}^{b} |B_i| \geq$$

$$\geq \left(1 - \left(\frac{g+3}{2} + (c+1)\left(\left\lfloor\frac{g+1}{8}\right\rfloor + \left\lfloor\frac{g+3}{8}\right\rfloor\right)\right)^{-1}\right) |V(G)|.$$

If G has girth g, Theorem 3.5 gives us

$$|V(H)| = |V(G)| - \sum_{i=1}^{b} |B_i^W| > |V(G)| - \frac{1}{(c+2)^{(g-3)/2}} \sum_{i=1}^{b} |B_i|$$

$$\geq \frac{(c+2)^{(g-3)/2} - 1}{(c+2)^{(g-3)/2}} |V(G)|.$$

□

A triangle-free graph is a graph that contains no K_3 as a subgraph. As a special case of Theorem 3.6 for $c = 0$ and $g = 5$ we have:

Corollary 3.7. *For every triangle-free graph G without a minor of K_{t+1} there is a t-colourable induced subgraph H of G such that $|V(H)| \geq \frac{4}{5}|V(G)|$.*

In the same settings, Dvořák and Yepremyan [2019] recently showed the following.

Theorem 3.8. *For every positive $0 < \varepsilon < 1/26$ there exists a positive integer t_0 such that for every $t \geq t_0$, if G is a triangle-free graph on n vertices with no K_t-minor then $\alpha(G) \geq \frac{n}{t^{1-\varepsilon}}$.*

As discussed in the Introduction, we can t times pick the maximum independent set and colour it with a single colour. This could lead to an improvement of Corollary 3.7 and better understanding of results mentioned in Chapter 2, but we do not investigate this direction here.

For $c = 0$, Theorem 3.6 gives a lower bound on the maximum sized t-colourable induced subgraph of a K_{t+1}-minor-free graph with (odd) girth g. But we can also use a small number of additional colours to cover almost the whole graph.

Theorem 3.9. *For every $\varepsilon, \delta > 0$ there exists t_0 such that for every $t > t_0$, every triangle-free and K_{t+1}-minor-free graph G admits a $(1+\delta)t$-colourable induced subgraph on at least $(1-\varepsilon)|V(G)|$ vertices.*

Proof. Given ε, δ, let us choose

$$c := \frac{1}{\varepsilon} - 5; \;\; t_0 := \frac{c}{\delta}.$$

Since G is triangle-free, it has odd girth 5, and from Theorem 3.6 we have an induced subgraph H coloured by $t+c$ colours. We have

$$t + c = \left(1 + \frac{c}{t}\right)t < \left(1 + \frac{c}{t_0}\right)t = (1+\delta)t.$$

Graph H has at least $\left(1 - \frac{1}{c+5}\right)|V(G)| = (1-\varepsilon)|V(G)|$ vertices. □

3.4 Conclusion

We used a rather technical, but fairly simple and constructive method of finding a blob decomposition to prove that we can colour a big fraction of vertices of every K_{t+1}-minor-free graph on n vertices with (odd) girth g.

The parameter g improves the ratio of covered vertices as it grows. Moreover, for a fixed value of g, we can leverage a small number of additional colours to colour almost the whole graph. Triangle-free graphs are the simplest object in this chapter, and because they are well known, there is a hope we can do significantly better for them than in general case.

Question 3.10. *Can we prove that there is a constant $\beta \geq 1$ such that every triangle-free graph without a K_{t+1} minor can be coloured with $\beta \cdot t$ colours?*

4

Maker-Breaker on multigraphs

In this chapter, by a graph we mean a multigraph, i.e. multiple edges between any pair of vertices are allowed. For simplicity of our proofs we forbid loops, that is, there is no edge with both endpoints in the same vertex. While this is non-standard, in the games we investigate, it is never useful for any player to pick a loop in their turn, so we may just assume there are no loops in the first place. Additionally, our definition of edge contracting does not create any loops.

Definition 4.1. *A* contraction of an edge $uv \in E(G)$ *is an operation in which we add to G an edge uw for every vertex w adjacent to v where $w \neq u$, and delete the vertex v from the graph.*

4.1 The (2,2) game

A $(2,2)$ Connectivity Maker-Breaker game is a game of two players, Maker and Breaker, in which both players take turns claiming previously unclaimed edges of a given (multi)graph G. Breaker plays first, and in each turn he first claims 2 edges, and then Maker claims 2 edges. If the edges Maker claims throughout the game form a spanning connected subgraph of G, she wins; otherwise, Breaker wins.

Notice that if Breaker wins, it is because he claimed all edges of some edge cut set. If both players play optimally and Maker has a winning strategy, we say that the graph is *Maker's win*; otherwise, it is *Breaker's win*.

The problem is well known as stated. For convenience we alter what players do in their steps, but immediately observe that it does not change the outcome of the game. Instead of claiming 2 edges in his turn, Breaker deletes 2 edges. Instead of claiming 2 edges in her turn, Maker contracts 2 edges. If the graph becomes disconnected, Breaker wins, and if it is contracted into a single vertex, Maker wins.

Observation 4.2. *Every graph G is Maker's win if and only if Maker has a winning strategy in the corresponding contraction/deletion game.*

Proof. If G is Maker's win, Maker can claim edges to end up with a spanning connected subgraph of G. If instead of claiming edges she contracts them, she ends up with the whole graph contracted into a single vertex.

If G is Breaker's win, Breaker can claim edges to end up with a subgraph containing an edge cut set of G. If instead of claiming edges he deletes them, he disconnects the graph. □

The notions of contraction and deletion of edges is more suitable for our proofs and we stick to it.

Similar to other versions mentioned in Chapter 1, also the $(2,2)$ game is tightly connected to the number of edge-disjoint spanning trees.

Proposition 4.3. *For every graph G, if G has 3 edge-disjoint spanning trees, then G is Maker's win.*

Proof. We proceed by induction on n, the number of vertices in G. If $n = 1$, Maker wins trivially, and if $n = 2$, there are at least 3 parallel edges between the 2 vertices of G, so Breaker can delete at most 2 in his step, and Maker wins by contracting the remaining one.

Let $n > 2$ and let T_1, T_2, T_3 be 3 edge-disjoint spanning trees in G. If Breaker deletes no edge from any of T_1, T_2, and T_3, Maker can contract any 2 edges and finish with graph G'. Because T_1, T_2, T_3 (possibly with some contracted edges) are still 3 edge-disjoint spanning trees of G', Maker wins by the induction hypothesis. So it is not useful for Breaker to choose any edge not belonging to any of the trees.

Assume, without loss of generality, that Breaker deletes 2 edges from T_1. This disconnects T_1 into 3 disjoint sets V_1, V_2, V_3 such that $V_1 \cup V_2 \cup V_3 = V(G)$, and

$T_1[V_1], T_1[V_2]$ and $T_1[V_3]$ are connected graphs. Because T_2 is a spanning tree, there are edges $e, f \in E(T_2)$ such that, up to symmetry, $e = v_1v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$, and $f = v_2'v_3$ for $v_2' \in V_2$ and $v_3 \in V_3$. Maker contracts e and f. After this contraction, the edges of T_1, T_2, T_3 that survived the deletion and contraction still form 3 edge-disjoint spanning trees, and Maker wins by the induction hypothesis.

For the last case, assume, without loss of generality, that Breaker deletes an edge from T_1 and an edge from T_2. It disconnects T_1 into sets V_1, V_2 and T_2 into sets V_3, V_4. Because T_3 is a spanning tree, there are edges $e, f \in E(T_3)$ such that $e = v_1v_2$ for $v_1 \in V_1$ and $v_2 \in V_2$, and f similarly connects V_3 with V_4. Maker contracts e, f and we again have a graph with less vertices and 3 edge-disjoint spanning trees. □

On the other hand, 3 edge-disjoint spanning trees are not necessary for a graph to be Maker's win. Let K_4 have vertices v_1, v_2, v_3, v_4 and add edges v_1v_2 and v_3v_4. It is a simple case analysis that this graph is Maker's win. But it only has 8 edges, while we need at least 9 to have 3 disjoint-spanning trees. Even though 3 edge-disjoint spanning trees are not necessary, having exactly 2 of them is not sufficient.

Proposition 4.4. *For all $n > 1$, if G has n vertices and at most $2(n-1)$ edges, then G is Breaker's win.*

Proof. We proceed by induction on n. If $n = 2$ then G has 2 edges, Breaker can delete both and win. If $n = 3$, a vertex of G only has 2 edges incident with it, Breaker deletes both and disconnects the graph.

Let $n > 3$. In each single round, Breaker deletes 2 edges and Maker contracts 2 edges. This decreases the number of vertices by 2 and the number of edges by at least 4. The number of edges is at most $2(n-1) - 4 = 2(n-3)$, so we can use the induction hypothesis, making Breaker win. □

We move away from spanning trees, and consider a family of graphs with an unfortunate property related to Maker-Breaker games. Let G_k for $k \geq 0$ be the following graph on $k+2$ vertices. There are 2 special vertices u_1 and u_2 connected by 2 edges (an edge of multiplicity 2), and each remaining vertex v is connected to u_1 by a single edge and to u_2 by 2 edges.

Lemma 4.5. *For every even k, G_k is Breaker's win.*

Proof. We proceed by induction on even values of k. If $k = 0$ then Breaker deletes both edges u_1u_2 and disconnects the graph.

For $k > 0$, see Figure 4.1 and consider 2 different vertices $v_1, v_2 \in V \setminus \{u_1, u_2\}$. If Breaker deletes edges u_1v_1 and u_1v_2, both v_1 and v_2 only have 2 edges incident with them. If Maker does not contract at least 1 edge incident with v_1 (and v_2) in her next step, Breaker deletes both remaining edges, and disconnects the graph. Therefore, Maker has to contract both u_2v_1 and u_2v_2. The resulting graph is G_{k-2}, which is Breaker's win by the induction hypothesis.

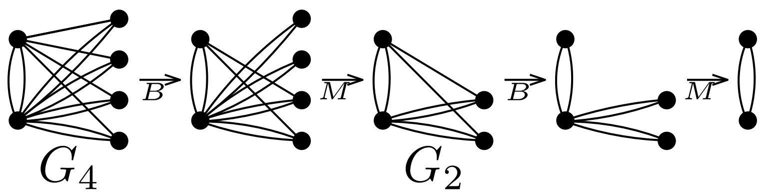

Figure 4.1: Breaker's win graphs G_4 and G_2 together with Breaker's winning strategy.

□

The following is a simple observation which we only state here to simplify the later proofs.

Observation 4.6. *If G is a connected graph where each edge has multiplicity at least three then G is Maker's win.*

Proof. We proceed by induction on the number of vertices. If there is only 1 vertex, Maker wins by default. If there are 2 vertices, there are at least 3 parallel edges between them, Breaker can only delete 2 in his turn and Maker wins by contracting the last edge.

If the number of vertices is at least 3, Breaker either deletes 2 edges between the same pair vertices, in which case Maker contracts a third edge and any other edge in the graph, decreasing the number of vertices by 2, and wins by the induction hypothesis. Otherwise, Breaker deletes edges $uv, u'v'$ but then Maker contracts $uv, u'v'$ and achieves the same outcome as in the previous case. □

Lemma 4.7. *For every odd k, G_k is Maker's win.*

Proof. We proceed by induction on odd values of k. For each Breaker's choice of edges to delete, see Figure 4.2, we describe Maker's counter-play:

1. Breaker deletes $u_i v_1$ and $u_j v_2$ for $1 \leq i, j \leq 2$:

   Maker contracts any edge incident with v_1 and any edge incident with v_2. This yields a graph G_{k-2} with some additional multiedges $u_1 u_2$ that can only help Maker and she wins by induction.

2. Breaker deletes $u_i v_1$ and $u_2 v_1$, or $u_i v_1$ and $u_1 u_2$ for $1 \leq i \leq 2$:

   Maker contracts an edge incident with v_1 and an edge $u_1 u_2$. After this operation the graph is connected and only consists of edges of multiplicity at least three and by Observation 4.6 Maker wins.

3. Breaker deletes $u_1 u_2$ and $u_1 u_2$:

   Maker contracts edges $u_1 v_1$ and $u_2 v_1$ for any $v_1 \in V \setminus \{u_1, u_2\}$. This de facto contracts $u_1 u_2$ and yields a graph that is connected and only consists of edges of multiplicity at least three, which is by Observation 4.6 Maker's win.

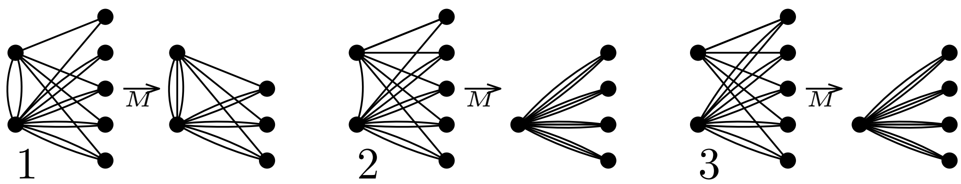

Figure 4.2: Maker's win graphs G_5 after Breaker's turn, and Maker's turn.

□

These lemmata together give us the following:

Theorem 4.8. *Graph G_k is Maker's win if and only if k is odd.*

The deletion of a vertex is not a monotone operation for G_k with respect to being Maker's or Breaker's win. The deletion of a vertex is not even monotone for the game in which both players only claim (delete/contract) 1 edge in their turn. Consider any Maker's win graph G and add a new vertex u connected to any other vertex v by a single edge. This graph $G+u+uv$ is Breaker's win, but deleting vertex u makes it Maker's win. On the other hand, a graph on 3 vertices v_1, v_2, v_3 and edges $v_1v_2, v_1v_2, v_1v_3, v_1v_3, v_2v_3$ is Maker's win, but deleting the vertex v_1 makes it Breaker's win.

4.2 The (1/2,1/2) game

Even more unfortunate is that if we take 2 graphs G_3 and connect them by a vertex (take a vertex of each graph and unify them), we get a Breaker's win graph despite both copies of G_3 being Maker's win. We introduce a variant of the $(2,2)$ game that addresses this last downfalls.

A $(1/2,1/2)$ Connectivity Maker-Breaker game is a game of two players, Maker and Breaker, played on a graph G. Breaker goes first and in each turn he chooses the value $c \in \{1,2\}$. Then he deletes c edges, and Maker contracts c edges. The game ends when the graph becomes disconnected, in which case Breaker wins and G is Breaker's win, or the graph becomes contracted into a single vertex, in which case Maker wins and G is Maker's win.

In the rest of the chapter we only consider the $(1/2,1/2)$ game and properties of being Maker's or Breaker's win as stated here.

It is easy to see that if Breaker wins the $(2,2)$ game, he also wins the $(1/2,1/2)$ game. Consequently, Proposition 4.4 follows without change. The proof of Proposition 4.3 can be slightly modified to show that even for this game, if we have a graph with 3 edge-disjoint spanning trees, then it is Maker's win.

The ability to choose how many edges both players claim in a step gives Breaker more power and G_k becomes Breaker's win even for odd values of k, because he can switch from the odd case to the even case by choosing to remove only one edge in his first turn and follow his winning strategy for even k afterwards.

Even if deleting vertices can change which player wins, the property of being Maker's win or Breaker's win is monotone under other operations.

Observation 4.9. *Let $G=(V,E)$ be Breaker's win. Then for every subset $E' \subseteq E$ it holds that $G'=(V,E')$ is Breaker's win.*

Proof. For contradiction assume G' is Maker's win for some $E' \subseteq E$. We show how Maker can adapt her strategy for G' to also win on G.

Let $e, f \in E$ be edges chosen by Breaker in his turn. Let $e' = e$ in case $e \in E'$; otherwise, let e' be any edge in E' not yet chosen by either player. Similarly, let $f' = f$ in case $f \in E'$; otherwise, let f' be any edge in E' not yet chosen by either player. Maker can assume Breaker deleted all edges e, e', f, f', exactly two of which belong to G'. She can then follow her strategy for G' and react to Breaker's move. If Breaker only chooses a single edge in his turn, we can consider previous case with $e = f$, in which case we also get $e' = f'$, so edges e, e', f, f' have exactly one edge from G' among them. Maker never deviates from her winning strategy for G', so she contracts the whole graph into a single vertex, and wins the game. □

Observation 4.10. *Let G be Maker's win. Then for every graph H obtained from G by contracting edges it holds that H is Maker's win.*

Proof. For contradiction assume H is Breaker's win. Then Breaker can follow the same strategy (delete the same edges) also for G and disconnect it, which is a contradiction with G being Maker's win. □

The proof of the following proposition is an example of splitting the gameplay.

Proposition 4.11. *Let H be a subgraph of G and let H be Maker's win. Then G is Maker's win if and only if G/H is Maker's win.*

Proof. Let G/H be Maker's win and consider Maker's winning strategies for both G/H and H. We describe a winning strategy for Maker on G as follows. If in some turn Breaker only deletes a single edge, or two edges both in either G/H or H, Maker follows her winning strategy for the respective graph. Otherwise, Breaker chooses edges e, f from G/H and H respectively. Maker behaves as if Breaker only deleted the edge e according to her strategy for G/H, and then as if Breaker only deleted f according to her winning strategy for H. Neither on G/H nor H Maker deviates from her winning strategy and so she wins on both, contracting the whole G into a single vertex.

The other implication follows from Observation 4.10. □

4.3 Vertex connectivity

Assume a graph G has a vertex cut set of size 1 (an articulation). Then it can be split into graphs G_1, G_2 such that these intersect in a single vertex and each edge of G belongs to either G_1 or G_2. We show that deciding whether it is Maker's win or Breaker's win can be done by solving for each subgraph independently.

Theorem 4.12. *Let graphs G, G_1, G_2 be as above. Then G is Maker's win if and only if both G_1 and G_2 are Maker's win.*

Proof. If G_1 (or G_2) is a Breaker's win, in G Breaker can delete only edges belonging to G_1 and disconnect the graph.

If G_1 and G_2 are Maker's win, consider Maker's winning strategies for both graphs. We describe a winning strategy for Maker on G as follows. If in some turn Breaker only deletes edge(s) in either G_1 or G_2, Maker follows her winning strategy for the respective graph. Otherwise, Breaker chooses edges e_1, e_2 from G_1 and G_2 respectively. Maker behaves as if Breaker only deleted the edge e_1 according to her strategy for G_1, and then as if Breaker only deleted e_2 according to her winning strategy for G_2. Neither on G_1 nor G_2 Maker deviates from her winning strategy and so she wins on both, contracting the whole G into a single vertex. □

Assume G has a vertex cut set of size 2. Then it can be split into graphs G_1 and G_2 such that these intersect in vertices u, v with $u \neq v$, and each edge of G belongs to either G_1 or G_2. Even in this case we can observe some Maker-Breaker properties induced by the game properties for G_1 and G_2. If there is an edge $uv \in E(G)$ we do not mind very much if this edge ends up in G_1 or G_2, but in our case analysis we sometimes add an additional edge uv to G_1 or G_2 in which case we denote these graphs as G_1^+ and G_2^+ respectively.

In the rest of this section we let $G, G_1, G_1^+, G_2, G_2^+, u$ and v be as above.

We introduce graphs M and B which we use in the case analysis later in this section.

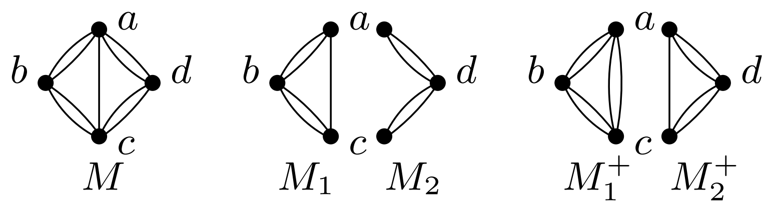

Figure 4.3: A Maker's win graph *M* with a vertex cut set of size 2 that can be split, as suggested, to Breaker's win graphs.

Proposition 4.13. *See Figure 4.3. The graphs M and M_1^+ are Maker's win. The graphs M_1, M_2 and M_2^+ are Breaker's win.*

Proof. Let us first consider M_1^+ and any move of Breaker. Up to symmetry, Breaker can only choose to delete the edge *ab*, both *ab* edges, or edges *ab* and *ac*. In no case Breaker disconnects the graph, so if Maker has 2 moves, she contracts the graph into a single vertex and wins. If Breaker only deletes an edge *ab*, Maker contracts the other *ab* edge which results in a graph on 2 vertices and 4 parallel edges, which is Maker's win by Observation 4.6.

Consider the graph *M*. If Breaker only deletes 1 edge, Maker can always contract an edge to end up with M_1^+, or M_1^+ with an additional edge. Both are Maker's win. If Breaker deletes 2 edges, Maker can always contract edges to end up with a graph on 2 vertices and 4 parallel edges, which is Maker's win by Observation 4.6.

The graph M_1 can be won by Breaker by deleting *ac*. After Maker's move we have a graph on 2 vertices and 2 parallel edges, and Breaker can delete both to disconnect the graph. The graph M_2^+ is the same as M_1, so it is also Breaker's win.

The graph M_2 is the same as M_1 without an edge and by Observation 4.9, it is Breaker's win. □

Proposition 4.14. *The graph B_1^+ in Figure 4.4 is Maker's win. The graphs B, B_1, B_2 and B_2^+ are Breaker's win.*

Proof. Consider the graph B_1^+. If Breaker deletes both edges *ac*, Maker contracts *ab* and *bd* leading to a graph on 2 vertices and 3 parallel edges which is Maker's win by Observation 4.6. If he deletes both edges *bd*, Maker contracts *ab* and *ad* resulting in a Maker's win graph on 2 vertices and 5 parallel edges. If Breaker

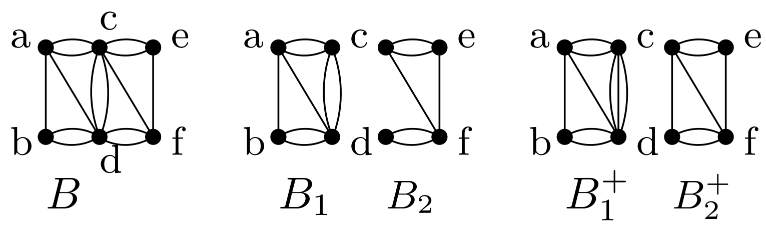

Figure 4.4: A Breaker's win graph B with a vertex cut set of size 2 that can be split, as suggested, to Breaker's win graphs.

deletes any other 2 edges, Maker contracts ac and bd which also leads to a Maker's win graph on 2 vertices and at least 3 parallel edges. We leave the case when Breaker only deletes one edge as an exercise.

Consider a vertex v that has at most 2 incident edges. If it is Breaker's turn, he can delete both these edges and disconnect the graph. As a result, if it is Maker's turn, she has to contract at least one of the edges not to lose in the next turn.

Let us consider the graph B_1. Breaker deletes ab and ad. Maker has to contract ac and bd, but then e only has 2 incident edges and Breaker wins. By Observation 4.9 B_2 and B_2^+ are also Breaker's win.

For the graph B, Breaker can delete ab and ad after which Maker has to contract ac and bd. Then Breaker deletes cf and ef, Maker has to contract ce and df, only to leave c with 2 incident edges, so Breaker wins. □

In order for Breaker to win, he needs to disconnect the graph. When playing on G it may happen that he disconnects G_1, but Maker contracts G_2 together with both disconnected components of G_1. This cannot occur if Breaker can disconnect G_1 in such a way that there is a connected component containing no cut vertex.

Definition 4.15. *A graph G with $u, v \in V(G)$ is uv-*broken *if Breaker has a winning strategy in which he disconnects the graph into nonempty $V_1, V_2 \subset V(G)$ such that $u, v \notin V_1$.*

We are ready to investigate what can be deduced about the game for G if we know the outcome for both G_1 and G_2.

Theorem 4.16. *Let G, G_1, G_2, u, v be as above:*

1. *If G_1 is Maker's win and G_2 is Maker's win, then G is Maker's win.*
2. *If G_1 is Maker's win and G_2 is Breaker's win, then G is Breaker's win if and only if G_2 is uv-broken.*
3. *If G_1 is Breaker's win and G_2 is Maker's win, then G is Breaker's win if and only if G_1 is uv-broken.*
4. *If G_1 is Breaker's win and G_2 is Breaker's win, then G can be Maker's or Breaker's win.*

Proof.

1. Maker has winning strategies S_1 for G_1 and S_2 for G_2. We show how to combine them into a winning strategy for G. If in his step Breaker only deletes edge(s) from G_1 (or G_2), then Maker responds by following S_1 (S_2) in her step. If Breaker deletes edges $e_1 \in E(G_1)$ and $e_2 \in E(G_2)$, Maker behaves as if Breaker only deleted the edge e_1 according to her strategy S_1, and then as if Breaker only deleted e_2 according to her winning strategy S_2. Neither on G_1 nor G_2 Maker deviates from S_1 or S_2 respectively, and she contracts both, and indeed the whole G, into a single vertex.
2. If G_2 is uv-broken, Breaker follows his winning strategy also on G to win by disconnecting some component from both u and v. If G_2 is not uv-broken, let Maker follow her winning strategy for G_1 and the strategy that does not allow G_2 to be uv-broken for G_2 as in the previous case. She contracts G_1 into a single vertex, so vertices u, v become one. In G_2, each vertex is either contracted to u or to v and so the whole graph is contracted to a single vertex.
3. Same as 2. with G_1 and G_2 swapped.
4. By Proposition 4.14, B is a Breaker's graph win with both B_1 and B_2 being Breaker's win. By Proposition 4.13, M is a Maker's win graph with both M_1 and M_2 being Breaker's win.

□

Theorem 4.17. *Let G, G_1^+, G_2, u, v be as above:*

1. *If G_1^+ is Maker's win and G_2 is Maker's win, then G is Maker's win.*
2. *If G_1^+ is Maker's win and G_2 is Breaker's win, then G can be Maker's or Breaker's win.*
3. *If G_1^+ is Breaker's win and G_2 is Maker's win, then G is Breaker's win if and only if G_1^+ is uv-broken.*
4. *If G_1^+ is Breaker's win and G_2 is Breaker's win, then G can be Maker's or Breaker's win.*

Proof.

1. By Observation 4.10, the graph G_1' created from G_1^+ by contracting the edge uv is Maker's win. Maker has winning strategies S_1 and S_2 for G_1' and G_2 respectively. If Breaker chooses an edge or both edges from G_2 in his step, Maker responds according to S_2. It follows that G_2, and in particular u and v, will be contracted into a single vertex. Therefore, Maker can play as if the game is played on G_1' and G_2 and since he wins on both, he wins on G.

2. By Proposition 4.14, B is a Breaker's win graph with B_1^+ being Maker's win and B_2 being Breaker's win. By Proposition 4.13, M is a Maker's win graph with M_1^+ being Maker's win and M_2 being Breaker's win.

3. If G_1^+ is uv-broken, Breaker can follows his winning strategy also on G to win by disconnecting some component from both u and v. If G_1^+ is not uv-broken, let Maker follow her winning strategy for G_2 and the strategy S_1 that does not allow G_1^+ to be uv-broken for G_1^+. She contracts G_2 into a single vertex, so vertices u, v become one. If at some turn, Maker is supposed to contract the extra edge uv in G_1^+. She can rely on this being done in G_2 and contract any other edge instead. At the end, each vertex in G_1^+ is either contracted to u or to v and so the whole graph is contracted to a single vertex.

4. By Proposition 4.14, B is a Breaker's win graph with both B_2^+ and B_1 being Breaker's win. By Proposition 4.13, M is a Maker's win graph with both M_2^+ and M_1 being Breaker's win.

□

Theorem 4.18. *Let G, G_1^+, G_2^+, u, v be as above:*

1. *If G_1^+ is Maker's win and G_2^+ is Maker's win, then G is Maker's win.*
2. *If G_1^+ is Maker's win and G_2^+ is Breaker's win, then G can be Maker's or Breaker's win.*
3. *If G_1^+ is Breaker's win and G_2^+ is Maker's win, then G can be Maker's or Breaker's win.*

Proof.

1. Maker has winning strategies S_1 for G_1^+ and S_2 for G_2^+. Because of symmetry, we only consider the gameplay for G_1. If Breaker chooses an edge from G_1, Maker chooses an edge according to S_1. The only problem occurs when Maker's response is to play the artificial edge uv which appears in G_1^+, but not in G_1. In that case, Maker instead behaves as if Breaker played the edge uv in G_2^+ and contracts the edge that she would play according to S_2. That way no player chooses uv in G_2^+ so the remaining game on the remainder of G_2 is Maker's win and u, v is contracted into a single vertex. Knowing that, we can assume that on the remainder of G_1 vertices u, v will become unified and continue as if Maker really contracted uv. Following the rest of the turns according to S_1 and S_2 deems G Maker's win.
2. By Proposition 4.14, B is a Breaker's win graph with B_1^+ being Maker's win and B_2^+ being Breaker's win. By Proposition 4.13, M is a Maker's win graph with M_1^+ being Maker's win and M_2^+ being Breaker's win.
3. Same as 2. with G_1^+ and G_2^+ swapped.

□

In the first case of Theorem 4.18 it is feasible for both G_1 and G_2 to be Breaker's win in a Maker's win graph G. What makes this possible is that once Maker looses on G_1 he has an extra move on G_2 thanks to which she actually wins there and contracts u and v into a single vertex, also contracting the components of G_1. This feature of Maker's play prevents us from establishing the behaviour of the only undetermined case: when both G_1^+ and G_2^+ are Breaker's win it remains open whether G can be Maker's win.

4.4 Simulations

To help us in our research we developed a computer program (Kučera [2019]). For a given graph it answers whether it is Maker's win or Breaker's win (and provides a winning strategy) for the $(1/2, 1/2)$ game. Because it is mostly brute force, we are only able to use it on rather small graphs. With the help of a database of non-isomorphic simple graphs (no loops or parallel edges) by McKay and Piperno [2014], we established the minimum number of edges m a Maker's win simple graph on n vertices can have and computed the number of such graphs. See table 4.1.

n	2	3	4	5	6	7	8	9	10
m	3	6	8	10	13	15	17	20	22
#				1	2	18	48	≥ 4732	≥ 3

Table 4.1: The minimum number of edges m in Maker's win graphs on n vertices and the number of non-isomorphic *simple* graphs of that size.

4.5 Conclusion

We studied 2 variants of the Maker-Breaker connectivity game. The second allows Breaker to choose whether both players claim 1 or 2 edges for each coming round. We noted there are several operations that maintain winning strategies for Maker or Breaker, and attempted to simplify the game for graphs with a vertex cut set of size at most 2. It may be useful for getting a deeper understanding of the problem to answer the only completely unsolved (and unstated) case of Theorem 4.18.

Question 4.19. *If G, G_1^+, G_2^+, u and v are as in Section 4.3, and both G_1^+, G_2^+ are Breaker's win, can G be Maker's win?*

Both variants are connected with the number of edge-disjoint spanning trees. To allow Maker's win, 3 are enough, while 2 are not. This implies that any graph on n vertices that is Maker's win and minimizes the number of edges m, satisfies $2(n-1) < m \leq 3(n-1)$. Figure 4.5 shows minimum Maker's win graphs for each small, fixed number of vertices.

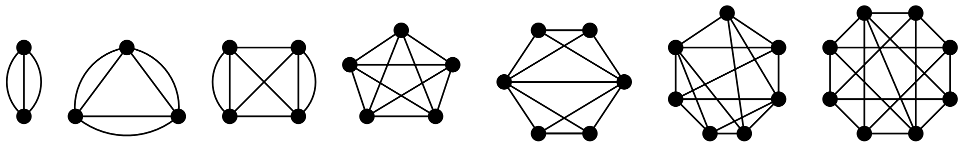

Figure 4.5: Examples of Maker's win graphs on the minimum number of edges.

Using the few values of m from Table 4.1, we can hope the game fits a pattern:

Question 4.20. *Is the minimum number of edges of a Maker's win graph on $n > 1$ vertices equal to $2n + \left\lfloor \frac{n-3}{3} \right\rfloor$?*

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