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A Class of Two-dimensional Strong Markov Processes and A Continuous-Time Principal-Agent Problem with Costly Renegotiation

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of 79 pages.

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Abstract

In the first part of this thesis, we study a continuous-time principal-agent model without precommitment. The agent runs an economic project on behalf of the principal. To this end, the agents apply effort that is costly to them and unobservable by the principal. In return, the agent receives compensation from the principal. The agent is strictly risk-averse and their objective is to maximize their expected utility of compensation minus their expected disutility of effort. The principal is risk-neutral and their objective is to maximize their expected utility of income generated by the project minus the compensation paid to the agent. The optimal contract should maximize the principal's expected utility subject to the constraint that it should induce a contractual environment in which it is optimal for the agent to always be truthful. To exclude the requirement of precommitment, the contract allows for costly renegotiation. The optimal contract is fully determined by deriving the explicit solution to a suitable control problem that combines regular stochastic control with singular stochastic control.

In the second part of this thesis, we present a study of two-dimensional strong Markov processes whose second component is the running maximum of the first one. The study of such processes has been motivated by recent development in financial mathematics, such as the introduction and the analysis of the π and the watermark options. We first introduce a suitable concept of regularity that generalises the standard regularity assumption of the theory of one-dimensional diffusions/strong Markov process to the two-dimensional setting that we study. Next, we characterise the class of scale functions, namely, the functions that yield local martingales when composed with a Markov process in the family we study. We then show that such a process in natural scale can be represented as a time-changed Brownian motion and its running maximum. Finally, we present a study of associated r-invariant functions. Our analysis makes heavy use of the standard theory of one-dimensional diffusions. The main difficulties arise from the behaviour of the processes on the diagonal where their two components coincide.

Contents

Chapter 1

Introduction and Literature Review

Principal-agent problems study the interaction between a principal and an agent. The principal hires the agent to accomplish a project and receives the output, while the agent receives compensation from the principal and makes effort until the termination of the project. The principal chooses a contract which incentivises the agent to accept and work until termination. The agent maximizes over his actions while the principal maximizes over admissible contracts based on the agent's incentives. In the second best setting, the action of the agent is not directly observable by the principal due to the noise in the output, which is observable only by the agent.

The first principal-agent problem model in continuous time setting was studied by Holmstrom and Milgrom [15]. The principal pays the agent at the terminal time T. The agents try to maximize their expected utility function of terminal payment from the principal minus their own effort cost, while the principal try to maximize a utility function of final output minus her terminal payment to the agent. This problem can be solved either via first order methods or a BSDE comparison principle (see also Cvitanic and Zhang [8]). Later, Cvitanic, Wan and Zhang [7] studied a similar problem of Holmstrom and Milgrom [15] using FBSDE and stochastic maximum principle (see also Cvitanic and Zhang [8]). Early seminal papers in continuous time setting are due to DeMarzo and Sannikov [12] and Sannikov [27]. They represent the agent's continuation value as a stochastic process and solve both of the agent's and principal's problems using standard HJB equation approach. The former paper considers the situation where the agent is able to steal from the output process without being discovered by the agent. The agent is offered a continuous contract payment from the principal and tries to maximize this payment plus the stole benefit. On the other hand, the principal tries to maximize her cashflow minus the contract payment and the stolen loss. The latter paper consider the situation when the agent receives continuous payment from the principal and the principal benefits from the agent's continuous effort. The agents try to maximize their discounted utility stream from the payment minus the discounted disutility stream from their efforts. On the other hand, the principals try to maximize their discounted effort stream minus the discounted payment stream.

With the exception of DeMarzo and Sannikov [12], all of the above mentioned papers are based on the so called "weak formulation" in which the agent determines the drift of the output process SDE by choosing an equivalent probability measure. Evans, Miller and Yang [13] revisited the problem of Sannikov [27] in the strong formulation in which strong solutions to the output process' SDE are considered and the agent chooses a controlled process that affects the drift. This paper makes suitable convexity assumptions and solves the resulting control problem by means of PDE techniques.

Anderson and Zervos [2] also adopted a strong formulation. So far, all models allow the agent to control the drift of the output process SDE. It is worth noting that Cvitanic, Possamai and Touzi [9], [10] also studied the problem in which the agent can control the volatility of the output process SDE using the theory of quadratic BSDE.

In the first chapter of the thesis, we consider the problem of Sannikov [27] in the strong formulation. On the top of that, we add two new features: 1) we allow for an exit option for the principal. 2) we allow for costless or costly renegotiation to take place. Renegotiation is defined as an update of current contract between the agent and the principal. From economical point of view, the two parties will happily accept the renegotiation if it benefits them both, possibly with a (proportional) cost. Therefore, it makes sense to consider only renegotiations that are beneficial to both. As we saw from Sannikov's solution in [27], the slope at 0 of the value function could be strictly positive. This means that if renegotiation is costless, with possible renegotiation between the agent and the principal, both of them could be better off. In some cases, it might not be optimal to renegotiate due to the cost, while in the costless case, the slope of the value function is nonpositive to exclude a better off situation for both parties.

To the best of our knowledge, this model is the very first one that studies a principal-agent problem with renegotiation. Our main contributions include solving the Sannikov's problem in this new setting and filling some details and gaps in Sannikov's original paper. (Sannikov's results are correct but several steps are not explicitly proved.)

One possible future continuing work could be to consider a fixed cost along with the already existing proportional cost. This extension is very practical in the real world applications and would give rise to an impulse control problem.

The theory of one-dimensional regular diffusions has been comprehensively studied by many contributors: Itô and McKean [16] and Roger Williams [26] are standard references for the theory. The law and boundary behaviours of these processes are fully determined by their scale function p and their speed measure m. The process $p(X)$ can be written as a time change of a Brownian motion on a possibly enriched probability space by a PCHAF (perfect continuous homogeneous additive functional) that depends on the speed measure m (see Theorem V.41.1 of Roger and Williams [26]). In other words, every one-dimensional regular diffusion can be obtained by time-changing a standard one-dimensional Brownian motion and then composing the resulting process with a scale function.

The notion of r-invariant functions and resolvents can then be introduced. These play important roles in solving several optimal stopping and the stochastic control problems. In Alvarez [1], a singular control problem of one-dimensional diffusion is studied. In Karatzas and Dayanik [11], the value functions of optimal stopping problems involving one-dimensional diffusions are characterised by r-excessive functions that are the smallest majorants of the terminal payoffs. The papers by Johnson and Zervos [17], Lamberton and Zervos [19] exploit solutions of some ODEs which are measures to solve optimal stopping problems and price some exotic derivatives related to one-dimensional diffusion.

In the second chapter of the thesis, we investigate the two-dimensional continuous strong Markov processes (X, S) whose second component is the running maximum of the first one on the state space E. This class includes those (X, S) where X is a one-dimension continuous strong Markov processes. Moreover, it also includes the Azema-Yor process which can be characterised as a solution to the Bachelier equation

$$dX_t = u(S_t)dM_t,$$

where $\frac{1}{u}$ is locally integrable and M is a continuous local martingale (see Obloj [23]). The Azema-Yor process can be used to solve Skorokhod problem and provide bounds for the law of running maximum process S according to Obloj and Yor [22]. It is also related to drawdown equation where the constraint $X \geq \kappa(S)$ is satisfied for some Borel functions κ such that $\kappa(s) < s$ for all s. The drawdown equation has application in stochastic control and portfolio optimization where we require (X, S) satisfy a similar constraint according to Carraro, El Karoui and Obloj [5]. Note that our class of two-dimensional continuous strong Markov processes contains the class Azema-Yor process as a proper subclass and the problem of its characterisation is interesting in its own right.

Our first objective is to show that such processes can also be obtained by time-changing a standard one-dimensional Brownian motion and its running maximum and then composing the resulting processes with a scale function. We start by defining a notion of regularity in our two dimensional setting for the strong Markov processes (X, S). In order to use probability tools such as Itô-Tanaka's formula or its extension, we introduce the family of (one to one) scale functions $p : E \to \mathbb{R}^2$ such that the process $(p(X_t, S_t), p(S_t, S_t))$ is also strong Markov while $p(X_t, S_t)$ is a local martingale. Then, we present an extended version of Itô-Tanaka formula by Lamberton and Zervos [20] concerning a continuous semimartingale X and its running maximum S. With the aid of this formula, we can show that after composing with a scale function, the process (X, S) can be identified as a time-changed of Brownian motion and its running maximum by a PCHAF. The PCHAF depends on the speed measure $m(\cdot; s)$ of the process X while it makes an excursion with running maximum s as well as a measure λ that is a kind of speed measure of process X while the process (X, S) is on the line $\{x = s\}$ in x-s plane. Hence, the law of (X, S) is completely determined by its scale function p, the speed measure m and λ similar to one-dimensional diffusion case. Finally, we consider the r-invariant functions corresponding to the two dimensional process (X, S) and derive a differential equation which they satisfy as in one dimensional diffusion case.

To the best of our knowledge, this is the first article characterising two-dimensional continuous strong Markov processes (X, S) whose second component is the running maximum of the first one. The main difficulties lie in how to construct scale functions in this two-dimensional setting and verify that all auxiliary functions we use do satisfy the requirements of our extension of Itô-Tanaka's formula.

The study of this family of stochastic processes has partly been motivated by some recent development in financial mathematics, in particular pricing derivatives involving running maximums, such as the introduction and the analysis of π options (Guo and Zervos [14]) and the watermark options (Rodosthenous and Zervos [25]). Similar solution to a PDE that are measures horizontally and diagonally analogous to one-dimensional case in Johnson and Zervos [17] may be established in the future work.

Chapter 2

A Continuous-Time Principal-Agent Problem with Costly Renegotiation

This chapter is based on joint work with Professor Mihail Zervos.

2.1 Outline

In this chapter, we study a continuous-time principal-agent model without precommitment. The agent runs an economic project on behalf of the principal. To this end, the agent applies effort that is costly to them and unobservable by the principal. In return, the agent receives compensation from the principal. The agent is strictly risk-averse and their objective is to maximize their expected utility of compensation minus their expected disutility of effort. The principal is risk-neutral and their objective is to maximize their expected utility of income generated by the project minus the compensation paid to the agent. The optimal contract should maximize the principal's expected utility subject to the constraint that it should induce a contractual environment in which it is optimal for the agent to always be truthful. To exclude the requirement of precommitment, the contract allows for costly renegotiation. The optimal contract is fully determined by deriving the explicit solution to a suitable control problem that combines regular stochastic control with singular stochastic control.

The chapter is organised as follows. In Section 2.2, we set up the maximization problems (without renegotiation) for both the agent and the principal. We make assumptions on relevant constants, utility and disutility functions that are very standard. Then, we define the concept of contracts. In Section 2.3, we introduce the notion of a renegotiation process, which can be discrete or continuous. In Section 2.4, we give a representation of the agent's continuation value process and motivate the definition of dynamic contract. We solve the agent problem by restricting the principal to choose from a class of incentive compatible contracts, which we refer to admissible contracts. Then, we state the optimisation problem of the principal and the corresponding HJB equation. In Section 2.5, we construct the solution to the HJB equation associated with the value function of the principal. In Section 2.6, we state the value function of the principal and we discuss the optimal contract for the principal under different cases. Then, we use a verification theorem to conclude our results.

2.2 The principal-agent problem model without renegotiation

We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a standard one-dimensional Brownian motion W. We denote by $(\mathcal{F}_t)$ the natural filtration of W. Here, as well as throughout the chapter, we refer to the filtration satisfying the usual conditions that is obtained by rendering right-continuous natural filtration of a given process and augmenting it by the $\mathbb{P}$-negligible sets in $\mathcal{F}$ simply as the process' "natural filtration".

The agent runs an economic project on behalf of the principal. The project generates the cashflow process Y given by

$$dY_t = A_t\, dt + \sigma\, dW_t, \tag{2.1}$$

where $\sigma > 0$ is a constant and A is a $\mathbb{R}_+$-valued process modelling the agent's effort in running the project. To compensate the agent's effort, the principal pays the agent at a rate modelled by an $\mathbb{R}_+$-valued process C. We assume that the agent's application of effort is unobservable by the principal, namely, the principal observes only the reported cashflow Y. Accordingly, the compensation process C is adapted to the natural filtration $(\mathcal{G}_t)$ of Y. The contractual agreement between the principal and the agent also involves a discretionary time τ at which the project is liquidated. This is a $(\mathcal{G}_t)$-stopping time.

The agent is risk-averse with limited liability and his objective is to maximize expected utility of compensation minus expected disutility of effort. In particular, the agent aims to maximize the performance criterion

$$\overline{\mathcal{I}}_a(C, \tau, A) = \mathbb{E}\left[\int_0^\tau e^{-rs}[u(C_s) - k(A_s)]\, ds\right], \tag{2.2}$$

where the discount rate $r > 0$ is a constant and u, k satisfy the following assumption

Assumption 2.2.1. *The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is C^2, strictly concave,*

$$u(0) = 0, \quad \lim_{c\to\infty} u(c) = \infty, \quad \lim_{c\to\infty} u'(c) = 0, \quad \lim_{c\to 0} u'(c) = \infty$$
$$\text{and } \liminf_{q\uparrow 0} \frac{d}{dq} u \circ (u')^{-1}\left(-\frac{1}{q}\right) > -\infty.$$

The disutility function $k : \mathbb{R}_+ \to \mathbb{R}_+$ is C^3 strictly convex,

$$k(0) = 0, \quad k'(0) = \gamma \quad \text{and} \quad k'''(x) \geq 0 \quad \text{for all } x > 0,$$

where $\gamma > 0$ is a constant.

Remark 2.2.1. *The setting we have considered up to this point, including Assumption 2.2.1, is the same as in Sannikov [27].*

Remark 2.2.2. *The concavity of u and $-k$ reveal that the agent is risk-averse. The C^2 differentiability and boundary conditions at 0 and ∞ ensure the existence of $C^1 \setminus \{0\}$, locally Lipschitz minimizer $c^\star$ and maximizer $z^\star$ to be defined later. We will see that these conditions guarantee that a specific ODE has a unique solution and a specific SDE has a unique strong solution. In turn, these results will ensure that the value function is C^1 and the optimal contract exist.*

Definition 2.2.1. *An $\mathbb{R}_+$-valued $(\mathcal{F}_t)$-progressively measurable process A is admissible if*

$$\mathbb{E}\left[\int_0^\infty e^{-rs}[k(A_s) + A_s^2]ds\right] < \infty.$$

We denote by $\mathcal{A}$ the family of all admissible effort processes.

Remark 2.2.3. *Here, the integrability conditions are imposed to make sure all integrals exist. They are the same as in Chapter 5 of Cvitanic and Zhang [8] and are implicitly assumed by Sannikov [27].*

The principal is risk-neutral and their objective is to maximize the expected income generated by the project minus the expected compensation paid to the agent. In particular, the principal aims to maximize the expected payoff

$$\begin{aligned} \overline{\mathcal{I}}_p(C,\tau,A) &= \mathbb{E}\left[\int_0^\tau e^{-rs}\, dY_s - \int_0^\tau e^{-rs}C_s ds + e^{-r\tau}L\right] \\ &= \mathbb{E}\left[\int_0^\tau e^{-rs}(A_s - C_s)ds + e^{-r\tau}L\right] \end{aligned} \tag{2.3}$$

over all compensation processes C and liquidation times τ. Here, the discounting rate $r > 0$ and the liquidation payoff $L \geq 0$ are given constants.

The following definition, which involves no renegotiation, provides a first step to formalising the contractual environment we consider.

Definition 2.2.2. *A contract without renegotiation $\Gamma = (C, \tau)$ is a function mapping each process A in the set of admissible effort processes $\mathcal{A}$ to a pair $(C(A), \tau(A))$, where $C(A)$ is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-progressively measurable process and $\tau(A)$ is a $(\mathcal{G}_t)$-stopping time, where $(\mathcal{G}_t) = (\mathcal{G}_t(A))$ is the natural filtration of the process Y given by* (2.1).

We need to impose suitable integrability conditions to the compensation processes $C(A)$ for the optimisation problems we consider to be well-defined. We will address such issues in the more specific context of Section 4.

In light of this definition, the problem of determining an optimal contract in the absence of renegotiation can be viewed as a Stackelberg game:

Agent: Given a contract Γ, the agent chooses an effort strategy $A^\star = A^\star(\Gamma)$ that maximizes their expected payoff $\overline{\mathcal{I}}_a(C(A), \tau(A), A)$ given by (2.2) subject to

$$\mathbf{1}_{\{t<\tau\}}\mathbb{E}\left[\int_t^\tau e^{-r(s-t)}[u(C_s) - k(A_s)]ds \,\middle|\, \mathcal{G}_t\right] \geq 0 \text{ for all } t.$$

Principal: Given the optimal response $A^\star(\Gamma)$ of the agent to each contract Γ, the principal chooses a contract $\Gamma^\star$ that maximizes their expected payoff $\overline{\mathcal{I}}_p(C, \tau, A^\star(\Gamma))$ given by (2.3).

2.3 The principal-agent problem model with renegotiation

We introduce the possibility of renegotiation in the contractual environment. To this end, we first consider a sequence of functions $T_j : \mathcal{A} \to [0, \infty]$, $j \geq 0$, such that

$$T_0(A) = 0, \quad T_j(A) < T_{j+1}(A) \text{ on } \{T_j(A) < \infty\}$$

and $T_j(A)$ is a $\mathcal{G}_t \equiv \mathcal{G}_t(A)$-stopping time for all $j \geq 0$ and all $A \in \mathcal{A}$. We assume that renegotiation takes place at each of the times T_j, $j \geq 1$, and a contract $\Gamma_j = (C_j, \tau_j)$ in the sense of Definition 2.2.2 prevails over the period $[T_{j-1}, T_j[$, for $j \geq 1$.

We can define a single contract $\Gamma = (C, \tau)$ in the sense of Definition 2.2.2 that represents the sequence of contracts arising from renegotiation as follows. First, we define

$$C_t(A) = \sum_{j=1}^{\infty} C_{j,t}(A) \mathbf{1}_{\{T_{j-1}(A) \leq t < T_j(A)\}}, \tag{2.4}$$

for $t \geq 0$ and $A \in \mathcal{A}$. Next, we define recursively the functions $\xi_j : \mathcal{A} \to [0, \infty]$, $j \geq 1$, by

$$\xi_1 = \tau_1 \mathbf{1}_{\{\tau_1 < T_1\}} + \infty \mathbf{1}_{\{\tau_1 \geq T_1\}}$$

and

$$\begin{aligned} \xi_{j+1} = {} & \xi_j \mathbf{1}_{\{\xi_j < \infty\}} + \tau_{j+1} \mathbf{1}_{\{\tau_{j+1} \in [T_j, T_{j+1}[\} \cap \{T_j < \xi_j\}} \\ & + \infty\, \mathbf{1}_{(\{\xi_j < \infty\} \cup (\{\tau_{j+1} \in [T_j, T_{j+1}[\} \cap \{T_j < \xi_j\}))^c}, \end{aligned}$$

for $j \geq 1$, where we have dropped the dependence as $A \in \mathcal{A}$ to simplify the notation. Finally, we define

$$\tau(A) = \bigwedge_{j=1}^{\infty} \xi_j(A), \quad \text{for } A \in \mathcal{A}. \tag{2.5}$$

In other words, this synthesized contract makes the payments and follows the termination strategy of the j-th contract between the renegotiation times T_{j-1} and T_j.

Lemma 2.3.1. *Given sequences of renegotiation times (T_j) and contracts Γ_j as above, (2.4) and (2.5) defines a contract without renegotiation in the sense of Definition 2.2.2.*

Proof. Throughout the proof, we consider an effort process $A \in \mathcal{A}$ fixed and we drop it from the notation to simplify the formulas.

The process C defined by (2.4) is $(\mathcal{G}_t)$-measurable because the process C_j, $j \geq 1$, are $(\mathcal{G}_t)$-measurable and the times T_j, $j \geq 0$ are $(\mathcal{G}_t)$-stopping times, we can show that ξ_j , $j \geq 1$ are $(\mathcal{G}_t)$-stopping times by induction as follows. Given any $t \geq 0$,

$$\begin{aligned}\{\xi_1 \leq t\} &= \{\tau_1 \leq t\} \cap \{\tau_1 < T_1\} \\ &= \{\tau_1 < T_1 \leq t\} \cup (\{\tau_1 \leq t\} \cap \{t < T_1\}) \in \mathcal{G}_t\end{aligned}$$

which proves that ξ_1 is a $(\mathcal{G}_t)$-stopping time. Assuming that ξ_j is a $(\mathcal{G}_t)$-stopping time, we can see that, given any $t \geq 0$,

$$\{\xi_{j+1} \leq t\} = \{\xi_j \leq t\} \cup (\{T_j \leq \tau_{j+1} \leq t\} \cap \{T_j < \xi_j\}) \in \mathcal{G}_t,$$

because

$$\begin{aligned}\{T_j < \xi_j\} \in \mathcal{G}_{T_j} &\Rightarrow \{T_j < \xi_j\} \cap \{T_j \leq t\} \in \mathcal{G}_t \\ \text{and } \{T_j \leq \tau_{j+1}\} \in \mathcal{G}_{T_j} &\Rightarrow \{T_j \leq \tau_{j+1}\} \cap \{\tau_{j+1} \leq t\} \in \mathcal{G}_t.\end{aligned}$$

The claim that ξ_{j+1} is a $(\mathcal{G}_t)$-stopping time follows. Finally, the time τ defined by (2.5) is a $(\mathcal{G}_t)$-stopping time by Exercise (4.17) in Chapter I of Revuz and Yor [24] because $(\mathcal{G}_t)$ is right-continuous. ∎

To proceed further, we define

$$\mathcal{P}(t, C, \tau, A) = \mathbf{1}_{\{t<\tau(A)\}}\mathbb{E}\left[\int_t^{\tau(A)} e^{-r(s-t)}[u(C(A_s)) - k(A_s)]ds \,\middle|\, \mathcal{G}_t\right],$$

for $t \geq 0$, where (C, τ) is any contract in the sense of Definition 2.2.2 and A is any admissible effort process. We next consider a sequence of contracts $\Gamma_j = (C_j, \tau_j)$, $j \geq 1$, arising from renegotiation at a sequence of renegotiation times T_j, $j \geq 0$, as well as the effective contract $\Gamma = (C, \tau)$ that is as in the previous lemma. In this context, the agent's running promise P_t associated with the contract prevailing at time t is given by

$$P_t = \mathcal{P}(t, C_j, \tau_j, A) \quad \text{on } \{T_{j-1} \leq t < \tau_j \wedge T_j\}. \tag{2.6}$$

On the other hand, the agent's effective promise $\bar{P}_t$ that reflects the effect of future renegotiations is given by

$$\bar{P}_t = \mathcal{P}(t, C, \tau, A).$$

In view of these definitions and the fact that

$$C_t = C_{l+1,t} \text{ for all } t \in [T_l, T_{l+1}[\quad \text{and } \{T_l \leq \tau < T_{l+1}\} = \{T_l \leq \tau_{l+1} < T_{l+1}\},$$

on the event $\{T_{j-1} \leq t < T_j\} \cap \{t < \tau\}$, we can see that

$$
\begin{aligned}
\bar{P}_t =& \mathbb{E}\bigg[\int_t^{\tau_j \wedge T_j} e^{-r(s-t)}[u(C_{j,s}) - k(A_s)]ds \\
&+ \sum_{l=j}^{\infty} \mathbf{1}_{\{T_l < \tau\}} \int_{T_l}^{\tau_{l+1} \wedge T_{l+1}} e^{-r(s-t)}[u(C_{l+1,s}) - k(A_s)]ds \,\bigg|\, \mathcal{G}_t\bigg] \\
=& \mathbb{E}\bigg[\int_t^{\tau_j} e^{-r(s-t)}[u(C_{j,s}) - k(A_s)]ds \\
&+ \sum_{l=j}^{\infty} \mathbf{1}_{\{T_l < \tau\}} e^{-r(T_l - t)} \mathbb{E}\Big[\int_{T_l}^{\tau_{l+1}} e^{-r(s-T_l)}[u(C_{l+1,s}) - k(A_s)]ds \\
&- \int_{T_l}^{\tau_l} e^{-r(s-T_l)}[u(C_{l,s}) - k(A_s)]ds \Big| \mathcal{G}_{T_l}\Big] \Big| \mathcal{G}_t\bigg] \\
=& P_t + \mathbb{E}\bigg[\sum_{l=j}^{\infty} \mathbf{1}_{\{T_l < \tau\}} e^{-r(T_l - t)} \Delta R_{T_l} \,\bigg|\, \mathcal{G}_t\bigg] \\
=& P_t + \mathbb{E}\bigg[\int_{]t,\tau]} e^{-r(s-t)} dR_s \,\bigg|\, \mathcal{G}_t\bigg],
\end{aligned}
\tag{2.7}
$$

where R is the piece-wise constant process

$$
\begin{aligned}
R_t &= \sum_{i=1}^{\infty} \Delta R_{T_i} \mathbf{1}_{\{T_i \leq t\}} \\
&= \sum_{i=1}^{\infty} \mathbf{1}_{\{T_i < \tau \wedge t\}} \mathbb{E}\Big[\int_{T_i}^{\tau_{i+1}} e^{-r(s-T_i)}[u(C_{i+1,s}) - k(A_s)]ds \\
&- \int_{T_i}^{\tau_i} e^{-r(s-T_i)}[u(C_{i,s}) - k(A_s)]ds \Big| \mathcal{G}_{T_i}\Big] \Big| \mathcal{G}_t\Big] \\
&= \sum_{i=1}^{\infty} (P_{T_i} - P_{T_{i-}}) \mathbf{1}_{\{T_i \leq t\}}.
\end{aligned}
$$

In summary, we have seen that the agent's running promise process admits the expression

$$
\begin{aligned}
P_t &= \mathbf{1}_{\{t<\tau\}} \mathbb{E}\bigg[\int_t^{\tau} e^{-r(s-t)}[u(C_s) - k(A_s)]ds - \int_{]t,\tau]} e^{-r(s-t)} dR_s \,\bigg|\, \mathcal{G}_t\bigg] \\
&= \bar{P}_t - \mathbf{1}_{\{t<\tau\}} \mathbb{E}\bigg[\int_{]t,\tau]} e^{-r(s-t)} dR_s \,\bigg|\, \mathcal{G}_t\bigg],
\end{aligned}
\tag{2.8}
$$

where the renegotiation process R captures all changes to the agent's running promise resulting from renegotiation.

At the renegotiation time T_j, the agent will be agreeable to replacing their current contract Γ_j with the new one Γ_{j+1} if and only if this results in an increase of their running promise, namely, if and only if

$$\Delta P_{T_j} = \Delta R_{T_j} \geq 0.$$

Furthermore, we allow for continuous renegotiation. In particular, we allow for R to be any increasing process representing the cumulative increase of the agent's running promise resulting from renegotiation.

Definition 2.3.1. *A contract with renegotiation $\Gamma = (C, \tau, R)$ is a function mapping each admissible effort process A to a triplet $(C(A), \tau(A), R(A))$, where $C(A)$ is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-progressively measurable process, $\tau(A)$ is a $(\mathcal{G}_t)$-stopping time and $R(A)$ is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-adapted process with increasing sample paths.*

Given such a contract, the agent's objective is to maximize the performance criterion

$$\mathcal{I}_a(C, \tau, R, A) = P_0 = \mathbb{E}\left[\int_0^\tau e^{-rs}\Big[u(C_s) - k(A_s)\Big]\,ds - \int_{]0,\tau]} e^{-rs}dR_s\right] \tag{2.9}$$

subject to

$$P_t \geq 0 \text{ for all } t. \tag{2.10}$$

Remark 2.3.1. *The performance criterion (2.9) is the agent's running promise at time* 0*. Dynamic programming suggests that an optimality for the agent's contract should remain optimal at any later time. Accordingly, a contract that maximizes the performance criterion (2.9) should maximize the agent's running promise at all times. By focusing on maximizing their running rather than their effective pay-off, this setting assumes that the agent is "myopic". In practice, it would be more appropriate to consider the agent maximizing the performance criterion*

$$\hat{\mathcal{I}}_a(C, \tau, R, A) = \bar{P}_0 = \mathbb{E}\left[\int_0^\tau e^{-rs}\Big[u(C_s) - k(A_s)\Big]\,ds\right], \tag{2.11}$$

which is their effective promise at time 0*. We discuss such issues further in Section 2.7.*

Renegotiation leading to the increases of the agent's running promise may be costly for the principal. We assume that such costs are proportional to the increases of the agent's running promise. Therefore, given a contract in the sense of Definition 2.3.1, the principal's objective is to maximize the performance index

$$\mathcal{I}_p(C, \tau, R, A) = \mathbb{E}\left[\int_0^\tau e^{-rs}(A_s - C_s)ds - \kappa\int_{]0,\tau]} e^{-rs}dR_s + e^{-r\tau}L\right], \tag{2.12}$$

where $\kappa \geq 0$ is a constant. Note that we allow for the possibility of costless renegotiation, which corresponds to the value $\kappa = 0$.

We conclude this section with the statement of the Stackelberg game whose solution can determine an optimal contract in the presence of renegotiations:

Agent: Given a contract Γ, the agent chooses an effort strategy $A^\star = A^\star(\Gamma)$ that maximizes their expected payoff $\mathcal{I}_a(C(A), \tau(A), R(A), A)$ given by (2.9) subject to (2.10).

Principal: Given the optimal response $A^\star(\Gamma)$ of the agent to each contract Γ, the principal chooses a contract $\Gamma^\star$ that maximizes their expected payoff $\mathcal{I}_p(C, \tau, R, A^\star(\Gamma))$ given by (2.12).

2.4 State space representation of contracts with renegotiation

The purpose of this section is to determine a class of contracts admitting a state space representation. To this end, we follow the standard approach to continuous time principal-agent theory that was pioneered by Sannikov [27]. The starting point is Lemma 2.4.1, which provides stochastic dynamics for the agent's (running) promise process P under the assumption that the agent is truthful to the principal. This result *motivates* restricting attention to contracts characterized by the state space representation given by (2.14). In turn, this representation *motivates* the class of contracts given by Definition 2.4.1 and characterized by the state process given by (2.16). The next step is *motivated* by Lemma 2.4.2, which provides sufficient conditions for a contract in the sense of Definition 2.4.1 to be incentive compatible, namely, to be such that it is optimal for the agent to be truthful to the principal. This result *motivates* restricting attention to the class of contracts introduced by Definition 2.4.2. The principal-agent problem then reduces to choosing the contract among the ones in Definition 2.4.2 that maximizes the principal's payoff. It is worth noting that this approach restricts the original class of contracts twice. To the best of our knowledge, there are no results in the literature on how "severe" such restrictions might be.

Lemma 2.4.1. *Consider a contract $\Gamma = (C, \tau, R)$ in the sense of Definition 2.3.1 and suppose that the agent adopts an effort process A that is observable by the principal, so that $(\mathcal{G}_t) = (\mathcal{F}_t)$. There exists a $(\mathcal{G}_t)$-progressively measurable process Z satisfying suitable integrability conditions such that the agent's running promise process P, which is given by (2.8), satisfies the stochastic equation*

$$dP_t = \big(rP_t - u(C_t) + k(A_t)\big)\, dt + dR_t + \sigma Z_t\, dW_t. \tag{2.13}$$

Proof. Since $(\mathcal{F}_t)$ is the natural filtration of the Brownian motion W, the martingale representation theorem implies that there exists an $(\mathcal{F}_t)$-progressively measurable process Z satisfying suitable integrability conditions such that

$$\mathbb{E}\left[\int_0^\tau e^{-rs}\, [(u(C_s) - k(A_s))\, ds - \int_{]0,\tau]} e^{-rs} dR_s \,\Big|\, \mathcal{G}_t\right] = P_0 + \sigma \int_0^t e^{-rs} Z_s\, dW_s,$$

This identity and (2.8) imply that

$$e^{-rt} P_t = P_0 - \int_0^t e^{-rs}[u(C_s) - k(A_s)]\, ds + \int_{]0,t]} e^{-rs}\, dR_s + \sigma \int_0^t e^{-rs} Z_s\, dW_s.$$

The stochastic dynamics (2.13) follow from this representation and an application of the integration by parts formula. ∎

This lemma suggests the possibility of characterising a contract by means of a state process X with dynamics

$$\begin{aligned} dX_t =& [rX_t - u(C_t) + k(A_t)]\, dt + dR_t + \sigma Z_t\, dW_t \\ =& [rX_t - u(C_t) + k(A_t) - A_t Z_t]\, dt + dR_t + Z_t\, dY_t. \end{aligned} \tag{2.14}$$

Instead of making a choice (C, τ, R) and determining the agent's running promise process P using (2.8), such contracts make a choice (C, τ, R, Z) and then determine the agent's running promise process P by identifying it with the solution X to the SDE (2.14). For such an identification, the definition (2.8) of the agent's running process implies that

$$X_{\tau+t} = 0 \quad \text{for all } t \geq 0. \tag{2.15}$$

To develop this new perspective in a way that makes the agent's optimisation problem in the Stackelberg game straightforward to solve, we need to introduce an additional element in the contract. This is a recommended effort process E that the principal would accept as the agent's effort process. Later, we will restrict the principal to focus on contracts that induce the agent to follow the recommended effort process, which makes the problem tractable.

Definition 2.4.1. *A dynamic contract $D = (C, \tau, R, Z, E)$ is a function mapping each admissible effort process $A \in \mathcal{A}$ to a quintuple $(C(A), \tau(A), R(A), Z(A), E(A))$, where E is called the recommended effort, with the following properties (here we drop the explicit dependence on A for notational simplicity):*

- *C is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-progressively measurable process such that*

$$\mathbb{E}\Big[\int_0^\infty e^{-rs} u(C_s)\, ds\Big] < \infty.$$

- *τ is a $(\mathcal{G}_t)$-stopping time.*

- *R is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-adapted process with increasing càdlàg sample paths such that*

$$R_0 = 0 \quad \textit{and} \quad \mathbb{E}\Big[\int_{]0,\infty[} e^{-rs}\, dR_s\Big] < \infty.$$

- *Z is an $\mathbb{R}$-valued $(\mathcal{G}_t)$-progressively measurable process such that*

$$\mathbb{E}\Big[\int_0^\infty e^{-rs} Z_s^2\, ds\Big] < \infty.$$

- *E is an $\mathbb{R}_+$-valued $(\mathcal{G}_t)$-progressively measurable process such that*

$$\mathbb{E}\left[\int_0^\infty e^{-rs}[k(E_s) + E_s^2]\, ds\right] < \infty.$$

The reason for introducing the recommended effort process in the contract is because the actual effort process A may be unobservable by the principal, in which case, the solution to (2.14) is not adapted to the reported information flow that is modelled by the filtration $(\mathcal{G}_t)$. On the other hand, the family of dynamic contracts introduced by the previous definition are associated with the state process X given by

$$dX_t = [rX_t - u(C_t) + k(E_t) - E_t Z_t]dt + dR_t + Z_t\, dY_t, \quad X_0 = x > 0, \tag{2.16}$$

which is adapted to the filtration $(\mathcal{G}_t)$. Notice that equation (2.16) is exactly (2.14) when $A = E$.

Assumption 2.2.1 implies that the function $\mathbb{R} \ni a \mapsto az - k(a)$ has a unique maximum for each fixed $z \in \mathbb{R}$ and

$$\alpha(z) = \arg\max_{a\in\mathbb{R}_+} \{az - k(a)\} = \begin{cases} (k')^{-1}(z) > 0 \ , & \text{for } z > \gamma, \\ 0 \ , & \text{for } z \le \gamma. \end{cases} \tag{2.17}$$

The following result shows that, if the recommended effort E identifies with $\alpha(Z)$, then the contract is incentive compatible, namely, it is optimal for the agent to adopt the recommended effort.

Lemma 2.4.2. *Given a dynamic contract in the sense of Definition 2.4.1, the corresponding solution to the SDE (2.16) is well-defined. Furthermore, if the contract is such that*

$$E(A) = \alpha(Z(A)) \quad \textit{for all } A \in \mathcal{A}, \tag{2.18}$$

there exists $A^\star \in \mathcal{A}$ such that

$$A^\star = \alpha(Z(A^\star)) \tag{2.19}$$

and the associated solution to the SDE (2.16) satisfies

$$X_{\tau(A)+t}(A)\mathbf{1}_{\{\tau(A)<\infty\}} = 0 \quad \textit{for all } t \ge 0 \textit{ and } \lim_{T\to\infty} \mathbb{E}[e^{-rT} X_T(A)\mathbf{1}_{\{T<\tau\}}] = 0 \tag{2.20}$$

then

$$\sup_{A\in\mathcal{A}} \mathcal{I}_a(C(A), \tau(A), R(A), A) = \mathcal{I}_a(C(A^\star), \tau(A^\star), R(A^\star), A^\star) = X_0 \tag{2.21}$$

Proof. Given a dynamic contract D as in the statement of the lemma, we denote by C, τ, Z, R, E its valuation at any given $A \in \mathcal{A}$, and we drop the explicit dependence on A itself for notational simplicity. In view of integrability conditions in Definition 2.2.1, 2.4.1 and Hölder's inequality, we can see that

$$\mathbb{E}\left[\int_0^\infty e^{-rs} A_s |Z_s|\, ds\right] \le \left(\mathbb{E}\left[\int_0^\infty e^{-rs} A_s^2\, ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^\infty e^{-rs} Z_s^2\, ds\right]\right)^{\frac{1}{2}}$$

and

$$\mathbb{E}\left[\int_0^\infty e^{-rs} E_s |Z_s|\, ds\right] \le \left(\mathbb{E}\left[\int_0^\infty e^{-rs} E_s^2\, ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^\infty e^{-rs} Z_s^2\, ds\right]\right)^{\frac{1}{2}}.$$

We can then define process $(e^{-rt}X_t)$ as a stochastic integral

$$e^{-rt}X_t = X_0 + \int_0^t e^{-rs}(-u(C_s) + k(E_s) - E_s Z_s)\, ds + \int_{[0,t]} e^{-rs}\, dR_s + \int_0^t e^{-rs} Z_s\, dY_s,$$

which is $(\mathcal{G}_t)$-adapted. By integration by parts, X is the solution to (2.16). Rearrange, we can see that

$$\begin{aligned} e^{-r(T\wedge\tau)}X_{T\wedge\tau} = X_0 &- \int_0^{T\wedge\tau} e^{-rs}[(E_s Z_s - k(E_s)) - (A_s Z_s - k(A_s))]\, ds \\ &- \int_0^{T\wedge\tau} e^{-rs}[u(C_s) - k(A_s)]\, ds + \int_{]0,T\wedge\tau]} e^{-rs}\, dR_s \\ &+ \int_0^{T\wedge\tau} \sigma e^{-rs} Z_s\, dW_s \end{aligned} \tag{2.22}$$

Now suppose that the contract is such that (2.18)-(2.20) hold true. The expression (2.17) and (2.22) imply that

$$\begin{aligned} &\int_0^{T\wedge\tau} e^{-rs}[u(C_s) - k(A_s)]ds - \int_{]0,T\wedge\tau]} e^{-rs} dR_s \\ \le &X_0 - e^{-rT}X_T \mathbf{1}_{\{T<\tau\}} + \int_0^{T\wedge\tau} \sigma e^{-rs} Z_s dW_s \end{aligned}$$

with equality if $A = \alpha(Z)$. The integrability condition on Z in Definition 2.4.1 implies that the stochastic integral is a martingale. In view of this observation and the relevant integrability conditions in Definition 2.4.1, we can take expectation and pass to the limit as $T \to \infty$ using the monotone convergence theorem to obtain (2.21). ∎

In view of the previous result, we now restrict attention to incentive compatible dynamic contracts such that the recommended effort E identifies with $\alpha(Z)$ and (2.20) holds. Such contracts are associated with the state process

$$\begin{aligned} dX_t &= [rX_t - u(C_t) + k(\alpha(Z_t)) - \alpha(Z_t)Z_t]\, dt + dR_t + Z_t\, dY_t \\ &= [rX_t - u(C_t) + k(\alpha(Z_t))]\, dt + dR_t + \sigma Z_t\, dW_t. \end{aligned} \tag{2.23}$$

The following definition summarises the discussion and analysis of the section thus far.

Definition 2.4.2. *An incentive compatible dynamic contract $D = (C, \tau, R, Z)$ is a function mapping each effort process $A \in \mathcal{A}$ to $(C(A), \tau(A), R(A), Z(A))$ with the following properties (here we drop the explicit dependence on A for simplicity):*

- *The processes C, R, Z and random time τ are as in Definition 2.4.1.*
- *The solution to (2.23) is such that*

$$X_{\tau+t}\mathbf{1}_{\{\tau<\infty\}} = 0 \textit{ for all } t \ge 0 \textit{ and } \lim_{T\to\infty} \mathbb{E}[e^{-rT}X_T\mathbf{1}_{\{T<\tau\}}] = 0.$$

We now restrict attention to admissible controls in the sense of Definition 2.4.2. Given such a contract, Lemma 2.4.2 implies that the agent will choose $A^\star = \alpha(Z(A^\star))$ as their effort process. This effort process is fully observable by the principal. Therefore, its associated reported information flow $(\mathcal{G}_t)$ identifies with the Brownian filtration $(\mathcal{F}_t)$. We are faced with the state process

$$dX_t = [rX_t - u(C_t) + k(\alpha(Z_t))]\, dt + dR_t + \sigma Z_t\, dW_t, \quad X_0 = x \geq 0. \tag{2.24}$$

In this context, the principal's performance index given by (2.12) takes the form

$$\mathcal{J}_p(C, \tau, R, Z) = \mathbb{E}\left[\int_0^\tau e^{-rs}(\alpha(Z_s) - C_s)\, ds - \kappa \int_{]0,\tau]} e^{-rs}\, dR_s + e^{-r\tau} L\right]. \tag{2.25}$$

To determine the optimal incentive compatible dynamic contracts, we need to solve the stochastic control problem defined by (2.24) and (2.25). In particular, we need to maximize the performance criterion given by (2.25) over all admissible controls.

Thus, we are faced with a control problem with mixed regular stochastic control and singular stochastic control. The value function of this problem is defined by

$$V(x) = \sup_{(C,\tau,R,Z)\in\mathcal{C}} \mathcal{J}_p(C, \tau, R, Z). \tag{2.26}$$

In view of standard stochastic control theory, we expect that the problem's value function should identify with a suitable solution to the HJB equation

$$\max\left\{\max_{c\geq 0, z\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2 z^2 f''(x) + (rx - u(c) + k(\alpha(z)))f'(x) - rf(x) + \alpha(z) - c\right\},\ f'(x) - \kappa\right\} = 0. \tag{2.27}$$

where u and k are defined in Assumption 2.2.1 and α is defined in (2.17).

To derive the solution to this HJB equation that identifies with value function, we need suitable boundary conditions. Depending on parameter values, we will encounter three different cases, each corresponding to a different optimal strategy of the principal.

In any of the three cases, we will later show that it is optimal to follow a deterministic strategy that requires 0 effort from the agent, involves no renegotiation and delivers the initial promise x to the agent if x is sufficiently large. Indeed, for all x in a closed neighbourhood of ∞, it is optimal to choose

$$C_t^\star = c^\dagger(x) \quad \text{and } Z_t^\star = R_t^\star = 0 \quad \text{for all } t \geq 0,$$

where $c^\dagger(x)$ is such that

$$x = \int_0^\infty e^{-rt} u(c^\dagger(x))dt \quad \Leftrightarrow \quad rx = u(c^\dagger(x))$$

These choices imply that the agent's promise process

$$dX_t = (rX_t - u(c^\dagger(X_t))dt = 0$$

so that $X_t = x$ for all $t \geq 0$. Accordingly, the closed half-infinite interval in which this strategy is optimal acts as an "absorbing" part of the state space. In this interval, the value function identifies with the function H we consider first in the next section.

2.5 The Construction of a Solution to the HJB Equation

In view of the discussion at the end of the previous section, we start our analysis with the function $H : \mathbb{R}_+ \to]-\infty, 0]$ that is defined by

$$H(x) = -\frac{u^{-1}(rx)}{r}. \tag{2.28}$$

We can check that

$$H'(x) = -\frac{1}{u'\Big(u^{-1}(rx)\Big)} < 0, \quad \lim_{x\to\infty} H'(x) = -\infty,$$

H is strictly concave and

$$\begin{aligned}&\min_{c\geq 0}\Big\{rH(x) + H'(x)\Big(u(c) - rx\Big) + c\Big\}\\ &= rH(x) + H'(x)\Big[u(c^\dagger(x)) - rx\Big] + c^\dagger(x) = 0,\end{aligned} \tag{2.29}$$

where

$$c^\dagger(x) = (u')^{-1}\Big(-\frac{1}{H'(x)}\Big) = u^{-1}(rx) = -rH(x), \quad \text{for } x > 0. \tag{2.30}$$

H also represents the principal's payoff if he pays constant $c^\dagger(x)$ to the agent for 0 effort forever because

$$\int_0^\infty e^{-rt}(-c^\dagger(x))dt = -\frac{1}{r}c^\dagger(x) = -\frac{u^{-1}(rx)}{r} = H(x).$$

There exists a unique straight line that passes through the point $(0, L)$ and is tangent to H. To see that, consider the line

$$l_y(x) = H'(y)(x - y)$$

for any $y \geq 0$. It is the line tangent to H at $(y, H(y))$. We know $y \mapsto l_y(0) = -yH'(y)$ is continuous and strictly increasing from 0 to ∞. Therefore there exist y_0 such that $L = l_{y_0}(0)$. We choose this line $l_{y_0}(x)$ and represent it as

$$l_{\min}(x) = \underline{q}x + L, \tag{2.31}$$

where $\underline{q}$ and $x_{\min}$ are the unique solution to the system of equations

$$H'(x_{\min}) = \underline{q} \quad \textit{and} \quad H(x_{\min}) - L = H'(x_{\min})x_{\min}. \tag{2.32}$$

Note that $\underline{q} = \underline{q}(L)$ is a function mapping $\mathbb{R}_+$ to $]-\infty, 0]$. Since H takes negative values and $L \geq 0$,

$$0 \leq x_{\min}, \quad \underline{q} \leq 0 \quad \text{and} \quad 0 = x_{\min} \Leftrightarrow \underline{q} = 0 \text{ and } L = 0. \tag{2.33}$$

To simplify the notation, we also define

$$c_{\min} = u^{-1}(rx_{\min}) = -rH(x_{\min}). \tag{2.34}$$

We also note that the second identity in (2.32) can be rewritten as

$$-\frac{c_{\min}}{r} - L = -\frac{1}{u'(c_{\min})}\frac{u(c_{\min})}{r},$$

which is equivalent to

$$u(c_{\min}) - u'(c_{\min})(rL + c_{\min}) = 0. \tag{2.35}$$

In our analysis, we will also need to consider the line

$$l_\gamma(x) = -\frac{1}{\gamma}(x - x_\gamma) + H(x_\gamma). \tag{2.36}$$

where $\gamma > 0$ is as in Assumption 2.2.1 and

$$x_\gamma := \frac{u \circ (u')^{-1}(\gamma)}{r}, \tag{2.37}$$

Furthermore, we note that that

$$H'(x_\gamma) = -\frac{1}{\gamma} \tag{2.38}$$

and we define

$$L_\gamma := l_\gamma(0) = -H'(x_\gamma)x_\gamma + H(x_\gamma). \tag{2.39}$$

Lemma 2.5.1. *The following equivalences hold true:*

$$\begin{aligned} &\underline{q} < -\frac{1}{\gamma} \quad \Leftrightarrow \quad x_{\min} > x_\gamma \quad \Leftrightarrow \quad L > L_\gamma \\ \text{and} \quad &\underline{q} > -\frac{1}{\gamma} \quad \Leftrightarrow \quad x_{\min} < x_\gamma \quad \Leftrightarrow \quad L < L_\gamma \end{aligned} \tag{2.40}$$

Proof. In view of the identity $H'(x_\gamma) = -\frac{1}{\gamma}$ and the strict concavity of H, we can see that

$$H'(x_{\min}) = \underline{q} < -\frac{1}{\gamma} = H'(x_\gamma) \quad \Leftrightarrow \quad x_{\min} > x_\gamma.$$

If we define

$$\Lambda(x) = -xH'(x) + H(x)$$

then Λ is strictly increasing because

$$\Lambda'(x) = -xH''(x) > 0 \quad \text{for all} \quad x > 0$$

Combining this observation with the identities $L = \Lambda(x_{\min})$ and $L_\gamma = \Lambda(x_\gamma)$, which follow from (2.32) and (2.39), we obtain

$$x_{\min} > x_\gamma \quad \Leftrightarrow \quad L > L_\gamma.$$

We can establish the equivalences involving the reverse inequalities similarly. ∎

Remark 2.5.1. *When* $\underline{q} = -\frac{1}{\gamma} \Leftrightarrow x_{\min} = x_\gamma \Leftrightarrow L = L_\gamma$*, the corresponding line* $l_{\min}$ *is* l_γ.

In the study of further properties of H as well as in several other proofs, we will need the properties of functions k and α summarised in the following result.

Lemma 2.5.2. *The function* k *is such that*

$$\sup_{a\geq 0}\Big\{a + qk(a)\Big\} \begin{cases} = 0, & \textit{if } q \leq -\frac{1}{\gamma}, \\ \in]0,\infty[, & \textit{if } -\frac{1}{\gamma} < q < 0. \end{cases} \tag{2.41}$$

The function α *is* C^2 *on* $\mathbb{R} \setminus \{\gamma\}$*. Furthermore, it satisfies the following properties:*

$$\liminf_{z\to\infty} \frac{-\alpha(z)}{z} > -\infty \quad \textit{and} \quad \liminf_{z\to\infty} \frac{-k(\alpha(z))}{z^2} > -\infty, \tag{2.42}$$

$$\alpha'(z) = \begin{cases} \frac{1}{k''\circ(k')^{-1}(z)} > 0, & \textit{for } z > \gamma, \quad \textit{and} \quad \lim_{z\downarrow\gamma}\alpha'(z) = \frac{1}{k''(0)} \\ 0, & \textit{for } z < \gamma, \end{cases} \tag{2.43}$$

$$\alpha''(z) = \begin{cases} -\frac{k'''\circ(k')^{-1}(z)}{(k''\circ(k')^{-1}(z))^3} \leq 0, & \textit{for } z > \gamma, \\ 0, & \textit{for } z < \gamma. \end{cases} \tag{2.44}$$

Proof. Notice that (2.41), (2.43) and (2.44) follow from straightforward differentiation.

The first limit in (2.42) follows from the concavity of α. To derive the second one, we use the integration by parts to obtain

$$\begin{aligned} k(\alpha(z)) &= k(0) + \int_\gamma^z s\alpha'(s)ds = z\alpha(z) - \int_\gamma^z \alpha(s)ds \\ &\leq z\alpha(z) \quad \text{for all } z \geq \gamma \end{aligned} \tag{2.45}$$

This calculation and the first limit in (2.42) imply the second limit in (2.42). ∎

Proposition 2.5.3. *Let* $x_{\min}$ *and* x_γ *be the points given by (2.32) and (2.37). The function* H *defined by (2.28) satisfies the HJB equation inside the interval* $[x_\gamma,\infty[$*. Furthermore,*

$$H(x) = \sup_{c\geq c^\dagger(x)} \left\{L - \frac{rL + c}{u(c)}x\right\} \quad \textit{for all } x \geq x_{\min}. \tag{2.46}$$

where the function $c^\dagger$ *is defined in (2.30).*

Proof. The strict concavity of H and (2.38) imply that

$$H'(x) \leq -\frac{1}{\gamma} \quad \text{for all } x \geq x_\gamma. \tag{2.47}$$

In view of this inequality and (2.41), we obtain

$$\sup_{a\geq 0}\{a + H'(x)k(a)\} = 0$$

Combining this result with (2.29) and the strict concavity of H, we can see that

$$\max_{c\geq 0, z\in\mathbb{R}} \left\{ \frac{1}{2}\sigma^2 z^2 H''(x) + \Big[rx - u(c) + k(\alpha(z))\Big] H'(x) - rH(x) + \alpha(z) - c \right\} = 0 \tag{2.48}$$

for all $x \geq x_\gamma$.

By (2.47) and (2.48), it follows that H satisfies the HJB equation (2.27) inside the interval $[x_\gamma, \infty[$.

To establish (2.46), we first note that the expressions (2.30) for $c^\dagger$ imply that

$$\begin{aligned} H(x) &= L - \Big(L + \frac{c^\dagger(x)}{r}\Big) \\ &= L - \frac{(rL + c^\dagger(x))x}{u(u^{-1}(rx))} = L - \zeta(c^\dagger(x))x \end{aligned} \tag{2.49}$$

for all $x > 0$, where

$$\zeta(c) = \frac{rL + c}{u(c)}, \quad \text{for } c > 0.$$

Combining (2.35) and the fact that

$$\frac{d}{dc}\Big[u(c) - u'(c)(rL + c)\Big] = -u''(c)(rL + c) > 0,$$

We obtain

$$u(c) - u'(c)(rL + c) > 0 \quad \text{for all } c > c_{\min}.$$

Therefore,

$$\zeta'(c) = \frac{u(c) - u'(c)(rL + c)}{u(c)^2} > 0 \quad \text{for all } c > c_{\min}$$

In view of this result, the fact that $c^\dagger(x) \geq c_{\min}$ for all $x \geq x_{\min}$ and (2.49), we can see that

$$H(x) = L - \zeta(c^\dagger(x))x = \sup_{c\geq c^\dagger(x)} \Big\{L - \zeta(c)x\Big\} \quad \text{for all } x \geq x_{\min}.$$

So (2.46) follows. ∎

In view of the calculation

$$\frac{\partial}{\partial c}\Big\{rp + q\Big[u(c) - rx - k(\alpha(z))\Big] - \alpha(z) + c\Big\} = qu'(c) + 1, \quad \text{for } c > 0,$$

and the strict concavity of u, we can see that

$$\begin{aligned} &\frac{rp + q\Big[u(c^\star(q)) - rx - k(\alpha(z))\Big] - \alpha(z) + c^\star(q)}{\frac{1}{2}\sigma^2 z^2} \\ &= \inf_{c\geq 0} \left\{ \frac{rp + q\Big[u(c) - rx - k(\alpha(z))\Big] - \alpha(z) + c}{\frac{1}{2}\sigma^2 z^2} \right\} \end{aligned}$$

where

$$c^\star(q) = \begin{cases} 0, & \text{for } q \geq 0, \\ (u')^{-1}\left(-\frac{1}{q}\right), & \text{for } q < 0. \end{cases} \tag{2.50}$$

We define

$$Q(x,p,q) = rp + q[u(c^\star(q)) - rx] + c^\star(q), \quad \text{for } (x,p,q) \in [0,\infty[\times\mathbb{R}\times\mathbb{R}. \tag{2.51}$$

Notice that Q is differentiable on $[0,\infty[\times\mathbb{R}\times\mathbb{R}\setminus\{0\}$ and the left and right derivative exists along $q = 0$, hence it is locally Lipschitz.

Lemma 2.5.4. *The function Q is such that*

$$Q(0,L,q) \begin{cases} = 0, & \text{if } q = \underline{q}, \\ \geq 0, & \text{if } q \in]\underline{q},\infty[, \end{cases}$$

where $\underline{q} \leq 0$ is as in (2.32) (see also (2.33)). Furthermore,

$$Q(x,H(x),H'(x)) = 0 \quad \text{for all } x \geq 0, \tag{2.52}$$

$$Q(x,l_{\min}(x),l'_{\min}(x)) = Q(x_{\min},H(x_{\min}),H'(x_{\min})) = 0 \quad \text{for all } x \geq 0$$

$$\text{and} \quad Q(x,l_\gamma(x),l'_\gamma(x)) = Q(x_\gamma,H(x_\gamma),H'(x_\gamma)) = 0 \quad \text{for all } x \geq 0, \tag{2.53}$$

where the functions $l_{\min}$ and l_γ are defined by (2.31) and (2.36).

Proof. To simplify the notation, we define $g(q) = Q(0,L,q)$. For $q \geq 0$, $c^\star(q) = 0$ and

$$g(q) = rL \tag{2.54}$$

To proceed further, we may assume that $\underline{q} < 0$. In view of (2.29) and the system of equations (2.32), we can see that

$$\begin{aligned} 0 &= \min_{c\geq 0}\left\{rH(x_{\min}) + H'(x_{\min})(u(c) - rx_{\min}) + c\right\} \\ &= \min_{c\geq 0}\left\{rL + \underline{q}u(c) + c\right\} = g(\underline{q}). \end{aligned} \tag{2.55}$$

On the other hand,

$$g'(q) = u \circ (u')^{-1}\left(-\frac{1}{q}\right) > 0 \quad \text{for all } q < 0 \tag{2.56}$$

Combining this result with (2.54) and (2.55), we conclude $g(q) > 0$ for all $q \in]\underline{q},0[$. Finally, (2.52) follows from a comparison of the definition of Q and (2.28), while (2.53) follows from straightforward calculation. ∎

We define

$$I_{x,p,q}(z) = \frac{Q(x,p,q) - qk(\alpha(z)) - \alpha(z)}{\frac{1}{2}\sigma^2 z^2} \tag{2.57}$$

for $(x,p,q) \in [0,\infty[\times\mathbb{R}^2$ and $z \in \mathbb{R}\setminus\{0\}$, as well as

$$K(x,p,q) = \inf_{z\geq\gamma} I_{x,p,q}(z). \tag{2.58}$$

It is well defined due to (2.42) of Lemma 2.5.2. Also, we consider the domain

$$D = \{(x,p,q) \in [0,\infty[\times\mathbb{R}^2 \big| K(x,p,q) < 0\}.$$

Lemma 2.5.5. *The following statements hold true:*

(i) $[0,\infty[\times\mathbb{R}\times\mathbb{R}^+\subset D$.

(ii) Given any point $(x,p,q)\in D$, *there exists a unique* $z^\star = z^\star(x,p,q)\in[\gamma,\infty[$ *such that*

$$K(x,p,q) = I_{x,p,q}(z^\star).$$

where $z^\star$ *is* C^1 *in* D *with bounded derivative up to* ∂D *and* K *is locally Lipschitz in* D *in the sense that for any positive integer* m, *the restriction on* $[0,m]\times[-m,m]^2\cap D$ *is Lipschitz.*

(iii) If $(x,p,q)\in D$ *and* $Q(x,p,q)\geq 0$,

$$K(x,p,q) = \inf_{z\in\mathbb{R}\setminus\{0\}} I_{x,p,q}(z).$$

(iv) When $\underline{q}\leq -\frac{1}{\gamma}\Leftrightarrow x_{\min}\geq x_\gamma \Leftrightarrow L\geq L_\gamma$,

$$K(x,l_{\min}(x),l'_{\min}(x)) = 0$$

for all $x\in[0,\infty[$.

Proof of (i). If $q\geq 0$, notice that $\lim_{z\to\infty}\alpha(z)=\infty$, we have $K(x,p,q)<0$. Hence the result.
Proof of (ii). Now for any $(x,p,q)\in D$, we use the identity $k'(\alpha(z))=z$, which is true for all $z>\gamma$ to calculate,

$$\frac{d}{dz}I_{x,p,q}(z) = \frac{1}{\frac{1}{2}\sigma^2 z^3}L_{x,p,q}(z),$$

where

$$L_{x,p,q}(z) = q\Big[2k(\alpha(z)) - z^2\alpha'(z)\Big] + 2\alpha(z) - z\alpha'(z) - 2Q(x,p,q).$$

Furthermore

$$\frac{d}{dz}L_{x,p,q}(z) = \alpha'(z) - z\alpha''(z)(1+qz).$$

Viewing $L_{x,p,q}(z)$ as a multivariable function $L(x,p,q,z)$, we further compute

$$\frac{\partial L}{\partial z} = \frac{d}{dz}L_{x,p,q}(z) = \alpha'(z) - z\alpha''(z)(1+qz), \tag{2.59}$$

$$\frac{\partial L}{\partial x} = 2rq,$$

$$\frac{\partial L}{\partial p} = -2r,$$

$$\frac{\partial L}{\partial q} = \begin{cases} 0, & \text{if } q\geq 0,\\ 2[k(\alpha(z)) - u\circ(u')^{-1}(-\frac{1}{q}) + rx] - z^2\alpha'(z), & \text{if } q<0. \end{cases} \tag{2.60}$$

Now we face three cases:

Case 1. $q \geq 0$
If this is the case, then the inequality in (2.44) implies that

$$\frac{d}{dz} L_{x,p,q}(z) \geq \alpha'(z) > 0 \quad \text{for all } z > \gamma, \tag{2.61}$$

which implies

$$\lim_{z\to\infty} L_{x,p,q}(z) = \infty \tag{2.62}$$

because $\lim_{z\to\infty} \alpha(z) = \infty$. We now have two sub-cases.

Sub-case 1.1 The first arise if $\lim_{z\downarrow\gamma} L_{x,p,q}(z) \geq 0$, which is equivalent to $Q(x,p,q) \leq -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}$, where α'_+ denotes the right derivative. In this case, (2.61) implies that the minimizer is at $z^\star = z^\star(x,p,q) = \gamma$.

Sub-case 1.2 The second case arises if $\lim_{z\downarrow\gamma} L_{x,p,q}(z) < 0$, which is equivalent to $Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}$. In this case, (2.62) and intermediate theorem imply that there exists $z^\star = z^\star(x,p,q) > \gamma$ such that $L_{x,p,q}(z^\star) = 0$ and it is unique by (2.61). We proceed to argue our $z^\star$ is locally bounded. Consider the function

$$\bar{z}(x,p,q) = \inf\{z \geq \gamma \big| \alpha(z) + L_{x,p,q}(\gamma) \geq 0\},$$

which is finite as $\lim_{z\to\infty} \alpha(z) = \infty$ and it is also continuous. Then by (2.61), we will have $L_{x,p,q}(z) \geq \alpha(z) + L_{x,p,q}(\gamma)$ for $z \geq \gamma$. Hence, $z^\star(x,p,q) \leq \bar{z}(x,p,q)$, which is locally bounded on $[0,\infty[\times\mathbb{R}\times\mathbb{R}^+$. By (2.59) and the implicit function theorem, we deduce that $z^\star(x,p,q)$ is C^1 on

$$\left\{(x,p,q) \in]0,\infty[\times\mathbb{R}^2 \big| Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}\right\},$$

with

$$\begin{aligned}
\frac{\partial z^\star}{\partial x}(x,p,q) &= -\left(\frac{\partial L}{\partial z}\right)^{-1} \frac{\partial L}{\partial x}\Big(x,p,q,z^\star(x,p,q)\Big),\\
\frac{\partial z^\star}{\partial p}(x,p,q) &= -\left(\frac{\partial L}{\partial z}\right)^{-1} \frac{\partial L}{\partial p}\Big(x,p,q,z^\star(x,p,q)\Big),\\
\frac{\partial z^\star}{\partial q}(x,p,q) &= -\left(\frac{\partial L}{\partial z}\right)^{-1} \frac{\partial L}{\partial q}\Big(x,p,q,z^\star(x,p,q)\Big),
\end{aligned} \tag{2.63}$$

and these derivatives are bounded as (x,p,q) tends to the boundary of

$$\left\{(x,p,q) | Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}\right\}.$$

Case 2. $-\frac{1}{\gamma} < q < 0$.
The fact that $K(x,p,q) < 0$ implies that

$$\max_{z\geq\gamma}\{qk(\alpha(z)) + \alpha(z)\} > Q(x,p,q),$$

where the maximum is attained at $-\frac{1}{q} > \gamma$. Furthermore, as

$$I_{x,p,q}(z) > I_{x,p,q}\left(-\frac{1}{q}\right) \quad \text{for all } z > -\frac{1}{q},$$

we can see that minimizer of $I_{x,p,q}(z)$ over $z \geq \gamma$ exists and belongs to$[\gamma, -\frac{1}{q}]$. Notice that

$$\frac{d}{dz}L_{x,p,q}(z) = \alpha'(z) - z\alpha''(z)(1+qz) > 0 \quad \text{for all } z \in]\gamma, -\frac{1}{q}],$$

we have the minimizer $z^\star$ of $I_{x,p,q}(z)$ is unique. Again, we have two sub-cases

Sub-case 2.1 The first case is when

$$\lim_{z\downarrow\gamma} L_{x,p,q}(z) \geq 0 \Leftrightarrow Q(x,p,q) \leq -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2},$$

then the minimizer is at $z^\star = \gamma$.

Sub-case 2.2 The second case is when

$$\lim_{z\downarrow\gamma} L_{x,p,q}(z) < 0 \Leftrightarrow Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2},$$

then $z^\star \in]\gamma, -\frac{1}{q}]$. By (2.59) and the implicit function theorem, we deduce that $z^\star(x,p,q)$ is C^1 on

$$\{(x,p,q) \in]0,\infty[\times\mathbb{R}^2 | Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}\},$$

and these derivatives (2.59) are bounded as (x,p,q) tends to the boundary of $\{(x,p,q)|Q(x,p,q) > -\frac{\alpha'_+(\gamma)(\gamma+q\gamma^2)}{2}\}$.

Case 3. $q < -\frac{1}{\gamma}$.
Then the minimizer is at $z^\star = \gamma$.

To sum up, we can see that $z^\star$ is C^1 in D with bounded derivative up to ∂D.

Suppose now we have $(x_1,p_1,q_1) \in [0,m]\times[-m,m]^2$, $(x_2,p_2,q_2) \in [0,m]\times[-m,m]^2 \cap D$ and $K(x_1,p_1,q_1) \geq K(x_2,p_2,q_2)$, we can see that

$$\begin{aligned}|K(x_1,p_1,q_1) - K(x_2,p_2,q_2)| &= K(x_1,p_1,q_1) - K(x_2,p_2,q_2)\\ &\leq I_{x_1,p_1,q_1}\big(z^\star(x_2,p_2,q_2)\big) - I_{x_2,p_2,q_2}\big(z^\star(x_2,p_2,q_2)\big)\\ &\leq C\left(|x_1-x_2| + |p_1-p_2| + |q_1-q_2|\right), \end{aligned} \tag{2.64}$$

for some constant C. Hence local Lipschitz is established.
Proof of (iii). This is because for $Q(x,p,q) \geq 0$, and any $z \in]-\infty,\gamma]\setminus\{0\}$, we have $I_{x,p,q}(z) \geq 0$.
Proof of (iv). This is due to (2.41), (2.53) and the definition of K. ∎

Remark 2.5.2. *The set $D \cup \partial D$ may be a strict subset of $[0,\infty[\times\mathbb{R}^2$ or the identity $D \cup \partial D = [0,\infty[\times\mathbb{R}^2$ may be true, depending on the choice of k. To see this claim, first we choose L such that $\underline{q} < -\frac{1}{\gamma}$. Then we have $Q(0, L, -\frac{1}{\gamma}) > 0$.*

1. *For $k(a) = a^2 + \gamma a$, we have*

$$\alpha(z) = \begin{cases} \frac{z-\gamma}{2}, & \textit{for}\, z > \gamma, \\ 0, & \textit{for}\, z < \gamma, \end{cases} \tag{2.65}$$

and

$$k(\alpha(z)) = \begin{cases} \left(\frac{z-\gamma}{2}\right)^2 + \gamma\left(\frac{z-\gamma}{2}\right), & \textit{for}\, z > \gamma, \\ 0, & \textit{for}\, z \le \gamma, \end{cases} \tag{2.66}$$

Now we can see that $I_{0,L,-\frac{1}{\gamma}}(z) > 0$ for all $z \ge \gamma$ and $\lim_{z\to\infty} I_{0,L,-\frac{1}{\gamma}}(z) = \frac{1}{4\gamma} > 0$. Hence $K(0, L, -\frac{1}{\gamma}) > 0$, $D \cup \partial D$ is a strict subset of $[0,\infty[\times\mathbb{R}^2$.

2. *Let $k(a) = a^3 + \gamma a$, we have*

$$\alpha(z) = \begin{cases} \left(\frac{z-\gamma}{3}\right)^{\frac{1}{2}}, & \textit{for}\, z > \gamma, \\ 0, & \textit{for}\, z \le \gamma, \end{cases} \tag{2.67}$$

and

$$k(\alpha(z)) = \begin{cases} \left(\frac{z-\gamma}{3}\right)^{\frac{3}{2}} + \gamma\left(\frac{z-\gamma}{3}\right)^{\frac{1}{2}}, & \textit{for}\, z > \gamma, \\ 0, & \textit{for}\, z \le \gamma, \end{cases} \tag{2.68}$$

Now we can see that $\lim_{z\to\infty} I_{x,p,q}(z) = 0$ so that $K(x,p,q) \le 0$ and $D \cup \partial D = [0,\infty[\times\mathbb{R}^2$.

Lemma 2.5.6. *Given L, suppose that $\underline{q} = \underline{q}(L) > -\frac{1}{\gamma}$, where $\underline{q}$ is as in (2.32) and γ is as in Assumption 2.2.1. Let*

$$\bar{q} = \bar{q}(L) = \inf\{q > \underline{q} | K(0, L, q) = 0\}.$$

Then for any $q \in [\underline{q}, \bar{q}] \cap \mathbb{R}$, the ODE

$$F''(x) = K(x, F(x), F'(x)) \tag{2.69}$$

with initial conditions

$$F(0) = L, \;\; F'(0) = q \tag{2.70}$$

has a unique concave solution $F_q :]0, \kappa_q[\to \mathbb{R}$, where $[0, \kappa_q[$ is the largest neighbourhood for the existence and uniqueness of solution to the initial value problem.

Furthermore, there exists $q^\star \in [\underline{q}, \bar{q}] \cap \mathbb{R}$ such that $\kappa_{q^\star} = \infty$, $F_{q^\star} \ge H$ on $[0,\infty[$ and $x_c = \inf\{x > 0; F_{q^\star}(x) = H(x)\} \le x_\gamma$ with $F'_{q^\star}(x_c) = H'(x_c)$.

Proof.

Step.1 Existence and uniqueness of a solution F_q to the ODE (2.69) with K replaced by $\tilde{K}$. Set

$$\tilde{K}(x,p,q) = \begin{cases} K(x,p,q), & \text{if } (x,p,q) \in D, \\ 0, & \text{if } (x,p,q) \in D^c. \end{cases}$$

We observe that $\tilde{K}$ is locally Lipschitz. To show it, it is enough to consider points $(x_1,p_1,q_1) \in [0,m] \times [-m,m]^2$ and $(x_1,p_1,q_1) \in [0,m] \times [-m,m]^2 \cap D$ with $K(x_1,p_1,q_1) \geq K(x_2,p_2,q_2)$. We know

$$\begin{aligned} \left| \tilde{K}(x_1,p_1,q_1) - \tilde{K}(x_2,p_2,q_2) \right| &\leq K(x_1,p_1,q_1) - K(x_2,p_2,q_2) \\ &\leq C\left(|x_1 - x_2| + |p_1 - p_2| + |q_1 - q_2|\right) \end{aligned}$$

by equation (2.64). Hence the result. Because of this, we can solve (2.69) with K replaced by $\tilde{K}$ given any initial condition at 0. In particular, for initial condition (2.70), there exists a unique solution F_q on $[0,\kappa_q[$, where $[0,\kappa_q[$ is the largest neighbourhood for the existence and uniqueness.

Step.2 Given any $q \in [\underline{q},\bar{q}] \cap \mathbb{R}$, we show that $F_q''(x) < 0$ for all $x \in [0,\kappa_q[$.

In the presence of the assumption $\underline{q} > -\frac{1}{\gamma}$, we can see that (2.41), (2.55) and the definition of K (2.58) imply that

$$K(0,L,q) < 0,$$

for $q = \underline{q}$ as well as for all $q \geq 0$. It follows that either $\bar{q} \in]\underline{q},0[$ or $\bar{q} = \infty$.

We have $F_q''(0) < 0$ for any $q \in [\underline{q},\bar{q}[$. From this, we further obtain $F_q''(x) < 0$ for $x \in [0,\kappa_q[$. To see this claim, we argue by contradiction and assume that

$$x_l := \inf\{x > 0 | F_q''(x) = \tilde{K}(x,F(x),F'(x)) = 0\} < \kappa_q$$

Since both K and $\tilde{K}$ are locally Lipschitz thus continuous on $\mathbb{R}^+ \times \mathbb{R}^2$, together with the fact that K and $\tilde{K}$ coincide in D, we have $K = \tilde{K}$ on the closure of D. In particular, $K(x_l,F(x_l),F'(x_l)) = \tilde{K}(x_l,F(x_l),F'(x_l)) = 0$

In view of the fact that

$$\begin{aligned} F''(x_l) &= \tilde{K}(x_l,F(x_l),F'(x_l)) \\ &= K(x_l,F(x_l),F'(x_l)) \\ &= \inf_{c\geq 0, z\geq\gamma} \frac{rF(x_l) + F'(x_l)(u(c) - rx_l - k(\alpha(z))) - \alpha(z) + c}{\frac{1}{2}\sigma^2 z^2} \\ &= 0, \end{aligned}$$

we can see that if we define

$$l(x) = F_q(x_l) + F_q'(x_l)(x - x_l),$$

then l satisfies the ODE (2.69) with K replaced by $\tilde{K}$ in the domain $]0,\infty[\supset]0,\kappa_q[$. Combing this observation with the fact that F_q is the unique solution of the

ODE(2.69) with K replaced by $\tilde{K}$, $l(x_l) = F_q(x_l)$ and $l'(x_l) = F_q'(x_l)$ in the neighborhood $[0, x_l]$, we have $F_q(x) = l(x)$ on $[0, x_l]$ which contradicts that $F_q''(x) < 0$ in $]0, x_l[$.

Now as $\tilde{K}(x, F_q(x), F_q'(x)) = K(x, F_q(x), F_q'(x)) < 0$ for all $q \in [\underline{q}, \bar{q}[$ and $x \in [0, \kappa_q[$, we conclude $F_q(x)$ is the unique concave solution to (2.69).

Step.3 Some comparison result.

We first show that given any real number q_1, q_2 such that

$$\underline{q} \le q_1 < q_2 \le \bar{q},$$

we have

$$F_{q_1}(x) < F_{q_2}(x) \, for\ all\ \ x \in]0, \kappa_{q_1} \wedge \kappa_{q_2}[. \tag{2.71}$$

In the case that $q_2 = \bar{q} < \infty$, we can check that

$$F_{q_2}(x) = q_2 x + L.$$

Therefore the concave function $F_{q_1}(x) < F_{q_2}(x)$ for all $x \in]0, \kappa_{q_1}[$.

Now assume that we are in the case $q_2 < \bar{q}$. In the view of the observations that

$$F_{q_1}(0) = F_{q_2}(0) = L \quad and \quad F_{q_1}'(0) = q_1 < q_2 = F_{q_2}'(0),$$

we can see that (2.71) will follow if we show that

$$x_l := \inf\{x \in]0, \kappa_{q_1} \wedge \kappa_{q_2}[| F_{q_1}'(x) \ge F_{q_2}'(x)\} \wedge \kappa_{q_1} \wedge \kappa_{q_2} = \kappa_{q_1} \wedge \kappa_{q_2}. \tag{2.72}$$

To this end, we argue by contradiction and we assume that $x_l < \kappa_{q_1} \wedge \kappa_{q_2}$. We already know that $F_{q_i}''(x_l)$ is strictly negative for $i = 1, 2$, so that

$$K(x_l, F_{q_i}(x_l), F_{q_i}'(x_l)) = I_{x_l, F_{q_i}(x_l), F_{q_i}'(x_l)}\Big(z^\star(x_l, F_{q_i}(x_l), F_{q_i}'(x_l))\Big) \quad \text{for } i = 1, 2.$$

The identities $F_{q_1}(x_l) < F_{q_2}(x_l)$ and $F_{q_1}'(x_l) = F_{q_2}'(x_l)$ imply that

$$\begin{aligned} F_{q_1}''(x_l) &= I_{x_l, F_{q_1}(x_l), F_{q_1}'(x_l)}\Big(z^\star(x_l, F_{q_1}(x_l), F_{q_1}'(x_l))\Big) \\ &\le I_{x_l, F_{q_1}(x_l), F_{q_1}'(x_l)}\Big(z^\star(x_l, F_{q_2}(x_l), F_{q_2}'(x_l))\Big) \\ &< I_{x_l, F_{q_2}(x_l), F_{q_2}'(x_l)}\Big(z^\star(x_l, F_{q_2}(x_l), F_{q_2}'(x_l))\Big) \\ &= F_{q_2}''(x_l). \end{aligned}$$

However, this inequality contradicts the definition of x_l.

Next we show that for $q \in [\underline{q}, \bar{q}[$ such that F_q satisfies the ODE (2.69) with

$$F(x_0) = l_\gamma(x_0), \;\; F'(x_0) > -\frac{1}{\gamma}, \text{for some } x_0 \in [0, \kappa_q[,$$

then $\kappa_q = \infty$ and $F_q(x) > l_\gamma(x)$ for all $x > x_0$, where l_γ is defined in (2.36).

To prove this, it is enough to show $F_q'(x) > -\frac{1}{\gamma}$ for all $x \in [x_0, \kappa_q[$. If this is not the case, we have that

$$x_l := \inf\{x > x_0 | F_q'(x) \leq -\frac{1}{\gamma}\} < \kappa_q.$$

But then,

$$\begin{aligned} F_q''(x_l) &= K(x_l, F_q(x_l), -\frac{1}{\gamma}) = I_{x_l, F_q(x_l), -\frac{1}{\gamma}}(\gamma) \\ &> I_{x_l, l_\gamma(x_l), -\frac{1}{\gamma}}(\gamma) = 0, \end{aligned}$$

which is a contradiction. So $F_q'(x) > -\frac{1}{\gamma}$ for all $x \in [x_0, \kappa_q[$.

From concavity, the fact that $F_q(x) > l_\gamma(x)$ for all $x \in]0, \kappa_q[$, we have $\kappa_q = \infty$ due to the continuation of ODE result in Chapter 1 Section 4 of Coddington and Levinson [6].

Step.4 We will show that given any $q \in [\underline{q}, \bar{q}[$,

$$\chi(q) := \inf\{x \in [0, \kappa_q[| F_q(x) \leq H(x)\} \in]0, x_\gamma[\cup \{\infty\}, \tag{2.73}$$

and in particular,

$$\chi(\underline{q}) \in [0, x_\gamma[. \tag{2.74}$$

In view of the strict concavity of F_q and the initial conditions $F_{\underline{q}}(0) = L \geq 0$, $F_{\underline{q}}'(0) = \underline{q}$, we can see that $F_{\underline{q}}(x) < l_{\min}(x)$ for all $x > 0$, where $l_{\min}$ is the straight line defined by (2.31). Combining this observation with the facts

$$x_{\min} < x_\gamma \quad \textit{and} \quad l_{\min}(x_{\min}) = H(x_{\min}),$$

which follow from (2.40) and the definition of $l_{\min}$, respectively, we obtain (2.74).

To show (2.73), we argue by contradiction. To this end, we assume that there exists q such that $\chi(q) \in [x_\gamma, \infty[$. Such assumption, the definition of $\chi(q)$ and the definition of l_γ imply that

$$F_q(x_\gamma) \geq H(x_\gamma) = l_\gamma(x_\gamma).$$

On the other hand the definition of (2.40) and l_γ imply that

$$F_q(0) = L < L_\gamma = l_\gamma(0).$$

In view of these inequalities and the strict concavity of the function $F_q - l_\gamma$, we can see that there exists a unique point $\tilde{x} \in]0, x_\gamma[$ such that

$$F_q(\tilde{x}) = l_\gamma(\tilde{x}) \quad \textit{and} \quad F_q'(\tilde{x}) > l_\gamma'(\tilde{x}).$$

By the second comparison result we developed in Step 3, we can see that

$$F_q(x) > l_\gamma(x) \geq H(x) \, \forall x > \tilde{x},$$

which contradicts the assumption $\chi(q) \in [x_\gamma, \infty[$.

Step.5 There exists a slope $q^\dagger \in]\underline{q}, \bar{q}] \cap \mathbb{R}$ such that $F_{q^\dagger}(x) > H(x)$ for all $x > 0$.

We first observe that from concavity, for q such that $F_q(x) > H(x)$ for all $x \in]0, \kappa_q[$, we have $\kappa_q = \infty$ due to the continuation of ODE result in Chapter 1 Section 4 of Coddington and Levinson [6].

If $\bar{q} < \infty$, note that $F''_{\bar{q}}(0) = 0$, we deduce that

$$F_{\bar{q}}(x) = \bar{q}x + L$$

is a solution to the ODE (2.69) on $[0, \infty[$.

Notice that since $\bar{q} > \underline{q}$, we have the line

$$F_{\bar{q}}(x) > L + \underline{q}x \geq H(x),$$

for any $x \in]0, \infty[$. So $q^\dagger := \bar{q}$ will do.

Now suppose that $\bar{q} = \infty$. Given any $q > 0$ fixed, we define

$$\bar{\chi}(q) := \inf\{x > 0 | F'_q(x) = \frac{q}{2}\} > 0.$$

If $\bar{\chi}(q) = \infty$, then we can take the value of q we consider for $q^\dagger$ because the inequality $F'_q(x) > \frac{q}{2}$ for all $x > 0$ and the initial condition $F_q(0) = L$ imply that $F_q(x) > L \geq 0 > H(x)$ for all $x > 0$. So, we may assume that $\bar{\chi}(q) < \infty$.

Given any $x \in [0, \bar{\chi}(q)]$,

$$rF_q(x) - rF'_q(x)x \geq rF_q(0) = rL$$

and $c^\star(F'_q(x)) = 0$ (see also 2.50). Combining these observations with the estimates given by Lemma 2.5.2, we obtain

$$F''_q(x) \geq \min_{z \geq \gamma} \frac{rL - qk(\alpha(z)) - \alpha(z)}{\frac{1}{2}\sigma^2 z^2} \geq -C(q+1).$$

Using this estimate, we calculate

$$\frac{q}{2} = F'_q(0) - F'_q(\bar{\chi}(q)) = -\int_0^{\bar{\chi}(q)} F''_q(y)dy \leq C(q+1)\bar{\chi}(q),$$

so that

$$\chi(q) > \bar{\chi}(q) \geq \frac{q}{2C(q+1)} > \frac{1}{4C},$$

for $q \geq 1$. In view of this inequality and the definition of $\bar{\chi}(q)$, we can see that

$$\begin{aligned} F_q(\frac{1}{4C}) &= F_q(0) + \int_0^{\frac{1}{4C}} F'_q(y)dy \\ &> L + \frac{q}{8C}. \end{aligned}$$

It follows that, if we choose any $q^\dagger > 8Cl_\gamma(\frac{1}{4C})$, then $F_{q^\dagger}(\frac{1}{4C}) > l_\gamma(\frac{1}{4C})$ where l_γ is defined by (2.36). By comparison, we have $F_{q^\dagger}(x) > l_\gamma(x) > H(x)$ for all $x \in]\frac{1}{4C}, \infty[$.

We also know $F_{q^\dagger}(x) > L > H(x)$ for all $x \in]0, \frac{1}{4C}] \subset]0, \bar{\chi}(q)]$. Together, we have established the result.

Step.6 Existence and Uniqueness of $q^\star \in]\underline{q}, \bar{q} \wedge q^\dagger[$ *such that* $F_{q^\star} \geq H$ *for all* $x \geq 0$ *and* $\chi(q^\star) \in]0, x_\gamma[$, *where* χ *is defined by (2.73).*

Define $S = \{q \in [\underline{q}, \bar{q}] \cap \mathbb{R} \mid \chi(q) \in [0, x_\gamma]\}$ and note that S is nonempty thanks to (2.74).

Given any $q \in S$, the comparison result we established in Step 3 implies that $[\underline{q}, q] \subset S$. Therefore, S is an interval. The required point is given by

$$q^\star = \sup S \leq q^\dagger.$$

To show $q^\star \in S$, we consider any strictly increasing sequence (q_n) such that $\lim_{n\to\infty} q_n = q^\star$. The corresponding sequence $(\chi(q_n))$ is increasing and bounded by x_γ by Step 3 and 4. Therefore $x_c := \lim_{n\to\infty} \chi(q_n)$ exists in $]0, x_\gamma]$. In view of the continuous dependence of a solution to an ODE with respect to initial parameters (see Theorem 7.5 in Chapter 1, Coddington and Levinson [6]), we can see that the identities $F_{q_n}(\chi(q_n)) = H(\chi(q_n))$ imply in the limit that $F_{q^\star}(x_c) = H(x_c)$.

We now consider any strictly decreasing sequence (q_n) such that $\lim_{n\to\infty} q_n = q^\star$. The definition of S implies that $\kappa_{q_n} = \infty$ and $F_{q_n}(x) > H(x)$ for all $x > 0$ and $n \geq 1$. By passing to the limit as $n \to \infty$, we obtain $\kappa_{q^\star} = \infty$ and $F_{q^\star}(x) \geq H(x)$ for all $x > 0$. ∎

From now on, F will denote the solution to ODE(2.69) with initial slope $q^\star$.

Lemma 2.5.7. *Suppose that* $q^\star > \kappa$, *where* $q^\star$ *refers to that of Lemma 2.5.6 and* κ *refers to (2.12), then for some* $L_\kappa \in]L, L_\gamma[$ *and* $\tilde{x}_c > x_c$, *there exists* $\tilde{F}$ *on* $[0, \infty[$ *satisfying the ODE (2.69) such that* $\tilde{F} \geq H$ *on* $[0, \infty[$ *and*

$$\tilde{F}(0) = L_\kappa, \tilde{F}'(0) = \kappa,$$
$$\tilde{F}(\tilde{x}_c) = H(\tilde{x}_c), \tilde{F}'(\tilde{x}_c) = H'(\tilde{x}_c).$$

Proof. For any $l \in [L, L_\gamma]$, we denote F_l to be the solution to ODE (2.69) such that $F_l \geq H$ on $[0, \infty[$,

$$F_l(0) = l, F(x_c^l) = H(x_c^l) \text{ and } F'(x_c^l) = H'(x_c^l),$$

for some $x_c^l \in [0, x_\gamma[$. The existence and uniqueness of the family of concave solutions on $[0, \infty[$ follows directly from Lemma 2.5.6 and the ODE continuation result in Chapter 1 Section 4 of Coddington and Levinson [6]. By the comparison result whose proof is similar to Step 3 of Lemma 2.5.6, we also have that $[L, L_\gamma] \ni l \mapsto F_l'(0)$ is strictly decreasing and continuous. The fact that $F_L'(0) > \kappa$ and $F_{L_\gamma}'(0) < 0$ implies that there exists L_κ such that $F_{L_\kappa}'(0) = \kappa$. We set $\tilde{F} = F_{L_\kappa}$ and $\tilde{x}_c = x_c^{L_\kappa}$ will do. ∎

Lemma 2.5.8. *Let* f *be a concave solution to ODE (2.69) on* $[0, \infty[$ *such that for some* $x_c > 0$, *we have*

$$f(x_c) = H(x_c), \quad f'(x_c) = H'(x_c)$$
$$f(x) > H(x) \quad \textit{for all } x \in [x_0, x_c[.$$

Suppose further that $Q(0, f(0), f'(0)) \geq 0$, *then we have*

$$Q(x, f(x), f'(x)) \geq 0 \quad \textit{for all } x \in [0, x_c].$$

Proof. Note that $Q(x_c, f(x_c), f'(x_c)) = Q(x_c, H(x_c), H'(x_c)) = 0$.

We argue by contradiction that if there exists $x' \in]0, x_c[$ such that

$$Q(x', f(x'), f'(x')) < 0.$$

Note that since

$$Q(x, f(x), f'(x)) = rf(x) + f'(x)\Big(u(c^\star(f'(x))) - rx\Big) + c^\star(f'(x)),$$

Q is a continuous function in x. Without loss of generality, suppose $Q(x, f(x), f'(x))$ attains minimum at x' in $]0, x_c[$.

If $f'(x') \geq 0$ so that $c^\star(f'(x')) = 0$, we would have

$$Q(x', f(x'), f'(x')) = rf(x') - rxf'(x') < 0.$$

But by concavity of f,

$$\begin{aligned} rf(x') - rx'f'(x') &\geq rf(0) \\ &= Q(0, f(0), f'(0)) \\ &\geq 0, \end{aligned}$$

which is a contradiction.

Else if $f'(x') < 0$ so that $c^\star(f'(x')) > 0$ is differentiable at x'. Differentiate $Q(x, f(x), f'(x))$ at x', we have $f''(x')\Big(u(c^\star(f'(x'))) - rx'\Big) = 0$ implies $u\Big(c^\star(f'(x'))\Big) = rx'$. Plug this back,

$$Q(x', f(x'), f'(x')) = rf(x') + c^\star(f'(x')) = rf(x') + u^{-1}(rx') < 0,$$

which is a contradiction to $f(x) > H(x), \forall x \in]0, x_c[$. ∎

Remark 2.5.3. *In particular, we have $Q(x, F(x), F'(x)) \geq 0$ for all $x \in [0, x_c]$ and $Q(x, \tilde{F}(x), \tilde{F}'(x)) \geq 0$ for all $x \in [0, \tilde{x}_c]$.*

2.6 The solution to the HJB equation that identifies with the value function

We now use the solution to the HJB equation that we derived in the previous section to derive the solution to the principal's optimisation problem. To this end, we need to consider three different cases that are determined by the points $\underline{q}$ and $q^\star$, which are as in (2.32) and as in Lemma 2.5.6, respectively.

If $\underline{q} \leq -\frac{1}{\gamma}$, then we define

$$S_0(x) = \begin{cases} L + \underline{q}x, & \text{for } x \in [0, x_{\min}[, \\ H(x), & \text{for } x \in [x_{\min}, \infty[, \end{cases} \tag{2.75}$$

where $x_{\min}$ is as in (2.32) and H is defined by (2.28).

If $\underline{q} > -\frac{1}{\gamma}$ and $q^\star \le \kappa$, then we define

$$S_1(x) = \begin{cases} F(x), & \text{for } x \in [0, x_c[, \\ H(x), & \text{for } x \in [x_c, \infty[, \end{cases} \tag{2.76}$$

where x_c is as in Lemma 2.5.6 and $F = F_{q^\star}$.

If $\underline{q} > -\frac{1}{\gamma}$ and $q^\star > \kappa$, then we define

$$S_2(x) = \begin{cases} \tilde{F}(x), & \text{for } x \in [0, \tilde{x}_c[, \\ H(x), & \text{for } x \in [\tilde{x}_c, \infty[, \end{cases} \tag{2.77}$$

where $\tilde{x}_c$ and $\tilde{F}$ are defined in Lemma 2.5.7.

Theorem 2.6.1. *Consider the stochastic control problem defined by (2.24)-(2.25) and suppose that the functions u, k satisfy the conditions in Assumption 2.2.1. The following cases hold true.*

Case 1. *If $\underline{q} \le -\frac{1}{\gamma}$, then $V = S_0$ and the optimal admissible control is deterministic and given by*

$$C_t^\star = \begin{cases} c_{\min}, & \text{if } X_t^\star \in [0, x_{\min}[, \\ u^{-1}(rx), & \text{if } X_0^\star = x \ge x_{\min}, \end{cases}$$

$$\tau^\star = \begin{cases} \frac{1}{r}\ln(1 + \frac{rx}{u(c_{\min}) - rx}), & \text{if } X_0^\star = x \in [0, x_{\min}[, \\ \infty, & \text{if } X_0^\star = x \ge x_{\min}, \end{cases}$$

and $Z_t^\star = R_t^\star = 0$.

Case 2. *If $\underline{q} > -\frac{1}{\gamma}$ and $q^\star \le \kappa$, then $V = S_1$ and the optimal admissible control is given by*

$$\begin{aligned} C_t^\star &= c^\star(F'(X_t^\star))\mathbf{1}_{\{X_t^\star \in [0, x_c[\}} + u^{-1}(rX_t^\star)\mathbf{1}_{\{X_t^\star \in [x_c, \infty[\}}, \\ \tau^\star &= \inf\{t \mid X_t^\star = 0\}, \\ Z_t^\star &= z^\star(X_t^\star, S_1(X_t^\star), S_1'(X_t^\star))\mathbf{1}_{\{X_t^\star \in [0, x_c[\}}, \end{aligned}$$

$R_t^\star = 0$, where $c^\star$ is as in (2.50) and $z^\star$ is as in Lemma 2.5.5.

Case 3. *If $\underline{q} > -\frac{1}{\gamma}$ and $q^\star > \kappa$, then $V = S_2$ and the optimal control is given by*

$$\begin{aligned} C_t^\star &= c^\star(\tilde{F}'(X_t^\star))\mathbf{1}_{\{X_t^\star \in [0, \tilde{x}_c[\}} + u^{-1}(rX_t^\star)\mathbf{1}_{\{X_t^\star \in [\tilde{x}_c, \infty[\}}, \\ \tau^\star &= \infty, \\ Z_t^\star &= z^\star(X_t^\star, S_2(X_t^\star), S_2'(X_t^\star))\mathbf{1}_{\{X_t^\star \in [0, \tilde{x}_c[\}} \end{aligned}$$

and $R_t^\star$ is the minimum process that reflects $X^\star$ at 0 in the positive direction.

To prove this result, we will use a mix of a verification argument with the following result.

Lemma 2.6.2. *Let $x_b = x_{\min} \vee x_\gamma$, where $x_{\min}$ and x_γ are given by (2.32) and (2.37) respectively. Then we have*

$$V(x) = \sup_{c \geq c^\dagger(x)} \Big\{ L - \frac{rL + c}{u(c)} x \Big\} = H(x) \quad \text{for all } x \geq x_b,$$

where $c^\dagger$ is as in (2.30). Furthermore, the choice $C_t^\star = u^{-1}(rx)$, $\tau^\star = \infty$ and $R_t^\star = Z_t^\star = 0$ for all $t \geq 0$ is the optimal admissible control for every initial $x \geq x_b$.

Proof. For all $x \geq x_b$, we choose any admissible control $(C, \tau, R, Z) \in \mathcal{C}$ and let X be the solution to the SDE (2.24) . Also, we define

$$c_x = u^{-1}\Big(\frac{x}{\mathbb{E}[\int_0^\tau e^{-rt} dt]}\Big) \tag{2.78}$$

and we note that the inequality

$$x = \mathbb{E}\left[\int_0^\tau e^{-rt} u(c_x) dt\right] \leq \mathbb{E}\left[\int_0^\tau e^{-rt} u(c_x) dt + e^{-r\tau} x\right] = x + \Big(u(c_x) - rx\Big) \mathbb{E}\Big[\int_0^\tau e^{-rt} dt\Big]$$

implies that

$$c_x \geq u^{-1}(rx) = c^\dagger(x). \tag{2.79}$$

Therefore,

$$u'(c_x) \leq u' \circ u^{-1}(rx) \leq u' \circ u^{-1}(r x_\gamma) = \gamma. \tag{2.80}$$

Using the integration by parts formula, we can see that

$$\begin{aligned} e^{-r(T\wedge\tau)} X_{T\wedge\tau} = x - \int_0^{T\wedge\tau} e^{-rs} [u(C_s) - k(\alpha(Z_s))] ds + \int_{]0, T\wedge\tau[} e^{-rs} dR_s \\ + \int_0^{T\wedge\tau} \sigma e^{-rs} Z_s dW_s \end{aligned}$$

Using admissibility conditions in Definition 2.4.1 and the monotone convergence theorem, we can take expectations and pass to the limit as $T \to \infty$ to obtain

$$x = \mathbb{E}\left[\int_0^\tau e^{-rs} [u(C_s) - k(\alpha(Z_s))] ds - \int_{]0,\tau[} e^{-rs} dR_s\right] \tag{2.81}$$

The concavity of u implies that

$$u(C_t) \leq u(c_x) + u'(c_x)(C_t - c_x)$$

Similarly, the convexity and the other assumptions on k implies that

$$-k(\alpha(Z_t)) \leq -k'(0)\alpha(Z_t) = -\gamma\alpha(Z_t)$$

In view of these inequalities, (2.78) and the fact R is an increasing process, we can see that (2.81) implies that

$$
\begin{aligned}
x &\le \mathbb{E}\left[\int_0^\tau e^{-rt}\Big(u(c_x)+u'(c_x)(C_t-c_x)-\gamma\alpha(Z_t)\Big)dt - e^{-rt}dR_t\right]\\
&\le x-u'(c_x)\left\{\mathbb{E}\left[\int_0^\tau e^{-rt}(\alpha(Z_t)-C_t)dt+e^{-r\tau}L\right]-\left(L-(rL+c_x)\mathbb{E}\left[\int_0^\tau e^{-rt}dt\right]\right)\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\mathcal{J}_p(C,R,Z) &\le \mathbb{E}\left[\int_0^\tau e^{-rt}(\alpha(Z_t)-C_t)dt+e^{-r\tau}L\right]\\
&\le L-(rL+c_x)\mathbb{E}\left[\int_0^\tau e^{-rt}dt\right]\\
&= L-\frac{rL+c_x}{u(c_x)}x.
\end{aligned}
$$

In view of (2.79) and Lemma 2.5.3, these inequalities imply that

$$
\mathcal{J}_p(C,R,Z) \le \sup_{c\ge c^\dagger(x)}\{L-\frac{rL+c}{u(c)}x\}=H(x).
$$

It follows that $V(x)\le H(x)$ for all $x\ge x_b$. On the other hand, it's straightforward to check that the choices $(C^\star,R^\star,Z^\star)$ as in the statement of the lemma are such that $\mathcal{J}_p(C^\star,R^\star,Z^\star)=H(x)$ for every initial promise $x\ge x_b$. ∎

Proof of Theorem 2.6.1. We first note that, in each of the cases, the candidate value functions are smooth solutions to the HJB equation 2.27.

Case 1. By construction, the concave function $S_0:\mathbb{R}_+\to\mathbb{R}$ is C^1 and its restriction in $\mathbb{R}^+\setminus\{x_{\min}\}$ is C^2. For $x\in[x_{\min},\infty[$, we have $V(x)=H(x)=S_0(x)$ satisfying (2.27) by Proposition 2.5.3. For $x\in[0,x_{\min}]$, we have $S_0'(x)=\underline{q}\le-\frac{1}{\gamma}$. By (2.41), we have

$$
\max_{z\in\mathbb{R}}\Big\{\alpha(z)+S_0'(x)k(\alpha(z))\Big\}=0,
$$

and by (2.53), we have

$$
Q(x,S_0(x),S_0'(x))=0.
$$

Thus, S_0 satisfies

$$
\max_{c\ge0,z\in\mathbb{R}}\left\{\frac{1}{2}\sigma^2z^2f''(x)+\Big[rx-u(c)+k(\alpha(z))\Big]f'(x)-rf(x)+\alpha(z)-c\right\}=0
$$

and the HJB equation (2.27).

Case 2. By construction, the concave function $S_1 : \mathbb{R}_+ \to \mathbb{R}$ is C^1 and its restriction in $\mathbb{R}^+ \setminus \{x_c\}$ is C^2. We would like to check that S_1 satisfies the HJB equation (2.27). For $x \in [x_\gamma, \infty[$, we have $V(x) = H(x) = S_1(x)$ satisfying (2.27) by Proposition 2.5.3.

For $x \in [0, x_c]$, we have by construction of F,

$$\max_{c\geq 0, z\geq\gamma} \left\{\frac{1}{2}\sigma^2 z^2 F''(x) + F'(x)\Big(rx + k\Big(\alpha(z)\Big) - u(c)\Big) - rF(x) + \alpha(z) - c\right\} = 0,$$

where the maximizer is attained at $z = z^\star(x, f(x), f'(x))$ as in Lemma 2.5.6. From Lemma 2.5.4, we have that $Q(0, L, q^\star) \geq 0$. Together with the fact that $Q(x_c, F(x_c), F'(x_c)) = 0$, we have that by Lemma 2.5.8,

$$Q(x, F(x), F'(x)) \geq 0 \quad \text{for all } x \in [0, x_c].$$

Then we have

$$\begin{aligned}&\max_{z\geq\gamma} \left\{\frac{1}{2}\sigma^2 z^2 F''(x) + F'(x)k(\alpha(z)) + \alpha(z)\right\}\\ =&\max_{z\in\mathbb{R}} \left\{\frac{1}{2}\sigma^2 z^2 F''(x) + F'(x)k(\alpha(z)) + \alpha(z)\right\} \geq 0 \quad \text{for all } x \in [0, x_c],\end{aligned}$$

so that the HJB equation (2.27) holds for $x \in [0, x_c]$. For $x \in [x_c, x_\gamma]$, we have $S_1(x) = H(x)$. Let F_l be as in Lemma 2.5.7, we know for any $x \in [x_c, x_\gamma]$, $x = x_c^l$ for some $l \in [L, L_\gamma]$. The facts that $F_l(x_c^l) = H(x_c^l)$, $F_l'(x_c^l) = H'(x_c^l)$ and $F_l''(x_c^l) \geq H''(x_c^l)$ imply that

$$\begin{aligned}0 &= \max_{z\in[\gamma,\infty]} \left\{\frac{1}{2}\sigma^2 z^2 F_l''(x_c^l) + F_l'(x_c^l)k(\alpha(z)) + \alpha(z)\right\}\\ &= \max_{z\in\mathbb{R}} \left\{\frac{1}{2}\sigma^2 z^2 F_l''(x_c^l) + F_l'(x_c^l)k(\alpha(z)) + \alpha(z)\right\}\\ &\geq \max_{z\in\mathbb{R}} \left\{\frac{1}{2}\sigma^2 z^2 H''(x_c^l) + H'(x_c^l)k(\alpha(z)) + \alpha(z)\right\}\\ &\geq 0.\end{aligned}$$

Together with (2.29), we have that the HJB equation (2.27) holds for $x \in [x_c, x_\gamma]$.

Case 3. By construction, the concave function $S_2 : \mathbb{R}_+ \to \mathbb{R}$ is C^1 and its restriction in $\mathbb{R}^+ \setminus \{\tilde{x}_c\}$ is C^2. S_2 satisfies the HJB equation (2.27) by the same argument as in Case 2 after replacing L by L_κ, x_c by $\tilde{x}_c$, F by $\tilde{F}$ and S_1 by S_2.

In each of the three cases, Lemma 2.6.2 implies that the restrictions of the functions S_j, $j = 0, 1, 2$, in $[x_b, \infty[$ identify with the restriction of the value function in $[x_b, \infty[$ and corresponding expressions for $(C^\star, \tau^\star, R^\star, Z^\star)$ in the statement of the theorem provide an optimal admissible control.

To establish the theorem, we can therefore restrict to initial promise $x \in [0, x_b[$. We use f to stand for S_0, S_1 or S_2, depending on the case. Given a fixed initial promise $x \in [0, x_b[$, we consider any admissible control $(C, \tau, R, Z) \in \mathcal{C}$ and let X be the associated solution to (2.24) and T_{x_b} be the first hitting time of $[x_b, \infty[$. The existence and uniqueness of the strong solution to the SDE follows from the fact

$u(c^\star(f'(x))) = u \circ (u')^{-1}(-\frac{1}{f'(x)})$ and $z^\star(x, f(x), f''(x))$ are Lipschitz on $[0, x_b]$ (See Assumption 2.2.1 and Lemma 2.5.5).

The observation that

$$X_t \in [0, x_b] \quad \text{for all } t \le T \wedge \tau \wedge T_{x_b}.$$

Itô's isometry, and the admissibility conditions on Z imply that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}\sigma Z_t f'(X_t) dW_t\right)^2\right] \\
=&\mathbb{E}\left[\int_0^T \mathbf{1}_{\{t<\tau\wedge T_{x_b}\}} e^{-rt}\sigma^2 Z_t^2 (f')^2(X_t)dt\right] \\
\le&\sigma^2 \sup_{s\in[0,x_b]} (f')^2(s)\mathbb{E}\left[\int_0^\infty e^{-rt}Z_t^2 dt\right] < \infty.
\end{aligned}$$

We use Itô's formula with generalised derivatives (See Krylov [18]) to obtain

$$\begin{aligned}
&e^{-r(T\wedge\tau\wedge T_{x_b})} f(X_{T\wedge\tau\wedge T_{x_b}}) \\
=&f(x) + \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}\left(\frac{1}{2}\sigma^2 Z_t^2 f''(X_t) + (rX_t - u(C_t) + k(\alpha(Z_t))f'(X_t) - rf(X_t)\right) dt \\
&+ \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt} f'(X_t)\, dR_t^c + \sum_{0\le t\le T\wedge\tau\wedge T_{x_b}} e^{-rt}(f(X_t) - f(X_{t-})) \\
&+ \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}\sigma Z_t f'(X_t)\, dW_t.
\end{aligned}$$

Rearranging terms, we can see that this expression implies that

$$\begin{aligned}
&\int_0^{T\wedge\tau} e^{-rt}(\alpha(Z_t) - C_t)dt - \kappa\int_0^{T\wedge\tau} e^{-rt}dR_t + e^{-r(T\wedge\tau)}L \\
&= f(x) + \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}\Big(\frac{1}{2}\sigma^2 Z_t^2 f''(X_t) + (rX_t - u(C_t) + k(\alpha(Z_t))f'(X_t) \\
&- rf(X_t) + \alpha(Z_t) - C_t\Big)dt \\
&+ \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}(f'(X_t) - \kappa)\, dR_t^c + \sum_{t\le T\wedge\tau\wedge T_{x_b}} e^{-rt}\left(f(X_t) - f(X_{t-}) - \kappa\Delta R_t\right) \\
&+ \Bigg[\int_{T\wedge\tau\wedge T_{x_b}}^{T\wedge\tau} e^{-rt}(\alpha(Z_t) - C_t)dt - \kappa\int_{T\wedge\tau\wedge T_{x_b}}^{T\wedge\tau} e^{-rt}dR_t + e^{-r(T\wedge\tau)}L \\
&- e^{-r(T\wedge\tau\wedge T_{x_b})} f(X_{T\wedge\tau\wedge T_{x_b}})\Bigg] + \int_0^{T\wedge\tau\wedge T_{x_b}} e^{-rt}\sigma Z_t f'(X_t)\, dW_t.
\end{aligned}$$

Using the relevant admissibility condition on (C, τ, R, Z) and the monotone as well as the dominated convergence theorem, we can pass to the limit as $T \to \infty$ in this inequality to obtatin

$$
\begin{aligned}
\mathcal{J}_p(C,R,Z) \le f(x) + \mathbb{E}\Big[\int_0^{\tau\wedge T_{x_b}} & e^{-rt}\Big(\frac{1}{2}\sigma^2 Z_t^2 f''(X_t) + (rX_t - u(C_t) + k(\alpha(Z_t))f'(X_t) \\
& - rf(X_t) + \alpha(Z_t) - C_t\Big)dt\Big] \\
& + \mathbb{E}\Big[\int_0^{\tau\wedge T_{x_b}} e^{-rt}(f'(X_t) - \kappa)\, dR_t^{\mathrm{c}} + \sum_{t\le\tau\wedge T_{x_b}} e^{-rt}\left(f(X_t) - f(X_{t-}) - \kappa\Delta R_t\right)\Big] \\
& + \mathbb{E}\Big[\mathbf{1}_{\{T_{x_b}<\tau\}}\int_{T_{x_b}}^{\tau} e^{-rt}(\alpha(Z_t) - C_t)dt - \kappa\int_{T_{x_b}}^{\tau} e^{-rt}dR_t + e^{-r(T\wedge\tau)}L \\
& - e^{-rT_{x_p}}f(X_{T_{x_b}})\Big] + \mathbb{E}[\mathbf{1}_{\{\tau<T_{x_b}\}}e^{-r\tau}(L - f(X_\tau))] \qquad (2.82)
\end{aligned}
$$

In view of Lemma 2.6.2, it follows that

$$
\mathcal{J}_p(C,R,\tau) \le f(x).
$$

To prove the reverse inequality and establish the optimality of the controls in the statement of the theorem, we need that, in each of the three cases, the controls are such that all of the inequalities of (2.82) hold with equality. Furthermore, it is straightforward to check that these controls are admissible: to this end, it is important to notice that the optimal controlled state process is confined in the bounded interval $[0, x_c]$ and $[0, \tilde{x}_c]$ in case 2 and 3. ∎

2.7 More realistic extensions of the model

The model that we have studied in this chapter assumes that the agent is "myopic" and maximizes their running rather than their effective promise (see Remark 2.3.1). The more realistic version of the problem in which the agent maximizes their effective promise is mathematically much more challenging. In particular, devising "incentive compatible" contracts that induce the agent to follow the principal's recommended strategy should involve a structure of recommended strategies that is most substantially complicated than the ones in Lemma 2.4.2. I have made great effort to develop the corresponding theory, which should replace (2.18) by

$$
E(A) = \beta(X(A), Z(A))
$$

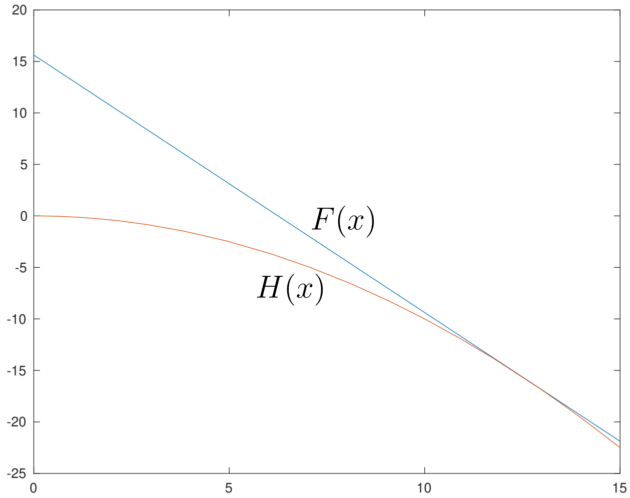
for some function β such that $\beta(\cdot, z)$ is not constant, and suitable other changes in Lemma 2.4.2. It turns out that determining the function β cannot be uncoupled from the principal's optimization problem but is an integral part of it. So far, this remains an open question and I hope to address it in the future.

Designing renegotiation-proof contracts by following the approach briefly discussed in Section IV.B of DeMarzo and Sannikov [12] is another line of future research. Such contracts should involve randomized termination of the contract. In the setting of Section 2.2-2.4, we would have to introduce an extra $(\mathcal{G}_t)$-adapted process that will represent the controlled hazard process of the random time at which the contract may be terminated, which is straightforward. The principal's stochastic control problem would then involve this extra process. The objective would then be to determine the optimal contract subject to the extra constraint that the choice $R = 0$ is optimal, which would force the agent's running payoff to be identical to the agent's effective payoff. As conjectured by DeMarzo and Sannikov [12], the optimal contract may involve a minimal value $\underline{x}$ for the agent's promise and the optimal hazard process will reflect the agent's promise in $\underline{x}$ in the positive direction. The solution to the resulting control problem's HJB equation is a very straightforward adaption of the analysis in Section 2.5, which makes this research direction very appealing. I shall certainly pursue it in the future.

2.8 Graphs and Interpretations

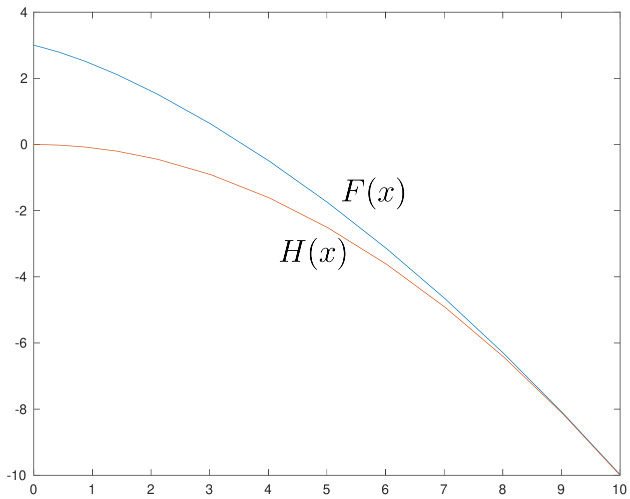
We choose a positive L, $r = 0.1$, $\kappa = 0$, $u(c) = \sqrt{c}$ and $k(a) = 0.5a^2 + 0.4a$ here. Below are Case 1, 2 and 3 in Theorem 2.6.1.

Case 1.



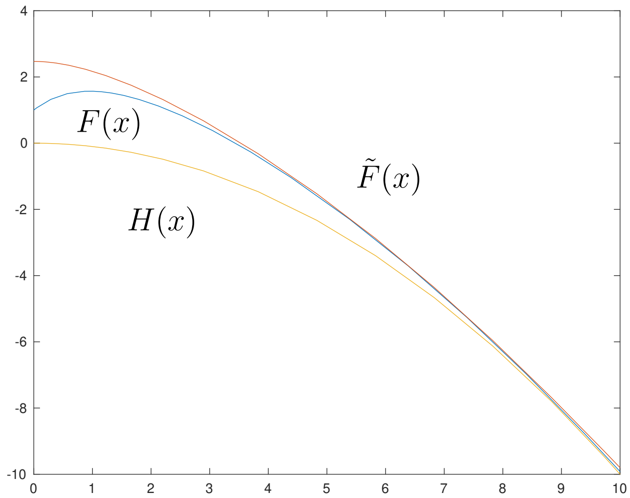
This is when principal's liquidation gain dominates the gain from agent's effort. The principal will recommend the agent to do nothing until the agent's running promise reaches 0. There is no renegotiation involved in this case.

Case 2.



This is when principal's liquidation gain and gain from agent's effort are both important. For x large, the agent is recommended to do nothing and enjoy constant compensation forever. For x is not so large, the agent follows a positive recommended effort until his running promise reaches 0. There is no renegotiation involved in this case.

Case 3.



This is when the gain from agent's effort dominates principal's liquidation gain. For x large, the agent is recommended to do nothing and enjoy constant compensation forever. For x is not so large, the agent follows a positive recommended effort and the principal is always going to renegotiate when the agent's running promise reaches 0. The consequence is that the contract will never terminate and the principal receives higher payoff at $x = 0$ than the liquidation gain L.

Chapter 3

A Class of Two-dimensional Strong Markov Processes

This chapter is based on joint work with Professor Mihail Zervos.

3.1 Outline

In this chapter, we present a study of two-dimensional strong Markov processes whose second component is the running maximum of the first one. The study of such processes has been motivated by recent developments in financial mathematics, such as the introduction and the analysis of the π and the watermark options. We first introduce a suitable concept of regularity that generalises the standard regularity assumption of the theory of one-dimensional diffusions to the two-dimensional setting that we study. Next, we characterise the class of scale functions, namely, the functions that yield local martingales when composed with a Markov process in the family we study. We then show that such a process in natural scale can be represented as a time-changed Brownian motion and its running maximum. Finally, we present a study of associated r-invariant functions. Our analysis makes heavy use of the standard theory of one-dimensional diffusions. The main difficulties arise from the behaviour of the processes on the diagonal where their two components coincide.

The chapter is organised as follows. In Section 3.2, we set up the notations, definitions and assumptions, and we prove some preliminary results. In Section 3.3, we extend the notion of scale function of one dimensional diffusion to our two dimensional case. In Section 3.4, we characterise the time change of the process to a standard Brownian motion and its running maximum. In Section 3.5, we collect and present illustrative examples. In Section 3.6, we introduce and characterise the corresponding two dimensional r-invariant functions and we derive a differential equation associated with them.

3.2 Set up

We let $\mathcal{I}$ be an interval with either $\mathcal{I} =]\alpha, \beta[$ or $\mathcal{I} = [\alpha, \beta[$, where $-\infty \leq \alpha < \beta \leq \infty$ are constants. We fix a Borel-measurable function $\kappa : \mathcal{I} \to \mathbb{R} \cup \{-\infty\}$ such that

$$\kappa(s) < s \quad \text{and} \quad \inf_{s \in [a,b]} \big\{ s - \kappa(s) \big\} > 0 \text{ for all } \alpha < a < b < \beta. \tag{3.1}$$

We set $\mathcal{I}_s =]\kappa(s), s]$ or $\mathcal{I}_s = [\kappa(s), s]$ for all $s \in \mathcal{I}$ and assume that the sets

$$\begin{aligned} &\{s \in \mathcal{I} \mid \mathcal{I}_s =]\kappa(s), s]\}, \\ C_\infty &= \{s \in \mathcal{I} \mid \mathcal{I}_s =]-\infty, s]\}, \\ C_1 &= \{s \in \mathcal{I} \setminus C_\infty \mid \mathcal{I}_s =]\kappa(s), s]\}, \\ C_2 &= \{s \in \mathcal{I} \setminus C_\infty \mid \mathcal{I}_s = [\kappa(s), s]\} \end{aligned} \tag{3.2}$$

are Borel-measurable. Also, we assume that $-\infty < \kappa(s)$ for all $s \in C_2$. The state space of the process we study is

$$E = \{(x, s) \in \mathbb{R}^2 \mid \ x \in \mathcal{I}_s \text{ and } s \in \mathcal{I}\}.$$

We denote

$$\mathcal{E} = \{A \in \mathcal{B}(\mathbb{R}^2) \mid A \subset E\}.$$

Lemma 3.2.1. *E is in $\mathcal{B}(\mathbb{R}^2)$.*

Proof. Consider the sets C_∞, C_1 and C_2 defined in (3.2) and the $\mathcal{B}(\mathbb{R} \times \mathcal{I})/\mathcal{B}(\mathbb{R})$-measurable function f defined by $f(x, s) = x - \kappa(s)$ for all $x \in \mathbb{R}$ and $s \in \mathcal{I}$. Then

$$E = \big(f^{-1}(]0, \infty[) \cup (f^{-1}(\{0\}) \cap C_2)\big) \cap \{(x, s) \in \mathbb{R}^2 \mid x \leq s\}$$

is in $\mathcal{B}(\mathbb{R}^2)$. ∎

We consider the canonical measurable space $(\Omega, \mathcal{F})$, where Ω is the family $C(\mathbb{R}^+, \mathbb{R}^2)$ of all continuous paths $\omega = (\omega_1, \omega_2)$, $\mathcal{F} = \sigma\left((\omega_1(t), \omega_2(t)), t \geq 0\right)$ and

$$\mathcal{F}_t = \bigcap_{\epsilon > 0} \sigma\left((\omega_1(u), \omega_2(u)), u \leq t + \epsilon\right) \quad \text{for } t \geq 0$$

is a right continuous filtration. We denote (X, S) the coordinate process defined by

$$X_t = \omega_1(t) \quad \text{and} \quad S_t = \omega_2(t) \quad \text{for all } \omega = (\omega_1, \omega_2) \in \Omega \text{ and } t \geq 0.$$

Furthermore, we denote by $\{\theta_t, t \geq 0\}$ the family of shift operators, which are defined by

$$\theta_t(\omega)(u) = \omega(u + t) \quad \text{for } \omega \in \Omega \text{ and } t \geq 0.$$

Definition 3.2.1. *A process and its running maximum that are jointly strong Markov (PRM-JSM) with state space E is a family of probability measures $\{\mathbb{P}^{x,s}; (x, s) \in E\}$ on $(\Omega, \mathcal{F})$ such that*

(i) $(x, s) \mapsto \mathbb{P}^{x,s}(C)$ is $\mathcal{E}/\mathcal{B}([0, 1])$measurable for all $C \in \mathcal{F}$,

(ii) $\mathbb{P}^{x,s}((X_0, S_0) = (x, s)) = 1$ *for all* $(x, s) \in E$*,*

(iii) $\mathbb{P}^{x,s}((X_t, S_t) \in E) = 1$ *for all* $t \geq 0$*,* $(x, s) \in E$*, PRM-JSM*

(iv) $\mathbb{P}^{x,s}(S_t = s \vee (\sup_{0 \leq u \leq t} X_u)) = 1$ *for all* $t \geq 0$*,* $(x, s) \in E$*,*

(v) the strong Markov property holds true, namely,

$$\mathbb{E}^{x,s}\big[Z \circ \theta_T \mid \mathcal{F}_T\big]\mathbf{1}_{\{T<\infty\}} = \mathbb{E}^{X_T,S_T}\big[Z\big]\mathbf{1}_{\{T<\infty\}}$$

for all bounded random variables Z*, all points* $(x, s) \in E$ *and all* $(\mathcal{F}_t)$*- stopping times* T*.*

Example 1 in Section 3.5 provides an example of a PRM-JSM.

Given $(x, s) \in E$ and a locally bounded Borel-measurable function χ such that $\kappa(s) \leq \chi(s) < s$, we denote by

$$\begin{aligned} T_{x,s} &= \inf\{t \geq 0 \mid \ (X_t, S_t) = (x, s)\} \\ T_s &= \inf\{t \geq 0 \mid \ (X_t, S_t) = (s, s)\} \\ \text{and } \mathcal{T}_\chi &= \inf\{t \geq 0 \mid \ X_t = \chi(S_t)\}. \end{aligned}$$

Definition 3.2.2. *We say* χ *is accessible if it is Borel measurable, locally bounded and*

(i) $\mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi) > 0$ *for all* $s \in \mathcal{I}$ *and* $\chi(s) < x \leq s \leq b < \beta$*,*

(ii) $\mathbb{P}^{x,s}(T_{\chi(s),s} < \infty) > 0$ *for all* $s \in \mathcal{I}$ *and* $\chi(s) \leq x < s < \beta$*,*

(iii) $\mathbb{P}^{s,s}(\mathcal{T}_\chi < T_b) > 0$ *for all* $s \in \mathcal{I}$ *and* $\chi(s) \leq s < b < \beta$*.*

There are examples where any of these conditions fail to hold. See Example 2 in Section 3.5.

Given constants $\alpha < a < b < \beta$, and an accessible function χ, we will use the notation

$$\begin{aligned} E_\chi &= \{(x, s) \in E \mid s \in \mathcal{I} \text{ and } \chi(s) \leq x \leq s\}, \\ \mathring{E}_\chi &= \{(x, s) \mid s \in \mathcal{I} \text{ and } \chi(s) < x < s\}, \\ E_{a,\chi} &= \{(x, s) \in E \mid a \leq s \in \mathcal{I} \text{ and } \chi(s) \leq x \leq s\}, \\ E^b_\chi &= \{(x, s) \mid \chi(s) \leq x \leq s \text{ and } \mathcal{I} \ni s < b\}, \\ \text{and } \quad E_{a,b,\chi} &= \{(x, s) \in E \mid a \leq s \leq b \text{ and } \chi(s) \leq x \leq s\}. \end{aligned}$$

Definition 3.2.3. *A PRM-JSM with state space* E *is regular if there exists a sequence* $\{\chi_n\}_{n=1}^\infty$ *of accessible functions on* $\mathcal{I}$ *such that*

$$\kappa(s) \leq \chi_{n+1}(s) \leq \chi_n(s) < s \text{ and } \mathcal{I}_s = \bigcup_n [\chi_n(s), s] \text{ for all } s \in \mathcal{I}, n \geq 1.$$

Remark 3.2.1. *For a regular PRM-JSM with state space* E *with* $\{\chi_n\}_{n=1}^\infty$ *as in Definition 3.2.3, we have* $E = \bigcup_n E_{\chi_n}$*.*

From now on, we assume the PRM-JSM with state space E that we consider are regular.

Lemma 3.2.2. *Given a PRM-JSM with state space E,*

$$\mathbb{P}^{s,s}\left(T_{s_1} < \infty \text{ and } \lim_{s_2>s_1, s_2\to s_1} T_{s_2} = T_{s_1}\right) = \mathbb{P}^{s,s}(T_{s_1} < \infty) \quad \text{for all } s \leq s_1 \in \mathcal{I}. \tag{3.3}$$

Furthermore,

$$\mathbb{P}^{s,s}\left(S_t > s \text{ for all } t > 0\right) = 1. \tag{3.4}$$

Proof. The definition of shift operators implies that

$$\begin{aligned}&\left\{\omega \in \Omega \mid T_{s_1}(\omega) < \infty \text{ and } \lim_{s_2>s_1, s_2\to s_1} T_{s_2}(\omega) = T_{s_1}(\omega)\right\}\\ =&\left\{\omega \in \Omega \mid T_{s_1}(\omega) < \infty \text{ and } \left(\lim_{s_2>s_1, s_2\to s_1} T_{s_2}\right)\circ\theta_{T_{s_1}}(\omega) = (T_{s_1}\circ\theta_{T_{s_1}})(\omega) = 0\right\}.\end{aligned}$$

This observation, the strong Markov property and the tower property of conditional expectation imply that

$$\begin{aligned}&\mathbb{P}^{s,s}\left(\left\{\omega \in \Omega \mid T_{s_1}(\omega) < \infty \text{ and } \lim_{s_2>s_1, s_2\to s_1} T_{s_2}(\omega) = T_{s_1}(\omega)\right\}\right)\\ =&\mathbb{E}^{s,s}\left[\mathbb{P}^{s,s}\left(\left(\lim_{s_2>s_1, s_2\to s_1} T_{s_2}\right)\circ\theta_{T_{s_1}} = 0 \mid \mathcal{F}_{T_{s_1}}\right)\mathbf{1}_{\{T_{s_1}<\infty\}}\right]\\ =&\mathbb{E}^{s,s}\left[\mathbb{P}^{s_1,s_1}\left(\lim_{s_2>s_1, s_2\to s_1} T_{s_2} = 0\right)\mathbf{1}_{\{T_{s_1}<\infty\}}\right]\\ =&\mathbb{P}^{s_1,s_1}(\Lambda)\mathbb{P}^{s,s}(T_{s_1} < \infty),\end{aligned} \tag{3.5}$$

where $\Lambda = \{\lim_{s_2>s_1, s_2\to s_1} T_{s_2} = 0\}$. The right continuity of $\mathcal{F}_t$ implies that $\Lambda \in \mathcal{F}_0$. Therefore, $\mathbb{P}^{s_1,s_1}(\Lambda) = 0$ or 1 by Blumenthal's 0-1 law. It follows that $\mathbb{P}^{s_1,s_1}(\Lambda) = 1$. To see this claim, we let $T = \lim_{s_2>s_1, s_2\to s_1} T_{s_2}$ so that $\Lambda^c = \{T > 0\}$. Notice that

$$\begin{aligned}0 &= \mathbb{E}^{s,s}\left[\mathbf{1}_{\Lambda^c}\circ\theta_T\mathbf{1}_{\{T<\infty\}}\right]\\ &= \mathbb{E}^{s,s}\left[\mathbb{P}^{s_1,s_1}(\Lambda^c)\mathbf{1}_{\{T<\infty\}}\right]\\ &= \mathbb{P}^{s_1,s_1}(\Lambda^c)\mathbb{P}^{s,s}(T < \infty).\end{aligned}$$

This is only possible for $\mathbb{P}^{s_1,s_1}(\Lambda^c) = 0$ because $\mathbb{P}^{s,s}(T < \infty) > 0$ by the regularity of the strong Markov process (see Definition 3.2.2 (i) in particular). Combine this with (3.5), we have that (3.3) holds.

Combining the identity $\mathbb{P}^{s,s}(\lim_{s_2>s, s_2\to s} T_{s_2} = T_s) = 1$, which follows from (3.3), with the observation that

$$\left\{\omega \in \Omega \mid \lim_{s_2>s, s_2\to s} T_{s_2} = 0\right\} = \{\omega \in \Omega \mid S_t > s \text{ for all } t > 0\},$$

we obtain (3.4). ∎

Corollary 3.2.3. $\mathbb{P}^{s,s}\left(\lim_{s_2\to s_1} T_{s_2} = T_{s_1}\right) = 1$ *for all $s \leq s_1 \in \mathcal{I}$.*

Proof. By continuity of paths, we have

$$\mathbb{P}^{s,s}\left(\lim_{s_2<s_1, s_2\to s_1} T_{s_2} = T_{s_1}\right) = 1.$$

Combine this and Lemma 3.2.2, we have the result. ∎

On $\mathcal{I}_s \times \{s\}$, one can check that the PRM-JSM (X, S) satisfies the regularity assumption as in one dimensional case. Namely for all $x \in \mathring{\mathcal{I}}_s$ (interior of $\mathcal{I}_s$) and $y \in \mathcal{I}_s$, we have

$$\mathbb{P}^{x,s}(T_{y,s} < \infty) > 0.$$

Given a function $p : E \to \mathbb{R}$, we note that $p(\cdot, s) : \mathcal{I}_s \to \mathbb{R}$ is a scale function on $\mathcal{I}_s$ if it is continuous and strictly increasing such that, for all $\kappa(s) < a \le x \le b \le s$ and $s \in \mathcal{I}$, we have

$$\frac{p(x,s) - p(a,s)}{p(b,s) - p(a,s)} = \mathbb{P}^{x,s}(T_{b,s} < T_{a,s}) \tag{3.6}$$

holds, or equivalently, $p(X_{t\wedge T_{a,s}\wedge T_{b,s}}, S_{t\wedge T_{a,s}\wedge T_{b,s}})$ is a $\mathbb{P}^{x,s}$ uniformly integrable martingale.

For any $s \in \mathcal{I}$, $J = [a, b] \subseteq]\kappa(s), s]$ and $x \in J$, we define the function $h_J(\cdot; s)$ by

$$h_J(x; s) = \mathbb{E}^{x,s}[T_{a,s} \wedge T_{b,s}] < \infty. \tag{3.7}$$

If X is a local martingale, then $h_J(\cdot; s)$ is concave which induces a positive measure

$$m(dz; s) = -\frac{1}{2}h''_J(dz; s) \text{ on } [a, b], \tag{3.8}$$

which can be extended throughout $\mathcal{I}_s$ independent of J. Furthermore,

$$0 < m([a, b[; s) < \infty, \quad \text{for all } \kappa(s) < a < b < s. \tag{3.9}$$

This measure $m(\cdot; s)$ which depends on s is precisely the speed measure on $\mathcal{I}_s$ as in the one dimensional case.

Next, we state the classification of left endpoint of $\mathcal{I}_s \times \{s\}$ which is exactly the same as in the one dimensional case. Indeed, all of the claims follow from V.44-47 of Roger and Williams [26]. The end point $(\kappa(s), s)$ of the interval $\mathcal{I}_s \times \{s\}$ is called inaccessible if $\kappa(s) \notin \mathcal{I}_s$. If $\kappa(s) \in \mathcal{I}_s$, the point $(\kappa(s), s)$ is called absorbing if $\mathbb{P}^{\kappa(s),s}(T_{y,s} < \infty) = 0$ for all $y \in \mathcal{I}_s \setminus \{\kappa(s)\}$ and the point $(\kappa(s), s)$ is called reflecting if $\mathbb{P}^{\kappa(s),s}(T_{y,s} < \infty) > 0$ for some $y \in \mathcal{I}_s \setminus \{\kappa(s)\}$.

We will be working with functions defined using $m(\cdot; s)$ and we need joint measurability of these functions. For any Borel-measurable χ such that $\kappa(s) < \chi(s) < s$ for all $s \in \mathcal{I}$, let $T_{\mathring{E}_\chi}$ be the exit time of the process (X, S) from $\mathring{E}_\chi$, and define g_χ to be the function

$$g_\chi(x, s) = \mathbb{E}^{x,s}[T_{\mathring{E}_\chi}] \text{ for all } (x, s) \in \{(x, s) \in \mathbb{R}^2 \mid x \le s\}.$$

We have that the map $(x, s) \mapsto g_\chi(x, s)$ is finite by (3.7) and is jointly measurable by (i) of Definition 3.2.1. Furthermore, if X is a local martingale, from one dimension theory, we know that the function $g_\chi(x,s)$ is concave in x variable for $x \in [\chi(s), s]$, with

$$m([a,b[;s) = -\frac{1}{2}\left[(g_\chi)_{x-}(b,s) - (g_\chi)_{x-}(a,s)\right] \quad \text{for all} \quad \chi(s) < a < b \le s, \tag{3.10}$$

where $m(\cdot;s)$ is defined in (3.8) and $(g_\chi)_{x-}$ is the left derivative of g_χ with respect to x variable.

Let us define

$$\mathfrak{G}_\chi(s) = \sup_{x\in[\chi(s),s]} g_\chi(x,s), \tag{3.11}$$

which is finite since $g_\chi(\cdot,s)$ is continuous on $[\chi(s), s]$. Note that for any constant c, the set

$$\{s \in \mathcal{I} \mid \exists x \in \mathcal{I}_s \text{ s.t. } g_\chi(x,s) > c\} = \bigcup_{x\in\mathbb{Q}} \{s \in \mathcal{I} \mid s \ge x \text{ and } g_\chi(x,s) > c\}$$

is measurable. Therefore, the finite function $\mathfrak{G}_\chi(s)$ is measurable. We also define

$$\mathfrak{M}_\chi(s) = (s - \chi(s))m(]\chi(s), s[;s). \tag{3.12}$$

Notice that

$$m(]\chi(s), s[;s) = -\frac{1}{2}\left[(g_\chi)_{x-}(s,s) - \lim_{a\downarrow\chi(s)} (g_\chi)_{x-}(a,s)\right]$$

is measurable as a function of s. By (3.12), so is $\mathfrak{M}_\chi(s)$. Moreover, we know from VII Theorem 3.6 of Revuz and Yor [24] that if X is a local martingale so that we can choose $p(x,s) = x$, for

$$K_\chi = \begin{cases} \frac{(x-\chi(s))(s-y)}{s-\chi(s)}, & \text{if } \chi(s) \le x \le y \le s, \\ \frac{(y-\chi(s))(s-x)}{s-\chi(s)}, & \text{if } \chi(s) \le y \le x \le s, \\ 0, & \text{otherwise,} \end{cases}$$

we have

$$\begin{aligned} g_\chi(s) &= \mathbb{E}^{x,s}[T_{\mathring{E}_\chi}] \\ &= \int K_\chi(x,z)m(dz;s) \\ &\le \int (s-\chi(s))m(dz;s) \\ &= (s-\chi(s))m(]\chi(s), s[;s), \end{aligned}$$

so that $\mathfrak{G}_\chi(s) \le \mathfrak{M}_\chi(s)$ for all accessible χ and $s \in \mathcal{I}$.

Remark 3.2.2. *For PRM-JSM (X,S) with X a local martingale, the measurable set $\{s \in \mathcal{I} \mid m([x,s[;s) < \infty$ for all $x \in \mathcal{I}_s\}$ might be nonempty, see Example 3 in Section 3.5.*

Remark 3.2.3. *For PRM-JSM (X,S), we can time change (X,S) in a way such that the measurable set $\{s \in \mathcal{I} \mid m([x,s[;s) < \infty \text{ for all } x \in \mathcal{I}_s\} = \emptyset$ and the time changed process remains a regular PRM-JSM. Let p be the function defined in Lemma 3.3.1 and Lemma 3.3.2. Let $\{s_i\}_{i\in\mathbb{Z}}$ be a strictly increasing in i such that $\cup_{i\in\mathbb{Z}}[s_i,s_{i+1}] = \mathcal{I}$. Let $N_i \geq 1$ be such that*

$$\frac{1}{N_i} < \inf_{s\in[s_i,s_{i+1}]} \big(s-\kappa(s)\big).$$

For $s \in [s_i,s_{i+1}]$, by regularity (See V.51 Theorem 2 of Roger and Williams [26]) we have that

$$\int_{]s-\frac{1}{N_i},s[} (p(s,s)-p(x,s))\, m(dx;s) < \infty,$$

so that

$$\sum_{n=N_i}^{\infty} \left(p(s,s)-p(s-\frac{1}{n},s)\right) m(]s-\frac{1}{n},s-\frac{1}{n+1}[;s) < \infty.$$

We let

$$C_t = \sum_{i\in\mathbb{Z},j\geq 0} \int_{[0,t]} \left(1+\frac{1}{p(s,s)-p(s-\frac{1}{N_i+j},s)}\mathbf{1}_{\{S_t\in[s_i,s_{i+1}[\}\cap\{X_t\in[S_t-\frac{1}{N_i+j},S_t-\frac{1}{N_i+j+1}[\}}\right)du$$

and set $\tilde{X}_t = X_{C_t}, \tilde{S}_t = S_{C_t}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{C_t}$. We can introduce a new family of probability measures $\{\tilde{\mathbb{P}}^{x,s}\}$ satisfying

$$\tilde{\mathbb{P}}^{x,s} = \mathbb{P}^{x,s}(\tilde{X},\tilde{S})^{-1}.$$

We know from III.21 of Roger and Williams [26] that $\{\tilde{\mathbb{P}}^{x,s}\}$ is a PRM-JSM. Let us denote

$$\tilde{T}_{x,s} = \inf\{t\geq 0 \mid (\tilde{X}_t,\tilde{S}_t)=(x,s)\} \text{ and } \tilde{T}_\chi = \inf\{t\geq 0 \mid \tilde{X}_t = \chi(\tilde{S}_t)\}.$$

Notice that

$$\begin{aligned}
&\mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi) = \mathbb{P}^{x,s}(\tilde{T}_b < \tilde{T}_\chi) = \tilde{\mathbb{P}}^{x,s}(T_b < \mathcal{T}_\chi),\\
&\mathbb{P}^{x,s}(T_{\chi(s),s} < \infty) = \mathbb{P}^{x,s}(\tilde{T}_{\chi(s),s} < \infty) = \tilde{\mathbb{P}}^{x,s}(T_{\chi(s),s} < \infty),\\
&\mathbb{P}^{s,s}(\mathcal{T}_\chi < T_b) = \mathbb{P}^{s,s}(\tilde{T}_\chi < \tilde{T}_b) = \tilde{\mathbb{P}}^{s,s}(\mathcal{T}_\chi < T_b),
\end{aligned}$$

we have that the regularity of $\{\tilde{P}^{x,s}\}$ inherits from the regularity of $\{P^{x,s}\}$. One can check the corresponding speed measure satisfies

$$\begin{aligned}
&\tilde{m}([s-\frac{1}{n},s-\frac{1}{n+1}[;s) = \left(p(s,s)-p(s-\frac{1}{n},s)\right) m([s-\frac{1}{n},s-\frac{1}{n+1}[;s)\\
&\text{for all} \quad n\geq N_i \text{ and } s\in[s_i,s_{i+1}[,\\
&\tilde{m}([a,b[;s) = m([a,b[;s) \quad \text{for all} \quad [a,b[\subset]\kappa(s),s-\frac{1}{N_i}[\text{ and } s\in[s_i,s_{i+1}[
\end{aligned}$$

so that

$$\tilde{m}([s-\frac{1}{N_i},s[;s) = \sum_{n=N_i}^{\infty}\left(p(s,s)-p(s-\frac{1}{n},s)\right) m([s-\frac{1}{n},s-\frac{1}{n+1}[) < \infty \quad \forall s\in[s_i,s_{i+1}[.$$

Assumption 3.2.1. *From now on, except in Lemma 3.3.1 and Lemma 3.3.2, we assume our regular PRM-JSMs satisfy*

$$m([x, s[; s) < \infty \text{ for all } x \in \mathcal{I}_s.$$

Lemma 3.2.4. *Consider a regular PRM-JSM $\{\mathbb{P}^{x,s}, (x, s) \in E\}$. There exists a sequence $\{\chi_n\}$ such that*

(i) χ_n is accessible for all $n \geq 1$,

(ii) $\mathfrak{G}_{\chi_n}(s)$ is locally bounded on $\mathcal{I}$,

(iii) and $\bigcup_{n=1}^{\infty} E_{\chi_n} = E$.

Proof. let $\{\tilde{\chi}_n\}$ be accessible sequence in the sense of Definition 3.2.3. We define

$$\chi_n^q = \begin{cases} \tilde{\chi}_n(s), & \text{if } q = 0, \\ \tilde{\chi}_n(s) \vee (s - \frac{1}{q}), & \text{if } q \geq 1. \end{cases}$$

If we define

$$A_n^{q,k} = \{s \mid \mathfrak{G}_{\chi_n^q}(s) \leq k\}, \text{ for } q \geq 0 \text{ and } k \geq 1,$$

$A_n^{q,k}$ is measurable and

$$\bigcup_{q=0}^{\infty} A_n^{q,k} = \mathcal{I}$$

due to Assumption 3.2.1. Let us define

$$\chi_{n,k} = \mathbf{1}_{\{s \in A_n^{0,k}\}} \chi_n^0(s) + \sum_{q \geq 1} \mathbf{1}_{\{s \in A_n^{q,k} \setminus A_n^{q-1,k}\}} \chi_n^q(s).$$

We have that

$$\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{\chi_{n,k}} = E.$$

Let $\{s_i\}_{i \in \mathbb{Z}}$ be a strictly increasing in i such that $s_{i+1} - s_i > 0$ and $\cup_{i \in \mathbb{Z}} [s_i, s_{i+1}] = \mathcal{I}$. Now for every n, there exists K_n^i large enough and increasing with respect to n such that $\mathbb{P}^{s_i, s_i}(T_{s_{i+1}} < \mathcal{T}_{\chi_{n, K_n^i}}) > 0$. Now set

$$\chi_n(s) = \sum_{i \in \mathbb{Z}} \mathbf{1}_{s \in [s_i, s_{i+1}[} \chi_{n, K_n^i}(s).$$

By a diagonal argument, we can see that

$$\bigcup_{n=1}^{\infty} E_{\chi_n} = E.$$

We can check accessibility of the sequence $\{\chi_n\}_{n=1}^{\infty}$ and conclude the sequence satisfies the conditions in Definition 3.2.3 as follows. Given $\kappa(s) < \chi_n(s) \leq x < s \leq b$, we will have $s \in [s_{i_1}, s_{i_1+1}]$ and $b \in [s_{i_2}, s_{i_2+1}]$ for some i_1 and i_2. Combine this with the fact that $\mathbb{P}^{s_i,s_i}(T_{s_{i+1}} < \mathcal{T}_{\chi_n}) > 0$ for all i, we have $\mathbb{P}^{x,s}(T_b < \mathcal{T}_{\chi_n}) > 0$. We also have

$$\mathbb{P}^{x,s}(T_{\chi_n(s),s} < \infty) > 0$$

and

$$\mathbb{P}^{s,s}(\mathcal{T}_{\chi_n} < T_b) \geq \mathbb{P}^{s,s}(\mathcal{T}_{\tilde{\chi}_n} < T_b) > 0$$

hold. ∎

Corollary 3.2.5. *Under Assumption 3.2.1, Lemma 3.2.4 holds with* $\mathfrak{G}$ *replaced by* $\mathfrak{M}$.

Proof. The proof follows from the same line as in Lemma 3.2.4. ∎

Lemma 3.2.6. *Consider any accessible function* χ *and any constants* $a < b \in \mathcal{I}$. *Also, suppose that there exists a constant* $K > 0$ *such that* $\mathfrak{G}_\chi(s) \leq K$ *for all* $s \in [a, b]$, *where* $\mathfrak{G}_\chi$ *is given by (3.11). If we define*

$$\mathcal{D} = E_{a,b,\chi} \quad \textit{and} \quad F_\mathcal{D}(x,s) = \mathbb{E}^{x,s}[T_b \wedge \mathcal{T}_\chi] \quad \textit{for } (x,s) \in \mathcal{D}, \tag{3.13}$$

then the function $F_\mathcal{D}$ *is finite. Furthermore, given any* $s \in [a, b]$, *if* X *is a local martingale, the function* $F_\mathcal{D}(\cdot, s)$ *is concave, its second derivative* $(F_\mathcal{D})_{xx}(\cdot, s)$ *in the sense of of distributions is a negative measure such that*

$$-\frac{1}{2}(F_\mathcal{D})_{xx}(\cdot, s) = m(\cdot; s) \quad \textit{on } (]\chi(s), s[, \mathcal{B}(]\chi(s), s[)) \textit{ for all } s \in [a, b].$$

where $m(\cdot; s)$ *is defined by (3.8).*

Proof. From the regularity assumption, there exist $v_1, \delta_1 > 0$ such that $\mathbb{P}^{a,a}(T_b \leq v_1) > \delta_1$. We have

$$\begin{aligned}\delta_1 &\leq \mathbb{P}^{a,a}(T_b \leq v_1) = \mathbb{P}^{a,a}(T_b \leq v_1, T_s < T_b)\\ &= \mathbb{E}^{a,a}[\mathbf{1}_{\{T_s + T_b\circ\theta_{T_s} \leq v_1\}}\mathbf{1}_{\{T_s < T_b\}}]\\ &\leq \mathbb{E}^{a,a}[\mathbf{1}_{\{T_b\circ\theta_{T_s} \leq v_1\}}] = \mathbb{P}^{s,s}(T_b \leq v_1) \quad \text{for all } s \in [a, b].\end{aligned}$$

The definition of (3.11) implies that

$$\mathbb{E}^{x,s}[T_s \wedge \mathcal{T}_\chi] \leq \mathfrak{G}_\chi(s) \leq K \quad \text{for all } (x, s) \in \mathcal{D}. \tag{3.14}$$

By Markov inequality, we have

$$\mathbb{P}^{x,s}(T_s \wedge \mathcal{T}_\chi \geq v_2) \leq \frac{K}{v_2}$$

for any $v_2 > 0$. We can then choose $v_2 \geq v_1$ such that

$$\mathbb{P}^{x,s}(T_s \wedge \mathcal{T}_\chi \leq v_2) \geq \frac{1}{2} \quad \text{for all } (x, s) \in E_{a,b,\chi}.$$

Now we have

$$\begin{aligned}\mathbb{P}^{x,s}(T_b \wedge \mathcal{T}_\chi \leq 2v_2) &= \mathbb{P}^{x,s}(T_s \wedge \mathcal{T}_\chi + (T_b \wedge \mathcal{T}_\chi) \circ \theta_{T_s\wedge\mathcal{T}_\chi} \leq 2v_2)\\ &\geq \mathbb{P}^{x,s}((T_b \wedge \mathcal{T}_\chi) \circ \theta_{T_s\wedge\mathcal{T}_\chi} \leq v_2, T_s \wedge \mathcal{T}_\chi \leq v_2)\\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_s\wedge\mathcal{T}_\chi \leq v_2\}}\mathbb{P}^{X_{T_s\wedge\mathcal{T}_\chi}, S_{T_s\wedge\mathcal{T}_\chi}}(T_b \wedge \mathcal{T}_\chi \leq v_2)]\\ &\geq \delta_1\mathbb{P}^{x,s}(T_s \wedge \mathcal{T}_\chi \leq v_2) \quad \text{for all } (x, s) \in E_{a,b,\chi}.\end{aligned}$$

Set $v = 2v_2$ and $\delta = \frac{1}{2}\delta_1$, we have that

$$\mathbb{P}^{x,s}(T_b \wedge \mathcal{T}_\chi \le v) \ge \delta \quad \text{for all } (x,s) \in E_{a,b,\chi}. \tag{3.15}$$

Notice that

$$\begin{aligned}\mathbb{P}^{x,s}(T_b \wedge \mathcal{T}_\chi \ge nv) &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b\wedge\mathcal{T}_\chi \ge nv\}}] \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b\wedge\mathcal{T}_\chi \ge (n-1)v\}}\mathbf{1}_{\{(n-1)v+(T_b\wedge\mathcal{T}_\chi)\circ\theta_{(n-1)v}\ge nv\}}] \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b\wedge\mathcal{T}_\chi \ge (n-1)v\}}\mathbf{1}_{\{(T_b\wedge\mathcal{T}_\chi)\circ\theta_{(n-1)v}\ge v\}}] \\ &= \mathbb{E}^{x_0,s_0}[\mathbf{1}_{\{T_b\wedge\mathcal{T}_\chi \ge (n-1)v\}}\mathbb{P}^{X_{(n-1)v},S_{(n-1)v}}(T_b\wedge\mathcal{T}_\chi \ge v)] \\ &\quad \text{for all } (x,s)\in\mathcal{D}. \end{aligned} \tag{3.16}$$

We observe on $\{(n-1)v \le T_b \wedge \mathcal{T}_\chi\}$, by (3.15),

$$\mathbb{P}^{X_{(n-1)v},S_{(n-1)v}}(T_b\wedge\mathcal{T}_\chi \ge v) \le 1-\delta,$$

so that by induction on (3.16),

$$\mathbb{P}^{x,s}(T_b \wedge \mathcal{T}_\chi \ge nv) \le (1-\delta)^{n-1} \quad \text{for all } (x,s)\in\mathcal{D},$$

which implies

$$\mathbb{E}^{x,s}[T_b\wedge\mathcal{T}_\chi] < \infty \quad \text{for all } (x,s)\in\mathcal{D}. \tag{3.17}$$

The remaining follows the same line of argument in the proof of (47.10) in Roger and Williams [26]. ∎

Remark 3.2.4. *We note that the result might not hold for accessible χ without restricting the speed measure. See Example 3 in Section 3.5.*

Lemma 3.2.7. *Fix any accessible function χ such that $\mathfrak{G}_\chi : \mathcal{I} \to \mathbb{R}^+$ is locally bounded, where $\mathfrak{G}$ is defined by (3.11). There exists a kernel μ_χ^r for any $r \ge 0$ such that*

$$\mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] = \int_{\mathcal{I}} f(u)\mu_\chi^r(du;s) \tag{3.18}$$

for all $s \in \mathcal{I}$ and all measurable integrable functions $f : \mathcal{I} \to \mathbb{R}$. Furthermore, this kernel is such that, given any $s \le s_1 < s_2 \in \mathcal{I}$,

$$\mu_\chi^r([s_1,s_2[;s) > 0 \quad \text{and } \mu_\chi^r(\{s_1\};s) = 0.$$

Proof. Fix any accessible function χ satisfying the requirement of the lemma. Also, consider any $f \in C_b(\mathcal{I})$, where $C_b(\mathcal{I})$ is the set of all bounded continuous functions on $\mathcal{I}$. We first show that the function $\mathcal{I} \ni s \mapsto \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}]$ is continuous. Lemma 3.2.6 implies that $T_s \wedge \mathcal{T}_\chi < \infty$, $\mathbb{P}^{s_0,s_0}$- a.s, for any points $s_0 < s$ in $\mathcal{I}$. In view of this result and the strong Markov property, we obtain

$$\begin{aligned}\mathbb{E}^{s_0,s_0}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] &= \mathbb{E}^{s_0,s_0}\left[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}\left(\mathbf{1}_{\{T_s<\mathcal{T}_\chi<\infty\}}+\mathbf{1}_{\{\mathcal{T}_\chi<T_s\}}\right)\right] \\ &= \mathbb{E}^{s_0,s_0}[e^{-rT_s}\mathbf{1}_{\{T_s<\mathcal{T}_\chi<\infty\}}]\mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] \\ &\quad + \mathbb{E}^{s_0,s_0}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<T_s\}}].\end{aligned}$$

By Corollary 3.2.3, we can see that the functions $s \mapsto \mathbb{E}^{s_0,s_0}[e^{-r\mathcal{T}_s}\mathbf{1}_{\{T_s<\mathcal{T}_\chi<\infty\}}] > 0$ and $s \mapsto \mathbb{E}^{s_0,s_0}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<T_s\}}]$ are continuous on $[s_0, \beta[$. Combining these observations, we can see that the function $s \mapsto \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}]$ is continuous on $[s_0, \beta[$ and the required continuity follows because $s_0 \in \mathcal{I}$ has been arbitrary. The mapping $C_b(\mathcal{I}) \ni f \mapsto \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}]$ is a positive linear functional mapping from $C_b(\mathcal{I})$ to $C_b(\mathcal{I})$. Therefore, Theorem X.11 in Meyer [21] implies that there exists a kernel $\mu^r_\chi(;s)$ such that

$$\mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] = \int_{\mathcal{I}} f(u)\mu^r_\chi(du;s)$$

for all $s \in \mathcal{I}$ and all measurable integrable functions $f : \mathcal{I} \to \mathbb{R}$.

Given any points $s \le s_1 \le s_2$ in $\mathcal{I}$, (3.18) implies that

$$\mu^r_\chi([s_1, s_2[;s) = \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\mathbf{1}_{\{T_{s_1}<\mathcal{T}_\chi<T_{s_2}\}}] > 0, \tag{3.19}$$

the inequality following by regularity of the strong Markov process (in particular, see Definition 3.2.2 (iii)). Furthermore, (3.3) in Lemma 3.2.2, (3.19) and the dominated convergence theorem imply that $\mu^r_\chi(\{s_1\};s) = 0$. ∎

Lemma 3.2.8. *Given χ an accessible function on $\mathcal{I}$ such that $\mathfrak{G}_\chi$ is locally bounded, for any locally bounded measurable $\tilde{\chi} : \mathcal{I} \to \mathbb{R}$ such that $\kappa(s) \le \tilde{\chi} \le \chi(s) < s$ with $\kappa(s) = \tilde{\chi}(s)$ possible only on $s \in C_2$ defined in Lemma 3.2.1, we have that $\tilde{\chi}$ is accessible. In particular, for any $d : \mathcal{I} \to \mathbb{R}$ locally bounded away from 0 such that $s - \kappa(s) > d(s)$ for all $s \in \mathcal{I}$, we have $\tilde{\chi} = \chi(s) \wedge (s - d(s))$ is accessible.*

Proof. To check (i) of Definition 3.2.2, we note that

$$\mathbb{P}^{x,s}(T_b < \mathcal{T}_{\tilde{\chi}}) \ge \mathbb{P}^{x,s}(T_b < T_\chi) > 0.$$

for any $\tilde{\chi}(s) \le \chi(s) < x \le s \le b$. Notice that $\mathbb{P}^{x,s}(T_s < T_{\tilde{\chi}(s),s}) > 0$ by regularity and the property of χ_n. Now, we have

$$\mathbb{P}^{x,s}(T_b < \mathcal{T}_{\tilde{\chi}}) = \mathbb{P}^{x,s}(T_s < T_{\tilde{\chi}(s),s})\mathbb{P}^{s,s}(T_b < \mathcal{T}_{\tilde{\chi}}) \ge \mathbb{P}^{x,s}(T_s < T_{\tilde{\chi}(s),s})\mathbb{P}^{s,s}(T_b < T_\chi) > 0$$

for $\tilde{\chi}(s) < x \le \chi(s) \le s \le b$.

(ii) in Definition 3.2.2 follows from

$$\mathbb{P}^{x,s}(T_{\chi_n(s),s}) > 0 \quad \text{and} \quad \chi_n(s) \to \kappa(s) \text{ as } n \to \infty.$$

for any $\chi_n(s) \le x < s$ where $\{\chi_n\}$ are as in Definition 3.2.2.

As for (iii) in Definition 3.2.2, we notice that

$$\begin{aligned}
\mathbb{P}^{s,s}(\mathcal{T}_{\tilde{\chi}} < T_b) &= \mathbb{E}^{s,s}[\mathbf{1}_{\{\mathcal{T}_{\tilde{\chi}}<T_b\}}] \\
&= \mathbb{E}^{s,s}\left[\mathbf{1}_{\{\mathcal{T}_\chi<T_b\}}\mathbf{1}_{\{\mathcal{T}_{\tilde{\chi}}<T_b\}}\right] \\
&= \mathbb{E}^{s,s}\left[\mathbf{1}_{\{\mathcal{T}_\chi<T_b\}}\mathbf{1}_{\{\mathcal{T}_{\tilde{\chi}}\circ\theta_{\mathcal{T}_\chi}<T_b\circ\theta_{\mathcal{T}_\chi}\}}\right] \\
&= \mathbb{E}^{s,s}\left[\mathbf{1}_{\{\mathcal{T}_\chi<T_b\}}\mathbb{P}^{\chi(S_{\mathcal{T}_\chi}),S_{\mathcal{T}_\chi}}(\mathcal{T}_{\tilde{\chi}} < T_b)\right] \\
&> \int_{[s,b[} \mathbb{P}^{\chi(u),u}(T_{\tilde{\chi}(u),u} < T_u)\mu^0_\chi(du;s) \\
&> 0,
\end{aligned}$$

where the last line follows from the fact $\mu^0_\chi(du;s)$ is a positive measure and $\mathbb{P}^{\chi(u),u}(T_{\tilde{\chi}(u),u} < T_u) > 0$ for $u \in [s, b[$. ∎

3.3 The scale function

In this section, we introduce the notion of scale function of the regular PRM-JSM defined in the previous section.

Definition 3.3.1. *A function $p : E \to \mathbb{R}$ is called a scale function for the regular PRM-JSM $\{\mathbb{P}^{x,s}, (x,s) \in E\}$ with coordinate process (X, S) if and only if*

(i) the map $(x,s) \mapsto (p(x,s), p(s,s))$ is one-to-one, the maps $\mathcal{I}_s \ni x \mapsto p(x,s)$ and $\mathcal{I} \ni s \mapsto p(s,s)$ are continuous,

(ii) given any $(x,s) \in E$, the process $p(X_{t\wedge\mathcal{T}_\kappa}, S_{t\wedge\mathcal{T}_\kappa})$ is a $(\mathcal{F}_t, \mathbb{P}^{x,s})$-local martingale.

Lemma 3.3.1. *Given $\alpha < a < b < \beta$ and an accessible function χ, define*

$$\mathcal{D} = E_{a,b,\chi} \text{ and } \quad p_{\mathcal{D}}(x,s) = \mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi), \text{ for all } (x,s) \in \mathcal{D}, \tag{3.20}$$

then $p_{\mathcal{D}}$ is measurable and the following statements hold true:

(i) Given any $(x,s) \in \mathcal{D}$, the process $(p_{\mathcal{D}}(X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi}), t \geq 0)$ is a $(\mathcal{F}_t, \mathbb{P}^{x,s})$ uniformly integrable martingale, in particular,

$$p_{\mathcal{D}}(X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi}) = \mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi \mid \mathcal{F}_t).$$

(ii) The function $\mathcal{D} \ni (x,s) \mapsto (p_{\mathcal{D}}(x,s), p_{\mathcal{D}}(s,s))$ is one-to-one,

$$p_{\mathcal{D}}(x,s) < p_{\mathcal{D}}(y,s) \quad \text{for all } s \in [a,b] \text{ and } \chi(s) \leq x < y \leq s \tag{3.21}$$

$$\text{and} \quad p_{\mathcal{D}}(s_1,s_1) < p_{\mathcal{D}}(s_2,s_2) \quad \text{for all } a \leq s_1 < s_2 \leq b. \tag{3.22}$$

(iii) The function $[\chi(s), s] \ni x \mapsto p_{\mathcal{D}}(x,s)$ is continuous for all $s \in [a,b]$. And the function $[a,b] \ni s \mapsto p_{\mathcal{D}}(s,s)$ is continuous.

Proof. We first note that the definition of the shift operators imply that

$$\big\{\omega \in \Omega \mid \; T_b(\omega) < \mathcal{T}_\chi(\omega)\big\} = \big\{\omega \in \Omega \mid \; T_b \circ \theta_{t\wedge T_b\wedge\mathcal{T}_\chi}(\omega) < \mathcal{T}_\chi(\omega) \circ \theta_{t\wedge T_b\wedge\mathcal{T}_\chi}\big\}.$$

In view of this observations and the strong Markov property (iii) in Definition 3.2.1, we can see that

$$\begin{aligned} p_{\mathcal{D}}(X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi}) &= \mathbb{P}^{X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi}}(T_b < \mathcal{T}_\chi) \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b<\mathcal{T}_\chi\}} \circ \theta_{t\wedge T_b\wedge\mathcal{T}_\chi} \mid \mathcal{F}_{t\wedge T_b\wedge\mathcal{T}_\chi}] \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b<\mathcal{T}_\chi\}} \mid \mathcal{F}_{t\wedge T_b\wedge\mathcal{T}_\chi}] \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b<\mathcal{T}_\chi\}}\mathbf{1}_{\{t\leq T_b\wedge\mathcal{T}_\chi\}} \mid \mathcal{F}_t] + \mathbf{1}_{\{T_b<\mathcal{T}_\chi\}}\mathbf{1}_{\{T_b\wedge\mathcal{T}_\chi\leq t\}} \\ &= \mathbb{E}^{x,s}[\mathbf{1}_{\{T_b<\mathcal{T}_\chi\}} \mid \mathcal{F}_t], \end{aligned}$$

and (i) follows.

Fix any points $s \in [a,b]$ and $\chi(s) \leq x < y \leq s$. The inequality (3.21) follows trivially if $x = \chi(s)$, because $p_{\mathcal{D}}(x,s) = 0$ and property (i) in Definition 3.2.2

implies that $p_{\mathcal{D}}(y,s) = \mathbb{P}^{y,s}(T_b < \mathcal{T}_\chi) > 0$. We therefore assume that $\chi(s) < x$ in what follows. The definition of the shift operators implies that

$$\big\{\omega \in \Omega \mid T_b(\omega) < \mathcal{T}_\chi(\omega)\big\} = \big\{\omega \in \Omega \mid (T_{y,s} + T_b \circ \theta_{T_{y,s}})(\omega) < (T_{y,s} + \mathcal{T}_\chi \circ \theta_{T_{y,s}})(\omega)\big\}$$

Also, property *(ii)* in Definition 3.2.2 implies that $\mathbb{P}^{x,s}(T_{y,s} < \mathcal{T}_\chi) < 1$. In view of this observation and the strong Markov property *(iii)* in Definition 3.2.1, we can see that

$$\begin{aligned}
p_{\mathcal{D}}(x,s) &= \mathbb{E}^{x,s}\big[\mathbf{1}_{\{T_{y,s}<\mathcal{T}_\chi\}}\mathbf{1}_{\{T_b\circ\theta_{T_{y,s}}<\mathcal{T}_\chi\circ\theta_{T_{y,s}}\}}\big] \\
&= \mathbb{E}^{x,s}\big[\mathbb{E}^{x,s}\big[\mathbf{1}_{\{T_b\circ\theta_{T_{y,s}}<\mathcal{T}_\chi\circ\theta_{T_{y,s}}\}} \mid \mathcal{F}_{T_{y,s}}\big]\mathbf{1}_{\{T_{y,s}<\mathcal{T}_\chi\}}\big] \\
&= \mathbb{E}^{x,s}\big[\mathbb{E}^{X_{T_{y,s}},S_{T_{y,s}}}\big[\mathbf{1}_{\{T_b<\mathcal{T}_\chi\}}\big]\mathbf{1}_{\{T_{y,s}<\mathcal{T}_\chi\}}\big] \\
&= \mathbb{P}^{y,s}(T_b < \mathcal{T}_\chi)\mathbb{P}^{x,s}(T_{y,s} < \mathcal{T}_\chi) \\
&= p_{\mathcal{D}}(y,s)\mathbb{P}^{x,s}(T_{y,s} < \mathcal{T}_\chi) \qquad (3.23) \\
&< p_{\mathcal{D}}(y,s).
\end{aligned}$$

Property *(iii)* of Definition 3.2.2 implies that

$$\mathbb{P}^{s_1,s_1}(T_{s_2} < \mathcal{T}_\chi) \le 1 - \mathbb{P}^{s_1,s_1}(\mathcal{T}_\chi < T_{s_2}) < 1.$$

Using these inequalities we can derive (3.22) by following exactly the same reasoning as in the proof of (3.21).

The continuity of $[\chi(s), s] \ni x \mapsto p_{\mathcal{D}}(x,s)$ follows from (3.23) and the fact $y \mapsto \mathbb{P}^{x,s}(T_{y,s} < \mathcal{T}_\chi)$ is continuous for $y \in [x,s]$ and $x \in [\chi(s), s[$ is arbitrary. For $s \in [a,b[$, notice that

$$\mathbb{P}^{a,a}(T_b < \mathcal{T}_\chi) = \mathbb{P}^{a,a}(T_s < \mathcal{T}_\chi)\mathbb{P}^{s,s}(T_b < \mathcal{T}_\chi).$$

The continuity of $[a,b] \ni s \mapsto p_{\mathcal{D}}(s,s)$ follows from the fact $s \mapsto \mathbb{P}^{a,a}(T_s < \mathcal{T}_\chi)$ is continuous for $s \in [a,b]$. ∎

We now extend $p_{\mathcal{D}}$ to a function p on E that is a scale function.

Lemma 3.3.2. *Let $\{\mathbb{P}^{x,s}, (x,s) \in E\}$ be a regular PRM-JSM with coordinate process (X,S). Then there exists a function $p : E \to \mathbb{R}$ that is a scale function for (X,S) in the sense of Definition 3.3.1.*

Proof. Fix any $a, b \in \mathbb{R}$ such that $\alpha \le a < b \le \beta$, and let $\{\chi_n\}$ be a sequence of accessible functions in the sense of Definition 3.2.2. Let $\mathcal{D}_n$ and $p_{\mathcal{D}_n} : \mathcal{D}_n \to \mathbb{R}^+$ be defined by (3.20) with χ_n in place of χ.

The expression (3.6) with χ_n in place of a and s in place of b imply that given any $1 \le n$, there exist functions $\alpha_n, \beta_n : [a,b] \mapsto \mathbb{R}$ such that

$$p_{\mathcal{D}_1}(x,s) = \alpha_n(s)p_{\mathcal{D}_n}(x,s) + \beta_n(s) \quad \text{for all } s \in [a,b] \text{ and } x \in [\chi_1(s), s]. \qquad (3.24)$$

Given any $s \in [a,b]$, we define

$$p_{a,b}(x,s) = \begin{cases} p_{\mathcal{D}_1}(x,s), & \text{if } x \in [\chi_1(s), s] \text{ and } s \in [a,b], \\ \alpha_n(s)p_{\mathcal{D}_n}(x,s) + \beta_n(s), & \text{if } x \in [\chi_n(s), \chi_{n-1}(s)[\text{ and } s \in [a,b]. \end{cases} \qquad (3.25)$$

In view of (3.24), we can see that

$$p_{a,b}(x,s) = \alpha_{1n}(s)p_{\mathcal{D}_n}(x,s) + \beta_{1n}, \text{ if } x \in [\chi_n(s), s] \text{ and } s \in [a,b].$$

We now proceed to check that $p_{a,b}(X_{t\wedge T_b \wedge \mathcal{T}_\kappa}, S_{t\wedge T_b\wedge\mathcal{T}_\kappa})$ is a local martingale for (X,S) starting from $(x,s) \in]\kappa(s), s] \times [a,b[$.

For $x < s$, the process $p_{a,b}(X_{t\wedge\mathcal{T}_\kappa \wedge T_s}, S_{t\wedge\mathcal{T}_\kappa\wedge T_s})$ starting from (x,s) is a local martingale as in the one dimensional case. And for (X,S) starting from $(s,s) \in \mathcal{D}_n$, the process $p_{a,b}(X_{t\wedge\mathcal{T}_{\chi_n}\wedge T_b}, S_{t\wedge\mathcal{T}_{\chi_n}\wedge T_b})$ is a local martingale by Lemma 3.3.1. Now let us define a sequence of stopping times,

$$\begin{aligned} R_0 &= \inf\{t \geq 0 \mid X_t = S_t\} \wedge \mathcal{T}_\kappa; \\ R_{2n-1} &= \inf\{t \geq R_{2n-2} \mid (X_t, S_t) = (\chi_n(S_t), S_t))\} \wedge T_b \quad , \text{for } n \geq 1; \\ R_{2n} &= \inf\{t \geq R_{2n-1} \mid X_t = S_t\} \wedge \mathcal{T}_\kappa, \quad \text{for } n \geq 1, \end{aligned}$$

we have that $R_n \uparrow T_b \wedge \mathcal{T}_\kappa$ and $p_{a,b}(X_{t\wedge R_n}, S_{t\wedge R_n})$ is a local martingale for all n. Hence $p_{a,b}(X_{t\wedge T_b\wedge\mathcal{T}_\kappa}, S_{t\wedge T_b\wedge\mathcal{T}_\kappa})$ is also a local martingale.

Now let $\{[a_n, b_n[\}_{n\in\mathbb{Z}}$ be a sequence of disjoint intervals such that $a_n = b_{n-1}$ for all $n \in \mathbb{Z}$ and $\bigcup_{n\in\mathbb{Z}}[a_n, b_n[= \mathcal{I}$.

Notice $p_{a_n,b_n}(s,s)$ is non-vanishing for $s \in [a_n, b_n[$, we can scale p_{a_n,b_n} such that, upon renaming, we have

$$p_{a_n,b_n}(b_n, b_n) = p_{a_{n+1},b_{n+1}}(b_n, b_n).$$

Now we define

$$p(x,s) = p_{a_n,b_n}(x,s) \quad \text{for} \quad x \in \mathcal{I}_s \text{ and } s \in [a_n, b_n[.$$

By construction, one can check p is a scale function. Also note that p is measurable since all α_{mn}, β_{mn} and p_{Dm} are measurable. ∎

Definition 3.3.2. *We say the PRM-JSM (X,S) is in the natural scale if $X_{\cdot\wedge\mathcal{T}_\kappa}$ is a local martingale, or equivalently, $p(x,s) = x$ is a scale function.*

Lemma 3.3.3. *Given$\{\mathbb{P}^{x,s}, (x,s) \in E\}$ a regular PRM-JSM with state space E and a scale function p. Let π be the map $\pi(x,s) = (p(x,s), p(s,s))$, $\tilde{E} = \{\pi(x,s) \mid (x,s) \in E\}$ and $\tilde{\mathcal{E}} = \tilde{E} \cap \mathcal{B}(\mathbb{R}^2)$. We have the bijection $\pi : E \to \tilde{E}$ and its inverse $\pi^{-1} : \tilde{E} \to E$ are $\mathcal{E}/\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}/\mathcal{E}$ measurable respectively. And for the probability measure defined by*

$$\tilde{\mathbb{P}}^{\pi(x,s)}((X_t, S_t) \in \pi(A)) = \mathbb{P}^{x,s}((X_t,S_t) \in A) \text{ for all } (x,s) \in E \text{ and } A \in \mathcal{E}, \quad (3.26)$$

we have $(\Omega, \mathcal{F}, \mathcal{F}_t, X, S, \tilde{\mathbb{P}}^{x,s})$ is a regular, canonical strong Markov process on $\tilde{E}$.

Proof. The map $\pi : E \to \tilde{E}$ defined by $\pi(x,s) = (p(x,s), p(s,s))$ is one-one and such that given any set of the form $A = \cup_{s\in[s_1,s_2]}[a,s] \times \{s\} \cap E$, which generates $\mathcal{E}$, we have $\pi(A) = \{\pi(x,s) \mid (x,s) \in A\}$ belongs to $\tilde{\mathcal{E}}$ because $p(x,s)$ is measurable. Thus, we know $\pi^{-1} : \tilde{E} \to E$ is $\tilde{\mathcal{E}}/\mathcal{E}$ measurable. A symmetrical argument shows that π is $\mathcal{E}/\tilde{\mathcal{E}}$ measurable.

We prove that $\{\tilde{\mathbb{P}}^{x,s}, (x,s) \in \tilde{E}\}$ is a regular, canonical strong Markov process with respect to $\tilde{E}$. (i) of Definition 3.2.1 follows from that for all $C = \{(X_t, S_t) \in A\}$, we have $(x,s) \mapsto \tilde{\mathbb{P}}^{x,s}(C)$ is $\mathcal{E}/\mathcal{B}([0,1])$ measurable by (3.26) and measurability results proved in previous paragraph. (ii)-(iv) can be checked directly. It remains to prove (v). Given a bounded measurable function f, let us define a function

$$\Psi_f(x,s) = \mathbb{E}^{x,s}[f(X_t, S_t)] = \tilde{\mathbb{E}}^{\pi(x,s)}[f(\pi^{-1}(X_t, S_t))].$$

To prove strong Markov property of $\{\mathbb{P}^{x,s}\}$, we want to show for all bounded measurable function f, all bounded stopping time T and some fixed $t > 0$, the function Ψ_f satisfies

$$\mathbb{E}^{x,s}[f(X_{T+t}, S_{T+t}) \mid \mathcal{F}_T] = \Psi_f(X_T, S_T).$$

By Lemma 1.3.3 of Stroock and Varadhan [28], we know $\mathcal{F}_T = \sigma(X_{u\wedge T}, u \geq 0)$, which can be generated by $\prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T} s) \in A\}}$ by continuity of paths. For

$$Z = \tilde{\mathbb{E}}^{\pi(x,s)}[f(\pi^{-1}(X_{T+t}, S_{T+t})) \mid \mathcal{F}_T],$$

we have

$$\begin{aligned}
\tilde{\mathbb{E}}^{\pi(x,s)}\left[Z \prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T}) \in \pi(A)\}}\right] &= \tilde{\mathbb{E}}^{\pi(x,s)}\left[f(\pi^{-1}(X_{T+t}, S_{T+t})) \prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T}) \in \pi(A)\}}\right] \\
&= \mathbb{E}^{x,s}\left[f(X_{T+t}, s_{T+t}) \prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T}) \in A\}}\right] \\
&= \mathbb{E}^{x,s}\left[\Psi_f(X_T, S_T) \prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T}) \in A\}}\right] \\
&= \tilde{\mathbb{E}}^{\pi(x,s)}\left[\Psi_f(\pi^{-1}(X_T, S_T)) \prod_i \mathbf{1}_{\{(X_{t_i\wedge T}, S_{t_i\wedge T}) \in \pi(A)\}}\right]
\end{aligned}$$

so that

$$\begin{aligned}
\tilde{\mathbb{E}}^{\pi(x,s)}[f(\pi^{-1}(X_{T+t}, S_{T+t})) \mid \mathcal{F}_T] &= \Psi_f(\pi^{-1}(X_T, S_T)) \\
&= \mathbb{E}^{\pi^{-1}(X_T, S_T)}[f(X_t, S_t)] \\
&= \tilde{\mathbb{E}}^{X_T, S_T}[f(\pi^{-1}(X_t, S_t))].
\end{aligned}$$

The result follows from last expression if we replace $f \circ \pi^{-1}$ by f. The regularity can be easily checked. ∎

We now give a brief characterisation of the family of scale functions.

We define the function

$$\nu_{g,\chi}(s, s_1, s_2) = g(s_1)\mathbb{P}^{s,s}(T_{s_1} < \mathcal{T}_\chi) - g(s_2)\mathbb{P}^{s,s}(T_{s_2} < \mathcal{T}_\chi) \quad \text{for all } s \leq s_1 \leq s_2 \in \mathcal{I}.$$

Definition 3.3.3. *An function $g : \mathcal{I} \to \mathbb{R}$ is called scale-generating if*

(i) it is continuous, strictly increasing and

(ii) *for every accessible* χ *such that* $\mathfrak{G}_\chi$ *is locally bounded, there exists a function* $f_{g,\chi} : \mathcal{I} \to \mathbb{R}$ *for all* $s \in \mathcal{I}$ *such that*

$$\nu_{g,\chi}(s, s_1, s_2) = \int_{]s_1,s_2[} f_{g,\chi}(u)\mu^0_\chi(du; s) \quad \text{for all } s \in \mathcal{I} \text{ and } s \le s_1 < s_2 \in \mathcal{I},$$

where μ^0_χ *is defined in Lemma 3.2.7.*

Remark 3.3.1. *Any two such functions* $f_{g,\chi}$ *are equal* $\mu_\chi(\cdot; s)$ *almost surely for all* $s \in \mathcal{I}$.

Theorem 3.3.4. *Consider any strong Markov process, the following statements hold true:*

(i) *If* p *is a scale function, then the function* $\mathcal{I} \ni s \mapsto p(s, s)$ *is scale-generating.*

(ii) *Given any scale-generating function* g*, there exists a scale function* p *of the strong Markov process such that* $p(s, s) = g(s)$ *for all* $s \in \mathcal{I}$.

Proof. Let χ be accessible such that $\mathfrak{G}_\chi$ is locally bounded. The existence of such χ follows from Assumption 3.2.1 and Lemma 3.2.4. By Lemma 3.2.6 (notice the fact that $\mathfrak{G}_\chi$ is locally bounded), we have $\mathbb{P}^{s,s}(T_b \wedge \mathcal{T}_\chi) = 1$ for all $s \le b \in \mathcal{I}$.

If p is a scale function, we know $p(s, s)$ is continuous and strictly increasing. Moreover, by optional sampling theorem, we have for all $s \le s_1 < s_2 \in \mathcal{I}$,

$$p(s, s) = \mathbb{P}^{s,s}(T_{s_i} < \mathcal{T}_\chi)p(s_i, s_i) + \mathbb{E}^{s,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_{s_i}\}} p(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})] \quad \text{for } i = 1, 2.$$

Subtracting, we obtain

$$\begin{aligned} p(s_1, s_1)\mathbb{P}^{s,s}(T_{s_1} < \mathcal{T}_\chi) - p(s_2, s_2)\mathbb{P}^{s,s}(T_{s_2} < \mathcal{T}_\chi) &= \mathbb{E}^{s,s}[\mathbf{1}_{\{T_{s_1} < \mathcal{T}_\chi < T_{s_2}\}} p(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})] \\ &= \int_{]s_1,s_2[} p(\chi(u), u)\mu_\chi(du; s). \end{aligned}$$

So for $g(s) = p(s, s)$, we can set $f_{g,\chi}(s) = p(\chi(s), s) < g(s)$ for all $s \in \mathcal{I}$.

Conversely, given any scale-generating function g and the corresponding $f_{g,\chi}$, we have

$$\nu_{g,\chi}(s, s_1, s_2) = \int_{]s_1,s_2[} f_{g,\chi}(u)\mu_\chi(du; s) = \mathbb{E}^{s,s}[\mathbf{1}_{\{T_{s_1} < \mathcal{T}_\chi < T_{s_2}\}} f_{g,\chi}(S_{\mathcal{T}_\chi})]$$
$$\text{for all } s \in \mathring{\mathcal{I}} \text{ and } \alpha < s \le s_1 < s_2 < \beta.$$

Notice that

$$\nu_{g,\chi}(s, s, b) = g(s) - \mathbb{P}^{s,s}(T_b < \mathcal{T}_\chi)g(b) = \mathbb{E}^{s,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f_{g,\chi}(S_{\mathcal{T}_\chi})] \quad \text{for all } s < b. \tag{3.27}$$

By the monotonicity of g, we would have

$$\mathbb{E}^{s,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f_{g,\chi}(S_{\mathcal{T}_\chi})] = g(s) - \mathbb{P}^{s,s}(T_b < \mathcal{T}_\chi)g(b) < \mathbb{E}^{s,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} g(S_{\mathcal{T}_\chi})] \quad \text{for all } s < b \in \mathcal{I}.$$

which implies $f_{g,\chi} < g$, $\mu_\chi(; s)$ almost surely. We then choose a version of $f_{g,\chi}$ such that $f_{g,\chi} < g$.

We define for $(x,s) \in E^b_\chi$ a function

$$p_{b,\chi}(x,s) = g(b)\mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi) + \mathbb{E}^{x,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f_{g,\chi}(S_{\mathcal{T}_\chi})]. \tag{3.28}$$

By (3.27), we have $p_{b,\chi}(s,s) = g(s)$ for all $s < b$.

Next we show $p_{b,\chi}(X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi})$ is a uniformly integrable martingale. It is enough to show the second term of the right hand side of (3.28) is a uniformly integrable martingale.

$$\begin{aligned}
&\mathbb{E}^{X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi}}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f(S_{\mathcal{T}_\chi})] \\
=&\mathbb{E}^{x,s}[\mathbf{1}_{\{\mathcal{T}_\chi\circ\theta_{t\wedge T_b\wedge\mathcal{T}_\chi} < T_b\circ\theta_{t\wedge T_b\wedge\mathcal{T}_\chi}\}} f(S_{\mathcal{T}_\chi\circ\theta_{t\wedge T_b\wedge\mathcal{T}_\chi}}) \mid \mathcal{F}_{t\wedge T_b\wedge\mathcal{T}_\chi}] \\
=&\mathbb{E}^{x,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f(S_{\mathcal{T}_\chi}) \mid \mathcal{F}_{t\wedge T_b\wedge\mathcal{T}_\chi}] \\
=&\mathbb{E}^{x,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f(S_{\mathcal{T}_\chi})\mathbf{1}_{\{t\le T_{b,b}\wedge\mathcal{T}_\chi\}} \mid \mathcal{F}_t] + \mathbf{1}_{\{\mathcal{T}_\chi < T_b < t\}} f(S_{\mathcal{T}_\chi}) \\
=&\mathbb{E}^{x,s}[\mathbf{1}_{\{\mathcal{T}_\chi < T_b\}} f(S_{\mathcal{T}_\chi}) \mid \mathcal{F}_t].
\end{aligned}$$

We can also check for $b_1 < b_2$, $p_{b_1,\chi}$ and $p_{b_2,\chi}$ agree on $E^{b_1}_\chi$ by the definition (3.28), the fact that $p_{b,\chi}(s,s) = g(s)$ for all $s < b \in \mathcal{I}$ and the local martingale property. Thus we can extend p to E_χ by defining $p_\chi(x,s) = p_{b,\chi}(x,s)$ for any $(x,s) \in E_\chi$.

We can then extend p_χ to a scale function p on E following the same manner as in Lemma 3.3.2, see (3.25). ∎

3.4 Time change characterisation

In this and subsequent section, we can work in natural scale assuming that X is a local martingale, (X,S) is regular with respect to E with $\{\chi_n\}$ satisfying the condition of Corollary 3.2.5. We show that under some characterised time change, (X,S) becomes $(B,\bar{B})$, a Brownian motion and its running maximum. We then conclude with boundary behaviours in E are similar as in one dimensional case.

Lemma 3.4.1. *Given $\mathcal{D}$ and $F_\mathcal{D}$ as in Lemma 3.2.6, the following statements hold true:*

(i) The function $F_{\mathcal{D},d}$ defined by

$$F_{\mathcal{D},d}(s) = F_\mathcal{D}(s,s), \text{ for } s \in [a,b]$$

is continuous and of bounded variation.

(ii) Suppose there exists a constant $K > 0$ such that

$$\sup_{s\in[a,b]} \mathfrak{M}_\chi(s) \le K \quad \text{and} \quad \inf_{s\in[a,b]} \{s - \chi(s)\} \ge d > 0, \tag{3.29}$$

where $\mathfrak{M}_\chi$ is the function defined by (3.12). The left derivative of $F_\mathcal{D}$ with respect to x, $[a,b] \ni s \mapsto (F_\mathcal{D})_{x-}(s,s)$, is bounded.

(iii) The process M defined by

$$M_t = t \wedge T_b \wedge \mathcal{T}_\chi + F_\mathcal{D}(X_{t\wedge T_b\wedge\mathcal{T}_\chi}, S_{t\wedge T_b\wedge\mathcal{T}_\chi})$$

is a uniformly integrable $(\mathcal{F}_t)$-martingale.

Proof of (i). Given $a \le s \le b$, we use the strong Markov property and the fact that $T_s \wedge \mathcal{T}_\chi < \infty$, $\mathbb{P}^{a,a}$ almost surely to obtain

$$F_{\mathcal{D},d}(a) = \mathbb{E}^{a,a}[T_s \wedge \mathcal{T}_\chi] + \mathbb{P}^{a,a}(T_s < \mathcal{T}_\chi) F_{\mathcal{D},d}(s).$$

It follows that

$$F_{\mathcal{D},d}(s) \le \frac{1}{\mathbb{P}^{a,a}(T_s < \mathcal{T}_\chi)} F_{\mathcal{D},d}(a) \le \frac{1}{\mathbb{P}^{a,a}(T_b < \mathcal{T}_\chi)} F_{\mathcal{D},d}(a) := C \quad \text{for all } s \in [a,b]$$

which establishes the boundedness of $F_{\mathcal{D},d}$. On the other hand, combining (3.30) with the fact that $\mathbb{E}^{a,a}[T_s \wedge \mathcal{T}_\chi]$ and $\mathbb{P}^{a,a}(T_s < \mathcal{T}_\chi)$ are continuous as functions of $s \in [a,b]$ (See Corollary 3.2.3), we obtain the continuity of $F_{\mathcal{D},d}$.

Now consider any $a = s_0 < s_1 < \cdots < s_N = b$, and define

$$q_{-1} = 1 \quad \text{and} \quad p_i = 1 - q_i = \mathbb{P}^{s_i,s_i}(\mathcal{T}_\chi < T_{s_{i+1}}) \quad \text{for } i = 0, \cdots, N-1.$$

In view of the regularity condition (i) of Definition 3.2.3, the strong Markov property and a simple inductive argument, we can see that

$$\begin{aligned}
0 < \mathbb{P}^{a,a}(T_b < \mathcal{T}_\chi) &= \mathbb{P}^{a,a}(T_{s_1} < \mathcal{T}_\chi, T_b < \mathcal{T}_\chi) \\
&= \mathbb{P}^{a,a}(T_{s_1} < \mathcal{T}_\chi, T_b \circ \theta_{T_{s_1}} < \mathcal{T}_\chi \circ \theta_{T_{s_1}}) \\
&= \mathbb{P}^{a,a}(T_{s_1} < \mathcal{T}_\chi)\mathbb{P}^{s_1,s_1}(T_b < \mathcal{T}_\chi) \\
&= \prod_{k=0}^{N-1} \mathbb{P}^{s_k,s_k}(T_{s_{k+1}} < \mathcal{T}_\chi) \\
&= \prod_{k=-1}^{N-1} q_k =: r < 1. \qquad (3.30)
\end{aligned}$$

On the other hand, we can use the strong Markov property and (3.30) to obtain

$$\begin{aligned}
\mathbb{E}^{a,a}[T_b \wedge \mathcal{T}_\chi] &= \mathbb{E}^{s_0,s_0}[T_{s_1} \wedge \mathcal{T}_\chi] + q_0 \mathbb{E}^{s_1,s_1}[T_b \wedge \mathcal{T}_\chi] \\
&= \sum_{i=0}^{N-1} \mathbb{E}^{s_i,s_i}[T_{s_{i+1}} \wedge \mathcal{T}_\chi] \prod_{k=-1}^{i-1} q_k \\
&\ge r \sum_{i=0}^{N-1} \mathbb{E}^{s_i,s_i}[T_{s_{i+1}} \wedge \mathcal{T}_\chi]. \qquad (3.31)
\end{aligned}$$

Furthermore, the arithmetic mean- geometric mean inequality implies that

$$\sum_{i=0}^{N-1} p_i = N - \sum_{i=0}^{N-1} q_i \le N - N\Big(\prod_{k=0}^{N} q_k\Big)^{\frac{1}{N}} = N(1 - r^{\frac{1}{N}}) \quad \text{for all } N \in \mathbb{N}.$$

Using (3.30) with a replaced by s_i, s replaced by s_{i+1}, (3.31) and (3.32) we obtain

$$\begin{aligned}\sum_{i=0}^{N-1}|F_{\mathcal{D},d}(s_{i+1})-F_{\mathcal{D},d}(s_i)| &\le \sum_{i=0}^{N-1}\mathbb{E}^{s_i,s_i}[T_{s_{i+1}}\wedge\mathcal{T}_\chi] \\ &\quad+\sum_{i=0}^{N-1}\mathbb{P}^{s_i,s_i}(\mathcal{T}_\chi<T_{s_{i+1}})F_{\mathcal{D},d}(s_{i+1}) \\ &\le \sum_{i=0}^{N-1}\mathbb{E}^{s_i,s_i}[T_{s_{i+1}}\wedge\mathcal{T}_\chi]+C\sum_{i=0}^{N-1}p_i \\ &\le \frac{1}{r}\mathbb{E}^{a,a}[T_b\wedge\mathcal{T}_\chi]+CN(1-r^{\frac{1}{N}}). \end{aligned}\tag{3.32}$$

This result and the fact that $\lim_{N\to\infty}N(1-r^{\frac{1}{N}})=-\ln r$ imply that $F_{\mathcal{D},d}$ is of bounded variation because $a=s_0<s_1<\cdots s_N=b$ has been arbitrary.
Proof of (ii). Given $s\in[a,b]$ and $x\in]\chi(s),s[$, the concavity of $F_{\mathcal{D}}(\cdot,s)$ implies that that $(F_{\mathcal{D}})_{x-}(x,s)$ is finite. In view of this observation, Lemma 3.2.6 and (3.29), we can see that

$$(F_{\mathcal{D}})_{x-}(s,s)=(F_{\mathcal{D}})_{x-}(x,s)+(F_{\mathcal{D}})_{x,x}([x,s[;s)=(F_{\mathcal{D}})_{x-}(x,s)-2m([x,s[;s)>-\infty$$

for all $s\in[a,b]$. Furthermore,

$$\begin{aligned}F_{\mathcal{D}}(s,s)&=F_{\mathcal{D}}(s,s)-F_{\mathcal{D}}(\chi(s),s)\\ &=(F_{\mathcal{D}})_{x-}(s,s)(s-\chi(s))+2\int_{\chi(s)}^{s}m([u,s[;s)du\quad\text{for all } s\in[a,b].\end{aligned}$$

It follows that

$$\begin{aligned}|(F_{\mathcal{D}})_{x-}(s,s)|&=\left|\frac{1}{s-\chi(s)}\Big((F_{\mathcal{D}})(s,s)-2\int_{\chi(s)}^{s}2m([u,s[;s)du\Big)\right|\\ &\le\frac{1}{d}(C+4K)<\infty\quad\text{for all } s\in[a,b],\end{aligned}$$

and the result follows.
Proof of (iii). By the definition of $F_{\mathcal{D}}$ and strong Markov property, we have

$$M_t=t\wedge T_b\wedge\mathcal{T}_\chi+F_{\mathcal{D}}(X_{t\wedge T_b\wedge\mathcal{T}_\chi},S_{t\wedge T_b\wedge\mathcal{T}_\chi})=\mathbb{E}^{x,s}[T_b\wedge\mathcal{T}_\chi\mid\mathcal{F}_t]$$

is a $(\mathcal{F}_t)$-uniformly integrable martingale. ∎

Remark 3.4.1. *Without the condition $s-\chi(s)\ge d>0$ for all $s\in[a,b]$, $(F_{\mathcal{D}})_{x-}(s,s)$ will explode in $[a,b]$. See Example 4.*

We now quote a result from Lamberton and Zervos [20] and apply it to obtain the time change. For an interval $J\subset\mathbb{R}$, denote

$$l(J)=\inf J \text{ and } r(J)=\sup J.$$

Theorem 3.4.2. *For any measurable $\chi : \mathcal{I} \to \mathbb{R}$ such that $s - \chi(s)$ is locally bounded away from 0, that is, for any $[a,b] \subset \mathcal{I}$, we have*

$$\inf_{s\in[a,b]} \big(s - \chi(s)\big) > 0, \tag{3.33}$$

consider any measurable function $F : E \to \mathbb{R}$ that satisfy
(1) the function $\mathcal{I}_s \ni x \mapsto F(x,s)$ is the difference of two convex functions for any $s \in \mathcal{I}$,
(2) the function $\mathcal{I} \ni s \ni F_{x-}(s,s)$ is locally bounded and Borel measurable.
(3) and given any $x_0 < a < b \in \mathcal{I}$, there exists a constant $K = K(x_0,a,b) > 0$ such that

$$|F_{x-}(x,s)| \le K, \ \forall x \in [x_0, s] \text{ and } s \in [a,b].$$

Then for $(X_0, S_0) \in E_\chi$, the Itô-Tanaka-Meyer formula

$$\begin{aligned} F(X_t \wedge \mathcal{T}_\chi, S_t \wedge \mathcal{T}_\chi) =& F(X_0,S_0) + F(S_t \wedge \mathcal{T}_\chi, S_t \wedge \mathcal{T}_\chi) - F(S_0,S_0) - \int_0^{t\wedge\mathcal{T}_\chi} F_{x-}(S_u,S_u)dS_u \\ &+ \frac{1}{2}A^F_{t\wedge\mathcal{T}_\chi} + \int_0^{t\wedge\mathcal{T}_\chi} F_{x-}(X_u,S_u)dX_u \end{aligned}$$

holds, where A^F is of finite variation with

$$A^F_t = \sum_{J\in\mathcal{J}} \Big\{ \int_{(\alpha, S_{l(J)})} (L^z_{t\wedge l(J)} - L^z_{t\wedge r(J)}) F_{xx}(dz, S_{l(J)}) \Big\}$$

and $\mathcal{J}$ is the collection of pairwise disjoint intervals such that

$$\{t \ge 0 | X_t < S_t\} = \bigcup_{J\in\mathcal{J}} J = \bigcup_{J\in\mathcal{J}}]l(J), r(J)[.$$

Theorem 3.4.3. *Let $\{\mathbb{P}^{x,s}, (x,s) \in E\}$ be a regular PRM-JSM in natural scale corresponding to accessible functions $\{\chi_n\}_{n=1}^\infty$ as in Corollary 3.2.5. In addition, we suppose that $s - \chi_1$ is locally bounded away from 0, that is, for any $[a,b] \subset \mathcal{I}$, we have*

$$\inf_{s\in[a,b]} \big(s - \chi_1(s)\big) > 0 \tag{3.34}$$

and

$$\mathfrak{M}_{\chi_n}(s) \le K \quad \text{for any } s \in [a,b] \text{ and some constant } K = K(a,b,n) > 0. \tag{3.35}$$

Given an initial point $(x,s) \in E$, there exists $\{\Omega, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, B_t, \bar{B}_t, \tilde{\mathbb{P}}^{x,s}\}$, a filtered probability space with coordinate map B_t a $(\tilde{\mathcal{F}}_t)$- Brownian motion B along with $\bar{B}$ such that $B_0 = x$ and $\bar{B}_t = \sup_{0\le u\le t} B_u \vee s$, $\tilde{\mathbb{P}}^{x,s}$ almost surely. Also denote by l^a_t the jointly continuous version of local time of B. There exists a positive measure $m(\cdot; s)$ on $\mathcal{I}_s$ for all $s \in \mathcal{I}$ that satisfies

$$\begin{aligned} &0 < m([a,b[; s) < \infty \quad \text{for all } \kappa(s) < a < b \le s, \\ &m(\{(\kappa(s), s)\}; s) \in [0,\infty] \quad \text{if } (\kappa(s), s) \in E, \end{aligned}$$

and is such that the law of (X,S) under $\mathbb{P}^{x,s}$ coincides with the law of the process $(B_\gamma, \bar{B}_\gamma)$ under $\tilde{\mathbb{P}}^{x,s}$ where γ is the right-continuous inverse to an increasing $[0,\infty]$-valued process A given by,

$$A_t = \lambda(s, \bar{B}_t) + \sum_{J\in\mathcal{J}} \Big\{ \int_{]\kappa(\bar{B}_{l(J)}), \bar{B}_{l(J)}[} (l^z_{t\wedge r(J)} - l^z_{t\wedge l(J)}) m(dz; \bar{B}_{l(J)}) \Big\},$$

where $\mathcal{J}$ is the collection of pairwise disjoint intervals such that

$$\{t \geq 0 | B_t < \bar{B}_t\} = \bigcup_{J\in\mathcal{J}} J = \bigcup_{J\in\mathcal{J}}]l(J), r(J)[,$$

λ is a function that takes $s_1 \leq s_2 \in \mathcal{I}$ to $\mathbb{R}$ that satisfies

$$\lambda(s_1, s_2) = -F_n(s_2, s_2) + F_n(s_1, s_1) + \int_{s_1}^{s_2} F_{n,x-}(u,u)du \tag{3.36}$$

for all n such that $s_1 \leq s_2 \leq b_n$, where $F_n(x,s) = \mathbb{E}^{x,s}[T_{b_n} \wedge \mathcal{T}_{\chi_n}]$ for $(x,s) \in E_{s_0, b_n, \chi_n}$ and $(b_n)_{n=1}^\infty$ is any fixed sequence tending to β.

Proof. Let η_n be a sequence of stopping times with

$$\begin{aligned} \eta_0 &= 0, \\ \eta_{2n+1} &= \inf\{t \geq \eta_{2n} \mid X_t = \kappa(S_t)\} \text{ for all } n \geq 0, \\ \text{and } \eta_{2n} &= \inf\{t \geq \eta_{2n-1} \mid X_t = S_t\} \text{ for all } n \geq 1. \end{aligned}$$

Let $\tilde{A}_t = \inf\{s \mid [X]_s = t\}$ and note that $\tilde{A} : [0, [X]_\infty[\to \mathbb{R}^+$.

We know on $\{\eta_{2n-1} \leq t\}$, $X_{t\wedge\eta_{2n}}$ is a submartingale (see (47.24) in V.47 of Roger and Williams [26]). Itô-Tanaka lemma implies that

$$\begin{aligned} X_{t\wedge\eta_{2n}} - X_{\eta_{2n-1}} &= X_{t\wedge\eta_{2n}} - \kappa(S_{\eta_{2n-1}}) \\ &= \big(X_{t\wedge\eta_{2n}} - \kappa(S_{\eta_{2n-1}})\big)^+ \\ &= \int_{\eta_{2n-1}}^{t\wedge\eta_{2n}} \mathbf{1}_{\{X_u > \kappa(S_u)\}} dX_u + L^{\kappa(S_{\eta_{2n-1}})}_{t\wedge\eta_{2n}} - L^{\kappa(S_{\eta_{2n-1}})}_{\eta_{2n-1}}, \end{aligned}$$

where L^x_t is the local time of X. On the other hand, on $\{\eta_{2n} \leq t\}$

$$X_{t\wedge\eta_{2n+1}} - X_{\eta_{2n}} = \int_{\eta_{2n}}^{t\wedge\eta_{2n+1}} \mathbf{1}_{\{X_u > \kappa(S_u)\}} dX_u.$$

Then we obtain that,

$$X_t = x + Z_t + \tilde{L}_t,$$

where

$$Z_t = \int_0^t \mathbf{1}_{\{X_u > \kappa(S_u)\}} dX_u$$

is a continuous local martingale (see (47.25) in V.47 of Roger and Williams [26]) and

$$\tilde{L}_t = \sum_{n=1}^{\infty} \left(L_{t\wedge\eta_{2n}}^{\kappa(S_{\eta_{2n-1}})} - L_{t\wedge\eta_{2n-1}}^{\kappa(S_{\eta_{2n-1}})} \right)$$

is an increasing process. On some enrichment of $(\Omega, \mathcal{F}, \mathbb{P}^{x,s})$, there exists a Brownian motion W such that $Z_t = W([Z]_t)$. We can define a local submartingale

$$\tilde{B}_t = x + W_t + \tilde{l}_t,$$

where $\tilde{l}_t = \tilde{L}_{\tilde{A}_t}$. The fact that $\tilde{B}_t$ is a local martingale with $d[\tilde{B}]_t = dt$ for all $t \in [[X]_{\eta_{2n}}, [X]_{\eta_{2n+1}}[$, $n \geq 0$ implies $\tilde{B}_t$ is a Brownian motion for $t \in [[X]_{\eta_{2n}}, [X]_{\eta_{2n+1}}[$ and $n \geq 0$. We have also that $\tilde{B}_t - \tilde{B}_{[X]_{\eta_{2n-1}}}$ behaves like a reflecting Brownian motion for $t \in [[X]_{\eta_{2n-1}}, [X]_{\eta_{2n}}[$ and $n \geq 1$.

Without loss of generality, we assume $(x_0, s_0) \in E_1$. Consider $F_n(x,s) := \mathbb{E}^{x,s}[T_{b_n} \wedge \mathcal{T}_{\chi_n}]$ for $(x,s) \in E_n$, by (iii) of Lemma 3.4.1, we have the

$$M_t^n := t \wedge T_{b_n} \wedge \mathcal{T}_{\chi_n} + F_n(X_{t\wedge T_{b_n}\wedge\mathcal{T}_{\chi_n}}, S_{t\wedge T_{b_n}\wedge\mathcal{T}_{\chi_n}}) = \mathbb{E}^{x,s}[T_{b_n} \wedge \mathcal{T}_{\chi_n} \mid \mathcal{F}_t]$$

is a $(\mathcal{F}_t)$-uniformly integrable martingale. After time change, we have that

$$\tilde{M}_t^n := M_{\tilde{A}_t}^n = \tilde{A}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_\chi}} + F_n(\tilde{B}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_\chi}}, \bar{\tilde{B}}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_\chi}})$$

is a $(\mathcal{F}_{\tilde{A}_t})$-uniformly integrable martingale.

Notice that the condition (1)-(3) for Theorem 3.4.2 are satisfied for $F_n, n \geq 1$ by Lemma 3.2.6. So we can apply Theorem 3.4.2 to get

$$\begin{aligned}
\tilde{M}^n_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} - \tilde{M}_0^n = {} & \tilde{A}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} + F_n(\bar{\tilde{B}}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}, \bar{\tilde{B}}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}) - F_n(s,s) \\
& + \int_0^{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} F_{n,x-}(\bar{\tilde{B}}_u, \bar{\tilde{B}}_u) d\bar{\tilde{B}}_u \\
& - \int_0^{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} F_{n,x-}(\tilde{B}_u, \bar{\tilde{B}}_u) d\tilde{B}_u \\
& - \sum_{J\in\tilde{\mathcal{J}}} \Big\{ \int_{]\kappa(\bar{\tilde{B}}_{l(J)}), \bar{\tilde{B}}_{l(J)}[} (\tilde{l}^z_{t\wedge r(J)\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} - \tilde{l}^z_{t\wedge l(J)\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}) \\
& \cdot m(dz; \bar{B}_{l(J)}) \Big\},
\end{aligned}$$

where $\tilde{\mathcal{J}}$ is the collection of pairwise disjoint intervals such that

$$\{t \geq 0 \mid \tilde{B}_t < \bar{\tilde{B}}_t\} = \bigcup_{J\in\tilde{\mathcal{J}}} J = \bigcup_{J\in\tilde{\mathcal{J}}}]l(J), r(J)[,$$

and $\tilde{l}_t^z$ is the local time of process $\tilde{B}_t$.

We can apply the Doobs-Meyer decomposition theorem to identify the finite variation part,

$$\begin{aligned}\tilde{A}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} =& - F_n(\bar{\tilde{B}}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}, \bar{\tilde{B}}_{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}) + F_n(s,s) \\ &+ \int_0^{t\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} F_{n,x-}(\bar{\tilde{B}}_u, \bar{\tilde{B}}_u) d\bar{\tilde{B}}_u \\ &+ \sum_{J\in\tilde{\mathcal{J}}} \Big\{ \int_{]\kappa(\bar{\tilde{B}}_{l(J)}),\bar{\tilde{B}}_{l(J)}[} (\tilde{l}^z_{t\wedge r(J)\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}} - \tilde{l}^z_{t\wedge l(J)\wedge[X]_{T_{b_n}}\wedge[X]_{\mathcal{T}_{\chi n}}}) \\ &\cdot m(dz; \bar{\tilde{B}}_{l(J)}) \Big\}.\end{aligned}$$

Notice that

$$\begin{aligned}&\Big(- F_m(\bar{\tilde{B}}_{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}}, \bar{\tilde{B}}_{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}}) + F_m(s_0,s_0) + \int_0^{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}} F_{m,x-}(\bar{\tilde{B}}_u, \bar{\tilde{B}}_u) d\bar{\tilde{B}}_u\Big) \\ =&\Big(- F_n(\bar{\tilde{B}}_{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}}, \bar{\tilde{B}}_{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}}) + F_n(s_0,s_0) + \int_0^{[X]_{T_s}\wedge[X]_{\mathcal{T}_{\chi m}}} F_{n,x-}(\bar{\tilde{B}}_u, \bar{\tilde{B}}_u) d\bar{\tilde{B}}_u\Big)\end{aligned}$$

for all $s_0 \le s \le b_m \le b_n$. Multiply $\mathbf{1}_{\{[X]_{T_s}<[X]_{\mathcal{T}_{\chi m}}\}}$ on both sides, apply change of variable formula to the integral terms and take expectations, it follows that

$$-F_m(s,s)+F_m(s_0,s_0)+\int_{s_0}^s F_{m,x-}(u,u)du = -F_n(s,s)+F_n(s_0,s_0)+\int_{s_0}^s F_{n,x-}(u,u)du$$

for all $s_0 \le s \le b_m \le b_n$. Hence we can define λ consistently as in (3.36).

As $[X]_{T_{b_n}} \wedge [X]_{\mathcal{T}_{\chi n}} \to [X]_{\eta_1}$, we have that

$$\tilde{A}_{t\wedge[X]_{\eta_1}} = \lambda(\bar{\tilde{B}}_{t\wedge[X]_{\eta_1}}, s_0)+\sum_{J\in\tilde{\mathcal{J}}} \Big\{ \int_{]\kappa(\bar{\tilde{B}}_{l(J)}),\bar{\tilde{B}}_{l(J)}[} (\tilde{l}^z_{t\wedge r(J)\wedge[X]_{\eta_1}} - \tilde{l}^z_{t\wedge l(J)\wedge[X]_{\eta_1}}) m(dz; \bar{B}_{l(J)}) \Big\}. \tag{3.37}$$

Combining result from V47.1 of Roger and Williams [26] with (3.37), we obtain

$$\tilde{A}_{t\wedge[X]_{\eta_2}} =\lambda(\bar{\tilde{B}}_{t\wedge[X]_{\eta_2}}, s_0) + \sum_{J\in\tilde{\mathcal{J}}} \Big\{ \int_{[\kappa(\bar{\tilde{B}}_{l(J)}),\bar{\tilde{B}}_{l(J)}[} (\tilde{l}^z_{t\wedge r(J)\wedge[X]_{\eta_2}} - \tilde{l}^z_{t\wedge l(J)\wedge[X]_{\eta_2}}) m(dz; \bar{\tilde{B}}_{l(J)}) \Big\}.$$

By induction and the fact that $[X]_{\eta_n} \to [X]_\infty$ as $n \to \infty$, we have

$$\tilde{A}_{t\wedge[X]_\infty} = \lambda(\bar{\tilde{B}}_{t\wedge[X]_\infty}, s_0)+\sum_{J\in\tilde{\mathcal{J}}} \Big\{ \int_{[\kappa(\bar{\tilde{B}}_{l(J)}),\bar{\tilde{B}}_{l(J)}[} (\tilde{l}^z_{t\wedge r(J)\wedge[X]_\infty} - \tilde{l}^z_{t\wedge l(J)\wedge[X]_\infty}) m(dz; \bar{\tilde{B}}_{l(J)}) \Big\}.$$

We define $\tilde{\gamma}_t$ to be the right continuous inverse of $\tilde{A}_t$. It follows that $X_t = \tilde{B}_{\tilde{\gamma}_t}$.

For any $\{\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, B_t, \bar{B}_t, \tilde{\mathbb{P}}^{x,s}\}$, where B is a Brownian motion under $\tilde{\mathbb{P}}^{x,s}$. Let $N_t = \int_0^t \mathbf{1}_{\{B_u \ge \kappa(\bar{B}_u)\}} du$ and Γ_t be the right continuous inverse of N_t. Then we have $\tilde{B}_t = B_{\Gamma_t}$ and $\tilde{l}^z_t = l^z_{\Gamma_t}$. We define

$$A_t = \lambda(\bar{B}_t, s) + \sum_{J\in\mathcal{J}} \Big\{ \int_{]\kappa(\bar{B}_{l(J)}),\bar{B}_{l(J)}[} (l^z_{t\wedge r(J)} - l^z_{t\wedge l(J)}) m(dz; \bar{B}_{l(J)}) \Big\},$$

where $\mathcal{J}$ is the collection of pairwise disjoint intervals such that

$$\{t \geq 0 \mid B_t < \bar{B}_t\} = \bigcup_{J \in \mathcal{J}} J = \bigcup_{J \in \mathcal{J}}]l(J), r(J)[,$$

γ_t and $\tilde{\gamma}_t$ are the right continuous inverse of A_t and $\tilde{A}_t$ respectively. Then we have that

$$\tilde{A}_t = A_{\Gamma_t} \Leftrightarrow \Gamma_t = \gamma_{A_{\Gamma_t}} = \gamma_{\tilde{A}_t} \Leftrightarrow \Gamma_{\tilde{\gamma}_t} = \gamma_{\tilde{A}_{\tilde{\gamma}_t}} = \gamma_t.$$

From this, it follows that

$$\tilde{B}_{\tilde{\gamma}_t} = B_{\Gamma_{\tilde{\gamma}_t}} = B_{\gamma_t}.$$

∎

Remark 3.4.2. *For the case that the PRM-JSM is regular but does not satisfy (3.34) or (3.35). We can proceed as follows. We consider $\{\mathbb{P}^{x,s}\}$ on E and $\{\chi_n\}$ satisfy the assumptions in Corollary 3.2.5. Let d and $\tilde{\chi}$ be as in Lemma 3.2.8. Notice that*

$$\mathfrak{M}_{\tilde{\chi}_n}(s) = \mathfrak{M}_{\chi_n}(s) \vee m([s - d(s), s[; s)d(s)$$

We are not sure if $\mathfrak{M}_{\tilde{\chi}_n}$ is locally bounded.

What we could do is that we time change to $\tilde{X}_t = X_{C_t}, \tilde{S}_t = S_{C_t}$ and $\tilde{\mathcal{F}}_t = \mathcal{F}_{C_t}$ where

$$C_t = \int_0^t (m([s - d(s), s[; s)d(s) \vee 1)\mathbf{1}_{\{X_t \in [S_t - d(S_t), S_t]\}} + \mathbf{1}_{\{X_t \notin [S_t - d(S_t), S_t]\}} dt$$

is $(\mathcal{F}_t)$-adapted. We can define a new PRM-JSM as in Remark 3.2.3 with the new C_t here. Now one can check that both (3.34) and (3.35) hold.

3.5 The Examples

Example 1. *Let Ω_1 be the family $C(\mathbb{R}_+, \mathcal{I})$ of continuous path ω_1, $\mathcal{G} = \sigma\big(\omega_1(t), t \geq 0\big)$ and*

$$\mathcal{G}_t = \bigcap_{\epsilon > 0} \sigma\big(\omega_1(u), u \leq t + \epsilon\big), \textit{for } t \geq 0.$$

We denote the coordinate process

$$X_t = \omega_1(t) \quad \textit{for all } \omega_1 \in \Omega_1 \textit{ and } t \geq 0.$$

and $\{\theta_t, t \geq 0\}$ the family of shift operators, which are defined by

$$\theta_t(\omega_1)(u) = \omega_1(u + t) \quad \textit{for } \omega_1 \in \Omega_1 \textit{ and } t \geq 0.$$

We first consider the case where $\mathcal{I} = \mathbb{R}$ and $\{\mathbb{P}^x\}$ are Wiener measures on $(\Omega_1, \mathcal{G})$ for $x \in \mathcal{I}$. The map $(X, s \vee (\sup_{0 \leq u \leq \cdot} X_u)) : \Omega_1 \to \Omega$ is $\mathcal{G}/\mathcal{F}$ measurable. We define a family of probability measure $\{\mathbb{P}^{x,s}\}$ on $(\Omega, \mathcal{F})$ by

$$\mathbb{P}^{x,s} = \mathbb{P}^x \circ \big(X, s \vee (\sup_{0 \leq u \leq \cdot} X_u)\big)^{-1}. \tag{3.38}$$

We can now prove that $\{\mathbb{P}^{x,s}\}$ is a canonical strong Markov process with respect to $\bigcup_{s\in\mathcal{I}}\mathcal{I}_s\times\{s\}$. To prove (i) of Definition 3.2.1 , we know for any bounded continuous f, the map

$$(x,s)\mapsto \mathbb{E}^x[f(X_t, s\vee(\sup_{0\le u\le t} X_u))]$$

is jointly measurable with respect to x and continuous with respect to s. Hence, the map is jointly measurable. As the set of bounded continuous functions are dense in L^1, passing to the limit, we have the map for any $A\in\mathcal{E}$,

$$(x,s)\mapsto \mathbb{P}^x((X_t, s\vee(\sup_{0\le u\le t} X_u))\in A)=\mathbb{P}^{x,s}((X_t,S_t)\in A)$$

is measurable. (ii), (iii) and (iv) can be checked directly from the definition of $\{\mathbb{P}^{x,s}\}$. To prove (v), it suffices to show for all bounded measurable function f, $(x,s)\in\{(x,y)\in\mathbb{R}^2 \mid x\le y\}$, $t\in\mathbb{R}_+$ and $\mathcal{F}_t$- stopping time T,

$$\mathbb{E}^{x,s}[f(X_{T+t},S_{T+t})\mid\mathcal{F}_T]=\mathbb{E}^{X_T,S_T}[f(X_t,S_t)] \tag{3.39}$$

Recall $\mathbb{E}^{X_T,S_T}[f(X_t,S_t)]$ is the function $\Psi_f(x,s)=\mathbb{E}^{x,s}[f(X_t,S_t)]$ with (X_T,S_T) inserted in place of (x,s). To this end, it suffices to prove a more general assertion that if $g(x,y,s,z)$ is a bounded measurable function, then

$$\mathbb{E}^{x,s}[g(X_T, X_{T+t}-X_T, S_T, \sup_{0\le u\le t}(X_{T+u}-X_T))\mid\mathcal{F}_T]=\Phi_g(X_T,S_T), \tag{3.40}$$

where

$$\Phi_g(x,s):=\int_0^\infty\int_{-\infty}^z g(x,y,s,z)\frac{2(2z-y)}{\sqrt{2\pi t^3}}\exp(\frac{-2(2z-y)^2}{2t})dydz.$$

The equality (3.39) follows from this if we set $g(x,y,s,z)=f(x+y,s\vee(x+z))$. By monotone class theorem, it suffice to prove (3.40) for g of the special form $g(x,y,s,z)=g_1(x)g_2(y)g_3(s)g_4(z)$. Under this assumption, the left hand side of (3.40) is equal to

$$g_1(X_T)g_3(S_T)\mathbb{E}^{x,s}[g_2(X_{T+t}-X_T)g_4(\sup_{0\le u\le t}(X_{T+u}-X_T))\mid\mathcal{F}_T]. \tag{3.41}$$

Note that $W_t:=X_{T+t}-X_T$ is a Brownian motion independent of $\mathcal{F}_T$ with running maximum $\bar{W}_t=\sup_{0\le u\le t}W_t=\sup_{0\le u\le t}(X_{T+u}-X_T)$. Thus, (3.41) equals to

$$g_1(X_T)g_3(S_T)\int_0^\infty\int_{-\infty}^z g_2(y)g_4(z)\frac{2(2z-y)}{\sqrt{2\pi t^3}}\exp(\frac{-2(2z-y)^2}{2t})dydz=\Phi_g(X_T,S_T)$$

and we are done in this case.

For the general case where $\mathbb{P}^x$ is strong Markov, we know $\mathbb{P}^x=\tilde{\mathbb{P}}^x\circ Z^{-1}$, where $\tilde{\mathbb{P}}^x$ is a standard Wiener measure and Z is a time substitution of X by a right continuous inverse of a PCHAF. We can define a family probability measure $\mathbb{P}^{x,s}=\tilde{\mathbb{P}}^x\circ(Z, s\vee\sup_{0\le u\le\cdot}Z_u)^{-1}$. By the result of III.21 in Roger and Williams [26], we have that $\{\mathbb{P}^{x,s}\}$ is strong Markov.

We let $\mathcal{I} =]-\infty, \infty[$ for Example 2, $\mathcal{I} = [0, \infty[$ for Example 3, 4 and 5, $E = \cup_{s\in\mathcal{I}}]-\infty, s] \times \{s\}$ and $(\Omega_1, \mathcal{G}, \mathcal{G}_t, W_t, \mathbb{P}^x)$ be as in Example 1, with W_t the coordinate mapping and $\mathbb{P}^x$ the Wiener measure on $\mathcal{I}$. Let $(Z_t, \bar{Z}_t)$ be a process on the filtered space $(\Omega_1, \mathcal{G}, \mathcal{G}_t)$ such that $Z_0 = x$ and $\bar{Z}_t = s \vee \sup_{0\le u\le t} Z_t$, $\mathbb{P}^x$ almost surely. For $(\Omega, \mathcal{F}, \mathcal{F}_t, X_t, S_t)$, we can define a family of probability measures $\mathbb{P}^{x,s} = \mathbb{P}^0 \circ (Z, \bar{Z})^{-1}$.

Example 2. *(i) For $(b, b) \in E$, for*

$$Z_t = \begin{cases} b - (b - x)\exp(W_t), & \textit{for } \bar{Z}_t < b, \\ x + W_t, & \textit{for } \bar{Z}_t \ge b. \end{cases}$$

Then $\mathbb{P}^{x,s}$ is strong Markov. Now, for $\chi(u) = u - 1$ for all $u \in \mathcal{I}$ and $x < s < b$, we have

$$\mathbb{P}^{x,s}(T_b < \mathcal{T}_\chi) = 0.$$

(ii) Fix some $(x_\star, s_\star) \in E^\circ$, for

$$Z_t = \begin{cases} x + W_t, & \textit{for } t < T_{x_\star, s_\star}, \\ x_\star, & \textit{for } t \ge T_{x_\star, s_\star}, \end{cases}$$

where $T_{x_\star,s_\star} = \inf\{t \ge 0 \mid (Z_t, \bar{Z}_t) = (x_\star, s_\star)\}$. Then $\mathbb{P}^{x,s}$ is strong Markov. Now, for χ such that $\chi(s_\star) < x_\star$, we have for all $x_\star < x < s_\star$,

$$\mathbb{P}^{x,s_\star}(T_{\chi(s_\star),s_\star} < \infty) = 0.$$

(iii) For $(b, b) \in E$,

$$Z_t = x + t,$$

then $\mathbb{P}^{x,s}$ is strong Markov. Now, for $\chi(u) = u - 1$ for all $u \in \mathcal{I}$ and $x \le s < b$, we have

$$\mathbb{P}^{s,s}(\mathcal{T}_\chi < T_b) = 0.$$

Example 3. *Take $\chi(s) = -5$ for all $s \in \mathbb{R}^+$, for the process with dynamic*

$$\begin{aligned} dZ_t &= (S_t - Z_t)^{\frac{1}{2}} dW_t, \\ Z_0 &= x, \end{aligned}$$

where W is a Brownian motion. One calculate to see that

$$m(]-5, s[; s) = \int_{]-5,s[} \frac{1}{s-u} du = \infty.$$

for all $s \in \mathbb{R}_+$.

Example 4. *Let a, b be constants and $\{s_k\}_{k=0}^\infty$ be a strictly increasing such that $a < s_0$ and $\lim_{k\to\infty} s_k = b$. Consider the process Z with dynamic*

$$\begin{aligned} dZ_t &= \sum_{n=0}^{\infty} \mathbf{1}_{\{\bar{Z}_t \in [s_n, s_{n+1}[\}} \sigma_n dW_t + \mathbf{1}_{\{\bar{Z}_t \in [b,\infty[\}} dW_t, \\ Z_0 &= x, \end{aligned}$$

where $\sigma_n, b > 0$ *for all* $n \geq 0$. *Then we can conclude* $\mathbb{P}^{x,s}$ *is strong Markov. We set* $\chi(s) = a$ *for all* $s \in [s_0, \infty[$. *We denote* $T_s = \inf\{t; \bar{Z}_t \geq s\}$ *and* $\mathcal{T}_\chi = \inf\{t; Z_t = \chi(\bar{Z}_t)\}$. *We will now check that the PRM-JSM is regular. We first check that* χ_1 *is accessible, (i) of Definition 3.2.2 is because*

$$\mathbb{P}^{x,s}(T_{y,y} < T_{\chi_1}) = \frac{x-a}{b-a} > 0 \quad \textit{for all } a \leq x \leq s \leq y,$$

(ii) and (iii) follow directly from the properties of Brownian motion. Hence the regularity holds as $\cup_n E_{\chi_n} = E$. *Now if we set*

$$s_0 = 0, s_{n+1} - s_n = \frac{1}{3^n} \textit{ and } \sigma_n = \frac{1}{2^n} \quad \textit{for all } n \geq 0,$$

we have $b = \frac{3}{2}$ *and*

$$\begin{aligned}
\mathbb{E}^{s_0,s_0}[T_b \wedge \mathcal{T}_\chi] &= \mathbb{E}^{s_0,s_0}[T_{s_1} \wedge \mathcal{T}_\chi] + \frac{s_0 - a}{s_1 - a}\mathbb{E}^{s_1,s_1}[T_b \wedge \mathcal{T}_\chi] \\
&= \mathbb{E}^{s_0,s_0}[T_{s_1} \wedge \mathcal{T}_\chi] + \frac{s_0 - a}{s_1 - a}\Big[\mathbb{E}^{s_1,s_1}[T_{s_2} \wedge \mathcal{T}_\chi] + \frac{s_1 - a}{s_2 - a}\mathbb{E}^{s_1,s_1}[T_b \wedge \mathcal{T}_\chi]\Big] \\
&\geq \sum_{n=0}^{\infty} \frac{s_0 - a}{s_n - a}\mathbb{E}^{s_n,s_n}[T_{s_{n+1},s_{n+1}} \wedge \mathcal{T}_\chi] \\
&= \sum_{n=0}^{\infty} \frac{(s_0 - a)(s_{n+1} - s_n)}{\sigma_n^2},
\end{aligned}$$

which diverges by our choice.

Example 5. *Let* a, b, c *be constants,* $\{a_n\}_{n=0}^\infty$ *and* $\{s_n\}_{n=0}^\infty$ *be a strictly increasing such that* $a \leq a_n < s_n < b < c$ *and* $\lim_{k\to\infty} s_k = b$. *Set*

$$\chi(s) = \sum_{n=0}^{\infty} \mathbf{1}_{\{s\in[s_n,s_{n+1}[\}} a_n + \mathbf{1}_{\{s\in[b,\infty[\}} a \quad \textit{for all } s \in \mathcal{I}$$

Consider the process Z *defined by*

$$\begin{aligned}
dZ_t &= \sum_{n=0}^{\infty} \mathbf{1}_{\{\bar{Z}_t\in[s_n,s_{n+1}[\}} \sigma_n dW_t + \mathbf{1}_{\{\bar{Z}_t\in[b,\infty[\}} dW_t, \\
Z_0 &= x.
\end{aligned}$$

If we set $a = -1$

$$s_0 = 0,\ \Delta_n := s_{n+1} - s_n = \frac{1}{2^{2n}(2^n - 1)},\ \sigma_n = \frac{1}{2^n} \textit{ and } a_n = s_n - \sigma_n^2 \quad \textit{for all } n \geq 0,$$

we have

$$m([\chi_(s), s[; s) = 2 \tag{3.42}$$

for $s \in [s_0, b[$. *Notice then,*

$$\begin{aligned}\mathbb{P}^{s_0,s_0}(T_b < T_\chi) &= \prod_{n=0}^{\infty} \mathbb{P}^{s_n,s_n}(T_{s_{n+1},s_{n+1}} < T_\chi) \\ &= \prod_{n=0}^{\infty} \frac{s_n - a_n}{s_{n+1} - a_n} \\ &= \prod_{n=0}^{\infty} \frac{\sigma_n^2}{\sigma_n^2 + \Delta_n} \\ &= \prod_{n=0}^{\infty} (1 - \frac{1}{2^{n+1}}) \\ &:= r > 0.\end{aligned}$$

We can use this and the properties of Brownian motions to prove that χ *is accessible. Notice that by Lemma 3.2.6, we have for* $\mathcal{D} = E_{s_0,c,\chi}$, $F_{\mathcal{D}}(x,s) = \mathbb{E}^{x,s}[T_c \wedge T_\chi] < \infty$. *We can choose* c *large enough such that*

$$\mathbb{E}^{b,b}[T_c \wedge T_\chi] = (c-b)(b-a) \geq \frac{1}{r}.$$

Then

$$\begin{aligned}F_{\mathcal{D}}(s_n, s_n) &= \mathbb{E}^{s_n,s_n}[T_b \wedge T_\chi] + \mathbb{P}^{s_n,s_n}(T_b < T_\chi)\mathbb{E}^{b,b}[T_c \wedge T_\chi] \\ &\geq \mathbb{P}^{s_0,s_0}(T_b < T_\chi)\mathbb{E}^{b,b}[T_c \wedge T_\chi] \\ &\geq 1.\end{aligned} \tag{3.43}$$

Equation (3.42) and (3.43) and the fact that $a_n = s_n - \sigma_n^2$ *imply*

$$(F_{\mathcal{D}})_{x-}(s_n, s_n) = \frac{-2\int m([u, s_n[; s_n)du}{s_n - a_n} + \frac{F_{\mathcal{D}}(s_n, s_n)}{s_n - a_n} \geq -4 + \frac{1}{\sigma_n^2} \uparrow \infty$$

and $F_{x-}(s,s)$ *is not locally bounded.*

3.6 The r-invariant functions

In this section, we will introduce the notion of an r-invariant function for PRM-JSM. The term *invariant* follows from chapter 2 of Borodin and Salminen [3].

Definition 3.6.1. *We say a nonnegative function* ϕ *on* E *is a left* r*-invariant function if for any accessible function* χ *such that* $\mathfrak{G}_\chi : \mathcal{I} \to \mathbb{R}^+$ *is locally bounded, we have*

$$\phi(x,s) = \mathbb{E}^{x,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi < \infty\}}] \quad \textit{for all} \quad (x,s) \in E_\chi. \tag{3.44}$$

We say a nonnegative function ψ *on* E *is a right* r*-invariant function if*

$$\psi(x,s) = \psi(z,z)\mathbb{E}^{x,s}[e^{-rT_z}\mathbf{1}_{\{T_z < \infty\}}] \quad \textit{for all} \quad (x,s) \in E \textit{ and } z \textit{ such that } x \leq s \leq z.$$

We will now prove the existence of r-invariant function. For any continuous function $g : \mathcal{I} \to \mathbb{R}_+$, we define

$$\nu^r_{g,\chi}(s, s_1, s_2) := g(s_1)\mathbb{E}^{s,s}[e^{-rT_{s_1}}\mathbf{1}_{\{T_{s_1}<\mathcal{T}_\chi\}}] - g(s_2)\mathbb{E}^{s,s}[e^{-rT_{s_2}}\mathbf{1}_{\{T_{s_2}<\mathcal{T}_\chi\}}] \tag{3.45}$$

for all $\alpha \le s \le s_1 \le s_2 < \beta$.

Definition 3.6.2. *We call a continuous function $g : \mathcal{I} \to \mathbb{R}^+$ is invariant-generating if for every accessible χ such that $\mathfrak{G}_\chi$ is locally bounded,*

(i) $\lim_{z\to\infty} g(z)\mathbb{E}^{s,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}}] = 0$ *for all* $s \in \mathcal{I}$,

(ii) there exists an integrable function $f_{g,\chi} : \mathcal{I} \to \mathbb{R}$ for all $s \in \mathcal{I}$ such that

$$\nu^r_{g,\chi}(s, s_1, s_2) = \int_{]s_1,s_2[} f_{g,\chi}(u)\mu^r_\chi(du; s) > 0 \quad \text{for all } s \le s_1 < s_2 \in \mathcal{I}. \tag{3.46}$$

Lemma 3.6.1. *Let f be any measurable integrable function on $\chi \cup \{b\}$ and $\mathcal{T}$ be $\mathcal{T}_\chi$, T_z or $T_z \wedge \mathcal{T}_\chi$. Consider the function defined as*

$$u(x, s) := \mathbb{E}^{x,s}[e^{-r\mathcal{T}} f(X_\mathcal{T}, S_\mathcal{T})\mathbf{1}_{\{\mathcal{T}<\infty\}}]$$

for $(x, s) \in E_\chi$. We have $e^{-r(t\wedge\mathcal{T})}u(X_{t\wedge\mathcal{T}}, S_{t\wedge\mathcal{T}})$ is a martingale.

Proof. For any bounded stopping time H, by definition of u and strong Markov property, we have

$$\begin{aligned} u(x, s) &= \mathbb{E}^{x,s}[e^{-r\mathcal{T}} f(X_\mathcal{T}, S_\mathcal{T})\mathbf{1}_{\{\mathcal{T}<\infty\}}] \\ &= \mathbb{E}^{x,s}[e^{-r(H\wedge\mathcal{T})}e^{-r\mathcal{T}\circ\theta_{H\wedge\mathcal{T}}} f(X_{\mathcal{T}\circ\theta_{H\wedge\mathcal{T}}}, S_{\mathcal{T}\circ\theta_{H\wedge\mathcal{T}}})\mathbf{1}_{\{\mathcal{T}\circ\theta_{H\wedge\mathcal{T}}<\infty\}}] \\ &= \mathbb{E}^{x,s}\left[e^{-r(H\wedge\mathcal{T})}\mathbb{E}^{X_{H\wedge\mathcal{T}},S_{H\wedge\mathcal{T}}}[e^{-r\mathcal{T}} f(X_\mathcal{T}, S_\mathcal{T})\mathbf{1}_{\{\mathcal{T}<\infty\}}]\right] \\ &= \mathbb{E}^{x,s}[e^{-r(H\wedge\mathcal{T})}u(X_{H\wedge\mathcal{T}}, S_{H\wedge\mathcal{T}})]. \end{aligned}$$

Hence the result. ∎

Corollary 3.6.2. *Both $e^{-r(t\wedge\mathcal{T})}\phi(X_{t\wedge\mathcal{T}}, S_{t\wedge\mathcal{T}})$ and $e^{-r(t\wedge\mathcal{T})}\psi(X_{t\wedge\mathcal{T}}, S_{t\wedge\mathcal{T}})$ are a local martingale, where ϕ and ψ are left and right r-invariant functions repesectively.*

Theorem 3.6.3. *Consider any strong Markov process, the following statements hold true:*

(i) If ϕ is a left r-invariant generating function. Then, the function $\mathcal{I} \ni s \mapsto \phi(s, s)$ is invariant generating.

(ii) Given any invariant generating function $g : \mathcal{I} \to \mathbb{R}^+$, there exists a left r-invariant function such that $\phi(s, s) = g(s)$ for all $s \in \mathcal{I}$.

Proof. Let χ be accessible such that $\mathfrak{G}_\chi$ is locally bounded. By Lemma 3.2.6 we have $\mathbb{P}^{s,s}(T_z \wedge \mathcal{T}_\chi) = 1$ for all $s \le z \in \mathcal{I}$. By this result, strong Markov property and definition of ϕ, we have

$$\begin{aligned}\phi(s,s) &= \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}]\\ &= \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}(\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}} + \mathbf{1}_{\{\mathcal{T}_\chi<T_z\}})]\\ &= \mathbb{E}^{s,s}[e^{-rT_z}e^{-r\mathcal{T}_\chi\circ\theta_{T_z}}\phi(X_{\mathcal{T}_\chi\circ\theta_{T_z}}, S_{\mathcal{T}_\chi\circ\theta_{T_z}})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}}]\\ &+ \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<T_z\}}]\\ &= \phi(z,z)\mathbb{E}^{s,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}}] + \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<T_z\}}] \qquad (3.47)\end{aligned}$$

for all $s \le z \in \mathcal{I}$. Continuity of the map $\mathcal{I} \ni z \mapsto \phi(z,z)$ follows from (3.47), the fact that $z \mapsto \mathbb{E}^{s,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}}] > 0$ and $z \mapsto \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<T_z\}}]$ are continuous as a function of $z \in [s,\beta[$. By (3.47) and the dominated convergence theorem, we have

$$\lim_{z\to\infty}\phi(z,z)\mathbb{E}^{x,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}}] = 0.$$

Again by (3.47), we have

$$\begin{aligned}&\mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}\phi(X_{\mathcal{T}_\chi}, S_{\mathcal{T}_\chi})\mathbf{1}_{\{T_{s_1}<\mathcal{T}_\chi<T_{s_2}\}}]\\ =&\phi(s_1,s_1)\mathbb{E}^{s,s}[e^{-rT_{s_1}}\mathbf{1}_{\{T_{s_1}<\mathcal{T}_\chi\}}] - \phi(s_2,s_2)\mathbb{E}^{s,s}[e^{-rT_{s_2}}\mathbf{1}_{\{T_{s_2}<\mathcal{T}_\chi\}}] \qquad (3.48)\end{aligned}$$

for all $s \le s_1 \le s_2 \in \mathcal{I}$. Hence it suffice to take $f_{g,\chi} = \phi(\chi(s), s)$.

Conversely, fix a choice of χ and a version of $f_{g,\chi}$, we can define

$$\phi(x,s) = \mathbb{E}^{x,s}[e^{-r\mathcal{T}_\chi}f_{g,\chi}(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] \quad \text{for all } (x,s) \in E_\chi.$$

By (3.45) and (3.46), let $s = s_1$ and $s_2 \to \infty$ in (3.46), we have

$$\phi(s,s) = \mathbb{E}^{s,s}[e^{-r\mathcal{T}_\chi}f_{g,\chi}(S_{\mathcal{T}_\chi})\mathbf{1}_{\{\mathcal{T}_\chi<\infty\}}] = g(s) \quad \text{for all } (x,s) \in E_\chi.$$

By Lemma 3.6.1, we can see that $e^{-r(t\wedge\mathcal{T}_\chi)}\phi(X_{t\wedge\mathcal{T}_\chi}, X_{t\wedge\mathcal{T}_\chi})$ is a local martingale on E_χ. Let $\{z_n\}_{n=0}^\infty$ be a sequence of decreasing functions on $\mathcal{I}$ such that $z_0(s) = \chi(s)$ and $\lim_{n\to\infty} z_n(s) = \kappa(s)$ for all $s \in \mathcal{I}$. We can define function ϕ_n on E_{z_n} by

$$\begin{aligned}&\phi_n(s,s) = \phi(s,s) = g(s),\\ &\phi_n(z_n(s),s) = \frac{1}{\mathbb{E}^{\chi(s),s}[e^{-rT_{z_n(s),s}}\mathbf{1}_{\{T_{z_n(s),s}<T_s\}}]}\left[f_{g,\chi}(s) - g(s)\mathbb{E}^{\chi(s),s}[e^{-rT_s}\mathbf{1}_{\{T_s<T_{z_n(s),s}\}}]\right]\end{aligned}$$

and

$$\phi_n(x,s) = \mathbb{E}^{x,s}[e^{-rT_{z_n(s),s}}\mathbf{1}_{\{T_{z_n(s),s}<T_s\}}\phi_n(z_n(s),s) + e^{-rT_s}\mathbf{1}_{\{T_s<T_{z_n(s),s}\}}g(s)].$$

By Lemma 3.6.1, one can check that $e^{-r(t\wedge T_s\wedge T_{z_n})}\phi_n(X_{t\wedge T_s\wedge T_{z_n}}, X_{t\wedge T_s\wedge T_{z_n}})$ is a uniformly integrable martingale for all $n \ge 0$. Combine this with the fact that for all $m \le n$,

$$\phi_m(z_m(s),s) = \phi_n(z_m(s),s) \text{ and } \phi_m(s,s) = \phi_n(s,s) = g(s), \quad \text{for all } s \in \mathcal{I},$$

we can conclude that $\phi_m = \phi_n$ on E_{z_m}. Therefore, one can extend ϕ to E by defining $\phi = \phi_n$ on E_{z_n}. Let us define a sequence of stopping times,

$$\begin{aligned} R_0 &= \inf\{t \geq 0 \mid X_t = S_t\} \wedge \mathcal{T}_\kappa; \\ R_{2n-1} &= \inf\{t \geq R_{2n-2} \mid X_t = \chi(S_t)\} \quad \text{for } n \geq 1; \\ R_{2n} &= \inf\{t \geq R_{2n-1} \mid X_t = S_t\} \wedge \mathcal{T}_\kappa \quad \text{for } n \geq 1. \end{aligned}$$

Upon localising with this sequence of stopping times, we can check that $e^{-rt}\phi(X_{t\wedge\mathcal{T}_\kappa}, S_{t\wedge\mathcal{T}_\kappa})$ is a local martingale. Then

$$\phi(x,s) = \mathbb{E}^{x,s}\left[e^{-r(T_z\wedge\mathcal{T}_\chi)}\phi(X_{T_z\wedge\mathcal{T}_\chi}, S_{T_z\wedge\mathcal{T}_\chi})(\mathbf{1}_{\{T_z<\mathcal{T}_\chi\}} + \mathbf{1}_{\{\mathcal{T}_\chi<T_z\}})\right]$$

holds for all χ such that $\mathfrak{G}_\chi$ is locally bounded. Let $z \to \infty$, by dominated convergence theorem we obtain (3.44). ∎

Lemma 3.6.4. *Both $\phi(\cdot, s)$ and $\psi(\cdot, s)$ are convex on $]\kappa(s), s[$, where ϕ and ψ are left and right r-invariant functions repesectively.*

Proof. For $x \in \mathcal{I}_s$, we choose a and b such that $\kappa(s) < a < x < b \leq s$. Then we have

$$\begin{aligned} \phi(x,s) &= \mathbb{E}^{x,s}[e^{-r(T_{a,s}\wedge T_{b,s})}e^{-r\mathcal{T}_\chi\circ\theta_{T_{a,s}\wedge T_{b,s}}}\phi(X_{\mathcal{T}_\chi\circ\theta_{T_{a,s}\wedge T_{b,s}}}, S_{\mathcal{T}_\chi\circ\theta_{T_{a,s}\wedge T_{b,s}}})] \\ &= \mathbb{E}^{x,s}[e^{-r(T_{a,s}\wedge T_{b,s})}(\mathbf{1}_{\{T_{a,s}<T_{b,s}\}}\phi(a,s) + \mathbf{1}_{\{T_{b,s}<T_{a,s}\}}\phi(b,s))] \\ &\leq \frac{b-x}{b-a}\phi(a,s) + \frac{x-a}{b-a}\phi(b,s) \end{aligned}$$

and the convexity follows for $x \in]\kappa(s), s[$.

For $a < x < b \leq s \leq z$, we can repeat the above argument with $\mathcal{T}_\chi$ replaced by T_z and ϕ replaced by ψ. Then we obtain the convexity of $\psi(\cdot, s)$ on $]\kappa(s), s[$. ∎

Lemma 3.6.5. *The functions $I \ni z \mapsto \phi(z,z)$ and $I \ni z \mapsto \psi(z,z)$ are continuous.*

Proof. The continuity of $z \mapsto \phi(z,z)$ was proved in Theorem 3.6.3. The continuity of $z \mapsto \psi(z,z)$ follows directly from

$$\psi(s,s) = \psi(z,z)\mathbb{E}^{s,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\infty\}}]$$

and $\mathbb{E}^{s,s}[e^{-rT_z}\mathbf{1}_{\{T_z<\infty\}}]$ is a positive continuous function of z. ∎

Theorem 3.6.6. *Suppose the left r-invariant function ϕ is such that $\phi(\chi(s), s)$ is locally bounded for some accessible χ with $s - \chi(s)$ locally bounded away from 0 and $\mathfrak{M}_\chi(s)$ also locally bounded. We have that ϕ solves the differential equation:*

$$\phi_{xx}(dx,s) - r\phi(x,s)m(dx,s) = 0, \tag{3.49}$$

$$d\phi(s,s) - \phi_{x-}(s,s)ds - r\phi(s,s)d\lambda(s,s_0) = 0 \quad \textit{for all } s \geq s_0. \tag{3.50}$$

Proof. Fix s. For any $\kappa(s) < a < x < b < s$, the process

$$\left(e^{-r(t\wedge T_{a,s}\wedge T_{b,s})}\phi(X_{t\wedge T_{a,s}\wedge T_{b,s}}, S_{t\wedge T_{a,s}\wedge T_{b,s}}),\ t \geq 0\right)$$

is a uniformly integrable martingale as in one dimensional case. Apply Itô-Tanaka's formula and equate the finite variation part, we obtain (3.49).

One can see that our assumptions on ϕ enables us to apply Theorem 3.4.2 with ϕ in place of F. Integration by parts implies that

$$\begin{aligned}e^{-r(t\wedge\mathcal{T}_\kappa)}\phi(X_{t\wedge\mathcal{T}_\kappa},S_{t\wedge\mathcal{T}_\kappa}) =&\phi(X_0,S_0)+\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\phi_{x-}(X_u,S_u)dX_u\\&+\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\big(d\phi(S_u,S_u)-\phi_{x-}(S_u,S_u)dS_u\big)\\&+\int_0^{t\wedge\mathcal{T}_\kappa}\frac{1}{2}\sum_{J\in\mathcal{J}}e^{-ru}\phi_{xx}(dx,S_{l(J)})(dL^x_{u\wedge r(J)}-dL^x_{u\wedge l(J)})\\&-\int_0^{t\wedge\mathcal{T}_\kappa}re^{-ru}\phi(X_u,S_u)du,\end{aligned}\tag{3.51}$$

where L^x_t is the local time for X.

Recall the PCHAF A, $(B,\bar{B})$ and l are defined in Theorem 3.4.3, we can see that

$$\begin{aligned}\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\phi(X_u,S_u)du&=\int_0^{\gamma_{t\wedge\mathcal{T}_\kappa}}e^{-rA_u}\phi(B_u,\bar{B}_u)dA_u\\&=\int_0^{\gamma_{t\wedge\mathcal{T}_\kappa}}e^{-rA_u}\phi(B_u,\bar{B}_u)\frac{1}{2}\sum_{J\in\mathcal{J}}m(dx,\bar{B}_{l(J)})(dl^x_{u\wedge r(J)}-dl^x_{u\wedge l(J)})\\&\quad+\int_0^{\gamma_{t\wedge\mathcal{T}_\kappa}}e^{-rA_u}\phi(\bar{B}_u,\bar{B}_u)d\lambda(\bar{B}_u,s_0)\\&=\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\phi(X_u,S_u)\frac{1}{2}\sum_{J\in\mathcal{J}}m(dx,S_{l(J)})(dL^x_{u\wedge r(J)}-dL^x_{u\wedge l(J)})\\&\quad+\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\phi(S_u,S_u)d\lambda(S_u,s_0).\end{aligned}$$

Plug this back into (3.51), use Doobs Meyer decomposition theorem and the fact (3.49), we can see that

$$\int_0^{t\wedge\mathcal{T}_\kappa}e^{-ru}\big(d\phi(S_u,S_u)-\phi_{x-}(S_u,S_u)dS_u-r\phi(S_u,S_u)d\lambda(S_u,s_0)\big)=0.$$

This further implies that

$$\phi(S_{t\wedge\mathcal{T}_\kappa},S_{t\wedge\mathcal{T}_\kappa})-\phi(s_0,s_0)-\int_0^{t\wedge\mathcal{T}_\kappa}\phi_{x-}(S_u,S_u)dS_u-\int_0^{t\wedge\mathcal{T}_\kappa}r\phi(S_u,S_u)d\lambda(S_u,s_0)=0.$$

We replace t by T_s, multiply both sides by $\mathbf{1}_{\{T_s<\mathcal{T}_\chi\}}$ and take expectations, it follows that (3.50) holds. ∎

Remark 3.6.1. *Itô We take $\chi=\chi_1$ in Theorem 3.4.3 and prescribe $\phi(\chi(s),s)=1$. Then the left r-invariant function ϕ defined via the same procedure as in Theorem 3.6.3 satisfies the assumption of this theorem.*

Remark 3.6.2. *The right r-invariant function ψ satisfies the same differential equations following the same line of proof replacing ϕ by ψ.*

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