London School of Economics and Political Science

ESSAYS ON CORPORATE FINANCE UNDER ASYMMETRIC INFORMATION

Yue Yuan

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Declaration

I certify that the thesis I have presented for examination for the Ph.D. degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I confirm that Chapter 3 is jointly co-authored with Philip Bond and Hongda Zhong and I contributed 33% of this work.

I declare that the thesis consists of 25,458 words excluding appendices.

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Abstract

This thesis contains three essays on corporate financial transactions when there is asymmetric information between firms and investors.

In the first essay, "Security Design under Common-Value Competition", I show common-value competition drives informed investors to propose debt financing, because debt protects them against the winner's curse. The information sensitiveness of investors' securities is positively related to their market power. Among different aspects of market power, monopolistic power is the primary determinant.

In the second essay, "Security Design with Two-Sided Asymmetric Information", I study a model in which a firm organizes a security-bid auction when both the firm and investors have private information. In equilibrium, the firm cannot credibly reveal its private valuation by security design, and requires payments in the most information-sensitive security family regardless of its valuation.

In the third essay, "Share Issues versus Share Repurchases", Philip Bond, Hongda Zhong and I study firms' share issues and repurchases in a unified framework with the informational friction that firms have superior information to investors. We find asymmetric outcomes of issue and repurchase methods: firms separate on different issue methods, but pool on the most efficient repurchase method. Moreover, firms use more efficient issue methods when raising larger amounts of capital. Both results are consistent with empirical evidences in the literature.

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Chapter 1

Security Design under Common-Value Competition

1.1 Introduction

One of the most influential explanations for the uniquitous use of debt financing is the signalling model of Myers and Majluf (1984) followed by Nachman and Noe (1994), DeMarzo and Duffie (1999) and others. In these models, firms with superior information issue the least information sensitive security, debt, to signal high firm value and minimize underprcing. Yet financial markets for private firms, which account for the majority of corporates, feature three substantial departures from these models. First, private firms are usually financed by professional investors who have developed proprietary expertise in project evaluation and are able to gain better insight into a firm's prospect than the entrepreneur. Second, securities for financing private firms are usually proposed by the informed investors rather than by the entrepreneurs. Third, it is natural for an investor designing a security to consider competition from other investors with almost identical resources – cash; while in the standard models, the security designer is the monopolistic owner of the investment opportunity – the firm.

In the only existing paper that studies such a situation, DeMarzo et al. (2005) predict the opposite of debt financing: an informed investor offers to finance a firm in exchange for the most information sensitive security – levered equity or a complete buyout, to signal high prospect of the firm under their investment and minimize undervaluation of their offer.[1] However, virtually all private firms have their outside capital raised through securities less information sensitive than even straight equity. Debt, convertible preferred equity or at most straight equity is the predominant solution.[2] I therefore argue the existing theories are incomplete and sometimes counterfactual.

This paper complements this literature by showing security design is sensitive to the structure of competition. DeMarzo et al. (2005) focus on competition in a private-value environment, in which an investor's private information coincides with her own value added to the firm.[3] This paper studies competition in a common-value environment, in which each investor has a private estimation of the firm's "intrinsic" value which is independent of who finances it. The intrinsic or common value may represent the firm's assets in place or the entrpereneur's skills. With common values, the model prediction is restored to debt financing. Since uncertainty in common values is a primary concern in many situations in the financial market, I argue this model reconciles the existing theories with the empirical evidence under a reasonable change of the assumption.

The model is set up as a security auction. A penniless entrepreneur has a project to start, which requires a fixed amount of investment. A finite number of investors compete for the opportunity to invest. The random value of the project

[1]In their paper, this result is presented as "an investor offers to pay debt or cash to the entrepreneur", which is equivalent to levered-equity financing or a buyout.

[2]Kaplan and Strömberg (2003) document 79.8% of venture capital investment contracts use convertible preferred equity, and the rest use combinations of convertible debt, preferred equity, convertible preferred equity and straight equity. In the 2004-2006 Kauffman Firm Survey of Entrepreneurs, among the 1710 reported incidences of outside financing, 93% are through debt and the remaining are through equity (Robb and Robinson, 2014). In the 2016 Annual Survey of Entrepreneurs conducted by the US Census Bureau, 98% incidences of outside financing are through debt (Hwang et al., 2019).

[3]DeMarzo et al. extend their analysis on "formal auctions" in which the entrepreneur pre-commits to a security design to common values, but restrict their study on "informal auctions" in which investors freely design the security to private values.

is common to all investors, about which each investor observes a private signal. Each investor offers to invest in exchange for a security that she freely designs subject to monotonicity and limited liability. The entrepreneur receives the offers in sealed bids and accepts the most attractive one. This is a signalling game because the entrepreneur may infer investors' private information from their offers when evaluating the offers. Moreover, the structure of the game implies the entrepreneur's evaluation of an offer and this investor's gain conditional on winning are both affected by other investors' private information, which is not a feature of the private-value model and makes the game difficult to solve. For this reason, this paper can show the existence or non-existence of only part of all possible equilibria.

With two investors, there is an equilibrium in which both investors offer debt. This equilibrium survives the D1 refinement of Cho and Kreps (1987), which implies it is supported by "reasonable" beliefs. There is no equilibrium in which one investor makes her offer from an ordered set, such as debt with different face values, and the other investor makes her offer from another ordered set that is more information sensitive, such as equity with different shares. With an arbitrary number of investors, there is no symmetric equilibrium in which offers are from an ordered set more information sensitive than a threshold.

The intuition is as follows. As in standard auctions, an investor is more likely to win when other investors have lower valuations of the firm. With common values, this exposes an investor to adverse selection from her competitors, which is known as the winner's curse. This brings a two-fold advantage to offering an information insensitive security. First, holding the set of scenarios in which an investor wins constant, if she lowers the sensitiveness of her offered security to decreases in other investors' private signals, she can reduce the effect of the winner's curse and increase her payoff conditional on winning. Second, if an investor's security is much less information sensitive than the others' bids, she may win when other investors have higher signals instead of lower signals, which reverses the winner's curse into the winner's blessing. This is because in a fully revealing equilibrium, the entrepreneur can infer investors' private signals and become better informed than any single investor. Holding an investor's signal constant, when the other investors have high signals such that the entrepreneur believes the project is valuable, the least information sensitive security among the offers may be considered the cheapest source of funding and chosen by the entrepreneur.

In comparison, there is no winner's curse in the private-value model of DeMarzo et al. (2005). The primary force that drives their result is signalling incentives. To make one's offer attractive, an investor signals high private value by volunteering to hold the most information sensitive security, levered equity. With common values, on the other hand, one's signalling incentive is not as straightforward and depends on the information sensitiveness of the security she plans to offer relative to others' offers. If an investor's offer is more sensitive than the others', her offer is a cheaper source of financing and more attractive to the entrepreneur when the entrepreneur believes in a low valuation. Therefore, the investor wants to signal pessimism. Similarly, if one's offer is less sensitive than the others', she wants to signal optimism. However, in both cases, the investor is doomed to send the undesired signal: the very choice to hold an information sensitive security signals optimism, and the choice to hold an information insensitive security signals pessimism. Therefore, the signalling incentive has no clear implication on the equilibrium security design other than deterring deviation from a given equilibrium strategy to both more information sensitive securities and less information sensitive securities.

The results have implications on the relation between the entrepreneur's revenue and the entrepreneur's commitment power. Auction theory has established that seller revenue increases if the rule of the auction requires payments to be contingent on other bidders' information or post-auction information (Milgrom and Weber, 1982; Hansen, 1985; Crémer, 1987; Samuelson, 1987; Riley, 1988; Rhodes-Kropf and Viswanathan, 2000; DeMarzo et al., 2005; Axelson, 2007). In particular, with affiliated values, which nests common values and private values as special cases, DeMarzo et al. show if the entrepreneur can precommit to only considering financing through securities from an ordered set, such as debt or equity, the entrepreneur's revenue decreases in the information sensitiveness of the precommitted ordered set (that is, increases in the information sensitiveness of the entrepreneur's payoff), and is highest when she is precommitted to debt financing.[4] Combined with the above, the result of this paper implies with common values, the entrepreneur without precommitment expects the same revenue as making the best precommitment. In contrast, with private values, DeMarzo et al. show investors offer levered equity financing or buyouts when they freely design securities, which implies the entrepreneur without precommitment receives the lowest revenue among auctions with precommitment, and thus benefits from committing to debt financing.

The results also shed light on the roles of different aspects of market power in security design. In a model of security design under information asymmetry, the relative market power between entrepreneurs and investors can be summarised in three aspects: the competition structure, informational advantage and the right to move first to design the security. A comparison of models with different assumptions in these aspects from this paper and the existing literature indicates the party with the larger market power receives the more information sensitive security. In particular, two patterns hold, which imply an order of importance among different aspects of market power. First, competition is the primary determinant of security design among the three aspects of market power. As a proxy for monopolistic power, I call an agent "indispensable" if the same project payoff cannot be realised without the participation of this agent. If agents in one party are indispensible while those in the other are not, the indispensable party has the larger monopolistic power and receives the more information sensitive security. For example, in the model of this paper, even though the investors have private information and design the securities, the entrepreneur is the sole indispensable party, and receives levered equity.[5] That competition dominates the

[4]DeMarzo et al. present this result in their paper as the entrepreneur's revenue is the highest when the entrepreneur precommits to receiving call options in payment.

[5]Other examples in which the entrepreneur is the only indispensable party and receives

other two aspects of market power is intuitive in this model: even though an investor is privately informed, she does not have access to her competitors' private information, which constitutes her informational disadvantage. Moreover, due to competition, all investors' private information is revealed to the entrepreneur through their offers, which further strengthens an investor's informational disadvantage. When designing the security, an investor is majorly concerned about minimising the adverse effect of her informational disadvantage, and therefore designs an information insensitive security.

The second pattern is, when both parties are indispensable, the party which moves first to design the security receives the more information sensitive security. For example, both parties are indispensable in the private-value model of DeMarzo et al. (2005). If the entrepreneur can precommit to a security design, she commits to debt financing and receives levered equity; if investors freely design the securities, they receive levered equity or the whole project. Intuitively, when the party with superior information designs the security, she retain an information sensitive security to signal high project value to encourage acceptance; when the party with inferior infromation designs the security, she retains the information sensitive security to minimise the counterparty's informational rent.

The paper is organised as follows. Section 1.2 sets up the security auction. Following DeMarzo et al. (2005), I call an auction in which the entrepreneur precommits to an ordered set of securities a "formal auction", and an auction in which investors freely design securities an "informal auction". Section 1.3 analyses a formal auction in preparation for the analysis of the informal auction. Section 1.4 characterises equilibria of the informal auction. Section 1.5 discusses the relation between different aspects of market power and security design. Section 1.6 concludes. All proofs are relegated to Appendix 1.7.2 and 1.7.3.

levered equity include the models of Myers and Majluf (1984), Axelson (2007), Burkart and Lee (2016) and Yang (2020), as well as the formal auction with common values of DeMarzo et al. (2005).

1.1.1 Related Literature

This paper contributes to the literature on security design under information asymmetry. While the first models in this literature consider adverse selection problems of an informed security issuer (Myers and Majluf, 1984; Nachman and Noe, 1994; DeMarzo and Duffie, 1999; Biais and Mariotti, 2005; DeMarzo, 2005; Vanasco, 2017), this paper is more closely related to those papers on security design by informed investors. As mentioned earlier, it is closest to DeMarzo et al. (2005), who study a similar setting but with private values. In his review on security auctions, Skrzypacz (2013) discusses the effects of the seller's selection and the winner's curse on security design in common-value environments. For each of the two effects, he provides an example in which the effect drives bidders to pay with securities rather than cash. Garmaise (2007) studies a common-value auction in which investors design securities after the entrepreneur announces and commits to a ranking of securities. Based on their private information, investors offer securities that are overvalued in the announced ranking. Fishman (1989) and Inderst and Mueller (2006) consider competition between an informed investor and uninformed investors. Fishman considers an informed investor who trades-off the advantage of offering a buyout to deter competition from uninformed investors, and the advantage of investing in levered equity which induces the entrepreneur to make an efficient accept/reject decision. In the model of Inderst and Mueller, the informed investor designs a security which to the largest extent commits herself to an efficient investment decision after screening the project. Burkart and Lee (2016) study a bilateral trade in which the informed investor designs an information insensitive security to signal low firm value and purchase the security at a low price.[6] Vladimirov (2015) considers securities for financing privately informed bidders in takeover contests, and shows financiers receive lev-

[6]Besides explicitly modelling competition, this paper differs from the model of Burkart and Lee in the content of private information. Burkart and Lee assume the investor has private information on the firm's assets in place, while this paper assumes the investors have private information on the project to start. As a result, in the former case, the investor offers an information insensitive security to signal low outside option of the entrepreneur, while in the bilateral version of this paper as analysed in Section 1.5, the investor offers an information sensitive security to signal high inside option of the entrepreneur.

ered equity if they have larger bargaining power than bidders, and receive debt otherwise. Inderst and Vladimirov (2019) consider the interaction between security design and holdup problems in later rounds of financing when the investor gains an informational advantage over outside investors.

Another strand of this literature studies issuers' security design when investors are privately informed. Hansen (1985), Crémer (1987), Rhodes-Kropf and Viswanathan (2000), DeMarzo et al. (2005), Axelson (2007), Abhishek et al. (2015) and Sogo et al. (2016) show that the issuer's optimal design of a security auction is an auction in which investors receive information insensitive securities. Gorbenko and Malenko (2011) show when issuers design security auctions under competition, they commit to more information sensitive securities when the competition intensifies. Liu and Bernhardt (ming) show a takeover target optimally commits to a menu of securities with different information sensitiveness levels to solve an adverse selection problem from acquirers privately informed about synergies and standalone values. Che and Kim (2010) demonstrate when investment costs are positively related to investors' private values, security auctions suffer from an adverse selection problem, which is more severe when investors are restricted to receiving less information sensitive securities. In takeover auctions with heterogeneous acquirers, Liu (2016) and Liu and Bernhardt (2021) show an auction restricted to equity payments may generate lower revenue for the target than a buyout auction, and provide the optimal mechanism of equity auctions and a close-to-optimal implementation, which restore the revenue-superiority of information insensitive securities. Dang et al. (2015), Yang and Zeng (2019) and Yang (2020) consider situations in which the investor can acquire information at a cost to facilitate the investment decision, and show the issuer designs an information insensitive security to discourage the investor from information acquisition when information acquisition is inefficient, and designs an information sensitive security when information is socially valuable.

1.2 Model Setup

Consider a penniless entrepreneur with a project which requires fixed initial investment $I > 0$ and yields random future payoff Z. The unconditional distribution of Z is common knowledge. The entrepreneur has no private information.

There are $N \geq 2$ investors who are interested in financing the project. Each investor observes a private signal about Z. Let $\boldsymbol{X} = (X_1, \cdots, X_N)$ denote the vector of private signals observed by individual investors. I will also refer to X_i as the type of investor i.

Assumption 1.1. *The project payoff Z and the private signals $\boldsymbol{X} = (X_1, \cdots, X_N)$ satisfy the following properties:*

(a) Conditional on $Z = z$, X_i for all i are independently and identically distributed with probability density $f_{X|Z}(\cdot|z)$ on support $[x_L, x_H]$;

(b) Conditional on $\boldsymbol{X} = \boldsymbol{x}$, Z has probability density $f_{Z|\boldsymbol{X}}(\cdot|\boldsymbol{x})$ on support $[0, \infty)$;

(c) (X_i, Z) satisfy the strict monotone likelihood ratio property (SMLRP), i.e., the likelihood ratio $f_{X|Z}(x|z)/f_{X|Z}(x'|z)$ is strictly increasing in z if $x > x'$;

(d) $E[Z|\boldsymbol{X} = x_L\mathbf{1}] - I > 0$;

(e) $f_{X|Z}(x|z)$ is differentiable in x, and the functions $\left|\frac{\partial f_{Z|\boldsymbol{X}}(z|\boldsymbol{x})}{\partial x_i}\right|$ and $\left|z\frac{\partial f_{Z|\boldsymbol{X}}(z|\boldsymbol{x})}{\partial x_i}\right|$ are integrable on $z \in [0, \infty)$.

Part (c) of the assumption guarantees that each investor's private signal is a positive signal of the project payoff Z. Part (d) restricts the attention to projects with positive NPV conditional on any profile of private signals. Part (e) ensures derivatives can be taken through expectation operators.

The project can be financed by only one investor. Each investor makes a sealed-bid investment offer to the entrepreneur, which specifies the security that the investor requires in return for investing I. A feasible security is defined in the following way:

Definition 1.1. *A feasible security is a function $S : [0, \infty) \mapsto \mathbb{R}$ such that*

(1) (Limited Liability of the Entrepreneur) $S(z) \leq z$*,*

(2) (Boundedness) $S(z)$ *is bounded from below,*

(3) (Dual Monotonicity) $S(z)$ *and* $z - S(z)$ *weakly increase in* z*.*

The limited liability constraint requires the entrepreneur can pledge no more than the project value. I impose no limited liability constraint on investors so that investors are allowed to offer to pay more cash than I.[7] Boundedness is assumed to guarantee integrability. The results of the model do not change if the limited liability constraint on investors, $S(z) \geq 0$, is imposed.

The dual monotonicity constraint requires that the stakes of the investor and the entrepreneur both weakly increase in the project value. This assumption can be justified by that non-monotonic or decreasing stakes may induce agents to take harmful actions on the project.

After seeing all the offers, the entrepreneur forms belief about Z, and accepts the offer that she believes requires a security of the lowest value. If several offers look equally valuable, the entrepreneur accepts each of them with equal probability. The winning investor invests I to implement the project and receives the security S that she requires in her offer. The winning investor has net payoff $S(Z) - I$. The entrepreneur has net payoff $Z - S(Z)$. The losing inestors have payoff 0.

1.2.1 Relations Among Securities

In order to facilitate the equilibrium characterisation, I introduce several concepts that describe the relations among feasible securities, most of which follow DeMarzo et al. (2005).

Definition 1.2. *(Unambiguously Ordered) Security* S^1 *is unambiguously larger than security* S^2 *if* $S^1(z) \geq S^2(z)$ *for all* z *and* $S^1(z) > S^2(z)$ *for some* z*. Two*

[7] If in addition to investing I, an investor offers to pay cash $C > 0$ to the entrepreneur, and requires security $\hat{S}(Z) \geq 0$ in return, then the offer can be represented by $S(Z) = \hat{S}(Z) - C$. If the investor offers to invest an additional amount C into the project instead of paying C to the entrepreneur, then this offer can be represented by $S(Z) = \hat{S}(Z + C) - C$. In both cases, S is feasible as long as $\hat{S}$ is feasible.

securities are unambiguously ordered if one is unambiguously larger than the other or if they are equal for all z.

Definition 1.3. *(Ordered Set of Securities) An ordered set of securities is a function $S(s,\cdot)$ with $s \in [s_0, s_1]$ such that*

(1) $S(s,\cdot)$ is a feasible security,

(2) $S(s,z)$ is continuous and almost everywhere differentiable in s, and $S_1(s,z)f_{Z|\boldsymbol{X}}(z|\boldsymbol{x})$ is integrable on $z \in [0,\infty)$,

(3) $S(s',\cdot)$ is unambiguously larger than $S(s,\cdot)$ for $s' > s$,

(4) $E[S(s_0,Z)|\boldsymbol{X} = x_H\mathbf{1}] = 0$ and $S(s_1,z) = z$.

Examples of ordered sets of securities include:
The set of debt: $S^D(s^D,z) = \min(s^D,z)$ with $s^D \geq 0$ being the face value;
The set of equity: $S^E(s^E,z) = s^E z$ with $s^E \in [0,1]$ being the equity share;
The set of levered equity: $S^{LE}(s^{LE},z) = \max(z-(-s^{LE}),0)$ with $-s^{LE} \geq 0$ being the face value of counterparty's debt;
The set of buyout: $S^{BO}(s^{BO},z) = z-(-s^{BO})$ with $-s^{BO} \geq 0$ being the buyout price.

The main goal of the paper is to study the informal auction, in which offers are *not* restricted to ordered sets. The concept of ordered sets is only to facilitate the derivation and statement of equilibrium results.

In general, unless they are unambiguously ordered, the relative magnitude of two securities depends on the project value Z. For example, an equity can be more valuable than a debt when the project value is high, but less valuable when the project value is low. The above relation is simple enough to be captured by the following concept of "cross from below".

Definition 1.4. *(Cross from Below) Security S^1 crosses S^2 from below if they are not unambiguously ordered, and there is z^* such that $S^1(z) \geq S^2(z)$ for $z > z^*$ and $S^1(z) \leq S^2(z)$ for $z < z^*$.*

A debt receives the whole project value when it is lower than the face value, and is thus more valuable than any other security in these scenarios. Once the project value exceeds the face value, the debt value remains constant, whereas a non-debt security may continue to increase in the project value and surpass the debt value. This implies debt has the following property, which as shown in later sections makes it an equilibrium outcome of the informal auction:

Lemma 1.1. *If a debt and a non-debt security are not unambiguously ordered, the non-debt security crosses the debt from below.*

Based on "cross from below", a partial order can be defined over ordered sets of securities:

Definition 1.5. *(Steeper) Ordered set of securities $S^A(\cdot,\cdot)$ is steeper than ordered set $S^B(\cdot,\cdot)$ if for any pair of (s^A, s^B), $S^A(s^A,\cdot)$ crosses $S^B(s^B,\cdot)$ from below if they are not unambiguously ordered.*

For instance, the set of levered equity is steeper than the set of equity, which is steeper than the set of debt. According to Lemma 1.1, any ordered set that includes a non-debt security is steeper than the set of debt.

1.3 Formal Auction

As a necessary step of analysing the informal auction, in which investors can choose among all feasible securities, I first adapt DeMarzo et al. (2005)'s results on formal auctions to the current setting.

A formal auction is one in which investors can only make offers from an ordered set of securities. Since offers are unambiguously ordered, the entrepreneur's decision is as simple as in a first-price cash auction. I define a "monotonic equilibrium" as an equilibrium in which each investor's offered security is weakly decreasing in their private signal, i.e., the stake left to the entrepreneur is weakly increasing. It is analogous to an increasing equilibrium in a cash auction.

Let $Y_i \equiv \max\{X_j : j \neq i\}$ denote the highest private signal among investors other than i. Let $G(\cdot|x)$ and $g(\cdot|x)$ denote the cumulative probability and the density of Y_i conditional on $X_i = x$. Let $F_{Y|Z}(\cdot|z)$ and $f_{Y|Z}(\cdot|z)$ denote the cumulative probability and the density of Y_i conditional on $Z_i = z$.

Lemma 1.2. *In any symmetric monotonic equilibrium of a formal auction restricted to ordered set S, the investor strategy s_x must be strictly decreasing, continuous and differentiable, and satisfy the differential equation*

$$\frac{ds_x}{dx} = -\frac{E[S(s_x, Z) - I|X_i = Y_i = x]}{E[S_1(s_x, Z)|X_i = x, Y_i \leq x]} \cdot \frac{g(x|x)}{G(x|x)} \tag{1.1}$$

and the boundary condition

$$E[S(s_{x_L}, Z) - I|X_i = Y_i = x_L] = 0. \tag{1.2}$$

The above strategy supports an equilibrium if for each x, there is z^* such that

$$[S(s_x, z) - I]f_{Y|Z}(x|z) + S_1(s_x, z)F_{Y|Z}(x|z)\frac{ds_x}{dx} \gtreqless 0 \tag{1.3}$$

for $z \gtreqless z^*$.

The left-hand side of inequality (1.3) is a type x investor's benefit from marginally decreasing the required security from $S(s_x, \cdot)$ conditional on $Z = z$. The first term is the investor's gain due to the increased probability of winning. The second term is the investor's loss due to the smaller security she receives upon winning. By requiring the marginal benefit be positive for higher z and negative for lower z, condition (1.3) guarantees that an investor with a higher signal, which implies a higher estimate of the project payoff, is more willing to decrease the requested security, and hence the existence of a monotonic equilibrium.

For S being the set of equity, condition (1.3) always holds. For S being the set of debt, condition (1.3) can be simplified to

$$(s_x - I)\frac{f_{Y|Z}(x|s_x)}{F_{Y|Z}(x|s_x)} > -\frac{ds_x}{dx}. \tag{1.4}$$

For the rest of the paper, I assume condition (1.3) holds for the set of debt. Whether it holds for other ordered sets does not affect the results.

Assumption 1.2. *For $S(s, \cdot)$ being the set of debt and s_x being the strategy that satisfies* (1.1) *and* (1.2)*, inequality* (1.4) *holds for each x.*

1.4 Informal Auction

In the informal auction, investors are allowed to offer any feasible securities. Since securities are in general not unambiguously ordered, the entrepreneur's comparison of offers depends on her valuation of the project. Although the entrepreneur has no private information, she may infer the investors' private signals from their offers. This makes the informal auction a signalling game.

This paper considers pure strategies only. Let $\mathbb{S}$ denote the set of feasible securities. Let $\sigma^{(i)} : [x_L, x_H] \mapsto \mathbb{S}$ be the strategy of investor i, that is investor i with private signal x offers security $\sigma^{(i)}(x)$. Let $\mu : [x_L, x_H]^N \times \mathbb{S}^N \mapsto [0, \infty)$ be the entrepreneur's belief, that is the entrepreneur attributes probability density $\mu(x_1, \cdots, x_N|S_1, \cdots, S_N)$ to the profile of private signals $(x_1, \cdots, x_N)$ when seeing a profile of offers $(S_1, \cdots, S_N)$. μ satisfies

$$\int \mu(x_1, \cdots, x_N|S_1, \cdots, S_N)dx_1 \cdots dx_N = 1.$$

Let $p^{(i)}(\mathbf{S}, \mu)$ denote the probability that the entrepreneur accepts investor i's offer under belief μ if the profile of offers is $\mathbf{S} = (S_1, \cdots, S_N)$. For vector $\boldsymbol{a} = (a_1, \cdots, a_N)$, let $\boldsymbol{a}_{-i}$ denote $(a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_N)$ and $\boldsymbol{a}_{m:n}$ denote $(a_m, \cdots, a_n)$. Let $\pi^{(i)}(S|\boldsymbol{S}_{-i}, \mu, \boldsymbol{x})$ denote the expected payoff of investor i if she offers S, the other $N-1$ investors offer $\boldsymbol{S}_{-i}$, the entrepreneur has belief μ, and $\boldsymbol{X} = \boldsymbol{x}$:

$$\pi^{(i)}(S|\boldsymbol{S}_{-i}, \mu, \boldsymbol{x}) = E[S(Z) - I|\boldsymbol{X} = \boldsymbol{x}]p^{(i)}(\boldsymbol{S}_{1:i-1}, S, \boldsymbol{S}_{i+1:N}, \mu).$$

Let $\Pi^{(i)}(S|x, \boldsymbol{\sigma}_{-i}, \mu)$ denote the expected payoff of type x investor i if she offers S, the other $N-1$ investors follow the profile of strategies $\boldsymbol{\sigma}_{-i}$, and the entrepreneur has belief μ:

$$\Pi^{(i)}(S|x, \boldsymbol{\sigma}_{-i}, \mu) \quad = E[\pi^{(i)}(S|\boldsymbol{\sigma}_{-i}(\boldsymbol{X}_{-i}), \mu, \boldsymbol{X}_{1:i-1}, x, \boldsymbol{X}_{i+1:N})|X_i = x]$$

where $\boldsymbol{\sigma}_{-i}(\boldsymbol{X}_{-i})$ is the point-wise evaluation of the element functions of $\boldsymbol{\sigma}_{-i}$.

Unlike traditional signalling games with one signal sender or multiple senders with independent signals, the common-value auction involves multiple signal senders whose signals are interdependent. With this feature, the entrepreneur's belief needs to acknowledge the unconditional joint distribution of private signals. In other words, the entrepreneur has the freedom to interpret the offers only up to interpreting the investors' strategies. This feature is captured by the concept of sequential equilibrium. A sequential equilibrium is a profile of strategies $\boldsymbol{\sigma}$ and a belief μ that satisfy:

Rationality:

$$\sigma^{(i)}(x) \in \arg\max_{S} \Pi^{(i)}(S|x, \boldsymbol{\sigma}_{-i}, \mu);$$

Consistency:

There are non-negative bounded functions $\hat{\mu}^{(1)}, \cdots, \hat{\mu}^{(N)}$ such that

$$\hat{\mu}^{(i)}(S|x) = \begin{cases} 1, & \sigma^{(i)}(x) = S \\ 0, & \sigma^{(i)}(x) \neq S \text{ and } \sigma^{(i)}(x') = S \, \exists \, x'. \end{cases} \tag{1.5}$$

and

$$\begin{aligned} &\mu(x_1, \cdots, x_N|S_1, \cdots, S_N) \\ &= \frac{\hat{\mu}^{(1)}(S_1|x_1) \cdots \hat{\mu}^{(N)}(S_N|x_N) f_{\boldsymbol{X}}(x_1, \cdots, x_N)}{\int \hat{\mu}^{(1)}(S_1|x'_1) \cdots \hat{\mu}^{(N)}(S_N|x'_N) f_{\boldsymbol{X}}(x'_1, \cdots, x'_N) dx'_1 \cdots dx'_N}, \end{aligned} \tag{1.6}$$

where $f_{\boldsymbol{X}}$ is the unconditional probability density of $\boldsymbol{X}$.

$\hat{\mu}^{(i)}(S|x)$ acts as the likelihood in the entrepreneur's belief that type x investor i offers S. Equation (1.5) requires when investor i offers a security that is the equilibrium offer of some type of investor i, the entrepreneur believes investor i has followed the equilibrium strategy. For security S that is not offered by any type of investor i in equilibrium, sequential equilibrium imposes no restriction on $\hat{\mu}^{(i)}(S|x)$ and only requires μ be consistent with some non-negative $\hat{\mu}^{(i)}(S|x)$ through equation (1.6).

To verify whether a sequential equilibrium has "reasonable" off-equilibrium beliefs, I apply the D1 refinement of Cho and Kreps (1987). D1 requires that an off-equilibrium action must only be associated with those types who are "most likely to deviate" to this action. A type is "most likely to deviate" to an action if there is no second type such that under any belief that makes this type weakly prefer to deviate, the second type strongly prefers to deviate. Formally, let $\Pi^{(i)*}(x)$ denote the equilibrium payoff of type x investor i. For fixed S and i, define

$$D_x \equiv \{\mu' : \Pi^{(i)}(S|x, \boldsymbol{\sigma}_{-i}, \mu') > \Pi^{(i)*}(x)\} \tag{1.7}$$

and

$$D_x^0 \equiv \{\mu' : \Pi^{(i)}(S|x, \boldsymbol{\sigma}_{-i}, \mu') = \Pi^{(i)*}(x)\}. \tag{1.8}$$

A D1 equilibrium is a sequential equilibrium that satisfies

D1 consistency: $\hat{\mu}^{(i)}(S|x) = 0$ *if there is* x' *such that* $D_x \cup D_x^0 \subseteq D_{x'}$.

As before, a "monotonic" equilibrium is defined as an equilibrium in which each investor's required security is unambiguously decreasing in her private signal.

Proposition 1.1. *With two investors, the informal auction has a symmetric monotonic D1 equilibrium (and hence a sequential equilibrium) in which all investors offer debt financing.*

I provide a sketch of proof here, and relegate the complete proof to Appendix 1.7.2. I will show that the symmetric monotonic equilibrium strategy of the formal auction restricted to debt, namely s_x^D that satisfies (1.1) and (1.2), supports a D1 equilibrium of the informal auction. That the strategy supports an equilibrium of the formal auction immediately implies in the conjectured debt equilibrium of the informal auction, no type of investor has an incentive to deviate to a different debt offer. What remains to show is that no type of investor has an incentive to deviate to any non-debt security.

Consider non-debt security S. In a D1 belief μ, S is believed to be offered by the "most-likely-to-deviate" type.[8] Suppose there is a type who has an incentive to deviate to S under μ. By definition, the "most-likely-to-deviate" type also has an incentive to deviate to S under μ. Therefore, the "most-likely-to-deviate" type must have an incentive to deviate to S when the entrepreneur correctly infers her type from the deviation.

However, as explained in the next two paragraphs, the above implies the "most-likely-to-deviate" type also has an incentive to deviate to some debt. This contradicts the earlier statement that no type benefits from deviating to a debt, and thus completes the proof.

Let x be the type "most-likely-to-deviate" to security S. Suppose investor 1 observes signal x and deviates to S, and investor 2 follows the equilibrium debt strategy. Under a D1 belief, the entrepreneur believes investor 1 has signal x. Since investor 2's debt offer s_y^D is strictly decreasing in her private signal y, the entrepreneur can infer investor 2's private signal with certainty. Therefore, the entrepreneur accepts investor 1's offer S if and only if

$$E[S(Z)|X_1 = x, X_2 = y] < E[S^D(s_y^D, Z)|X_1 = x, X_2 = y].$$

As shown in Figure 1.1, there is a cutoff value y^* such that the inequality holds if and only if investor 2's private signal y is below y^*. This is because $E[S(Z)|X_1 = x, X_2 = y]$ crosses $E[S^D(s_y^D, Z)|X_1 = x, X_2 = y]$ from below when varying y, which is due to the following two reasons. First, Lemma 1.1 implies S crosses $S^D(s_y^D, \cdot)$ from below for a fixed y if they are not unambiguously ordered. Second, s_y^D decreases in y. Therefore, investor 1's offer S is accepted if and only if $y < y^*$.

[8] There may be multiple "most-likely-to-deviate" types, which is covered in the proof. Here I simplify the argument by assuming there is only one "most-likely-to-deviate" type.

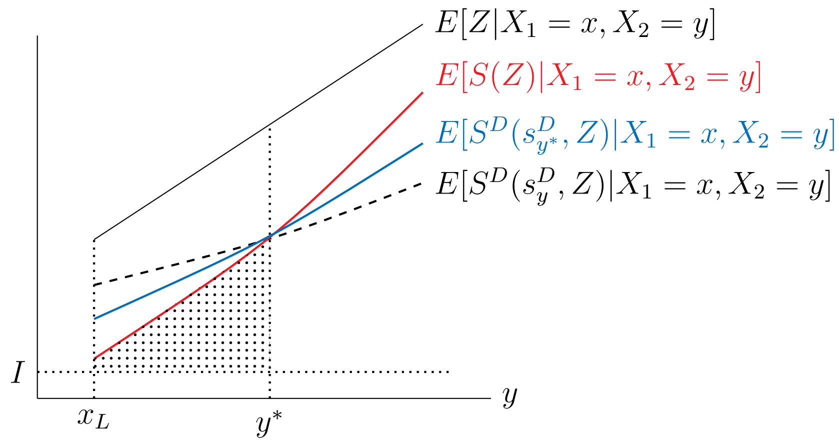


Figure 1.1: Key argument for Proposition 1.1

If investor 2 follows a monotonic strategy to offer debt $S^D(s^D_y, \cdot)$ when $X_2 = y$, and if the entrepreneur believes $X_1 = x$ when investor 1 offers non-debt security S, then type x investor 1 prefers offering debt $S^D(s^D_{y^*}, \cdot)$ to offering S.

The dotted area in Figure 1.1 is investor 1's deviation payoff.

Now consider the payoff of type x investor 1 if she instead offers debt $S^D(s^D_{y^*}, \cdot)$. Facing two debt offers, the entrepreneur simply accepts the one with the lower face value. Investor 1 again wins if and only if $y < y^*$. As shown in Figure 1.1, for $y < y^*$,

$$E[S(Z)|X_1 = x, X_2 = y] < E[S^D(s^D_{y^*}, Z)|X_1 = x, X_2 = y].$$

This is because the left-hand side is equal to the right-hand side for $y = y^*$, and S crosses $S^D(s^D_{y^*}, \cdot)$ from below. Therefore, by offering $S^D(s^D_{y^*}, \cdot)$, type x investor 1 wins for the same probability as offering S but receives a security more valuable than S conditional on winning. Her payoff is the dotted area in the figure plus the area above it between the blue curve and the red curve. If type x investor 1 benefits from deviating to S, she strictly benefits from deviating to $S^D(s^D_{y^*}, \cdot)$.

In summary, as in all common-value auctions, an investor faces the winner's curse. While her probability of winning is determined by the scenario in which she marginally wins, her payoff conditional on winning is determined by the scenarios in which she strictly wins, which implies others have lower signals than in the marginally-winning scenario. Among all the securities that lead to the same probability of winning, debt is the least information sensitive and thus the most valuable in strictly-winning scenarios. In order to acheive any probability of winning, the investor chooses debt to minimize the effect of the winner's curse.

With three or more investors, the entrepreneur's choice between two offers can depend on information revealed from a third offer. This paper does not reach a conclusion under this complication. This being said, Propositions 1.1 holds for three or more investors if the entrepreneur ignores information in offers that are unambiguously smaller than another offer.

In the sketch proof of Proposition 1.1, I have used the argument that an investor' s best reponse to her competitor's debt strategy is also a debt strategy. Next, I extend the argument to situations in which an investor's competitor uses a non-debt monotonic strategy, and show one' s best reponse cannot be a strategy steeper than her competitor's.

Proposition 1.2. *With two investors, there is no monotonic sequential equilibrium in which both investors make offers from ordered sets, of which one is steeper than the other, and the two offers are not always unambiguously ordered.*

For example, the proposition implies there is no monotonic sequential equilibrium in which one investor offers equity and the other offers debt.

By requiring a security that crosses her competitor's offer from below, an investor wins when her competitor has a lower private signal. The investor can instead mimic the type of her competitor that she marginally wins over, which allows her to win in the same scenarios but receive a more valuable security in these scenarios, because this flatter security provides better protection against the winner's curse than her original security.

The discussion so far has revolved around the argument that investors do not offer securities steeper than their competitors' offers. The next result is based on investors' incentive to offer securities flatter than their competitors' offers. For an ordered set S, let $s^\dagger$ be such that

$$E[S(s^\dagger, Z) - I|\boldsymbol{X} = x_L\mathbf{1}] = 0, \tag{1.9}$$

that is the value of $S(s^\dagger, \cdot)$ is expected to be equal to the investment required for the project if all investors have the lowest possible private signal x_L. An ordered set S is labelled as "too steep" if it satisfies[9]

$$\frac{d}{dx}E\big[S(s^\dagger, Z) - \frac{4(N-1)}{3(N-2)}S^D(s^{D\dagger}, Z)|\boldsymbol{X} = x\mathbf{1}\big]\big|_{x=x_L} > 0. \tag{1.10}$$

Proposition 1.3. *There is no symmetric monotonic sequential equilibrium in which investors make offers from a too steep ordered set.*

The key to the proof is that if investors follow a monotonic strategy in a too-steep ordered set S, then an investor with private signal x_L benefits from deviating to debt $S^D(s^{D\dagger}, \cdot)$. Suppose $N = 2$, and both investors follow monotonic strategy s_x, that is a type x investor offers $S(s_x, \cdot)$. According to Lemma 1.2, a type x_L investor loses the auction almost for sure and expects profit 0 when she marginally wins. On the other hand, the security value of $S^D(s^{D\dagger}, \cdot)$ is by definition equal to I and the value of $S(s_{x_L}, \cdot)$ when all investors have signal x_L. That S is too steep implies when investor 2's signal y increases, $E[S(s_y, Z)|X_1 = x_L, X_2 = y]$ crosses $E[S^D(s^{D\dagger}, Z)|X_1 = x_L, X_2 = y]$ from below at $y = x_L$ as shown in Figure 1.2. If investor 1 deviates to $S^D(s^{D\dagger}, \cdot)$ and the entrepreneur infers investor 1 has signal x_L, investor 1's offer is the cheaper source of financing and thus wins when investor 2's offer is slightly higher than x_L. If the entrepreneur holds any other belief, it will only make investor 1's debt offer look even cheaper and thus more attractive compared to investor 2's offer. Therefore, by deviating to $S^D(s^{D\dagger}, \cdot)$, type x_L investor 1 can win with strictly positive probability. Since the debt value is increasing in investor 2's private signal, type x_L investor 1 expects strictly

[9] An example set of parameters such that the sets of equity, levered equity and buyouts are too steep is provided in Appendix 1.7.1. The example differs from the model setup in that Z has a discrete distribution. The discrete distribution is assumed in order to derive closed-form verification of (1.10). All results in this article apply to the example despite the discrete distribution.

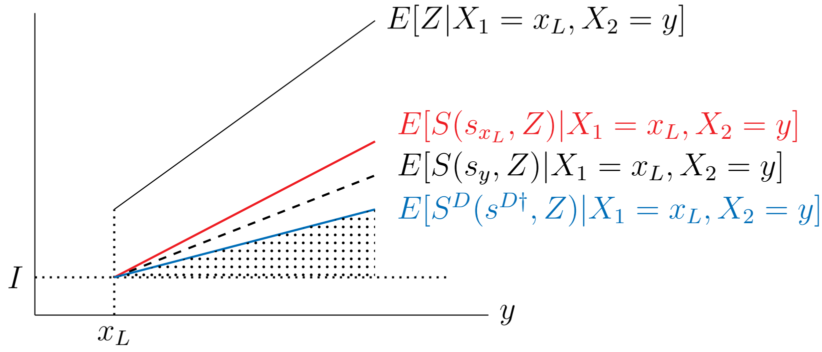


Figure 1.2: Key argument for Proposition 1.3

In a conejctured symmetric monotonic equilibrium in which both investors make offers from a too-steep ordered set S, type x_L investor 1 wins for probability zero in equilibrium, but wins with positive probability and expects positive payoff by deviating to debt $S^D(s^{D\dagger}, \cdot)$.

positive payoff $E[S^D(s^{D\dagger}, \cdot) - I|X_1 = x_L, X_2 = y]$ upon winning. Therefore, the deviation is profitable for a type x_L investor. The deviation payoff is the dotted area in Figure 1.2.

Intuitively, the entrepreneur can infer investors' private information from their offers and evaluate the project more precisely than any single investor in a fully-revealing equilibrium. The entrepreneur prefers to raise capital through a flat security and retain an information sensitive stake when she believes the project value is high. By offering a security that is flatter than her competitors' offers, an investor may select into winning a valuable project and reverse the winner's curse into the winner's blessing. Therefore, the best response to a too-steep strategy cannot be the same strategy.

In summary, by showing an equilibrium outcome with debt financing and ruling out two types of equilibrium outcomes that involve steeper securities, I argue investors tend to offer flat securities due to the advantage of flat offers in the presence of the winner's curse.

1.4.1 The Winner's Curse and Signalling

The above results are closely tied to the assumption that the project payoff is common when financed by different investors. In a very similar setup, DeMarzo et al. (2005) make the alternative assumption of private values that each investor has private information about their own influence on the project value, and show investors offer levered equity financing or buyouts, which are the steepest ordered sets. The drastically different security designs in the two settings are driven separately by the winner's curse and signalling.

The winner's curse, which leads to debt financing in the common-value setting, is absent with pure private values, in which case the losing investors' private signals are uninformative about the project payoff financed by the winning investor.

On the other hand, signalling incentive drives the design of steep securities in the private-value setting. Similar to the firm in Myers and Majluf (1984) which signals high value by issuing debt and retaining levered equity, an investor signals high private value to make her offer look more attractive by requiring to hold levered equity. With common values, the signalling incentive is more perplexing and is dominated by the concern of the winner's curse. Between a steeper security and a flatter security, an optimistic entrepreneur may prefer the flatter one while a pessimistic one may prefer the steeper one. As a result, an investor who requires a security flatter than the other investors wishes to signal high common value. If she signals high value by increasing the steepness of her offer, she is likely to stop when her offer is as steep as the others' offers. With such an offer, the investor no longer has an incentive to signal because the offers are unambiguously ordered and the entrepreneur's choice is not affected by belief. Similarly, an investor who requires a security steeper than the other investors wishes to signal low common value and can do so by flattening the security, which is likely to stop when her offer is as flat as the others' offers such that there is no more signalling incentive. The signalling effect pushes investors to offer securities from an ordered set, which implies one may not lose much generality by focusing on symmetric monotonic equilibria. On the other side of the coin, it deters deviation from a symmetric monotonic equilibrium to either a steeper or a flatter security, which blurs the implication of the winner's curse that flatter securities are superior choices, and makes it hard to eliminate multiple symmetric monotonic equilibria using D1 criterion.

1.4.2 Entrepreneur Revenue and Commitment Power

DeMarzo et al. (2005) show the entrepreneur's revenue in the formal auction restricted to the set of debt is the highest among formal auctions restricted to different ordered sets. The result applies to affliated values, which nests the common-value model of this paper.

Lemma 1.3. *(DeMarzo et al., 2005) The entrepreneur's revenue in the symmetric monotonic equilibrium of the formal auction restricted to the set of debt is higher than in the symmetric monotonic equilibrium of the formal auction restricted to another ordered set.*

Since investors offer debt in the informal auction, the entrepreneur does not have an incentive to organise a formal auction. Even without the entrepreneur's precommitment, competition among investors drives them to design the security that leads to a revenue for the entrepreneur as if she has made the optimal precommitment.

Proposition 1.4. *With two investors, the entrepreneur in an informal auction can acheive the highest revenue among formal auctions.*

1.5 Market Power and Security Design

In many models of security design, the party with superior information receives the more information sensitive security (Myers and Majluf, 1984; Nachman and

Noe, 1994; DeMarzo and Duffie, 1999; DeMarzo et al., 2005). This paper is one of those in which the informed party holds an information insensitive security, and the key to the explanation is the competition structure. In order to further manifest that competition has a higher-order effect than superior information, in this section I study variants of the benchmark model with modified assumptions on competition and information. I will put the discussion in the context of the existing literature and summarise the effects of different aspects of market power on security design near the end.

1.5.1 The Role of Information

In the first variant of the model, I change the assumption on the information structure. Instead of an uninformed entrepreneur, I consider an informed entrepreneur, who observes the vector of investors' private signals $\boldsymbol{X}$.

Proposition 1.5. *With an informed entrepreneur and two investors, there is a unique symmetric monotonic equilibrium, in which both investors offer debt.*

For the existence of the equilibrium, one can invoke the argument in the proof of Proposition 1.1 that no type of investor benefits from deviating to a non-debt security when the entrepreneur correctly infers her type. In the case with an informed entrepreneur, the entrepreneur observes the deviating investor's type, which implies no type of investor benefits from deviating to a non-debt security.

Whereas Proposition 1.3 only shows the non-existence of symmetric monotonic equilibria with too-steep offers, Proposition 1.5 rules out any symmetric monotonic equilibrium with a non-debt offer. It is hardly surprising that the stronger result holds with an informed entrepreneur. When the entrepreneur observes investors' private signals, an investor gains from keeping her probability of winning constant and increasing her payoff upon winning by deviating to a slightly flatter security. When the entrepreneur is uninformed, offering a flatter security signals lower valuation, which makes the offer look less attractive to the entrepreneur than it actually is. This signalling effect may potentially deter deviation from a steep offer, and makes it difficult to conclude whether debt financing is the unique symmetric monotonic equilibrium.

In summary, the result of debt financing is robust to assuming the entrepreneur instead of the investors is the better informed party.

1.5.2 The Role of Competition

To examine the role of competition, I consider a bilateral trade in which the only investor makes a take-it-or-leave-it offer to the entrepreneur.

Assumption 1.3. *In a bilateral trade,*

(1) $N = 1$*;*

(2) The entrepreneur has fixed outside option $b > 0$*.*

Part (2) of the assumption is needed because otherwise the monopolistic investor will request the whole project and leave nothing to the entrepreneur, which renders security design degenerate. The outside option b can represent the entrepreneur's salary from a regular job once the project is abandoned.

I consider both cases in which the investor and the entrepreneur respectively have superior information. For simplicity, assume the entrepreneur accepts the offer when she is indifferent.

Bilateral Trade with an Informed Investor

First consider the case in which the investor observes private signal X, while the entrepreneur does not have private information. To be comparable with the benchmark model, I assume $E[Z|X = x_L] > I + b$, so that the project has positive NPV conditional on any value of X.

Proposition 1.6. *In a bilateral trade with an informed investor, there is a unique sequential equilibrium up to variation in the offer of type x_L investor, in which all types of the investor except type x_L offer a buyout at price b.*

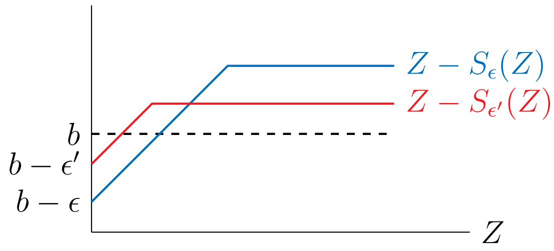
Intuitively, by offering a buyout, the investor can avoid her offer being undervalued because the value of cash is unambiguous. If the investor is cash constrained so that she can pay no cash in addition to investing I, then there is a unique D1 equilibrium outcome up to variation in the offer of type x_L, in which all types of the investor offer levered-equity financing. The investor signals high valuation by requiring the most information sensitive security, in a way similar to the informed firm in Myers and Majluf (1984) signals high valuation by issuing debt and retaining levered equity.

Bilateral Trade with an Informed Entrepreneur

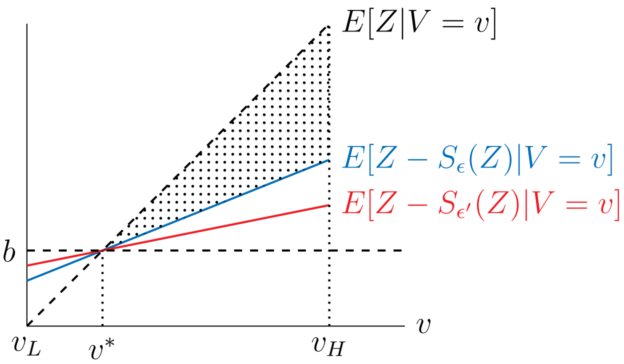
Next consider the case in which the entrepreneur observes private signal V while the investor has no private signal. V and Z satisfy the SMLRP. The entrepreneur accepts offer S if and only if $E[Z - S(Z)|V] \geq b$. This implies the security S determines a rule of project screening implemented by the entrepreneur. Meanwhile, S determines how the surplus is split when the offer is accepted. When designing the security, the investor aims to both induce efficient screening and minimise the entrepreneur's surplus.

Let v_L and v_H denote the maximum and minimum values of V. If $E[Z|V = v_L] \geq I + b$, then it is efficient to start the project for all values of V. By offering a buyout at price b, the investor can guarantee the offer is always accepted and she receives the full project NPV. Offering any other security is suboptimal.

If $E[Z|V = v_L] < I + b$, the investor can achieve a payoff infinitely close to the maximum total surplus by offering levered-equity financing which is infinitely close to a buyout at price b. Let v^* be such that it is efficient to start the project if and only if $V \geq v^*$: $E[Z|V = v^*] = I + b$. Let $S_\epsilon(Z) = \max(Z - C_\epsilon, 0) - b + \epsilon$



(a) The entrepreneur's payoff is $Z - S_\epsilon(Z) = b - \epsilon + \min(C_\epsilon, Z)$ if offer S_ϵ is accepted. $Z - S_\epsilon(Z)$ crosses $Z - S_{\epsilon'}(Z)$ from below for $\epsilon' < \epsilon$.



(b) For $\epsilon' < \epsilon$, both S_ϵ and $S_{\epsilon'}$ induce the efficient screening, and $S_{\epsilon'}$ leads to higher investor surplus than S_ϵ. The dotted area is the investor's surplus by offering S_ϵ.

Figure 1.3: Bilateral trade with an informed entrepreneur

be levered equity with additional cash payment $b - \epsilon$ to the entrepreneur, where C_ϵ is determined by $E[Z - S_\epsilon(Z)|V = v^*] = b$. For $\epsilon > 0$, the entrepreneur accepts offer S_ϵ if and only if $V \geq v^*$, which implies S_ϵ induces the efficient screening. As illustrated in Figure 1.3, as ϵ decreases and approaches 0, $Z - S_\epsilon(Z)$ becomes flatter, which implies $S_\epsilon(Z)$ becomes steeper, so that the investor's payoff increases and approaches the full project NPV. On the other hand, $S_\epsilon(Z)$ for $\epsilon = 0$ is a buyout with price b, which will be accepted by the entrepreneur under all values of V and thus fails to induce the efficient screening.

Proposition 1.7. *In a bilateral trade with an informed entrepreneur, the investor can receive payoff infinitely close to the maximum total surplus by offering a combination of levered-equity financing and cash payment which is infinitely close to a buyout offer at price b.*

Intuitively, since the entrepreneur accepts the offer when the project value is high enough, the investor faces advantageous selection. Within the spectrum of securities that induce the efficient screening, the investor chooses the steepest security, which as much as possible captures the project value extra to what is enough to cover the entrepreneur's opportunity cost, b, when it is greater than b. In this way, the entrepreneur's informational rent is diminished.

In summary, regardless of which party has superior information, the investor in a bilateral trade requires the most information sensitive security. The result is in contrast to the result of the benchmark model with multiple investors, which implies competition is crucial to the equilibrium security design. Moreover, the analysis of the bilateral trade implies if the entrepreneur instead of the investor proposes a security through a take-it-or-leave-it offer, she will propose debt financing (regardless of which party has superior information). This implies who has the power to commit to a security is cruicial to the result when both parties are monopolies.

1.5.3 Three Aspects of Market Power

As implicit in the analysis above, one can view private information, monopoly power and the right to move first to design the security as three aspects of market power relavent to security design. When monopoly power is carefully defined, two patterns can be summarised to hold in the models of this paper and the existing literature on security design under information asymmetry.

Let the concept of "indispensability" proxy for monopoly power. I call the group of entrepreneurs indispensable if each entrepreneur is indispensable for implementing their own project, that is another agent will not be able to realize the same project payoff without contracting with this entrepreneur. Similarly, the group of investors is indispensable if each investor is indispensable to realize their own project payoff. Under this definition, Table 1.1 summarises the key assumptions of different models on which parties are indispensable, which party moves first to propose the security and which party is privately informed. The last column of the table lists which party receives levered equity in equilibrium. Consistent with the conclusions from variants of the model of this paper, two patterns

	Indispensable	Security Design	Informed	Levered Equity
Informed entrepreneur's issuance (Myers and Majluf, 1984; Nachman and Noe, 1994; DeMarzo and Duffie, 1999)	E	E	E	E
Uninformed seller's design (Yang, 2020)	E	E	I	E
Formal auction w/ common values (DKS, Axelson, 2007; Sogo et al., 2016)	E	E	I	E
Informed buyer's design (Burkart and Lee, 2016)	E	I	I	E
Informal auction w/ common values (this paper)	E	I	E or I	E
Formal auction w/ private values (DKS)	E,I	E	I	E
Informal auction w/ private values (DKS)	E,I	I	I	I
Bilateral trade (this paper)	E,I	I	E or I	I

Table 1.1: Models on security design under information asymmetry

Assumptions and outcomes of each model in terms of which parties are indispensable, which party designs the security, which party is privately informed, and which party receives levered equity in equilibrium. E = Entrepreneur, I = Investor. DKS = DeMarzo et al. (2005).

can be summarised from this broader range of models:

(1) If only one of the two parties is indispensable, the indispensable party receives levered equity;

(2) If both parties are indispensable, the party that designs the security receives levered equity.

For example, in the informal auction with common values of this paper, the entrepreneur is indispensable while each investor is not, because the project payoff is not affected by the identity of the investor. The result that the entrepreneur receives levered equity is consistent with pattern (1). In the formal and informal auctions with private values of DeMarzo et al. (2005), each investor is indispensable for realising their private project payoff, which implies they have differentiable resources and face competition less direct than investors with common values. The result that the entrepreneur receives levered equity in the formal auction and investors receive levered equity in the informal auction is consistent with pattern (2).

An agent is not considered indispensable in the above definition if she is indispensable only for the private benefit she enjoys from owning part of the project. For example, the buyers in the models of Burkart and Lee (2016) and Yang (2020) are not indispensable even though they have lower discount rates on the the project than the seller, because their participation does not affect the value of the part of the project retained by the seller. This specification is necessary for the summarised patterns to be consistent with the result of Burkart and Lee (2016).

The patterns imply the party with the larger market power receives the steeper security. Among the three aspects of market power, competition is the primary determinant, and the right to move first to propose the security is secondary.

Some other models in this literature do not fit the patterns exactly, mostly due to assumptions more subtle than can be characterised by the binary concept of indispensability. These models are nevertheless consistent with the notion that steepness of a party's security is positively related to its monopolistic power or its irreplaceable contribution to the project payoff. Gorbenko and Malenko (2011) study design of security auctions by competing entrepreneurs. Indispensability is ambiguous in the model because each entrepreneur is at first not indispensable and becomes indispensable when an investor enters her auction. The result shows steepness of the securities retained by entrepreneurs increases in the ratio of investors to entrepreneurs. Inderst and Mueller (2006) and Yang and Zeng (2019) consider an investor who receives or can acquire information after the security design and makes the financing decision based on the information. In the model of Inderst and Mueller, even though the investor is not indispensable for financing the project due to competition from other uninformed investors, she is the only one that receives private information and can increase the expected project payoff by screening. In both models, as the probability that the project is profitable decreases, which implies the investor's screening becomes more socially valuable, she receives a steeper security.

1.6 Concluding Remarks

This paper studies security design by competing investors with private information on the common value of the project. The contribution is three-fold. First, competition with common values is a major concern in financial markets but has been understudied in the literature on security design. By modelling common-value competition, this paper provides a solution to reconcile theories of security design under information asymmetry with the prevalence of debt financing in private firms. Second, this paper makes the first attempt, though far from being complete, to systematically investigate the relationship between market power and security design under information asymmetry. By comparing the models in this paper and various models in the literature, I find a party's market power is positively related to the information sensitiveness of the security it receives. Among different aspects of market power, monopolistic power is primary and the power to commit to a security design is secondary. Third, the model contributes to theories of multi-dimensional bidding, and can be applied to situations other than corporate finance. For example, a publishing house may take competition from other publishing houses into account when deciding the advance payment and royalty rate to offer to an author; an employer inevitably considers the outside options of a successful job candidate when designing the fixed and performance-based components of the offer. In both situations, it is not hard to imagine each publishing house or employer's offer reflects their private information on the common value of the book or the candidate.

Several questions follow. First, since informal auctions with common values and private values result in opposite security designs, it is useful to understand the same with hybrids of private values and common values. A natural way to model hybrids is affiliated values which is introduced to auction theory by Milgrom and Weber (1982). The hybrid model may predict securities steeper than debt but flatter than levered equity, and has the potential to explain the use of straight equity and hybrid securities.

Second, this paper assumes only one investor can finance the project. If co-financing is allowed, investors may enjoy larger market power. It will be interesting to know whether investors will design securities steeper than debt in consequence.

Third, indispensablity, which is my proxy for monopolistic power, is related to effort provision once the project is started. Agents who are indispensable at the investment phase often continue to have on-going influence on the project performance. Parallel to my prediction that a party with monopolistic power tends to receive a steep security, models with endogenous effort provision predict an agent whose effort is crucial to the project performance tends to receive a steep security. It is useful to design empirical strategies to distinguish the effects of monopolistic power and endogenous effort provision when agents with big influence on the outcome of their investments are observed to receive steep securities.

Fourth, this paper considers take-it-or-leave-it offers, while actual bargainings between entrepreneurs and investors can be much more complex. An investor may want to offer a menu of securities instead of a single one; there can be back-and-forth negotiations instead of having someone committing to a take-it-or-leave-it offer. It is useful to understand whether and how these market structures shape security design.

Last but not least, this paper leaves out competition among entrepreneurs. In practice, an investor may simultaneously talk with multiple entrepreneurs and make bidding decisions subject to a budget constraint. Modelling such a situation is empirically relevant, and will provide an examination of the notion that private-value competition does not impede monopolistic power in security design.

1.7 Appendix

1.7.1 A Numerical Example

This part of the appendix provides a set of model parameters under which the set of equity, the set of levered equity and the set of buyout are too steep according to inequality (1.10).

Let V be a random variable whose value is either v_L or v_H for equal probability. $(Z, X_1, \cdots, X_N)$ are independent conditional on V. Let $f_{Z|V}(z|v)$ be the probability that $Z = z$ conditional on $V = v$:

$$f_{Z|V}(z|v_L) = \begin{cases} \frac{1}{2}, & z = 0 \\ \frac{1}{3}, & z = 3 \\ \frac{1}{6}, & z = z_H \end{cases}, \quad f_{Z|V}(z|v_H) = \begin{cases} \frac{1}{3}, & z = 0 \\ \frac{1}{3}, & z = 3 \\ \frac{1}{3}, & z = z_H. \end{cases}$$

where z_H is a constant larger than 3. Suppose $x_H > x_L \geq 0$. Let $f_{X|V}(x|v)$ for $x \in [x_L, x_H]$ be the probability density of $X_i = x$ conditional on $V = v$:

$$f_{X|V}(x|v) = \begin{cases} \dfrac{1}{\int_{x_L}^{x_H} dx} = \dfrac{1}{x_H - x_L}, & v = v_L \\ \dfrac{x}{\int_{x_L}^{x_H} x dx} = \dfrac{2x}{x_H^2 - x_L^2}, & v = v_H. \end{cases}$$

Let $I = 1$. These parameters satisfy Assumption 1.1 except that Z does not have full support on $[0, \infty)$, and hence functions of z can only be summed instead of integrated. The advantage of choosing a discrete distribution of Z is it allows me to verify the too-steep condition (1.10) in closed-form.

For concreteness, I first establish that a weaker version of Assumption 1.2 is satisfied, so that there is a symmetric monotonic equilibrium in a formal auction restricted to debt. I do not verify Assumption 1.2 directly due to the difficulty of solving s_x^D in closed form.

1. The debt stratey that satisfies equations (1.1) and (1.2) supports an equilibrium of a formal auction restricted to debt.

The following lemma provides a sufficient condition for the existence of a debt equilibrium in an auction restricted to debt.

Lemma 1.4. *If $E[S^D(s^D, Z)|V = v] - I$ is log-submodular in s^D and v when it is positive, that is*

$$\frac{E[S^D(s^D, Z)|V = v_H] - I}{E[S^D(s^D, Z)|V = v_L] - I} \text{ decreases in } s_D \text{ when } E[S^D(s^D, Z)|V = v_L] - I > 0, \tag{1.11}$$

then the debt stratey that satisfies equations (1.1) *and* (1.2) *supports an equilibrium of a formal auction restricted to debt.*

Inequality (1.11) is implied by Assumption 1.2 and not vice versa, but is still sufficient for the existence of a debt equilibrium. Therefore, I next verify condition (1.11). Let $S^D(s^D, \cdot)$ with $s^D \in [0, 12]$ be a debt with face value s^D:

$$\begin{aligned} S^D(s^D, z) &= \min(z, s^D) \\ &= \begin{cases} s^D, & \text{if } z = z_H, \text{ or } z = 3 \text{ and } s^D < 3 \\ 3, & \text{if } z = 3 \text{ and } s^D \geq 3 \\ 0, & \text{if } z = 0 \end{cases}. \end{aligned}$$

Since

$$E[S^D(s^D, Z)|V = v_H] = \begin{cases} \frac{5}{6}s^D, & \text{if } s^D < 3 \\ \frac{1}{2}s^D + 1, & \text{if } s^D > 3 \end{cases},$$

$$E[S^D(s^D, Z)|V = v_L] = \begin{cases} \frac{2}{3}s^D, & \text{if } s^D < 3 \\ \frac{1}{3}s^D + 1, & \text{if } s^D > 3 \end{cases},$$

$$\frac{E[S^D(s^D, Z)|V = v_H] - I}{E[S^D(s^D, Z)|V = v_L] - I} = \begin{cases} \frac{\frac{5}{6}s^D - 1}{\frac{2}{3}s^D - 1}, & \text{if } s^D < 3 \\ \frac{3}{2}, & \text{if } s^D > 3 \end{cases}$$

is weakly decreasing in s^D when $s^D \geq \frac{3}{2}$.

2. If $z_H = 12$, condition (1.10) holds for S being the sets of buyout, levered equity and equity when $N \geq 3$, $N \geq 4$ and $N \geq 9$ respectively.

The joint distribution of $(V, X_1, \cdots, X_N)$ implies

$$f_{V|X_1,\cdots,X_N}(v_L|x,\cdots,x) = \frac{a}{a + bx^N}$$

where $a \equiv \frac{1}{(x_H - x_L)^N}$ and $b \equiv \frac{2^N}{(x_H^2 - x_L^2)^N}$. Therefore,

$$f_{Z|\boldsymbol{X}}(z|\boldsymbol{X} = x\mathbf{1}) = \begin{cases} \frac{\frac{1}{2}a + \frac{1}{3}bx^N}{a + bx^N}, & z = 0 \\ \frac{\frac{1}{3}a + \frac{1}{3}bx^N}{a + bx^N}, & z = 3 \\ \frac{\frac{1}{6}a + \frac{1}{3}bx^N}{a + bx^N}, & z = z_H \end{cases}.$$

Let $S^E(s^E, z) = s^E z$ with $s^E \in [0, 1]$ be a share s^E of equity. Let $S^{LE}(s^{LE}, z)$ with $-s^{LE} \in [0, z_H]$ be levered equity with counterparty debt level $-s^{LE}$:

$$\begin{aligned} S^{LE}(s^{LE}, z) &= \max(0, z - (-s^{LE})) \\ &= \begin{cases} z_H - (-s^{LE}), & \text{if } z = z_H \\ 3 - (-s^{LE}), & \text{if } z = 3 \text{ and } -s^{LE} < 3 \\ 0, & \text{o/w} \end{cases}. \end{aligned}$$

Let $S^{BO}(s^{BO}, z) = z - (-s^{BO})$ with $-s^{BO} \in [0, z_H]$ is the buyout price. Let $q(x) \equiv f_{Z|\boldsymbol{X}}(z_H|\boldsymbol{X} = x\mathbf{1})$. Then

$$E[S^D(s^D, Z)|\boldsymbol{X} = x\mathbf{1}] = \begin{cases} [\frac{1}{3} + q(x)]s^D, & s^D < 3 \\ 1 + q(x)s^D, & s^D > 3, \end{cases}$$

$$E[S^E(s^E, Z)|\boldsymbol{X} = x\mathbf{1}] = s^E\left[z_H q(x) + 1\right],$$

$$E[S^{LE}(s^{LE}, Z)|\boldsymbol{X} = x\mathbf{1}] = \begin{cases} 1 - \frac{1}{3}(-s^{LE}) + q(x)[z_H - (-s^{LE})], & -s^{LE} < 3 \\ q(x)[z_H - (-s^{LE})], & -s^{LE} > 3, \end{cases}$$

$$E[S^{BO}(s^{BO}, Z)|\boldsymbol{X} = x\mathbf{1}] = q(x)z_H + 1 - (-s^{BO}).$$

According to (1.9),

$$s^{D\dagger} = \frac{3}{1+q(x_L)}, \quad s^{E\dagger} = \frac{1}{1+q(x_L)z_H}, \quad s^{LE\dagger} = -[z_H - \frac{1}{q(x_L)}], \quad s^{BO\dagger} = -q(x_L)z_H.$$

Moreover,

$$\frac{d}{dx}E[S^D(s^D, Z)|\boldsymbol{X} = x\mathbf{1}] = s^D q'(x),$$
$$\frac{d}{dx}E[S^E(s^E, Z)|\boldsymbol{X} = x\mathbf{1}] = z_H s^E q'(x),$$
$$\frac{d}{dx}E[S^{LE}(s^{LE}, Z)|\boldsymbol{X} = x\mathbf{1}] = [z_H - (-s^{LE})]q'(x),$$
$$\frac{d}{dx}E[S^{BO}(s^{BO}, Z)|\boldsymbol{X} = x\mathbf{1}] = z_H q'(x).$$

Let $x_L = 0$. Then $q(x_L) = \frac{1}{6}$. Condition (1.10) for S being the set of equity, levered equity and buyout are respectively equivalent to

$$\frac{s^{E\dagger}z_H}{s^{D\dagger}} = \frac{7}{3} \cdot \frac{z_H}{6+z_H} > \frac{4(N-1)}{3(N-2)}, \tag{1.12}$$

$$\frac{z_H - (-s^{LE\dagger})}{s^{D\dagger}} = \frac{7}{3} > \frac{4(N-1)}{3(N-2)}, \tag{1.13}$$

$$\frac{z_H}{s^{D\dagger}} = \frac{7}{18}z_H > \frac{4(N-1)}{3(N-2)}. \tag{1.14}$$

Let $z_H = 12$. Then conditions (1.12), (1.13) and (1.14) hold when $N \geq 9$, $N \geq 4$ and $N \geq 3$ respectively.

1.7.2 Proof of Propositions

Proof of propositions is provided in Appendix 1.7.2. Proof of lemmas is provided in Appendix 1.7.3.

Lemma 1.5. *(1) For any security S, $E[S(Z) - I|\mathbf{X} = \mathbf{x}] = 0$ implies*

$$\frac{d}{dx_i}E[S(Z) - I|\mathbf{X} = \mathbf{x}] > 0.$$

(2) If security S^1 crosses S^2 from below, $E[S^1(Z) - S^2(Z)|\mathbf{X} = \mathbf{x}] = 0$ *implies*

$$\frac{d}{dx_i} E[S^1(Z) - S^2(Z)|\mathbf{X} = \mathbf{x}] > 0.$$

Lemma 1.6. *Consider $E^{\mu,S,y}[f(Z)]$ defined by equation* (1.16) *in the proof of Proposition 1.1. If security S^1 crosses security S^2 from below, then*

$$E^{\mu,S,y}[S^1(Z) - S^2(Z)] = 0$$

implies

$$\frac{d}{dy} E^{\mu,S,y}[S^1(Z) - S^2(Z)] > 0.$$

Proof of Proposition 1.1: Let s_x^D be the investor strategy in the monotonic symmetric equilibrium of the formal auction restricted to debt, as described in Lemma 1.2. The optimality of the strategy in the formal auction implies that in an informal auction, if all investors follow this strategy, an investor does not benefit from deviating to any debt, that is

$$\Pi^*(x) \geq \Pi(S^D(s_{x'}^D, \cdot)|x, \mu) \tag{1.15}$$

for all x, x' and μ.[10] To show that s_x^D supports an equilibrium under a D1-consistent belief, it suffices to show that no type of investor 1 benefits from deviating to an arbitrary non-debt security S under any D1-consistent belief. According to Lemma 1.1, S crosses $S^D(s_x^D, \cdot)$ from below if they are not unambiguously ordered.

Step 1. Under any belief μ, there is y^μ such that if investor 1 deviates to S, she wins if and only if investor 2 has private signal below y^μ.

Let $E^{\mu,S,y}[f(Z)]$ be the entrepreneur's evaluation of function $f(Z)$ under belief μ, if investor 1 offers S and investor 2 offers debt $S^D(s_y^D, \cdot)$. Since s_y^D is strictly

[10] In this proof, superscripts of $\Pi^{(i)}$, $\Pi^{*(i)}$ and $\hat{\mu}^{(i)}$ are omitted with the understanding that the superscript is 1. The argument $\boldsymbol{\sigma}_2$ of function Π is omitted with the understanding that investor 2 plays the equilibrium strategy.

decreasing, y is the only type of investor 2 that offers $S^D(s_y^D,\cdot)$ in equilibrium. According to equations (1.6) and (1.5), there is $\hat{\mu}(S|x)$ such that

$$\mu(x,y'|S,S^D(s_y^D,\cdot)) = \begin{cases} \frac{\hat{\mu}(S|x)f_{\mathbf{X}}(x,y)}{\int \hat{\mu}(S|u)f_{\mathbf{X}}(u,y)du}, & y'=y \\ 0, & y' \neq y \end{cases}$$

Therefore,

$$E^{\mu,S,y}[f(Z)] = \int E[f(Z)|X_1=x,X_2=y]\frac{\hat{\mu}(S|x)f_{\mathbf{X}}(x,y)}{\int \hat{\mu}(S|u)f_{\mathbf{X}}(u,y)du}dx. \tag{1.16}$$

According to Lemma 1.1 and 1.6, $E^{\mu,S,y}[S(Z)-S^D(s_y^D,Z)]=0$ implies

$$\begin{aligned}
&\frac{d}{dy}E^{\mu,S,y}[S(Z)-S^D(s_y^D,Z)] \\
=&\frac{d}{dy}E^{\mu,S,y}[S(Z)-S^D(s_{y'}^D,Z)]\big|_{y'=y} - E^{\mu,S,y}[S_1^D(s_y^D,Z)]\frac{ds_y^D}{dy} \\
\geq&\frac{d}{dy}E^{\mu,S,y}[S(Z)-S^D(s_{y'}^D,Z)]\big|_{y'=y} \\
>&0.
\end{aligned}$$

Therefore, there is y^μ such that

$$E^{\mu,S,y}[S(Z)-S^D(s_y^D,Z)] \gtrless 0 \tag{1.17}$$

for $y \gtrless y^\mu$. This implies the entrepreneur accepts investor 1's offer S when investor 2 has signal below y^μ.

Step 2. Suppose a type strictly benefits from deviating to S under a D1-consistent belief μ^*. Then all types in the support of $\hat{\mu}^*(\cdot|S)$ strictly benefit from deviating to S under μ^*.

Consider D_x and D_x^0 for security S as defined in equation (1.7) and (1.8), which stand for the sets of beliefs under which type x investor 1 strictly benefits from and is indifferent about deviating to S. If $\mu \in D_x \cup D_x^0$, then (1) $\mu' \in D_x^0$ for μ' such that $y^{\mu'} = y^\mu$ and (2) $\mu'' \in D_x$ for μ'' such that $y^{\mu''} > y^\mu$. Part (2) is because according to part (1) of Lemma 1.5, $\Pi(S|x,\mu'') > 0$ implies $E[S(Z) - I|X_1 = x, X_2 = y^{\mu''}] > 0$, which implies $\Pi(S|x,\mu'') - \Pi(S|x,\mu) > 0$.

Therefore, for each x, there is y_x such that $D_x = \{\mu : y^\mu > y_x\}$ and $D_x^0 = \{\mu : y^\mu = y_x\}$. If any type of investor 1 strictly benefits from deviating to S under a D1-consistent belief μ^*, then all types of investor 1 in the support of $\hat{\mu}^*(\cdot|S)$ strictly benefit from deviating to S under μ^*.

Step 3. That all types in the support of $\hat{\mu}^*(\cdot|S)$ strictly benefit from deviating to S under μ^* contradicts that no type benefits from deviating to a debt.

Suppose type x is in the support of $\hat{\mu}^*(\cdot|S)$. That she strictly benefits from deviating to S under μ^* implies $\Pi(S|x,\mu^*) > \Pi^*(x^*)$. According to inequality (1.15), this implies

$$\Pi(S|x,\mu^*) > \Pi(S^D(s^D_{y^{\mu^*}},\cdot)|x,\mu^*),$$

that is the investor expects higher payoff by offering S than by offering debt $S^D(s^D_{y^{\mu^*}},\cdot)$. With either offer, the investor wins if and only if the other investor has a private signal below y^{μ^*}. Therefore, there must be $y < y^{\mu^*}$ such that

$$E[S(Z) - S^D(s^D_{y^{\mu^*}}, Z)|X_1 = x, X_2 = y] > 0.$$

According to Lemma 1.5, this implies

$$E[S(Z) - S^D(s^D_{y^{\mu^*}}, Z)|X_1 = x, X_2 = y^{\mu^*}] > 0. \tag{1.18}$$

Since (1.18) holds for all x in the support of $\hat{\mu}^*(\cdot|S)$, multiplying it by $\mu^*(x, y^{\mu^*}|S, s^D_{y^{\mu^*}})$ and integrating over x in the support of $\hat{\mu}^*(\cdot|S)$ implies

$$E^{\mu^*,S,y^{\mu^*}}[S(Z) - S^D(s^D_{y^{\mu^*}}, Z)] > 0.$$

Since $E^{\mu^*,S,y}[S(Z) - S^D(s^D_y, Z)]$ is continuous in y, the above contradicts the definition of y^{μ^*} in inequality (1.17). This completes the proof.

Proof of Proposition 1.2: Suppose ordered set S^A is steeper than ordered set S^B. I conjecture an equilibrium in which investor 1 offers securities from S^A and investor 2 offers securities from S^B, and I will show that there is a type of investor 1 who benefits from deviating to offering a security from S^B.

Step 1. There is strictly positive measure of x such that type x investor 1 wins with strictly positive probability in equilibrium.

Suppose the opposite is true, that is for measure 1 of x, type x investor 1 wins with probability 0. This implies for measure 1 of x, type x investor 2 wins with probability 1. If investor 2 can win with probability 1 by offering a certain security, it is suboptimal for her to offer another security unambiguously smaller. Therefore, for measure 1 of x, type x investor 2 makes an identical offer. Denote this offer by $S^B(s^{B*}, \cdot)$.

That investor 2 participates with measure 1 of private signals implies she expects non-negative profit by offering $S^B(s^{*B}, \cdot)$ when $X_2 = x_L$, i.e.,

$$E[S^B(s^{B*}, Z) - I | X_1 \in \chi_1, X_2 = x_L] \geq 0.$$

where χ_1 denotes the set of x such that type x investor 1 wins with probability 0. Lemma 1.5 part (1) implies

$$E[S^B(s^{B*}, Z) - I | X_1 \in \chi_1, X_2 \geq x_L] > 0.$$

Therefore, there is strictly positive measure of $x \in \chi_1$ such that

$$E[S^B(s^{B*}, Z) - I | X_1 = x, X_2 \geq x_L] > 0.$$

which implies type x investor 1 benefits from deviating to a security in S^B that is slightly smaller than s^{B*}. The deviation allows her to win with probability 1 and expect strictly positive payoff.

Therefore, there must be a strictly positive measure of x such that type x investor

1 wins with strictly positive probability in equilibrium.

Step 2. There is x^* such that type x^* investor 1 expects strictly positive payoff.

Consider two types $x < x^*$ such that type x and type x^* investor 1 both win with strictly positive probability. In the function $\Pi^{(1)}$, we temporarily omit the arguments $\sigma^{(2)}$ and μ with the understanding that $\sigma^{(2)}$ is the equilibrium strategy of investor 2 and μ is the equilibrium belief.

$$\Pi^{(1)}(\sigma^{(1)}(x^*)|x^*) \geq \Pi^{(1)}(\sigma^{(1)}(x)|x^*) > \Pi^{(1)}(\sigma^{(1)}(x)|x) > 0.$$

The first inequality follows the optimality of equilibrium offers. The second inequality follows Lemma 1.5 part (1).

Step 3. Type x^* investor 1 benefits from deviating to a security in S^B.

Let $S^A(s_x^A, \cdot)$ and $S^B(s_x^B, \cdot)$ denote the equilibrium offers of a type x investor 1 and a type x investor 2. Then for $y_2 > y_1$,

$$E[S^A(s_{x^*}^A, Z) - S^B(s_{y_1}^B, Z)|X_1 = x^*, X_2 = y_1] \geq 0$$

implies

$$E[S^A(s_{x^*}^A, Z) - S^B(s_{y_2}^B, Z)|X_1 = x^*, X_2 = y_2] > 0.$$

This is because

$$\begin{aligned} &E[S^A(s_{x^*}^A, Z)|X_1 = x^*, X_2 = y_2] \\ >&E[S^B(s_{y_1}^B, Z)|X_1 = x^*, X_2 = y_2] \\ \geq&E[S^B(s_{y_2}^B, Z)|X_1 = x^*, X_2 = y_2]. \end{aligned}$$

The first inequality is implied by Lemma 1.5 part (2) and the second inequality is because s_x^A is assumed to decrease in x.

Therefore, there is y^* such that for $y \gtreqless y^*$,

$$E[S^A(s_{x^*}^A, Z) - S^B(s_y^B, Z)|X_1 = x^*, X_2 = y] \gtreqless 0. \tag{1.19}$$

This implies if s_y^B is fully-revealing, type x^* investor 1 wins in equilibrium if and only if $X_2 < y^*$.

(1) Consider the case in which investor 2 with private signal $y > y^*$ and $y < y^*$ offer different securities. Then type x^* investor 1 wins when $X_2 < y^*$. By offering $S^B(s^{B*}, \cdot)$ with $s^{B*} = \sup\{s_y^B : y < y^*\}$, investor 1 also wins when $X_2 < y^*$. According to (1.19) and Lemma 1.5 part (2),

$$E[S^A(s_{x^*}^A, Z) - S^B(s^{B*}, Z)|X_1 = x^*, X_2 = y] < 0$$

for $y < y^*$. Therefore, type x^* investor 1 benefits from deviating to offering $S^B(s^{B*}, \cdot)$.

(2) Consider the case in which there is a range of x such that $s_x^B = s_{y^*}^B$. Define $\chi_2 \equiv \{x : s_x^B = s_{y^*}^B\}$. Consider each possibility regarding the sign of

$$E[S^A(s_{x^*}^A, Z) - S^B(s_{y^*}^B, Z)|X_1 = x^*, X_2 \in \chi_2]. \tag{1.20}$$

(a) Suppose (1.20) > 0. Then when investor 1 offers $S^A(s_{x^*}^A, \cdot)$ and investor 2 offers $S^B(s_{y^*}^B, \cdot)$, the entrepreneur accepts the offer of investor 2. Therefore, investor 1 with private signal x^* wins only when $X_2 < \inf \chi_2$.

$$\Pi^{(1)*}(x^*) = \int^{\inf \chi_2} E[S^A(s_{x^*}^A, Z) - I|X_1 = x^*, X_2 = y]g(y|x^*)dy.$$

Let $s^{B**} = \inf\{s_y^B : y < \inf \chi_2\}$. According to (1.19) and Lemma 1.5 part (2), for $y < \inf \chi_2$,

$$E[S^A(s_{x^*}^A, Z) - S^B(s^{B**}, Z)|X_1 = x^*, X_2 = y] < 0.$$

Therefore,

$$\int^{\inf \chi_2} E[S^A(s_{x^*}^A, Z) - S^B(s^{B**}, Z)|X_1 = x^*, X_2 = y]g(y|x^*)dy < 0. \tag{1.21}$$

First, suppose $s^{B**} > s^B_{y^*}$.

Then by offering $s^B \in (s^B_{y^*}, s^{B**})$, investor 1 also wins when $X_2 < \inf \chi_2$. By the continuity of (1.21) in s^{B**}, there is $s^B \in (s^B_{y^*}, s^{B**})$ such that

$$\int^{\inf \chi_2} E[S^A(s^A_{x^*}, Z) - S^B(s^B, Z)|X_1 = x^*, X_2 = y]g(y|x^*)dy < 0.$$

Therefore, a type x^* investor 1 benefits from deviating to $S^B(s^B, \cdot)$.

second, suppose $s^{B**} = s^B_{y^*}$. The definition of χ_2 implies $s^B_y > s^B_{y^*} = s^{B**}$ for $y < \inf \chi_2$, and the definition of s^{B**} implies s^B_y is left-continuous at $y = \inf \chi_2$. Therefore, by offering $S^B(s^B, \cdot)$ with s^B marginally higher than $s^B_{y^*} = s^{B**}$, investor 1 wins when $X_2 < y'$ for y' marginally smaller than $\inf \chi_2$.

$$\Pi^{(1)}(S^B(s^B, \cdot)|x^*) = \int^{y'} E[S^B(s^B, Z) - I|X_1 = x^*, X_2 = y]g(y|x^*)dy.$$

Since (1.21) is continuous in $\inf \chi_2$,

$$\Pi^{(1)*}(x^*) < \Pi^{(1)}(S^B(s^B, \cdot)|x^*),$$

that is type x^* investor 1 benefits from deviating to $S^B(s^B, \cdot)$.

(b) Suppose (1.20) $= 0$. Then by offering $S^A(S^A_{x^*}, \cdot)$, investor 1 wins when $X_2 < \inf \chi_2$, and wins with probability $1/2$ when $X_2 \in \chi_2$. If investor 1 offers $S^B(s^B_{y^*}, \cdot)$, she also wins in the same scenarios. (1.20) $= 0$ and inequality (1.19) imply that type x^* investor 1 cannot do worse by offering $S^B(s^B_{y^*}, \cdot)$. Since her equilibrium payoff is strictly positive, she strictly prefers offering $S^B(s^B, \cdot)$ with s^B slightly smaller than $s^B_{y^*}$ over offering $S^B(s^B_{y^*}, \cdot)$, because it increases the chance of winning discretely and decreases the payoff conditional on winning marginally. Therefore, type x^* investor 1 benefits from deviating to $S^B(s^B, \cdot)$.

(c) Suppose (1.20) < 0. Then by offering $S^A(S^A_{x^*}, \cdot)$, investor 1 wins when

$X_2 < \sup \chi_2$.

$$\Pi^{(1)*}(x^*) = \int^{\sup \chi_2} E[S^A(s^A_{x^*}, Z) - I | X_1 = x^*, X_2 = y] g(y|x^*) dy.$$

By offering $S^B(s^B, \cdot)$ marginally smaller than $S^B(s^B_{y^*}, \cdot)$, investor 1 wins when $X_1 < y'$ with y' equal to or marginally larger than $\sup \chi_2$.

$$\Pi^{(1)}(S^B(s^B, \cdot)|x^*) = \int^{y'} E[S^B(s^B, Z) - I | X_1 = x^*, X_2 = y] g(y|x^*) dy.$$

That (1.20) < 0 and inequality (1.19) imply

$$\int^{\sup \chi_2} E[S^A(s^A_{x^*}, Z) - S^B(s^B_{y^*}, Z) | X_1 = x^*, X_2 = y] g(y|x^*) dy < 0.$$

Since the above inequality is continuous in $\sup \chi_2$,

$$\Pi^{(1)*}(x^*) < \Pi^{(1)}(S^B(s^B, \cdot)|x^*),$$

that is a type x^* investor 1 benefits from deviating to $S^B(s^B, \cdot)$.

Lemma 1.7. *For an ordered set S and the investor strategy s_x that satisfies equation* (1.1) *and* (1.2),

$$\begin{aligned} &E[S_1(s_{x_L}, Z) | X_i = Y_i = x_L] \frac{ds_x}{dx}|_{x=x_L} \\ &= -\frac{N+2}{4} \frac{d}{dx} E[S(s_{x_L}, Z) | X_i = x, Y_i = x_L]\big|_{x=x_L}. \end{aligned} \tag{1.22}$$

Proof of Proposition 1.3: Suppose there is a symmetric monotonic sequential equilibrium in which all investors offer securities in an ordered set S that satisfies Condition 1.10. According to Lemma 1.2, the investor strategy s_x must satisfy equation (1.1) and (1.2). I will show that type x_L investor benefits from deviating to offering $S^D(s^{D\dagger}, \cdot)$ as defined in (1.9) under any belief of the entrepreneur.

Step 1. Condition (1.10) *implies*

$$\frac{d}{dy}E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x_L, \boldsymbol{X}_{-i} = y\mathbf{1}]\big|_{y=x_L} < 0, \tag{1.23}$$

where $\boldsymbol{X}_{-i}$ denotes the vector of investor private signals except X_i.

Rewriting the left hand side of(1.23),

$$\begin{aligned}
&\frac{d}{dy}E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x_L, \boldsymbol{X}_{-i} = y\mathbf{1}]\big|_{y=x_L} \\
&= (N-1)\frac{d}{dx}E[S^D(s^{D\dagger}, Z) - S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L} \\
&\quad - E[S_1(s_{x_L}, Z)|X_i = Y_i = x_L]\frac{ds_x}{dx}\big|_{x=x_L} \\
&= (N-1)\frac{d}{dx}E[S^D(s^{D\dagger}, Z) - S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L} \\
&\quad + \frac{N+2}{4}\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L}.
\end{aligned} \tag{1.24}$$

The second equation is due to Lemma 1.7. Regarranging (1.24) and substituting s_{x_L} by $s^\dagger$ (which are equal according to equation (1.2) and (1.9)) show that inequality (1.23) is equivalent to Condition (1.10).

Step 2. A type x_L investor can win with strictly positive probability by deviating to $S^D(s^{D\dagger}, \cdot)$.

According to equation (1.2) and (1.9),

$$E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|\boldsymbol{X} = x_L\mathbf{1}] = 0.$$

Inequality (1.23) implies there is $\bar{y} > x_L$ such that for $y \in (x_L, \bar{y})$,

$$E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x_L, \boldsymbol{X_{-i}} = y\mathbf{1}] < 0.$$

Let $Y_{i,m}$ be the m-th highest private signal among investors other than i. For each $y \in (x_L, \bar{y})$, there is $\epsilon > 0$ such that for $y_2, \cdots, y_{N-1} \in (y - \epsilon, y)$,

$$E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x_L, Y_i = y, Y_{i,2} = y_2, \cdots, Y_{i,N-1} = y_{N-1}] < 0.$$

Therefore, for each $y \in (x_L, \bar{y})$,

$$Pr(E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|\boldsymbol{X}] < 0|X_i = x_L, Y_i = y) > 0.$$

Integrating the above with respect to $G(y|x_L)$,

$$Pr(E[E[S^D(s^{D\dagger}, Z) - S(s_{Y_i}, Z)|\boldsymbol{X}] < 0|X_i = x_L) > 0,$$

i.e., by offering $S^D(s^{D\dagger}, \cdot)$, type x_L investor wins with strictly positive probabiltiy, if the entrepreneur believes $S^D(s^{D\dagger}, \cdot)$ is offered by type x_L investor.

According to Lemma 1.5 part (2), for $x > x_L$,

$$E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x_L, \boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}] < 0$$

implies

$$E[S^D(s^{D\dagger}, Z) - S(s_y, Z)|X_i = x, \boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}] < 0.$$

Therefore, if the entrepreneur believes $S^D(s^{D\dagger}, \cdot)$ is offered by a type higher than x_L with positive probability, type x_L investor can still win with strictly positive probabiltiy by offering $S^D(s^{D\dagger}, \cdot)$.

Step 3. Type x_L investor expects strictly positive payoff by offering $S^D(s^{D\dagger}, \cdot)$.

Equation (1.9) and Lemma 1.5 part (1) imply

$$E[S^D(s^{D\dagger}, Z) - I|X_i = x_L, \boldsymbol{X}_{-i} = \boldsymbol{x}_{-i}] > 0$$

for $\boldsymbol{x}_{-i} \neq x_L\mathbf{1}$. Therefore, an investor with private signal x_L can expect strictly positive payoff by offering $S^D(s^{D\dagger}, \cdot)$. Since she expects zero payoff in equilibrium, she benefits from deviating to $S^D(s^{D\dagger}, \cdot)$.

Proof of Proposition 1.4: Proposition 1.1 has established the entrepreneur in an informal auction can receive the same payoff as in a formal auction restricted to debt financing. It follows Lemma 1.3 that the entrepreneur revenue in a formal auction restricted to debt financing is the highest among formal auctions.

Proof of Proposition 1.5: I first show the debt strategy s_x^D that satisfies (1.1) and (1.2) supports a symmetric monotonic equilibrium. Since the strategy supports a symmetric monotonic equilibrium of a formal auction restricted to debt, no type benefits from deviating to another debt. It suffices to show no type benefits from deviating to an arbitrary non-debt security S.

Suppose type x investor 1 deviates to S while investor 2 follows the equilibrium strategy. Lemma 1.1 implies S crosses $S^D(s_y^D, \cdot)$ from below for any y if they are not unambiguously ordered. Therefore,

$$E[S(Z) - S^D(s_y^D, Z)|X_1 = x, X_2 = y] = 0$$

implies

$$\begin{aligned}
&\frac{d}{dy}E[S(Z) - S^D(s_y^D, Z)|X_1 = x, X_2 = y] \\
&= \frac{d}{dy}E[S(Z) - S^D(s_{y'}^D, Z)|X_1 = x, X_2 = y]|_{y'=y} \\
&\quad - E[S_1^D(s_y^D, Z)|X_1 = x, X_2 = y]\frac{ds_y^D}{dy} \\
&> 0.
\end{aligned}$$

The inequality is due to Lemma 1.5 and $\frac{ds_y^D}{dy} < 0$. Therefore, there is y^* such that for $y \gtrless y^*$,

$$E[S(Z) - S^D(s_y^D, Z)|X_1 = x, X_2 = y] \gtrless 0.$$

Since the entrepreneur observes both investors' private signals, she chooses investor 1's deviated offer if and only if $X_2 < y^*$. If type x investor 1 alternatively offers $S^D(s_{y^*}^D, \cdot)$, her offer is unambiguously ordered with investor 2's offer, and she again wins if and only if $X_2 < y^*$. By definition of y^*,

$$E[S(Z) - S^D(s_{y^*}^D, Z)|X_1 = x, X_2 = y^*] = 0.$$

Lemma 1.5 implies for $y < y^*$,

$$E[S(Z) - S^D(s_{y^*}^D, Z)|X_1 = x, X_2 = y] < 0,$$

that is conditional on $X_1 = x$ and $X_2 = y < y^*$, S is less valuable than $S^D(s_{y^*}^D, \cdot)$. Since type x investor 1 does not benefit from deviating to $S^D(s_{y^*}^D, \cdot)$, it cannot benefit from deviating to S.

I next show there is no symmetric monotonic equilibrium in which some type offers a non-debt security. Consider a symmetric monotonic equilibrium strategy in which type x investor offers $S(s_x, \cdot)$ from ordered set S. Lemma 1.2 implies s_x satisfies (1.1) and (1.2). Suppose there is type x such that $S(s_x, \cdot)$ is not debt. Then there is debt S^D such that $S(s_x, \cdot)$ crosses S^D from below and

$$E[S(s_x, Z) - S^D(Z)|X_1 = X_2 = x] = 0.$$

Let S^ϵ for $\epsilon \in (0, 1)$ be the combination of $S(s_x, \cdot)$ and S^D:

$$S^\epsilon(Z) = \epsilon S^D(Z) + (1 - \epsilon)S(s_x, Z).$$

This implies $S(s_x, \cdot)$ crosses S^ϵ from below, that is for $y \gtrless x$,

$$E[S^\epsilon(Z) - S(s_x, Z)|X_1 = x, X_2 = y] \lesseqgtr 0.$$

For $y \gtrless x$, since

$$E[S(s_x, Z) - S(s_y, Z)|X_1 = x, X_2 = y] \gtrless 0,$$

for ϵ is small enough,

$$E[S^\epsilon(Z) - S(s_y, Z)|X_1 = x, X_2 = y] \gtrless 0.$$

If type x investor 1 deviates to offering S^ϵ, it still wins when $X_2 < x$. Conditional on $X_2 < x$, security S^ϵ is strictly more valuable than $S(s_x, \cdot)$. Therefore, type x

investor 1 benefits from deviating to offering S^ϵ.

Proof of Proposition 1.6: I first show that the strategy that type $x > x_L$ investor offers to buy out the project at price b and type x_L investor offers security S^* such that $E[Z - S^*(Z)|X = x_L] = b$ supports an equilibrium. Under this strategy, type x investor has payoff $E[Z|X = x] - I - b$ for all x. Consider the entrepreneur belief that all securities except buyouts are offered by type x_L investor. Under this belief, the entrepreneur accepts a security S if and only if $E[Z - S(Z)|X = x_L] \geq b$, which implies $E[Z - S(Z)|X = x] > b$ for $x > x_L$ according to Lemma 1.5. By deviating to a security S that is not buyout at price b, type $x > x_L$ investor is either rejected or accepted with payoff

$$E[S(Z)|X = x] - I < E[Z|X = x] - I - b.$$

Therefore, type $x > x_L$ investor does not benefit from deviating. It is obvious that type x_L investor does not deviate.

I next show that there is no equilibrium in which some type other than x_L offers a security that is not buyout at price b. Since under any entrepreneur belief, type x investor can offer a buyout at price b and guarantee payoff $E[Z|X = x] - I - b$, her equilibrium payoff must be no smaller. This implies the entrepreneur always accepts the offer in equilibrium. It is obvious that no type offers a buyout at a different price. Suppose type $x^* > x_L$ investor offers a non-buyout security S. That she prefers S to buyout at price b implies

$$E[S(Z)|X = x^*] - I \geq E[Z|X = x^*] - I - b,$$

which implies $E[Z - S(Z)|X = x^*] \leq b$. Lemma 1.5 implies $E[Z - S(Z)|X = x] < b$ for $x < x^*$. Therefore, type $x < x^*$ investor can guarantee payoff

$$E[S(Z)|X = x] - I > E[Z|X = x] - I - b$$

by offering S. This implies the payoff of type x investor is no smaller than $E[Z|X = x] - I - b$ for all x and strictly larger for positive measure of x. This implies the entrepreneur's expected payoff is smaller than b, which contradicts that the entrepreneur always accepts the offer in equilibrium.

Proof of Proposition 1.7: The maximum total surplus is achieved when the project is started if and only if $V \geq v^*$ with v^* defined by $E[Z|V = v^*] = I + b$. The total surplus in this case is

$$\int_{v^*} E[Z - I - b|V = v] f_V(v) dv.$$

Since the entrepreneur cannot expect negative payoff in equilibrium, the investor payoff cannot exceed the total surplus.

According to the definition of C_ϵ, for $\epsilon > 0$, the entrepreneur accepts the offer S_ϵ if and only if $V \geq v^*$. Since

$$\lim_{\epsilon \to 0} S_\epsilon(Z) = Z - b,$$

the investor's payoff conditional on $V = v \geq v^*$ approaches $E[Z - I - b|V = v]$ when ϵ approaches zero.

1.7.3 Proof of Lemmas

Proof of Lemma 1.1: Let S be an arbitrary non-debt security, and $S^D(s^D, \cdot)$ be a debt with face value s^D. Suppose S and $S^D(s^D, \cdot)$ are not unambiguously ordered. To show S crosses $S^D(s^D, \cdot)$ from below, it suffices to show that $S(z^*) > S^D(s^D, z^*)$ implies $S(z) \geq S^D(s^D, z)$ for $z > z^*$, and $S(z^*) < S^D(s^D, z^*)$ implies $S(z) \leq S^D(s^D, z)$ for $z < z^*$.

Limited liability of the entrepreneur implies $S(z) \leq z = S^D(s^D, z)$ for $z \leq s^D$. Therefore, $S(z^*) > S^D(s^D, z^*)$ implies $z^* > s^D$. Monotonicity of $S(\cdot)$ implies

$S(z) \geq S(z^*)$ for $z > z^*$. Meanwhile, $S^D(s^D, z) = s^D = S^D(s^D, z^*)$ for $z > z^* > s^D$. Therefore, $S(z) > S^D(s^D, z)$ for $z > z^*$.

On the other hand, suppose $S(z^*) < S^D(s^D, z^*)$. Limited liability of the entrepreneur implies $S(z) \leq z = S^D(s^D, z)$ for $z \leq s^D$. If $z^* > s^D$, monotonoicity of $S(\cdot)$ implies $S(z) \leq S(z^*) < S^D(s^D, z^*) = s^D = S^D(s^D, z)$ for $z \in (s^D, z^*)$.

Lemma 1.8. *(Y_i, Z) satisfy the SMLRP.*

Proof. Since $F_{Y|Z}(y|z) = F_{X|Z}(y|z)^{N-1}$,

$$f_{Y|Z}(y|z) = (N-1)F_{X|Z}(y|z)^{N-2} f_{X|Z}(y|z).$$

Suppose $y_1 > y_2$. Since (X_i, Z) satisfy the SMLRP, $\frac{f_{X|Z}(y_1|z)}{f_{X|Z}(y_2|z)}$ strictly increases in z. Moreover,

$$f_{X|Z}(x_1|z_1)f_{X|Z}(x_2|z_2) > f_{X|Z}(x_1|z_2)f_{X|Z}(x_2|z_1)$$

for $x_1 > x_2$ and $z_1 > z_2$, which implies

$$\int_{y_2}^{y_1} f_{X|Z}(x_1|z_1)dx_1 \int^{y_2} f_{X|Z}(x_2|z_2)dx_2 > \int_{y_2}^{y_1} f_{X|Z}(x_1|z_2)dx_1 \int^{y_2} f_{X|Z}(x_2|z_1)dx_2.$$

Therefore ,

$$\frac{F_{X|Z}(y_1|z_1)}{F_{X|Z}(y_2|z_1)} > \frac{F_{X|Z}(y_1|z_2)}{F_{X|Z}(y_2|z_2)},$$

that is $\frac{F_{X|Z}(y_1|z)}{F_{X|Z}(y_2|z)}$ strictly increase in z. This implies $\frac{f_{Y|Z}(y_1|z)}{f_{Y|Z}(y_2|z)}$ strictly increases in z. □

Lemma 1.9. $\frac{\partial \ln f_{Z|\mathbf{X}}(z|\mathbf{x})}{\partial x_i}$, $\frac{\partial \ln f_{Z|X,Y}(z|x,y)}{\partial x}$ *and* $\frac{\partial \ln f_{Z|X,Y}(z|x,y)}{\partial y}$ *strictly increase in z.*

Proof.

$$\begin{aligned}\frac{\partial \ln f_{Z|X,Y}(z|x,y)}{\partial y} &= \frac{\partial \ln f_{Z,X|Y}(z,x|y)}{\partial y} - \frac{\partial \ln f_{X|Y}(x|y)}{\partial y} \\ &= \frac{\partial \ln f_{Z|Y}(z|y)}{\partial y} - \frac{\partial \ln f_{X|Y}(x|y)}{\partial y}\end{aligned}$$

The first equation is from the Bayes' rule. The second equation is because $f_{Z,X|Y}(z,x|y) = f_{X|Z}(x|z)f_{Z|Y}(z|y)$ due to the Bayes' rule and that X and Y are independent conditional on Z. According to Lemma 1.8, $\partial \ln \frac{f_{Z|Y}(z_1|y)}{f_{Z|Y}(z_2|y)}/\partial y > 0$ for $z_1 > z_2$, which implies $\frac{\partial \ln f_{Z|Y}(z|y)}{\partial y}$ strictly increases in z. Therefore, $\frac{\partial \ln f_{Z|X,Y}(z|x,y)}{\partial y}$ strictly increases in z.

That $\frac{\partial \ln f_{Z|\mathbf{X}}(z|\mathbf{x})}{\partial x_i}$ and $\frac{\partial \ln f_{Z|X,Y}(z|x,y)}{\partial x}$ strictly increase in z are proved in similar ways. □

Proof of Lemma 1.5:

(1) Since $M \leq S(z) \leq z$ for some finite number M and under part (e) of Assumption 1.1, the Dominated Convergence Theorem implies

$$\begin{aligned}
&\frac{d}{dx_i}E[S(Z)-I|\mathbf{X}=\mathbf{x}] \\
&= \int [S(z)-I]\frac{\partial f_{Z|\mathbf{X}}(z|\mathbf{x})}{\partial x_i}dz \\
&= \int [S(z)-I]\left[\frac{\partial f_{Z|\mathbf{X}}(z|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z|\mathbf{x})} - \frac{\partial f_{Z|\mathbf{X}}(z^*|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z^*|\mathbf{x})}\right] f_{Z|\mathbf{X}}(z|\mathbf{x})dz
\end{aligned}$$

for all z^*. The second equation is because $E[S(Z)-I|\mathbf{X}=\mathbf{x}] = 0$. Since $S(z)$ is increasing, we can pick z^* such that $S(z)-I \geq 0$ for $z > z^*$ and $S(z)-I \leq 0$ for $z < z^*$. Since $S(z) \leq z$, there is positive measure of z such that $S(z) \neq I$. According to Lemma 1.9, $\frac{\partial f_{Z|\mathbf{X}}(z|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z|\mathbf{x})} \gtrless \frac{\partial f_{Z|\mathbf{X}}(z^*|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z^*|\mathbf{x})}$ for $z \gtrless z^*$. Therefore,

$$[S(z)-I]\left[\frac{\partial f_{Z|\mathbf{X}}(z|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z|\mathbf{x})} - \frac{\partial f_{Z|\mathbf{X}}(z^*|\mathbf{x})/\partial x_i}{f_{Z|\mathbf{X}}(z^*|\mathbf{x})}\right] \geq 0$$

for $z \neq z^*$, with the inequality holds strictly for positive measure of z. This implies

$$\frac{d}{dx_i}E[S(Z)-I|\mathbf{X}=\mathbf{x}] > 0.$$

(2) The proof is identical to that of part (1), with $S(z)-I$ substituted by $S^1(z)-S^2(z)$. Since S^1 crosses S^2 from below, there is z^* such that $S^1(z)-$

$S^2(z) \geq 0$ for $z > z^*$ and $S^1(z) - S^2(z) \leq 0$ for $z < z^*$, and continuity of $S^1(z) - S^2(z)$ guarantees both inequalities hold strictly for positive measures of z.

Proof of Lemma 1.2: In step 1 I show the necessary conditions of a symmetric monotonic equilibrium. In step 2 I show the existence of a symmetric monotonic equilibrium under Condition (1.3).

Step 1. Necessary conditions of a symmetric monotonic equilibrium.

Let s_x be the equilibrium strategy in a symmetric monotonic equilibrium. In equilibrium, the entrepreneur accepts the offer with the lowest required s.

Step 1.1. s_x is strictly increasing.

Suppose s_x were constant on an interval. Then investor types in that interval expect to win for strictly positive probability. Participation implies every type in that interval expects non-negative profit upon winning. Part (1) of Lemma 1.5 implies some type on that interval expects strictly positive profit upon winning. Such a type strictly benefits from lowering her required security marginally, which decreases the winning payoff marginally but increases the probability of winning discretely.

Step 1.2. s_x is continuous.

Otherwise a type that requires a security just below a discontinuity could gain by increasing her bid.

Step 1.3. s_x is differentiable.

Let $\Pi(s, x)$ denote the expected payoff of a type x investor by bidding security $S(s, \cdot)$:

$$\begin{aligned} \Pi(s_{x'}, x) &= \int^{x'} E[S(s_{x'}, Z) - I | X_i = x, Y_i = y] g(y|x) dy \\ &= E[S(s_{x'}, Z) - I | X_i = x, Y_i \leq x'] G(x'|x). \end{aligned} \tag{1.25}$$

According to part (2) of Definition 1.3,

$$\frac{\partial}{\partial s}E[S(s,Z)|X_i=x,Y_i\le x']=E[S_1(s,Z)|X_i=x,Y_i\le x'].$$

Suppose $x'>x$, which implies $s_{x'}<s_x$. By the Mean Value Theorem, there is $s^*\in[s_{x'},s_x]$ such that

$$\begin{aligned}&E[S(s_x,Z)|X_i=x,Y_i\le x']-E[S(s_{x'},Z)|X_i=x,Y_i\le x']\\&=(s_x-s_{x'})E[S_1(s^*,Z)|X_i=x,Y_i\le x'].\end{aligned}$$

Therefore,

$$\begin{aligned}\Pi(s_{x'},x)=&\int^{x'}E[S(s_x,Z)-I|X_i=x,Y_i=y]g(y|x)dy\\&-(s_x-s_{x'})E[S_1(s^*,Z)|X_i=x,Y_i\le x']G(x'|x)\end{aligned}$$

The optimality of s_x implies $\Pi(s_x,x)\ge\Pi(s_{x'},x)$, which can be rewritten as

$$\frac{s_{x'}-s_x}{x'-x}\le-\frac{\int_x^{x'}E[S(s_x,Z)-I|X_i=x,Y_i=y]g(y|x)dy}{(x'-x)E[S_1(s^*,Z)|X_i=x,Y_i\le x']G(x'|x)}. \tag{1.26}$$

Similarly, there is $s^{**}\in[s_{x'},s_x]$ such that

$$\begin{aligned}&E[S(s_x,Z)|X_i=x',Y_i\le x']-E[S(s_{x'},Z)|X_i=x',Y_i\le x']\\&=E[S_1(s^{**},Z)|X_i=x',Y_i\le x'](s_x-s_{x'})\end{aligned}$$

and

$$\begin{aligned}\Pi(s_x,x')=&\int^{x}E[S(s_{x'},Z)-I|X_i=x',Y_i=y]g(y|x')dy\\&+(s_x-s_{x'})E[S_1(s^{**},Z)|X_i=x',Y_i\le x]G(x|x').\end{aligned}$$

The optimality of $s_{x'}$ implies $\Pi(s_{x'},x')\ge\Pi(s_x,x')$, which implies

$$\frac{s_{x'}-s_x}{x'-x}\ge-\frac{\int_x^{x'}E[S(s_{x'},Z)-I|X_i=x',Y_i=y]g(y|x')dy}{(x'-x)E[ES_1(s^{**},Z)|X_i=x',Y_i\le x]G(x|x')}. \tag{1.27}$$

Taking limits of (1.26) and (1.27) establishes differential equation (1.1).

Step 1.4. Boundary condition (1.2) *holds.*

Let $\underline{x}$ be the lowest type of investor that makes an offer. If $\underline{x} > x_L$, then a type $\underline{x}$ investor expects to win with strictly positive probability $G(\underline{x}|\underline{x})$, and thus strictly benefits from requiring a slightly larger security. Therefore $\underline{x} = x_L$.

A type x_L investor has zero chance to win. If $E[S(s_{x_L}, Z) - I|X_i = Y_i = x_L] > 0$, a type x_L investor benefits from marginally decreasing the required security, which allows her to win with strictly positive probability and expect a strictly positive profit upon winning. If $E[S(s_{x_L}, Z) - I|X_i = Y_i = x_L] < 0$, an investor with a signal slightly higher than x_L earns negative profit which violates participation.

Step 2. There is a symmetric monotonic equilibrium under condition (1.3).

It suffices to show that if investors follow strategy s_x in step 1, a type x investor's payoff by mimicking type x', $\Pi(s_{x'}, x)$, is decreasing (resp. increasing) in x' for $x' \gtrless x$. Since s_x is differentiable, $\Pi(s_{x'}, x)$ is differentiable in x'. It suffices to show $\frac{\partial \Pi(s_{x'},x)}{\partial x'} \lesseqgtr 0$ for $x' \gtrless x$.

$$\begin{aligned}
&\frac{\partial \Pi(s_{x'}, x)}{\partial x'} \\
&= E[S(s_{x'}, Z) - I|X_i = x, Y_i = x']g(x'|x) + E[S_1(s_{x'}, Z)|X_i = x, Y_i \leq x']G(x'|x)\frac{ds_{x'}}{dx'} \\
&= \int [S(s_{x'}, z) - I]f_{Y|Z}(x'|z)f_{Z|X}(z|x)dz + \int S_1(s_{x'}, z)F_{Y|Z}(x'|z)f_{Z|X}(z|x)dz\frac{ds_{x'}}{dx'} \\
&= \int a(z)f_{Z|X}(z|x)dz
\end{aligned}$$

where $f_{Z|X}(\cdot|x)$ denotes the density of Z conditional on $X_i = x$, and

$$a(z) \equiv [S(s_{x'}, z) - I]f_{Y|Z}(x'|z) + S_1(s_{x'}, z)F_{Y|Z}(x'|z)\frac{ds_{x'}}{dx'}.$$

According to condition (1.3), there is z^* such that $a(z) \gtrless 0$ for $z \gtrless z^*$.

According to equation (1.1), $\int a(z)f_{Z|X}(z|x')dz = 0$. Therefore,

$$\frac{\partial \Pi(s_{x'}, x)}{\partial x'} = \int a(z)\left[\frac{f_{Z|X}(z|x)}{f_{Z|X}(z|x')} - \frac{f_{Z|X}(z^*|x)}{f_{Z|X}(z^*|x')}\right]f_{Z|X}(z|x')dz.$$

Suppose $x' < x$. The SMLRP of (Z, X_i) implies $\frac{f_{Z|X}(z|x)}{f_{Z|X}(z|x')} - \frac{f_{Z|X}(z^*|x)}{f_{Z|X}(z^*|x')} \gtreqless 0$ for $z \gtreqless z^*$. Therefore, $\frac{\partial \Pi(s_{x'},x)}{\partial x'} > 0$. Similarly, $\frac{\partial \Pi(s_{x'},x)}{\partial x'} < 0$ for $x' > x$.

Proof of Lemma 1.6: According to equation (1.16),

$$\begin{aligned} & E^{\mu,S,y}[S^1(Z) - S^2(Z)] \\ & = \int [S^1(z) - S^2(z)] \underbrace{\int f_{Z|X,Y}(z|x,y) \frac{\hat{\mu}(S|x) f_{X|Y}(x|y)}{\int \hat{\mu}(S|u) f_{X|Y}(u|y) du} dx}_{\equiv \mu_{Z|Y,S}(z|y,S)} dz. \end{aligned}$$

Denote the underbraced part by $\mu_{Z|Y,X}(z|y,S)$.

Step 1. $\dfrac{\partial^2 \ln \mu_{Z|Y,S}(z|y,S)}{\partial y \partial z} > 0$.

That X_i and Y_i are independent conditional on Z implies

$$f_{Z|X,Y}(z|x,y) = \frac{f_{X|Z}(x|z) f_{Z|Y}(z|y)}{f_{X|Y}(x|y)}.$$

Therefore

$$\mu_{Z|Y,S}(z|y,S) = f_{Z|Y}(z|y) \cdot \frac{\int \hat{\mu}(S|x) f_{X|Z}(x|z) dx}{\int \hat{\mu}(S|u) f_{X|Y}(u|y) du}.$$

According to Lemma 1.8, (Y_i, Z) satisfies the SMLRP. Therefore

$$\frac{\partial^2 \ln \mu_{Z|Y,X}(z|y,S)}{\partial y \partial z} = \frac{\partial^2 \ln f_{Z|Y}(z|y)}{\partial y \partial z} > 0.$$

Step 2. Show the lemma.

Suppose $E^{\mu,S,y}[S^1(Z) - S^2(Z)] = 0$. Then

$$\begin{aligned} & \frac{d}{dy} E^{\mu,S,y}[S^1(Z) - S^2(Z)] \\ & = \int [S^1(z) - S^2(z)] \left[\frac{\partial \mu_{Z|Y,S}(z|y,S)/\partial y}{\mu_{Z|Y,S}(z|y,S)} - \frac{\partial \mu_{Z|Y,S}(z^*|y,S)/\partial y}{\mu_{Z|Y,S}(z^*|y,S)} \right] \mu_{Z|Y,S}(z|y,S) dz. \end{aligned}$$

for any z^*. Since S^1 crosses S^2 from below, let z^* be such that $S^1(z) \geq S^2(z)$ for $z > z^*$ and $S^1(z) \leq S^2(z)$ for $z < z^*$, and both inequalities hold strictly for positive measures of z. According to step 1, $\frac{\partial\mu_{Z|Y,S}(z|y,S)/\partial y}{\mu_{Z|Y,S}(z|y,S)}$ strictly increases in z. Therefore,

$$[S^1(z) - S^2(z)]\left[\frac{\partial\mu_{Z|Y,S}(z|y,S)/\partial y}{\mu_{Z|Y,S}(z|y,S)} - \frac{\partial\mu_{Z|Y,S}(z^*|y,S)/\partial y}{\mu_{Z|Y,S}(z^*|y,S)}\right] \geq 0,$$

and the inequality holds strictly for positive measure of z. This implies

$$\frac{d}{dy}E^{\mu,S,y}[S^1(Z) - S^2(Z)] > 0.$$

Proof of Lemma 1.7: *Step 1. Show that*

$$\begin{aligned} &\frac{d}{dx}E[S(s_{x_L}, Z)|X_i \leq x, Y_i = x_L]\big|_{x=x_L} \\ &= \frac{1}{2}\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L}. \end{aligned} \tag{1.28}$$

The left-hand side can be calculated as

$$\begin{aligned} &\frac{d}{dx}E[S(s_{x_L}, Z)|X_i \leq x, Y_i = x_L]\big|_{x=x_L} \\ &= \frac{d}{dx}\frac{\int^x E[S(s_{x_L}, Z)|X_i = u, Y_i = x_L]f_{X|Y}(u|x_L)du}{F_{X|Y}(x|x_L)}\Big|_{x=x_L} \\ &= \lim_{x\to x_L}\frac{f_{X|Y}(x|x_L)}{F_{X|Y}(x|x_L)}\bigg[E[S(s_{x_L}, Z) - I|X_i = x, Y_i = x_L] \\ &\quad - E[S(s_{x_L}, Z) - I|X_i \leq x, Y_i = x_L]\bigg] \\ &= \frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L} \\ &\quad - \frac{d}{dx}E[S(s_{x_L}, Z)|X_i \leq x, Y_i = x_L]\big|_{x=x_L}, \end{aligned}$$

which implies (1.28). The third equation is from the L'Hopital's rule.

Step 2. Show that

$$\begin{aligned} &\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x_L, Y_i = x]\big|_{x=x_L} \\ &= \frac{N}{2}\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\big|_{x=x_L}. \end{aligned} \tag{1.29}$$

The left-hand side can be calculated as

$$
\begin{aligned}
&\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x_L, Y_i = x]\Big|_{x=x_L} \\
&= \frac{d}{dx}E[S(s_{x_L}, Z)|X_1 = x_L, X_2 = x, X_3, \cdots, X_N \le x]\Big|_{x=x_L} \\
&= \frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\Big|_{x=x_L} \\
&\quad + (N-2)\frac{d}{dx}E[S(s_{x_L}, Z)|X_i \le x, Y_i = x_L]\Big|_{x=x_L}.
\end{aligned}
$$

Plugging (1.28) into the above leads to (1.29).

Step 3. Show that

$$
\begin{aligned}
&\frac{d}{dx}E[S(s_x, Z)|X_i = Y_i = x]\Big|_{x=x_L} \\
&= E[S_1(s_{x_L}, Z)|X_i = Y_i = x_L]\frac{ds_x}{dx}\Big|_{x=x_L} \\
&\quad + \frac{N+2}{2}\frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\Big|_{x=x_L}.
\end{aligned}
\tag{1.30}
$$

The left-hand side can be calculated as

$$
\begin{aligned}
&\frac{d}{dx}E[S(s_x, Z)|X_i = Y_i = x]\Big|_{x=x_L} \\
&= E[S_1(s_{x_L}, Z)|X_i = Y_i = x_L]\frac{ds_x}{dx}\Big|_{x=x_L} + \frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x, Y_i = x_L]\Big|_{x=x_L} \\
&\quad + \frac{d}{dx}E[S(s_{x_L}, Z)|X_i = x_L, Y_i = x]\Big|_{x=x_L}.
\end{aligned}
$$

Plugging (1.29) into the above leads to (1.30).

Step 4. Show (1.22).

Take the limit of equation (1.1) at $x = x_L$:

$$
\begin{aligned}
&E[S_1(s_{x_L}, Z)|X_i = Y_i = x_L]\frac{ds_x}{dx}\Big|_{x=x_L} \\
&= -\lim_{x\to x_L} E[S(s_x, Z) - I|X_i = Y_i = x]\frac{g(x|x)}{G(x|x)} \\
&= -\frac{\lim_{x\to x_L} dE[S(s_x, Z)|X_i = Y_i = x]/dx}{\lim_{x\to x_L} dG(x|x)/dx}g(x_L|x_L).
\end{aligned}
\tag{1.31}
$$

The second equation is from the L'Hopital's rule. Plugging (1.30) and

$$\lim_{x \to x_L} \frac{dG(x|x)}{dx}\Big|_{x=x_L} = g(x_L|x_L) + \underbrace{\frac{dG(x_L|x)}{dx}\Big|_{x=x_L}}_{0}$$

into (1.31) and rearranging lead to (1.22).

Chapter 2

Security Design with Two-Sided Asymmetric Information

2.1 Introduction

When firms raise capital under asymmetric information, security design has non-trivial implications on firm revenues. Firms may have inside information that is not available to investors, which leads to inevitable mispricing when firms issue securities to raise capital. Myers and Majluf (1984); Nachman and Noe (1994); DeMarzo and Duffie (1999) show informed firms pool on the same security design that minimizes mispricing, under which the firm can retain an information-sensitive stake. In particular, Nachman and Noe (1994) incorporate the market's belief about securities not issued in equilibrium, and establish conditions under which all firms pool on the same security due to the market's unfavourable belief about any off-equilibrium security design.

On the other hand, professional investors are repeated players in capital markets and may gain experience and expertise on evaluating businesses, which gives them private insight about firms' financial prospects. There is a literature on firms' design of security-bid auctions (Hansen, 1985; Crémer, 1987; Rhodes-Kropf and Viswanathan, 2000; DeMarzo et al., 2005; Axelson, 2007; Abhishek et al., 2015; Sogo et al., 2016)[1], which shows an uninformed firm optimally designs an auction in which bids are restricted to leaving information-sensitive stakes to the firm. Under the firm's optimal auction design, competition among investors is the most intense, which minimises their informational rent and maximises the firm's revenue.

In principle, both types of informational frictions can coexist in a market. A natural question follows: when both the firm and investors have private information, how does the firm's signalling incentive interact with its incentive to enhance competition among investors? This paper studies security design under this type of informational friction. The result shows that among a set of security designs whose information sensitivenss can be ranked by the concept of "strong steepness" first used by Abhishek et al. (2015), all firms pool on the strongly steepest security design. For example, among combinations of equity and cash payments, the set of equity with a smaller fixed cash payment is strongly steeper than the set of equity with a larger fixed cash payment, and the firm optimally forbids any cash payment.

The model considers a firm that needs to raise capital for an investment project. The project value is uncertain, about which both the firm and each potential investor has private information. The firm designs a second-price auction in which investors bid with securities from an ordered set chosen by the firm, such as the set of different shares of equity. The investor with the highest bid wins the investment opportunity and pays the security specified in second-highest bid to the firm. Based on the firm's choice of securities to allow in the auction, investors infer the firm's private information and decide their bidding strategies accordingly. If investors have positive belief about the firm's private information, they will bid more aggressively by offering to pay a larger security to the firm (or equivalently, requiring a smaller security in return for the investment). As a result, firms wish to signal high private valuation. On the other hand, as in

[1]These models can be extended to nest auctions to sell a fixed security in which investors bid prices.

models in which firms have no private information, conditional on its valuation being truly revealed, a firm expects higher revenue by restricting bids to steeper securities. This makes a flat security design a potential costly signalling device. A firm with high valuation may potentially be able to distinguish itself by choosing flatter securities which deter worse firms from mimicking.

However, when the firm's choice of ordered sets can be ordered by strong steepness, such an attempt to separate from worse firms cannot succeed in equilibrium. Consider a simple example as follows. Suppose the firm's private signal is either *high* or *low*. The firm can choose to either restrict all bids to *equity*, or restrict all bids to *cash* (which is equivalent to buyout offers). *Equity* is strongly steeper than *cash*. Conjecture an equilibrium in which a firm with signal *high* chooses *cash* and a firm with signal *low* chooses *equity*. When investors bid with *cash* under the belief that the firm' signal is *high*, the firm's revenue is independent of its true value since the value of cash is not sensitive to firm value. That no firm deviates implies a *low*-type firm's revenue from holding an *equity* auction is higher than either type of firm's revenue from a *cash* auction, which is in turn higher than a *high*-type firm's revenue from holding an *equity* auction. However, this leads to contradiction because the value of any equity offer is increasing in the firm's true signal.

Intuitively, the revenue-enhancing effect of choosing a strongly steeper ordered set is more significant if the true firm value is higher. Compared with a firm with a lower private valuation, it is even more costly for a better firm to choose a strongly flatter ordered set. If the equilibrium belief about a strongly flatter ordered set is good enough to attract a better firm, a worse firm is always willing to mimic. This destroys any separating equilibrium.

In addition to ruling out separating equilibria, I deploy the D1 refinement criterion of Cho and Kreps (1987) to select among pooling equilibria. Since a better type is more likely to prefer a strongly steeper ordered set, D1 regulates investors believe deviation to any ordered set strongly steeper than the equilibrium security design is made by the best firm, and deviation to any strongly flatter ordered set is made by the worst firm. Given the favourable beliefs attached to strongly steep securities, and that it is revenue enhancing to use steeper securities conditional on the firm's information is truly revealed, it is hardly surprising that all types of firms choose the strongly steepest ordered set in the unique D1 equilibrium.

2.2 Model Setup

This paper considers a model in which a penniless entrepreneur designs a security auction to finance an investment project. The project requires fixed investment I and generates random future payoff Z. There are N investors, each of whom has abundant capital for the investment. Each of the entrepreneur and investors observes a private signal. Let V denote the entrepreneur's private signal, and X_i for $i = 1, \cdots, N$ denote the private signal of investor i. Let $\boldsymbol{X} = (X_1, \cdots, X_N)$ denote the vector of all investors' private signals and $\boldsymbol{x} = (x_1, \cdots, x_N)$ the vector of signal values. The value of a private signal will also be referred to as the type of the entrepreneur or an investor.

Assumption 2.1. *The project payoff Z and the private signals $(V, X_1, \cdots, X_N)$ satisfy the following properties:*

(1) The entrepreneur's private signal V is distributed with probability density $f_V(\cdot)$ on support $[v_L, v_H]$;

(2) Each investor's private signal X_i is distributed with probability density $f_X(\cdot)$ on support $[x_L, x_H]$;

(3) All private signals $(V, X_1, \cdots, X_N)$ are independently distributed;

(4) Conditional on the values of private signals $(V, X_1, \cdots, X_N) = (v, x_1, \cdots, x_N)$, Z is distributed with probability density $f_{Z|V,\boldsymbol{X}}(\cdot|v, x_1, \cdots, x_N)$, which is symmetric in the last N arguments, and has full support on $[0, \infty)$.

(5) Each private signal (V or X_i) and Z satisfy the strict monotone likelihood ratio property (SMLRP) conditional on other private signals, i.e., the likelihood ratio $f_{Z|V,\boldsymbol{X}}(z|v, x_1, \cdots, x_N)/f_{Z|V,\boldsymbol{X}}(z'|v, x_1, \cdots, x_N)$ is strictly increasing in the last $N + 1$ arguments if $z > z'$;

(6) $E[Z|V = v_L, \boldsymbol{X} = x_L\mathbf{1}] - I > 0$.

Part (5) of the assumption implies each private signal is a positive signal of the project payoff Z. As part (6) implies, this paper focuses on projects that have positive NPV conditional on any profile of private signals.

The project can be financed by only one investor. The entrepreneur holds a second-price auction to decide which investor makes the investment and how the project payoff is allocated between the entrepreneur and the investor. In the auction, each investor makes a security offer s from a predetermined ordered set S. The investor that offers the highest s wins - she makes investment I, pays the entrepreneur the security offered in the second-highest bid, and own the remaining of the project. The definitions of security and ordered set of securities follow those in DeMarzo et al. (2005):

Definition 2.1. *A feasible security is a function* $S : [0, \infty) \mapsto \mathbb{R}$ *such that*

(1) (Limited Liability of the Entrepreneur) $S(z) \geq 0$*,*
(2) (Dual Monotonicity) $S(z)$ *and* $z - S(z)$ *weakly increase in* z*.*

Definition 2.2. *An ordered set of securities is a function* $S(s, \cdot)$ *with* $s \in [s_0, s_1]$ *such that*

(1) $S(s, \cdot)$ *is a feasible security,*
(2) $S(s, z)$ *is continuous and almost everywhere differentiable in* s*, and* $S_1(s, z) f_{Z|V,\boldsymbol{X}}(z|v, \boldsymbol{x})$ *is integrable on* $z \in [0, \infty)$*,*
(3) For $s' > s$*,* $S(s', z) \geq S(s, z)$ *for all* z *and* $S(s', z) > S(s, z)$ *for some* z*,*
(4) $E[Z - S(s_0, Z)|V = v_L, \boldsymbol{X} = x_L\mathbf{1}] > I$ *and* $E[Z - S(s_1, Z)|V = v_H, \boldsymbol{X} = x_H\mathbf{1}] < I$*.*

To describe the entrepreneur's choices in designing the security auction, I define "strong steepness" following Abhishek et al. (2015), which is a stronger version of the commonly-used concept of "steepness" first defined by DeMarzo et al. (2005).

Definition 2.3. *Ordered set* S^A *is strongly steeper than ordered set* S^B *if there are* s^A *and* s^B *such that* $S^A(s^A, z) - S^B(s^B, z)$ *assumes both positive and nevative* *values over $z \in [0, \infty)$, and for any such s^A and s^B, $S^A(s^A, z) - S^B(s^B, z)$ is non-decreasing in z.*

Let $\mathbb{S} = \{S(\lambda, s, z), \lambda \in [\lambda_L, \lambda_H]\}$ be a set of ordered sets such that $S(\lambda, \cdot, \cdot)$ is strongly steeper than $S(\lambda', \cdot, \cdot)$ if and only if $\lambda > \lambda'$. For examples of $\mathbb{S}$, consider a "slope-increasing" ordered set of securities S:

Definition 2.4. *An ordered set of securities S is slope-increasing if for any two securities $S(s, \cdot)$ and $S(s', \cdot)$ with $s > s'$, $S_2(s, z) \geq S_2(s', z)$ for all z and $S_2(s, z) > S_2(s', z)$ for some z.*

The set of debt, the set of equity and the set of levered equity are all slope-increasing ordered sets. If $S(\cdot, \cdot)$ is a slope-increasing ordered set, and $S(\lambda, s, z)$ with $-\lambda \geq 0$ represents an offer to invest I and pay cash $-\lambda$ and security $S(s, \cdot)$ to the entrepreneur ($S(\lambda, s, z) = S(s, z) + (-\lambda)$), then ordered set $S(\lambda, \cdot, \cdot)$ is strongly steeper than $S(\lambda', \cdot, \cdot)$ if and only if the component of cash payment is smaller: $-\lambda < -\lambda'$. The strongly steepest ordered set $\lambda_H = 0$ is the original ordered set $S(\cdot, \cdot)$ without any cash component.

The entrepreneur can choose an ordered set λ from set $\mathbb{S}$, so that offers in the auction are restricted to securities from ordered set $S(\lambda, \cdot, \cdot)$. For example, if $S(\lambda, s, z) = sz + (-\lambda)$ with $-\lambda \geq 0$ and $s \in [0, 1]$, then $S(\lambda, s, \cdot)$ represents the combination of cash $-\lambda$ and fraction s of the project's equity. In this case, $\mathbb{S}$ includes all combinations of equity payments and cash payments. The entrepreneur's auction design is to require a fixed cash payment $-\lambda$ in every offer. Investors decide the equity share s in their offers in the auction.

2.3 Equilibrium Characterisation

I first present investors' bidding strategies in a partial equilibrium in subsection 2.3.1, and then solve the entrepreneur's security design problem and characterise a full equilibrium.

2.3.1 Bidding Strategies

Consider investors' bidding strategies in an auction restricted to ordered set $S(\lambda, \cdot, \cdot)$. Suppose it is common knowledge among investors that the entrepreneur's private signal V has probability density $\mu(\cdot)$. I focus on a symmetric monotonic equilibrium of the auction, that is an equilibrium in which investors follow the same bidding strategy and an investor with a larger private signal offers a weakly larger security. As in a second-price cash auction with common values, investors offer their true values conditional on marginally winning:

Lemma 2.1. *An auction in which offers are restricted to ordered set $S(\lambda, \cdot, \cdot)$ and investors believe it is common knowledge that V has probability density $\mu(\cdot)$ has a symmetric monotonic equilibrium. An investor with private signal x offers security $S(\lambda, s_x^{\lambda,\mu}, \cdot)$ such that*

$$\int E\left[Z - S(\lambda, s_x^{\lambda,\mu}, Z)|X_i = Y_i = x, V = v\right]\mu(v)dv = I \tag{2.1}$$

where Y_i denotes the highest private signal among investors other than i.

2.3.2 Entrepreneur Revenue

If the true entrepreneur signal is v, while investors believe V follows distribution μ, the entrepreneur's expected payoff from an auction restricted to $S(\lambda, \cdot, \cdot)$ is

$$\Pi(\lambda, \mu, v) = \int\int S(\lambda, s_y^{\lambda,\mu}, z) f_{Z|V,X^{(2)}}(z|v, y)dz f_{X^{(2)}}(y)dy \quad . \tag{2.2}$$

where $X^{(i)}$ denotes the i-th highest private signal among $(X_1, \cdots, X_N)$, $f_{Z|V,X^{(2)}}(\cdot|v, y)$ denotes the probability density of Z conditional on $V = v$ and $X^{(2)} = y$, and $f_{X^{(2)}}(\cdot)$ is the probability density of $X^{(2)}$.

For convenience of comparing bidding strategies and entrepreneur payoffs under different beliefs, I assume that the probability density of Z conditional on $V = v$ and $\boldsymbol{X} = \boldsymbol{x}$, $f_{Z|V,\boldsymbol{X}}(\cdot|v, \boldsymbol{x})$, satisfies the following property:

Assumption 2.2. *For any $z, \boldsymbol{x}$ and probability density function μ,*

$$\int f_{Z|V,\boldsymbol{X}}(z|v,\boldsymbol{x})\mu(v)dv = f_{Z|V,\boldsymbol{X}}(z|\int v\mu(v)dv, \boldsymbol{x}).$$

Assumption 2.2 requires investors' posterior believed distribution of Z is affected by μ, their believed distribution of the entrepreneur's signal V, only through the believed expectation of V, $\int v\mu(v)dv$.

Lemma 2.2. *A type-v entrepreneur's revenue from an auction restricted to ordered set $S(\lambda,\cdot,\cdot)$ under investors' belief that V follows distribution μ, denoted by $\Pi(\lambda,\mu,v)$, satisfies:*

(1) $\Pi(\lambda,\mu,v)$ strictly increases in $\int v'\mu(v')dv'$;
(2) $\Pi(\lambda,\mu,v)$ is strictly increasing v unless $S(\lambda, s_x^{\lambda,\mu},\cdot)$ is purely cash for all x;
(3) $\Pi(\lambda,\mu,v)$ is strictly increasing in λ if $\int v'\mu(v')dv' \leq v$.

$\Pi(\lambda,\mu,v)$ increases in $\int v'\mu(v')dv'$ because $s_y^{\lambda,\mu}$ increases in $\int v'\mu(v')dv'$ according to (2.1) and Assumption 2.2. $\Pi(\lambda,\mu,v)$ is increasing in v because security value is increasing in Z and (V,Z) satisfy the SMLRP.

Suppose $\int v'\mu(v')dv' \leq v$. To compare entrepreneur revenue from two ordered sets $\lambda^1 > \lambda^2$, rewrite (2.2) as

$$\Pi(\lambda,\mu,v) = \int E\left[S(\lambda, s_y^{\lambda,\mu}, Z)|X_i \geq y, Y_i = y, V = v\right] f_{X^{(2)}}(y)dy \quad . \tag{2.3}$$

For a fixed y, notice that (2.1) implies the expected values of $S(\lambda^1, s_y^{\lambda^1,\mu},\cdot)$ and $S(\lambda^2, s_y^{\lambda^2,\mu},\cdot)$ are the same conditional on $X_i = Y_i = y$ and V follows distribution μ. Compared with these values of X_i, Y_i and V, the conditions $X_i \geq y$, $Y_i = y$ and $V = v$ imply an improvement on the distribution of Z. Since λ^1 is strongly steeper than λ^2, the increase in the value of $S(\lambda^1, s_y^{\lambda^1,\mu},\cdot)$ is larger than that in the value of $S(\lambda^2, s_y^{\lambda^2,\mu},\cdot)$. Therefore, $\Pi(\lambda^1,\mu,v) > \Pi(\lambda^2,\mu,v)$.[2]

[2]If $v < \int v\mu(v)dv$, it is unclear whether $X_i \geq y$, $Y_i = y$ and $V = v$ improves the distribution of Z over the conditions $X_i = Y_i = y$ and V follows μ. Therefore, $\Pi(\lambda,\mu,v)$ may or may not be monotonic in λ.

Under the assumption that the entrepreneur has no private information, DeMarzo et al. (2005) show the entrepreneur expects higher revenue by restricting offers to a steeper ordered set. Part (3) of Lemma 2.2 extends the result to situations where investors' belief is kept constant at a distribution that is worse than the entrepreneur's true type. In both cases, it is sub-optimal for the entrepreneur to restrict offers to an ordered set flatter than the steepest, $\lambda < \lambda_H$. The essence of the property is the Linkage Principle first formalized by Milgrom and Weber (1982): the more the payment to the entrepreneur is linked to the project value, the less the winning investor's payoff is linked to the same, which leads to more intense competition among investors and higher entrepreneur revenue.

2.3.3 Single-Crossing Property

This ranking of entrepreneur revenue opens up the question whether the entrepreneur may credibly signal high project valuation by choosing a flat ordered set of securities. As shown in Lemma 2.2, a better belief encourages investors to make higher offers in the auction and enhances the entrepreneur's revenue. If a flatter ordered set can induce a better belief, which outweighs the negative effect on revenue due to reduced competition, the entrepreneur may be willing to hold an auction with the flatter ordered set.

As in all signalling games, a signal sender may be able to credibly reveal its type with a costly signal only if a single crossing condition is satisfied, which guarantees a worse type does not mimic. In this game, the needed condition is that between an auction restricted to a steeper ordered set associated with a belief and an auction restricted to a flatter ordered set associated with another belief, if a lower-type entrepreneur prefers the former, a higher-type entrepreneur has the same preference strictly. However, the following lemma shows the opposite is true:

Lemma 2.3. *Consider two ordered sets $\lambda^1 > \lambda^2$. If a type v entrepreneur weakly prefers to hold an auction restricted to λ^1 under investors' belief that V follows* *distribution μ^1 rather than hold an auction restricted to λ^2 under investors' belief that V follows distribution μ^2, then a higher-type entrepreneur $v' > v$ has the same preference strictly.*

Intuitively, a higher-type entrepreneur puts more weight on high values of Z. For any realiasation of $X^{(2)}$, strategy $s_x^{\lambda,\mu}$ defined by (2.1) implies there is z such that

$$S(\lambda^1, s_{X^{(2)}}^{\lambda^1,\mu^1}, z) = S(\lambda^2, s_{X^{(2)}}^{\lambda^2,\mu^2}, z) = z - I.$$

This combined with the definition of strongly steeper implies $S(\lambda^1, s_{X^{(2)}}^{\lambda^1,\mu^1}, z) - S(\lambda^2, s_{X^{(2)}}^{\lambda^2,\mu^2}, z)$ increases in z. Therefore, compared with a lower-type entrepreneur, a higher-type entrepreneur has higher preference for $S(\lambda^1, s_{X^{(2)}}^{\lambda^1,\mu^1}, \cdot)$ over $S(\lambda^2, s_{X^{(2)}}^{\lambda^2,\mu^2}, \cdot)$ for each realisation of $X^{(2)}$.

2.3.4 Equilibrium Design of Security Auction

Let $\lambda^* : [v_L, v_H] \mapsto [\lambda_L, \lambda_H]$ denote the entrepreneur's auction design strategy, that is the entrepreneur chooses ordered set $\lambda^*(v)$ if her private signal is $V = v$. Let $\mu^* : [\lambda_L, \lambda_H] \times [v_L, v_H] \mapsto [0, \infty)$ denote investors' belief, that is investors assign probability density $\mu^*(\lambda, v)$ to entrepreneur type v when the entrepreneur chooses ordered set λ. An equilibrium is a pair of (λ^*, μ^*) that satisfies

$$\lambda^*(v) \in \arg\max_{\lambda} \Pi(\lambda, \mu^*(\lambda, \cdot), v), \tag{2.4}$$

and

$$\mu^*(\lambda, v) = \begin{cases} \frac{f_V(v)}{\int_{\lambda^*(v)=\lambda} f_V(v)dv}, & \text{if } \lambda^*(v) = \lambda \\ 0, & \text{o/w} \end{cases} \tag{2.5}$$

for λ such that $\{v : \lambda^*(v) = \lambda\}$ is not empty.

The single crossing property given in Lemma 2.3 is in the opposite direction of what supports separation in a signalling game. It implies all types of the entrepreneur will pool on the same ordered set λ in equilibrium.

Suppose there is an equilibrium in which ordered sets λ^1 and λ^2 with $\lambda^1 > \lambda^2$ are both chosen by non-empty sets of types of the entrepreneur. Consider type $\bar{v}$, the best type that chooses λ^2. It is either fairly valued or undervalued in equilibrium, and Lemma 2.2 implies it prefers λ^1 to λ^2 if both choices induce the equilibrium belief associated with λ^2. Lemma (2.3) implies an entrepreneur who chooses λ^1 in equilibrium has higher private signal than an entrepreneur who chooses λ^2, which implies λ^1 must be associated with better belief than λ^2. Lemma 2.2 implies by deviating to λ^1, type $\bar{v}$ can expect a payoff even higher than the hypothetical payoff from choosing λ^1 under the equilibrium belief about λ^2. As a result, this entrepreneur benefits from the deviation, which destroys such an equilibrium.

Proposition 2.1. *In any equilibrium, all types of the entrepreneur choose the same ordered set.*

Multiple pooling equilibria may exist due to arbitrarily unfavourable beliefs about off-equilibrium ordered sets. To select among these equilibria, I deploy the D1 refinement criterion of Cho and Kreps (1987) which prune "unreasonable" off-equilibrium beliefs. Fixing an equilibrium, let $\Pi^*(v)$ denote the equilibrium payoff of type v entrepreneur. For a fixed ordered set λ, let

$$D_v \equiv \{\mu : \Pi(\lambda, \mu, v) > \Pi^*(v)\}$$

and

$$D_v^0 \equiv \{\mu : \Pi(\lambda, \mu, v) = \Pi^*(v)\}.$$

In addition to (2.5), a D1 belief requires $\mu^*(\lambda, v) = 0$ if there is a type v' such that $D_v \cup D_v^0 \subseteq D_{v'}$ (where D_v, D_v^0 and $D_{v'}$ are with respect to λ). An equilibrium outcome is a D1 outcome if it can be supported by a D1 belief. Intuitively, D1 requires investors to believe a deviation to an off-equilibrium ordered set is done by those types of the entrepreneur that are most likely to make the deviation.

Proposition 2.2. *A unique D1 equilibrium outcome exists, in which all types of the entrepreneur choose the strongly steepest ordered set λ_H.*

D1 rules out pooling on any ordered set strongly flatter than λ_H. As shown in Lemma 2.2, the entrepreneur strictly prefers λ_H to λ if investors can observe V. In a conjectured equilibrium in which the entrepreneur always chooses ordered set $\lambda < \lambda_H$, Lemma 2.3 implies investors believe a deviation to λ_H is made by the best type of the entrepreneur under a D1 belief. Such a favourable belief attracts deviation. For example, the average type of the entrepreneur deviates to profit from both overvaluation and more intense competition among investors.

2.4 Appendix

In this appendix, proofs of propositions and lemmas are provided.

Proof of Lemma 2.1:

Suppose all investors other than investor i follow strategy $s_x^{\lambda,\mu}$ which satisfies (2.1). Conditional on $X_i = x$ and $Y_i = y$, investor i has expected payoff

$$U(x,y) = \int E[Z - S(\lambda, s_y^{\lambda,\mu}, Z)|X_i = x, Y_i = y, V = v]\mu(v)dv - I$$

from winning the project. Since (Z, X_i, Y_i, V) are affiliated, and $Z - S(\lambda, s_y^{\lambda,\mu}, Z)$ is strictly increasing in Z, $U(x,y)$ is strictly increasing in x. Since (2.1) implies $U(y,y) = 0$, $U(x,y) \gtreqless U(y,y) = 0$ when $x \gtreqless y$. It is optimal for investor i to win if and only if $x > y$. The only way to acheive this is to follow strategy $s_x^{\lambda,\mu}$.

Proof of Lemma 2.2:

1. Assumption 2.2 implies (2.1) can be rewritten as

$$E\left[Z - S(\lambda, s_x^{\lambda,\mu}, Z)|X_i = Y_i = x, V = \int v\mu(v)dv\right] = I. \tag{2.6}$$

Since (Z, X_i, Y_i, V) are affiliated, and $Z - S(\lambda, s_y^{\lambda,\mu}, Z)$ is strictly increasing in Z, $s_x^{\lambda,\mu}$ is strictly increasing in $\int v\mu(v)dv$. Therefore, $\Pi(\lambda, \mu, v)$ strictly increases in $\int v\mu(v)dv$.

2. Consider $v^1 > v^2$.

$$\begin{aligned}
&\Pi(\lambda, \mu, v^1) - \Pi(\lambda, \mu, v^2) \\
&= \int\int S(\lambda, s_y^{\lambda,\mu}, z) \left[f_{Z|V,X^{(2)}}(z|v^1, y) - f_{Z|V,X^{(2)}}(z|v^2, y)\right] dz f_{X^{(2)}}(y) dy
\end{aligned}.$$

For each y there is $z^*(y)$ such that

$$f_{Z|V,X^{(2)}}(z^*(y)|v^1, y) = f_{Z|V,X^{(2)}}(z^*(y)|v^2, y).$$

Since (Z, X_i, Y_i, V) are affiliated, $f_{Z|V,X^{(2)}}(z|v^1, y) \gtrless f_{Z|V,X^{(2)}}(z|v^2, y)$ for $z \gtrless z^*(y)$. Since

$$\begin{aligned}
&\int S(\lambda, s_y^{\lambda,\mu}, z^*(y)) \left[f_{Z|V,X^{(2)}}(z|v^1, y) - f_{Z|V,X^{(2)}}(z|v^2, y)\right] dz \\
&= S(\lambda, s_y^{\lambda,\mu}, z^*(y)) \int \left[f_{Z|V,X^{(2)}}(z|v^1, y) - f_{Z|V,X^{(2)}}(z|v^2, y)\right] dz, \\
&= 0
\end{aligned}$$

we have

$$\begin{aligned}
&\Pi(\lambda, \mu, v^1) - \Pi(\lambda, \mu, v^2) \\
&= \int\int \left[S(\lambda, s_y^{\lambda,\mu}, z) - S(\lambda, s_y^{\lambda,\mu}, z^*(y))\right] \left[f_{Z|V,X^{(2)}}(z|v^1, y) - f_{Z|V,X^{(2)}}(z|v^2, y)\right] dz f_{X^{(2)}}(y) dy
\end{aligned}.$$

Since $S(\lambda, s_y^{\lambda,\mu}, z)$ is increasing in z, $S(\lambda, s_y^{\lambda,\mu}, z) - S(\lambda, s_y^{\lambda,\mu}, z^*(y))$ and $f_{Z|V,X^{(2)}}(z|v^1, y) - f_{Z|V,X^{(2)}}(z|v^2, y)$ have the same sign. Therefore, $\Pi(\lambda, \mu, v^1) > \Pi(\lambda, \mu, v^2)$.

3. Suppose $\int v\mu(v)dv \leq v$. (2.6) implies

$$\int S(\lambda, s_y^{\lambda,\mu}, z) f_{Z|V,X,Y}\left(z| \int v\mu(v)dv, y, y\right) dz$$

is the same for any λ. Let

$$\Pi(\lambda, \mu, v, y) \equiv \int S(\lambda, s_y^{\lambda,\mu}, z) f_{Z|V,X^{(2)}}(z|v, y) dz \quad ,$$

then

$$
\begin{aligned}
&\Pi(\lambda,\mu,v,y) - \int S(\lambda, s_y^{\lambda,\mu}, z) f_{Z|V,X,Y}\left(z|\int v\mu(v)dv, y, y\right) dz \\
&= \int S(\lambda, s_y^{\lambda,\mu}, z)\left[f_{Z|V,X^{(2)}}(z|v,y) - f_{Z|V,X,Y}\left(z|\int v\mu(v)dv, y, y\right)\right] dz \\
&= \int S(\lambda, s_y^{\lambda,\mu}, z)\left[\frac{\int_y f_{Z|V,X,Y}\left(z|\int v\mu(v)dv, x, y\right) f_X(x)dx}{\int_y f_X(x)dx} - f_{Z|V,X,Y}(z|y, y, \int v\mu(v)dv)\right] dz \\
&= \int_y \int S(\lambda, s_y^{\lambda,\mu}, z) b(x,y,z) dz \frac{f_X(x)}{\int_y f_X(x)dx} dx
\end{aligned}
$$

where

$$
b(x,y,z) = f_{Z|V,X,Y}(z|x,y,v) - f_{Z|V,X,Y}(z|y,y,\int v\mu(v)dv).
$$

Let $z^*(x,y)$ be such that $b(x,y,z^*(x,y)) = 0$. Since (Z, X_i, Y_i, V) are affiliated, $b(x,y,z) \gtrless 0$ for $z \gtrless z^*(x,y)$.

For $\lambda^1 > \lambda^2$,

$$
\begin{aligned}
&\Pi(\lambda^1,\mu,v,y) - \Pi(\lambda^2,\mu,v,y) \\
&= \int_y \int \left[S(\lambda, s_y^{\lambda^1,\mu}, z) - S(\lambda, s_y^{\lambda^2,\mu}, z)\right] b(x,y,z) dz \frac{f_X(x)}{\int_y f_X(x)dx} dx \\
&= \int_y \int c(x,y,z) b(x,y,z) dz \frac{f_X(x)}{\int_y f_X(x)dx} dx
\end{aligned}
$$

where

$$
c(x,y,z) = \left[S(\lambda, s_y^{\lambda^1,\mu}, z) - S(\lambda, s_y^{\lambda^1,\mu}, z^*(x,y))\right] - \left[S(\lambda, s_y^{\lambda^2,\mu}, z) - S(\lambda, s_y^{\lambda^2,\mu}, z^*(x,y))\right].
$$

The second equation is because

$$
\begin{aligned}
&\int \left[S(\lambda, s_y^{\lambda^1,\mu}, z^*(x,y)) - S(\lambda, s_y^{\lambda^2,\mu}, z^*(x,y))\right] b(x,y,z) dz \\
&= \left[S(\lambda, s_y^{\lambda^1,\mu}, z^*(x,y)) - S(\lambda, s_y^{\lambda^2,\mu}, z^*(x,y))\right] \int b(x,y,z) dz \cdot \\
&= 0
\end{aligned}
$$

Since λ^1 is strongly steeper than λ^2, and (2.1) implies there is z such that

$S(\lambda, s_y^{\lambda^1,\mu}, z) = S(\lambda, s_y^{\lambda^2,\mu}, z)$, $S(\lambda, s_y^{\lambda^1,\mu}, z) - S(\lambda, s_y^{\lambda^1,\mu}, z^*(x,y))$ is larger than $S(\lambda, s_y^{\lambda^2,\mu}, z) - S(\lambda, s_y^{\lambda^2,\mu}, z^*(x,y))$ when $z > z^*(x,y)$ and smaller when $z < z^*(x,y)$. Therefore, $c(x,y,z)$ and $b(x,y,z)$ always have the same sign. This implies $\Pi(\lambda^1,\mu,v,y) > \Pi(\lambda^2,\mu,v,y)$. Since $\Pi(\lambda,\mu,v) = \int \Pi(\lambda,\mu,v,y) f_{X^{(2)}}(y)dy$, $\Pi(\lambda^1,\mu,v) > \Pi(\lambda^2,\mu,v)$.

Proof of Lemma 2.3:

That type v entrepreneur weakly prefers λ^1 under belief V follows distribution μ^1 to λ^2 under belief V follows distribution μ^2 implies

$$
\begin{aligned}
&\Pi(\lambda^1,\mu^1,v) - \Pi(\lambda^2,\mu^2,v) \\
&= \int\int \left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z\right)\right] f_{Z|V,X^{(2)}}(z|v,y)dz f_{X^{(2)}}(y)dy. \\
&\geq 0
\end{aligned}
$$

For $v' > v$,

$$
\begin{aligned}
&\Pi(\lambda^1,\mu^1,v') - \Pi(\lambda^2,\mu^2,v') \\
&\geq \left[\Pi(\lambda^1,\mu^1,v') - \Pi(\lambda^2,\mu^2,v')\right] - \left[\Pi(\lambda^1,v^1,v) - \Pi(\lambda^2,v^2,v)\right] \\
&= \int\int \left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z\right)\right] \\
&\quad \cdot \left[f_{Z|V,X^{(2)}}(z|v',y) - f_{Z|V,X^{(2)}}(z|v,y)\right] dz f_{X^{(2)}}(y)dy
\end{aligned}
. \quad (2.7)
$$

Since $\int f_{Z|V,X^{(2)}}(z|v',y)dz = \int f_{Z|V,X^{(2)}}(z|v,y)dz = 1$, for each y there is $z^*(y)$ such that

$$f_{Z|V,X^{(2)}}(z^*(y)|v',y) = f_{Z|V,X^{(2)}}(z^*(y)|v,y).$$

Since (Z,V) and (Z,X_i) both satisfy the SLMRP, $(Z,V,X^{(2)})$ are affiliated, which implies $f_{Z|V,X^{(2)}}(z|v',y) \gtrless f_{Z|V,X^{(2)}}(z|v,y)$ for $z \gtrless z^*(y)$. Since

$$\begin{aligned}
&\int \left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z^*(y)\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z^*(y)\right)\right]\left[f_{Z|V,X^{(2)}}(z|v',y) - f_{Z|V,X^{(2)}}(z|v,y)\right] dz \\
&= \left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z^*(y)\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z^*(y)\right)\right]\int \left[f_{Z|V,X^{(2)}}(z|v',y) - f_{Z|V,X^{(2)}}(z|v,y)\right] dz \\
&= \left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z^*(y)\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z^*(y)\right)\right] \cdot 0 \\
&= 0
\end{aligned},$$

(2.7) implies

$$\begin{aligned}
&\Pi(\lambda^1, \mu^1, v') - \Pi(\lambda^2, \mu^2, v') \\
&\geq \int\int a(z,y)\left[f_{Z|V,X^{(2)}}(z|v',y) - f_{Z|V,X^{(2)}}(z|v,y)\right] dz f_{X^{(2)}}(y) dy
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
a(z,y) = &\left[S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z\right) - S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z^*(y)\right)\right] \\
&- \left[S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z^*(y)\right)\right]
\end{aligned}.$$

Since $S(\lambda^1, \cdot, \cdot)$ is strongly steeper than $S(\lambda^2, \cdot, \cdot)$, $S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z\right) - S\left(\lambda^1, s_y^{\lambda^1,\mu^1}, z^*(y)\right)$ is larger than $S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z\right) - S\left(\lambda^2, s_y^{\lambda^2,\mu^2}, z^*(y)\right)$ when $z > z^*(y)$ and smaller when $z < z^*(y)$. This implies $a(z,y)$ in (2.8) has the same sign as $f_{Z|V,X^{(2)}}(z|v',y) - f_{Z|V,X^{(2)}}(z|v,y)$ for all z and y. Therefore, $\Pi(\lambda^1, \mu^1, v') - \Pi(\lambda^2, \mu^2, v') > 0$, which implies type v' entrepreneur strictly prefers λ^1 under belief μ^1 to λ^2 under belief μ^2.

Proof of Proposition 2.1:

Suppose in contrast, there is an equilibrium in which there two ordered sets $\lambda^1 > \lambda^2$ that are both chosen by non-empty sets of types of the entrepreneur. Let μ^1 and μ^2 denote the equilibrium belief associated with λ^1 and λ^2. Let $\bar{v}$ be a type of the entrepreneur that chooses λ^2 that satisfies $\bar{v} \geq \int v\mu(\lambda^2, v)dv$. Lemma 2.2 implies

$$\Pi(\lambda^1, \mu(\lambda^2, \cdot), \bar{v}) > \Pi(\lambda^2, \mu(\lambda^2, \cdot), \bar{v}).$$

Lemma (2.3) implies an entrepreneur who chooses λ^1 in equilibrium has higher private signal than an entrepreneur who chooses λ^2, which implies $\int v\mu(\lambda^1, v)dv \geq \int v\mu(\lambda^2, v)dv$. Lemma 2.2 implies

$$\Pi(\lambda^1, \mu(\lambda^1, \cdot), \bar{v}) > \Pi(\lambda^1, \mu(\lambda^2, \cdot), \bar{v}).$$

This implies

$$\Pi(\lambda^1, \mu(\lambda^1, \cdot), \bar{v}) > \Pi(\lambda^2, \mu(\lambda^2, \cdot), \bar{v}) = \Pi^*(\bar{v}),$$

that is type $\bar{v}$ benefits from deviating to λ^1, which contradicts that this is an equilibrium.

Proof of Proposition 2.2:

Consider an equilibrium in which all types of the entrepreneur choose ordered set $\lambda < \lambda_H$. Lemma 2.3 implies with respect to a deviation to λ_H, $D_v \cup D_v^0 \subseteq D_{v_H}$. Therefore, D1 requires $\mu^*(\lambda_H, \cdot)$ has support only on v_H. For type v_H, Lemma 2.2 implies

$$\Pi\left(\lambda_H, \mu^*(\lambda_H, \cdot), v_H\right) > \Pi\left(\lambda, \mu^*(\lambda_H, \cdot), v_H\right) > \Pi^*\left(v_H\right),$$

that is type v_H benefits from deviating to λ_H. Therefore, $\lambda < \lambda_H$ cannot supported by a D1 belief.

Consider an equilibrium in which all types of the entrepreneur choose ordered set λ_H. Lemma 2.3 implies with respect to a deviation to any ordered set $\lambda < \lambda_H$, $D_v \cup D_v^0 \subseteq D_{v_L}$. Therefore, D1 requires $\mu^*(\lambda, \cdot)$ has support only on v_L. For any type v, Lemma 2.2 implies

$$\Pi^*\left(v\right) > \Pi\left(\lambda_H, \mu^*(\lambda, \cdot), v\right) > \Pi\left(\lambda, \mu^*(\lambda, \cdot), v\right),$$

that is type v does not benefit from deviating to λ. Therefore, no type of the entrepreneur deviates under the belief that any deviation is made by type v_L, which is a D1 belief.

Chapter 3

Share Issues versus Share Repurchases

3.1 Introduction

Public firms often tap into the equity market – they issue new shares to fund valuable investment opportunities and repurchase existing shares, returning surplus cash to investors who may utilize it more efficiently. In many ways, issuing and repurchasing shares are mirror images of each other. Both directions of equity transaction frequently occur under the informational friction of firms knowing more than outside investors. And for both issues and repurchases, firms choose both transaction size and method (speed).

In this paper we show that equilibrium *outcomes* in issue and repurchase transactions *are not* mirror images of each other. We obtain three main results. First, issues and repurchases exhibit a sharp asymmetry in the use of method: issuing firms signal via method, generating heterogeneity in equilibrium issue methods, while repurchasing firms do not, so that repurchase methods are homogeneous. Second, there is no corresponding asymmetry in transaction size choices: for both transaction types, transaction size is related to firm quality. Third: Since issuing firms signal via both method and size, how are the two choices related? We establish an unambiguous ordering: firms prefer to inefficiently reduce transaction size rather than to choose a more inefficient method. All three predictions are consistent with empirical observation. Looking ahead, the central economic force behind the asymmetry result is that issuing firms want to raise investors' perceptions of their value, while repurchasing firms instead want the opposite.

In more detail, we build such a unified framework based on Myers and Majluf (1984), allowing firms to choose both transaction size and method when they issue or repurchase shares. More specifically, firms privately know the value of their assets in place, whereas investors in the market only know the distribution. Firms have a positive NPV "project" that can only be implemented through trading equity. If the project requires a positive investment, the firm needs to raise capital by issuing shares. In contrast, if the "investment" is negative, then the firm needs to pay out capital by repurchasing equity. The positive NPV in the latter case can be interpreted as the part of free cash flows that the managers will waste if the capital is not paid out. The project is scalable and generates higher NPV if more capital is deployed (raised or paid out) up to a boundary. Firms can choose the size of the project (or equivalently equity transaction) and the transaction method which differs in its efficiency – the NPV generated per unit of capital deployed in the project. The firm's objective is to maximize its long-term shareholders' payoff. Investors in the market price shares competitively upon observing the transaction size and method.

While the specific labeling is not crucial for our theoretical insight, for concreteness and empirical relevance, we consider four methods, representing fast/slow ways to issue/repurchase equity, respectively. Firms can raise equity quickly in an SEO, which typically completes in 2-8 weeks (Gao and Ritter, 2010). Alternatively, they can issue gradually through at-the-market offerings (ATM) over a couple of years. Billett et al. (2019) provides a nice review of this growing popular issue method. On the flip side, repurchases can be carried out either swiftly in tender offers (henceforth, TOR, often lasting for a month (Masulis, 1980)) or slowly via open market repurchase (OMR) programs.

The model generates several surprising yet empirically consistent predictions. First, despite our symmetric fashion to model share issues and repurchases (as negative issues), speed asymmetry arises robustly in equilibrium between the two directions. Specifically, firms with different qualities can choose different issue methods (efficiency) to separate from one another, but all pool on the same method when they repurchase shares. This prediction maps well to the empirical observation that both SEOs and ATMs coexist as frequently observed issue methods, whereas OMR dominates the repurchase market.[1] The reason why issuing firms can separate by having different transaction efficiency is standard. Firms prefer to issue at a higher price, and better firms therefore have incentive to sacrifice NPV (by using a less efficient method) and reveal their superior types. These better firms are also more capable to bear this signaling cost as the NPV sacrificed features a smaller fraction of the total firm value. The more interesting insight is why repurchasing firms cannot do the same, namely, sacrificing NPV in exchange for a more favorable price. This is because when repurchasing, it is the worse firms who would like to reveal themselves and enjoy a lower repurchase price. However, they cannot achieve separation in equilibrium by sacrificing efficiency, as better firms who are less averse to sacrificing NPV would always mimic.

On the flip side, the transaction size can be a viable signal in both issues and repurchases. The result for the issue game again follows from the standard retention signaling intuition in the literature (Leland and Pyle, 1977; Krasker, 1986; DeMarzo and Duffie, 1999): Good firms issue less to signal their types and receive a higher issue price. However, the repurchase equilibrium is more interesting: Worse firms repurchase less at a lower price. As two possible signals in the repurchase game, how is size different from efficiency, such that firms can separate on size but not on efficiency? Like a less efficient repurchase method, a smaller

[1] Billett et al. (2019) document that ATMs represent 63% incidence and 26% issue proceeds of those for SEOs. In contrast, there are 1212 cases of open market repurchases in 1999, with a total size of 137 billion dollars, and tender offers and dutch auctions only account for 21 and 19 cases, and 1.7 billion dollar and 3.8 billion dollar proceeds respectively (see Grullon and Ikenberry (2000)).

size reduces the NPV thereby making it a possible signal, but unlike efficiency, repurchasing a smaller amount increases firm value as the firm retains more cash. Worse firms value the extra cash more as it represents a bigger fraction of the firm. Consequently, worse firms have both higher preference for repurchasing less as a signal and the incentive to separate from better firms to enjoy a lower repurchase price, sustaining the in equilibrium separation on size.

This comparison between the two signaling channels delivers the third insight of our model: Firms use transaction size as the primary signal and only use transaction efficiency when the separation on size becomes infeasible. Intuitively, both a decrease in transaction size and a decrease in transaction efficiency reduce NPV and thus may deter other firms from mimicking. Keeping the project NPV and issue price fixed, shares are likely to be overpriced for a worse firm but underpriced for a better firm, and thus a worse firm is more reluctant to decrease the issue size. Therefore, a lower issue size deters mimicking from worse firms on top of the effect of reduced NPV. The NPV sacrifice sufficient to deter worse firms from mimicking is smaller if the firm makes the sacrifice by lowering issue size rather than by lowering transaction efficiency. Size is thus given priority over transaction efficiency as a signaling tool. We view this result as a valuable contribution to the literature as the majority features only one dimensional signaling, that is either retention (size) such as in Leland and Pyle (1977); Krasker (1986); DeMarzo and Duffie (1999) or efficiency, but not both.

Next, for empirical relevance, we microfound the efficiency associated with the transaction methods (SEO and ATM when firms issue, OMR and TOR when repurchase) in Section 3.4. The faster issue method SEO is more efficient than ATM as the former allows the firm to immediately implement the project, whose NPV might disappear over time. The gradual repurchase method OMR is more efficient relative to TOR since the former allows the firm to pay out free cash flows as soon as they become available, whereas a lumpy TOR until all cash flows are realized risks managers' wasting of some cash flows. Under this interpretation, the model delivers the empirical prediction that larger issues are usually carried out more quickly in SEOs.[2] This empirical fact could be surprising at first glance as one might imagine firms may prefer to divide large issues into smaller pieces and issue more gradually. In the context of our model, firms tend to use size as a signal first. Worst firms issue the maximum amount in the most efficient method SEO. As the quality of the firm improves, they issue less until reaching the minimum issue size, at which point they start to reduce efficiency by switching to ATMs with different speeds. As a result, large sizes are correlated with faster issue methods, consistent with the empirical observation.

Finally, we appreciate that firms' preference solely for long-term shareholders' payoff may be extreme, as it implies that firms would prefer their share price to crash in repurchases which seems unnatural. We alleviate this concern by extending our model and allow the firm to favor for both a higher share price directly and the payoff to long-term shareholders. Our key results remain robust.

3.2 Model Setup

We construct a model that accommodates both share issuances and repurchases in a unified framework. Consider a firm with assets in place a and an opportunity to invest i in a new project. The value of assets in place a is the firm's private information, whereas others only know the value is drawn from a distribution $F(a)$ that is differentiable everywhere and has support on $[a_{\min}, a_{\max}]$. We refer to a as the firm's type.

The firm chooses project size i to lie in the closed interval between I_L and I, where I_L and I are exogenous constants that are common knowledge. Either $I > I_L \geq 0$, in which case the project is an investment project; or $I < I_L \leq 0$, in which case the project is a divestment project. The case $|I_L| > 0$ corresponds to

[2] We calculate from Table 2 of Billett et al. (2019) that the average proceeds per SEO are 256 million dollars, whereas average proceeds per ATM program are 92 million dollars. Even though the ratio of proceeds to market equity is roughly the same between the two methods (18% for SEO and 20% for ATM), it is significantly smaller for ATM than for SEO after controlling for other factors.

a minimum project size, which arises for investment projects if the project has a minimum scale, and divestment projects if the firm is compelled to pay out at least a minimum amount of cash (for example, if retaining cash above some level would lead to extremely wasteful spending).[3]

The investment choice i is associated with equity transactions: Investment projects ($i > 0$) require funding and hence share issues, while divestment projects ($i < 0$) produce cash to be paid out via repurchases. (For unmodeled reasons, the firm prefers to raise funding via equity to other securities, and to pay out cash via repurchases rather than dividends.)

In addition to project size i, the firm can also choose among equity transaction methods with different levels of efficiencies, captured by the variable $\theta \in [0, 1]$, with efficiency increasing in θ. For concreteness, we interpret different efficiency levels as various speeds that firms can choose to issue or repurchase equity. For instance, firms can transact quickly through standard SEOs and tender offer repurchases or slowly through at-the-market offerings and open market repurchases. We provide more details on the efficiency of different speeds and draw empirical implications in Section 3.4.

Denote by $V(a, i, \theta)$ the total value of a type a firm after choosing project size i and the efficiency level of the equity transaction θ. While we can accommodate more general function forms, for simplicity, we assume

$$V(a, i, \theta) = a + i + |i|\theta b,$$

[3]If $I_L > 0$, so that investment projects are being analyzed, then one might also want to allow the possibility of the firm simply doing nothing, i.e., $i \in \{0\} \cup [I_L, I]$. We have fully analyzed this case, and it does not yield any additional economic insights relative to $i \in [I_L, I]$. Both to avoid distracting complexity in the statements of our results, and also to preserve symmetry across issuance and repurchase analysis, we present our results for the case in which $|I_L| > 0$ indeed precludes the possibility of doing nothing. Effectively, for $I_L > 0$ (the issue setting) we are assuming, in terms of formal objects defined below, that

$$\frac{V(a_{\max}, I_L, 1)}{1 + \frac{I_L}{V(a_{\min}, I_L, 1) - I_L}} > V(a_{\max}, 0, 1), \tag{3.1}$$

i.e., the best firm prefers issuing I_L at full efficiency but at the most unfavorable price that can be supported in equilibrium over the alternative of doing nothing; along with the analogous assumption for repurchase ($I_L < 0$).

where $b \in (0,1)$ is a constant that captures the most efficient level of NPV per unit project. For the repurchase case ($I < I_L \leq 0$), value creation b stems from cash being more valuable in the hands of shareholders than the firm, either because of internal agency problems in the firm, or because of shareholders' liquidity needs.

The number of shares outstanding before any issue or repurchase is normalized to 1. Given an equity transaction price p, the firm needs to issue $\frac{i}{p}$ shares (or repurchase $\frac{-i}{p}$ shares if $i < 0$) in order to implement the project. The firm maximizes the payoff of its long-term investors, which is given by

$$\Pi(a, i, \theta, p) = \frac{V(a, i, \theta)}{1 + \frac{i}{p}}. \tag{3.2}$$

Upon observing the firm's investment choice i and transaction efficiency choice θ, competitive investors in the market update their beliefs about the firm type a and set the transaction price $P(i, \theta)$, so that they expect to break even.

We focus on pure strategy equilibria, which consist of each firm-type's choices of transaction size $i^*(a)$ and efficiency $\theta^*(a)$ and competitive investors' pricing function $P^*(i, \theta)$ such that the following two conditions hold. 1. Given $P^*(i, \theta)$, a type a firm's equilibrium strategy $i^*(a)$ and $\theta^*(a)$ maximizes its long-term shareholders' value:

$$(i^*(a), \theta^*(a)) \in \arg\max_{i,\theta} \Pi(a, i, \theta, P^*(i, \theta))$$

2. The pricing function $P^*(i, \theta)$ is consistent with firms' strategies, i.e., $P^*(i, \theta) = E\left[\Pi(a, i, \theta, P^*(i, \theta)) | i, \theta\right]$, or equivalently, given (3.2),

$$P^*(i, \theta) = E[V(a, i, \theta) | i, \theta] - i.$$

As in many signaling models, we have multiple equilibria. We employ the widely accepted D1 criterion (Cho and Kreps, 1987) to eliminate those equilibrium outcomes that are supported by "unreasonable" off-equilibrium beliefs. Broadly speaking, the D1 criterion requires that the belief associated with any off-equilibrium action must be supported on the set of types that are most likely to deviate to that action. Formally, let $\Pi^*(a)$ denote the equilibrium payoff of a type a firm. Given investment size i and efficiency θ, define

$$D_a = \{p : \Pi(a, i, \theta, p) > \Pi^*(a)\}$$

and

$$D_a^0 = \{p : \Pi(a, i, \theta, p) = \Pi^*(a)\}.$$

Let $\mu(a|i, \theta)$ denote the probability density that investors attribute to firm type a if the firm chooses investment i and efficiency θ. For type a, if there exists a second type a' such that $D_a \cup D_a^0 \subseteq D_{a'}$, then a D1 belief must have $\mu(a|i, \theta) = 0$. An equilibrium strategy (i^*, θ^*) is a D1 equilibrium outcome if it can be supported by a pricing function P such that $P(i, \theta) = \int (V(a, i, \theta) - i)\mu(a|i, \theta)da$ under a D1 belief μ. It is worth noting one of our key results – the speed (efficiency) asymmetry between share issue and repurchases (Proposition 3.3) – does not rely on equilibrium refinement.

3.3 Equilibrium Characterization

Unlike retention signaling models (Leland and Pyle, 1977; Myers and Majluf, 1984; DeMarzo and Duffie, 1999), we allow firms to use both the transaction size and efficiency to signal their types. We first show that firm types tend to use size to separate from one another before sacrificing transaction efficiency – the key result of Section 3.3.1. Next, we fully characterize the equilibrium outcomes of both the issue (Section 3.3.2) and repurchase games (Section 3.3.3). The key finding is that separation in transaction efficiency is possible only when firms issue shares but not when they repurchase.

3.3.1 Size v.s. Efficiency

We first show that under D1 refinement, firms prefer to signal by lowering transaction size rather than sacrificing transaction efficiency. This is true for both the issue and repurchase settings.

Proposition 3.1. *In any D1 equilibrium any firms that choose a transaction size* $|i| > |I_L|$ *use the most efficient method* $\theta = 1$.

Proposition 3.1 implies that any firm that uses efficiency to signal its type by choosing some $\theta < 1$ must have exhausted the option to signal through size, i.e., the transaction is already at the minimum size $|I_L|$. This result follows from the D1 refinement criterion. We explain the intuition for a type that fully separates in the issuance case, noting that the argument for pooling and repurchases is similar. Suppose to the contrary that a D1 equilibrium exists in which some firm a chooses a higher than minimum issue size $i > I_L$ and a less than fully efficient method $\theta < 1$. Consider any off-equilibrium deviation (i', θ') that leads to a lower firm value $V(a, i', \theta') < V(a, i, \theta)$. By D1, the beliefs associated with this deviation are no worse than a. The reason is that with lower firm value, such a deviation is attractive only if it induces strictly less dilution, $i'/p' < i/p$; and less dilution is in turn strictly more valuable for better firms. Hence, such deviations are at least fairly priced for firm a, and so firm a can profitably deviate by choosing a deviation from this class that also raises NPV ($V(a, i', \theta') - i' > V(a, i, \theta) - i$).

Intuitively, both a decrease in transaction size and a decrease in transaction efficiency reduce NPV and thus may deter other firms from mimicking. Keeping the project NPV and issue price fixed, shares are likely to be overpriced for a worse firm but underpriced for a better firm, and thus a worse firm is more reluctant to decrease the issue size. Therefore, a lower issue size deters mimicking from worse firms on top of the effect of reduced NPV. The NPV sacrifice sufficient to deter worse firms from mimicking is smaller if the firm makes the sacrifice by lowering issue size rather than by lowering transaction efficiency. Size is thus given priority over transaction efficiency as a signaling tool. An analogous intuition applies to repurchases.

3.3.2 Issuance

In this subsection, we characterize the equilibrium of the issuance setting ($I > I_L \geq 0$). The key is Proposition 3.1's statement that firms signal by scaling down the project in preference to signaling by adopting inefficient methods. From this result, there is an interval of firms that separate by issue size, potentially followed by an interval of firms that issue the minimum amount $i = I_L$, and separate by inefficient issue methods. The details of the equilibrium construction are essentially standard:

First, there is no distortion at the bottom: the worst firm $a_{\min}$ issues the maximum size ($i = I$) at maximum efficiency.

Second, firms in the interval above $a_{\min}$ separate by scaling down the project, while retaining maximal issuance efficiency $\theta = 1$. Given separation, issues are fairly priced, i.e., $P = V - i$. Writing $\hat{\imath}(a)$ for firm a's issue strategy, firm a's payoff from mimicking the issue strategy of firm $\tilde{a}$ is

$$\frac{V\left(a, \hat{\imath}\left(\tilde{a}\right), 1\right)}{1 + \frac{\hat{\imath}(\tilde{a})}{V\left(\tilde{a}, \hat{\imath}(\tilde{a}), 1\right) - \hat{\imath}(\tilde{a})}}.$$

As standard, the equilibrium condition is that firm a doesn't gain from mimicking neighboring firms, so that equilibrium strategy $\hat{\imath}\left(a\right)$ solves the differential equation

$$\frac{\partial}{\partial \tilde{a}} \left(\frac{V\left(a, \hat{\imath}\left(\tilde{a}\right), 1\right)}{1 + \frac{\hat{\imath}(\tilde{a})}{V\left(\tilde{a}, \hat{\imath}(\tilde{a}), 1\right) - \hat{\imath}(\tilde{a})}} \right)_{\tilde{a}=a} = 0. \tag{3.3}$$

subject to the boundary condition $\hat{\imath}\left(a_{\min}\right) = I$. By straightforward manipulation, (3.3) simplifies to

$$\frac{\partial \hat{i}(a)}{\partial a} = -\frac{\hat{i}(a)}{V_i(a,\hat{i}(a),1)-1}\frac{V_a(a,\hat{i}(a),1)}{V(a,\hat{i}(a),1)} = -\frac{\hat{i}(a)}{V(a,\hat{i}(a),1)b}, \tag{3.4}$$

where the second equality simply reflects the functional form of V.

The economic force behind separation is similar to in Leland and Pyle (1977), viz., better firms separate by retaining a larger fraction of equity, which is less costly for them.

Third, separation on issue size according to (3.4) continues as long as there is room. Specifically, if $\hat{i}\left(a_{\max}\right) \geq I_L$, all firms issue, separating on issue size, and the equilibrium characterization is complete; for use in Proposition 3.2, define $\hat{a} = a_{\max}$. If instead there is a such that $\hat{i}(a) = I_L$, define $\hat{a}$ as the smallest value of a such that $\hat{i}\left(a\right) = I_L$.

Firms in the interval above $\hat{a}$ issue the minimum amount I_L, and instead separate by adopting more inefficient methods. Writing $\hat{\theta}\left(a\right)$ for firm a's issuing strategy, for firms $a > \hat{a}$ the equilibrium strategy $\hat{\theta}\left(a\right)$ solves the differential equation

$$\frac{\partial}{\partial \tilde{a}}\left(\frac{V\left(a, I_L, \hat{\theta}\left(\tilde{a}\right)\right)}{1+\frac{I_L}{V\left(\tilde{a}, I_L, \hat{\theta}(\tilde{a})\right)-I_L}}\right)_{\tilde{a}=a} = 0, \tag{3.5}$$

subject to the boundary condition $\hat{\theta}\left(\hat{a}\right) = 1$. Equation (3.5) simplifies to

$$\frac{\partial \hat{\theta}(a)}{\partial a} = -\frac{I_L}{V_\theta\left(a, I_L, \hat{\theta}(a)\right)}\frac{V_a\left(a, I_L, \hat{\theta}(a)\right)}{V\left(a, I_L, \hat{\theta}(a)\right)} = -\frac{1}{V\left(a, I_L, \hat{\theta}(a)\right)b}. \tag{3.6}$$

Under the assumption that the best firm prefers issuing I_L with method $\theta = 1$ under the worst belief to doing nothing, there is enough room on efficiency θ for all types above $\hat{a}$ to fully separate.

Summarizing:

Proposition 3.2. *The issue game ($I > I_L \geq 0$) has a unique D1 equilibrium, in* *which firms* $a \in [a_{\min}, \hat{a}]$ *issue* $\hat{\imath}(a)$ *in the most efficient way (*$\theta = 1$*), and firms* $i \in (\hat{a}, \bar{a}]$ *issue* $i = I_L$ *at efficiency* $\hat{\theta}(a)$*, where* $\hat{a}$*,* $\hat{\imath}(\cdot)$*, and* $\hat{\theta}(\cdot)$ *are as defined above.)*

We highlight that D1 rules out pooling on any issue size and efficiency level. Consider a candidate equilibrium that entails pooling and an off-equilibrium path deviation that leads to a lower firm value (by either reducing size or efficiency). Intuitively, because better firms care less about change in firm value and are more concerned about dilution, the set of prices that make such a deviation attractive is larger for better firms. Accordingly, D1 means that beliefs associated with relevant off-path deviations heavily weight good firms. Consequently, the better firms associated with any pooling action would deviate. In contrast, as we show below, pooling is possible in the repurchase game.

One of the key takeaways when firms issue equity is that they can separate using issue methods with different efficiency levels (in addition to size). As we will show in the next subsection, such a separation becomes impossible when firms repurchase equity, and the only signaling possibility is through the size of repurchase.

Another takeaway is that firms primarily signal with retention from trading and secondarily with transaction efficiency. Up to our knowledge, this is a new insight in the literature on multiple signals. Besides the different signals under study, a prominent difference between our paper and most literature on multiple signals is the equilibrium concept we use. A signaling game usually has multiple equilibria due to arbitrarily unfavorable off-equilibrium beliefs, and equilibrium refinement criterions like D1 can sometimes select a unique outcome by pruning unreasonable off-equilibrium beliefs. When agents can send multiple signals, the task of determining "reasonable" off-equilibrium beliefs becomes particularly challenging due to the multi-dimensional (and hence large) space of off-equilibrium actions. Instead of regulating off-equilibrium beliefs, most of the literature on multiple signals look for the Pareto-optimal one among separating equilibria[4] (John and Williams, 1985; Ambarish et al., 1987; Besanko and Thakor, 1987; Ofer and Thakor, 1987; Viswanathan, 1987; Williams, 1988). On the other hand, in a general signaling model with multiple signals, Engers (1987); Cho and Sobel (1990); Ramey (1996) establish the existence and uniqueness of a D1 equilibrium, and show it coincides with the Pareto-optimal separating equilibrium. Our exposition to find the unique D1 equilibrium resembles that of (Ramey, 1996), but is not a direct application of Ramey's work. An important part of our work is to characterize the transition between the use of the two signals (transaction size and efficiency) when the space of one is exhausted, whereas Ramey's model precludes this possibility by assuming there is no upper boundary (analogous to the lower boundary on investment size in our model) on signals. In other words, the literature provides the interior solution while we derive a corner solution with meaningful economic implications.

Finally, we characterize the analytical solutions to the ODEs given by (3.4) and (3.6):

$$\hat{\imath}(a)^b \left(a + \hat{\imath}(a)b\right) = I^b \left(a_{\min} + Ib\right), \tag{3.7}$$

and

$$e^{b\hat{\theta}(a)} \left(a + I_L b\hat{\theta}(a)\right) = e^b \left(\hat{a} + I_L b\right). \tag{3.8}$$

As a technical note, when $I_L = 0$, the solution $\hat{\imath}$ in (3.7) never reaches $I_L = 0$ for any domain $[a_{\min}, a_{\max}]$. Therefore, the cutoff type $\hat{a}$ stated in Proposition 3.2 is $a_{\max}$, and all firms separate on issue size according to $\hat{\imath}$.

3.3.3 Repurchase

We now turn our attention to the case in which firms wish to pay out capital by repurchasing shares ($I < I_L \leq 0$). While it may be tempting to conjecture that the repurchase equilibrium is a mirror-image of the issue equilibrium, this

[4] The Pareto-optimal separating equilibrium is also called the "Riley equilibrium".

is not the case. In particular, repurchasing firms are unable to separate using transaction efficiency.

Proposition 3.3. *In the repurchase game ($I < I_L \leq 0$), all firms that repurchase the same size i choose the same efficiency θ.*

We highlight that Proposition 3.3 covers all equilibria, and is independent of the D1 refinement.

Recall that when firms issue equity, those firms that issue the minimum amount I_L separate by choosing different efficiency levels. Proposition 3.3 rules out such a possibility when firms repurchase, that is no firms in equilibrium can repurchase the same amount of equity with different transaction efficiency. Underlying the separating outcome in share issuances is the property that firms that are more eager to reveal their types (good types) are also more willing to sacrifice NPV by reducing efficiency (or size) in exchange for a more favorable price. This property, known as the single-crossing condition or Spence-Mirrlees condition (Mirrlees, 1971; Spence, 1973), enables efficiency (or size) as costly signals to distinguish types in equilibrium. To understand Proposition 3.3, it is useful to decompose the firms' objective function,

$$\ln \Pi(a, i, \theta, P^*(i, \theta)) = \ln V(a, i, \theta) - \ln\left(1 - \frac{-i}{P^*(i, \theta)}\right). \tag{3.9}$$

The first term corresponds to the percentage change in total firm value, while the second corresponds to the percentage change in the number of shares. When repurchasing shares, it is the worse firms who prefer to reveal their types and buy back shares at lower prices. Suppose that worse firms attempt to separate by adopting some less efficient method $\theta' \equiv \theta - \Delta\theta < \theta$ in exchange for a lower repurchase price $P(i, \theta')$. On the one hand, the resulting sacrifice in total firm value V is $\Delta\theta|i|b$, which represents a smaller fraction of a better firm. On the other hand, the percentage change in the number of shares is independent of firm type. Consequently, the lower efficiency choice θ' is more attractive for good firms than bad firms, and so separation of this type is impossible in equilibrium.

In contrast, firms can still separate on transaction size when they repurchase, just as in the issue game (Proposition 3.2).

Proposition 3.4. *In a D1 equilibrium outcome of the repurchase game ($I < I_L \leq 0$), if $a_{\min}$ and $a_{\max}$ are sufficiently close, then firms separate on size according to*

$$\frac{\partial \tilde{i}(a)}{\partial a} = \frac{\tilde{i}(a)}{V(a, \tilde{i}(a), 1)b}, \tag{3.10}$$

with the boundary condition

$$\tilde{i}(a_{max}) = I. \tag{3.11}$$

When $a_{\min}$ and $a_{\max}$ are sufficiently different such that (3.10) and (3.11) imply $\tilde{i}(a) = I_L$ for some $a \in (a_{\min}, a_{\max})$, only an upper interval of firms separate on size according to $\tilde{i}$ – a case we fully characterize in Proposition 3.5.

In words, Proposition 3.4 states: The best firm $a_{\max}$ repurchases the maximum amount I. Worse firms separate by scaling down the divestment "project," in order to lower the repurchase price. Note that the differential equation (3.10) coincides with the first equality in the analogous separation condition (3.4) in the issue game, but has a different sign because $V_i(a, i, \theta) - 1 = -\theta b$ in the repurchase setting (as opposed to $V_i(a, i, \theta) - 1 = \theta b$ in the issue setting). Proposition 3.4 also builds on Proposition 3.1's result that firms repurchasing strictly more than the minimum level I_L choose maximal efficiency $\theta = 1$.

Why can repurchasing firms separate using size i though they cannot separate using efficiency θ (Proposition 3.3)? In the repurchase setting, it is worse firms that wish to separate themselves from better firms so as to be able to acquire shares at a lower price. Consider a firm that offers a smaller repurchase size $|i'| \equiv |i| - |\Delta i|$, in order to obtain a lower price. While this smaller repurchase lowers the NPV of the divestment project by $\theta b|\Delta i|$, it raises firm value by $|\Delta i|(1 - \theta b)$, since the firm retains more cash. From the decomposition (3.9), this represents a larger fraction of firm value for worse firms. Since the effect on the number of shares is the same for all firm types, this makes the smaller repurchase size most attractive for worse firms.

Propositions 3.3 and 3.4 represent the principle insights of this subsection. First, there is a sharp asymmetry between equilibrium efficiency choices in issue and repurchase settings, namely that efficiency is not a viable means of separation for repurchasing firms. Second, there is no corresponding asymmetry with respect to the choice of transaction size; in particular, both repurchasing and issuing firms modulate transaction size in order to separate from other firms and obtain more favorable transaction prices.

The remainder of the subsection completes the characterization of repurchase equilibria. Issue equilibria and repurchase equilibria share the features that there are upper (repurchase) and lower (issue) intervals of firms that transact, using transaction size as a separation device; and potentially lower (repurchase) and upper (issue) intervals of firms that pool at the minimum transaction size I_L. The difference between issue and repurchase equilibria is that issuing firms that pool on I_L then use efficiency as a further means of separation, while repurchasing firms do not.

Equation (3.10) characterizes the form that separation on repurchase size takes. If $a_{\min}$ is sufficiently close to $a_{\max}$ that (3.10) leads to repurchases above the minimum size I_L ($i < I_L$, i.e., $|i| > I_L$) for all firms $a > a_{\min}$, then Proposition 3.4 is already a complete description of the repurchase equilibrium. For use in Proposition 3.5, define $\hat{a} = a_{\min}$.

The remaining case in which $\tilde{i}(\cdot)$ hits this minimum repurchase level $I_L < 0$ before $a_{\min}$ is reached is more complicated. As a first step, it is instructive to note that it *cannot* be an equilibrium for separation to continue according to (3.10) all the way until the minimum repurchase size $I_L < 0$ is hit. The reason is that in such a case, there is an interval of firms immediately below the separating firms that pool on the minimum repurchase size I_L. But firms marginally better than the firms pooling on I_L would gain by deviating and reducing their repurchases very slightly to I_L, since doing so generates a discrete price reduction.

Instead, the equilibrium consists of a cutoff type $\hat{a}$. Firms better than $\hat{a}$ separate according to (3.10). As discussed immediately above, the separation region ends before the minimum repurchase I_L is hit, i.e., $\tilde{i}(\hat{a}) < I_L$. Firms below $\hat{a}$ pool and repurchase the minimum amount, I_L; and so in particular, repurchase discretely less than all the separating firms above $\hat{a}$. The cutoff $\hat{a}$ is determined by that firm $\hat{a}$ is indifferent between repurchasing $\tilde{i}(\hat{a}) < I_L$ at the separating price $P = V\left(\hat{a}, \tilde{i}(\hat{a}), 1\right) - \tilde{i}(\hat{a})$ and repurchasing I_L at the pooling price $P = E\left[V(a, I_L, 1) | a \in (\bar{a}, \hat{a})\right] - I_L$:

$$V\left(\hat{a}, \tilde{i}(\hat{a}), 1\right) - \tilde{i}(\hat{a}) = \frac{V(\hat{a}, I_L, 1)}{1 + \frac{I_L}{E[V(a, I_L, 1)|a \in [a_{\min}, \hat{a})] - I_L}}. \tag{3.12}$$

If there is no $\hat{a}$ with $\tilde{i}(\hat{a}) < I_L$ that satisfies (3.12), it implies all types prefer repurchasing I_L at the price pooled with lower types to repurchasing its separating amount $\tilde{i}(a)$ at the fair price $V\left(a, \tilde{i}(a), 1\right) - \tilde{i}(a)$. In this case, $\hat{a} = a_{\max}$, and all types repurchase I_L.

Summarizing:

Proposition 3.5. *The repurchase game ($I < I_L \leq 0$) has a unique D1 equilibrium, in which firms with $a > \hat{a}$ separate and repurchase according to $\tilde{i}(\cdot)$ defined by (3.10) and (3.11), and firms $a < \hat{a}$ pool at the minimum repurchase size I_L. All repurchases take place at maximal efficiency, $\theta = 1$.*

Finally, we characterize the analytical solution to the ODE given by (3.10) and (3.11),

$$\tilde{i}(a)^{-b}\left(a - \tilde{i}(a)b\right) = I^{-b}\left(a_{\max} - Ib\right). \tag{3.13}$$

Similar to the issue game, we note a technical observation that when $I_L = 0$, the solution $\tilde{i}$ in (3.13) never reaches $I_L = 0$. In this case, the cutoff type $\hat{a}$ is $a_{\min}$, and all firms separate on their size choice according to $\tilde{i}$.

3.4 Empirical Implications

In this section, we explore the empirical implications of our model. There are broadly speaking two ways to issue seasoned equity in practice. The first method is a fast one-off SEO which is typically completed within several weeks.[5] A lesser known but growing popular method is at-the-market offering (henceforth, ATM). (Billett et al., 2019) provides a nice review of ATMs. In an ATM, the firm first registers new shares with the SEC, and then anonymously sell these shares gradually in the secondary market. Compared to SEOs, ATMs take much longer to complete, on average 6.2 quarters, and the proceeds are 20% of the firm's market value of equity. Similarly, firms can repurchase equity in a quick one-off fashion through tender offer repurchase (henceforth TOR) within a month.[6] Alternatively, they can carry out an open-market repurchase program (henceforth, OMR) over the horizon of several years.[7]

To map the equity transaction methods in our model to those in practice, we build a microfoundation of transaction efficiency associated with each issue method. The key implication of this exercise is that the fast issue method (SEO) and the gradual repurchase method (OMR) are more efficient compared with ATM and TOR respectively. Intuitively, once firms decide on the scale of the project, they can only implement it after the required capital for investment is raised. A prompt SEO therefore guarantees an investment project is implemented with no delay, whereas a gradual ATM risks losing the valuable investment opportunity. In contrast, when firms need to efficiently pay out free cash flow in the form of share repurchases, they can only do so when the cash flow is realized. To the extend that cash flow is typically generated gradually, a long-horizon OMR

[5]A nonshelf bookbuilt SEO often takes 2-8 weeks, while an accelerated bookbuilt SEO often takes 2 days from announcement to completion (Gao and Ritter, 2010; Huang and Zhang, 2011). SEO proceeds is on average 18% of market value of equity. (Billett et al., 2019)

[6]In a TOR, firms on average repurchase 16% of outstanding shares (1962-1986) (Lakonishok and Vermaelen, 1990). It takes 25 days on average from announcement of an TOR to the expiration of the offer (Masulis, 1980).

[7]On average, firms target to repurchase 7% of outstanding shares in three years, and 46.2%, 66.9%, and 73.9% of the target amount is completed by end of the first, second, and third year, respectively (Stephens and Weisbach, 1998).

program ensures a timely payout. If firms wait until all free cash flows are realized and then pay out in a TOR, some of the early free cash flows could have been wasted. The following microfoundation of the transaction efficiency θ details this intuition.

First, consider a firm that faces an investment opportunity at time 0 which requires capital outlay i. The firm needs to raise i through equity issuance. Suppose it can choose a timeframe t to complete the issuance, and the project can be implemented only after capital i is fully raised. Every instant before the project is implemented, a competitor may arrive with probability (intensity) α, reducing the project's NPV to a negligible level. As such, if the firm raises the required equity capital over a period of t, the expected NPV is $\theta(t)ib$ where

$$\theta(t) = Prob(\text{a competitor does not arrive before } t) = e^{-\alpha t}.$$

In this specification, $\theta(t) \in (0, 1]$ is strictly decreasing in t with $\theta(0) = 1$ and $\lim_{t\to\infty} \theta(t) = 0$. A faster issue method (i.e., a smaller t) corresponds to higher efficiency. SEO corresponds to fast issuance in a short period, which yields high project NPV, and ATM corresponds to slow issuance over a long period, during which the investment is delayed and profitability of the project has been decreased.

Consider a firm that generates free cash flows continuously through time at speed λ. If not paid out to shareholders, the free cash flow is deployed in bad projects and decays exponentially at rate β. This implies the free cash flow generated during period t accumulates to $\int_0^t \lambda e^{-\beta(t-s)} ds = \frac{\lambda}{\beta}(1 - e^{-\beta t})$ at the end of the period if it is not paid out. Suppose over the period between 0 and T, the firm chooses a total amount $|i| \leq \frac{\lambda}{\beta}(1 - e^{-\beta T})$ to pay out through repurchases. The firm can choose the frequency of repurchase $x \in \mathbb{N}^+$, such that it pays out $\frac{|i|}{x}$ at time t, $2t$, $\cdots$, T for $t = \frac{T}{x}$. At time T, the firm will have paid out $|i|$ and accumulated cash balance $\frac{\lambda}{\beta}(1 - e^{-\beta T}) - \frac{|i|}{x} \frac{1-e^{-\beta T}}{1-e^{-\frac{\beta T}{x}}}$. Compared with the scenario in which the firm does not repurchase, the repurchase program generates NPV

$\theta(x) = |i|\left(1 - \frac{1}{x} \cdot \frac{1-e^{-\beta T}}{1-e^{-\frac{\beta T}{x}}}\right) > 0$. Notice $\theta(x)$ increases in x. $\theta(1) = 0$ and $\lim_{x\to\infty} \theta(x) = |i|\left(1 - \frac{1-e^{-\beta T}}{\beta T}\right)$. An OMR repurchase program corresponds to a frequent repurchase program with a large x, which leads to a higher NPV than an infrequent repurchase such as a TOR. Based on our interpretation of the transaction method θ, the model generates several empirically consistent predictions.

Prediction 1: Both SEO and ATM coexist when firms issue equity, whereas OMR dominates when firms repurchase equity.

Lemma 3.5 shows that firms can separate using transaction method θ when they issue equity. Instead, Proposition 3.3 shows such separation is impossible when firms repurchase, and all repurchasing firms pool on the most efficient method $\theta = 1$, namely open market repurchase.

Empirically, when firms issue equity, both SEOs and ATMs are widely adopted with the more efficient method SEOs being more common. Billett et al. (2019) document that ATMs represent 63% incidence and 26% issue proceeds of those for SEOs. In contrast, almost all firms use the efficient method, i.e., open market repurchases, to buy back equity. For example, in 1999, there are 1212 cases of open market repurchases with a total size of 137 billion dollars. In comparison, tender offers and dutch auctions account for 21 and 19 cases, and 1.7 billion dollar and 3.8 billion dollar proceeds respectively (see Grullon and Ikenberry (2000)).

Prediction 2: Larger issues are carried out more quickly.

Proposition 3.2 implies that large issues $i > I_L$ are carried out in the most efficient method ($\theta = 1$), which is interpreted as SEO in our model. Firms conducting smaller issues $i = I_L$ can separate using different efficiency, interpreted as ATM with different transaction speeds in our model.

Empirically, Billett et al. (2019) document average proceeds per SEO is 256 million dollars, whereas average proceeds per ATM program is 92 million dollars (calculated from Table 2). Even though the ratio of proceeds to market equity is roughly the same between the two methods (18% for SEO and 20% for ATM), the ratio is significantly smaller for ATM than for SEO in their regression that controls for other factors (Table 4).

Prediction 3: SEO firms have lower asymmetric information than ATM firms (Billett et al., 2019) (Table 4).

3.5 Robustness: Preference for Share Price

So far in our model, firms only care about their long-term shareholders' payoff, resulting in an asymmetric preference about transaction price. Specifically, firms favor higher share price when issuing new equity and lower price when repurchasing. One could argue that even when firms repurchase shares, they may not wish their share price to collapse, even though a lower price enables them to repurchase more shares. In this section, we perturb the firm's objective function (3.2) to allow for explicit preference for share price:

$$\Pi(a, i, \theta, p) = p^{\epsilon} \left(\frac{V(a, i, \theta)}{1 + \frac{i}{p}} \right)^{1-\epsilon}, \tag{3.14}$$

where the parameter $\epsilon \in [0, 1]$ reflects the degree to which firms care about their share prices directly. When $\epsilon = 0$, this preference reduces to the original model (3.2). All other ingredients are the same.

Similar to (3.4) in the original model, when firms separate on issue size, they follow the modified ODE

$$\frac{d\hat{i}(a)}{da} = -\frac{\epsilon V(a, \hat{i}, 1) + (1-\epsilon)\hat{i}}{V(a, \hat{i}, 1) b}, \tag{3.15}$$

with the same boundary condition $\hat{i}(a_{\min}) = I$ and the fully efficient issue method $\theta = 1$.

As in the original model, when there is a minimum transaction size $I_L > 0$ binding in equilibrium, in the sense that the solution $\hat{i}$ to the ODE (3.15) reaches I_L before

a reaches $a_{\max}$, firms start to separate on transaction efficiency θ. The modified ODE that characterizes firms' efficiency strategy is given by

$$\frac{d\hat{\theta}(a)}{da} = -\frac{\epsilon V(a, I_L, \hat{\theta}) + (1-\epsilon) I_L}{V(a, I_L, \hat{\theta}) I_L b}. \tag{3.16}$$

Our key insight that firms cannot separate on θ when they repurchase equity remains robust to this ϵ-modification when ϵ is small. Specifically, when a minimum repurchase size $I_L > 0$ binds, firms can only pool on repurchasing I_L using the most efficient method. The modified ODE that gives the equilibrium size strategy is

$$\frac{d\tilde{i}(a)}{da} = \frac{\epsilon V(a, \tilde{i}, 1) + (1-\epsilon)\tilde{i}}{bV(a, \tilde{i}, 1)}. \tag{3.17}$$

3.6 Appendix

In the appendix, we provide proofs of all propositions and lemmas.

Lemma 3.1. *In the issue game ($I > I_L \geq 0$), if type a of the firm chooses the pair of issue size and method (i, θ), then for (i', θ') such that $i'(1 + \theta' b) < i(1 + \theta b)$, no type $a' < a$ chooses the pair (i', θ'), and a D1 belief satisfies $P^*(i', \theta') \geq V(a, i', \theta') - i'$. For (i'', θ'') such that $i''(1 + \theta'' b) > i(1 + \theta b)$, no type $a' > a$ chooses the pair (i'', θ''), and a D1 belief satisfies $P^*(i'', \theta'') \leq V(a, i'', \theta'') - i''$.*

In the repurchase game ($I < I_L \leq 0$), if type a of the firm chooses the pair of repurchase size and method (i, θ), then for (i', θ') such that $|i'(1-\theta' b)| < |i(1-\theta b)|$, no type $a' > a$ chooses the pair (i', θ'), and a D1 belief satisfies $P^*(i', \theta') \leq V(a, i', \theta') + (-i')$. For (i'', θ'') such that $|i''(1-\theta'' b)| > |i(1-\theta b)|$, no type $a'' < a$ chooses the pair (i'', θ''), and a D1 belief satisfies $P^*(i'', \theta'') \geq V(a, i'', \theta'') + (-i'')$.

Proof. We first prove for the issue game ($I > I_L \geq 0$). Consider types a and a' such that $a' < a$. Suppose type a of the firm chooses (i, θ), and there is (i', θ') such that $i'(1 + \theta' b) < i(1 + \theta b)$.

Suppose $p \in D^0_{a'} \cup D_{a'}$ with respect to (i', θ'), which implies

$$\Pi(a', i', \theta', p) \geq \Pi^*(a') \geq \Pi(a', i, \theta, P^*(i, \theta)).$$

This implies

$$\frac{\Pi(a', i', \theta', p)}{\Pi(a', i, \theta, P^*(i, \theta))} = \frac{a' + i'(1 + \theta' b)}{a' + i(1 + \theta b)} \cdot \frac{1 + \frac{i}{P^*(i,\theta)}}{1 + \frac{i'}{p}} \geq 1.$$

Since $i'(1 + \theta' b) < i(1 + \theta b)$, $\frac{\Pi(a,i',\theta',p)}{\Pi(a,i,\theta,P^*(i,\theta))}$ strictly increases in a. This implies $\frac{\Pi(a,i',\theta',p)}{\Pi(a,i,\theta,P^*(i,\theta))} > 1$. Since $\Pi^*(a) = \Pi(a, i, \theta, P^*(i, \theta))$, $p \in D_a$. Therefore, $D_{a'} \cup D^0_{a'} \subseteq D_a$, and D1 requires (i', θ') cannot be associated with any type $a' < a$. This implies $P^*(i', \theta') \geq V(a, i', \theta') - i'$.

Suppose type a' chooses (i', θ') in equilibrium. This implies $P^*(i', \theta') \in D^0_{a'}$, which implies $P^*(i', \theta') \in D_a$, which contradicts that type a chooses (i, θ) in equilibrium. Therefore, type a' does not choose (i', θ') in equilibrium.

For similar reasons, if type a of the firm chooses (i, θ), and there is (i'', θ'') such that $i''(1 + \theta'' b) > i(1 + \theta b)$, then $p \in D^0_{a''} \cup D_{a''}$ for type $a'' > a$ with respect to (i'', θ'') implies $p \in D_a$. This implies $P^*(i'', \theta'') \leq V(a, i'', \theta'') - i''$, and no type $a'' > a$ chooses (i'', θ'') in equilibrium.

We next prove for the repurchase game ($I < I_L \leq 0$). Consider types a and a' such that $a' > a$. Suppose type a of the firm chooses (i, θ), and there is (i', θ') such that $|i'(1 - \theta' b)| < |i(1 - \theta b)|$.

Suppose $p \in D^0_{a'} \cup D_{a'}$ with respect to (i', θ'), which implies

$$\Pi(a', i', \theta', p) \geq \Pi^*(a') \geq \Pi(a', i, \theta, P^*(i, \theta)).$$

This implies

$$\frac{\Pi(a', i', \theta', p)}{\Pi(a', i, \theta, P^*(i, \theta))} = \frac{a' - |i'(1 - \theta' b)|}{a' - |i(1 - \theta b)|} \cdot \frac{1 - \frac{|i|}{P^*(i,\theta)}}{1 - \frac{|i'|}{p}} \geq 1.$$

Since $|i'(1-\theta' b)| < |i(1-\theta b)|$, $\frac{\Pi(a,i',\theta',p)}{\Pi(a,i,\theta,P^*(i,\theta))}$ strictly decreases in a. This implies $\frac{\Pi(a,i',\theta',p)}{\Pi(a,i,\theta,P^*(i,\theta))} > 1$. Since $\Pi^*(a) = \Pi(a,i,\theta,P^*(i,\theta))$, $p \in D_a$. Therefore, $D_{a'} \cup D^0_{a'} \subseteq D_a$, and D1 requires (i',θ') cannot be associated with any type $a' > a$. This implies

$$P^*(i',\theta') \leq V(a,i',\theta') + (-i').$$

Suppose type a' chooses (i',θ') in equilibrium. This implies $\frac{\Pi(a',i',\theta',P^*(i',\theta'))}{\Pi(a',i,\theta,P^*(i,\theta))} \geq 1$. As shown above, this implies $\frac{\Pi(a,i',\theta',P^*(i',\theta'))}{\Pi(a,i,\theta,P^*(i,\theta))} > 1$, which contradicts that type a chooses (i,θ) in equilibrium. Therefore, type a' does not choose (i',θ') in equilibrium.

For similar reasons, if type a of the firm chooses (i,θ), and there is (i'',θ'') such that $|i'(1-\theta' b)| > |i(1-\theta b)|$, then $p \in D^0_{a''} \cup D_{a''}$ for type $a'' < a$ with respect to (i'',θ'') implies $p \in D_a$. This implies

$$P^*(i'',\theta'') \geq V(a,i'',\theta'') + (-i''),$$

and no type $a'' < a$ chooses (i'',θ'') in equilibrium. □

Lemma 3.2. *In a D1 outcome of the repurchase game ($I < I_L \leq 0$), all firms use the most efficient method $\theta = 1$.*

Proof. Suppose in a D1 equilibrium outcome, types in a non-empty set A of the firm choose (i,θ) with $\theta < 1$. Let $a' \in A$ be such that $V(a',i,\theta) \leq E[V(a,i,\theta)|a \in A]$. We show type a' strictly benefits from deviating to $(i,1)$.

Let a^* be such that $V(a^*,i,\theta) = E[V(a,i,\theta)|a \in A]$. Type a^* will be fairly priced if it chooses (i,θ), which implies

$$\Pi(a^*,i,\theta,P^*(i,\theta)) = V(a^*,i,\theta) + (-i).$$

If type a^* chooses $(i, 1)$, it has payoff

$$\begin{aligned}\Pi(a^*, i, 1, P^*(i, 1)) &= \frac{V(a^*, i, 1)}{1 - \frac{-i}{P^*(i,1)}} \\ &\geq \frac{V(a^*, i, 1)}{1 - \frac{-i}{V(a^*,i,1)+(-i)}} \; . \\ &= V(a^*, i, 1) + (-i) \\ &> V(a^*, i, \theta) + (-i)\end{aligned}$$

The first inequality is because $|i(1 - b)| < |i(1 - \theta b)|$ and Lemma 3.1 implies $P^*(i, 1) \leq V(a^*, i, 1) + (-i)$, and the second is due to $|i| > |i\theta|$. This implies $\frac{\Pi(a^*,i,1,P^*(i,1))}{\Pi(a^*,i,\theta,P^*(i,\theta))} > 1$. Since

$$\frac{\Pi(a, i, 1, P^*(i, 1))}{\Pi(a, i, \theta, P^*(i, \theta))} = \frac{a - |i(1 - b)|}{a - |i(1 - \theta b)|} \cdot \frac{1 - \frac{-i}{P^*(i,\theta)}}{1 - \frac{-i}{P^*(i,1)}}$$

strictly decreases in a and $a' \leq a^*$, $\frac{\Pi(a',i,1,P^*(i,1))}{\Pi(a',i,\theta,P^*(i,\theta))} > 1$. This implies type a' strictly benefits from deviating to $(i, 1)$. □

Proof of Proposition 3.1:

We first prove for the issue game ($I > I_L \geq 0$). Suppose in a D1 equilibrium outcome, types in a non-empty set A of the firm choose (i, θ) with $i > I_L$ and $\theta < 1$. Then there is (i'', θ') such that $i'' \in (I_L, i)$ and $i''\theta' = i\theta$. This implies $i''(1 + \theta' b) < i(1 + \theta b)$. There is i' slightly larger than i'' such that $i'\theta' > i\theta$ and $i'(1 + \theta' b) < i(1 + \theta b)$. Let $a' \in A$ be such that $V(a', i, \theta) \geq E[V(a, i, \theta)|a \in A]$. We show type a' strictly benefits from deviating to (i', θ').

Let a^* be such that $V(a^*, i, \theta) = E[V(a, i, \theta)|a \in A]$. Type a^* will be fairly priced if it chooses (i, θ), which implies

$$\Pi(a^*, i, \theta, P^*(i, \theta)) = V(a^*, i, \theta) - i.$$

If type a^* chooses (i', θ'), it has payoff

$$\begin{aligned}\Pi(a^*, i', \theta', P^*(i', \theta')) &= \frac{V(a^*, i', \theta')}{1 + \frac{i'}{P^*(i', \theta')}} \\ &\geq \frac{V(a^*, i', \theta')}{1 + \frac{i'}{V(a^*, i', \theta') - i'}}. \\ &= V(a^*, i', \theta') - i' \\ &> V(a^*, i, \theta) - i\end{aligned}$$

The first inequality is because Lemma 3.1 implies

$$P^*(i', \theta') \geq V(a^*, i', \theta') - i',$$

and the second is due to $i'\theta' > i\theta$. This implies $\frac{\Pi(a^*, i', \theta', P^*(i', \theta'))}{\Pi(a^*, i, \theta, P^*(i, \theta))} > 1$. Since

$$\frac{\Pi(a, i', \theta', P^*(i', \theta'))}{\Pi(a, i, \theta, P^*(i, \theta))} = \frac{a + i'(1 + \theta' b)}{a + i(1 + \theta b)} \cdot \frac{1 + \frac{i}{P^*(i, \theta)}}{1 + \frac{i'}{P^*(i', \theta')}}$$

strictly increases in a and $a' \geq a^*$, $\frac{\Pi(a', i', \theta', P^*(i', \theta'))}{\Pi(a', i, \theta, P^*(i, \theta))} > 1$. This implies type a' strictly benefits from deviating to (i', θ').

For the repurchase game ($I < I_L \leq 0$), it follows Lemma 3.2 that any type that repurchases a positive amount ($i < 0$) chooses the most efficient method $\theta = 1$.

Lemma 3.3. *When the project requires firms to raise equity ($I > I_L \geq 0$), a D1 equilibrium outcome has a cutoff firm type $\hat{a}$ such that firms with $a < \hat{a}$ issue strictly more than the minimum size $i^*(a) > I_L$, and firms with $a \in (\hat{a}, a_{\max}]$ issue equity I_L.*

When firms repurchase equity ($I < I_L \leq 0$), a D1 equilibrium outcome has a cutoff firm type $\hat{a}$ such that firms with $a > \hat{a}$ repurchase strictly more than the minimum size $|i^*(a)| > |I_L|$, and types with $a \in [a_{\min}, \hat{a})$ repurchase exactly $|i^*(a)| = |I_L|$.

Proof. We first prove for the issue game ($I > I_L \geq 0$).

According to Proposition 3.1, in a D1 outcome, if type a chooses issue size $i > I_L$, then it chooses speed $\theta = 1$. Suppose type a chooses $(i, 1)$ and type a' chooses (I_L, θ) for an arbitrary θ. Since $I_L(1+\theta b) < i(1+b)$, Lemma 3.1 implies $a < a'$.

Therefore, there must be cutoff type $\hat{a}$ such that $i^*(a) > I_L$ for types with $a < \hat{a}$ and $i^*(a) = I_L$ for types with $a \in (\hat{a}, a_{\max}]$.

We next prove for the repurchase game ($I < I_L \leq 0$).

According to Lemma 3.2, in a D1 outcome, any firm that repurchases a positive amount $|i| > 0$ chooses speed $\theta = 1$. Suppose for two repurchase sizes i and i' such that $|i| > |i'| \geq |I_L|$, type a chooses $(i, 1)$ and type a' chooses $(i', 1)$. Since $|i'(1-b)| < |i(1-b)|$, Lemma 3.1 implies $a > a'$.

Therefore, there must be cutoff type $\hat{a}$ such that $|i^*(a)| > |I_L|$ for types with $a > \hat{a}$ and $|i^*(a)| = |I_L|$ for types with $a \in [a_{\min}, \hat{a})$. □

Lemma 3.4. *In a D1 equilibrium outcome of the issue game ($I > I_L \geq 0$), for firms that issue strictly more than I_L, their size strategy is $i^*(a) = \hat{i}(a)$ which satisfies the differential equation (3.4) and the boundary condition $\hat{i}\,(a_{\min}) = I$.*

Proof. Define $A \equiv \{a : |i^*(a)| > I_L\}$. According to 3.3, $A = [a_{\min}, \hat{a})$ or $[a_{\min}, \hat{a}]$. According to Proposition 3.1, $\theta^*(a) = 1$ for $a \in A$.

(1) $i^*(a)$ is strictly decreasing on A.
If $i' < i$, then $i'(1+b) < i(1+b)$. Lemma 3.1 implies $i^*(a)$ is decreasing on A. To show $i^*(a)$ is strictly decreasing on A, it suffices to show there is no $i > I_L$ that is chosen by an interval of types.
Suppose types in an interval A' choose $(i, 1)$ in equilibrium. This implies

$$P^*(i, 1) = E[V(a, i, 1)|a \in A'] - i < V(\sup A', i, 1) - i.$$

If type $\sup A'$ chooses $(i, 1)$, it has payoff

$$\Pi(\sup A', i, 1, P^*(i, 1)) < V(\sup A', i, 1) - i.$$

For $i' < i$, Lemma 3.1 implies

$$P^*(i', 1) \geq V(\sup A', i', 1) - i'.$$

If type $\sup B$ chooses $(i', 1)$, it has payoff

$$\Pi(\sup A', i', 1, P^*(i', 1)) \geq V(\sup A', i', 1) - i'.$$

Since $\lim_{i' \uparrow i} V(\sup A', i', 1) - i' = V(\sup A', i, 1) - i$, there is i' such that

$$\Pi(\sup A', i', 1, P^*(i', 1)) > \Pi(\sup A', i, 1, P^*(i, 1)).$$

Since Π is continuous in a, there is $a \in A'$ such that

$$\Pi(a, i', 1, P^*(i', 1)) > \Pi(a, i, 1, P^*(i, 1)),$$

which contradicts that types in A' choose $(i, 1)$ in equilibrium.

(2) $i^*(a)$ is continuous on A.

Since $i^*(a)$ is decreasing, it suffices to rule out jump discontinuity on A. Consider a discontinuity at type $a^* \in A$.

Suppose $a^* > \inf A$, and there is $i > i^*(a^*)$ such that $i^*(a) > i$ for any $a \in A$ with $a < a^*$. This implies $i(1+b) < i^*(a)(1+b)$ for any $a \in A$ with $a < a^*$, and Lemma 3.1 implies

$$P^*(i, 1) \geq V(a, i, 1) - i$$

for any $a \in A$ with $a < a^*$. Therefore,

$$P^*(i, 1) \geq V(a^*, i, 1) - i.$$

This implies type a^* benefits from deviating to $(i, 1)$, which gives it higher

NPV than its equilibrium choice and at least fair pricing:

$$\begin{aligned}\Pi(a^*, i, 1, P^*(i,1)) &= \frac{V(a^*, i, 1)}{1 + \frac{i}{P^*(i,1)}} \\ &\geq \frac{V(a^*, i, 1)}{1 + \frac{i}{V(a^*,i,1)-i}} \\ &= V(a^*, i, 1) - i \\ &> V(a^*, i^*(a), 1) - i^*(a) \\ &= \Pi^*(a^*)\end{aligned}.$$

Suppose $a^* < \sup A$, and there is $i < i^*(a^*)$ such that $i^*(a) < i$ for $a \in A$ with $a > a^*$. Since $i^*(a)$ is strictly monotonic on A,

$$P^*(i^*(a^*), 1) = V(a^*, i^*(a^*), 1) - i^*(a^*).$$

This implies for $a > a^*$,

$$\Pi(a, i^*(a^*), 1, P^*(i^*(a^*), 1)) > V(a^*, i^*(a^*), 1) - i^*(a^*).$$

On the other hand, since $i^*(a)$ is strictly monotonic on A,

$$\Pi^*(a) = V(a, i^*(a), 1) - i^*(a) < V(a, i, 1) - i$$

for $a \in A$ with $a > a^*$. Since $V(a, i, 1)$ is continuous in a and i,

$$\begin{aligned}\lim_{a \downarrow a^*} V\left(a, i, 1\right) - i &= V\left(a^*, i, 1\right) - i \\ &< V\left(a^*, i^*(a^*), 1\right) - i^*(a^*)\end{aligned}.$$

Therefore, there is $a \in A$ with $a > a^*$ such that $V(a, i, 1) - i < V(a^*, i^*(a^*), 1) - i^*(a^*)$. This implies

$$\Pi^*(a) < \Pi(a, i^*(a^*), 1, P^*(i^*(a^*), 1)),$$

that is type a strictly benefits from deviating to $(i^*(a^*), 1)$.

(3) $i^*(a)$ is satisfies ODE (3.4) (substituting $\hat{i}$ by i^*) for $a \in A$.

That $i^*(a)$ is strictly decreasing on A implies

$$P^*(i^*(a), 1) = V(a, i^*(a), 1) - i^*(a)$$

and

$$\Pi^*(a) = V(a, i^*(a), 1) - i^*(a)$$

for $a \in A$. Consider types $a_1, a_2 \in A$ such that $a_1 < a_2$. Their equilibrium choices imply

$$\Pi(a_1, i^*(a_2), 1, P^*(i^*(a_2), 1)) \leq \Pi^*(a_1),$$

$$\Pi(a_2, i^*(a_1), 1, P^*(i^*(a_1), 1)) \leq \Pi^*(a_2).$$

These imply

$$\begin{aligned} & V(a_2, i^*(a_2), 1)\frac{V(a_1, i^*(a_2), 1) - V(a_1, i^*(a_1), 1)}{a_2 - a_1} \\ & + i^*(a_1)i^*(a_2)\frac{\frac{V(a_2, i^*(a_2), 1)}{i^*(a_2)} - \frac{V(a_1, i^*(a_2), 1)}{i^*(a_1)}}{a_2 - a_1} \\ & \leq 0 \end{aligned}, \tag{3.18}$$

$$\begin{aligned} & V(a_1, i^*(a_1), 1)\frac{V(a_2, i^*(a_2), 1) - V(a_2, i^*(a_1), 1)}{a_2 - a_1} \\ & + i^*(a_1)i^*(a_2)\frac{\frac{V(a_2, i^*(a_2), 1)}{i^*(a_2)} - \frac{V(a_1, i^*(a_1), 1)}{i^*(a_1)}}{a_2 - a_1} \\ & \geq 0 \end{aligned}. \tag{3.19}$$

Since $i^*(a)$ is continuous on A, taking the limits of (3.18) and (3.19) results in

$$V(a, i^*(a), 1)\left[V_i(a, i^*(a), 1) - 1\right] i^{*\prime}(a) + i^*(a)V_a(a, i^*(a), 1) = 0,$$

which can be simplified into ODE (3.4) (substituting $\hat{i}$ by i^*).

(4) If A is not empty, $i^*(a_{\min}) = I$.

Suppose type $a_{\min}$ chooses issue size $i \in (I_L, I)$. Since types in A are fully revealed in equilibrium, $\Pi^*(a_{\min}) = V(a_{\min}, i, 1) - i$. If type $a_{\min}$ deviates to size and speed $(I, 1)$, it has payoff

$$\begin{aligned}\Pi(a_{\min}, I, 1, P^*(I,1)) &= \frac{V(a_{\min}, I, 1)}{1 + \frac{I}{P^*(I,1)}} \\ &\geq \frac{V(a_{\min}, I, 1)}{1 + \frac{I}{V(a_{\min},I,1)-I}} . \\ &= V(a_{\min}, I, 1) - I \\ &> V(a_{\min}, i, 1) - i\end{aligned}$$

The first inequality is because

$$P^*(I, 1) \geq V(a_{\min}, I, 1) - I,$$

and the second inequality is because $I > i$. Therefore,

$$\Pi(a_{\min}, I, 1, P^*(I, 1)) > \Pi^*(a_{\min}),$$

that is type $a_{\min}$ strictly benefits from deviating to $(I, 1)$.

□

Lemma 3.5. *In a D1 equilibrium outcome of the issue game ($I > I_L \geq 0$), for firms that issue exactly I_L, their method strategy is $\theta^*(a) = \hat{\theta}(a)$ which satisfies the differential equation (3.6) with the boundary condition $\hat{\theta}(\hat{a}) = 1$.*

Proof. Lemma 3.3 implies the set $B \equiv \{a : i^*(a) = I_L\}$ is an interval from $\hat{a}$ to $\bar{a}$.

(1) $\theta^*(a)$ is strictly decreasing on B.
If $\theta' < \theta$, then $I_L(1+\theta' b) < I_L(1+\theta b)$. Lemma 3.1 implies $\theta^*(a)$ is decreasing on B. To show $\theta^*(a)$ is strictly decreasing on A, it suffices to show there is no θ such that (I_L, θ) is chosen by an interval of types.

Suppose types in an interval B' choose (I_L, θ) in equilibrium. This implies

$$\begin{aligned} P^*(I_L,\theta) &= E[V(a,I_L,\theta)|a \in B'] - I_L \\ &< V(\sup B', I_L, \theta) - I_L \end{aligned}.$$

If type $\sup B'$ chooses (I_L, θ), it has payoff

$$\Pi(\sup B', I_L, \theta, P^*(I_L,\theta)) < V(\sup B', I_L, \theta) - I_L.$$

For $\theta' < \theta$, Lemma 3.1 implies

$$P^*(I_L,\theta') \geq V(\sup B', I_L, \theta') - I_L.$$

If type $\sup B'$ chooses (I_L, θ'), it has payoff

$$\Pi(\sup B', I_L, \theta', P^*(I_L,\theta')) \geq V(\sup B', I_L, \theta') - I_L.$$

Since

$$\lim_{\theta' \uparrow \theta} V(\sup B', I_L, \theta') - I_L = V(\sup B', I_L, \theta) - I_L,$$

there is θ' such that

$$\Pi(\sup B', I_L, \theta', P^*(I_L,\theta')) > \Pi(\sup B', I_L, \theta, P^*(I_L,\theta)).$$

Since Π is continuous in a, there is $a \in B'$ such that

$$\Pi(a, I_L, \theta', P^*(I_L,\theta')) > \Pi(a, I_L, \theta, P^*(I_L,\theta)),$$

which contradicts that types in B' choose (I_L, θ) in equilibrium.

(2) $\theta^*(a)$ is continuous on B.

Since $\theta^*(a)$ is decreasing, it suffices to rule out jump discontinuity on B. Consider a discontinuity at type $a^* \in A$.

Suppose $a^* > \inf B$, and there is $\theta > \theta^*(a^*)$ such that $\theta^*(a) > \theta$ for any $a \in B$ with $a < a^*$. This implies $\theta(1+b) < \theta^*(a)(1+b)$ for any $a \in B$ with

$a < a^*$, and Lemma 3.1 implies

$$P^*(I_L, \theta) \geq V(a, I_L, \theta) - I_L$$

for any $a \in B$ with $a < a^*$. Therefore,

$$P^*(I_L, \theta) \geq V(a^*, I_L, \theta) - I_L.$$

This implies type a^* benefits from deviating to (I_L, θ), which gives it higher NPV than its equilibrium choice and at least fair pricing:

$$\begin{aligned}
\Pi(a^*, I_L, \theta, P^*(I_L, \theta)) &= \frac{V(a^*, I_L, \theta)}{1 + \frac{I_L}{P^*(I_L, \theta)}} \\
&\geq \frac{V(a^*, I_L, \theta)}{1 + \frac{I_L}{V(a^*, I_L, \theta) - I_L}} \\
&= V(a^*, I_L, \theta) - I_L \\
&> V(a^*, I_L, \theta^*(a)) - I_L \\
&= \Pi^*(a^*)
\end{aligned}.$$

Suppose $a^* < \sup B$, and there is $\theta < \theta^*(a^*)$ such that $\theta^*(a) < \theta$ for $a \in B$ with $a > a^*$. Since $\theta^*(a)$ is strictly monotonic on B,

$$P^*(I_L, \theta^*(a^*)) = V(a^*, I_L, \theta^*(a^*)) - I_L.$$

This implies for $a > a^*$,

$$\Pi(a, I_L, \theta^*(a^*), P^*(I_L, \theta^*(a^*))) > V(a^*, I_L, \theta^*(a^*))) - I_L.$$

On the other hand, since $\theta^*(a)$ is strictly monotonic on B,

$$\Pi^*(a) = V(a, I_L, \theta^*(a)) - I_L < V(a, I_L, \theta) - I_L$$

for $a \in B$ with $a > a^*$. Since $V(a, I_L, \theta)$ is continuous in a and θ,

$$\begin{aligned} \lim_{a \downarrow a^*} V(a, I_L, \theta) - I_L &= V(a^*, I_L, \theta) - I_L \\ &< V(a^*, I_L, \theta^*(a^*))) - I_L \end{aligned}.$$

Therefore, there is $a \in B$ with $a > a^*$ such that

$$V(a, I_L, \theta) - I_L < V(a^*, I_L, \theta^*(a^*))) - I_L.$$

This implies

$$\Pi^*(a) < \Pi(a, I_L, \theta^*(a^*), P^*(I_L, \theta^*(a^*))),$$

that is type a strictly benefits from deviating to $(I_L, \theta^*(a^*))$.

(3) $\theta^*(a)$ satisfies (3.6) with $\hat{\theta}(a)$ substituted by $\theta^*(a)$.

That $\theta^*(a)$ is strictly decreasing on B implies

$$P^*(I_L, \theta^*(a)) = V(a, I_L, \theta^*(a)) - I_L$$

and

$$\Pi^*(a) = V(a, I_L, \theta^*(a)) - I_L$$

for $a \in B$. Consider types $a_1, a_2 \in B$ such that $a_1 < a_2$. Their equilibrium choices imply

$$\Pi(a_1, I_L, \theta^*(a_2), P^*(I_L, \theta^*(a_2))) \leq \Pi^*(a_1),$$

$$\Pi(a_2, I_L, \theta^*(a_1), P^*(I_L, \theta^*(a_1))) \leq \Pi^*(a_2).$$

These imply

$$\begin{aligned} & V(a_2, I_L, \theta^*(a_2)) \frac{V(a_1, I_L, \theta^*(a_2)) - V(a_1, I_L, \theta^*(a_1))}{a_2 - a_1} \\ & + I_L \frac{V(a_2, I_L, \theta^*(a_2)) - V(a_1, I_L, \theta^*(a_2))}{a_2 - a_1} \\ & \leq 0 \end{aligned}, \tag{3.20}$$

$$
\begin{aligned}
&V(a_1, I_L, \theta^*(a_1))\frac{V(a_2, I_L, \theta^*(a_2)) - V(a_2, I_L, \theta^*(a_1))}{a_2 - a_1} \\
&+ I_L \frac{V(a_2, I_L, \theta^*(a_1)) - V(a_1, I_L, \theta^*(a_1))}{a_2 - a_1} \\
&\geq 0
\end{aligned}
. \tag{3.21}
$$

Taking the limits of (3.20) and (3.21) results in

$$
V(a, I_L, \theta^*(a))V_\theta(a, I_L, \theta^*(a))\theta^{*\prime}(a) + I_L V_a(a, I_L, \theta^*(a)) = 0,
$$

which can be simplified into (3.6).

(4) If B is not empty, $\lim_{a \downarrow \hat{a}} \theta^*(\hat{a}) = 1$.

Suppose B is not empty and $\lim_{a \downarrow \hat{a}} \theta^*(\hat{a}) = \theta < 1$. Let $\theta' \in (\theta, 1]$. Since $i^*(a) > I_L$ and $\theta^*(a) = 1$ for $a < \hat{a}$, Lemma (3.1) implies

$$
P^*(I_L, \theta') \geq V(\hat{a}, I_L, \theta') - I_L.
$$

. Type $\hat{a}$ strictly prefers (I_L, θ') at the equilibrium price to (I_L, θ) at price $p = V(\hat{a}, I_L, \theta) - I_L$:

$$
\begin{aligned}
\Pi(\hat{a}, I_L, \theta', P^*(I_L, \theta')) &= \frac{V(\hat{a}, I_L, \theta')}{1 + \frac{I_L}{P^*(I_L, \theta')}} \\
&\geq \frac{V(\hat{a}, I_L, \theta')}{1 + \frac{I_L}{V(\hat{a}, I_L, \theta') - I_L}} \\
&= V(\hat{a}, I_L, \theta') - I_L \\
&> V(\hat{a}, I_L, \theta) - I_L \\
&= \Pi(\hat{a}, I_L, \theta, p)
\end{aligned}
.
$$

Since $\lim_{a \downarrow \hat{a}} P^*(I_L, \theta^*(a)) = p$, the above implies

$$
\lim_{a \downarrow \hat{a}} \Pi(a, I_L, \theta', P^*(I_L, \theta')) > \lim_{a \downarrow \hat{a}} \Pi(a, I_L, \theta^*(a), P^*(I_L, \theta^*(a))).
$$

There is $a \in B$ that strictly prefers deviating to (I_L, θ').

□

Proof of Proposition 3.2:

We first prove the uniqueness and then the existence of the D1 equilibrium outcome.

For the uniqueness of the D1 equilibrium outcome, following Proposition 3.1 and Lemma 3.3, 3.4 and 3.5, it suffices to show $i^*(a)$ is continuous at $\hat{a}$.

Lemma 3.4 and 3.5 imply every issuing type is fairly priced, that is for all a,

$$\Pi^*(a) = P^*(i^*(a), \theta^*(a)) = V(a, i^*(a), \theta^*(a)) - i^*(a).$$

Suppose $\hat{a} < a_{\max}$. Lemma 3.3 and 3.4 imply $\lim_{a\uparrow\hat{a}} = \hat{i}(\hat{a})$ and $\lim_{a\downarrow\hat{a}} = I_L$. For continuity of $i^*(a)$ at $\hat{a}$, it suffices to show $\hat{i}(\hat{a}) = I_L$.

Suppose $\hat{i}(\hat{a}) > I_L$. Fix $i \in (I_L, \hat{i}(\hat{a}))$. Since $i^*(a) = \hat{i}(a) > \hat{i}(\hat{a})$ and $\theta^*(a) = 1$ for $a < \hat{a}$, Lemma 3.1 implies

$$P^*(i, 1) \geq V(\hat{a}, i, 1) - i.$$

Type $\hat{a}$ strictly prefers $(i, 1)$ to $(I_L, 1)$ at its fair price $p = V(\hat{a}, I_L, 1) - I_L$:

$$\begin{aligned}
\Pi(\hat{a}, i, 1, P^*(i, 1)) &= \frac{V(\hat{a}, i, 1)}{1 + \frac{i}{P^*(i,1)}} \\
&\geq \frac{V(\hat{a}, i, 1)}{1 + \frac{i}{V(\hat{a},i,1)-i}} \\
&= V(\hat{a}, i, 1) - i \\
&> V(\hat{a}, I_L, 1) - I_L \\
&= \Pi(\hat{a}, I_L, 1, p)
\end{aligned}.$$

According to Lemma 3.5, $\lim_{a\downarrow\hat{a}} \theta^*(a) = 1$. Therefore,

$$\lim_{a\downarrow\hat{a}} P^*(I_L, \theta^*(a)) = \lim_{a\downarrow\hat{a}} V(a, I_L, \theta^*(a)) - I_L = p.$$

The above implies

$$\lim_{a \downarrow \hat{a}} \Pi(a, i, 1, P^*(i, 1)) > \lim_{a \downarrow \hat{a}} \Pi(a, I_L, \theta^*(a), P^*(I_L, \theta^*(a))),$$

which implies there is type $a > \hat{a}$ that chooses $(I_L, \theta^*(a))$ in equilibrium benefits from deviating to $(i, 1)$. Therefore, $i^*(a)$ is continuous at $\hat{a}$.

We next prove the strategy described in the proposition indeed supports a D1 equilibrium outcome.

(1) No type benefits from mimicking another type.
Since $i^*(a)$ and $\theta^*(a)$ are continuous and almost everywhere differentiable, it is sufficient to show

$$\frac{d}{da'} \Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a'))\right) \lesseqgtr 0$$

for $a' \gtrless a$ and $a' \neq \hat{a}$.

$$\begin{aligned} &\frac{d}{da'} \Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a'))\right) \\ &= \Pi_i i^{*\prime}(a') + \Pi_\theta \theta^{*\prime}(a') + \Pi_p \frac{dP^*(i^*(a'), \theta^*(a'))}{da'} \end{aligned}$$

where Π_i, Π_θ and Π_p stand for the derivatives of $\Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a')\right)$ with respect to $i^*(a')$, $\theta^*(a')$ and $P^*(i^*(a'), \theta^*(a'))$.

$$\frac{\Pi_i}{\Pi} = \left(\frac{V_i\left(a, i^*(a'), \theta^*(a')\right)}{V\left(a, i^*(a'), \theta^*(a')\right)} - \frac{1}{P^*(i^*(a'), \theta^*(a'))}\right)$$

where Π stands for $\Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a'))\right)$, and

$$\frac{\Pi_\theta}{\Pi} = \frac{V_i\left(a, i^*(a'), \theta^*(a')\right)}{V\left(a, i^*(a'), \theta^*(a')\right)},$$

$$\frac{\Pi_p}{\Pi} = \frac{i^*(a')}{P^*(i^*(a'), \theta^*(a'))\left[P^*(i^*(a'), \theta^*(a')) + i^*(a')\right]}.$$

According to ODEs (3.4) and (3.6),

$$\frac{d}{da'}\Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a'))\right) = 0$$

if $a = a'$. Since $\frac{\Pi_i}{\Pi}$ and $\frac{\Pi_\theta}{\Pi}$ strictly decrease in a, and either $i^{*\prime}(a') < 0$ and $\theta^{*\prime}(a') = 0$ or $i^{*\prime}(a') = 0$ and $\theta^{*\prime}(a') < 0$,

$$\frac{d}{da'}\Pi\left(a, i^*(a'), \theta^*(a'), P^*(i^*(a'), \theta^*(a'))\right) \gtreqless 0$$

for any $a \gtreqless a'$. This implies type a does not benefit from mimicking another type.

(2) $\hat{\theta}(a) > 0$ for all a.

Suppose ODE (3.6) and the boundary condition $\hat{\theta}(\hat{a}) = 1$ imply there is $a' > \hat{a}$ such that $\hat{\theta}(a') = 0$. According to step 1, type a' strictly prefers $(I_L, 0)$ at price $V(a', I_L, 0) - I_L$ to $(I_L, 1)$ at price $V(\hat{a}, I_L, 1) - I_L$. Lemma 3.1 implies $a_{\max} \geq a'$ has the same preference. This implies

$$\begin{aligned} a_{\max} &\geq \frac{V(a_{\max}, I_L, 0)}{1 + \frac{I_L}{V(a', I_L, 0) - I_L}} \\ &= \Pi(a_{\max}, I_L, 0, V(a', I_L, 0) - I_L) \\ &> \Pi(a_{\max}, I_L, 1, V(\hat{a}, I_L, 1) - I_L) \\ &> \Pi(a_{\max}, I_L, 1, V(a_{\min}, I_L, 1) - I_L) \end{aligned}.$$

However, this leads to contradiction with (3.1), the assumption that type $a_{\max}$ prefers issuing I_L with method $\theta = 1$ under the worst belief to doing nothing.

(3) No type benefits from deviating to an off-equilibrium action (i, θ) such that

$$i(1 + \theta b) > i^*(a_{\max})(1 + \theta^*(a_{\max})b). \tag{3.22}$$

Let D_a and D_a^0 respectively denote the set of prices of (i, θ) that makes type a strictly prefer to deviate to (i, θ) and indifferent. There is $p(a)$ such that $D_a^0 = \{p(a)\}$ and $D_a = \{p > p(a)\}$. D1 belief about (i, θ) is supported on those types that deviate under the largest set of prices, $\arg\min_a p(a)$. If under the equilibrium belief, a type in $\arg\min_a p(a)$ does not benefit from deviating to (i, θ), that is $P^*(i, \theta) \leq \min_a p(a)$, then no type benefits from deviating to (i, θ).

Let type a'' be a type in the support of the D1 belief on (i, θ) such that $P^*(i, \theta) \leq V(a'', i, \theta) - i$. Inequality (3.22) implies there is $a' < \bar{a}$ such that its equilibrium choice (i', θ') satisfies $i'(1 + \theta' b) < i(1 + \theta b)$ and $i'\theta' > i\theta$. Lemma 3.1 implies (i, θ) cannot be associated with any type $a > a'$ in a D1 belief. Therefore, $a'' \leq a'$. Since type a'' does not benefit from mimicking type a' as shown in step 2, it does not benefit from deviating to (i, θ):

$$\begin{aligned}
\Pi(a'', i, \theta, P^*(i, \theta)) &= \frac{V(a'', i, \theta)}{1 + \frac{i}{P^*(i,\theta)}} \\
&\leq V(a'', i, \theta) - i \\
&< V(a'', i', \theta') - i' \\
&\leq \frac{V(a'', i', \theta')}{1 + \frac{i'}{V(a', i', \theta') - i'}} \\
&= \Pi(a'', i', \theta', P^*(i', \theta'))
\end{aligned} .$$

The second inequality is because $i\theta < i'\theta'$, and the third inequality is because $V(a', i', \theta') \geq V(a'', i', \theta')$. This implies no type a benefits from deviating to (i, θ).

(4) No type benefits from deviating to an off-equilibrium action (i, θ) such that

$$i(1 + \theta b) \leq i^*(a_{\max})(1 + \theta^*(a_{\max})b). \tag{3.23}$$

(3.23) implies $i\theta < i^*(a_{\max})\theta^*(a_{\max})$. Since no type benefits from mimicking type $a_{\max}$, it follows that no type benefits from deviating to (i, θ):

$$\begin{aligned}
\Pi(a, i, \theta, P^*(i, \theta)) &= \frac{V(a, i, \theta)}{1 + \frac{i}{P^*(i,\theta)}} \\
&\leq \frac{V(a, i, \theta)}{1 + \frac{i}{V(a_{\max}, i, \theta) - i}} \\
&= \frac{V(a, i, \theta)}{V(a_{\max}, i, \theta)} \left(V(a_{\max}, i, \theta) - i\right) \\
&< \frac{V(a, i^*(a_{\max}), \theta^*(a_{\max}))}{V(a_{\max}, i^*(a_{\max}), \theta^*(a_{\max}))} \left(V(a_{\max}, i^*(a_{\max}), \theta^*(a_{\max})) - i^*(a_{\max})\right) \\
&= \Pi\left(a, i^*(a_{\max}), \theta^*(a_{\max}), P^*\left(i^*(a_{\max}), \theta^*(a_{\max})\right)\right) \\
&\leq \Pi^*(a)
\end{aligned} .$$

The first inequality is because $P^*(i, \theta) \leq V(a_{\max}, i, \theta) - i$, the second inequality is due to $a < a_{\max}$, (3.23) and $i\theta < i^*(a_{\max})\theta^*(a_{\max})$, and the third equality is because

$$P^*\left(i^*(a_{\max}), \theta^*(a_{\max})\right) = V(a_{\max}, i^*(a_{\max}), \theta^*(a_{\max})) - i^*(a_{\max}).$$

Proof of Proposition 3.3:

Suppose types in A_1 repurchase i at efficiency level θ_1, and types in A_2 repurchase the same size i with a more efficient method $\theta_2 > \theta_1$. Lemma 3.1 implies types in A_1 are weakly better than types in A_2. This implies $P^*(i, \theta_2) \leq V(a, i, \theta_2) + (-i)$ for $a \in A_1$. Type $a_1 \in A_1$ such that $a_1 \leq E[a|a \in A_1]$ strictly benefits from deviating to (i, θ_2):

$$\begin{aligned}\Pi(a_1, i, \theta_2, P^*(i, \theta_2)) &= \frac{V(a_1, i, \theta_2)}{1 - \frac{-i}{P^*(i,\theta_2)}} \\ &\geq \frac{V(a_1, i, \theta_2)}{1 - \frac{-i}{V(a_1,i,\theta_2)+(-i)}} \\ &= V(a_1, i, \theta_2) + (-i) \\ &> V(a_1, i, \theta_1) + (-i) \\ &\geq \frac{V(a_1, i, \theta_1)}{1 - \frac{-i}{P^*(i,\theta_1)}} \\ &= \Pi^*(a_1)\end{aligned}.$$

The last inequality is because

$$\begin{aligned}P^*(i, \theta_1) &= E[V(a, i, \theta_1)|a \in A_1] + (-i) \\ &\geq V(a_1, i, \theta_1) + (-i)\end{aligned}.$$

Lemma 3.6. *In a D1 equilibrium outcome of the repurchase game ($I < I_L \leq 0$), firms that repurchase strictly more than the minimum size $|i^*(a)| > |I_L|$ ($a > \hat{a}$) separate on size according to (3.10) and (3.11).*

Proof. Define $A \equiv \{a : |i^*(a)| > I_L\}$. According to 3.3, $A = (\hat{a}, a_{\max}]$ or $[\hat{a}, a_{\max}]$ in a D1 equilibrium outcome. According to Proposition 3.1, $\theta^*(a) = 1$ for $a \in A$.

(1) $|i^*(a)|$ is strictly increasing on A.

If $|i'| < |i|$, then $|i'(1-b)| < |i(1-b)|$. Lemma 3.1 implies $|i^*(a)|$ is increasing on A. To show $i^*(a)$ is strictly increasing on A, it suffices to show there is no i with $|i| > I_L$ that is chosen by an interval of types.

Suppose types in an interval A' choose $(i, 1)$ in equilibrium. This implies

$$\begin{aligned}P^*(i, 1) &= E[V(a, i, 1)|a \in A'] + (-i) \\ &> V(\inf A', i, 1) + (-i)\end{aligned}.$$

If type $\inf A'$ chooses $(i, 1)$, it has payoff

$$\Pi(\inf A', i, 1, P^*(i, 1)) < V(\inf A', i, 1) + (-i).$$

For i' with $|i'| < |i|$, Lemma 3.1 implies

$$P^*(i', 1) \leq V(\inf A', i', 1) + (-i').$$

If type $\inf A'$ chooses $(i', 1)$, it has payoff

$$\Pi(\inf A', i', 1, P^*(i', 1)) \geq V(\inf A', i', 1) + (-i').$$

Since

$$\lim_{i' \uparrow i} V(\inf A', i', 1) = V(\inf A', i, 1) + (-i),$$

there is i' such that

$$\Pi(\inf A', i', 1, P^*(i', 1)) > \Pi(\inf A', i, 1, P^*(i, 1)).$$

Since Π is continuous in a, there is $a \in A'$ such that

$$\Pi(a, i', 1, P^*(i', 1)) > \Pi(a, i, 1, P^*(i, 1)),$$

which contradicts that types in B choose $(i, 1)$ in equilibrium.

(2) $|i^*(a)|$ is continuous on A.
Since $|i^*(a)|$ is increasing, it suffices to rule out jump discontinuity on A. Consider a discontinuity at type $a^* \in A$.
Suppose $a^* < \sup A$, and there is i with $|i| > |i^*(a^*)|$such that $|i^*(a)| > |i|$ for any $a \in A$ with $a > a^*$. This implies $|i(1-b)| < |i^*(a)(1-b)|$ for any $a \in A$ with $a > a^*$, and Lemma 3.1 implies

$$P^*(i, 1) \leq V(a, i, 1) + (-i)$$

for any $a \in A$ with $a > a^*$. Therefore,

$$P^*(i, 1) \leq V(a^*, i, 1) + (-i).$$

This implies type a^* benefits from deviating to $(i, 1)$, which gives it higher NPV than its equilibrium choice and allows it to repurchase at most at the fair price:

$$\begin{aligned}
\Pi(a^*, i, 1, P^*(i, 1)) &= \frac{V(a^*, i, 1)}{1 - \frac{-i}{P^*(i,1)}} \\
&\geq \frac{V(a^*, i, 1)}{1 - \frac{-i}{V(a^*,i,1)+(-i)}} \\
&= V(a^*, i, 1) + (-i) \\
&> V(a^*, i^*(a), 1) + (-i^*(a)) \\
&= \Pi^*(a^*)
\end{aligned}.$$

Suppose $a^* > \inf A$, and there is i with $|i| < |i^*(a^*)|$ such that $|i^*(a)| < |i|$ (that is $i^*(a) > i$) for $a \in A$ with $a < a^*$. Since $i^*(a)$ is strictly monotonic on A,

$$P^*(i^*(a^*), 1) = V(a^*, i^*(a^*), 1) + (-i^*(a^*)).$$

This implies

$$\lim_{a \uparrow a^*} \Pi(a, i^*(a^*), 1, P^*(i^*(a^*), 1)) = V(a^*, i^*(a^*), 1) + (-i^*(a^*)).$$

On the other hand, since $|i^*(a)|$ is strictly monotonic on A,

$$\begin{aligned}
\Pi^*(a) &= V(a, i^*(a), 1) + (-i^*(a)) \\
&< V(a, i, 1) + (-i)
\end{aligned}$$

for $a \in A$ with $a > a^*$. Since $V(a, i, 1)$ is continuous in a and i,

$$\begin{aligned}
\lim_{a \downarrow a^*} V(a, i, 1) + (-i) &= V(a^*, i, 1) + (-i) \\
&< V(a^*, i^*(a^*), 1) + (-i^*(a^*))
\end{aligned}.$$

Therefore, there is $a \in A$ with $a > a^*$ such that

$$V(a, i, 1) + (-i) < \Pi(a, i^*(a^*), 1, P^*(i^*(a^*), 1)).$$

This implies

$$\Pi^*(a) < \Pi(a, i^*(a^*), 1, P^*(i^*(a^*), 1)),$$

that is type a strictly benefits from deviating to $(i^*(a^*), 1)$.

(3) $i^*(a)$ satisfies (3.10) with $\tilde{i}(a)$ substituted by $i^*(a)$ for $a \in A$.

That $|i^*(a)|$ is strictly increasing on A implies

$$P^*(i^*(a), 1) = V(a, i^*(a), 1) + (-i^*(a))$$

and

$$\Pi^*(a) = V(a, i^*(a), 1) + (-i^*(a))$$

for $a \in A$. Consider types $a_1, a_2 \in A$ such that $a_1 < a_2$. Their equilibrium choices imply

$$\Pi(a_1, i^*(a_2), 1, P^*(i^*(a_2), 1)) \leq \Pi^*(a_1),$$

$$\Pi(a_2, i^*(a_1), 1, P^*(i^*(a_1), 1)) \leq \Pi^*(a_2).$$

These imply

$$\begin{aligned} & V(a_2, i^*(a_2), 1)\frac{V(a_1, i^*(a_2), 1) - V(a_1, i^*(a_1), 1)}{a_2 - a_1} \\ & + i^*(a_1)i^*(a_2)\frac{\frac{V(a_2,i^*(a_2),1)}{i^*(a_2)} - \frac{V(a_1,i^*(a_2),1)}{i^*(a_1)}}{a_2 - a_1} \\ & \leq 0 \end{aligned} , \tag{3.24}$$

$$\begin{aligned} & V(a_1, i^*(a_1), 1)\frac{V(a_2, i^*(a_2), 1) - V(a_2, i^*(a_1), 1)}{a_2 - a_1} \\ & + i^*(a_1)i^*(a_2)\frac{\frac{V(a_2,i^*(a_2),1)}{i^*(a_2)} - \frac{V(a_1,i^*(a_1),1)}{i^*(a_1)}}{a_2 - a_1} \\ & \geq 0 \end{aligned} . \tag{3.25}$$

Taking the limits of (3.24) and (3.25) results in

$$V(a, i^*(a), 1)\,[V_i(a, i^*(a), 1) - 1]\, i^{*\prime}(a) + i^*(a) V_a(a, i^*(a), 1) = 0,$$

which can be simplified into (3.10).

(4) If A is not empty, $i^*(a_{\max}) = I$.

Suppose type $a_{\max}$ chooses repurchase size i such that $|i| \in (I_L, I)$. Since $i^*(a)$ is strictly increasing on A, type $a_{\max}$ is fairly priced, which implies

$$\Pi^*(a_{\max}) = P^*(i, 1) = V(a_{\max}, i, 1) + (-i).$$

Then type $a_{\max}$ benefits from deviating to $(I, 1)$:

$$\begin{aligned}
\Pi(a_{\max}, I, 1, P^*(I, 1)) &= \frac{V(a_{\max}, I, 1)}{1 - \frac{-I}{P^*(I,1)}} \\
&\geq \frac{V(a_{\max}, I, 1)}{1 - \frac{-I}{V(a_{\max}, I, 1) + (-I)}} \quad . \\
&= V(a_{\max}, I, 1) + (-I) \\
&> V(a_{\max}, i, 1) + (-i)
\end{aligned}$$

The first inequality is because

$$P^*(I, 1) \leq V(a_{\max}, I, 1) + (-I),$$

and the second inequality is because $|I| > |i|$. Therefore, type $a_{\max}$ must choose the maximum repurchase size I in a D1 equilibrium outcome.

□

Proof of Proposition 3.4: it follows Proposition 3.5**.**

Proof of Proposition 3.5:

We first prove the uniqueness of the D1 equilibrium outcome. It follows Proposition 3.3 that all repurchasing types of the firm use speed $\theta = 1$. It follows Lemma

3.3 and Proposition 3.4 that there is cutoff type $\hat{a}$ such that types with $a > \hat{a}$ repurchase equity $\tilde{i}(a)$, and types with $a < \hat{a}$ repurchase I_L. It suffices to show

(1) If $\tilde{i}(a) < I_L$ for all $a > a_{\min}$, then $\hat{a} = a_{\min}$;

(2) If there is $a > a_{\min}$ such that $\tilde{i}(a) = I_L$, then $\hat{a}$ satisfies

$$\hat{a} = \max\left\{\tilde{a} : \frac{V(\tilde{a}, I_L, 1)}{1 + \frac{I_L}{E[V(a,I_L,1)|a\in(a_{\min},\tilde{a})]-I_L}} \geq V\left(\tilde{a}, \tilde{i}\left(\tilde{a}\right), 1\right) - \tilde{i}\left(\tilde{a}\right)\right\}. \qquad (3.26)$$

(1) If $\tilde{i}(a) < I_L$ for all $a > a_{\min}$, then all types repurchase $\tilde{i}(a) < I_L$, that is $\hat{a} = a_{\min}$.

(a) Each type a prefers choosing $(\tilde{i}(a), 1)$ at the price $V(a, \tilde{i}(a), 1) + (-\tilde{i}(a))$ to choosing $(\tilde{i}(a_{\min}), 1)$ at price $V(a_{\min}, \tilde{i}(a_{\min}), 1) + (-\tilde{i}(a_{\min}))$.

For $a' \gtrless a$,

$$\begin{aligned}&\frac{d}{da'}\Pi\left(a, \tilde{i}(a'), 1, V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right)\\&= \Pi_i\frac{\partial\tilde{i}(a')}{\partial a'} + \Pi_p\frac{\partial\left[V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right]}{\partial a'}\end{aligned}$$

where Π_i and Π_p stand for the derivatives of $\Pi\left(a, \tilde{i}(a'), 1, V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right)$ with respect to the second and fourth inputs.

$$\frac{\Pi_i}{\Pi} = \left(\frac{V_i\left(a, \tilde{i}(a'), 1\right)}{V\left(a, \tilde{i}(a'), 1\right)} - \frac{1}{V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))}\right)$$

where Π stands for $\Pi\left(a, \tilde{i}(a'), 1, V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right)$, and

$$\frac{\Pi_p}{\Pi} = \frac{\tilde{i}(a')}{\left[V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right]V(a', \tilde{i}(a'), 1)}.$$

According to (3.10),

$$\frac{d}{da'}\Pi\left(a, \tilde{i}(a'), 1, V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right) = 0$$

if $a = a'$. Since $\frac{\Pi_i}{\Pi}$ strictly decreases in a, and $\frac{\partial \tilde{i}(a')}{\partial a'} < 0$,

$$\frac{d}{da'}\Pi\left(a, \tilde{i}(a'), 1, V(a', \tilde{i}(a'), 1) + (-\tilde{i}(a'))\right) \gtreqless 0. \tag{3.27}$$

for $a \gtreqless a'$.

(b) $(a_{\min}, \hat{a})$ is empty.

Since $|\tilde{i}(a_{\min})| > |I_L|$, type $\hat{a}$ strictly prefers choosing $(\tilde{i}(a_{\min}), 1)$ at price $V(a_{\min}, \tilde{i}(a_{\min}), 1) + (-\tilde{i}(a_{\min}))$ to choosing $(I_L, 1)$ at the equilibrium price:

$$\begin{aligned}
&\Pi\left(\hat{a}, \tilde{i}(a_{\min}), 1, V(a_{\min}, \tilde{i}(a_{\min}), 1) + (-\tilde{i}(a_{\min}))\right) \\
&= \frac{V(\hat{a}, \tilde{i}(a_{\min}), 1)}{1 - \frac{-\tilde{i}(a_{\min})}{V(a_{\min}, \tilde{i}(a_{\min}), 1) + (-\tilde{i}(a_{\min}))}} \\
&= \frac{\hat{a} - (-\tilde{i}(a_{\min}))(1-b)}{a_{\min} - (-\tilde{i}(a_{\min}))(1-b)}\left[a_{\min} + (-\tilde{i}(a_{\min}))b\right] \\
&\geq \frac{\hat{a} - (-I_L)(1-b)}{a_{\min} - (-I_L)(1-b)}\left[a_{\min} + (-\tilde{i}(a_{\min}))b\right] \\
&> \frac{\hat{a} - (-I_L)(1-b)}{a_{\min} - (-I_L)(1-b)}\left[a_{\min} + (-I_L)b\right] \\
&= \Pi\left(\hat{a}, I_L, 1, V(a_{\min}, I_L, 1) + (-I_L)\right) \\
&\geq \Pi\left(\hat{a}, I_L, 1, P^*(I_L, 1)\right)
\end{aligned}.$$

The first inequality is due to $-\tilde{i}(a_{\min}) > -I_L$ and $\hat{a} \leq a_{\min}$, the second inequality is due to $-\tilde{i}(a_{\min}) > -I_L$, and the last inequality is due to

$$P^*(I_L, 1) \geq V(a_{\min}, I_L, 1) + (-I_L).$$

On the other hand, since either $\hat{a} = a_{\max}$ or types with $a > \hat{a}$ choose $\tilde{i}(a) < \tilde{i}(\hat{a})$, Lemma 3.1 implies

$$P^*(\tilde{i}(\hat{a}), 1) \leq V(\hat{a}, \tilde{i}(\hat{a}), 1) - \tilde{i}(\hat{a}).$$

Step (a) implies type $\hat{a}$ prefers choosing $(\tilde{i}(\hat{a}), 1)$ at price $V(\hat{a}, \tilde{i}(\hat{a}), 1) + (-\tilde{i}(\hat{a}))$ to choosing $(\tilde{i}(a_{\min}), 1)$ at price $V(a_{\min}, \tilde{i}(a_{\min}), 1) + (-\tilde{i}(a_{\min}))$.

Therefore,

$$\begin{aligned}\Pi\left(\hat{a},\tilde{i}(\hat{a}),1,P^*(\tilde{i}(\hat{a}),1)\right) &\geq \Pi\left(\hat{a},\tilde{i}(\hat{a}),1,V(\hat{a},\tilde{i}(\hat{a}),1)+(-\tilde{i}(\hat{a}))\right)\\ &\geq \Pi(\hat{a},\tilde{i}(a_{\min}),1,V(a_{\min},\tilde{i}(a_{\min}),1)+(-\tilde{i}(a_{\min})),\\ &> \Pi\left(\hat{a},I_L,1,P^*(I_L,1)\right)\end{aligned}$$

that is type $\hat{a}$ strictly prefers $(\tilde{i}(\hat{a}),1)$ to $(I_L,1)$. By continuity of $\Pi(a,i,\theta,p)$ in a, if $(a_{\min},\hat{a})$ is not empty, there is type $a'\in(a_{\min},\hat{a})$ that deviates to $(\tilde{i}(\hat{a}),1)$.

(2) If there is $a>a_{\min}$ such that $\tilde{i}(a)=I_L$, then $\hat{a}$ satisfies (3.26).

If there is $a>a_{\min}$ such that $\tilde{i}(a)=I_L$, then $\hat{a}>a_{\min}$, and types $a\in[a_{\min},\hat{a})$ repurchase I_L with method $\theta=1$.

Suppose $\hat{a}<a_{\max}$. Types with $a>\hat{a}$ prefer $(\tilde{i}(a),1)$ to $(I_L,1)$:

$$V(a,\tilde{i}(a),1)+(-\tilde{i}(a))\geq\Pi(a,I_L,1,P^*(I_L,1)).$$

Taking limits implies

$$V(\hat{a},\tilde{i}(\hat{a}),1)+(-\tilde{i}(\hat{a}))\geq\Pi(\hat{a},I_L,1,P^*(I_L,1)).$$

Types with $a'\in(\bar{a},\hat{a})$ prefer $(I_L,1)$ to $(\tilde{i}(a),1)$ for any $a>\hat{a}$:

$$\Pi(a',I_L,1,P^*(I_L,1))\geq\Pi(a',\tilde{i}(a),1,P^*(\tilde{i}(a),1)).$$

Taking limits of a' and a leads to

$$\Pi(\hat{a},I_L,1,P^*(I_L,1))\geq V(\hat{a},\tilde{i}(\hat{a}),1)+(-\tilde{i}(\hat{a})).$$

Therefore,

$$V(\hat{a},\tilde{i}(\hat{a}),1)+(-\tilde{i}(\hat{a}))=\Pi(\hat{a},I_L,1,P^*(I_L,1))$$

where

$$P^*(I_L, 1) = E\left[V(a, I_L, 1) | a \in [a_{\min}, \hat{a})\right] + (-I_L).$$

Suppose $\hat{a} = a_{\max}$. Then Lemma 3.1 implies $(\tilde{i}(\hat{a}), 1) = (I, 1)$ is associated with $a_{\max}$:

$$P^*(I, 1) = V(a_{\max}, I, 1) + (-I).$$

That types with $a \in [a_{\min}, \hat{a})$ prefer $(I_L, 1)$ to $(\tilde{i}(\hat{a}), 1)$ implies

$$\Pi(a, I_L, 1, P^*(I_L, 1)) \geq \Pi(a, \tilde{i}(\hat{a}), 1, P^*(\tilde{i}(a), 1)).$$

Taking limits implies

$$\Pi(\hat{a}, I_L, 1, P^*(I_L, 1)) \geq V(\hat{a}, \tilde{i}(\hat{a}), 1) + (-\tilde{i}(a)).$$

We next prove the existence of a D1 equilibrium outcome.

(1) If (3.10) implies $\tilde{i}(a) < I_L$ for all a, then the strategy to repurchase $\tilde{i}(a)$ for all a supports a D1 equilibrium outcome.

(a) No type benefits from mimicking another type.
Inequality (3.27) implies $\Pi\left(a, i^*(a'), 1, P^*(i^*(a'), 1)\right)$ is quasi-concave in a'.

(b) No type benefits from deviating to an off-equilibrium (i, θ) with

$$|i(1 - \theta b)| \leq |i^*(a_{\min})(1 - b)|. \tag{3.28}$$

Inequality (3.28) implies $|i\theta| \leq |i^*(a_{\min})|$. Since type a does not benefit from mimicking type $a_{\min}$, it does not benefit from deviating to (i,θ):

$$\begin{aligned}\Pi(a,i,\theta,P^*(i,\theta)) &= \frac{V(a,i,\theta)}{1-\frac{-i}{P^*(i,\theta)}} \\ &\leq \frac{V(a,i,\theta)}{1-\frac{-i}{V(a_{\min},i,\theta)+(-i)}} \\ &= \frac{V(a,i,\theta)}{V(a_{\min},i,\theta)}\left(V(a_{\min},i,\theta)+(-i)\right) \\ &\leq \frac{V(a,i^*(a_{\min}),1)}{V(a_{\min},i^*(a_{\min}),1)}P^*(i^*(a_{\min}),1) \\ &= \Pi(a,i^*(a_{\min}),1,P^*(i^*(a_{\min}),1))\end{aligned}$$

The second inequality is because $\frac{V(a,i,\theta)}{V(a_{\min},i,\theta)}$ weakly increases in $|i(1-\theta b)|$,

$$P^*(i^*(a_{\min}),1) = V(a_{\min},i^*(a_{\min}),1)+(-i^*(a_{\min}))$$

and $|i\theta| \leq |i^*(a_{\min})|$.

(c) No type benefits from deviating to (i,θ) with $\theta < 1$ and

$$|i(1-\theta b)| > |i^*(a_{\min})(1-b)|. \tag{3.29}$$

Let D_a and D_a^0 respectively denote the set of prices of (i,θ) that makes type a strictly prefer to deviate to (i,θ) and indifferent. There is $p(a)$ such that $D_a^0 = \{p(a)\}$ and $D_a = \{p < p(a)\}$. D1 belief about (i,θ) is supported on those types that deviate under the largest set of prices, $\arg\max_a p(a)$. If under the equilibrium belief, a type in $\arg\max_a p(a)$ does not benefit from deviating to (i,θ), that is

$$P^*(i,\theta) \geq \max_a p(a),$$

then no type benefits from deviating to (i,θ).

Let type a'' be a type in the support of the D1 belief on (i,θ) such that

$$P^*(i,\theta) \geq V(a'',i,\theta) - i.$$

Inequality (3.29) implies there is a' such that $|i(1-\theta b)| > |i^*(a')(1-b)|$ and $|i\theta| < |i^*(a')|$. Lemma 3.1 implies (i, θ) cannot be associated with any type $a > a'$ in a D1 belief. Therefore, there is type $a'' \leq a'$ such that a'' is associated with (i, θ) and

$$P^*(i, \theta) \geq V(a'', i, \theta) + (-i).$$

Since type a'' does not benefit from mimicking type a' as shown in step (c), it does not benefit from deviating to (i, θ):

$$\begin{aligned}
\Pi(a'', i, \theta, P^*(i, \theta)) &= \frac{V(a'', i, \theta)}{1 - \frac{-i}{P^*(i,\theta)}} \\
&\leq V(a'', i, \theta) + (-i) \\
&< V(a'', i^*(a'), 1) + (-i^*(a')) \ . \\
&< \frac{V(a'', i^*(a'), 1)}{1 - \frac{-i^*(a')}{V(a', i^*(a'), 1) + (-i^*(a'))}} \\
&= \Pi(a'', i^*(a'), 1, P^*(i^*(a'), 1))
\end{aligned}$$

The second inequality is because $|i\theta| < |i^*(a')|$, and the third inequality is because $V(a'', i', \theta') \leq V(a', i', \theta')$. This implies no type a benefits from deviating to (i, θ).

(2) Suppose (3.10) implies there is type $a^* > a_{\min}$ such that $|\tilde{i}(a^*)| = |I_L|$. Define $p(\hat{a})$, and $\hat{a}(p)$:

$$p(\hat{a}) \equiv E\left[V(a, I_L, 1) | a \in [a_{\min}, \hat{a})\right] + (-I_L),$$

$$\hat{a}(p) \equiv \sup\left\{a : \Pi(a, I_L, 1, p) \geq V(a, \tilde{i}(a), 1) + (-\tilde{i}(a))\right\}.$$

$\hat{a}$ satisfies (3.26) if and only if $\hat{a} = \hat{a}(p(\hat{a}))$.

(a) There is $\hat{a}$ such that $\hat{a} = \hat{a}(p(\hat{a}))$, or equivalently, there is p such that $p = p(\hat{a}(p))$.

Let

$$f(p) \equiv p - p(\hat{a}(p)).$$

Let

$$p_1 = V(a^*, I_L, 1) + (-I_L),$$

then $\hat{a}(p_1) = a^*$. $p(a^*) < p_1$,which implies $f(p_1) > 0$.

Let $p_2 = p(a^*)$. Then $\hat{a}(p_2) > a^*$. This implies $f(p_2) < 0$.

Since $f(p)$ is continuous in p, there must be $p^* \in (p_2, p_1)$ such that $f(p^*) = 0$. This implies $p^* = p(\hat{a}(p^*))$, and $\hat{a} = \hat{a}(p^*) > a^*$.

(b) This $\hat{a}$ supports a D1 equilibrium outcome.

i. The definition of $\hat{a}(p)$ implies no type in $(\hat{a}, a_{\max}]$ benefits from deviating to $(I_L, 1)$.

ii. It follows inequality (3.27) that $\Pi(a, i^*(a'), 1, P^*(i^*(a'), 1))$ is quasi-concave in a'. This implies no type in $(\hat{a}, a_{\max}]$ benefits from mimicking another type in $(\hat{a}, a_{\max}]$.

iii. If $\hat{a} < a_{\max}$, no type in $[a_{\min}, \hat{a})$ benefits from mimicking a type in $(\hat{a}, a_{\max}]$.

The definition of $\hat{a}(p)$ implies type $\hat{a}$ is indifferent between $(i^*(\hat{a}), 1)$ at price $V(\hat{a}, i^*(\hat{a}), 1) + (-i^*(\hat{a}))$ and $(I_L, 1)$:

$$\begin{aligned}
&\frac{\Pi(\hat{a}, i^*(\hat{a}), 1, V(\hat{a}, i^*(\hat{a}), 1) + (-i^*(\hat{a})))}{\Pi(\hat{a}, I_L, 1, P^*(I_L, 1))} \\
&= \frac{\hat{a} - (-i^*(\hat{a}))(1-b)}{\hat{a} - (-I_L)(1-b)} \cdot \frac{1 - \frac{-I_L}{P^*(I_L,1)}}{1 - \frac{-i^*(\hat{a})}{V(\hat{a}, i^*(\hat{a}), 1) + (-i^*(\hat{a}))}}. \\
&= 1
\end{aligned}$$

Since $\frac{a-(-i^*(\hat{a}))(1-b)}{a-(-I_L)(1-b)}$ increases in a, for $a \in [\bar{a}, \hat{a})$,

$$\frac{\Pi(a, i^*(\hat{a}), 1, V(\hat{a}, i^*(\hat{a}), 1) + (-i^*(\hat{a})))}{\Pi(a, I_L, 1, P^*(I_L, 1))} \leq 1.$$

Therefore, no type in $[a_{\min}, \hat{a})$ benefits from deviating to $(i^*(\hat{a}), 1)$ at price $V(\hat{a}, i^*(\hat{a}), 1) + (-i^*(\hat{a}))$. Inequality (3.27) implies $\Pi(a, i^*(a'), 1, P^*(i^*(a')$ is quasi-concave in a'. Therefore, no type in $[a_{\min}, \hat{a})$ benefits from mimicking a type in $(\hat{a}, a_{\max}]$.

iv. That no type benefits from deviating to (i, θ) with $\theta < 1$ and

$|i(1-\theta b)| > |\tilde{i}(\hat{a})(1-b)|$ follows an argument similar to 1(c).

v. No type benefits from deviating to $(i,\theta) \neq (I_L, 1)$ such that

$$|i(1-\theta b)| \leq |\tilde{i}(\hat{a})(1-b)|. \tag{3.30}$$

Inequality (3.30) implies $|i| \leq |\tilde{i}(\hat{a})|$ and $|i\theta| \leq |i^*(a)|$. Lemma 3.1 implies

$$P^*(i,\theta) = V(\hat{a}, i, \theta) + (-i)$$

and

$$P^*(\tilde{i}(\hat{a}), 1) = V(\hat{a}, \tilde{i}(\hat{a}), 1) + (-\tilde{i}(\hat{a})).$$

Type $\hat{a}$ weakly prefers $(i^*(\hat{a}), 1)$ to (i,θ) because both give it fair pricing and the former leads to higher NPV:

$$\begin{aligned} \Pi\left(\hat{a}, i, \theta, P^*(i,\theta)\right) &= V(\hat{a}, i, \theta) - (-i) \\ &\leq V(\hat{a}, \tilde{i}(\hat{a}), 1) - (-\tilde{i}(\hat{a})) \quad . \\ &= \Pi\left(\hat{a}, \tilde{i}(\hat{a}), 1, P^*(\tilde{i}(\hat{a}), 1)\right) \end{aligned}$$

Due to (3.30),

$$\frac{\Pi\left(a, i, \theta, P^*(i,\theta)\right)}{\Pi\left(a, \tilde{i}(\hat{a}), 1, P^*(\tilde{i}(\hat{a}), 1)\right)} = \frac{a - (-i)(1-\theta b)}{a - (-\tilde{i}(\hat{a}))(1-b)} \cdot \frac{1 - \frac{-\tilde{i}(\hat{a})}{P^*(\tilde{i}(\hat{a}),1)}}{1 - \frac{-i}{P^*(i,\theta)}}$$

is weakly decreasing in a, for $a > \hat{a}$, $\frac{\Pi(a,i,\theta,P^*(i,\theta))}{\Pi\left(a,\tilde{i}(\hat{a}),1,P^*(\tilde{i}(\hat{a}),1)\right)} \leq 1$. Inequality (3.27) implies types with $a > \hat{a}$ do not benefit from deviating to $(\tilde{i}(\hat{a}), 1)$. Therefore, they do not benefit from deviating to (i,θ).

On the other hand, the definition of $\hat{a}(p)$ implies type $\hat{a}$ weakly prefers $(I_L, 1)$ to $(i^*(\hat{a}), 1)$. Therefore, the above anlysis implies type $\hat{a}$ weakly prefers $(I_L, 1)$ to (i,θ). Since $(i,\theta) \neq (I_L, 1)$, $|i(1-$

$\theta b)| > |I_L(1-b)|$, which implies

$$\frac{\Pi\left(a, i, \theta, P^*(i, \theta)\right)}{\Pi\left(a, I_L, 1, P^*(I_L, 1)\right)} = \frac{a-(-i)(1-\theta b)}{a-(-I_L)(1-b)} \cdot \frac{1-\frac{-I_L}{P^*(I_L,1)}}{1-\frac{-i}{P^*(i,\theta)}}$$

strictly increases in a. This implies for $a < \hat{a}$, $\frac{\Pi(a,i,\theta,P^*(i,\theta))}{\Pi(a,I_L,1,P^*(I_L,1))} < 1$, which implies types with $a < \hat{a}$ prefer $(I_L, 1)$ to (i, θ). Since types with $a < \hat{a}$ do not benefit from deviating to $(-I_L, 1)$, they do not benefit from deviating to (i, θ).

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