# Nash Welfare, Valuated Matroids, and Gross Substitues

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# Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work, with the exceptions outlined in the section below.



# Statement of co-authored work

Section 1.1, Chapters 2 and 3 are based on the paper "Auction algorithms for market equilibrium with weak gross substitute demands and their applications" co-authored with Jugal Garg and László Végh [56].

Section 1.2 and Chapter 4 are based on the paper "Approximating Nash social welfare under Rado valuations" co-authored with Jugal Garg and László Végh [55].

Section 1.3, Chapters 5, 6 and 7 are based on the paper "On complete classes of valuated matroids" co-authored with Ben Smith, Georg Loho, and László Végh [69].

During my degree, I have also co-authored the following papers that are not part of this thesis:

- "FPT algorithms for finding near-cliques in $c$-closed graphs" with Balaram Behera, Shweta Jain, Tim Roughgarden and Comandur Seshadhri [15].
- "Safety in multi-assembly via paths appearing in all path covers of a DAG" with Manuel Cáceres, Brendan Mumey, Romeo Rizzi, Massimo Cairo, Kristoffer Sahlin, and Alexandru I. Tomescu [21].
- "Tractable fragments of the maximum Nash welfare problem" with Jugal Garg, Aniket Murhekar, and László Végh [54].

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## Abstract


We study computational aspects of equilibria and fair division problems with a focus on demand and valuation functions that satisfy the (weak) gross substitutes property.

We study the Arrow-Debreu exchange market model with divisible goods where agents' demands satisfy the weak gross substitutes (WGS) property. We give an auction algorithm that obtains an approximate market equilibrium for WGS demands. Previously, such algorithms were known only for restricted classes of WGS demands. We also derive the implications of our technique for spending-restricted market equilibrium for budget-separable piecewise linear concave (budget-SPLC) utilities. Spending-restricted equilibrium was introduced as a continuous relaxation of the Nash Social Welfare (NSW) problem.

Next, we present the first polynomial-time constant-factor approximation algorithm for the NSW problem under Rado valuations. Rado valuations form a general class of valuation functions that arise from maximum cost independent matching problems. They include as special cases assignment (OXS) valuations and weighted matroid rank functions. Our approach also gives the first polynomial-time constant-factor approximation algorithm for the asymmetric NSW problem under Rado valuations, provided that the maximum ratio between the weights is bounded by a constant.

We examine the Matroid Based Valuation (MBV) conjecture by Ostrovsky and Paes Leme (Theoretical Economics 2015). It asserts that every (discrete) gross substitute valuation is a matroid based valuation—a valuation obtained from weighted matroid rank functions by repeated applications of merge and endowment operations. Each matroid based valuation turns out to be an endowment of some Rado valuation. By introducing complete classes of valuated matroids, we exhibit a family of valuations that are gross substitutes but not endowed Rado valuations. This refutes the MBV conjecture. The family is defined via sparse paving matroids.

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# Contents

# 1 Introduction

Algorithmic game theory emerged alongside the rise of decentralized computer networks such as the Internet. The Internet created a new economy for exchange and commerce that allowed for the use of computational tools. Decentralized computer networks, including the Internet, arise from the interaction between many agents (network operators, service providers, users, etc.) each acting in their own self-interest. A natural way to view such selfish behaviour, both human and mechanical, is through the lens of game theory. Combining algorithmic thinking with game-theoretic is the main source of questions and techniques for the algorithmic game theory.

Both computer science and economics have benefited from this fruitful interaction that produced many deep connections between seemingly unrelated ideas. Fundamental concepts from economics like equilibria, auctions, and incentive compatible mechanisms are now central in the applications of computer science involving strategic agents. In the other direction, theoretical computer science has built on the classical economic theory by studying algorithms, approximability, and complexity of game-theoretic concepts.

In this thesis, we study the algorithmic aspects of three related economic topics.

Firstly, we study market equilibria in markets with divisible goods where agents have *weak gross substitutes demands*. Demand is a function that specifies the preffered bundles of an agent at given prices, and weak gross substitutes property states that increasing the price of a good does not reduce the demand for all the other goods. We give an auction-type algorithm for finding an approximate market equilibrium under such demands. In auction algorithms, the main idea is to set-up a set of simple "ground-rules" and let the agents outbid each other as long as they are willing to spend more money. The hope is that this process converges to an equilibrium. While the overall approach is arguably simple, new technical ideas are needed to give an auction algorithm that works for all weak gross substitutes demands.

Secondly, we study the Nash social welfare (NSW) problem: Allocate a set of indivisible items to a set of agents while maximizing the (weighted) geometric mean of agents valuations. This is a central problem in fair division and computational social choice, and it is known that the optimal allocations satisfy desirable fairness and efficiency properties. (Interestingly, the relaxation of the Nash social welfare problem is a market equilibrium problem which allows us to use our auction algorithm to find solutions to the relaxation.)

Since the NSW problem is computationally hard, the focus is on finding constant-factor approximation algorithms.[1] Our main contribution here is a constant-factor approximation algorithm for the problem under a class of submodular valuations that we call *Rado valuations*.

Thirdly, we study the relationship between Rado valuations and (discrete) gross substitutes valuations. Gross substitutes valuations are crucial for the existence and computability of equilibria in markets with indivisible items, mechanism design, and auctions of multiple items. The two classes of valuations are closely related through the *Matroid Based Valuations (MBV)* conjecture. The conjecture states that every gross substitutes valuation arises from weighted matroid rank functions via *endowment* and *merge* operations. If true, the MBV conjecture would imply that every gross substitute valuations is an endowment of some Rado valuation.

A surprising and strong connection was discovered between gross substitutes valuations and valuated matroids. Valuted matroids are a valuated generalization of the classical concept of matroids in discrete mathematics and computer science. We exploit this connection and disprove the MBV conjecture by studying valuated matroids instead. In particular, we introduce the notion of *complete classes* of valuated matroids. We show that the smallest complete class containing R-induced and R-minor valuated matroids (valuated matroids corresponding to Rado valuations and endowed Rado valuations) is not the class of all valuated matoids. This answers negatively a question of Frank, and as a corollary shows that there are gross substitute valuations that cannot be obtained as an endowment of a Rado valuation. This disproves the MBV conjecture.

## 1.1 Auction Algorithm for Market Equilibrium under WGS demands

A *Fisher market* consists of a set of divisible goods and a set of agents each with some budget and preferences over bundles of goods. A market equilibrium comprises a set of prices and allocations of goods to the agents such that each agent spends all their money on a demanded bundle at these prices, and the market clears: the full amount of each good is allocated.

Formally, we define a market equilibrium using *demand systems*. Let $[k] := \{1, 2, \ldots, k\}$. Let $A = [n]$ be a set of agents and let $G = [m]$ be a set of divisible goods. Without loss of generality we assume that the supply of each good is one unit. Each agent $i \in [n]$ arrives at the market with a budget $b_i \in \mathbb{R}_+$. A *bundle* $x$ is non-negative vector $x \in \mathbb{R}^m_+$. A *demand system* is a function $D : \mathbb{R}^{m+1}_+ \to 2^{\mathbb{R}^m_+}$; where $D(p, b)$ denotes the (possibly infinite) set

[1] Throughout the thesis, by an approximation algorithm we mean an approximation algorithm running in polynomial time, unless stated otherwise.

of *optimal* or *demand* bundles at prices $p$ and budget $b$. Here $2^{\mathbb{R}^m_+}$ denotes the family of al subsets of $\mathbb{R}^m_+$.

**Definition 1.1.1** (Market equilibrium). *Let $D_i$ denote the demand system and $b_i$ the budget of agent $i \in A$ in a Fisher market with goods $G$. We say that the prices $p \in \mathbb{R}^m_+$ and bundles $x^{(i)} \in \mathbb{R}^m_+$ form a* market equilibrium *if*

- $x^{(i)} \in D_i(p, b_i)$, *and*
- $\sum_{i=1}^n x_j^{(i)} \leq 1$, *with equality whenever $p_j > 0$, for all $j \in G$.*

The existence of a market equilibrium is always guaranteed under some weak assumptions, as shown by Arrow and Debreu [5], using Kakutani's fixed point theorem. The computational aspects of finding a market equilibrium have been extensively studied in the theoretical computer science community over the last two decades, establishing hardness results as well as polynomial-time algorithms for certain cases [20, 26, 31, 39, 44, 58, 74, 116, 120].

**Utility functions** A standard way to implement a demand system is via an explicitly given utility function. Assume agent $i$ is equipped with a concave utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}_+$. Utility function measures agents satisfaction with a certain bundle of goods. In this case, the set of demand bundles at prices $p$ and budget $b_i$ is the set of bundles maximizing the utility subject to the budget constraint, i.e., the optimal solutions of the following program

$$\begin{aligned} \max \ & u_i(x) \\ \text{s.t.} \ & p^\top x \leq b_i \\ & x \geq 0\,. \end{aligned} \tag{Max-utility}$$

Formally, $D_i(p, b) := D^{u_i}(p, b) = \arg\max_{x \in \mathbb{R}^m_+} \{u_i(x) : p^\top x \leq b_i\}$. Most models studied in the literature assume strictly concave utilities and thus have a unique optimal solution; a notable exception is the case of linear utility functions. A utility function $u$ is linear if, for some $v \in \mathbb{R}^m_+$, it holds $u(x) = v^\top x$ for all $x \in \mathbb{R}_+$.

**Eisenberg-Gale program** A particularly remarkable connection between market equilibria and convex programming was discovered by Eisenberg and Gale [46]. In the case of linear utilities, the market equilibria are exactly the optimal solutions to the following convex program

$$\begin{aligned} \max \sum_{i=1}^{n} b_i \log v_i^\top x^{(i)} \qquad & \left(= \sum_{i=1}^{n} b_i \log u_i(x^{(i)})\right) \\ \sum_{i=1}^{n} x_j^{(i)} \le 1, \quad & \forall j \in [m]. \\ x^{(i)} \ge 0, \quad & \forall i \in [n]. \end{aligned} \tag{EG}$$

Let $p \in \mathbb{R}_+^m$ denote the Lagrange multipliers of the constraints. By the Karush–Kuhn–Tucker (KKT) optimality conditions, $x$ and $p$ are primal and dual optimal solution if and only if

- $b_i \frac{v_{ij}}{v_i^\top x^{(i)}} \le p_j$ where equality holds whenever $x_{ij} > 0$; and
- $\sum_{i \in [n]} x_j^{(i)} \le 1$ where equality holds whenever $p_j > 0$.

The first conditions implies that $x^{(i)} \in D^{u_i}$ for agent $i$, that is, $x^{(i)}$ maximizes (Max-utility). In case of linear utilities, $x^{(i)}$ is a maximizer of (Max-utility) if and only if $i$ spends all budget $b_i$ on the goods with the highest $\frac{v_{ij}}{p_j}$ – called *maximum bang per buck* (MBB) goods. With a bit of algebraic manipulation we can see that this is exactly what the first condition states. The second condition is the same as in the definition of market equilibrium.

More generally, Eisenberg [45] showed that the optimal solutions of the above program are in one-to-one correspondence with the market equilibria whenever the utility functions are homogenous of degree one, that is, $u_i(\alpha x) = \alpha u_i(x)$ for any $\alpha > 0$. In particular, in these cases we can find a market equilibrium with standard convex programming approaches.

**Weak Gross Sustitutability** The first idea for an algorithm or dynamics for finding an equilibrium comes from Walras in 1874 [119]. He informally described the following process, called *tâtonnement*, after observing the stock market. Start with arbitrary prices. If the total demand for the goods is the same as the supply, we have an equilibrium. Otherwise, pick an arbitrary good and "fix" its price: adjust the price of this good such that the demand is equal to its supply. The adjustment might interfere with the demand of other goods, but we are interested in the limit of such process.

The process does not always converge to an equilibrium [109] but a continuous version of the process converges to an equilibrium whenever the utility functions satisfy *weak gross substitutability (WGS)* [4, 7].[2]

Gross substitutability captures the following type of interaction between prices and demands for goods. At given prices, an agent demands a certain amount of goods. If the

[2] Arrow and Hurwitz [7] first studied the local stability of an equilibrium under WGS utilities, and then together with Block [4] they showed that the stability is global [3].

price of a single good increases then we expect that demand for this good decreases. Consequently, more money can be spent on the goods with the unchanged price and thereby the demand for such goods should not decrease. The formal definition follows.

**Definition 1.1.2** (Weak Gross Substitutes)**.** *Let* $(p, b) \in \mathbb{R}_+^{m+1}$ *and* $x \in D(p, b)$*. If for any* $p' \geq p$ *and* $b' \geq b$ *there exists* $y \in D(p', b')$ *such that* $y_j \geq x_j$ *whenever* $p'_j = p_j$*, we say that the demand system* $D$ *satisfies the* weak gross substitutes (WGS) *property.*

The demand system arising from linear utilities satisfies the WGS property. When the demand system is given by a utility function as in (Max-utility), we will simply say that the utility function satisfies the WGS property.

**Computational complexity** The polynomial-time computability of market equilibrium for WGS utilities was first established by Codenotti, Pemmaraju, and Varadarajan [32]. Later, a simple ascending-price algorithm using *global demand queries* was given by Bei, Garg, and Hoefer [16]. Further, Codenotti, McCune, and Varadarajan [30] have shown that a simple discrete variant of the tâtonnement algorithm converges to an approximate equilibrium (see also [102, Section 6.3]). This was followed by a number of papers providing tâtonnement algorithms for various classes of utility functions and restricted models, some of them substantially weakening the need for central coordination among agents, see e.g., [9, 27, 28, 34, 48].

However, most of these algorithms still rely on global demand queries. In a sense, they require a central authority (responsible for updating prices) to have some general information about the demands of all agents in the market.

**Auction algorithms** *Auction algorithms* form a subclass of tâtonnement-type algorithms. Whereas prices in tâtonnement may increase as well as decrease, in auctions prices may only go up. The first such algorithms have been established for markets in which agents have linear utilities by Garg and Kapoor [59] (see also [102, Section 5.12]). The algorithm was later improved [60] and generalized to separable concave gross substitute utility functions [62], to a subclass of non-separable gross-substitutes called *uniformly separable* [61], and to a production model with linear production constraints and linear utilities [77].

There is a long history of auction algorithms both in the optimization and in the economics literature. Bertsekas [17, 18] has introduced auction algorithms for assignment and transportation problems. Closely related algorithms were introduced for markets with indivisible items, by Kelso and Crawford [81], and Demange, Gale, and Sotomayor [38]. We will discuss markets with indivisible items later in this section. In both contexts, the appeal of auction algorithms is their simplicity and distributed nature: under simple "ground rules" the agents outbid each other and in the process converge to an approximate market equilibrium. These algorithms do not require a central authority (e.g., to update the prices) and need only minimal coordination between the agents. Further, these algorithmic frameworks are quite robust and easily allow for various extensions and generalizations as we demonstrate in later sections.

**First main result** The first main contribution is an auction algorithm that computes an approximate market equilibrium for WGS demand oracles, settling an open question from [61]. Our auction algorithm works for more general exchange markets and is presented in Chapter 2. This result shows that for WGS demands, this restricted class of tâtonnement algorithms already suffices to obtain a market equilibrium. The result affirms the natural intuition that the WGS property is geared for auction algorithms. A main invariant in auction algorithms is that at every price increase, the agents will still hold on to the goods they have purchased previously at the lower prices. This property is almost identical to the definition of the WGS property; nevertheless, making an auction algorithm work for general WGS utilities requires some careful technical ideas.

The previously mentioned auction algorithms operate with two prices for each good, a lower price $p_j$ and a higher price $(1+\epsilon)p_j$. For linear utilities, [59] maintains that all purchases are maximum bang-per-buck goods with respect to the lower or higher price. This idea can be extended to separable [60] and to uniformly separable utilities [62], but does not work if the utilities are genuinely non-separable. For this general case, our main technical idea is to maintain subsets of optimal bundles for each agent with respect to some individual prices. These individual prices can be different for the agents but fall between the higher and lower prices $p$ and $(1+\epsilon)p$.

**(Discrete) Gross Substitutes** We have already mentioned WGS utilities in the case of divisible goods. In the case of discrete (indivisible) items, an analogous concept of *gross substitutes* valuations is crucial for the existence and computation of the so-called Walrasian equilibrium. This concept was defined by Kelso and Crawford in 1982 [81]. For a price vector $p \in \mathbb{R}^V$ and a subset $X \subseteq V$, we let $p(X) = \sum_{j\in X} p_j$. A *valuation* $v : 2^V \to \mathbb{R}_+$ is a monotone function taking value 0 on the empty set. The set of *optimal bundles* at prices $p$ is called *demand correspondence* and is the set $D(v,p)$,

$$D(v,p) := \arg\max_{X\subseteq V} v(X) - p(X)\,.$$

**Definition 1.1.3** (Gross Substitutes)**.** *The valuation function $v : 2^V \to \mathbb{R}_+$ is a* gross substitutes (GS) valuation *if for any $p, p' \in \mathbb{R}^V$ such that $p' \geq p$ and any $X \in D(v,p)$, there exists an $X' \in D(v,p')$ such that $X \cap \{j : p_j = p'_j\} \subseteq X'$.*

That is, if we have an optimal bundle at prices $p$ and increase some of the prices, then there will be an optimal bundle that contains all items whose price remained unchanged.

For brevity, in this thesis, we differentiate *divisible goods* and *indivisible* (discrete) *items*. The terms *utility* functions and *WGS* is reserved for continuous utility functions $u : \mathbb{R}^m \to \mathbb{R}_+$ over the set of goods and the terms *GS* and *valuation* for the indivisible items. Additionally, we use *additive* for valuations and *linear* for utilities.

**Auction algorithms and discrete gross substitutability** Auction algorithms have been widely studied in the context of markets with discrete items. An equilibrium may not always exist in such markets. When agents have GS valuations, an equilibrium is guaranteed to exist, and an approximate equilibrium can be efficiently found via a simple auction algorithm, extending [37]. It turns out that the discrete gross substitutes property is essentially a necessary and sufficient condition for the auction algorithm to work and for an equilibrium to exist [65].

Whereas the definitions of discrete gross substitutes and continuous WGS utilities is very similar, there does not appear to be a direct connection between these notions. The main difference is in the utility concepts: for indivisible markets, the standard model is to maximize the valuation minus the price of the set at given prices, whereas the standard divisible market models operate with *fiat money*: the prices appear via the budget constraints but not in the utility value. Still, our first result can be interpreted as a continuous analogue of the strong link between auction algorithms and the gross substitutes property for markets with indivisible items: we show that auction algorithms are applicable for the entire class of WGS utilities for markets with divisible goods. We suspect that the converse should also be true, namely, that the applicability of auction algorithms should be limited to WGS utilities/demands. In contrast, tâtonnement algorithms have been successfully applied beyond the WGS class [27, 28, 48].

## 1.2 Approximating Nash social welfare

In the discrete Nash social welfare (NSW) problem, we need to allocate a set $\mathcal{G}$ of $m$ indivisible items to a set $\mathcal{A}$ of $n$ agents where each agent $i$ has a valuation function $v_i : 2^{\mathcal{G}} \to \mathbb{R}_+$ and weight (entitlement) $w_i > 0$. The goal is to find an allocation maximizing the NSW, defined as the weighted geometric mean of the valuations:

$$\max \left\{ \left( \prod_{i\in\mathcal{A}} v_i(S_i)^{w_i} \right)^{1/\sum_{i\in\mathcal{A}} w_i} : \{S_i : i \in \mathcal{A}\} \text{ forms a partition of } \mathcal{G} \right\} .$$

We refer to the special case when all agents have equal weight (i.e., $w_i = 1$) as the *symmetric* NSW problem, and call the general case the *asymmetric* NSW problem. By taking the logarithm of the objective function we see that the NSW problem is a discrete version of the Eisenberg-Gale program (EG). The objective was first discovered by Nash as the unique solution to a bargaining game [76, 100]. It also coincides with a notion of proportional fairness in networking [80], and with the competitive equilibrium from equal incomes [115].

Fair and efficient allocation of resources is a fundamental problem in many disciplines, including computer science, economics, and social choice theory. The Nash social welfare (NSW) is a popular objective that provides a balanced tradeoff between fairness and efficiency.

A common measure of efficiency is maximizing the *utilitarian social welfare*, i.e., finding an allocation $(S_1, \dots, S_n)$ that maximizes $\sum_{i\in\mathcal{A}} v_i(S_i)$. Naturally, efficiency comes at the expense of fairness: in an optimal utilitarian social welfare allocation some agents might receive no items.

On the other side, maximizing fairness is often associated with maximizing the minimum value across all agents, i.e., $\max_{(S_1,S_2,\dots,S_n)} \min_{i\in\mathcal{A}} v_i(S_i)$. This is also known as *max-min fairness* or the *Santa Claus* problem. Fairness comes at the expense of efficiency: we might have to assign most of the items to an agent with low valuation of all subsets of $\mathcal{G}$ when compared to the valuations of other agents.

The Nash social welfare balances the two above objectives and has provable fairness and efficiency guarantees. When agents are symmetric, the optimal NSW allocation satisfies a relaxation of envy-freeness called *envy-freeness up to one good* and is Pareto-optimal [22]. These properties also carry over to the asymmetric case, where the optimal NSW satisfies a weighted relaxation of envy-freeness and is Pareto-optimal [23].

A distinctive feature of the NSW objective is its invariance under scaling of the valuations. That is, unlike the utilitarian social welfare and the max-min fairness, the set of optimal allocations in the NSW problem remains unchanged even if the valuations of the agents are scaled by arbitrary positive constants.

Finding an optimum of the NSW problem is NP-hard already for two agents with additive valuations (by a reduction from the subset sum problem); a valuation $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ is additive if $v(S) = \sum_{j\in S} v(j)$ for all $S \subseteq \mathcal{G}$. Moreover, the problem is APX-hard for additive valuations [52]. The focus is then on finding constant-factor approximation algorithms for the problem. Naturally, the approximability depends on the class of valuations function we allow. We give two approximation algorithms for the NSW problem. The first one gives a constant-factor approximation algorithm for the symmetric NSW under budget-SPLC valuations, and uses a slightly modified version of our auction algorithm as the starting point (Section 1.2.1). The second is an approximation algorithm for the asymmetric NSW under Rado valuations with the approximation guarantee depending on $\max_{i\in\mathcal{A}} w_i$ (Sections 1.2.2 and 1.2.3). In the symmetric case, this gives the first constant-factor approximation algorithm for Rado valuations.

### 1.2.1 Spending restricted equilibrium and auction algorithm

In a break-through result, Cole and Gkatzelis [35, 36] gave the first constant-factor approximation algorithm for the symmetric NSW problem under additive valuations. They first solve a relaxed continuous problem and round the fractional solution. A natural relaxation is (EG) with linear utilities and $b_i = 1$ for all agents $i$. However, that relaxation has an unbounded integrality gap (e.g., if there is only one item, the optimal NSW value is zero while (EG) has a non-zero solution). To circumvent this issue, the algorithm in [36] first computes a *spending restricted equilibrium* and rounds such an equilibrium to an integer solution of value at least $1/2e^{1/e}$ times the optimal NSW value. We now define spending restricted equilibrium for arbitrary demand systems and explain how we used our auction algorithm to obtain a constant-factor approximation algorithm for the NSW problem under budget-SPLC valuations.

**Definition 1.2.1** (SR-equilibrium)**.** *Let $A$ be a set of agents with demand systems $D_i(p, b_i)$ and fixed budgets $b_i \in \mathbb{R}_+$ for all $i \in A$. We say that the prices $p \in \mathbb{R}^m$ and allocations $x^{(i)} \in D_i(p, b_i)$ form a* Spending Restricted (SR) *equilibrium, if $\sum_{i\in A} x_j^{(i)} = \min\{1, 1/p_j\}$ for all $j \in [m]$.*

It is clear that the amount of money spent on good $j$ is bounded by 1. Note that the spending restrictions cannot be directly added to (EG) as they involve the Lagrange multipliers $p$. An SR-equilibrium in [36] was found via an extension of [39, 103].

The same approach was extended to *separable, piecewise-linear concave (SPLC)* valuations [2], and *budget-additive* valuations [53]. Both papers find the corresponding SR-equilibra (exact or approximate) via fairly complex combinatorial algorithms.

**Approximating NSW under budget-SPLC valuations** We show that auction algorithms are particularly well-suited for SR-equilibrium computation: once the price of a good goes above one, we can naturally decrease the total available amount of these goods. (Surprisingly, here we do not make the standard non-satiation assumption that requires each agent to fully spend her budget on each demanded bundle.) Hence, we obtain a simple approximation algorithm for SR-equilibrium under WGS demands.

Next, we consider the NSW problem with $n$ agents and $m$ items, in which we have $D_j$ units (copies) of item $j$. Each agent $i$ has a budget-SPLC valuation function defined as follows. For every good $j$, agent $i$ has $k_{ij}$ *segments* with *strictly* decreasing utility rates $u_{ij1} > u_{ij2} > \ldots > u_{ijk_{ij}} \geq 0$. Segment $t \in [k_{ij}]$ has length $d_{ijt}$ and agent $i$ values at $u_{ijt}$ each of the units in the $t$-th segment of good $j$. We assume that $\sum_{t\in[k_{ij}]} d_{ijt} = D_j$. Furthermore, agent $i$'s value is capped at $U_i$, i.e., their value for a subset of items is the minimum of $U_i$ and the sum of the values accumulated from the items.

After showing that budget-SPLC *utilities* are WGS under the Gale demand (see Section 3.2.1), we apply the auction algorithm to the relaxed problem to find an approximate

SR-equilibrium. Using a similar rounding as in [53], we obtain a 2.404-approximation algorithm for maximizing NSW in polynomial time when agents have budget-SPLC valuations. The previous approximation algorithm for this setting in [24] runs in pseudopolynomial time (polynomial dependence on $\max_i \sum_j k_{ij}$). These results are presented in Chapter 3.

### 1.2.2 Rado valuations

In game theory, *valuations* or *valuation functions* model user preferences. The term *valuation function* or just *valuation* is used for discrete functions $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ over the set of items $\mathcal{G}$. In particular, a valuation assigns a numerical value to each subset of items. We assume that every valuation function is *monotone*: $v(S) \leq v(T)$ for all $S \subseteq T \subseteq \mathcal{G}$ (also called *free-disposal*); and we that the value of the empty set is 0: $v(\emptyset) = 0$ (also called *normalized* valuation).

A central place occupy submodular functions/valuations. A valuation $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ is *submodular* if

$$v(S) + v(T) \geq v(S \cap T) + v(S \cup T) \quad \forall S, T \subseteq \mathcal{G} .$$

Or equivalently, via *decreasing marginals*, $v$ is submodular if and only if $v(S \cup \{e\}) - v(S) \leq v(T \cup \{e\}) - v(T)$ for all $S \subseteq T \subset \mathcal{G}$ and $e \in \mathcal{G} \setminus T$.

Thus, submodular valuations have a natural diminishing returns making them particularly suitable for applications in economics [112]. Submodular functions are also unavoidable in computer science in areas such are combinatorial optimization [90], discrete convex analysis [98], machine learning [84], and computer vision [75].

Another important class of valuations are subadditive valuations. We say that $v : 2^{\mathcal{G}} \to \mathbb{R}_+$ is *subadditive* if

$$v(S) + v(T) \geq v(S \cup T) \quad \forall S, T \subseteq \mathcal{G} .$$

Trivially, every submodular valution is also subadditive. Subadditve functions are also very common in mathematics, economics and related areas.

**Matroids** To introduce Rado valuations we need to recall the notion of a matroid. A *matroid* on a finite ground set $V$ is given as $\mathcal{M} = (V, \mathcal{I})$, where $\mathcal{I} \subseteq 2^V$ is a nonempty collection of *independent sets*. This collection is required to satisfy the *independence axioms*:

(I1) *Monotonicity:* if $X \in \mathcal{I}$ then $Y \in \mathcal{I}$ for all $Y \subseteq X$, and

(I2) *Exchange property:* if $X, Y \in \mathcal{I}$, $|X| < |Y|$, then there exists a $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

The *rank function* $r_{\mathcal{M}} : 2^V \to \mathbb{Z}_+$ associated with the matroid $\mathcal{M}$ is defined with $r_{\mathcal{M}}(X)$ denoting the size of the largest independent subset of $X \subseteq V$. A fundamental property implied by (I2) is that every maximal independent set in $X$ has size $r_{\mathcal{M}}(X)$. The value $r_{\mathcal{M}}(V)$ is called the rank of the matroid, and the maximal independent sets are called *bases*. A set $X \subseteq V$ is in $\mathcal{I}$ if and only if $r(X) = |X|$; and we can equivalently define a matroid by its rank function justifying an alternative notation $\mathcal{M} = (V, r)$. We refer the reader to [110, Part IV] for matroids and their role in optimization. For other characterizations of matroids, see e.g., [105].

It is easy to check that every rank function is submodular. Moreover, every integer valued monotone submodular set function on $V$ with $v(X) \leq |X|$ arises as the rank function of a matroid. Given a weighting $g \in \mathbb{R}_+^V$, the *weighted rank function* $r_g(X)$ is the maximum $g$-weight of a maximal independent set in $X$; this function is also submodular.

**Rado valuations** The key class of valuations for our next main results is that of Rado valuations. We propose the name "Rado valuations" in honor of Richard Rado, who first studied the independent matching problem [107].[3] We denote a bipartite graph by $(\mathcal{G}, V; E)$, where $\mathcal{G}, V$ are the partitioned node sets and $E$ the edge set.

**Definition 1.2.2** (Rado valuation)**.** *Assume we are given a bipartite graph $(\mathcal{G}, V; E)$ with a cost function $c : E \to \mathbb{R}_+$ on the edges, and a matroid $\mathcal{M} = (V, \mathcal{I})$. For a subset of items $S \subseteq \mathcal{G}$, the* Rado valuation function *$v(S)$ is defined as the maximum cost of a matching $M$ in $(\mathcal{G}, V; E)$ such that $\partial_{\mathcal{G}}(M) \subseteq S$ and $\partial_V(M) \in \mathcal{I}$, i.e.,*

$$v(S) := \max \left\{ \sum_{e \in M} c(e) : M \textit{ is a matching}, \partial_{\mathcal{G}}(M) \subseteq S, \partial_V(M) \in \mathcal{I} \right\} .$$

*Here, $\partial_X(M)$ denotes the set of endpoints of $M$ in a vertex-set $X$.*

Let us consider the special case where the matroid $\mathcal{M}$ is the free matroid on $V$, i.e., $\mathcal{I} = 2^V$. In this case, the matroid constraints $\partial_V(M) \in \mathcal{I}$ are vacuous. The value of a set $S$ is then the maximum cost matching in the bipartite subgraph induced by $S \cup V$. Such valuations are called *assignment valuations* by Shapley [111], and *OXS valuations* by Lehmann, Lehmann, and Nisan [85].

Shapley [111] gave a nice interpretation of assignment valuations. Assume that the agent is a company. Furthermore, assume that the items $\mathcal{G}$ are workers and $V$ is the set of jobs within the company. The edge set represents the possibilities (willingness) of assigning workers to jobs, and the cost $c_{jk}$ is the value the company gets by assigning worker $j$ to job $k$. By the definition of assignment valuations, the value of a subset $S \subseteq \mathcal{G}$ of workers for the company is the maximum possible value the company gets by assigning workers $S$ to jobs $V$.

[3]These functions previously appeared as *valuations arising via bipartite matching with a matroid constraint* [89] and *independent assignment valuations* [98].

The same interpretation extends to Rado valuations with the additional possibility that the occupied set of jobs must be an independent set in matroid $\mathcal{M}$. For example, the company may partition the set of all jobs $V$ into certain types, require that at most one job of each type to be assigned and additionally limit the total number of employed workers—a laminar matroid constraint.

As another example of Rado valuations, consider the case where $V$ is a copy of the set of items $\mathcal{G}$, with each $j \in \mathcal{G}$ having a corresponding $j' \in V$, and let $E = \{(j, j') : j \in \mathcal{G}\}$. Let $g : \mathcal{G} \to \mathbb{R}$, and $c_{jj'} = g_j$ for all $j \in \mathcal{G}$, and let $r$ be the rank function of $\mathcal{M}$. In this case $v(S)$ equals the weighted matroid rank function $r_g(S)$, i.e., the maximum $g$-weight of an independent subset of $S$.

In Section 6.4, we prove that every Rado valuation on a ground set $\mathcal{G}$ of $m$ elements admits a representation with a bipartite graph of size $O(m^2)$.

The relation between popular classes of valuation functions is given in Figure 1.1.

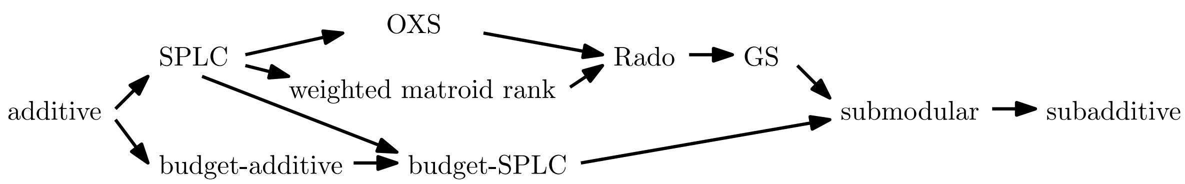


Figure 1.1: Relation between classes of valuation functions. Arrows represent *strict* inclusion. If an arrow is not present, the classes are incomparable; see [85, 102]. Strict containment between Rado valuations and gross substitute valuations is proved in Section 4.7. Note that SPLC and budget-SPLC valuations are defined over $\mathbb{Z}^{\mathcal{G}}$, but an equivalent formulation over $2^{\mathcal{G}}$ can be easily obtained by considering the copies as individual items, and grouping them into "item-types".

### 1.2.3 Approximating Asymmetric Nash Social Welfare under Rado valuations

Next to the approach of using SR-equilibrium, other innovative approaches for approximating the symmetric NSW problem under additive valuations were also developed. Anari et al. [1] gave a constant-factor approximation algorithm using the theory of real stable polynomials. Barman et al. [14] developed another approach based on local search (price envy-freeness) that provides the state-of-the-art approximation factor of $1.45$.

All three approaches have also been extended to obtain constant-factor approximation algorithms for mild generalizations of additive, namely, budget-additive [53], SPLC [2], and budget-SPLC valuations [24]. All these approaches heavily exploit the symmetry of agents and the characteristics of 'additive-like' valuations (such as MBB) which make them hard to extend to significantly more general settings. Moreover, as we have already mentioned the corresponding continuous utility functions satisfy the WGS property.

For more general valuations or the asymmetric NSW problem, the best approximation algorithms achieve an $O(n)$-approximation factor [13, 25, 57], and these algorithms work for the asymmetric NSW under subadditive valuations. However, their analysis is based on averaging arguments, making them hard to yield a factor better than $O(n)$ even for the special cases, e.g., OXS valuations, or only two types of agents with weights 1 or 2 under additive valuations. Therefore, with the following exception, $O(n)$ remained the best approximation factor for the symmetric NSW problem beyond 'additive-like' valuations or for the asymmetric NSW problem.

Li and Vondrák [88] gave a $\frac{e^3}{(e-1)^2}$-estimation algorithm for the optimal value of the *symmetric* NSW problem under valuations arising as conic combinations of Rado valuations. The paper extends the real stable polynomial approach [1]. The algorithm approximates the objective value only and does not find a near-optimal allocation, as the randomized rounding finds an approximate solution with exponentially small probability.

**Second main result** We make significant progress towards both symmetric and asymmetric NSW by developing a novel approach for approximating the problem. In particular, we give a constant-factor approximation algorithm for the symmetric NSW problem under Rado valuations.

**Theorem 1.2.3.** *There exists a polynomial-time* $256e^{3/e}{\approx}772$*-approximation algorithm for the symmetric Nash social welfare problem under Rado valuations.*

Next, we obtain a constant-factor approximation for the asymmetric NSW problem under Rado valuations, provided that the maximum ratio between the weights is bounded by a constant. We note that no such result was known even for additive valuations. Assume the weights $w_i$ of the agents fall in the interval $[1, \gamma - 1]$ for some $\gamma \geq 2$.

**Theorem 1.2.4.** *There exists a polynomial-time* $256\gamma^3$*-approximation algorithm for the asymmetric Nash social welfare problem with Rado valuations. For additive valuations, there exists a polynomial-time* $16\gamma$*-approximation algorithm.*

We note that $\gamma$ in the theorem can be replaced by $\min\left\{O\left(\frac{\gamma}{\log\gamma}\right), n\right\}$ as we explain in Section 4.1.2.

The algorithm carefully combines techniques from convex programming and bipartite matching. It is a modular algorithm presented in five phases. We give an overview of these phases in Section 4.2 with further details given throughout Chapter 4.

Table 1.1 summarizes the updated best approximation guarantees for the problem under various valuation functions.

We note that our approach can be easily modified (in **Phase IV**) to give similar results for the class of *budget-Rado* valuations, where the value of subsets of items is given by a Rado valuation but not more than a given threshold.

| Valuations | Symmetric | Asymmetric |
|---|---|---|
| Additive | 1.45 [14] | $O(\gamma)$ [Theorem 1.2.4] |
| SPLC | 1.45 [24] | $O(\gamma^3)$ [Theorem 1.2.4] |
| budget-SPLC | 1.45 [24], 2.404 [Theorem 3.2.7] | $O(n)$ [13, 25] |
| Rado | $O(1)$ [Theorem 1.2.3] | $O(\gamma^3)$ [Theorem 1.2.4] |
| Subadditive | $O(n)$ [13, 25] | $O(n)$ [13, 25] |

Table 1.1: Summary of the best approximation algorithms for the NSW problem. The table excludes the $\frac{e^3}{(e-1)^2}$-estimation algorithm for the cone of Rado valuations [88]. The results for budget-SPLC valutations are pseudopolynomial-time and polynomial-time algorithms, respectively.

**Subsequent work** In a subsequent work, Li and Vondrák [89] obtained a 380-approximation algorithm for the symmetric NSW problem under submodular valuations. This is obtained by strengthening and extending our approach while introducing important new techniques. This settles the constant-factor approximability of the symmetric NSW problem as an $O(n^{1-\varepsilon})$-approximation algorithm for the problem under subadditive valuations requires an exponential number of oracle queries for any fixed $\varepsilon > 0$ [13].

The constant-factor approximability of the asymmetric NSW problem remains open even for additive valuations. We note that in our approach, for additive valuations, the factor $\gamma$ only appears in a single reduction step (**Phase II**).

**Difficulties in approximating asymmetric NSW** We note that even if the weights of the agents are bounded, an $O(1)$-approximation algorithm for the symmetric case does not yield an $O(1)$-approximation algorithm to the asymmetric case. To illustrate this point, consider two items and two agents with weights $w_1 = 2$, $w_2 = 1$ and additive valuations $v_1(\{a\}) = M$, $v_1(\{b\}) = 1$, $v_2(\{a\}) = M + 1$, $v_2(\{b\}) = 1$, where $M$ is an arbitrarily large number. The unique optimal solution to the symmetric case (by setting $w'_1 = w'_2 = 1$) is allocating good $b$ to agent 1 and good $a$ to agent 2. However, this returns an NSW value $(M+1)^{1/3}$ for the original weights. This can be worse by an arbitrary factor than the value $M^{2/3}$ obtainable by assigning good $a$ to agent 1 and good $b$ to agent 2.

The same example shows another difficulty and illustrates the limit of equilibrium-based approaches when approximating asymmetric NSW. Namely, any constant-factor approximation algorithm for the asymmetric NSW under additive valuations, cannot just round an equilibrium only on the MBB edges (the support of an equilibrium). Consider any prices $p_a$ and $p_b$ for our two items $a$ and $b$. Then, regardless of the choice of $p_a$ and $p_b$ it cannot be the case that both item $a$ is MBB for agent $1$ and that item $b$ is MBB for agent $2$. Thus, any assignment via the MBB edges will assign $b$ to $1$ and $a$ to $2$; note that this is regardless of agents budgets, prices, and spending limits on the items. As we have seen above, this can be arbitrarily worse than the optimum.

## 1.3 Constructions of substitutes and complete classes of valuated matroids

**Constructions of substitutes** Gross substitutes valuations play a central role in algorithmic game theory and especially in auction and mechanism design. In such settings an important issue is the way in which agents represent their valuations. This raised a quest for a constructive characterization of all GS valuations. Constructive characterization of GS valuations would specify a language in which agents can represent their valuations in a compact and expressible way [86]. Moreover, valuations with a constructive description facilitate more algorithmic techniques, especially linear programming as we will see in Sections 4.1.1, 4.4, and 6.2. While we know many characterizations of GS functions (Balkanski and Paes Leme [10] mention eight), finding a constructive one remains elusive.

The first attempt to "construct" all GS valuations was by Hatfield and Milgrom [67]. After observing that most examples of GS valuations arising in applications are built from assignment valuations and the endowment operation, they asked if this is true for all GS valuations. Ostrovsky and Paes Leme [104] showed that this is not the case: some matroid rank functions cannot be constructed as endowed assignment valuations while all (weighted) matroid rank functions are GS valuations. Instead, Ostrovsky and Paes Leme proposed the *matroid based valuations (MBV)* conjecture. Matroid based valuations are those that arise from weighted matroid rank functions by repeatedly applying the operations of *merge* and *endowment*; the conjecture states that all GS valuations arise in this way. Tran [114] showed that using only merge but no endowment operations does not suffice, but the conjecture remained open.

Given a valuation $v : 2^{V'} \to \mathbb{R}$ and $W \subseteq V'$, we can define the *endowed* valuation $v' : 2^{V' \setminus W} \to \mathbb{R}_+$ as $v'(X) = v(X \cup W) - v(W)$. We define the *merge* of the valuations $v_1, v_2 : 2^V \to \mathbb{R}_+$ as $v^*(X) = \max_{T \subseteq X} v_1(T) + v_2(X \setminus T)$ for all $X \subseteq V$. In economics terms, the merge is the valuation of the company formed by the two agents; the endowment is the valuation of an agent who already has $W$ and measures the marginal contribution of the items in $V' \setminus W$.

It turns out that every matroid based valuation is an endowed Rado valuation, so the MBV conjecture would imply that all GS valuations are endowed Rado valuations. We disprove the MBV conjecture by exhibiting a class of GS valuations that are not endowed Rado valuations. This is achieved by studying the complete classes of valuated matroids. We note that it is unclear if the class of MBV valuations is the same as the class of endowed Rado valuations, or it is a strict subclass.

**Valuated matroids** Valuated (generalized) matroids capture a quantitative version of the exchange axiom(s) for matroids. They were first introduced by Dress and Wenzel [42], motivated by questions related to number theory and the greedy algorithm. Later, Murota [95] identified them as a fundamental building block for discrete convex analysis. They play important roles across different areas of mathematics and computer science, with particularly many applications in algorithmic game theory.

Valuated (generalized) matroids can be defined in many different ways [50, 86, 98]. We follow [51, 99], and say that a function $f : 2^V \to \mathbb{R} \cup \{-\infty\}$ is a *valuated generalized matroid* if two properties hold:

$$\begin{aligned} &\forall X, Y \subseteq V \text{ with } |X| < |Y| : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X+j) + f(Y-j)\} \end{aligned} \tag{1.1a}$$

$$\begin{aligned} &\forall X, Y \subseteq V \text{ with } |X| = |Y| \text{ and } \forall i \in X \setminus Y : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X-i+j) + f(Y+i-j)\}. \end{aligned} \tag{1.1b}$$

For fixed $r \leq |V|$, those functions $\binom{V}{r} \to \mathbb{R} \cup \{-\infty\}$ fulfilling (1.1b) are *valuated matroids*. This means that each layer of a valuated generalized matroid (a gross substitutes valuation) is a valuated matroid. Valuated matroids with codomain $\{0, -\infty\}$ coincide with usual matroids as the sets taking value $0$ form the bases of a matroid; we call them *trivially valuated matroids*. In this context, (1.1b) corresponds to the strong basis exchange property. Valuated *generalized* matroids that are monotone and take value $0$ on the empty set are exactly GS valuations.

**R-minor valuated matroids and Frank's question** We are interested in the following classes of valuated matroids arising from independent matchings in bipartite graphs. The name is inspired by the work of **R**ado [107], similarly as **R**ado valuations.

**Definition 1.3.1** (R-minor, R-induced). *Let $G = (V \cup W, U; E)$ be a bipartite graph with edge weights $c \in \mathbb{R}^E$, and a matroid $\mathcal{M}$ on $U$ of rank $d + |W|$. We define an* R-minor *valuated matroid $f : \binom{V}{d} \to \mathbb{R}$ for $X \in \binom{V}{d}$ as follows.*

*The value $f(X)$ is the maximum weight of a matching in $G$ whose endpoints in $V \cup W$ are $X \cup W$, and the endpoints in $U$ form a basis in $\mathcal{M}$. For $W = \emptyset$, the function $f$ is called an* R-induced *valuated matroid.*

This concept naturally extends to valuated generalized matroids (which we denote by $\mathrm{R}^\natural$ instead of R, as is often the case in discrete convex analysis [98]): the endpoints in $U$ should not form a basis but a set in a *generalized matroid*. [4] It is immediate to see that Rado valuations (resp. endowed Rado valuations) are exactly monotone $\mathrm{R}^\natural$-induced (resp. $\mathrm{R}^\natural$-minor) valuated generalized matroids, taking value zero on the empty set.

[4] Generalized matroids are defined as the effective domain of a $\{0, -\infty\}$-valued valuated generalized matroid, see Section 7.1. The canonical examples are independent sets of matroids.

In 2003, Frank [68] (see also lectures by Murota [96, 97], and Paes Leme [87]) asked if all valuated matroids arise as *R-induced* valuated matroids. The corresponding version of this question for valuations asks if all GS valuations are Rado. If true, this would imply that our approximation algorithm for the NSW problem discussed in Section 1.2 works for all GS valuations. Unfortunately, in Section 4.7, we show that not all GS valuations are Rado. The reason is that GS valuations (resp. valuated generalized matroids) are closed under endowment (resp. contraction), whereas Rado valuations (resp. R$^\natural$-induced valuated generalized matroids) are not.

Noting that R-minor valuated matroids are precisely the contractions of R-induced valuated matroids, this suggests a natural refinement of the original question:

*Do all valuated matroids arise as R-minor valuated matroids?*

We will show that *(i)* R-minor valuated matroids form a *complete class* closed under several fundamental operations, yet *(ii)* not all valuated matroids are R-minor.

**Complete classes** Let us consider R-induced and R-minor valuated matroids where $\mathcal{M}$ is the free matroid and $c \equiv 0$. Valuated matroids $f$ arising in such forms are the $\{0, -\infty\}$ indicator functions of *transversal matroids* and *gammoids* respectively. In 1977, Ingleton [71] studied representations of transversal matroids and gammoids. He observed that gammoids arise via this simple construction yet form a rich class closed under several fundamental matroid operations. This motivated the definition of a *complete class* of matroids by requiring closure under the operations *restriction, dual, direct sum, principal extension*. Closure under principal extension combined with restriction implies closure under induction by bipartite graphs which encompasses many other natural matroid operations, including matroid union. Closure under this operation is what creates the rich structure of complete classes, even when one starts from very basic matroids; e.g., gammoids arise as the smallest complete class by taking the closure of the matroid on one element.

We extend the notion of complete classes to valuated matroids (Chapter 5). These are classes of valuated matroids closed under the valuated generalizations of the fundamental operations *restriction, dual, direct sum, principal extension*. The crucial ingredient going beyond the basic operations already introduced in [42] is (valuated) principal extension. Analogously as in the case of matroids, valuated gammoids from the smallest complete class of valuated matroids (Theorem 5.2.6).

After introducing complete classes, we show that the smallest class of valuated matroids containing all trivially valuated matroids and that is closed for the mentioned operations is exactly the class of *R-minor* valuated matroids (Section 5.3). Hence, the refined question by Frank is equivalent to the following: does the smallest complete class containing trivially valuated matroids cover all valuated matroids?

**Third main result** The third main contribution of the thesis is proving that there are valuated matroids that are not R-minor valuated matroids.

We can use an information-theoretic argument to show that not all valuated matroids are *R-induced* by constructing valuated matroids with many independent values (Section 6.4). However, such an argument does not seem extendable to R-minor valuated matroids. Instead, disproving the more general claim relies on a well-chosen family of valuated matroids. In Chapter 6, we show that none of the valuated matroids in the following family is R-minor.

**Definition 1.3.2.** *For $n \geq 2$, we define $\mathcal{F}_n$ as the following family of functions $\binom{[2n]}{4} \to \mathbb{R}$. Let $V = [2n]$, $P_i = \{2i-1, 2i\}$ for $i \in [n]$, and let*

$$\mathcal{H} = \{P_i \cup P_j : ij \equiv 0 \mod 2\} \qquad (\mathcal{H}\text{-def})$$

*i.e. we take pairs such that at least one of $i, j$ is even. Let $X^* = P_1 \cup P_2 = \{1, 2, 3, 4\}$. A function $h : \binom{V}{4} \to \mathbb{R} \cup \{-\infty\}$ is in the family $\mathcal{F}_n$ if and only if the following hold:*

- *$h(X) = 0$ if $X \in \binom{V}{4} \setminus \mathcal{H}$,*
- *$h(X) < 0$ if $X \in \mathcal{H}$, and*
- *$h(X^*)$ is the unique largest nonzero value of the function.*

**Theorem 1.3.3.** *If $n \geq 2$, then all functions in $\mathcal{F}_n$ are valuated matroids. If $n \geq 16$, then no function in $\mathcal{F}_n$ arises as an $R$-minor valuated matroid.*

**Sparse paving matroids** A matroid of rank $d$ is *paving* if all circuits are of size $d$ or $d+1$, and *sparse paving* if in addition the intersection of any two $d$-element circuits is of size at most $d-2$. The functions in $\mathcal{F}_n$ are derived from sparse paving matroids. Namely, if $B$ is the family of bases of a sparse paving matroid of rank $d$, then any function $h : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ with $h(X) = 0$ if $X \in \mathcal{B}$ and $h(X) < 0$ otherwise gives a valuated matroid, see Section 6.3.1. In particular, this implies that all functions in $\mathcal{F}_n$ are valuated matroids.

Our construction is inspired by Knuth [82]. He gave an elegant construction of a doubly exponentially large family of sparse paving matroids; the strongest lower bound on the number of matroids on $n$ elements. In fact, it was conjectured in [93] that asymptotically almost all matroids are sparse paving.

**Refuting the Matroid Based Valuation Conjecture** Building on Theorem 1.3.3, we also refute the MBV conjecture by Ostrovsky and Paes Leme [104]. This is done by considering R$^\natural$-minor valuated generalized matroids and reducing to Theorem 1.3.3.

First, we show that every function that can be obtained from weighted matroid rank functions by repeatedly applying merge and endowment is an R$^\natural$-minor valuated generalized matroid.

Then, we show that the function $h^\natural : 2^V \to \mathbb{R}_{\geq 0}$ defined as follows is a valuated generalized matroid but not R$^\natural$-minor. This disproves the MBV conjecture. For an arbitrary valuated matroid $h \in \mathcal{F}_n$ taking values only in $(-1, 0]$ we define

$$h^\natural(X) := \begin{cases} |X| & \text{for } |X| \leq 3, \\ 4 + h(X) & \text{for } |X| = 4, \\ 4 & \text{for } |X| \geq 5. \end{cases}$$

We achieve this by focusing on the function restricted to all $4$-subsets of $V$. This is an R-minor valuated matroid and therefore allows us to apply Theorem 1.3.3. Note that the function $h^\natural$ has the additional structure of being *monotone* and only taking value zero on the empty set, as the MBV conjecture refers to valuations.

# 2 Auction algorithm for market equilibrium with weak gross substitute demands

We give an auction algorithm for finding an approximate market equilibrium in the exchange markets where the demands of each agent satisfied the WGS property. In Chapter 1 we have introduced Fisher markets. Here, we introduce and work with more general Arrow-Debreu exchange markets.

The main idea in auction algorithms is the following: we set low initial prices, we let the agents outbid each other for parts of the goods while obeying simple "ground rules"; in the process they converge to an approximate market-equilibrium.

The rest of this chapter is organized as follows. We first formally define the market model and the assumptions in Section 2.1. Then, we give examples of WGS demands systems and the notion of price elasticity are given in Section 2.2. The algorithm is presented in Section 2.3. To present the algorithm we will rely on a subroutine `FindNewPrices`. The subroutine allows agents to update the prices and demands based on which the outbidding occurs. We give different way of implement this procedure after we present the main algorithm, in Section 2.3.2 and Section 2.3.3.

## 2.1 The exchange market model

We consider a market with a set of agents $A = [n]$ and divisible goods $G = [m]$. Each agent $i \in [n]$ arrives at the market with an initial endowment of goods $e^{(i)} \in \mathbb{R}^m_+$. Thus, the total amount of each good $j \in [m]$ is $e_j$ where $e = \sum_{i=1}^n e^{(i)}$; w.l.o.g. $e_j > 0$. Given a non-negative price vector $p \in \mathbb{R}^m_+$, the budget of agent $i$ at prices $p$ is defined as $b_i = b_i(p) = p^\top e^{(i)}$. It follows that $p^\top e = \sum_i p^\top e^{(i)} = \sum_i b_i$.

We recall that a *demand system* is a function $D : \mathbb{R}^{m+1}_+ \to 2^{\mathbb{R}^m_+}$; $D(p, b)$ denotes the set of demanded bundles of an agent at prices $p$ and budget $b$. If $|D(p, b)| = 1$ for all $(p, b) \in \mathbb{R}^{m+1}$ we say that the demand system is *simple*, and we will also use $D(p, b)$ to denote this single bundle.

We include the budget $b$ in the definition of the demand system, even though for exchange markets the budget of agent $i$ is uniquely defined by the prices as $p^\top e^{(i)}$. This formalism will be useful for our algorithm where the budgets are defined according to a slightly different set of prices.

We extend the definition of the market equilibrium to exchange markets.

**Definition 2.1.1** (Market equilibrium)**.** *Let $D_i$ denote the demand system of agent $i \in A$. We say that the prices $p \in \mathbb{R}^m_+$ and bundles $x^{(i)} \in \mathbb{R}^m_+$ form a* market equilibrium *if*

- $x^{(i)} \in D_i(p, p^\top e^{(i)})$*, and*
- $\sum_{i=1}^n x_j^{(i)} \le e_j$*, with equality whenever $p_j > 0$, for all $j \in G$.*

That is, $p$ and optimal bundles $x^{(i)}$ form an equilibrium if no good is overdemanded and goods at a positive price are fully sold. Note that this implies that every agent fully spends their budget.

When a demand system satisfies the WGS property we also say that it is a WGS demand system. In the context of the tâtonnement process, the weak gross substitutes property is usually defined with respect to the *aggregate* excess demand function of all agents. We use the stronger requirement of having a WGS demand system for each individual agent. The previous auction algorithms [61, 62] have also used WGS on the level of agents as this seems to be the necessary condition that allows agents to update their bundles individually, as opposed to tâtonnement, where the prices adjustments react to the aggregate demands. We note that WGS demands for individual agents are also assumed in the context of indivisible goods.

Next, we formally describe the orcale access to the demands our algorithm uses.

**Definition 2.1.2** (Demand oracle)**.** *For a WGS demand system $D(p, b)$, a* demand oracle *requires two vectors $(p, b), (p', b') \in \mathbb{R}^{m+1}_+$ such that $(p', b') \ge (p, b)$, and a vector $x \in D(p, b)$. The output is a vector $y \in D(p', b')$ such that that $y_j \ge x_j$ whenever $p'_j = p_j$.*

In other words, the oracle provides the allocations guaranteed by the definitions of WGS systems. The complex form of the definition is due to the possible non-uniqueness of demand bundles. For simple demand systems, the input to the oracle is simply a vector $(p', b') \in \mathbb{R}^{m+1}_+$, and the output is the unique vector $y \in D(p', b')$.

The auction algorithm relies on the more powerful `FindNewPrices` subroutine, which can be seen as a strengthening of the demand oracle, incorporating a mechanism for price increments. There are various ways to implement such a subroutine: in Section 2.3.2 we use a simple iterative application of the demand oracle for the case of bounded price elasticities; in Section 2.3.3 we use a convex programming approach for Gale demand systems; and in Section 3.2.1 we devise a combinatorial algorithm for budget-SPLC utilities.

For exchange markets, we will make the following assumptions:

**Assumption 1** (Scale invariance)**.** *For every agent $i$, $D_i(p, b_i) = D_i(\alpha p, \alpha b_i)$ for all $\alpha > 0$.*

That is, we require that the demand is homogeneous of degree $0$; informally, the demand does not depend on the currency. This is a standard assumption in microeconomics and exchange markets, see e.g. [6, 41, 45, 92].

**Assumption 2** (Non-satiation)**.** *For all demand systems, and for every $(p, b) \in \mathbb{R}_+^{m+1}$, and every $x \in D(p, b)$, we have $p^\top x = b$.*

That is, in every optimal bundle the agents must fully spend their budgets. This is a standard assumption for exchange markets as it is necessary for the fundamental theorems of welfare economics (see e.g. [91, Chapter 16]). However, we note that we will not require this assumption in Section 3.1 for spending restricted Fisher markets.

**Approximate equilibria** Let us now define the concept of an $\epsilon$-equilibrium in exchange markets, which is what our algorithm will find. We require that each agent gets an approximate optimal bundle and market clears approximately.

**Definition 2.1.3** (Approximate equilibrium)**.** *For an $\epsilon > 0$, the prices $p \in \mathbb{R}^m$ and bundles $x^{(i)} \in \mathbb{R}_+^m$ form an $\epsilon$-*approximate market equilibrium *if*

*(i)* $x^{(i)} \le z^{(i)}$ *for some* $z^{(i)} \in D_i(p^{(i)}, p^\top e^{(i)})$, *where* $p \le p^{(i)} \le (1+\epsilon)p$,

*(ii)* $\sum_{i=1}^n x_j^{(i)} \le e_j$, *and*

*(iii)* $\sum_{j=1}^m p_j \left( e_j - \sum_{i=1}^n x_j^{(i)} \right) \le \epsilon p^\top e$.

That is, every agent owns a subset of their optimal bundle at prices that are within a factor $(1+\epsilon)$ from $p$, and all goods are nearly sold: the value of the unsold goods is at most an $\epsilon$ fraction of the total value of the goods. The total value of the goods "taken away" from the near-optimal bundles of the agents is $\sum_{i=1}^n p^\top (z^{(i)} - x^{(i)})$. Parts (i) and (iii), together with the fact that ${p^{(i)}}^\top z^{(i)} \le p^\top e^{(i)}$ for all $i$, imply that this amount is $\le 2\epsilon p^\top e$.

Definition 2.1.3 can be seen as a natural extension of the corresponding approximate KKT conditions in [59, 61, 62]. For linear utilities, [59] requires the approximate maximum bang-per-buck condition $v_{ij}/p_j \le (1+\epsilon) v_{ik}/p_k$ for any agent $i$, goods $j$ and $k$ such that $x_{ik} > 0$. Thus, one can set approximate prices $p \le p^{(i)} \le (1+\epsilon)p$ for each agent with respect to which they purchase maximum bang-per-buck goods.

Condition (iii) corresponds to the definition of approximate equilibrium in [40] and [63]. This notion is weaker than the ones used in [59, 61, 62]. The most important difference is that the latter papers guarantee that each agent recovers approximately their optimal utility. Such a property could be achieved by strengthening the bound in (iii) from $\epsilon p^\top e$ to $\epsilon p_{\min} e_{\min}$, where $p_{\min}$ is the minimum price and $e_{\min}$ is the smallest total fractional amount in the initial endowment of any agent. However, this would come at the expense of substantially worse running time guarantees in our algorithmic framework, in particular the running time would not be polynomial anymore.

## 2.2 Examples and properties of WGS demand systems

We now present some classical examples of WGS utilities previously studied in the literature.

- For $v \in \mathbb{R}^m_+$ the *linear utility* is given by $u(x) = v^\top x$. By definition, $D^u(p,b) = \arg\max\{v^\top x : p^\top x \le b\}$.
- The *constant elasticity of substitution (CES)* utility is defined by $u(x) = \left(\sum_j \beta_j^{\frac{1}{\sigma}} x_j^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$, where $\sum_j \beta_j = 1$. Then, $D(b,p) = \{x\}$ for the unique optimal bundle $x$ given by $x_j = \dfrac{\beta_j p_j^{-\sigma} b}{\sum_k \beta_k p_k^{1-\sigma}}$. It is well-known that a CES demand system satisfies the WGS property if and only if $\sigma > 1$.
- The *Cobb-Douglas* utility function is given by $u(x) = \prod_j x_j^{\alpha_j}$ where $\sum_j \alpha_j = 1$, $\alpha \ge 0$. The unique optimal bundle is therefore $x_j = b\alpha_j/p_j$ and $D^u(p,b) = \{x\}$. The Cobb-Douglas utility function satisfies the WGS property for any parameter choices.
- The *nested CES* utility function is defined recursively (see [74] for more details). Any CES function is a nested CES function. If $g, h_1, \ldots, h_t$ are nested CES functions, then $f(x) = \max g(h_1(x^1), \ldots, h_t(x^t))$ over all $x^1, \ldots, x^t$ such that $\sum_{k=1}^t x^k = x$, is a nested CES function. In a well-studied special case (see e.g., [79]), each good $j$ can only be used in at most one of the $h_i$'s.

**Conic combinations of demand systems** Given two WGS utility functions $u$ and $u'$, the demand system corresponding to their sum $u + u'$ may not be WGS. On the other hand, consider two simple WGS demand systems $D$ and $D'$ and nonnegative coefficients $\lambda, \lambda'$. Then it is easy to see that $\lambda D + \lambda' D'$ is also a simple WGS demand system. This enables the construction of some interesting demand systems. For example, [92] has studied hybrids of CES and Cobb-Douglas demands, where the demand system can be given as

$$x_j = \frac{b}{p_j}\left[\epsilon\alpha_j + (1-\epsilon)\frac{\beta_j p_j^{1-\sigma}}{\sum_k \beta_k p_k^{1-\sigma}}\right] ,$$

for some $0 \le \epsilon \le 1$ and $\sigma > 1$.[1]

[1] We note that this demand function does not seem to correspond to a nested CES utility function.

Note that if $D = D^u$ and $D' = D^{u'}$ for some concave utility functions $u$ and $u'$, the demand system $\lambda D + \lambda' D'$ will in general *not* correspond to the utility function $\lambda u + \lambda' u'$. In fact, it seems unclear if one can explicitly write utility functions corresponding to such convex combinations. Our model does not require the demand system to be given in the form $D = D^u$ for some concave utility function $u$.

**The Gale demand systems** For applications of our auction algorithm for the Nash social welfare problem, we will use *Gale demand systems* instead of (4.3), defined as

$$G^u(p, b) = \arg\max_{x \in \mathbb{R}^m_+} b \log u(x) - p^\top x\,. \tag{2.1}$$

We call $b \log u(x) - p^\top x$ the *Gale objective function*. It is easy to verify using Lagrangian duality that if all $u_i$'s are concave functions, and the utility functions correspond to the Gale demand systems $D_i(p, b) = G^{u_i}(p, b)$, then the program (EG) always finds a market equilibrium; see [101] for details. Moreover, if the utilities are homogenous of degree one, then this equilibrium coincides with the equilibrium for the "standard" demand systems given by (Max-utility). For general concave utility functions, the optimal bundles stay within the budget $b$ (that is, $p^\top x \le b$), but may not exhaust it.

We refer the reader to the paper by Nesterov and Shikhman [101] on Gale demand systems as well as the more general concept of Fisher-Gale equilibrium; they also give a tâtonnement type algorithm for finding such an equilibrium.

**Price elasticity of demands** One possible implementation of the key subroutine `Find-NewPrices` (Section 2.3) relies on the *(price) elasticity of the demands*.[2] The standard definition of the elasticity for good $j$ with respect to the price of good $k$ is $e_{j,k} = \partial \log x_j(p, b)/\partial \log p_k$, where $x_j(p, b)$ is the (unique) demand for good $j$ at prices $p$ and budget $b$. The WGS property guarantees that $e_{j,k} \ge 0$ if $j \ne k$, and consequently, $e_{k,k} \le 0$. The definition below corresponds to $e_{k,k} \ge -f$ for all $k \in [m]$, for the more general model of non-simple demand systems.

**Definition 2.2.1.** *Consider a WGS demand system $D(p, b)$. For some $f > 0$, we say that the* elasticity of $D(p, b)$ is at least $-f$, *if for any $\mu \ge 0$, $j \in [m]$, $(p, b) \in \mathbb{R}^{m+1}_+$ and $x \in D(p, b)$, if we define $p'$ as $p'_j = p_j(1 + \mu)$ and $p'_k = p_k$ for $k \in [m] \setminus \{j\}$, then there exists a bundle $x' \in D(p', b)$ such that $x'_j \ge \frac{1}{(1+\mu)^f} x_j$.*

[2] No finite lower bound can be given on the elasticity of linear demand systems. If we are buying a positive amount of good $j$, that means that $j$ maximizes $v_k/p_k$. If there is another good $\ell$ with $v_j/p_j = v_\ell/p_\ell$, then if we increase $p_j$ but leave the other prices unchanged, then $x'_j = 0$ for every optimal bundle $x'$ with respect to the new prices. Consequently, for this case, we have another way to implement `Find-NewPrices` in Lemma 2.3.8.

For the CES utilities and Cobb-Douglas utilities we prove the following easy bounds on the elasticity of the demands.

**Lemma 2.2.2.** *The CES demand system with parameter $\sigma > 1$ has elasticity at least $-\sigma$, and the Cobb-Douglas demand system has elasticity at least $-1$.*

*Proof.* Using the form of CES utilities described above, the demand at prices $p$ is $x_j = \frac{\beta_j p_j^{-\sigma} b}{\sum_k \beta_k p_k^{1-\sigma}}$. Fix a good $j$. Denote with $x'$ the optimal bundle where we increase the price of good $j$ by factor $(1+\mu)$. Since CES satisfies the WGS property for $\sigma > 1$, we have

$$\begin{aligned} x'_j &= \frac{\beta_j (1+\mu)^{-\sigma} p_j^{-\sigma} b}{\sum_{k\neq j} \beta_k p_k^{1-\sigma} + \beta_j (1+\mu)^{1-\sigma} p_k^{1-\sigma}} \\ &= \frac{\beta_j p_j^{-\sigma} b}{(1+\mu)^{\sigma} \sum_{k\neq j} \beta_k p_k^{1-\sigma} + \beta_j (1+\mu) p_k^{1-\sigma}} \\ &> \frac{\beta_j p_j^{-\sigma} b}{(1+\mu)^{\sigma} \sum_{k} \beta_k p_k^{1-\sigma}} = \frac{1}{(1+\mu)^{\sigma}} x_j \,. \end{aligned}$$

For *Cobb-Douglas* utility function is given by $u(x) = \prod_j x_j^{\alpha_j}$ where $\sum_j \alpha_j = 1, \alpha \geq 0$, the optimal bundle is $x_j = \frac{b\alpha_j}{p_j}$. Hence, increasing the price of a good by some factor leads to the decrease in demand for that good by the same factor. □

## 2.3 Auction algorithm for exchange markets

The algorithm (shown in Algorithm 1) uses the accuracy parameter $0 < \epsilon < 0.25$, and returns a $4\epsilon$-approximate equilibrium. We initialize all prices $p_j = 1$ and the prices will only increase during the execution of the algorithm, in increments by a factor $(1+\epsilon)$. This initialization is enabled by Assumption 1 that guarantees the existence of market clearing prices where all positive prices are $\geq 1$.[3]

We maintain a price vector $p$ called the *market prices*; the budget of agent $i \in [n]$ is $b_i = p^\top e^{(i)}$ at the current prices. Further, every agent $i \in [n]$ maintains individual prices $p^{(i)}$ such that $p \leq p^{(i)} \leq (1+\epsilon)p$. At any point of the algorithm, agent $i$ owns a bundle $c^{(i)}$ of the goods such that $c^{(i)} \leq x^{(i)}$ for some $x^{(i)} \in D_i(p^{(i)}, b_i)$. For each good $j$ an agent is paying either the lower price $p_j$ or the higher price $(1+\epsilon)p_j$. The price agent $i$ has to pay for good $j$ is the higher price $(1+\epsilon)p_j$ if $p_j^{(i)} = (1+\epsilon)p_j$ and the lower price $p_j$ otherwise.[4]

We consider the agents one-by-one. If an agent $i$ has surplus money, they use the subroutine `FindNewPrices` to update their prices $p^{(i)}$ and bundle $x^{(i)}$, by maintaining

[3] Even though there might be goods priced at 0 in an equilibrium, we can always find an $\epsilon$-approximate equilibrium where all prices are positive.

[4] Note that this is in contrast with [59] and the other previous auction algorithms where $i$ may pay $p_j$ for some amount of good $j$ and $(1+\epsilon)p_j$ for another amount.

**Algorithm 1:** Auction algorithm for exchange markets

**Input:** Demand systems $D_i$, and the endowment vectors $e^{(i)}$, and $\epsilon \in (0, 0.25)$.
**Output:** A $4\epsilon$-approximate market equilibrium.

**Initialization:** for all $i, j$ set $p_j \leftarrow 1$, $p_j^{(i)} \leftarrow 1$, $c_j^{(i)} \leftarrow 0$, $w_j = e_j = \sum_i e_j^{(i)}$, and $l_j = 0$
NewIt **for** $i \in [n]$ **do** `// recompute the budgets and surpluses`
  $b_i \leftarrow p^\top e^{(i)}$; $s_i \leftarrow b_i - \sum_{j \in L_i} c_j^{(i)} p_j - \sum_{j \in H_i} c_j^{(i)} (1+\epsilon) p_j$
**end**
**if** $\sum_{i=1}^n s_i \le 3\epsilon p^\top e$ **then return** $p$, $\{p^{(i)}\}_{i \in [n]}$ and $\{c^{(i)}\}_{i \in [n]}$
NewStp **for** $i \in [n]$ *with* $s_i > 0$ **do** `// step for agent` $i$
  $(\tilde{p}, y) \leftarrow$ `FindNewPrices`$(i, p^{(i)}, p, \epsilon, c^{(i)}, b_i)$
  **for** $j = 1$ **to** $m$ **do**
    **if** $p_j^{(i)} < (1+\epsilon)p_j$ *and* $\tilde{p}_j = (1+\epsilon)p_j$ **then** `// Case 1`
      $s_i \leftarrow s_i - c_j^{(i)} \cdot \epsilon p_j$; $l_j \leftarrow l_j - c_j^{(i)}$ `//` $i$ `pays` $(1+\epsilon)p_j$ `instead of` $p_j$
      `Outbid`$(i, j, y_j - c_j^{(i)})$
    **else if** $p_j^{(i)} = (1+\epsilon)p_j$ *and* $\tilde{p}_j = (1+\epsilon)p_j$ **then** `// Case 2`
      `Outbid`$(i, j, y_j - c_j^{(i)})$
    **end**
    `// Skip the goods with` $p_j^{(i)} < (1+\epsilon)p_j$ `&` $\tilde{p}_j < (1+\epsilon)p_j$`. Case 3`
  **end**
  $p^{(i)} \leftarrow \tilde{p}$; flag $\leftarrow 0$
  **for** $j \in [m]$ *with* $w_j + l_j = 0$ **do**
    $p_j \leftarrow (1+\epsilon)p_j$; $l_j = e_j$; `// price increase`
    **foreach** $k \in [n]$ **do** $p_j^{(k)} \leftarrow (1+\epsilon)p_j$
    flag $\leftarrow 1$
  **end**
  **if** *flag = 1* **then Go To** NewIt
**end**

$x_j^{(i)} \ge c_j^{(i)}$ — this latter requirement turns out to be the main challenge. They will then try to purchase $x_j^{(i)} - c_j^{(i)}$ amount of good $j$ in the `Outbid` procedure. They start by purchasing any unsold amount of good at price $p_j$. If they still need more, then they will outbid other agents who have been paying the lower price $p_j$ for this good, by offering the higher price $(1+\epsilon)p_j$ (if $p_j^{(i)} < (1+\epsilon)p_j$ this does not happen). Once good $j$ is sold only at the higher price $(1+\epsilon)p_j$, we increase the price of the good. If no price is increased, we move to the next agent. Otherwise, we announce the new prices $p$ and repeat. The algorithm terminates once the total surplus of the agents drops below $3\epsilon p^\top e$. At this point, we can conclude that the current prices and allocations form a $4\epsilon$-approximate equilibrium.

We express the running time of the algorithm in terms of the running time $T_F$ of the subroutine `FindNewPrices`, as well as the upper bound on the ratio $p_{\max}/p_{\min}$ of the largest and smallest nonzero prices at an $\epsilon$-equilibrium. Such an upper bound may be

**Procedure** Outbid($i, j, t$)

// $t$ is the amount of good $j$ for which agent $i$ wants to outbid.
**if** $w_j > 0$ **then** // a part of $j$ is unsold
  $\tau = \min\{w_j, t\}$
  $w_j \leftarrow w_j - \tau$; $c_j^{(i)} \leftarrow c_j^{(i)} + \tau$; $t \leftarrow t - \tau$
  $s_i \leftarrow s_i - \tau \cdot (1+\epsilon)p_j$ // here $p_j = 1$ always
**end**
**while** $t > 0$ *and* $l_j > 0$ **do**
  Let $k \in [n]$ be such that $c_j^{(k)} > 0$ and $p_j^{(k)} = p_j$. Set $\tau = \min\{c_j^{(k)}, t\}$
  $c_j^{(k)} \leftarrow c_j^{(k)} - \tau$; $c_j^{(i)} \leftarrow c_j^{(i)} + \tau$ // $i$ outbids $k$
  $s_k \leftarrow s_k + \tau \cdot p_j$; $s_i \leftarrow s_i - \tau \cdot (1+\epsilon)p_j$; $l_j \leftarrow l_j - \tau$; $t \leftarrow t - \tau$
**end**

obtained for specific demand systems.[5] Alternatively, one can follow the approach of the papers [30, 32] by adding a dummy agent with a Cobb-Douglas demand system and an initial endowment of a small fraction of all goods. In the presence of such an agent, we can obtain a strong bound on $p_{\max}/p_{\min}$, at the expense of obtaining a slightly worse approximation guarantee. We describe the construction in Section 2.3.4.

Note that for (approximate-)equilibrium prices $p$, $\alpha p$ also gives (approximate-) equilibrium prices with the same allocation, for any $\alpha > 0$. In our algorithm, the minimum price will remain at most $1+\epsilon$ throughout, see Lemma 2.3.4.

**Theorem 2.3.1.** *Let $T_F$ be an upper bound on the running time of the subroutine* `FindNewPrices`. *Algorithm 1 finds a $4\epsilon$-approximate market equilibrium in time $O\left(\frac{nmT_F}{\epsilon^2} \cdot \log\left(\frac{p_{\max}}{p_{\min}}\right)\right)$.*

We assume that $T_F = \Omega(m)$, since the output needs to return an $m$-dimensional vector of goods. There are various options for implementing `FindNewPrices`. In Section 2.3.2 we present a simple price increment procedure for the case of bounded elasticities; recall the elasticity bound $f$ from Definition 2.2.1. Using this subroutine and Lemma 2.3.7, we obtain the following overall bound.

**Theorem 2.3.2.** *If all agents have elasticity at least $-f$ for some $f > 0$, then an $\epsilon$-approximate equilibrium can be computed in time $O\left(\frac{nm^2 f \cdot T_D}{\epsilon^2} \cdot \log\left(\frac{p_{\max}}{p_{\min}}\right)\right)$, where $T_D$ is the time needed for one call to the demand oracle.*

As noted earlier, there are simple demand systems such as linear demand systems where the flexibility parameter cannot be bounded. However, in case the demand system is given in the form (Max-utility) via a utility function that is homogeneous of degree one, we can obtain an implementation of `FindNewPrices` by solving a convex pro-

[5] For demand systems given by an explicit utility function in the form (Max-utility), we give such a bound for spending-restricted Fisher-equilibria in Section 3.1.2.

gram. This is described in Section 2.3.3. In particular, this applies to for CES utilities with $\sigma > 1$ and Cobb-Douglas utilities. One could find further possible ways for implementing `FindNewPrices` for particular demand systems; for example, we give a simple direct procedure for linear utilities in Lemma 2.3.8, and a procedure for budget-SPLC utilities in Section 3.2.

We give an overview of the running times of the previous auction algorithms in Section 2.3.5.

**Invariants** Let us now summarize the invariant properties maintained throughout the algorithm. We say that a bundle $y$ dominates the bundle $x$ if $x \leq y$.

(a) Each good is partitioned into three parts according to the price it is being sold at:
- amount $w_j$ is the unsold part of the good,
- amount $l_j$ is sold at the lower price $p_j$, and
- amount $h_j$ is sold at the higher price $(1+\epsilon)p_j$.

Moreover, $w_j + l_j > 0$ at the end of each step, i.e., after each agents' turn there is always a part of the good that is unsold or owned by an agent at the lower price.

(b) The unsold amount $w_j$ of each good is non-increasing. If $w_j > 0$ then $p_j = 1$.

(c) The budget of agent $i$ is $b_i = p^\top e^{(i)}$. Each agent $i$ maintains prices $p^{(i)}$ such that $p \leq p^{(i)} \leq (1+\epsilon)p$, and owns a bundle $c^{(i)}$ that is dominated by a bundle $x^{(i)} \in D_i(p^{(i)}, b_i)$.

(d) For the amount $c_j^{(i)}$ of good $j$, agent $i$ pays
- price $p_j$ for goods in $L_i := \{j \in [m] : p_j^{(i)} < (1+\epsilon)p_j\}$, and
- the price $(1+\epsilon)p_j$ for goods in $H_i := \{j \in [m] : p_j^{(i)} = (1+\epsilon)p_j\} = [m] \setminus L_i$.

In accordance with (d), the *surplus* of agent $i$ is defined as

$$s_i := b_i - \sum_{j \in L_i} c_j^{(i)} p_j - \sum_{j \in H_i} c_j^{(i)} (1+\epsilon) p_j \ .$$

We note that the surplus could be negative.

**The Outbid subroutine** An important subroutine, described in Procedure `Outbid`, controls how the ownership of goods may change. If agent $k$ has paid price $p_j$ on a certain amount of good $j$, then agent $i$ may take over some of this amount by offering a higher price $(1+\epsilon)p_j$. Possibly $i = k$, in which case the agent outbids herself. We also incorporate into the procedure the case when a certain amount of a good is being purchased for the first time. Note that $p_j = 1$ at this point due to invariant (b).

**Main iterations** The algorithm is partitioned into iterations. Each iteration finishes when the price of a good increases from $p_j$ to $(1+\epsilon)p_j$. At every such event, the budgets $b_i$ of the agents also increase. Therefore, at the start of an iteration each agent $i$ recomputes their budget at line NewIt. An iteration is further partitioned into steps, which are single executions of the main for loop in Algorithm 1. The algorithm terminates as soon as the total surplus drops below $3\epsilon p^\top e$.

**Steps** Suppose we are considering agent $i$. By invariant (c), the agent is buying a bundle $c^{(i)} \leq x^{(i)}$ for some $x^{(i)} \in D_i(p^{(i)}, b_i)$. The subroutine `FindNewPrices`$(i, p^{(i)}, p, \epsilon, c^{(i)}, b_i)$ delivers new prices $\tilde{p}$ and a bundle $y$ such that

(A) $y \geq c^{(i)}$ and $y \in D_i(\tilde{p}, b_i)$; in addition,

(B) $p^{(i)} \leq \tilde{p} \leq (1+\epsilon)p$, and $\tilde{p}_j = (1+\epsilon)p_j$ whenever $y_j > (1+\epsilon)\, c_j^{(i)}$.

In other words, Condition (A) says that agent $i$ still wants whatever they own even at the increased prices $\tilde{p}$. Condition (B) is the crucial one for the outbid. It guarantees that $\tilde{p} \geq p^{(i)}$, and whenever an agent wants to buy more of some good than they already own at least by a factor $1+\epsilon$, then they are willing to pay the higher price $(1+\epsilon)p_j$ for it. (They might already be paying the increased price to start with if $p_j^{(i)} = (1+\epsilon)p_j$. In this case $\tilde{p}_j = (1+\epsilon)p_j = p_j^{(i)}$.) The description of this subroutine is postponed to Section 2.3.2. Observe that `FindNewPrices` will make progress whenever $c^{(i)}$ is far from $x^{(i)}$ for some agent $i$. When they are very close for each agent $i$, then we have already reached an approximate equilibrium.

The above properties suggest the following update rules for each good $j \in [m]$.

*Case 1.* $p_j^{(i)} < (1+\epsilon)p_j$ *and* $\tilde{p}_j = (1+\epsilon)p_j$. The good $j$ was in $L_i$ and needs to be moved to $H_i$, i.e., agent $i$ used to pay $p_j$ but now is willing to pay the higher price for $j$. Agent $i$ first outbids themselves for the amount $c_j^{(i)}$ they already own and starts paying $p_j(1+\epsilon)$ for this amount. Additionally, agent $i$ outbids on good $j$ up to the amount they want and that is available from the other agents.

*Case 2.* $p_j^{(i)} = (1+\epsilon)p_j$ *and* $\tilde{p}_j = (1+\epsilon)p_j$. The good $j$ was in $H_i$ and stays in $H_i$, i.e., agent $i$ continues to pay the higher price. The agent $i$ still keeps the amount $c_j^{(i)}$ of good $j$ that they already had and outbids for as much as they can from the other agents.

*Case 3.* $p_j^{(i)} < (1+\epsilon)p_j$ *and* $\tilde{p}_j < (1+\epsilon)p_j$. The good $j$ remains in $L_i$, i.e., agent $i$ continues to pay the lower price. By (B), we must have $c_j^{(i)} \leq y_j \leq (1+\epsilon)c_j^{(i)}$; the agent will not seek to buy more of these goods.

The cases above have covered all possibilities since $p_j^{(i)} \leq \tilde{p}_j$. Note that, at the end of the step, in the first two cases the agent will own $\min(y_j, l_j + w_j)$ amount of good $j$, whereas they will own $c_j^{(i)}$ amount in the third case. Once all of the goods have been considered we set $p^{(i)} = \tilde{p}$, $x^{(i)} = y$, and update $c^{(i)}$ as the current allocation. If $w_j + l_j = 0$ for some $j$ then $h_j = e_j$, i.e., the whole $j$ is sold at the higher price $p_j(1+\epsilon)$. For each such good $j$ we increase the market price $p_j$ to $(1+\epsilon)p_j$, and for all agents $k$ we set $p_j^{(k)} = p_j$ for the new increased $p_j$; finally, we set $l_j = e_j$ and $h_j = 0$. The step ends.

### 2.3.1 Analysis

For the correctness of the algorithm, we need to show that all invariants are maintained.

**Lemma 2.3.3.** *If all agents have WGS demand systems, then the invariants (a)-(d) are maintained throughout the algorithm.*

*Proof.* **(a)** We always sell the goods at either price $p_j$ or at price $(1+\epsilon)p_j$. Moreover, at the end of the step if we have a good with $w_j + l_j = 0$, we increase its price and set $l_j = e_j$ and hence, $w_j + l_j > 0$ holds again at the end of a step.

**(b)** Once a part of some good is sold to some agent, it remains being sold to the agents until the end of the algorithm. This is guaranteed by property (A) of the procedure `FindNewPrices`, and the fact that $c_j^{(i)}$ may only decrease if another $c_j^{(k)}$ increases by the same amount. Prices can be increased only for goods with $w_j + l_j = 0$. Consequently, a good with $w_j > 0$ must still be at the initial price $p_j = 1$.

**(c)** Suppose these properties hold for every agent before a step of agent $i$. The requirements (A) and (B) guarantee that $c^{(i)}$ is dominated by a bundle $x^{(i)} \in D_i(p^{(i)}, b_i)$ and prices satisfy $p \leq p^{(i)} \leq (1+\epsilon)p$, for each agent $i$.

Now, consider an agent $k$ different from $i$. In the step, $k$ could lose a part a good through the outbid only and hence $c^{(k)}$ does not increase. As long as the prices $p^{(k)}$ do not change, (c) holds trivially. The only time $p^{(k)}$ can change is the price increase step, namely, if $p_j$ increases to $(1+\epsilon)p_j$, it forces $p_j^{(k)} = (1+\epsilon)p_j$. Note that the price increase only happens once $l_j = 0$. Assume we had $p_j^{(k)} < (1+\epsilon)p_j$ before the price increase, that is, agent $k$ was buying good $j$ at the lower price $p_j$. By $l_j = 0$ and invariant (d), it follows that $c_j^{(k)} = 0$ at this point. The WGS property implies that after increasing $p_j^{(k)}$, the bundle $c^{(k)}$ will be still dominated by an optimal bundle.

To complete the proof of (c), it remains to show that it is maintained at the beginning of the iteration, when the budgets are recomputed. Since the budgets may only increase, this again follows by the WGS property.

**(d)** Straightforward to check for each case. □

**Lemma 2.3.4.** *The smallest price* $\min\{p_j : j \in G\}$ *remains at most* $(1+\epsilon)$ *throughout the algorithm.*

*Proof.* As long as $w_j > 0$ for at least one good $j$, then the minimum price is 1 according to invariant (b). Assume that at a certain iteration, $w_j = 0$ for all $j \in G$, and consider the first iteration when this happens. This iteration may raise the minimum price to $(1+\epsilon)$. We will show that the algorithm must terminate in the next iteration in line 5.

Let $\bar{s}_i$ be the excess resulting from charging the lower price $p_j$ for all goods (both in $H_i$ and $L_i$). Clearly, $\bar{s}_i \geq s_i$.

We claim that $\bar{s}_i \geq 0$. In the subroutine `FindNewPrices`, we had $c^{(i)} \leq y \in D_i(\tilde{p}, b_i)$ and $p \leq \tilde{p}$. Since the subroutine was last executed for $i$, prices may have increased. However, this can only increased $b_i$, and a price $p_j$ may have increased to $(1+\epsilon)p_j$ if we already had $\tilde{p}_j = (1+\epsilon)p_j$.

Since all goods are fully sold by invariant (b), $\sum_{i \in A} \bar{s}_i = \sum_{i \in A} b_i - p^\top e = 0$. Consequently, $\bar{s}_i = 0$ and therefore $s_i \leq 0$ for all $i \in A$. □

Next, we give a bound on the total number of iterations, using the same basic idea of organizing the steps into rounds as in [59]. A *round* consists of going over all agents exactly once in the main 'for' loop and doing a step for each of them; that is, every round except the last one comprises of exactly $n$ steps.

**Lemma 2.3.5.** *The number of rounds in an iteration is at most $2/\epsilon$.*

*Proof.* We fix an iteration and let $p$ denote the market prices at the start of the iteration. Consider a step of an agent $i$ within the iteration. If $i$ buys the remaining available portion of a good $j$, $i$ buys everything that is available at the cheaper price $p_j$, then the market price of $j$ increases and the iteration finishes. So for the rest of the proof we assume that the market price increase does not happen; consequently, the budget of each agent is unchanged and agent $i$ gets the amount of each good it desires.

Let $\varphi$ denote the total money spent at a certain point of this iteration that is spent by the agents on higher price goods. That is,

$$\varphi = (1+\epsilon) \sum_{i=1}^{n} \sum_{j \in H_i} c_j^{(i)} p_j .$$

**Claim 2.3.6.** *Let $s_i$ denote the surplus of agent $i$ at the beginning of their step. Then the value of $\varphi$ increases at least by $s_i - 2.25\epsilon b_i$ during agents $i$'s step.*

*Proof of Claim.* Recall Cases 1-3 in the description of the step. Let $T_k$ be the set of goods that fall into case $k$, that is, $T_1 \cup T_2 \cup T_3 = [m]$.

- If $j \in T_1$, then $(1+\epsilon)p_j y_j$ will be added to $\varphi$ in the `Outbid` subroutine: In this case, the agent also outbids itself, moving the good from $L_i$ to $H_i$.

- If $j \in T_2$, then $(1+\epsilon)p_j(y_j - c_j^{(i)})$ will be added to $\varphi$ in the `Outbid` subroutine.
- If $j \in T_3$, then we do not increase $\varphi$. Nevertheless, (B) guarantees that $\tilde{p}_j(y_j - c_j^{(i)}) \leq \epsilon \tilde{p}_j c_j^{(i)}$. Consequently,

$$\sum_{j \in T_3} \tilde{p}_j(y_j - c_j^{(i)}) \leq \epsilon \tilde{p}^\top c^{(i)}. \tag{2.2}$$

Also note that $\tilde{p}_j = (1+\epsilon)p_j$ if $j \in T_1 \cup T_2$. Assumption 2 on non-satiation guarantees that $\tilde{p}^\top y = b_i$. Let $\Delta\varphi$ denote the increment in $\varphi$; this can be lower bounded as

$$\begin{aligned}\Delta\varphi &= \sum_{j \in T_1} \tilde{p}_j y_j + \sum_{j \in T_2} \tilde{p}_j(y_j - c_j^{(i)}) = \tilde{p}^\top y - \sum_{j \in T_3} \tilde{p}_j y_j - \sum_{j \in T_2} \tilde{p}_j c_j^{(i)} \\ &\geq b_i - \sum_{j \in T_3} \tilde{p}_j(y_j - c_j^{(i)}) - \tilde{p}^\top c^{(i)} \geq b_i - (1+\epsilon)\tilde{p}^\top c^{(i)} \quad ,\end{aligned}$$

using (2.2). The money spent by the agent at the beginning of the step is $b_i - s_i$. Good $j$ is purchased at price at least $p_j$ according to (d), and $\tilde{p}_j \leq (1+\epsilon)p_j$. Consequently, $\tilde{p}^\top c^{(i)} \leq (1+\epsilon)(b_i - s_i)$. With the above inequality, we obtain

$$\Delta\varphi \geq b_i - (1+\epsilon)^2(b_i - s_i) \geq s_i - (2\epsilon + \epsilon^2)(b_i - s_i) \geq s_i - 2.25\epsilon b_i,$$

as $\epsilon < 0.25$. This completes the proof. ■

As long as $\sum_{i=1}^n s_i > 3\epsilon p^\top e$, the claim guarantees that $\varphi$ increases in every round by at least $3\epsilon p^\top e - 2.25\epsilon \sum_{i=1}^n b_i \geq 0.75\epsilon p^\top e > 0.5\epsilon p^\top e$. Since $\varphi \leq p^\top e$, the number of rounds is bounded by $2/\epsilon$. □

*Proof of Theorem 2.3.1.* In their steps, agents use their surpluses to outbid for the goods. Let us now bound the number of repeats in the 'while' loop (lines 6–9) in all calls to `Outbid` in a given iteration. When the `Outbid`$(i, j, t)$ is called, the 'while' loop is repeated until $t$ is set to 0 or $l_j$ is set to 0. If $l_j$ is set to zero then the iteration finishes; and hence, there is one such event per iteration. Let us count the number of times 'while' loop is repeated until $t$ is set to 0. Before this happens, some $c_j^{(k)}$ value must be set to zero. The total number of such zeroing events within a single iteration is bounded by $nm$ — each agent loses a good through the outbid at most once.

Hence, the number of 'while' loops is at most $nm$ plus the total number of calls to `Outbid`. This is at most $m$ in each step, and thus $nm$ in each round. According to Lemma 2.3.5, the number of 'while' loops in every iteration is at most $2nm/\epsilon$; each repeat takes $O(1)$ time. The same bound holds for the 'if' calls in lines 1–4 in `Outbid`.

Every step calls the procedure `FindNewPrices` exactly once. Consequently, the cost of an iteration is $O(\frac{nm}{\epsilon} + \frac{nT_F}{\epsilon}) = O(\frac{nT_F}{\epsilon})$, using the assumption that $T_F = \Omega(m)$. Therefore, the time taken by `FindNewPrices` in an iteration is $O(nT_F/\epsilon)$. By Lemma 2.3.4,

**Algorithm 2:** `FindNewPrices`

**Input:** $i, p^{(i)}, p, \epsilon, c^{(i)}, f, b_i$.
**Output:** Prices $\tilde{p}$ and bundle $y$.
Initialization: $\tilde{p} \leftarrow p^{(i)}$
Obtain $y \in D_i(\tilde{p}, b_i)$ from the demand oracle with $y \geq c^{(i)}$
**while** $\exists j : \tilde{p}_j < (1+\epsilon)p_j$ *and* $y_j > (1+\epsilon)c_j^{(i)}$ **do**
    $\tilde{p}_j \leftarrow \min\{(1+\epsilon)^{1/f}\tilde{p}_j, (1+\epsilon)p_j\}$
    Obtain $y' \in D_i(\tilde{p}, b_i)$ from the demand oracle such that $y'_k \geq y_k$ for $k \neq j$
    $y \leftarrow y'$
**end**
**return** $(\tilde{p}, y)$

the minimum price is always at most $1+\epsilon$ throughout and therefore $p_{\max}$ equals $\frac{p_{\max}}{p_{\min}}$ or $\frac{(1+\epsilon)p_{\max}}{p_{\min}}$. Hence, the number of iterations is bounded by $O(m \log_{1+\epsilon}(p_{\max}/p_{\min})) = O(\frac{m}{\epsilon}\log(p_{\max}/p_{\min}))$. The claimed running time bound follows, using also the assumption $T_F = \Omega(m)$.

It is left to show that the prices $p$ and bundles $c^{(i)}$ form a $4\epsilon$-approximate market equilibrium. The first two properties in the definition are clear: $c^{(i)}$ is dominated by an optimal bundle with respect to the prices $p^{(i)}$, and no good is oversold. At termination, the total surplus of the agents is bounded by $3\epsilon p^\top e$. However, this surplus is computed assuming that some goods are sold at price $p_j$ and others at price $(1+\epsilon)p_j$. Decreasing the price of the latter goods to $p_j$ releases an additional excess of at most $\epsilon p^\top e$. Consequently, $\sum_{j=1}^m p_j(e - \sum_{i=1}^n c_j^{(i)}) \leq 4\epsilon p^\top e$. □

### 2.3.2 Implementing `FindNewPrices` for bounded elasticities

We now describe the subroutine `FindNewPrices`$(i, p^{(i)}, p, \epsilon, c^{(i)}, b_i)$. Recall that the outputs are new prices $\tilde{p} \geq p^{(i)}$ and a bundle $y$ with

(A) $y \geq c^{(i)}$ and $y \in D_i(\tilde{p}, b_i)$; in addition

(B) $p^{(i)} \leq \tilde{p} \leq (1+\epsilon)p$, and $\tilde{p}_j = (1+\epsilon)p_j$ whenever $y_j > (1+\epsilon)\, c_j^{(i)}$.

Let us assume that the demand system $D_i$ has elasticity at least $-f$ for some $f > 0$. Our Algorithm 2 for this case is a simple price increment procedure. First, we obtain $y \in D_i(p^{(i)}, b_i)$ from the demand oracle with $y \geq c^{(i)}$. This is possible due to invariant (c), which guarantees that $c^{(i)} \leq x^{(i)}$ for some $x^{(i)} \leq D_i(p^{(i)}, b_i)$. Then, the demand oracle is able to return a bundle $y$ such that $y \geq x^{(i)} \geq c^{(i)}$. Then, we iterate the following step. As long as (B) is violated for a good $j$, we increase its price by a factor $(1+\epsilon)^{1/f}$ until it reaches the upper bound $(1+\epsilon)p_j$.

**Lemma 2.3.7.** *Assume the demand system $D_i$ has elasticity at least $-f$ for some $f > 0$. Algorithm 2 terminates with $\tilde{p}$ and $y$ satisfying (A) and (B) in time $O(mf \cdot T_D)$, where $T_D$ is the time for a call to the demand oracle.*

We will assume that $T_D = \Omega(m)$, since the demand oracle needs to output an $m$-dimensional vector.

*Proof.* The bound on the number of iterations is clear: since we have $p \leq \tilde{p} \leq (1+\epsilon)p$ throughout, the price of every good can increase at most $f$ times. Condition (A) is satisfied due to the WGS property and the bound on the demand elasticity. When increasing $\tilde{p}_j$, the demand $y_k$ for $k \neq j$ is non-decreasing as guaranteed by the demand oracle. Further, $y_j$ may decrease only by a factor $(1+\epsilon)$, and since we had $y_j > (1+\epsilon)c_j^{(i)}$ before the price update, we still have $y_j > c_j^{(i)}$ after the price update. Condition (B) is satisfied at termination since the while loop keeps running as long as it is violated. Checking the while condition each time requires $O(m)$ time; however, this will be dominated by the time $T_D$ according to the comment on $T_D \geq m$ above. □

As explained in Section 2.3, this is only one of the possible ways of implementing `FindNewPrices`. Section 2.3.3 presents a convex programming approach for utilities that are homogeneous of degree 1. For example, for CES with parameter $\sigma > 1$, the running time of Algorithm 2 depends linearly on $\sigma$ (Lemma 2.2.2), whereas the running time in Section 2.3.3 is independent on this parameter. Nevertheless, for small values of $\sigma$ the simple price increment procedure may be preferable to solving a convex program.

Further, more direct approaches for implementing `FindNewPrices` may be possible for particular demand systems. For Cobb-Douglas demands with parameter vector $\alpha^{(i)}$, it is easy to devise an $O(m)$ time algorithm implementing the procedure. The algorithm relies on the fact that the optimal bundle is the bundle that allocates $\alpha_j^{(i)} b_i$ money for good $j$. Hence, each price can be set independently of the others. The next lemma shows an implementation of `FindNewPrices` for linear utilities; recall from Section 2.2 that the elasticity is unbounded in this case.

**Lemma 2.3.8.** *`FindNewPrices` can be implemented in $O(m)$ for a linear demand system corresponding to the utility function $u(x) = v^\top x$.*

*Proof.* Recall that for linear utilities $y \in D_i(\tilde{p}, b)$ $y_j > 0$ if and only if $j \in \arg\max_k v_k/p_k$, called maximum bang-per-buck goods (MBB). We initialize $\tilde{p} = p^{(i)}$, and let $S \subseteq [m]$ denote the set of MBB goods. We start increasing the prices of all goods $j \in S$ at the same rate $\alpha$. Once a good outside $S$ becomes MBB, we include it in the set $S$ and also start raising its price. We terminate when the budget is exhausted or when the price $\tilde{p}_k$ for a good $k \in S$ reaches the upper bound $(1+\epsilon)p_k$. In the latter case, we return the bundle $y_j = c_j^{(i)}$ if $j \neq k$, and set $y_k = (b_i - \sum_{j\neq k} \tilde{p}_j c_j)/\tilde{p}_k$; clearly, $y_k \geq c_k^{(i)}$. These prices and allocations satisfy (A) and (B); in fact, we obtain (B) in the stronger form that $\tilde{p}_j = (1+\epsilon)p_j$ whenever $y_j > c_j^{(i)}$. We need to add a good to $S$ at most $m$ times, and thus we can implement the procedure in $O(m)$ time. □

### 2.3.3 Implementing `FindNewPrices` for Gale demand systems

We now show that the subroutine `FindNewPrices` can be implemented for Gale demand systems via convex programming. As previously noted, this result is also applicable for demand systems given in the form (Max-utility) for utility functions that are homogeneous of degree one, for which the optimal solutions to (Max-utility) and (2.1) coincide.

Let $u : \mathbb{R}^m_+ \to \mathbb{R}_+$ be a monotone concave differentiable function. Let us further assume that $u$ is strictly concave, and therefore we have unique demands: $|G^u(p,b)| = 1$ for all $(p,b) \in \mathbb{R}^m_+$.

We show that a stronger version of the subroutine can be implemented, replacing the condition $y_j > (1+\epsilon)c_j$ by $y_j > c_j$ in (B). We formulate the problem in a slightly more general form where the vector of higher prices $(1+\epsilon)p$ is replaced by an arbitrary price vector $q$.

Let $p, q, c \in \mathbb{R}^m_+$ and $x \in G^u(p,b)$ such that $p \le q$ and $c \le x$. The goal is to find $\tilde{p}$ and $y$ such that

(A′) $y \ge c$ where $y \in G^u(\tilde{p}, b)$, and

(B′) $p \le \tilde{p} \le q$ and $\tilde{p}_j = q_j$ whenever $y_j > c_j$.

The following convex program captures the idea that an agent is allowed to buy a good $j$ at two prices: amount $y'_j$ at price $p_j$ and amount $y''_j$ at price $q_j$. Moreover, the amount $c_j$ of good $j$ is offered at price $p_j$ and for the rest an agent pays the higher price $q_j$.[6]

$$\begin{aligned} \max \quad & b \ln u(y) - p^\top y' - q^\top y'' \\ & y = y' + y'' \\ & y' \le c \\ & y', y'' \ge 0\,. \end{aligned} \tag{2.3}$$

We show that the optimal solution to this program, along with the prices obtained from the KKT conditions satisfy the requirements.

Since all constraints are linear, strong duality holds. Let $y^* = y' + y''$ be an optimal solution of (2.3). Then, by the KKT conditions, there exists $\alpha \in \mathbb{R}^m_+$ such that for any $j \in [m]$,

[6] Trivially, if $p_j < q_j$ and $y' < c_j$ then $y''_j = 0$ in any optimal solution. For the goods where $p_j = q_j$ we assume that $y'_j < c_j$ implies $y''_j = 0$, i.e., we always give priority to $y'_j$.

(i) $b \cdot \frac{\partial_j u(y^*)}{u(y^*)} \leq \min\{\alpha_j + p_j, q_j\}$,

(ii) $b \cdot \frac{\partial_j u(y^*)}{u(y^*)} = \alpha_j + p_j$ whenever $y'_j > 0$,

(iii) $b \cdot \frac{\partial_j u(y^*)}{u(y^*)} = q_j$ whenever $y''_j > 0$, and

(iv) $y'_j = c_j$ whenever $\alpha_j > 0$.

Let us define $\tilde{p}_j := \alpha_j + p_j$.

**Lemma 2.3.9.** *The allocations $y^*$ and prices $\tilde{p}$ satisfy (A′) and (B′).*

*Proof.* Since all constraints are linear, strong duality holds for (2.1) as well as for (2.3). Let us start with (B′). First note that (ii) implies that $\tilde{p}_j = q_j$ whenever $y^*_j > c_j$. Moreover, from (i), (ii), and (iv) it follows that $\tilde{p}_j \leq q_j$.

For (A′), let us start by showing $y^* \in G^u(\tilde{p}, b)$. By the KKT conditions this is equivalent to that $\frac{b\partial_j u(y^*)}{u(y^*)} \leq \tilde{p}_j$ and equality holds whenever $y^*_j > 0$. This is immediate from (i), (ii), and the definition of $\tilde{p}_j$.

It remains to show that $y^* \geq c$. We prove by contradiction: assume that $y^*_j < c_j$ for a good $j$. This implies $y''_j = 0$ and $\alpha_j = 0$ by the optimality conditions, yielding $\tilde{p}_j = p_j$. By the strict concavity assumption, $y^*$ is the unique optimal bundle in $G^u(\tilde{p}, b)$. Using the WGS property for $(p, b)$ and $(\tilde{p}, b)$ we have $y^*_j \geq x_j$ since $p_j = \tilde{p}_j$. We obtain a contradiction to $y^*_j < c_j \leq x_j$. □

### 2.3.4 Adding a dummy agent to bound the prices

Recall that our bounds on the running time of the auction algorithm depend on the ratio of the maximum price and minimum price at the termination, i.e., at an $\epsilon$-approximate equilibrium. A convenient way of bounding this ratio is to work with a slightly modified market. We can use the same idea as [32, 30]. Given an exchange market $M$ with agents $A$ and goods $G$, we transform it to another market $\hat{M}$ with $n+1$ agents as follows. Let $\eta \leq 1$ be a parameter such that $\frac{\eta}{1+\eta} > \epsilon(1+\epsilon)m$ (and $\epsilon(1+\epsilon)m \leq 1/2$). For $i \in A$ we keep the same demand systems $D_i$ and the same initial endowments $e^{(i)}$. The market $\hat{M}$ has an extra agent $n+1$ with initial endowment $e^{(n+1)} = \eta e$ and whose demand bundle is given via Cobb-Douglas utility function $\left(\prod_j x_j^{(n+1)}\right)^{1/m}$. Agent $n+1$ spends exactly $\frac{1}{m}$ of the budget on any good $j$ since its unique demand bundle $x^{(n+1)}$ is given by $x_j^{(n+1)} = \dfrac{\eta p^\top e}{m p_j}$.

The lemma below shows that adding such an agent can be used to bound $\frac{p_{\max}}{p_{\min}}$, at the expense of working on an modified market.

**Lemma 2.3.10.** *(i) For an $\epsilon$-equilibrium of $\hat{M}$, formed by prices $p$ and bundles $x^{(i)}$ we have* $\dfrac{p_{\max}}{p_{\min}} \leq \dfrac{(1+\epsilon)m}{\eta - \epsilon m(1+\epsilon)(1+\eta)} \cdot \dfrac{e_{\max}}{e_{\min}}$, *where $e_{\max} = \max_j e_j$ and $e_{\min} = \min_j e_j$.*

*(ii) An $\epsilon$-equilibrium in $\hat{M}$ gives an $\epsilon(1+\eta)$-equilibrium in $M$.*

*Proof.* Consider an $\epsilon$-equilibrium in $\hat{M}$ formed by $p$ and bundles $x^{(i)}$. By definition, there exists $z^{(n+1)} \in D_{n+1}\left(p^{(n+1)}, \eta p^\top e\right)$ such that $x^{(n+1)} \leq z^{(n+1)}$ and $p \leq p^{(n+1)} \leq (1+\epsilon)p$. We have $z_j^{(n+1)} = \frac{\eta p^\top e}{m p_j^{(n+1)}}$, and therefore, $p_j z_j^{(n+1)} \geq \frac{\eta}{(1+\epsilon)m} p^\top e$. On the other hand, from the third condition of the definition of $\epsilon$-equilibrium it follows that $p_j(z_j^{(n+1)} - x_j^{(n+1)}) \leq \epsilon p^\top e(1+\eta)$. Hence, $p_j x_j^{(n+1)} \geq \left(\frac{\eta}{(1+\epsilon)m} - \epsilon(1+\eta)\right) p^\top e$ for all $j$. In particular, $x_j^{(n+1)} \geq \left(\frac{\eta}{(1+\epsilon)m} - \epsilon(1+\eta)\right) \frac{p_{\max} e_{\min}}{p_j}$ for all $j$. Since $x_j^{(n+1)} \leq e_j \leq e_{\max}$ in an $\epsilon$-equilibrium, we have

$$\frac{p_{\max}}{p_{\min}} \leq \left(\frac{\eta}{(1+\epsilon)m} - \epsilon(1+\eta)\right)^{-1} \frac{e_{\max}}{e_{\min}}.$$

The second part of the lemma follows easily from the definition of an approximate equilibrium. □

### 2.3.5 Running times of existing auction algorithms

We conclude this chapter by reviewing the running time bounds given in previous auction algorithms and comparing them to our bounds.

**Linear utility functions** The paper [59] includes two algorithms. The running time of the first algorithm is $O\left(\frac{nm}{\epsilon^2} \log \frac{p_{\max} \mathbb{1}^\top e}{\epsilon e_{\min}} \log p_{\max}\right)$, and for the second one it is $O\left(\frac{nm}{\epsilon}(n+m) \log p_{\max}\right)$. The running time in Theorem 2.3.1, with the bound $T_F = O(m)$ for linear utilities from Lemma 2.3.8, gives an additional factor $(m+n)$ bound, while removing the first log factor (or term). We note that we are using a weaker notion of equilibrium in our result. The additional factor is due to our global update step: due to the more general, nonseparable nature of our framework, we consider all goods when updating an agent, while [59] considers only one good for an update.

The paper also gives the price bound $p_{\max} \leq (1+\epsilon)\frac{v_{\max}}{v_{\min}}$ (assuming the algorithm is initialized with all prices equal to 1) where $v_{\max} = \max_{i,j} v_{i,j}$ and $v_{\min} = \min_{i,j} v_{i,j}$ are the highest utility and the lowest utility and agent has for a good, as well as a more general bound for the case when $v_{\min} = 0$ is possible. These bounds are comparable to our bounds in Section 3.1.2 for SR-equilibria.

**Separable WGS** In [62], the running time bound is presented only for the Fisher market case, given as $O\left(\frac{nm}{\epsilon} \log \frac{1}{\epsilon} \log \frac{v v_{\max} \mathbb{1}^\top b}{b_{\min} v_{\min}} \log m\right)$. Here, $v_{\max}$ and $v_{\min}$ are upper and lower bounds on the slopes of the functions (analogous to those we define in (3.1)), $b_{\min}$ is the smallest budget, and $v$ is the total utility an agent would get from owning the full amount of all goods. It is mentioned that the result could be extended to exchange markets, similarly to [59], but no details or running time estimation are provided.

**Uniformly separable WGS** The paper [61], gives essentially the same bound as above; the analysis is limited and mainly refers to [62]. A problematic issue is that the main motivation for the paper is to give bounds on CES and Cobb-Douglas utilities, but $v_{\max} = \infty$ for these particular utilities.

# 3 Auction algorithm, spending restricted equilibrium, and Nash social welfare

We show how to modify the auction algorithm given in Chapter 2 for finding a spending restricted (SR) equilibrium in Fisher markets when agents have weak gross substitute (WGS) demands. Recall the definition of an SR-equilibrium: For agents $A$ with demand systems $D_i(p, b_i)$ and fixed budgets $b_i \in \mathbb{R}_+$ for all $i \in A$, we say that the prices $p \in \mathbb{R}^m$ and allocations $x^{(i)} \in D_i(p, b_i)$ form a *Spending Restricted (SR)* equilibrium, if $\sum_{i \in A} x_j^{(i)} = \min\{1, 1/p_j\}$ for all $j \in [m]$.

The modified algorithm is given in Section 3.1. As an application, we then give a polynomial-time constant-factor approximation algorithm for the symmetric Nash social welfare (NSW) under budget separable piecewise-linear concave (budget-SPLC) valuations (Section 3.2).

## 3.1 Auction algorithm for spending restricted equilibrium

We present a modification of Algorithm 1 for finding an approximate SR-equilibrium in a Fisher market where each agent satisfies the WGS property.

We allow for a more general notion of SR-equilibrium, where we can restrict the amount of money spend on good $j$ to be any positive number $t_j \in [0, \infty]$ and not just one. Note that in case of SR-equilibrium we require that the available amount of each good is fully sold as opposed to the approximate equilibrium in exchange markets.

**Definition 3.1.1** (Approximate SR-equilibrium). *Let $t \in [1, \infty]^m$. For an $\epsilon > 0$, the prices $p \in \mathbb{R}^m$ and bundles $x^{(i)} \in \mathbb{R}^m_+$ form an $\epsilon$-*approximate SR-equilibrium *w.r.t. $t$ if*

*(i)* $x^{(i)} \le z^{(i)}$ *for some* $z^{(i)} \in D_i(p^{(i)}, b_i)$, *where* $p \le p^{(i)} \le (1+\epsilon)p$,

*(ii)* $\sum_{i=1}^n x_j^{(i)} = a_j := \min\{1, t_j/p_j\}$ *for all $j$, and*

*(iii)* $\sum_{j=1}^m p_j \left( \sum_{i=1}^n z_j^{(i)} - a_j \right) \le \epsilon \sum_{i=1}^n b_i$.

**Changes to the algorithm** To adopt the auction algorithm for the SR-equilibrium in Fisher markets we make four changes. *First,* the budgets $b_i$ are constant throughout the algorithm and are part of the input. As such, they do not depend on the prices of goods in the market. *Second,* we need to account for the fact that in an SR-equilibrium exactly $\min\{1, t_j/p_j\}$ of a good is sold. *Third,* the initialization must be changed since the prices cannot be scaled up as for exchange markets: we cannot assume that there exists an SR-equilibrium with $p_j \geq 1$ for all $j$. *Fourth,* we do not make Assumption 2 on non-satiation. We only use the following weaker assumption, namely that after the prices increase, the spending of every agent is non-decreasing.

**Assumption 3.** *Let $(p, b) \in \mathbb{R}^{m+1}$ and $x \in D(p, b)$. If $q \geq p$ and $y \in D(q, b)$, then $q^\top y \geq p^\top x$.*

For Gale demand systems arising in the NSW problem with budget-additive valuations (as in [52]), Assumption 2 does not hold, whereas this weaker assumption is true. The same is trivially true for the budget-SPLC valuations we study in Section 3.2.

We use exactly the same variables as before, except that $w$ is not used; we will have $w = 0$ throughout, i.e., all goods remain fully sold. We change the invariants (a) and (b) slightly. The invariants (c) and (d) remain the same.

(a) The *available amount* $a_j$ of each good is partitioned into two parts according to the price it is being sold at:

- amount $l_j$ is sold at the lower price $p_j$, and
- amount $h_j$ is sold at the higher price $(1 + \epsilon)p_j$.

Moreover, $l_j > 0$ at the end of every step, i.e., after every step there is always a part of the good owned by an agent at the lower price. It holds $l_j + h_j = a_j$.

(b) The amount of each good $j$ being sold is exactly $a_j = \min\{1, t_j/p_j\}$.

Recall the definition of the surplus $s_i = b_i - \sum_{j \in L_i} c_j^{(i)} p_j - \sum_{j \in H_i} c_j^{(i)}(1 + \epsilon)p_j$. In the modified algorithm, we will use the *relative surplus* $s_i^r$ instead, defined as

$$s_i^r := {p^{(i)}}^\top x^{(i)} - \sum_{j \in L_i} c_j^{(i)} p_j - \sum_{j \in H_i} c_j^{(i)}(1 + \epsilon)p_j \,.$$

This is the difference between the money the agent would like to spend and what they are actually spending (in accordance with (c) and (d)). Under Assumption 2, $s_i^r = s_i$ holds; we need to make the distinction since we do not assume non-satiation.

**Initialization** In the case of exchange markets, we used Assumption 1 to state that approximate equilibrium prices $\geq \mathbb{1}$ exist, and then we were able to initialize the algorithm by setting all prices to 1. This is not viable for Fisher markets, where even the total budget might be smaller than $m$. Instead, we assume that we are given some initial, small enough prices $\bar{p} < t$ and optimal bundles $x^{(i)} \in D_i(\bar{p}, b_i)$ such that $\sum_{i=1}^n x^{(i)} \geq \mathbb{1}$. A simple way to achieve this is to have a single agent that overdemands all the goods, i.e., there is $i \in [n]$ and $\bar{p} < t$ such that $x^{(i)} \geq \mathbb{1}$ for $x^{(i)} \in D_i(\bar{p}, b_i)$. Such an initialization would need to be given for the particular demands.

Given such prices and allocations, we initialize $p^{(i)} = \bar{p}$ for all $i$, and set all $c^{(i)}$'s such that $c^{(i)} \leq x^{(i)}$ and $\sum_i c^{(i)} = \mathbb{1}$. One can readily check that all invariants are satisfied after the initialization. In particular, $l_j = 1, h_j = 0$ for all $j \in [m]$.

**Changes to the algorithm** In procedure Outbid, lines 1-4 are redundant as $w_j = 0$ for all $j$. In the main part of the algorithm, one needs to make the following changes besides the initialization.

- Every occurrence of $s_i$ is replaced by $s_i^r$.
- We do not need to recompute the budgets and surpluses at line NewIt.
- We need to add a new line between lines 18-19. The new line decreases the amount of good sold to exactly $\min\{1, t_j/p_j\}$ by decreasing $c_j^{(i)}$ for all agents $i$ proportionally to the values $c_j^{(i)}$. This will decrease the amount of goods sold whenever $p_j > t_j$.

**Remark 3.1.2.** *A simple alternative initialization is to set the price of good $j$ as $p_j = \min\{\frac{\epsilon}{m}\sum_i b_i, t_j\}$, and start with allocations $c^{(i)} = 0$. The drawback is that we would obtain a slightly weaker equilibrium at termination. Part (ii) of Definition 3.1.1 requires that all goods are fully sold; we would need to weaken this property to saying that the total price of all unsold goods would be $\leq \epsilon \sum_i b_i$. Below, we describe the analysis for the case where initially all goods are fully sold, but it can be easily adapted to this version.*

### 3.1.1 Analysis

As previously mentioned, an ($\epsilon$-)SR equilibrium may not exist at all. In such cases, our algorithm will never terminate, increasing the prices unlimitedly. We give the running time in terms of the ratio $p_{SR\max}/p_{\min}$. Here, $p_{\min} = \min_j \bar{p}_j$, the smallest one among the initial prices, and $p_{SR\max}$ is an upper bound on the prices in the algorithm; note that we may have $p_{SR\max} = \infty$. In Section 3.1.2, we give a bound in terms of the maximum and minimum values of the partial derivatives of the utility function.

**Theorem 3.1.3.** *Let $T_F$ be an upper bound on the running time of the subroutine `FindNew-Prices`. Then there exists an auction algorithm that finds a $4\epsilon$-approximate SR equilibrium in time*

$$O\left(\frac{nmT_F}{\epsilon^2}\log\left(\frac{p_{SR\max}}{p_{\min}}\right)\right).$$

**Lemma 3.1.4.** *If all agents have WGS demand systems, then the invariants (a)-(d) are maintained throughout the algorithm.*

*Proof.* The justification for invariants (c) and (d) is similar to the one given in the proof of Lemma 2.3.3. We only present the proofs of (a) and (b) which need some small modifications.

**(a)** For each case it is clear that every good is being sold at either $p_j$ or $(1+\epsilon)p_j$. Using the invariant (b) it is also clear that exactly $a_j$ of good $j$ is sold. Moreover, at the end of the step if we have a good with $l_j = 0$ we increase its price at line 18 and set $l_j = a_j$. Hence $l_j > 0$ is also satisfied.

**(b)** We need to show that exactly $a_j = \max\{1, t_j/p_j\}$ of good $j$ is sold at any point. Suppose that the invariant holds at the beginning of a step. After the for loop, the invariant is still satisfied since the outbid only changes the owner of the good. The invariant could be violated only at line 18 when we increase the price at the end of the step. Trivially, if the price increases to $p_j$ and $p_j \leq t_j$ the invariant remains valid. So, we only need to deal with the case when the price of good $j$ increases to $p_j$ and $p_j > t_j$. Then, in the new line we added between 18 and 19, we will immediately take away some of good $j$ from the agents to restore the invariant. □

The bound on the number of rounds within an iteration is exactly the same as for Algorithm 1, while the proof differs slightly due to using Assumption 3 instead of Assumption 2.

**Lemma 3.1.5.** *The number of rounds in an iteration is at most* $2/\epsilon$.

*Proof.* The only change arises at the end of the proof of Claim 2.3.6. We state the new claim and show how the end of the proof changes.

**Claim 3.1.6.** *Let $s_i^r$ denote the relative surplus of agent $i$ at the beginning of their step. Then the value of $\varphi$ increases by at least $s_i^r - 2.25\epsilon b_i$ during agent $i$'s step.*

Let $\Delta\varphi$ denote the increment in $\varphi$; this can be lower bounded as

$$\begin{aligned}\Delta\varphi &= \sum_{j\in T_1} \tilde{p}_j y_j + \sum_{j\in T_2} \tilde{p}_j (y_j - c_j^{(i)}) = \tilde{p}^\top y - \sum_{j\in T_3} \tilde{p}_j y_j - \sum_{j\in T_2} \tilde{p}_j c_j^{(i)} \\ &\geq \tilde{p}^\top y - \sum_{j\in T_3} \tilde{p}_j (y_j - c_j^{(i)}) - \tilde{p}^\top c^{(i)} \geq \tilde{p}^\top y - (1+\epsilon)\tilde{p}^\top c^{(i)} \ ,\end{aligned}$$

using (2.2). The money spent by the agent at the beginning of the step is ${p^{(i)}}^\top x^{(i)} - s_i^r$. Good $j$ is purchased at price at least $p_j$ according to (d), and $\tilde{p}_j \leq (1+\epsilon)p_j$. Consequently, $\tilde{p}^\top c^{(i)} \leq (1+\epsilon)({p^{(i)}}^\top x^{(i)} - s_i^r)$. Assumption 3 yields $\tilde{p}^\top y \geq {p^{(i)}}^\top x^{(i)}$. Therefore, using $\epsilon < 0.25$, we obtain

$$\Delta\varphi \geq \tilde{p}^\top y - (1+\epsilon)^2({p^{(i)}}^\top x^{(i)} - s_i^r) \geq s_i^r + \tilde{p}^\top y - (1+\epsilon)^2 {p^{(i)}}^\top x^{(i)} \geq s_i^r - 2.25\epsilon\tilde{p}^\top y \geq s_i^r - 2.25\epsilon b_i \,,$$

The claim follows. □

*Proof of Theorem 3.1.3.* The running times follows similarly as in the proof of Theorem 2.3.1. All that remain is to show that the prices $p$ and bundles $c^{(i)}$ form a $4\epsilon$-approximate SR equilibrium. The first two properties in the definition are clear: $c^{(i)}$ is dominated by an optimal bundle $x^{(i)}$ with respect to the prices $p^{(i)}$, and exactly $a_j = \min\{1, t_j/p_j\}$ of each good $j$ is sold. At termination, the total relative surplus of the agents is bounded by $3\epsilon \sum_i b_i$. Moreover,

$$\begin{aligned}\sum_{i=1}^{n} s_i^r &= \sum_{i=1}^{n} {p^{(i)}}^\top x^{(i)} - \sum_{i=1}^{n} \left( \sum_{j\in L_i} c_j^{(i)} p_j + \sum_{j\in H_i} c_j^{(i)}(1+\epsilon)p_j \right) \\ &\geq \sum p^\top x^{(i)} - (1+\epsilon)\sum_{i=1}^{n} p^\top c^{(i)} \geq -\epsilon \sum_{i=1}^{n} b_i + \sum_{j=1}^{m} p_j \left( \sum_{i=1}^{n} x^{(i)} - a_j \right).\end{aligned}$$

Therefore, $\sum_{j=1}^m p_j(\sum_{i=1}^n x_j^{(i)} - a_j) \leq 4\epsilon \sum_i b_i$. □

### 3.1.2 Conditions on the existence of SR-equilibria

We now present a general bound on the value of $p_{SR\max}$. Suppose that the demand system of each agent $i$ is provided in terms of a monotone concave and differentiable utility function $u_i$ in the form (Max-utility). We now assume that each $u_i$ is differentiable. The arguments here can be easily adopted for the non-differentiable case by using subgradients. We let

$$\begin{aligned} D := \frac{\max_i b_i}{p_{\min}}, \quad v_{i\max} := \max_j \partial_j u_i(0), \quad v_{i\min} &:= \min_j\{\partial_j u_i(D\cdot \mathbb{1}) : \partial_j u_i(0) > 0\}, \\ V_{\max} &:= \max_i \frac{v_{i\max}}{v_{i\min}}, \\ t_{\max} &:= \max_j t_j. \end{aligned} \tag{3.1}$$

Note that if $\partial_j u_i(0) = 0$, then agent $i$ is not interested in good $j$ at all. In case $\partial_j u_i(0) > 0$ we say that agent $i$ *is interested* in good $j$. Note that $D$ is an upper bound on the amount of any single good that any agent could buy during the algorithm.

We note that $t_{\max} = \infty$ could be possible. However, we can truncate the value of every $t_j$ to $\min\{t_j, \sum_i b_i\}$ without changing the problem, since the total spending is at most the total budget; the price of a good can never rise above this value in the algorithm or in an SR-equilibrium. Thus, we may assume $t_{\max} \leq \sum_i b_i$ in the bounds below.

**A necessary condition on the existence of SR-equilibria** The condition $\sum_i b_i \leq \sum_j t_j$ is necessary for the existence of an SR-equilibrium, since $\sum_j t_j$ is the total amount of money that can be spent on the goods. One can formulate an extension of this, that amounts to Hall's condition in a certain graph. Let $(A \cup G, E)$ denote the bipartite graph where the two classes $A$ and $G$ represent the agents and goods, respectively. We add an edge $(i, j) \in E$ if $\partial_j u_i(0) > 0$, that is, if agent $i$ is interested in good $j$. For a subset $S \subseteq A$, we let $\Gamma(S) \subseteq G$ denote the set of neighbors in this graph. Then, Hall's condition, that is,

$$\sum_{i \in S} b_i \leq \sum_{j \in \Gamma(S)} t_j, \quad \forall S \subseteq A \tag{3.2}$$

is a necessary condition on the existence of an SR-equilibrium. Note however that this condition is not sufficient: it holds for the example of Cobb-Douglas utilities, where no SR-equilibrium exists, as explained after Definition 3.1.1.

**Upper bounds on the prices** We now give a bound on $p_{SR\max}$ in terms of $V_{\max}$ and $t_{\max}$. We first consider the case when every agent is interested in every good. In this case, (3.2) reduces to the case when $S$ contains every good. Note that the bounds are finite only if $v_{i\min} > 0$, and $v_{i\max}$ is finite. For the Cobb-Douglas utilities, $v_{i\max} = \infty$.

**Lemma 3.1.7.** *Assume the demand systems of the agents are given in form* (Max-utility) *for monotone concave and differentiable utility functions $u_i$.*

*(i) Suppose that every agent is interested in every good, that is, $\partial_j u_i(0) > 0$ for every agent $i$ and every good $j$. Assume that $\sum_i b_i \leq \sum_j t_j$. Then, the prices throughout the auction algorithm remain bounded by $(1+\epsilon)^2 t_{\max} V_{\max}$.*

*(ii) Assume condition* (3.2) *holds with strict inequality for all $S \subseteq B$. Then, the prices throughout the auction algorithm remain bounded by $(1+\epsilon)^n t_{\max} V_{\max}^{n-1}$.*

*The same bounds are valid for any $\epsilon$-SR equilibrium.*

*Proof.* Let us first consider *(i)*. Let $p$ denote the market prices at a certain point of the algorithm, or at an $\epsilon$-SR equilibrium, and let $p_{SR\min}$ be the minimal price among those. Observe that this might be different from $p_{\min}$, since $p_{\min}$ is the minimal price at initialization. Let $\ell$ be a good with $p_\ell = p_{SR\min}$.

We use the KKT conditions of convex program (Max-utility). We let $\beta^{(i)}$ denote the Lagrange multiplier of the budget constraint for agent $i$. Then, $\partial_j u_i(x^{(i)}) \leq \beta^{(i)} p_j^{(i)}$ for all goods $j$; and equality holds whenever $x_j^{(i)} > 0$. Recall that each good $j$ is owned by some agent during the algorithm as well as in an $\epsilon$-SR-equilibrium.

Consider a good $j$, and let $k$ be an agent buying $j$, i.e., $c_j^{(k)} > 0$ and therefore $x_j^{(k)} > 0$. By the above, $p_j^{(k)}/p_\ell^{(k)} \leq \partial_j u_k(x^{(k)})/\partial_\ell u_k(x^{(k)})$. The assumption that every agent is interested in every good means that $v_{i\min} = \min_j \partial_j u_i(D \cdot \mathbb{1})$. Since $x^{(\ell)} \leq D \cdot 1$, concavity implies $\partial_\ell u_k(x^{(k)}) \geq v_{i\min}$. We also get $\partial_j u_k(x^{(k)}) \leq v_{k\max}$. Consequently, $p_j^{(k)}/p_l^{(k)} \leq$

$v_{k\max}/v_{k\min} \le V_{\max}$. Finally, since $p \le p^{(k)} \le (1+\epsilon)p$ we have $p_j \le (1+\epsilon)p_{SR\min}V_{\max}$ for any good $j$.

The proof is complete by showing that $p_{SR\min} \le (1+\epsilon)t_{\max}$. To prove this, we first show that once $p \ge t$, the algorithm terminates. Indeed, if $p \ge t$, then the agents spend $\sum_j t_j$ in total, since the amount $a_j = \min\{1, t_j/p_j\}$ is always fully sold. The condition $\sum_i b_i \le \sum_j t_j$ shows that agents cannot have any surplus at this point. Thus, once the lowest price rises above $t_{\max}$, the algorithm terminates. Since the prices increase in steps of $(1+\epsilon)$, we get that $p_{SR\min} \le (1+\epsilon)t_{\max}$.

Let us now consider part *(ii)*. We take the bipartite graph $(A \cup G, E)$, and on the same set of nodes we define a directed graph as follows. We orient all edges in $E$ from $A$ to $G$, and also add the arc $(j, i)$ whenever $x_j^{(i)} > 0$. Fix any good $j$, and let $S$ be the set of agents in $A$ reachable from $j$ in this directed graph. Note that the set of goods reachable from $j$ will be precisely $\Gamma(S)$. Let $\ell \in \Gamma(S)$ be the good with the lowest price $p_\ell$. As above, we can show that $p_\ell \le (1+\epsilon)t_{\max}$, since $p \ge t$ is not possible. Indeed, once $p \ge t$, then all the available amounts of goods in $\Gamma(S)$ are fully sold, and their total value is $\sum_{j\in\Gamma(S)} t_j > \sum_{i\in S} b_i$ by the assumption. By the definition of $S$, no agent outside $S$ pays for goods in $\Gamma(S)$, leading to a contradiction.

The directed graph contains a path of length $\le 2(n-1)$ from $p_j$ to $p_\ell$. As in the proof of part *(i)*, one can argue that for any two consecutive goods $j'$ and $j''$ on this path, $p_{j'}/p_{j''} \le (1+\epsilon)V_{\max}$. This implies the bound. □

**Bounding the prices for Gale demand systems** Consider now the demand system $G^{u_i}(p, b_i)$ defined from a monotone concave utility function by (2.1). An important difference is that agent $i$ may not exhaust their full budget $b_i$; however, the concavity implies that they will never spend more than $b_i$ in the optimal bundle. Consequently, even $\sum_i b_i \le \sum_j t_j$ is not a necessary condition for the existence of an equilibrium.

Still, we can obtain the same bounds as in Lemma 3.1.7 on the prices. The proof is identical, noting that the KKT conditions for (2.1) also imply $p_j^{(k)}/p_\ell^{(k)} \le \partial_j u_k(x^{(k)})/\partial_\ell u_k(x^{(k)})$ if $x_j^{(k)} > 0$, and the fact that agent $i$ spends at most $b_i$ in their optimal bundle.

**Lemma 3.1.8.** *Assume every agent has a Gale demand system* (2.1) *for monotone concave and differentiable utility functions $u_i$.*

*(i) Suppose that every agent is interested in every good, that is, $\partial_j u_i(0) > 0$ for every agent $i$ and every good $j$. Assume that $\sum_i b_i \le \sum_j t_j$. Then, the prices throughout the auction algorithm remain bounded by $(1+\epsilon)^2 t_{\max}V_{\max}$.*

*(ii) Assume condition* (3.2) *holds with strict inequality for all $S \subseteq B$. Then, the prices throughout the auction algorithm remain bounded by $(1+\epsilon)^n t_{\max}V_{\max}^{n-1}$.*

*The same bounds are valid for any $\epsilon$-SR equilibrium.*

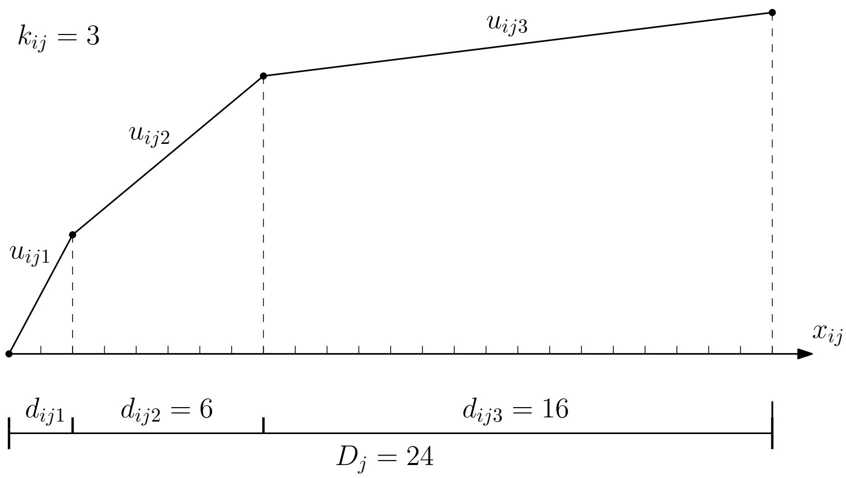


Figure 3.1: Agent $i$'s utility for good $j$.

## 3.2 Approximating Nash social welfare under budget-SPLC valuations

As an application of the spending restricted auction algorithm in Section 3.1, we give a polynomial-time $(2e^{1/2e} + \epsilon) \approx 2.404$-approximation algorithm for the NSW problem under budget-separable piecewise linear concave (SPLC) valuations—the common generalization of the models in [2] and [53]. We consider an instance of the NSW problem with $n$ agents and $m$ items, in which we have $D_j$ units (copies) of item $j$. Each agent $i$ has a budget-SPLC valuation function defined as follows (see Figure 3.1). For every good $j$, agent $i$ has $k_{ij}$ *segments* with *strictly* decreasing utility rates $u_{ij1} > u_{ij2} > \ldots > u_{ijk_{ij}} \geq 0$. Segment $t \in [k_{ij}]$ has length $d_{ijt}$ and agent $i$ values at $u_{ijt}$ each of the units in the segment. We assume that $\sum_{t\in[k_{ij}]} d_{ijt} = D_j$. Furthermore, agent $i$'s value is capped at $U_i$, i.e., their value for a subset of items is the minimum of $U_i$ and the sum of the values accumulated from the items.

Chaudhury et al. [24] gave a $e^{1/(1+\epsilon)e} \approx 1.45$-approximation algorithm for the problem, while Anari et al. [2] studied the problem with SPLC utilities ($U_i = \infty$) and gave a 2-approximation algorithm. The running times of these algorithms depend linearly on $M$, where $M = \sum_{j\in[m]} D_j$. In other words, [2] and [24] use segments of length 1. Therefore, when multiple copies of a good have the same utility rate, their algorithms run in pseudopolynomial time. Using the auction algorithm, we give an approximation algorithm running in polynomial time: the valuation function is specified by the utility rate and the length of a segment rather than $d_{ijt}$ segments of length one with the same value. The approach consists of three parts:

- Finding an SR-equilibrium for the instance of Fisher market arising as a relaxation of the NSW problem. The natural relaxation of the NSW problem uses the SR-equilibrium with respect to the Gale demand system, where each agent has budget $1$. We use the auction algorithm to find such an approximate SR-equilibrium $(x, p)$. It is worth pointing out that this is the main reason why we obtain a better running time guarantee than the existing approaches.

- Upper bound on the optimal value of the NSW in terms of prices $p$.
- Rounding the allocation $x$.

The last two rely on the ideas originally given by Cole and Gkatzelis [35] and extended in [2, 53]. More precisely, for the upper bound we follow [2] and we explain how the rounding reduces to the case of budget-additive linear utilities [53]. For the sake of simplicity, we present an upper bound and the rounding for an exact SR-equilibrium similar to the one in [53]. The modification to an approximate SR-equilibrium is straightforward. For the upper bound and rounding we make the assumption that $u_{ijt} \leq U_i$, as we could redefine the values to $u_{ijt} \leftarrow \min\{u_{ijt}, U_i\}$ without changing the objective value of the feasible allocations for the NSW instance.

**Gale demand and NSW** The demand systems of the market models in [2, 53] do not exactly correspond to (Max-utility). In [53] one needs additional conditions on the agents being "thrifty"; in [2] a "utility market model" is used. In both cases, the total spending of the agents can be below their budgets, i.e., they violate the non-satiation assumption. A natural unified way of capturing these equilibrium concepts is via Finding a spending-restricted equilibrium for Gale demand systems appears to be the right setting for NSW; in fact, the concepts used by [2] and [53] correspond to the Gale equilibrium in these settings, and moreover, these Gale demand systems admit the WGS property, see Section 3.2. On contrary, the demand systems arising from the previously mentioned utility functions do not satisfy the WGS property in the usual setting (Max-utility).

### 3.2.1 SR equilibrium under Gale demand systems of a budget-SPLC

We now consider the Gale demand system for *budget-SPLC*. We first show that the corresponding demand system is WGS—thus we can use the auction algorithm; and then we give an implementation of the `FindNewPrices` subroutine for this demand system. Note that the convex programming approach does not immediately apply, since the utility function is not differentiable, and the optimal bundle is not unique. Instead, we give a simple price increment procedure, an extension of that in Lemma 2.3.8 for linear utilities. As both the WGS property and `FindNewPrices` refer to a fixed agent, we drop the term $i$ denoting the agent in the subscripts.

The Gale demand system $G^u(p,b)$ is defined as the set of optimal solutions to the following formulation.

$$\begin{aligned}
\max \quad b\log\left(\sum_j\sum_t x_{jt}u_{jt}\right) - \sum_j p_j \sum_{t=1}^{k_j} x_{jt}& \\
\text{s.t.} \qquad x_{jt} \le d_{jt} \qquad & \forall j\in[m], t\in[k_j] \\
\sum_{j=1}^{m}\sum_{t=1}^{k_j} x_{jt}u_{jt} \le U & \\
x \ge 0\,. &
\end{aligned} \tag{3.3}$$

It can be easily verified, using the KKT conditions given below, that *admissible spendings* in [2] correspond to the case when $U=\infty$, and *modest and thrifty* demand bundles in [53] to the case when $k_j = 1$ for all $j$ with $d_{j1} = \infty$.

Let us now present the KKT conditions characterizing the optimal solution $x^*$. Let $r_{jt}$ be the Lagrange multipliers of the constraint $x_{jt} \le d_{jt}$ and $\gamma$ the Lagrange multiplier of the utility constraint. Recall that $u(x^*) = \sum_j\sum_t u_{jt}x^*_{jt}$. We have the following:

(i) $\frac{bu_{jt}}{u(x^*)} \le r_{jt} + p_j + u_{jt}\gamma$ for all $j,t$,

(ii) $\frac{bu_{jt}}{u(x^*)} = r_{jt} + p_j + u_{jt}\gamma$ whenever $x^*_{jt} > 0$,

(iii) $x^*_{jt} = d_{jt}$ whenever $r_{jt} > 0$, and

(iv) $\sum_j\sum_t x^*_{jt}u_{jt} = U$ whenever $\gamma > 0$.

**Lemma 3.2.1** (WGS property)**.** *The Gale demand system for budget-SPLC utilities satisfies the WGS property.*

*Proof.* Let us consider prices $p'$ defined as $p'_j = p_j$ for $j\in[m]\setminus\{\ell\}$ and $p'_\ell > p_\ell$. We show that there is an optimal bundle $x'$ at prices $p'$ such that $x'_{jt} \ge x^*_{jt}$ for all $j\ne\ell$ and all $t\in[k_j]$. For prices $p'$, let $\bar u$ be the optimal utility in (3.3) and let $\gamma'$ be the Lagrange multiplier for the constraint on the maximum utility achieved. We consider two cases.

*Case 1:* $\bar u < u(x^*_i)$. By (ii), $x^*_{jt} > 0$ implies $\frac{u_{jt}}{p_j} \ge \frac{u(x^*)}{b}$. Thus, we have $\frac{u_{jt}}{p'_j} = \frac{u_{jt}}{p_j} \ge \frac{u(x^*_i)}{b} > \frac{\bar u}{b}$ for all $j,t$ with $x^*_{jt} > 0$ and $j\ne\ell$.

Moreover, by (ii) and (iii), if $\frac{u_{jt}}{p'_j} > \frac{\bar u}{b}\cdot\left(1+\gamma'\cdot\frac{u_{jt}}{p'_j}\right)$ then $x'_{jt} = d_{jt}$. By (iv), $\bar u < u(x^*_i) \le U$ implies that $\gamma' = 0$, and hence $x'_{jt} = d_{jt}$ for all $j,t$ with $x^*_{jt} > 0$ and $j\ne\ell$. In other words, for every item $j$, $j\ne\ell$, every segment of the good that the agent was buying at prices $p$ is fully bought at prices $p'$. The lemma follows.

*Case 2:* $\bar u = u(x^*_i)$. It suffices to prove that the optimal solutions of the following knapsack linear program satisfy the WGS property.

$$\begin{aligned} \min \quad & \sum_j p_j \sum_{t=1}^{k_j} x_{jt} \\ \text{s.t.} \quad & x_{jt} \le d_{jt} \qquad \forall j \in [m], t \in [k_j] \\ & \sum_{j=1}^{m} \sum_{t=1}^{k_j} x_{jt} u_{jt} = \bar{u} \\ & x \ge 0\,. \end{aligned} \tag{3.4}$$

Suppose that the optimal solution $x$ is unique, then it can be build in a greedy fashion. Order the segments of all items in a decreasing order of the fractions $\frac{u_{jt}}{p_j}$. Then $x$ is obtained by purchasing the segments (i.e. allocating $x_{jt} = d_{jt}$) in the above order until the utility becomes $\bar{u}$; having in mind that the last purchased segment might be purchased only partially.

To prove the WGS property we consider increasing price $p_\ell$ of an item $\ell$. The price increase will cause the segments corresponding to good $\ell$ to move further back in the ordering while the relative order of all rest of the segments remains unchanged. Hence, by the greedy argument above, one can find an optimal solution $x'$ with $x'_{jt} \ge x_{jt}$ for all $j \ne \ell$ and $t \in [k_j]$.

In the case there are multiple optimal solutions, a similar argument holds since two optimal solution differ only on a set of goods with the same ratio $\frac{u_{jt}}{p_j}$. □

As in Section 2.3.3, we show that the following slightly more general version of `FindNewPrices` can be implemented. Let $p, q, c \in \mathbb{R}^m_+$ and $x \in G^u(p, b)$ such that $p \le q$ and $c \le x$. Find $\tilde{p}$ and $y$ such that

(A′) $y \ge c$ where $y \in G^u(\tilde{p}, b)$, and

(B′) $p \le \tilde{p} \le q$ and $\tilde{p}_j = q_j$ whenever $y_j > c_j$.

**Lemma 3.2.2** (`FindNewPrices`)**.** *The procedure `FindNewPrices` can be implemented in time $O(K)$ for Gale demand systems with budget-SPLC utilities, where $K = \sum_{j \in [m]} k_j$ is the number of segments with* different *marginal utility.*

The proof is via an algorithm that is an extension of the one in the proof of Lemma 2.3.8 for linear utilities.

*Proof.* We present an algorithm for finding such prices $\tilde{p}$ and bundle $y$. The algorithm initializes $\tilde{p} = p$ and $y = c$. The prices as well as the allocations are non-decreasing throughout the algorithm. Note that $u(y) < U$ at the initialization; otherwise, $c = x$ would follow and we can simply output $y = x$ and $\tilde{p} = p$. We maintain $p \le \tilde{p} \le q$

throughout. For each $j \in [m]$, let $t_j \in [k_j]$ denote the first segment of a good $j$ that is not completely sold in $y$, i.e., the minimal $t_j$ such that $y_{jt_j} < d_{jt_j}$. We call this the *active segment* for $j$.

Consider the optimal bundle $x$ such that $c \leq x$, and let $\gamma$ be the Lagrange multiplier for the utility cap constraint for $x$. We initialize $\beta = (b/u(x) - \gamma)^{-1}$. Then, from (i)-(iii) we see that if $x_{jt} = 0$ then $u_{jt}/p_j \leq \beta$, if $0 < x_{jt} < d_{jt}$ then $u_{jt}/p_j = \beta$, and if $x_{jt} = d_{jt}$ then $u_{jt}/p_j \geq \beta$.

**Stage I: enforcing the complementarity conditions** The algorithm proceeds in two stages. In the first stage, we consider the goods for which $u_{jt_j}/\tilde{p}_j > \beta$ yet $y_{jt_j} < d_{jt}$. (Recall that we initialized $y = c$ and $\tilde{p} = p$.) For each such good, we increase $\tilde{p}_j$ until either $u_{jt_j}/\tilde{p}_j = \beta$, or $\tilde{p}_j = q_j$. In the latter case, we buy the entire active segment of $j$, that is, we increase to $y_{jt_j} = d_{jt_j}$. Thus, $t_j$ increases by 1. If we still have $u_{jt_j}/q_j > \beta$, we again buy the entire active segment, and continue until $u_{jt_j}/q_j \leq \beta$ for the current active segment. This finishes the description of the first stage.

From the KKT optimality conditions on $x$, it is easy to see that $y \leq x$ at the end of the first stage. We claim that the following conditions are satisfied at this point:

$$y_{jt} = 0 \Rightarrow u_{jt}/\tilde{p}_j \leq \beta, \qquad 0 < y_{jt} < d_{jt} \Rightarrow u_{jt}/\tilde{p}_j = \beta, \quad y_{jt} = d_{jt} \Rightarrow u_{jt}/\tilde{p}_j \geq \beta \tag{3.5}$$

$$u(y) \leq \min\{U, b\beta\} \tag{3.6}$$

$$y_{jt} > c_{jt} \Rightarrow \tilde{p}_j = q_j \tag{3.7}$$

The conditions (3.5) and (3.7) are immediate from the algorithm. The bound (3.6) follows since $y \leq x$; $u(y) \leq u(x) \leq U$ by the feasibility of $x$ and $u(x) \leq b\beta$ by the definition of $\beta$.

**Stage II: price increases** In the second stage we continue increasing $y$ and $\tilde{p}$, as well as decreasing $\beta$ so that (3.5), (3.6), and (3.7) are maintained. The algorithm terminates once (3.6) holds at equality. In this case, one can verify from the optimality conditions that $y \in G^u(\tilde{p}, b)$. Together with (3.7), we see that the output satisfies (A') and (B').

The algorithm performs the following iterations. We let $A$ denote the set of goods for which $u_{jt_j}/\tilde{p}_j = \beta$. If there is a good $j \in A$ with $\tilde{p}_j = q_j$, then we start increasing $y_{jt_j}$ until either

1. $y_{jt_j} = d_{jt_j}$. Note that $t_j$ increases by one in this case, and $j$ leaves $A$.

2. The inequality (3.6) becomes binding. In this case, the algorithm terminates.

We now turn to the case when $\tilde{p}_j < q_j$ for all $j \in A$. During the iteration we multiplicatively increase the price of every good in $A$ by the same factor $\alpha > 0$, as well as decrease

$\beta$ by the factor $\alpha$. We choose the smallest value of $\alpha$ when one of the following events happen:

1. For some $j \in A$ we reach $\tilde{p}_j = q_j$. We change the allocations as described above.
2. The inequality (3.6) becomes binding (due to the decrease in $\beta$). In this case, the algorithm terminates.
3. For some good $\ell \notin A$, $\frac{u_{\ell t_\ell}}{p_\ell} = \beta$. In this case, we add $\ell$ to $A$, and iterate with the larger set.

It is easy to see that all three properties (3.5), (3.6), and (3.7) are maintained throughout the algorithm. We claim that the number of price change steps is at most $\sum_j k_j$. Indeed, a price increase step always ends when a good $j$ with $\tilde{p}_j = q_j$ enters $A$, either in case 1 or case 3. Once this happens, we increase $y_{jt_j}$; if the algorithm does not terminate, then we saturate the segment to $y_{jt_j} = d_j$. This shows that the number of price augmentation steps is bounded by the total number of segments $\sum_j k_j$. □

**Bound on $p_{SR\max}$** While the budget-SPLC utilities are not strictly monotone nor differentiable, the same bound as in Lemma 3.1.7 (or Lemma 3.1.8) can be similarly proved for $v_{i\max} = \max_{j\in[m],t\in[k_{ij}]} u_{ijt}$ and $v_{i\min} = \min_{j\in[m],t\in[k_{ij}]}\{u_{ijt} : u_{ijt} > 0\}$. The value $u_{ijt}$ represent the utility rate of agent $i$ for the $t$-th segment of item $j$.

Recalling that $D_j$ is the number of available units of good $j$, we have the following theorem.

**Theorem 3.2.3.** *Consider the Fisher market instance arising from the NSW problem where agents have budget-SPLC utilities. Let $K = \max_{i\in A}\sum_{j\in G} k_{ij}$ be the minimum number of segments needed to specify the utility of any agent. We can find an $\epsilon$-SR equilibrium with respect to the Gale demand systems and bounds $t_j := D_j$ in time $O\left(\frac{n^3mK}{\epsilon^2}\log\left(\frac{D_{\max}V_{\max}}{\epsilon}\right)\right)$.*

*Proof.* We start by adding a dummy agent $0$ to the market with budget $\epsilon$. The utility of agent $0$ is additive, meaning that for each good $j$, there is only one segment of length $D_j$ and $u_{0,j,1} = 1$. We initialize the auction algorithm by setting each price $p_j$ to $\frac{\epsilon}{\sum_j D_j}$ and assigning all goods to $0$. By running the auction algorithm for SR-equilibrium we obtain $\frac{4\epsilon}{5}$-approximate equilibrium. Now, we can remove the agent. As this agent could be buying the goods in amount at most $\epsilon$, by removing the dummy agent we are left with a slightly weaker notion of $\epsilon$-approximate equilibrium. Namely, the first and third condition in Definiton 3.1.1 are satisfied by the choice of the precision parameter, but the second condition is not satisfied exactly. Rather, we can only guarantee that $\sum_{i=1}^n x_j^{(i)} \le a_j$ and $\sum_{j\in[m]} p_j(a_j - \sum_{i=1}^n x_j^{(i)}) \le \epsilon$. In words, the total price of unsold available amounts of all goods is at most $\epsilon$.

By Theorem 3.1.3 the auction algorithm runs in $O\left(\frac{nmT_F}{\epsilon^2}\log\left(\frac{p_{SR\max}}{p_{\min}}\right)\right)$. Recall that $T_F$ is time needed to implement `FindNewPrices`. By Lemma 3.2.2, in this case $T_F$ is $O(K)$. By construction, $p_{\min} = \frac{\epsilon}{\sum_j D_j}$. By Lemma 3.1.8 we have that $p_{SR\max} \leq (1+\epsilon)^n D_{\max} V_{\max}^{n-1}$. □

### 3.2.2 Upper bound on the optimal NSW value

Let $(x, p)$ an SR-equilibrium in the Fisher market arising from an instance of NSW (with respect to the Gale demand system) and with bounds $(D_j)_{j\in[m]}$. In other words, $x_i \in G^{u_i}(p, 1)$ for each agent $i \in A$, and it holds $\sum_{i\in[n]} x_{ij} = \sum_{i\in[n],j\in[m],t\in[k_{ij}]} x_{ijt} = D_j \cdot \min\{1, 1/p_j\}$ for all $j \in G$. As $x_i \in G^{u_i}(p, 1)$ we have the following KKT conditions, see Section 3.2.1:

(i) $\frac{u_{ijt}}{u_i(x_i)} \leq r_{ijt} + p_j + u_{ijt}\gamma_i$,

(ii) $\frac{u_{ijt}}{u_i(x_i)} = r_{ijt} + p_j + u_{ijt}\gamma_i$ whenever $x_{ijt} > 0$,

(iii) $x_{ijt} = d_{ijt}$ whenever $r_{ijt} > 0$, and

(iv) $\sum_j \sum_t x_{ijt}u_{ijt} = U_i$ whenever $\gamma_i > 0$.

Let us describe some properties of SR-equilibrium $(x, p)$ that the above KKT conditions imply. By property (ii), $\frac{u_{ijt}}{r_{ijt}+p_j} = \frac{u_i(x_i)}{1-\gamma_i u_i(x_i)}$ whenever $x_{ijt} > 0$. This justifies defining $\mathrm{mbb}_i := \frac{u_i(x_i)}{1-\gamma_i u_i(x_i)}$. Since the SR-equilibrium as well as NSW are invariant under scaling each agent's utilities $u_{ijt}$ and $U_i$, we can assume that $\mathrm{mbb}_i = 1$ for all agents $i$. (This implies an appropriate implicit scaling of each $\gamma_i$ as well.) Then by property (iii) we obtain:

**Proposition 3.2.4.** *If $x_{ijt} > 0$ then $\frac{u_{ijt}}{p_j} \geq 1$. If $\frac{u_{ijt}}{p_j} > \mathrm{mbb}_i = 1$ then $x_{ijt} = d_{ijt}$.*

In other words, an agent only buys copies of goods with utility at least as much as their price, and if an agent values some copy of a good strictly more than its price then she also gets this copy in $x$.

We say that an agent $i$ is *capped* if $u_i(x_i) = U_i$ and *non-capped* otherwise. Let $H(p) = \{j \in [m] : p_j > 1\}$ be the set of *expensive* goods.

**Proposition 3.2.5.** *Assume $\mathrm{mbb}_i = 1$ for all agents $i$. For all capped agents $i$ it holds $x_{ijt} = 0$ for all $j \in H(p)$ and all $t \in [k_{ij}]$, and $u_i(x) = U_i \leq 1$. Each non-capped agent $i$ receives exactly one unit of utility, i.e., $u_i(x) = 1$.*

*Proof.* Suppose not and let $x_{ijt} > 0$ for some $j \in H(p)$. Then $u_{ijt} \geq p_j > 1$. Since $U_i \geq u_{ijt}$ it also holds that $U_i > 1$. A contradiction since $1 < \frac{U_i}{1-\gamma_i u_i(x_i)} = \frac{u_i(x_i)}{1-\gamma_i u_i(x_i)} = 1$ holds.

Since $\frac{u_i(x_i)}{1-\gamma_i u_i(x_i)} = 1$ and $\gamma_i u_i(x_i) \geq 0$ it follows that $u_i(x_i) \leq 1$. The property (iv) implies that $\gamma_i = 0$ for non-capped agents, , and therefore $u_i(x) = 1$. □

In order to prove an upper bound we may assume that $U_i = \infty$ for all non-capped agents. Such an assumption can only increase the optimal NSW, so if we prove the upper bound under the assumption it also holds in the original instance. Since "cap inequality" is ineffective for every non-capped agent, by the KKT conditions we can see that $(x, p)$ remains an SR-equilibrium. Denote with $A_c$ (resp. $A_u$) the set of capped (resp. non-capped) agents in the equilibrium $(x, p)$.

**Lemma 3.2.6.** *Let $p$ be a vector of $SR$-equilibrium prices and $x^*$ an optimal NSW allocation. Then*

$$\mathrm{NSW}(x^*) \leq \left( \prod_{i \in A_c} U_i \cdot \prod_{j \in H(p)} p_j^{D_j} \right)^{1/n} .$$

*Proof.* First we give a bound on the sum of the agents' utilities in any integer allocation $z$ as a function of prices $p$. Recall that $x$ is an SR-equilibrium allocation for prices $p$. Since valuations of the agents are scaled to have $\mathrm{mbb}_i = 1$, by Proposition 3.2.5 each non-capped agent receives exactly 1 unit of utility in $x$. Each capped agent receives $U_i$ utility in $x$ by definition. However, if there are some expensive goods then $x$ does not fully allocate all the goods. Each copy of the expensive goods generates 1 unit of utility in $x$ since the total spending on it is precisely 1 and since no capped agent buys expensive goods (Proposition 3.2.5).

Let $\bar{x}$ be the allocation in which we allocate every copy of each expensive good $j$ to a single agent spending on it in $x$. We can do so since the spending is exactly $D_j$ and thus, there are at least as many agents buying good $j$ as the copies. As all of these agents are non-capped and we assume that for such agents $U_i = \infty$, it follows that each copy of an expensive item generates exactly $p_j$ utility to the agents in $\bar{x}$. By Proposition 3.2.4, it is at least $p_j$ as $x_{ijt} > 0$ implies that $u_{ijt} \geq p_j$; it is at most $p_j$ by the contraposition of: $u_{ijt} > p_j$ implies that $x_{ijt} = d_{ijt} \geq 1$. Therefore, the total utility that all the items in $\bar{x}$ generate is:

$$\sum_{i \in A_c} U_i + |A_u| + \sum_{j \in H(p)} D_j(p_j - 1) = \sum_{i \in A_c} U_i + |A_u| - \sum_{j \in H(p)} D_j + \sum_{j \in H(p)} D_j p_j .$$

We claim that the total utility of all the agents in any integer allocation is not larger than the above sum. Consider the copies of item $j$. In $\bar{x}$, each one of those items generates either $p_j$ or more than $p_j$ utility. Moreover, any agent that can derive more than $p_j$ utility from a copy of a good actually receives the copy in $\bar{x}$. Therefore, $\bar{x}$ allocates the copy of goods to the agents such that the total utility all the goods generate is maximized. It follows that for any integral allocation $z$ the total utility all agents receive is at most

$$\sum_{i \in [n]} u_i(z) \leq \sum_{i \in A_c} U_i + |A_u| - \sum_{j \in H(p)} D_j + \sum_{j \in H(p)} D_j p_j .$$

At this point, suppose that we are given the above amount of utility and we can freely distribute it among agents to maximize NSW, regardless of what the utility function of each agent is, but only respecting the fact that the capped agents cannot get more than their cap, and that expensive goods are indivisible. By Proposition 3.2.5, all caps of the capped agents are at most $1$. Then, it is not too hard to see that the optimal way of distributing our lump sum of utility is to assign: each expensive copy to a non-capped agent and nothing else to those agents, exactly $U_i$ to each capped agent, and $1$ to everyone else. In this case, the NSW is exactly $\left(\prod_{i\in A_c} U_i \cdot \prod_{j\in H(p)} p_j^{D_j}\right)^{1/n}$. □

### 3.2.3 Rounding

As in the previous section, we assume that the utilities are scaled such that $\mathrm{mbb}_i = 1$. Moreover, we use that $u_{ijt} \le U_i$. We reduce our rounding to the case of budget-additive utilities in [53]. It is convenient to present the rounding in terms of the *spending graph*. For an SR-equilibrium $(x, p)$ the spending graph is a bipartite graph $(A, G; E)$ where an agent $i$ is adjacent to a good $j$ if and only if $x_{ij} > 0$. We show how to round $x$ to an integral allocation $x'$.

By the KKT conditions, whenever $\frac{u_{ijt}}{p_j} > \mathrm{mbb}_i$ then $x_{ijt} = d_{ijt}$ – in this case we allocate $d_{ijt}$ copies of good $j$ to $i$ by setting $x'_{ijt} \leftarrow d_{ijt}$. Moreover, if for for some triple $i, j, t$ we have $x_{ijt} > 1$ then we allocate $\lfloor x_{ijt} \rfloor$ units of good $j$ to agent $i$. Formally, we set $x'_{ijt} \leftarrow \lfloor x_{ijt} \rfloor$. Once we do this for all goods and all agents, any agent can have up to one unit of a good that she is buying in the SR-equilibrium but that is not yet allocated in $x'$. Hence there are at most $n$ units of each good $j$ that are still to be allocated. By the first rule for allocating goods, for these remaining copies of a good $j$, if an agent $i$ is buying a fraction of it, then $\frac{u_{ijt_i}}{p_j} = 1$ (where $t_i$ is the first non-saturated segment of agent $i$). By assuming that $u_{ijt} = 0$ for all $t > t_i$, we can transform the instance into an instance in which the utility of every agent is budget-additive. The only issue is that we could have several copies of a good. Since there are at most $n$ copies of each good that are unassigned and the utilities are budget-additive, we can simply split each good into the appropriate number of goods with a single copy. Then, the rest of the rounding follows the exact same steps as the rounding for budget-additive utilities in [53]. The analysis reduces in the same way. By choosing a suitable $\epsilon$ we obtain the following theorem.

**Theorem 3.2.7.** *Consider an instance of NSW problem where agents have budget-SPLC utilities. Let $K = \max_{i\in A} \sum_{j\in G} k_{ij}$ be the minimum number of segments needed to specify the utility of any agent. Then there is an algorithm running in time $O\left(n^3 mK \log\left(D_{\max} V_{\max}\right)\right)$ which produces a solution that is at most $2.404$ times worse than the optimum.*

# 4 Approximating asymmetric Nash social welfare under Rado valuations

We recall that the discrete Nash social welfare problem asks to solve the following problem

$$\max \left\{ \left( \prod_{i \in \mathcal{A}} v_i(S_i)^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} : \{S_i : i \in \mathcal{A}\} \text{ forms a partition of } \mathcal{G} \right\} . \tag{4.1}$$

where $v_i : 2^{\mathcal{G}} \to \mathbb{R}_+$ is the valuation function of agent $i$ and $w_i > 0$ is $i$'s weight (entitlement). In this chapter, we present an approximation algorithm for the asymmetric NSW problem under Rado valuations. The approximation ratio depends on $\gamma = 1 + \max_{i \in \mathcal{A}} w_i$, i.e., all weights fall into the interval $[1, \gamma - 1]$ for some $\gamma \geq 2$.

The valuation functions $v_i$ in the Nash social welfare problem are defined on subsets of $\mathcal{G}$. Our arguments are based on convex relaxations, which requires a continuous extension of the valuation functions to $\mathbb{R}_+^{\mathcal{G}}$. Thus, our first tasks is to provide a suitable extension for Rado valuations which we do in Section 4.1.1.

We present the overall approach and the main phases of our approximation algorithm in Section 4.2. The details and proofs required for the individual phases are then presented in the later sections. In Section 4.6, we show how our approach connects to the SR-equilibrium. Finally, in Section 4.7 we give an example separating Rado valuations and GS valuations.

## 4.1 Preliminaries

### 4.1.1 Concave extensions of discrete valuations

For any discrete valuation function $v : 2^{\mathcal{G}} \to \mathbb{R}$, we can define the *concave closure* $\bar{v} : [0,1]^{\mathcal{G}} \to \mathbb{R}$ as

$$\bar{v}(x) := \inf_{p \in \mathbb{R}^{\mathcal{G}}, \alpha \in \mathbb{R}} \{\langle p, x \rangle + \alpha : p(S) + \alpha \geq v(S) \quad \forall S \subseteq \mathcal{G}\} , \tag{4.2}$$

see e.g. [95, Section 3.4]. As the infimum of linear functions, $\bar{v}$ is always concave. Note that it provides the concave upper envelope of the function $v$ defined on the discrete set $\{0,1\}^{\mathcal{G}}$, meaning that $\bar{v} \le f$ for every concave function $f : \mathbb{R}_+^{\mathcal{G}} \to \mathbb{R}$ such that $v(S) \le f(\chi_S)$ for all $S \subseteq \mathcal{G}$, and where $\chi_S$ is an indicator vector in $\mathbb{R}^{\mathcal{G}}$ of a subset $S$. Moreover, whenever $v$ is monotone then $\bar{v}$ is a concave extension of $v$. By an *extension* we mean that the two functions coincide on the integer points, i.e., $\bar{v}(\chi_S) = v(S)$ for all $S \subseteq \mathcal{G}$. Namely, when $v$ is monotone the value $\bar{v}(\chi_S)$ is lower-bounded by $v(S)$ and it is also achieved by $\alpha = v(S)$ and $p$ set to $0$ in $S$ and to $v(\mathcal{G})$ outside of $S$, for any $S \subseteq \mathcal{G}$ .

Whereas the concave extension $\bar{v}$ can be defined for every monotone valuation function $v$, evaluating $\bar{v}(x)$ can be a hard problem. For example, in the case of submodular valuations, deciding whether $p(S) + \alpha \ge v(S)$ holds for all $S \subseteq \mathcal{G}$ amounts to submodular maximization and is thus NP-hard. Computing $\bar{v}(x)$ amounts to minimization over a polyhedron $P$ where separation is NP-hard; by the polynomial equivalence of optimization and separation [64], it follows that evaluating $\bar{v}(x)$ is NP-hard for submodular functions (see also [72, Lemma 6.15]).

**The concave extension of Rado valuations** Unlike with the submodular functions, the concave closure can be evaluated with polynomially many value oracle calls for any GS valuation. This is since, in contrast with general submodular functions, GS functions (and the difference of a GS function and an additive function) can be efficiently maximized with a simple greedy algorithm. Rado valuations are a subclass of GS valuations and thus their concave closure can be evaluated efficiently. Moreover, for Rado valuations the concave closure/extension is captured by an explicit a linear program. This representation of the concave extension is at the core of the arguments in Section 4.4, where we argue about the existence of a sparse optimal solution of a particular convex program.

**Theorem 4.1.1.** *Consider a Rado valuation $v : 2^{\mathcal{G}} \to \mathbb{R}$ given by a bipartite graph $(\mathcal{G}, V; E)$ with costs on the edges $c : E \to \mathbb{R}$, and a matroid $\mathcal{M} = (V, \mathcal{I})$ with a rank function $r = r_{\mathcal{M}}$ as in Definition 1.2.2. For $x \in [0,1]^{\mathcal{G}}$, let us define*

$$
\begin{aligned}
\nu(x) := \quad \max \quad & \sum_{(j,k)\in E} c_{jk} z_{jk} \\
\text{s.t.:} \quad & \sum_{k\in V} z_{jk} \le x_j \qquad \forall j \in \mathcal{G} \\
& \sum_{j\in\mathcal{G}, k\in T} z_{jk} \le r(T) \qquad \forall T \subseteq V \\
& z \ge 0 \,.
\end{aligned}
\tag{4.3}
$$

*Then, $\nu = \bar{v}$ is the concave extension of $v$, and $\bar{v}$ is monotone and subadditive*[1].

[1] Extending notions from discrete valuations, a function $f : \mathbb{R}_+^{\mathcal{G}} \to \mathbb{R}_+$ is *monotone* if $f(x) \le f(y)$ for $x \le y$, $x, y \in \mathbb{R}_+^{\mathcal{G}}$, and *subadditive* if $f(x+y) \le f(x) + f(y)$ for any $x, y \in [0,1]^{\mathcal{G}}$ such that $x + y \in [0,1]^{\mathcal{G}}$.

*Proof.* Monotonicity is immediate and concavity is implied once we prove $\nu = \bar{v}$. For subadditivity, if $z$ is the optimal solution in the program defining $\nu(x+y)$ for some $x, y \in [0,1]^{\mathcal{G}}$, then we can easily decompose $z = z' + z''$ such that $z'$ is feasible to the program defining $\nu(x)$ and $z''$ if feasible for $y$. Thus, $\nu(x+y) \le \nu(x) + \nu(y)$ follows.

It is left to show that $\nu = \bar{v}$. The value $\overline{v}(x)$ for $x \in [0,1]^m$ is defined by a linear program (4.2). We will use the dual LP:

$$\begin{aligned} \bar{v}(x) = \min \quad & p^\top x + \alpha \\ \text{s.t.:} \quad & p(S) + \alpha \ge v(S) \quad \forall S \subseteq \mathcal{G} \\ & (p, \alpha) \in \mathbb{R}^{m+1} \end{aligned} \qquad \begin{aligned} \max \quad & \sum_{S \subseteq G} \lambda_S v(S) \\ \text{s.t.:} \quad & \sum_{S \subseteq G} \lambda_S \chi_S = x \\ & \sum_{S \subseteq G} \lambda_S = 1 \\ & \lambda \ge 0 \end{aligned} \tag{4.4}$$

We let $\mathcal{M}(x)$ denote the set of feasible solutions of (4.3). Fix any $x \in \mathbb{R}^m$. We first show that $\overline{v}(x) \le \nu(x)$.

Consider an optimal solution $\lambda$ for the dual LP in (4.4) such that $\overline{v}(x) = \sum_{S \subseteq \mathcal{G}} \lambda_S v(S)$. For every $S \subseteq \mathcal{G}$, we have an integral allocation $M_S$ of the goods in $\mathcal{M}(\chi_S)$ that is optimal in the linear program (4.3) defining $\nu(\chi_S) = v(S)$. It is easy to see that $\sum_{S \subseteq \mathcal{G}} \lambda_S M_S \in \mathcal{M}(x)$. Thus, $\overline{v}(x) \le \nu(x)$.

For the other direction $\overline{v}(x) \ge \nu(x)$, let $z$ be the optimal solution defining $\nu(x)$ in (4.3). By the integrality of the bipartite matching polytope, we can write the fractional matching $z$ as a convex combinations of integral allocations $M_S$ for $S \subseteq \mathcal{G}$, i.e., $z = \sum_{S \subseteq \mathcal{G}} \lambda_S M_S$ for some $\lambda \ge 0$ with $\sum \lambda_S = 1$. The dual of (4.3) is

$$\begin{aligned} \min \quad & \sum_{j \in \mathcal{G}} x_j \pi_j + \sum_{T \subseteq V} \rho_T \\ \text{s.t.:} \quad & \pi_j + \sum_{T : k \in T} \rho_T \ge c_{jk} \qquad \forall j \in \mathcal{G}, \forall T \subseteq V \\ & \pi \in \mathbb{R}_+^{\mathcal{G}}, \quad \rho \in \mathbb{R}_+^{2^V}. \end{aligned}$$

Consider an optimal dual solution $(\pi, \rho)$. By complementarity, $\pi_i + \sum_{S:k \in T} \rho_T = c_{jk}$ for every $(j,k) \in \text{supp}(z)$; if $\rho_T > 0$ for $T \subseteq V$ then $z(\partial(T)) = r(T)$, and if $\pi_j > 0$ for $j \in \mathcal{G}$ then $z(\partial(j)) = x_j$.

Since $z = \sum_S \lambda_S M_S$, we have $M_S \subseteq \text{supp}(z)$, and $\partial_{M_S}(S) = r(S)$ whenever $z(\partial(S)) = r(S)$. Further, $z(\partial(j)) = x_j$ implies that every matching $M_S$ with $j \in S$ covers $j$. We see that $\chi_{M_S}$ and $(\pi, \rho)$ satisfy complementary slackness in (4.3) for every set $S$ with $\lambda_S > 0$.

Thus, $c(M_S) = \nu(\chi_S)$, and $\nu(\chi_S) = v(S)$. We can thus conclude that

$$\nu(x) = \sum_{S \subseteq \mathcal{G}} \lambda_S c(M_S) = \sum_{S \subseteq \mathcal{G}} \lambda_S v(S) \leq \bar{v}(x)\,,$$

completing the first part of the proof. $\square$

In the light of this theorem, in the rest of this chapter we will denote by $v : [0,1]^{\mathcal{G}} \to \mathbb{R}$ the continuous extension of Rado valuation $v$ defined in (4.3). Whereas our overall result requires the continuous extension of *Rado valuations*, much weaker assumptions suffice for most parts of the argument, as formulated next.

**Assumption 4.** *For every agent $i \in \mathcal{A}$ the continuous extension of the valuation function $v_i : [0,1]^{\mathcal{G}} \to \mathbb{R}_+$ is monotone, concave, and subadditive.*

### 4.1.2 Simple upper bounds

We will often use the following simple bounds.

**Lemma 4.1.2.** *Let $n, c \in \mathbb{N}$, $S \subseteq [n]$, and $1 \leq w_1, \ldots, w_n \leq \gamma - 1$. For $i \in S$ let $k_i \in \mathbb{R}_+$ such that $\sum_{i \in S} k_i \leq c \cdot n$. Then*

$$\left(\prod_{i \in S} k_i^{w_i}\right)^{1/\sum_{i=1}^n w_i} \leq c \cdot \gamma\,.$$

*Proof.* By the (weighted) arithmetic-geometric we have:

$$\begin{aligned}\left(\prod_{i \in S} k_i^{w_i}\right)^{1/\sum_{i=1}^n w_i} &= \prod_{i \in S} k_i^{\frac{w_i}{\sum_{i=1}^n w_i}} \cdot \prod_{i \in [n]\setminus S} 1^{\frac{w_i}{\sum_{i=1}^n w_i}} \\ &\leq \sum_{i \in S} \frac{w_i k_i}{\sum_{i=1}^n w_i} + \sum_{i \in [n]\setminus S} \frac{w_i}{\sum_{i=1}^n w_i} \leq (\gamma - 1)\frac{\sum_{i \in S} k_i}{\sum_{i=1}^n w_i} + 1 \leq c \cdot \gamma\,. \end{aligned}$$ $\square$

**Lemma 4.1.3.** *Let $n, c \in \mathbb{N}$, $S \subseteq [n]$. For $i \in S$ let $k_i \in \mathbb{R}_+$ such that $\sum_{i \in S} k_i \leq c \cdot n$. Then*

$$\left(\prod_{i \in S} k_i\right)^{1/n} \leq c \cdot e^{1/e}\,.$$

*Proof.* We present the proof for $c = 1$, the general cases easily reduces to $c = 1$ by scaling. Without loss of generality, we assume that $k_i \geq 1$ for $i \in S$. For fixed size of $S$ ($k = |S|$), the product $\prod_{i \in S} k_i$ is maximized when all $k_i$ are the same. Hence, $\left(\prod_{i \in S} k_i\right)^{1/n} \leq \left(\frac{n}{k}\right)^{k/n}$. Let $\xi = \frac{n}{k}$ then $\left(\frac{n}{k}\right)^{k/n} = \xi^{1/\xi}$. By the first order conditions, the value $\xi^{1/\xi}$ achieves the maximum for $\xi = e$. Hence, $\left(\prod_{i \in S} k_i\right)^{1/n} \leq e^{1/e}$. $\square$

We show that the bound in Lemma 4.1.2 can be slightly improved using the similar approach as in the proof of Lemma 4.1.3 Throughout the section the base of the logarithm is $e$. We recall that the Lamberth function $\mathcal{W}$ is the inverse of $t \mapsto t \ln t$ for $t \in \mathbb{R}_+$. For $x > e$ it holds $\mathcal{W}(x) < \log x$; and for $x > 41.19$ it holds $\mathcal{W}(x) > \log x - \log(\log x)$, see [66]. Let $\psi(x) = \left(\frac{x-2}{\mathcal{W}(\frac{x-2}{e})}\right)^{x/\left(x-2+\frac{x-2}{\mathcal{W}(\frac{x-2}{e})}\right)}$ (for $x > 2$). Then, by the above bound we get $\psi(x) \le \max\left\{\bar{c}, \frac{x-2}{\log(\frac{x-2}{e})-\log\log(\frac{x-2}{e})}\right\}$ for some constant $\bar{c}$ that depends on $41.19$. Now, we can prove our lemma.

**Lemma 4.1.4.** *Let $n \in \mathbb{N}$, $S \subseteq [n]$, and $1 \le w_1, \ldots, w_n < \gamma - 1$. For $i \in S$ let $k_i \in \mathbb{R}_+$ such that $\sum_{i \in S} k_i \le c \cdot n$. Assuming $c$ is a constant we have*

$$\left(\prod_{i\in S} k_i^{w_i}\right)^{1/\sum_{i=1}^n w_i} \le c \cdot \psi(\gamma) = O\left(\frac{\gamma}{\log(\gamma)}\right).$$

*Proof.* We present the proof for $c = 1$, the general cases easily reduces to $c = 1$ by scaling. Since $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$ we have $e^{\mathcal{W}(\frac{x-2}{e})+1} = e \cdot \frac{x-2}{e} \cdot \frac{1}{\mathcal{W}(\frac{x-2}{e})} = \frac{x-2}{\mathcal{W}(\frac{x-2}{e})}$. Hence,

$$\left(e^{\mathcal{W}(\frac{x-2}{e})+1}\right)^{x/\left(x-2+e^{\mathcal{W}(\frac{x-2}{e})+1}\right)} = \left(\frac{x-1}{\mathcal{W}(\frac{x-2}{e})}\right)^{x/\left(x-2+\frac{x-2}{\mathcal{W}(\frac{x-2}{e})}\right)},$$

for $x > 2$. It suffices to prove that

$$\left(\prod_{i\in S} k_i^{w_i}\right)^{1/\sum_{i=1}^n w_i} \le \left(e^{\mathcal{W}(\frac{\gamma-2}{e})+1}\right)^{\gamma/\left(\gamma-2+e^{\mathcal{W}(\frac{\gamma-2}{e})+1}\right)}.$$

Without loss of generality we can assume that $k_i \ge 1$. Then the worst case is if $w_i = \gamma - 1$ for all $i \in S$ and $w_i = 1$ for $i \in [n] \setminus S$. For fixed size of $S$ ($k = |S|$), the product $\prod_{i\in S} k_i^{\gamma-1}$ is maximized when all $k_i$ are the same. Hence, $\left(\prod_{i\in S} k_i^{w_i}\right)^{1/\sum_{i=1}^n w_i}$ is upper-bounded by $\left(\frac{n}{k}\right)^{k(\gamma-1)/(k(\gamma-1)+n-k)}$. Let $\xi = \frac{n}{k}$ then $\left(\frac{n}{k}\right)^{k(\gamma-1)/(k(\gamma-1)+n-k)} = \xi^{(\gamma-1)/(\gamma-2+\xi)}$. By the first order conditions, the value $\xi^{(\gamma-1)/(\gamma-2+\xi)}$ achieves the maximum for $\xi = e^{\mathcal{W}(\frac{\gamma-2}{e})+1}$. Hence,

$$\left(\frac{n}{k}\right)^{k\gamma/(k\gamma+n-k)} \le \left(e^{\mathcal{W}(\frac{\gamma-2}{e})+1}\right)^{(\gamma-1)/\left(\gamma-2+e^{\mathcal{W}(\frac{\gamma-2}{e})+1}\right)}.$$

□

## 4.2 Overview of the approach

Let $v_i$ be the extensions of the valuation function and $w_i > 0$ be the weight for each $i \in \mathcal{A}$. Given a fractional allocation $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^{\mathcal{A}\times\mathcal{G}}$, we let

$$\mathrm{NSW}(x) := \left(\prod_{i\in\mathcal{A}} v_i(x_i)^{w_i}\right)^{1/\sum_i w_i} .$$

Then, the asymmetric Nash social welfare program is captured by the following integer program.

$$\max \mathrm{NSW}(x) \quad \text{s.t.} \sum_{i\in\mathcal{A}} x_{ij} \le 1 \; \forall j \in \mathcal{G}, x \in \{0,1\}^E . \tag{NSW-IP}$$

Let OPT denote the optimum value. The natural relaxation is (NSW-IP) is

$$\max \mathrm{NSW}(x) \quad \text{s.t.} \sum_{i\in\mathcal{A}} x_{ij} \le 1 \; \forall j \in \mathcal{G}, x \ge 0 . \tag{4.5}$$

The objective is log-concave assuming the $v_i$'s are concave functions. However, Cole and Gkatzelis [35, Lemma 3.1] showed that this relaxation has unbounded integrality gap already for additive valuations.

We propose a mixed integer programming relaxation instead of (4.5). Consider a set of items $\mathcal{H} \subseteq \mathcal{G}$. Our mixed relaxation requires the items in $\mathcal{H}$ to be allocated integrally and the rest can be allocated fractionally.

$$\begin{aligned} & \max \quad \mathrm{NSW}(x) \\ \text{s.t.:} \quad & \sum_{i\in\mathcal{A}} x_{ij} \le 1 \qquad \forall j \in \mathcal{G} \\ & x_{ij} \in \{0,1\} \quad \forall j \in \mathcal{H}, \forall i \in \mathcal{A} \\ & x \ge 0 . \end{aligned} \tag{Mixed relaxation}$$

This clearly gives a relaxation of (NSW-IP): $\mathrm{OPT}_\mathcal{H} \ge \mathrm{OPT}$ where $\mathrm{OPT}_\mathcal{H}$ is optimal value of (Mixed relaxation) for any set of items $\mathcal{H}$. Theorem 1.2.4 is shown by constructing an integer allocation $x \in \{0,1\}^{\mathcal{A}\times\mathcal{G}}$ and an item set $\mathcal{H}$ such that $\mathrm{NSW}(x) \ge \mathrm{OPT}_\mathcal{H} / (256\gamma^3)$. This is proved in five phases:

**Phase I** Find an appropriate item set $\mathcal{H}$.

**Phase II** Approximate (Mixed relaxation) by another integer program (Mixed+matching).

**Phase III** Find an approximate mixed integer solution to (Mixed+matching).

**Phase IV** Find a *sparse* approximate mixed integer solution to (Mixed+matching).

**Phase V** Round the mixed integer solution to an integer solution.

We note that phases are not necessarily algorithmic phases but also conceptional reductions of the problem. Regardless, we call them phases for the sake of presentation. We now give an overview of all the phases; most proofs are deferred to later sections.

### 4.2.1 Phase I: Finding the item set $\mathcal{H}$

We solve a maximum weight matching problem that achieves the highest Nash social welfare value under the restriction that each agent may only receive a single item. This can be achieved by assigning an edge weight $\omega_{ij} = w_i \log(v_{ij})$ for every $i \in \mathcal{A}$, $j \in \mathcal{G}$, and solving the maximum weight assignment problem in the complete bipartite graph between $\mathcal{A}$ and $\mathcal{G}$; we recall the notation $v_{ij} = v_i(\{j\})$. We let $\tau : \mathcal{A} \to \mathcal{G}$ denote the optimal matching represented as a mapping, i.e. $\tau(i)$ is the item matched to agent $i \in \mathcal{A}$. We define $\mathcal{H}$ as the set of items assigned by $\tau$, i.e., $\mathcal{H} := \tau(\mathcal{A})$. We will refer to this set $\mathcal{H}$ as the *set of most preferred items*.

Interestingly, in case of symmetric agents endowed with additive valuations the set $\mathcal{H}$ contains all items with price at least one in any spending restricted equilibrium as in [35]; see Section 4.6.

The existence of $\tau$ with finite weight proves that the instance is feasible, i.e., there is a way of allocating one item to each agent such that agent values the assigned item positively. On the other hand, if no finite weight matching exists, the optimum value to (NSW-IP) is 0. Henceforth, we assume without loss of generality that the optimal NSW is non-zero.

### 4.2.2 Phase II: Reduction to the mixed matching relaxation

We approximate (Mixed relaxation) by a second mixed integer program. We use variables $y \in \mathbb{R}_+^{\mathcal{A}\times(\mathcal{G}\setminus\mathcal{H})}$ representing the fractional allocations of the items in $\mathcal{G} \setminus \mathcal{H}$. Even though the valuation functions $v_i$ are defined on $\mathbb{R}_+^{\mathcal{G}}$, we use $v_i(y_i)$ to denote $v_i(x_i)$, where $x_i$ is obtained from $y_i$ by setting $x_{ij} = 0$ for $j \in \mathcal{H}$ and $x_{ij} = y_{ij}$ for $j \in \mathcal{G} \setminus \mathcal{H}$.

$$
\begin{aligned}
\max \quad & \left( \prod_{i\in\mathcal{A}} \left(v_i(y_i) + v_{i\sigma(i)}\right)^{w_i} \right)^{1/\sum_i w_i} & & \\
\text{s.t.:} \quad & \sum_{i\in\mathcal{A}} y_{ij} \le 1 & \forall j \in \mathcal{G}\setminus\mathcal{H} & \\
& y_{ij} \ge 0 & \forall j \in \mathcal{G}\setminus\mathcal{H}, \forall i \in \mathcal{A} & \\
& \sigma : \mathcal{A} \to \mathcal{H} \text{ is a matching.} & &
\end{aligned}
\tag{Mixed+matching}
$$

We will refer to this program as the *mixed matching relaxation*. The program (Mixed+matching) differs from (Mixed relaxation) in two respects. Firstly, the objective differs from $\mathrm{NSW}(x)$: for each agent, the value of each agent in (Mixed relaxation) is given by the Rado valuation while in (Mixed+matching) we evaluate the utility of each agent separately on $\mathcal{H}$ and $\mathcal{G} \setminus \mathcal{H}$ and take the sum of these two values. Secondly, and more importantly, we require that the items in $\mathcal{H}$ are allocated to the agents by a matching. Unlike (Mixed relaxation), this will not be a relaxation of (NSW-IP): the optimal integer solution may allocate multiple items in $\mathcal{H}$ to the same agent. We show that the effect of both these changes is limited.

Let $(y, \sigma)$ be a feasible solution to (Mixed+matching). We define $\overline{\mathrm{NSW}}(y, \sigma)$ as the objective function value in (Mixed+matching), and let $\overline{\mathrm{OPT}}_{\mathcal{H}}$ denote the optimum value. Let us define $\mathrm{NSW}(y, \sigma)$ as the Nash social welfare of the same allocation. Namely, $\mathrm{NSW}(y, \sigma) = \mathrm{NSW}(x)$, where $x_{ij} = y_{ij}$ if $j \in \mathcal{G} \setminus \mathcal{H}$, and for $j \in \mathcal{H}$ we have $x_{ij} = 1$ if $j = \sigma(i)$, and $x_{ij} = 0$ otherwise. The next lemma is an easy consequence of monotonicity and subadditivity.

**Lemma 4.2.1.** *For any feasible solution $(y, \sigma)$ to (Mixed+matching), we have*

$$\overline{\mathrm{NSW}}(y, \sigma) \geq \mathrm{NSW}(y, \sigma) \geq \frac{1}{2}\overline{\mathrm{NSW}}(y, \sigma) \,.$$

*Proof.* We have $\overline{\mathrm{NSW}}(y, \sigma) \geq \mathrm{NSW}(y, \sigma)$ by subadditivity. By monotonicity: $2\,\mathrm{NSW}(y, \sigma) \geq \mathrm{NSW}(y, \emptyset) + \mathrm{NSW}(0, \sigma) = \overline{\mathrm{NSW}}(y, \sigma)$. □

Using this lemma, as well as Lemma 4.1.2, we can relate the optimum values and approximate solutions of (Mixed relaxation) and (Mixed+matching).

**Theorem 4.2.2.** *Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. For the optimum values $\mathrm{OPT}_{\mathcal{H}}$ to (Mixed relaxation) and $\overline{\mathrm{OPT}}_{\mathcal{H}}$ to (Mixed+matching), we have*

$$\overline{\mathrm{OPT}}_{\mathcal{H}} \geq \frac{1}{\gamma}\,\mathrm{OPT}_{\mathcal{H}} \,.$$

*Let $(y, \sigma)$ be an $\alpha$-approximate optimal solution to (Mixed+matching), that is, $\overline{\mathrm{NSW}}(y, \sigma) \geq \frac{1}{\alpha}\overline{\mathrm{OPT}}_{\mathcal{H}}$. Then, $\mathrm{NSW}(y, \sigma) \geq \frac{1}{2\alpha\gamma}\,\mathrm{OPT}_{\mathcal{H}}$. If the valuation functions $v_i$ are additive, then the stronger bound $\mathrm{NSW}(y, \sigma) \geq \frac{1}{\alpha\gamma}\,\mathrm{OPT}_{\mathcal{H}}$ applies.*

*Proof.* We first show that $\overline{\mathrm{OPT}}_{\mathcal{H}} \geq \frac{1}{\gamma}\,\mathrm{OPT}_{\mathcal{H}}$. Let $x$ be an optimal solution to (Mixed relaxation). For each agent $i$, let $K_i$ be the set of items agent $i$ receives from $\mathcal{H}$ under $x$; and let $y$ be the restriction of $x$ on $\mathcal{G} \setminus \mathcal{H}$ defined as $y_{ij} = x_{ij}$ for $j \in \mathcal{G} \setminus \mathcal{H}$ and $y_{ij} = 0$ otherwise. Let $k_i := |K_i|$. Denote with $S$ the set of agents that receive at least one items from $\mathcal{H}$, i.e., $S = \{i \in \mathcal{A} : k_i \geq 1\}$. For each agent $i \in S$ let $\sigma(i) = \max_{j \in K_i}\{v_{ij}\}$, and define $\sigma(i) = \emptyset$ for $i \in \mathcal{A} \setminus S$. Then, $(y, \sigma)$ is a feasible solution of (Mixed+matching).

In other words, $(y, \sigma)$ is obtained from $x$ once each agent $i \in S$ discards all items from $K_i$ except the most valuable one. By monotonicity and subadditivity, for all $i \in S$, we have

$$v_i(x_i) \leq v_i(y) + \sum_{j \in K_i} v_{ij} \leq k_i \cdot (v_i(y) + v_{i\sigma(i)}) \, .$$

Therefore,

$$\frac{\mathrm{OPT}_{\mathcal{H}}}{\overline{\mathrm{OPT}}_{\mathcal{H}}} \leq \frac{\mathrm{NSW}(x)}{\overline{\mathrm{NSW}}(y,\sigma)} = \left( \prod_{i \in S} \frac{v_i(x_i)^{w_i}}{(v_i(y) + v_{i\sigma(i)})^{w_i}} \right)^{1/\sum_i w_i} \leq \left( \prod_{i \in S} k_i^{w_i} \right)^{1/\sum_i w_i} \, .$$

Moreover, $\sum_{i \in S} k_i \leq |\mathcal{H}| = |\mathcal{A}| = n$. Then, the bound follows by Lemma 4.1.2. The second part of the theorem follows by Lemma 4.2.1. □

### 4.2.3 Phase III: Approximating the mixed matching relaxation

Our next goal is to find a 2-approximation solution to (Mixed+matching); we do not know whether this problem is polynomial-time solvable. By Theorem 4.2.2, this yields a $(4\gamma)$-approximation to (Mixed relaxation).

Let us first remove all items in $\mathcal{H}$. Some agents may only value positively the items $\mathcal{H}$. We let $\mathcal{A}'$ the subset of agents who have positive values for the items $\mathcal{G} \setminus \mathcal{H}$, that is, $\mathcal{A}' := \{i \in \mathcal{A} : v_i(\mathcal{G} \setminus \mathcal{H}) > 0\}$. Consider the "naïve" relaxation (4.5) on the instance restricted to $\mathcal{A}'$ and $\mathcal{G} \setminus \mathcal{H}$, and taking the logarithm of the objective

$$\begin{aligned} \max \quad & \sum_{i \in \mathcal{A}'} w_i \log(v_i(y_i)) \\ \text{s.t.:} \quad & \sum_{i \in \mathcal{A}'} y_{ij} \leq 1 \qquad \forall j \in \mathcal{G} \setminus \mathcal{H} \\ & y \geq 0. \end{aligned} \tag{EG-NSW}$$

This is the classical Eisenberg–Gale convex program that computes an equilibrium in Fisher markets with divisible items for homogeneous concave valuation functions [45]. Given an optimal solution $y^* \in \mathbb{R}_+^{\mathcal{A}' \times (\mathcal{G} \setminus \mathcal{H})}$ of (EG-NSW) we can find an approximate solution to (Mixed+matching).

**Theorem 4.2.3.** *Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. Let $\pi^*$ be maximum weight assignment in the complete bipartite graph between $\mathcal{A}$ and $\mathcal{H}$, with edge weights $\omega_{ij} = w_i \log(v_i(y_i^*) + v_{ij})$ for $i \in \mathcal{A}$, $j \in \mathcal{H}$. Then, $\overline{\mathrm{NSW}}(y^*, \pi^*) \geq \frac{1}{2}\overline{\mathrm{OPT}}_{\mathcal{H}}$.*

Theorem 4.2.3 is an immediate consequence of the following lemma.

**Lemma 4.2.4.** *Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. Let $\alpha > 0$ and $y^*$ be an optimal and $y$ a feasible solution of* (EG-NSW) *such that $v_i(y_i) \geq \frac{1}{\alpha} v_i(y_i^*)$ for all $i \in \mathcal{A}'$. Let $\pi$ be maximum* *weight assignment in the bipartite graph with colour classes $\mathcal{A}$ and $\mathcal{H}$, and edge weights $\omega_{ij} = w_i \log(v_i(y_i) + v_{ij})$ for $i \in \mathcal{A}$, $j \in \mathcal{H}$. Then,*

$$\overline{\mathrm{NSW}}(y, \pi) \geq \frac{1}{2\alpha} \overline{\mathrm{OPT}}_{\mathcal{H}} .$$

Since valuations $v_i$ are concave, (EG-NSW) is a convex program. For any $\varepsilon > 0$, we can find an $(1-\varepsilon)$-approximate solution in polynomial-time, where the running time depends on $\log(1/\varepsilon)$. It turns out that approximation of the objective function might not be enough. In Lemma 4.2.4 we require an agent-wise approximate solution: each agent gets at least a constant fraction of her value in the optimum. It is not clear if finding such agent-wise approximation is possible in polynomial time for general concave valuations $v_i$, but as we will see in the next section we can find an exact optimal solution for Rado valuations.

The proof of Lemma 4.2.4 is deferred to Section 4.3. It does not depend on the choice of $\mathcal{H}$ but only requires $|\mathcal{H}| = |\mathcal{A}|$.

### 4.2.4 Phase IV: A sparse approximate solution for the mixed matching relaxation

In this section we exploit the properties of Rado valuations. Assuming the agents have Rado valuation functions, we can find an approximate solution of (Mixed+matching) with a strong sparsity property. Even though the approximation ratio is weaker than given in Theorem 4.2.3, sparsity will be essential for the rounding in **Phase V**.

**Theorem 4.2.5.** *Suppose the functions $v_i$ are Rado valuations. Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. We can find a feasible solution $(y, \pi)$ to (Mixed+matching) such that*

*(i)* $\overline{\mathrm{NSW}}(y, \pi) \geq \frac{1}{4}\overline{\mathrm{OPT}}_{\mathcal{H}}$,

*(ii)* $\mathrm{supp}(y) \leq 2|\mathcal{A}| + |\mathcal{L}^+|$ *where $\mathcal{L}^+ = \{j \in \mathcal{G} \setminus \mathcal{H} : \sum_{i \in \mathcal{A}'} y_{ij} > 0\}$, that is, $\mathcal{L}^+$ is the set of allocated items in $y$.*

*Moreover, for additive valuation functions, we can strengthen* (i) *to* $\mathrm{NSW}(y, \sigma) \geq \frac{1}{2}\,\mathrm{OPT}_{\mathcal{H}}$ *and* (ii) *to* $\mathrm{supp}(y) \leq |\mathcal{A}| + |\mathcal{L}^+|$.

Let us start with the special case of additive valuations. In this case, an exact solution $y^*$ to the Eisenberg–Gale convex program (EG-NSW) can be found in strongly polynomial time [103, 118].

**Theorem 4.2.6.** *Assuming the valuations $v_i$ are additive, we can find an optimal solution $y^*$ of* (EG-NSW) *in strongly polynomial time such that the support* $\mathrm{supp}(y^*)$ *is a forest.*

The claim on the support follows easily by showing that any cycles in $\mathrm{supp}(y^*)$ can be eliminated, see e.g., [35, 43, 103]. Consequently, $|\mathrm{supp}(y^*)| \le |\mathcal{A}'| + |\mathcal{L}^+| - 1$. Together with Lemma 4.2.4, this proves the statement in Theorem 4.2.5 for additive valuations.

For Rado valuations, we first prove that an optimal solution of (EG-NSW) can be found in polynomial time, see Section 4.4.1. We first show that this is a rational convex program, and use the variant of the ellipsoid method for rational polyhedron [64].

**Lemma 4.2.7.** *Suppose that for each agent $i \in \mathcal{A}$, $v_i$ is a Rado valuation given by a bipartite graph $(\mathcal{G}, V_i; E_i)$, integer costs $c_i : E_i \to \mathbb{Z}$ and a matroid $\mathcal{M}_i = (V_i, \mathcal{I}_i)$ as in Definition 1.2.2. Let $T = \max_{i\in\mathcal{A}} |V_i|$, and $C = \max_{i\in\mathcal{A}} \|c_i\|_\infty$. Let the weights $w_i > 0$ be rational numbers given as quotients of two integers at most $U$. Assume the matroids $\mathcal{M}_i$ are given by rank oracles. Then,* (EG-NSW) *has a rational solution with $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ bit-complexity, and such a solution can be found in $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ arithmetic operations and calls to the matroid rank oracles.*

Our next lemma shows that any feasible solution to (EG-NSW) can be sparsified by losing at most the half of the value for each agent, see Section 4.4.2. This is achieved in two steps, using the sparsity of basic feasible solutions to linear programs. Half of the valuation may be lost in the second step, where for the fractionally allocated items we aim to remove one of the fractional edges. The set to be deleted is identified by writing an auxiliary linear program.

**Lemma 4.2.8.** *Suppose the functions $v_i$ are Rado valuations, and let $\hat{y}$ be a feasible solution to* (EG-NSW). *Then, in polynomial time we can find a feasible solution $y$ such that*

*(i)* $v_i(y) \ge \frac{1}{2} v_i(\hat{y})$,

*(ii)* $|\mathrm{supp}(y)| \le 2|\mathcal{A}'| + |\mathcal{L}^+|$ *where* $\mathcal{L}^+ := \mathcal{L}^+(y) = \{j \in \mathcal{G} \setminus \mathcal{H} : \sum_{i\in\mathcal{A}'} y_{ij} > 0\}$.

By combining Lemmas 4.2.4, 4.2.7, 4.2.8, we obtain Theorem 4.2.5 for Rado valuations.

### 4.2.5 Phase V: Rounding the mixed integer solution

For this phase of the algorithm, we require a sparse approximate solution as in Theorem 4.2.5, and exploit the choice of $\mathcal{H}$ as the set of most preferred items in **Phase I**. We start with a mixed integer solution $(y, \pi)$ as in Theorem 4.2.5. By a *reduction* of $(y, \pi)$ we mean a mixed integer solution $(y^r, \pi)$ obtained as follows. For each $j \in \mathcal{L}^+$, we pick an arbitrary agent $\kappa(j) \in \mathcal{A}$ such that $y_{\kappa(j)j} > 0$. We set $y^r_{\kappa(j)j} = y_{\kappa(j)j}$, and set $y^r_{ij} = 0$ if $i \ne \kappa(j)$. By the bound on $\mathrm{supp}(y)$, this amounts to setting $\le 2|\mathcal{A}|$ values $y_{ij}$ to 0. The proof of the next lemma is given in Section 4.5.

**Lemma 4.2.9.** *Let $\mathcal{H}$ be the set of most preferred items, and let $(y,\pi)$ be a solution to (Mixed+matching) as in Theorem 4.2.5. Let $(y^r,\pi)$ be a reduction of $(y,\pi)$. Then in polynomial-time we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that*

$$\overline{\mathrm{NSW}}(y^r,\rho) \geq \frac{1}{32\gamma^2}\overline{\mathrm{NSW}}(y,\pi)\,.$$

*Further, if the valuations are linear, then we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that $\overline{\mathrm{NSW}}(y^r,\rho) \geq \frac{1}{8}\overline{\mathrm{NSW}}(y,\pi)$.*

Such a matching $\rho$ can be found by combining the matching $\pi$ in the solution $(y,\pi)$, and the initial matching $\tau$ from **Phase I** that delivers the highest NSW value such that every agent may receive only one item. We swap from $\pi$ to $\tau$ on certain alternating paths and cycles.

We are ready to prove the main results.

**Theorem 1.2.4.** *There exists a polynomial-time $256\gamma^3$-approximation algorithm for the asymmetric Nash social welfare problem with Rado valuations. For additive valuations, there exists a polynomial-time $16\gamma$-approximation algorithm.*

*Proof.* From Theorem 4.2.5 and Lemma 4.2.9, we can obtain a solution an $(128\gamma^2)$-approximate solution $(y^r,\rho)$ to (Mixed+matching) such that for each item $\mathcal{L}^+$ there is exactly one incident edge in $\mathrm{supp}(y^r)$. We can obtain a 0–1 valued solution $x$ to (NSW-IP) by assigning each item in $\mathcal{H}$ according to $\rho$ and each item $j \in \mathcal{L}^+$ to the unique agent $i$ with $y^r_{ij} > 0$. Clearly, $\mathrm{NSW}(x) \geq \mathrm{NSW}(y^r,\rho)$. We obtain $\mathrm{NSW}(x) \geq \mathrm{OPT}_{\mathcal{H}}/(256\gamma^3) \geq \mathrm{OPT}/(256\gamma^3)$ using Theorem 4.2.2. For additive valuations, we use the stronger bounds in the same results.

□

**Theorem 1.2.3.** *There exists a polynomial-time $256e^{3/e}{\approx}772$-approximation algorithm for the symmetric Nash social welfare problem under Rado valuations.*

*Proof.* The proof follows exactly as the proof of Theorem 1.2.4 once we replace $\gamma$ by $e^{1/e}$. Such a change is justified as in the symmetric case we can use Lemma 4.1.3 instead of the bound given by Lemma 4.1.2. □

### 4.2.6 Approximating NSW under submodular valuations

As we already mentioned, following our work, Li and Vondrák [89] gave a 380-approximation algorithm for the symmetric NSW problem under *arbitrary* monotone submodular valuations. This is obtained by strengthening and extending our approach. Among others, the key two new ingredients and techniques are needed in **Phase III** and **Phase IV** to deal with multilinear extension of submodular valuations. We briefly mention these below.

Submodular valuation functions do not have a known concave extension that can be evaluated efficiently. Nevertheless, submodular valuations can be extended via the multilinear extension that is not concave but is concave along any line. Moreover, the multilinear extension can be efficiently approximated. Even so, the first major issue is then how to solve the corresponding fractional program in **Phase III** as this program is not convex anymore. Li and Vondrák here employ an iterated version of a continuous greedy algorithm and show that the fractional program can be 2-approximated. The second major issue appears in **Phase IV**. Namely, now the extension does not have a LP formulation that can be used for sparsifying feasible fractional solutions. They overcome this issue by using a randomized rounding. While the randomized rounding appears simple, the analysis is involved.

## 4.3 Phase III: Approximating the mixed matching relaxation

**Phase III** presents a general way of obtaining a 2-approximation to (Mixed+matching). By Theorem 4.2.2, this gives a $(4\gamma)$-approximation to (Mixed relaxation), a mixed integer relaxation of the ANSW problem. Recall that (Mixed+matching) is the following mixed integer program

$$\begin{aligned}
\max \quad & \left(\prod_{i\in\mathcal{A}} \left(v_i(y_i) + v_{i\sigma(i)}\right)^{w_i}\right)^{1/\sum_i w_i} \\
\text{s.t.:} \quad & \sum_{i\in\mathcal{A}} y_{ij} \le 1 \qquad \forall j \in \mathcal{G}\setminus\mathcal{H} \\
& y_{ij} \ge 0 \qquad \forall j \in \mathcal{G}\setminus\mathcal{H}, \forall i \in \mathcal{A} \\
& \sigma : \mathcal{A} \to \mathcal{H} \text{ is a matching.}
\end{aligned} \tag{Mixed+matching}$$

In the above problem, we need to allocate items $\mathcal{G}$ to the agents in $\mathcal{A}$ in order to maximize an objective function that is an approximation of the NSW. Items in $\mathcal{G}\setminus\mathcal{H}$ can be allocated fractionally to the agents without any constraints. The items in $\mathcal{H}$ have to be allocated integrally via an assignment, thereby allocating exactly one item from $\mathcal{H}$ to each agent $\mathcal{A}$.

While the exact computational complexity of (Mixed+matching) remains unresolved, we show that we can 2-approximate it.

Denote $\mathcal{L} = \mathcal{G}\setminus\mathcal{H}$. Let $\mathcal{A}'$ be the subset of agents that have positive value for the items in $\mathcal{G}\setminus\mathcal{H}$, $\mathcal{A}' := \{i \in \mathcal{A} : v_i(\mathcal{G}\setminus\mathcal{H}) > 0\}$, as some agents may only have positive value for the items in $\mathcal{H}$. Restricting (Mixed+matching) to the items $\mathcal{L}$ and agents $\mathcal{A}'$ and taking the objective yields an instance of (EG-NSW):

$$\begin{aligned} \max \quad & \sum_{i\in\mathcal{A}'} w_i \log v_i(y_i) \\ \text{s.t.:} \quad & \sum_{i\in\mathcal{A}'} y_{ij} \le 1 \qquad \forall j \in \mathcal{L} \\ & y_{ij} \ge 0 \qquad \forall j \in \mathcal{L}, \forall i \in \mathcal{A}'. \end{aligned}$$

The above is a convex program whenever the valuations $v_i(.)$ are concave, and we can solve it to an arbitrary precision in polynomial time if we have access to a supergradient oracle to the objective function. On the other hand, suppose that the variables $y$ are fixed in (Mixed+matching). Under the fixed $y$, we can find an optimal assignment $\sigma$. Namely, an optimal assignment is exactly a maximum weight assignment in the bipartite graph $(\mathcal{A}, \mathcal{H}; E)$ where the weight of an edge $ij$ for $i \in \mathcal{A}$, $j \in \mathcal{H}$ is $\omega_{ij} := w_i \log(v_i(y_i) + v_{ij})$.

Informally, (Mixed+matching) is a combination of two tractable problems. We show that an optimal solution $y^*$ to the restriction of the problem to $\mathcal{L}$ and $\mathcal{A}'$, and an optimal assignment with respect to the fixed $y^*$ gives a 2-approximation for (Mixed+matching).

In Section 4.3.1 we discuss the restriction of the problem to $\mathcal{L}$ and $\mathcal{A}'$ and give a technical lemma. The main result of the section is presented in Section 4.3.2.

### 4.3.1 Properties of Eisenberg–Gale program

Let us now consider the Eisenberg–Gale program (EG-NSW). An optimal solution $y^*$ and the optimal Lagrange multipliers $p_j$ for $j \in \mathcal{L}$ can be interpreted as the so-called *Gale equilibrium* in the market with divisible items $\mathcal{L}$, agents $\mathcal{A}'$, and where agent $i$ has valuation $v_i$ and budget $w_i$. In particular, $y^*$ represent the allocations and $p_j$ for $j \in \mathcal{L}$, specify the prices in the market equilibrium, see e.g., [56, 101].[2]

Our technical lemma relates the combined difference in valuations of each agent in the optimal solution $y^*$ and any other allocation $y'$. The rest of Section 4.3.1 is devoted to its proof.

**Lemma 4.3.1.** *Let $y^*$ be an optimal solution to* (EG-NSW)*. Then for any feasible solution $y'$ and any $\mathcal{A}'' \subseteq \mathcal{A}'$ it holds*

$$\sum_{i\in\mathcal{A}''} w_i \frac{v_i(y'_i)}{v_i(y^*_i)} \le \sum_{i\in\mathcal{A}''} w_i + \sum_{i\in\mathcal{A}'} w_i \,.$$

We recall some definitions and the Karush–Kuhn–Tucker (KKT) optimality conditions in terms of subgradients; see [108, Chapter 2 and Theorem 3.27]. Given a *convex* function $f : \mathbb{R}^M \to \mathbb{R}$, we say that $g$ is a *subgradient* of $h$ at $y^* \in \mathbb{R}^M$ if $f(y) \ge f(y^*) + g^\top(y - y^*)$

[2]In case of homogeneous valuations this can be used to find a *Fisher* equilibrium, since Fisher and Gale equilibria coincide under homogeneous valuations [46, 101].

for all $y \in \mathbb{R}^M$. The set of all subgradients at a point $y^*$ is called *subdifferential* and denoted as $\partial f(y^*)$. If the function is differentiable then $\partial f(y^*) = \{\nabla f(y^*)\}$. Consider the convex program

$$\begin{aligned} \min \quad & f_0(y) \\ \text{s.t.:} \quad & f_j(y) \le 0 \qquad \forall j \in \mathcal{L} \\ & y \ge 0\,, \end{aligned}$$

where $f_j$ for $j \in \{0\} \cup \mathcal{L}$ is convex. Assume that the there exists a strict feasible point (Slater's condition). Then, $y^*$ is a an optimal solution with the Lagrange multipliers $p$, if and only if the following conditions hold

- $f_j(y^*) \le 0$, $p_j \ge 0$ for all $j \in \mathcal{L}$ (primal and dual feasibility),
- $0 \in \partial f_0(y^*) + \sum_{j \in \mathcal{L}} p_j \partial f_j(y^*) + \{\mu \in \mathbb{R}^M_- : \mu^\top y^* = 0\}$ (stationarity), and
- $p_j f_j(y^*) = 0$ (complementary slackness).

We say that $g$ is a *supergradient* of the concave function $f$ if $-g$ is a subgradient of $-f$. The following proposition guarantees the existence of supergradients.

**Proposition 4.3.2.** *The function $f : \mathbb{R}^M_+ \to \mathbb{R}$ is concave if and only if $\forall y^* \in \mathbb{R}^M_+$ it has a non-empty superdifferential at $y^*$. In other words, there is $g \in \mathbb{R}^M$ such that*

$$f(y) \le f(y^*) + g^\top (y - y^*)\,.$$

We can interpret the Lagrange multipliers in (EG-NSW) as prices; the next claim states that no agent spends more that her budget in a Gale–equilibrium.

**Claim 4.3.3.** *Let $y^*$ be an optimum and $p$ be the optimal Lagrange multipliers of* (EG-NSW). *For all $i \in \mathcal{A}'$ it holds $p^\top y_i^* \le w_i$.*

*Proof.* Let us apply the KKT conditions to the concave maximization program (EG-NSW). The stationary condition can be written such that for each agent $i \in \mathcal{A}'$

$$0 \in \partial\left(-w_i \log(v_i(y_i^*))\right) + p + \{\mu_i \in \mathbb{R}^{\mathcal{L}}_- : \mu_i^\top y_i^* = 0\}\,.$$

By the composition rules for subgradients we have

$$0 \in -\frac{w_i \partial v_i(y_i^*)}{v_i(y_i^*)} + p + \{\mu_i \in \mathbb{R}^{\mathcal{L}}_- : \mu_i^\top y_i^* = 0\}\,.$$

Therefore, there exists a supergradient $g_i \in \partial v_i(y_i^*)$ such that $w_i g_i^\top = v_i(y_i^*) \cdot (p^\top + \mu_i^\top)$ where $\mu_i \le 0$ and $\mu_i^\top y_i^* = 0$.

By definition of subgradient (supergradient) at $y_i^*$, we have that $g_i^\top y_i^* \le v_i(y_i^*)$ for all $i \in \mathcal{A}'$. It follows that $p^\top y_i^* \le w_i$ for all $i \in \mathcal{A}'$. $\square$

*Proof of Lemma 4.3.1.* By the KKT conditions, for each $i \in \mathcal{A}'$, we have a supergradient $g_i \in \partial v_i(y_i^*)$ such that $\frac{w_i g_i}{v_i(y_i^*)} \leq p$ holds. By complementarity slackness, if $p_j > 0$ then $\sum_{i \in \mathcal{A}'} y_{ij}^* = 1$. Let $\overline{y}_{ij} = \max\{y_{ij}^*, y_{ij}'\}$. Then we obtain:

$$v_i(y_i') \leq v_i(\overline{y}_i) \leq v_i(y_i^*) + g_i^\top (\overline{y}_i - y_i^*) \leq v_i(y_i^*) + \frac{v_i(y_i^*) p^\top}{w_i} \cdot (\overline{y}_i - y_i^*) \,.$$

The first inequality is by monotonicity, the second by the definition of the supergradient, and the third from the KKT conditions as noted above. After rearranging we obtain $\frac{w_i v_i(y_i')}{v_i(y_i^*)} \leq w_i + p^\top (\overline{y}_i - y_i^*)$. Summing the previous inequality for each agent $i \in \mathcal{A}''$ for a subset $\mathcal{A}'' \subseteq \mathcal{A}'$, and by definition of $\overline{y}_i$, we have

$$\sum_{i \in \mathcal{A}''} \frac{w_i v_i(y_i')}{v_i(y_i^*)} \leq \sum_{i \in \mathcal{A}''} w_i + \sum_{i \in \mathcal{A}''} p^\top (\overline{y}_i - y_i^*) \leq \sum_{i \in \mathcal{A}''} w_i + p^\top \mathbb{1} \,.$$

Since $p_j > 0$ implies $\sum_{i \in \mathcal{A}'} y_{ij}^* = 1$ we have that $p^\top \mathbb{1} = p^\top \sum_{i \in \mathcal{A}'} y_i^*$. Then, by Claim 4.3.3 we have

$$\sum_{i \in \mathcal{A}''} w_i \frac{v_i(y_i')}{v_i(y_i^*)} \leq \sum_{i \in \mathcal{A}''} w_i + \sum_{i \in \mathcal{A}'} w_i$$ □

### 4.3.2 The approximation guarantee for the mixed matching relaxation

**Lemma 4.2.4.** *Let $\mathcal{H} \subseteq \mathcal{G}$ with $|\mathcal{H}| = |\mathcal{A}|$. Let $\alpha > 0$ and $y^*$ be an optimal and $y$ a feasible solution of (EG-NSW) such that $v_i(y_i) \geq \frac{1}{\alpha} v_i(y_i^*)$ for all $i \in \mathcal{A}'$. Let $\pi$ be maximum weight assignment in the bipartite graph with colour classes $\mathcal{A}$ and $\mathcal{H}$, and edge weights $\omega_{ij} = w_i \log(v_i(y_i) + v_{ij})$ for $i \in \mathcal{A}$, $j \in \mathcal{H}$. Then,*

$$\overline{\mathrm{NSW}}(y, \pi) \geq \frac{1}{2\alpha} \overline{\mathrm{OPT}}_{\mathcal{H}} \,.$$

*Proof.* Let $\pi^*$ be a maximum weight matching in the bipartite graph with colour classes $\mathcal{A}$ and $\mathcal{H}$ and with edge weights $q_i^* = w_i \log(v_i(y^*) + v_{ij})$. Equivalently, $\pi^*$ is a matching maximizing

$$\left( \prod_{i \in \mathcal{A}'} \left( v_i(y_i^*) + v_{i\pi^*(i)} \right)^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

We have the bounds

$$\overline{\mathrm{NSW}}(y, \pi) \geq \overline{\mathrm{NSW}}(y, \pi^*) \geq \frac{1}{\alpha} \overline{\mathrm{NSW}}(y^*, \pi^*) \,. \tag{4.6}$$

The first inequality is by the definition of $\pi$ as the maximum weight matching. The second inequality follows from the assumption $v_i(y_i) \geq \frac{1}{\alpha} v_i(y_i^*)$ for each $i \in \mathcal{A}'$.

The rest of the proof is devoted to proving that $\overline{\mathrm{NSW}}(y^*, \pi^*) \geq \frac{1}{2} \overline{\mathrm{OPT}}_{\mathcal{H}}$; together with

(4.6), this implies the statement. Let us introduce some notation. For an agent $i \in \mathcal{A}$, let $Y_i^* = v_i(y_i^*)$ be the value agent $i$ gets from the optimal fractional bundle $y^*$. Then,

$$\overline{\mathrm{NSW}}(y^*, \pi^*) = \left( \prod_{i \in \mathcal{A}'} (Y_i^* + v_{i\pi^*(i)})^{w_i} \prod_{i \in A \setminus \mathcal{A}'} v_{i\pi^*(i)}^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

Let $(y', \varrho)$ be an optimal solution achieving $\overline{\mathrm{OPT}}_{\mathcal{H}}$. For an agent $i \in \mathcal{A}$ let $Y_i = v_i(y_i')$ be the value agent $i$ gets from the fractional allocation $y'$. Then $\overline{\mathrm{OPT}}_{\mathcal{H}} = \overline{\mathrm{NSW}}(y', \varrho) = \left(\prod_{i \in \mathcal{A}} (Y_i + v_{i\varrho(i)})^{w_i}\right)^{1/\sum_{i \in \mathcal{A}} w_i}$. By definition of the set $\mathcal{A}'$, the agents in $\mathcal{A} \setminus \mathcal{A}'$ do not value the items in $\mathcal{L}$. Thus, by monotonicity

$$\overline{\mathrm{NSW}}(y', \varrho) = \left( \prod_{i \in \mathcal{A}'} (Y_i + v_{i\varrho(i)})^{w_i} \prod_{i \in \mathcal{A} \setminus \mathcal{A}'} v_{i\varrho(i)}^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

By the choice of $\pi^*$, we have

$$\overline{\mathrm{NSW}}(y^*, \pi^*) \geq \overline{\mathrm{NSW}}(y^*, \varrho) = \left( \prod_{i \in \mathcal{A}'} (Y_i^* + v_{i\varrho(i)})^{w_i} \prod_{i \in \mathcal{A} \setminus \mathcal{A}'} v_{i\varrho(i)}^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

Combining the last two we have: $\dfrac{\overline{\mathrm{NSW}}(y', \varrho)}{\overline{\mathrm{NSW}}(y^*, \pi^*)} \leq \left( \prod_{i \in \mathcal{A}'} \left( \dfrac{Y_i + v_{i\varrho(i)}}{Y_i^* + v_{i\varrho(i)}} \right)^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i}$ .

Let $\mathcal{A}'' = \{i \in \mathcal{A}' : Y_i > Y_i^*\}$ be the set of agents that get more value from $y'$ than $y^*$. Then, for $i \in \mathcal{A}' \setminus \mathcal{A}''$ the fraction $\dfrac{Y_i + v_{i\varrho(i)}}{Y_i^* + v_{i\varrho(i)}}$ is trivially bounded by 1. On the other hand, for $i \in \mathcal{A}''$ we have $\dfrac{Y_i + v_{i\varrho(i)}}{Y_i^* + v_{i\varrho(i)}} \leq \dfrac{Y_i}{Y_i^*}$. Since $\overline{\mathrm{OPT}}_{\mathcal{H}} = \overline{\mathrm{NSW}}(y', \varrho)$ it follows

$$\frac{\overline{\mathrm{OPT}}_{\mathcal{H}}}{\overline{\mathrm{NSW}}(y^*, \pi^*)} \leq \left( \prod_{i \in \mathcal{A}'} \left( \frac{Y_i + v_{i\varrho(i)}}{Y_i^* + v_{i\varrho(i)}} \right)^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} \leq \left( \prod_{i \in \mathcal{A}''} \left( \frac{Y_i}{Y_i^*} \right)^{w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

We claim that the last expression is bounded by 2. By Lemma 4.3.1 we have $\sum_{i \in \mathcal{A}''} w_i \frac{Y_i}{Y_i^*} \leq \sum_{i \in \mathcal{A}''} w_i + \sum_{i \in \mathcal{A}'} w_i$. Then by the inequality between weighted arithmetic and geometric mean we have

$$\prod_{i \in \mathcal{A}''} \left( \frac{Y_i}{Y_i^*} \right)^{w_i / \sum_{i \in \mathcal{A}} w_i} \leq \frac{\sum_{i \in \mathcal{A}''} w_i \frac{Y_i}{Y_i^*} + \sum_{i \in \mathcal{A} \setminus \mathcal{A}''} 1}{\sum_{i \in \mathcal{A}} w_i} \leq \frac{\sum_{i \in \mathcal{A}''} w_i + \sum_{i \in \mathcal{A}'} w_i + |\mathcal{A} \setminus \mathcal{A}''|}{\sum_{i \in \mathcal{A}} w_i} \leq 2 .$$

The lemma follows. $\square$

## 4.4 Phase IV: Obtaining a sparse approximate solution

Recall that a continuous Rado valuation is defined as an optimum of the LP (4.3). For the valuation $v_i$ of agent $i \in \mathcal{A}$, this is defined by a bipartite graph $(\mathcal{G}, V_i; E_i)$ with costs on the edges $c_i : E_i \to \mathbb{R}_+$, and a matroid $\mathcal{M}_i = (V_i, \mathcal{I}_i)$ with a rank function $r_i = r_{\mathcal{M}_i}$.

The program (EG-NSW) for $\mathcal{A}'$ and $\mathcal{L} = \mathcal{G} \setminus \mathcal{H}$ can be thus written as follows.

$$
\begin{aligned}
\max \quad & \sum_{i \in \mathcal{A}'} w_i \log \left( \sum_{j \in \mathcal{L}} \sum_{k \in V_i} c_{ijk} z_{ijk} \right) & \\
\text{s.t.:} \quad & \sum_{i \in \mathcal{A}'} y_{ij} \le 1 & \forall j \in \mathcal{L} \\
& \sum_{k \in V_i} z_{ijk} \le y_{ij} & \forall i \in \mathcal{A}', \forall j \in \mathcal{L} \\
& \sum_{j \in \mathcal{L}} \sum_{k \in S} z_{ijk} \le r_i(S) & \forall i \in \mathcal{A}', \forall S \subseteq V_i \\
& y \ge 0\,, \quad z \ge 0\,. &
\end{aligned}
$$

Without loss of generality we can assume that the second set of constraints always holds with equality, i.e., $y_{ij} = \sum_{k \in V_i} z_{ijk}$ for $j \in \mathcal{L}$ and $i \in \mathcal{A}'$. By eliminating the variables $y$, the program (EG-NSW) becomes:

$$
\begin{aligned}
\max \quad & \sum_{i \in \mathcal{A}'} w_i \log \left( \sum_{j \in \mathcal{L}} \sum_{k \in V_i} c_{ijk} z_{ijk} \right) & \\
\text{s.t.:} \quad & \sum_{i \in \mathcal{A}'} \sum_{k \in V_i} z_{ijk} \le 1 & \forall j \in \mathcal{L} \\
& \sum_{j \in \mathcal{L}} \sum_{k \in S} z_{ijk} \le r_i(S) & \forall i \in \mathcal{A}', \forall S \subseteq V_i \\
& z \ge 0\,, &
\end{aligned}
\tag{EG-Rado}
$$

Using this formulation, we first show that the Eisenberg–Gale type convex program (EG-NSW) can be solved exactly in polynomial time for Rado valuations (Section 4.4.1). We then transform the optimal solution to a sparse approximate solution (Section 4.4.2).

### 4.4.1 Solving the Eisenberg-Gale relaxation

In this section, we prove the following lemma.

**Lemma 4.2.7.** *Suppose that for each agent $i \in \mathcal{A}$, $v_i$ is a Rado valuation given by a bipartite graph $(\mathcal{G}, V_i; E_i)$, integer costs $c_i : E_i \to \mathbb{Z}$ and a matroid $\mathcal{M}_i = (V_i, \mathcal{I}_i)$ as in Definition 1.2.2. Let*

$T = \max_{i \in \mathcal{A}} |V_i|$, *and* $C = \max_{i \in \mathcal{A}} \|c_i\|_\infty$. *Let the weights* $w_i > 0$ *be rational numbers given as quotients of two integers at most* $U$. *Assume the matroids* $\mathcal{M}_i$ *are given by rank oracles. Then,* (EG-NSW) *has a rational solution with* $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ *bit-complexity, and such a solution can be found in* $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ *arithmetic operations and calls to the matroid rank oracles.*

As noted above, (EG-NSW) with Rado valuations for the set of agents $\mathcal{A}'$ and set of goods $\mathcal{L}$ is equivalent to (EG-Rado). Throughout, we assume this program is feasible, i.e. it has a solution with finite objective value. This is a mild condition only requiring the existence of at least one edge $(j,k) \in E_i$ with $c_{ijk} > 0$ and $r_i(\{k\}) = 1$ for every $i \in \mathcal{A}'$.

In general, one can only expect to solve convex programs approximately: no rational solution may even exist. Vazirani [117] defines rational convex programs where a finite optimum exists with bounded bit-complexity in the input size, where the input is described by a finite set of parameters. This model is not directly applicable for our program (EG-Rado) as it is described with an exponential number of constraints. The bound $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ does not take into account the matroidal constraints; it is polynomial in the amount of information needed to describe the objective function.[3]

We first show that the set of optimal solutions is a polytope where the vertices have polynomially bounded bit-complexity.

**Lemma 4.4.1.** *For an NSW problem instance with Rado valuations as in Lemma 4.2.7, the set of optimal solutions forms a polytope. The bit-complexity of each vertex of this polytope is bounded as* $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$.

To prove the above lemma we use the KKT conditions for (EG-Rado). Let $p_j$'s and $\alpha_i(S)$'s denote the Lagrange multipliers corresponding to the first and second sets of the constraints, respectively. It holds:

$(i)$ $\forall j \in \mathcal{L} : p_j \geq 0.$

$(ii)$ $\forall i \in \mathcal{A}', \forall S \subseteq V_i : \alpha_i(S) \geq 0.$

$(iii)$ $\forall j \in \mathcal{L} : p_j > 0 \implies \sum_{i \in \mathcal{A}', k \in V_i} z_{ijk} = 1.$

$(iv)$ $\forall i \in \mathcal{A}', \forall S \subseteq V_i : \alpha_i(S) > 0 \implies \sum_{j \in \mathcal{L}, k \in S} z_{ijk} = r_i(S).$

$(v)$ $\forall i \in \mathcal{A}', \forall j \in \mathcal{L}, \forall k' \in V_i : \dfrac{c_{ijk}}{p_j + \sum_{S : k \in S} \alpha_i(S)} \leq \dfrac{\sum_{j \in \mathcal{L}, k' \in V_i} c_{ijk'} z_{ijk'}}{w_i}.$

[3]We note that for exponential size linear programs, a standard way to bound the encoding size is giving bounds on facet/vertex-complexity, defined later in this section. The program (EG-Rado) maximizes a concave function over a polytope that has facet complexity $O(|\mathcal{A}|T)$.

$$(vi)\ \forall i \in \mathcal{A}', \forall j \in \mathcal{L} : z_{ijk} > 0 \implies \frac{c_{ijk}}{p_j + \sum_{S:k\in S} \alpha_i(S)} = \frac{\sum_{j\in\mathcal{L},k\in V_i} c_{ijk'} z_{ijk'}}{w_i}.$$

In $(v)$ and $(vi)$, we have divided the conditions by $p_j + \sum_{S:k\in S} \alpha_i(S)$ and multiplied by $\sum_{j\in\mathcal{L},k'\in V_i} c_{ijk'} z_{ijk'}$. By the feasibility assumption, both these must be positive.

We say that $(p, \alpha)$ are *optimal Lagrange multipliers* if they satisfy $(i)$–$(vi)$ together with any optimal solution $z$ to (EG-Rado).

**Claim 4.4.2.** *There exists an optimal solution $z$ with optimal Lagrange multipliers $(p, \alpha)$ with the following property: for every agent $i \in \mathcal{A}'$, the support of the vector $\alpha_i$ is a chain of sets $S_1^{(i)} \subset \cdots \subset S_{h_i}^{(i)} \subseteq V_i$ for some $h_i \in \mathbb{N}$.*

*Proof of Claim.* We use a standard uncrossing argument. Let $z$ be an optimal solution to (EG-Rado). Let us consider the set of optimal Lagrange multipliers $(p, \alpha)$. For a fixed $z$, the set of vectors $(p, \alpha)$ satisfying the constraints $(i)$–$(vi)$ forms a polytope, since each constraint can be equivalently written as a linear constraint, and $(iii)$, $(iv)$, and $(vi)$ imply boundedness. Thus, there exists a solution $(p, \alpha)$ that maximizes the objective

$$\varphi(p, \alpha) := \sum_{i\in\mathcal{A}'} \sum_{S\subseteq V_i} |S|^2 \alpha_i(S)\,.$$

We claim that such a solution satisfies the conditions. This follows by showing that for each $i \in \mathcal{A}'$, if $\alpha_i(X), \alpha_i(Y) > 0$ then either $X \subseteq Y$ or $Y \subseteq X$.

For a contradiction, assume $X \setminus Y, Y \setminus X \neq \emptyset$, and let $\varepsilon := \min\{\alpha_i(X), \alpha_i(Y)\} > 0$. Let us define $\alpha'$ as follows:

- $\alpha'_i(X \cup Y) = \alpha_i(X \cup Y) + \varepsilon$;
- $\alpha'(X) = \alpha(X) - \varepsilon$ and $\alpha'(Y) = \alpha(Y) - \varepsilon$;
- if $X \cap Y \neq \emptyset$, then $\alpha'_i(X \cap Y) = \alpha_i(X \cap Y) + \varepsilon$;
- if $S \notin \{X, Y, X \cup Y, X \cap Y\}$ then $\alpha'_i(S) = \alpha_i(S)$; and
- if $j \neq i$ then $\alpha'_j(S) = \alpha_j(S)$ for all $S$.

We claim that $(p, \alpha')$ are also optimal Lagrange multipliers. This gives a contradiction, since $\varphi(p, \alpha') > \varphi(p, \alpha)$. Constraints $(i)$–$(iii)$ are immediate. Constraints $(v)$ and $(vi)$ follow since $\sum_{S:k\in S} \alpha'_i(S) = \sum_{S:k\in S} \alpha_i(S)$ holds for all $i \in \mathcal{A}'$ and all $k \in V_i$. Finally, $(iv)$ follows by observing that for any $i \in \mathcal{A}'$ and any $j \in \mathcal{L}$,

$$\begin{aligned}
&\sum_{j\in\mathcal{L},k\in X} z_{ijk} + \sum_{j\in\mathcal{L},k\in Y} z_{ijk} = r_i(X) + r_i(Y) \geq r_i(X \cup Y) + r_i(X \cap Y) \\
&\geq \sum_{j\in\mathcal{L},k\in X\cap Y} z_{ijk} + \sum_{j\in\mathcal{L},k\in X\cup Y} z_{ijk} = \sum_{j\in\mathcal{L},k\in X} z_{ijk} + \sum_{j\in\mathcal{L},k\in Y} z_{ijk}\,,
\end{aligned}$$

using the submodularity of $r_i$. We must have equality throughout, implying $(iv)$ for $S = X \cup Y$ and $S = X \cap Y$. ∎

*Proof of Lemma 4.4.1.* Let $z$ be any optimal solution to (EG-Rado) and let $(p, \alpha)$ be any optimal Lagrange multipliers as in Claim 4.4.2, with $\alpha_i$ supported on the chain $S_1^{(i)} \subset S_2^{(i)} \subset \ldots \subset S_{h_i}^{(i)}$.

Let $\mathcal{L}' \subseteq \mathcal{L}$ be the subset of goods with $p_j > 0$, and let $E_i' \subseteq E_i$ be the set of edges $(j,k)$ for which $c_{ijk}/(p_j + \sum_{S:k\in S}\alpha_i(S))$ is maximized. Clearly, $z_{ijk} > 0$ only if $(j,k) \in E_i'$.

We perform the following variable substitution:

$$q_j := \frac{1}{p_j} \quad \forall j \in \mathcal{L}, \qquad \text{and} \qquad Q_{jt}^{(i)} := \frac{1}{p_j + \sum_{b=t}^{h_i} \alpha_i\left(S_b^{(i)}\right)} \quad \forall i \in \mathcal{A}',\ \forall t \in [h_i]\,. \tag{4.7}$$

We show that, provided the supports $\mathcal{L}'$, $E_i'$, we can define a linear program in the variables $q_j$'s, $Q_{jt}^{(i)}$'s, and $z_{ijk}$ as follows. We include all feasibility constraints on $z_{ijk}$ from (EG-Rado) and the following additional constraints:

$$\begin{aligned}
\sum_{i\in\mathcal{A}',k\in V_i} z_{ijk} &= 1 && \forall j \in \mathcal{L}' \\
\sum_{j\in\mathcal{L},k\in S} z_{ijk} &= r_i(S) && \forall i \in A', \forall S \subseteq V_i \\
w_i c_{ijk} Q_{jt}^{(i)} &\le \sum_{j\in\mathcal{L},k'\in V_i} c_{ijk'} z_{ijk'} && \forall i \in \mathcal{A}', \forall (j,k) \in E_i,\ \text{and } t \text{ s.t. } k \in S_t^{(i)} \setminus S_{t-1}^{(i)} \\
w_i c_{ijk} Q_{jt}^{(i)} &= \sum_{j\in\mathcal{L},k'\in V_i} c_{ijk'} z_{ijk'} && \forall i \in \mathcal{A}', \forall (j,k) \in E_i',\ \text{and } t \text{ s.t. } k \in S_t^{(i)} \setminus S_{t-1}^{(i)} \\
Q_{jt}^{(i)} &\le Q_{j(t+1)}^{(i)} && \forall i \in \mathcal{A}', j \in \mathcal{L}', t \in [h_i - 1] \\
q_j &= 0 && \forall j \in \mathcal{L} \setminus \mathcal{L}' \\
z_{ijk} &= 0 && \forall i \in \mathcal{A}', (j,k) \in E_i \setminus E_i' \\
Q, q &\ge 0
\end{aligned}$$

Let $P \in \mathbb{R}^{(\sum_{i\in\mathcal{A}'}|E_i|)\times\mathcal{L}'\times(\sum_{j\in F'} h_i)}$ be the set of feasible solutions to this LP. According to $(i)$–$(vi)$, $(z, q, Q) \in P$, where $(q, Q)$ is obtained from $(p, \alpha)$ as in (4.7). Conversely, if $(z', q', Q') \in P$, then we can map $(q', Q')$ to a nonnegative $(p', \alpha')$ such that (4.7) holds and $(z', p', \alpha')$ satisfy $(i)$–$(vi)$.

Since all coefficients in the system are rational numbers from the input, and the feasible region $P$ is bounded, it follows that $P$ is a polytope where all basic feasible solutions are rational vectors with encoding size polynomially bounded in the input.

Let us fix $(q', Q')$ in a basic feasible solution, and let $P'' = \{z'' : (z'', q', Q') \in P\}$. Then, $z'' \in P''$ if and only if $z''$ is optimal with respect to (EG-Rado). Further, $P''$ is a polytope defined by linear constraints with polynomially bounded coefficients. Thus, the claim follows. □

**The Ellipsoid Method for Rational Polyhedra** We quickly recall some relevant concepts for the Ellipsoid Method from the book [64] by Grötschel, Lovász, and Schrijver. A *strong separation oracle* for the convex set $K \subseteq \mathbb{R}^n$ takes as input a vector $x \in \mathbb{R}^n$, and either returns the answer $x \in K$, or returns a vector $a \in \mathbb{R}^n$ such that $\langle a, x\rangle > \max\{\langle a, z\rangle : z \in K\}$.

Let us recall the definitions of facet and vertex complexity. We only include the definitions for polytopes, instead of general polyhedra.

**Definition 4.4.3** ([64, Definition (6.2.2)]). *Let $P \subseteq \mathbb{R}^n$ be a polytope.*

1. *We say that $P$ has* facet-complexity at most $\varphi$, *if $P$ can be defined by a system of linear inequalities with rational coefficients such that each inequality has encoding length at most $\varphi$. If $P = \mathbb{R}^n$, we require $\varphi \geq n + 1$. The triple $(P; n, \varphi)$ is called a* well-described polytope*.*

2. *We say that $P$ has* vertex-complexity at most $\nu$, *if $P$ is the convex hull of a finite set of rational vectors, all having encoding length at most $\nu$. $P = \emptyset$, then we require $\nu \geq n$.*

**Lemma 4.4.4** ([64, Lemma (6.2.4)]). *If $P$ has vertex-complexity at most $\nu$, then $P$ has facet-complexity at most $3n^2\nu$.*

**Theorem 4.4.5** ([64, Theorems (6.4.9), (6.5.7)]). *For a well-described polyhedron $(P; n, \varphi)$ given by a strong separation oracle, there exists oracle-polynomial time algorithm that either returns a vertex solution $x \in P$, or concludes that $P = \emptyset$. Given a linear objective function $\langle c, x\rangle$, if $P \neq \emptyset$ then there exists an oracle-polynomial time algorithm that finds an optimal vertex solution to $\max\ \langle c, x\rangle$ s.t. $x \in P$.*

An oracle-polynomial time algorithm means that the number of arithmetic operations and calls to the strong separation oracle is bounded as poly$(\varphi)$; note that $\varphi \geq n$.

*Proof of Lemma 4.2.7.* Let $P$ be the set of feasible solutions and $P^*$ the set of optimal solutions to (EG-Rado). We note that $P \neq \emptyset$ since $z = 0$ is a feasible solution. Further, $P^* \neq \emptyset$ since $P$ is bounded. Lemma 4.4.1 asserts that this is a nonempty polytope with vertex-complexity $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$; thus $(P^*, \sum_{i\in\mathcal{A}} |E_i|, \varphi)$ is a well-described polytope for some $\varphi \in \mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$ by Lemma 4.4.4.

We now describe the strong separation oracle to $P^*$. For a vector $z \in \mathbb{R}^{\times_{i\in\mathcal{A}} E_i}$, we first check whether $z \in P$. Checking the first set of $|\mathcal{A}|$ constraints is straightforward. The submodular constraints can be verified by solving $|\mathcal{A}|$ submodular function minimization problems. We either conclude $z \in P$, or obtain a separating hyperplane for $z$ and $P$ that is also a separating hyperplane for $z$ and $P^*$.

If $z \in P$, the we compute the gradient $\nabla f(z)$, where $f(z)$ denotes the objective function. We then solve the linear optimization problem $\max\langle \nabla f(z), x\rangle$ s.t. $x \in P$. $(P^*, \sum_{i\in\mathcal{A}} |E_i|, \sum_{i\in\mathcal{A}} |E_i| + \log T)$ is a well-described polytope since all coefficients are 0 and 1 and the left hand side values are at most $T$. Using the strong separation oracle for $P$ we just described, the second half of Theorem 4.4.5 shows that we can find an optimal solution $x^* \in P$ in time $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$.

If $\max\langle \nabla f(z), x^*\rangle = \max\langle \nabla f(z), z\rangle$, i.e., if $z$ itself is an optimal solution, then we conclude that $z \in P^*$. Otherwise, $\langle \nabla f(z), x\rangle > \langle \nabla f(z), z\rangle$ is a valid separating hyperplane.

Thus, by the first half of Theorem 4.4.5, we can find an optimal solution $x \in P^*$ in time $\mathrm{poly}(|\mathcal{A}|, |\mathcal{G}|, T, \log C, \log U)$.

This method requires the implementation of the ellipsoid method for linear optimization inside the separation oracle. We now show that this can be easily avoided by always using the hyperplane $\langle \nabla f(z), x\rangle > \langle \nabla f(z), z\rangle$, without solving the LP. If $z \in P \setminus P^*$, then this is always valid, but if $z \in P^*$, then this holds with equality instead of strict inequality.

Nevertheless, we can run the ellipsoid method using the gradients as separating directions (without solving the LP). This ultimately leads to concluding $P^* = \emptyset$, since the algorithm returns a separating hyperplane for every $z \in \mathbb{R}^{\times_{i\in\mathcal{A}} E_i}$. At this point, we consider the feasible solution $z \in P$ with the largest objective value $f(z)$ visited by the algorithm, and conclude that this solution must have been optimal. This is true since if no optimal solutions would have been visited, then every separating hyperplane we used would be a valid strong separator for $P^*$, and thus, we could not have reached the false conclusion $P^* = \emptyset$. □

**Remark 4.4.6.** *We note that a similar argument was used by Jain [73, Theorem 12], showing that whenever a convex set is given with a strong separation oracle and is guaranteed to contain a point of bit-complexity at most $\nu$, then a feasible solution can be found in polynomial time, using simultaneous Diophantine approximation. Our proof leverages the stronger property that the optimal solution set $P^*$ is a well-described polytope.*

### 4.4.2 Sparse solutions to Eisenberg-Gale relaxation

In this section we prove Lemma 4.2.8. Recall that the polytope $P^*$ is the set of optimal solutions to (EG-Rado) as in Lemma 4.4.1. In Lemma 4.4.7 and Corollary 4.4.8, we show that the solution of every vertex solution of $P^*$ is sparse. In Lemma 4.2.8 we further sparsify such a solution by losing at most half of the value for each agent. The arguments in both steps rely on bounding the number of non-zero variables in particular linear systems.

Consider an optimal solution $z$ for (EG-Rado) that is also a basic solution to $P^*$. According to Theorem 4.4.5, we can require that the optimal solution found in Lemma 4.2.7 is a basic solution. We define $v_i^* := \sum_{k\in V_i} c_{ijk} z_{ijk}$ as the optimum utility value attained by agent $i \in \mathcal{A}'$; by strict convexity of the objective, these values are the same for all optimal solutions.

**Lemma 4.4.7.** *Every optimal solution $z \in P^*$ satisfies $|\mathrm{supp}(z')| \le |\mathcal{A}'| + 2|\mathcal{L}^+(z')| - |R_1| - |R_2|$, where*

$$\mathcal{L}^+(z) = \left\{ j \in \mathcal{L} : \sum_{i \in \mathcal{A}'} \sum_{k \in V_i} z_{ijk} > 0 \right\},$$
$$R_1 = \{ j \in \mathcal{L} : \exists!\, i \in \mathcal{A}' \text{ such that } 0 < \sum_{k \in V_i} z_{ijk} < 1 \},$$
$$R_2 = \{ j \in \mathcal{L} : \exists!\, i \in \mathcal{A}' \text{ such that } z_{ijk} = 1 \text{ for some } k \in V_i \}.$$

The set $\mathcal{L}^+$ is the set of allocated items in $\mathcal{L}$ by $z$; $R_1$ is the set of items in $\mathcal{L}$ each of which is allocated to one agent only, but the item is not fully allocated; and $R_2$ is the set of items in $\mathcal{L}$ each of which is fully allocated to agent via single edge of the graph $(\mathcal{G}, V_i; E_i)$. Obviously, $R_1$ and $R_2$ are disjoint.

*Proof of Lemma 4.4.7.* The following LP gives a description of $P^*$. We note that this is a different description from the extended system in the proof of Lemma 4.4.1: here, we can make use of the optimal values $v_i^*$ and thus do not require the dual variables. Note that the notion of vertex solutions is independent of the describing system.

$$\begin{aligned}
\sum_{j \in \mathcal{L}, k \in V_i} c_{ijk} z_{ijk} &\ge \hat{v}_i^* && \forall i \in \mathcal{A}' \\
\sum_{i \in \mathcal{A}', k \in V_i} z_{ijk} &\le 1 && \forall j \in \mathcal{L} \\
\sum_{j \in \mathcal{L}, k \in S} z_{ijk} &\le r_i(S) && \forall i \in \mathcal{A}', \forall S \subseteq V_i \\
z &\ge 0\,.
\end{aligned}$$

In order to prove the bound on the support of a vertex (basic feasible) solution to $P^*$, we upper-bound the number of linearly independent *tight* constraints. Trivially, there are at most $|\mathcal{A}'|$ tight constraints of the first type. By definition of sets $\mathcal{L}^+$ and $R_1$ there are at most $|\mathcal{L}^+| - |R_1|$ tight constraints of the second type.

Let us bound the maximal number of tight submodular constraints. By Claim 4.4.2, for each agent $i \in \mathcal{A}'$, the maximal set of linearly independent tight submodular constraints forms a chain. Formally, for $i \in \mathcal{A}'$ there exist sets $S_1^i \subset S_2^i \subset \cdots \subset S_{h_i}^i \subseteq V_i$, such that the set of constraints $\{\sum_{j \in \mathcal{G}, k \in S_t^i} z_{ijk} \le r_i(S_t^i)\}_{t=1}^{h_i}$ generates all the tight submodular constraints for agent $i$. All together, there are at most $|\mathcal{A}'| + |\mathcal{L}^+| - |R_1| + \sum_{i \in \mathcal{A}'} h_i$ tight constraints.

Now, let us consider an element $j \in R_2$ and let $i$ be the agent such that $z_{ijk} = 1$ for some

$k \in V_i$. Since $r_i$ is rank function we have $z_{ijk} = 1 = r_i(\{k\})$. Let $S^i_b$ be the smallest set in the $i$-th chain containing $k$. Since $\{k\}$ is also tight we can assume that $k = S^i_b \setminus S^i_{b-1}$. Therefore, the tight inequalities corresponding to $S^i_b, S^i_{b-1}$ and $z_{ijk} \leq 1$ (or equivalently $\sum_{k \in V_i} z_{ijk} \leq 1$) are not linearly independent and we can drop the inequality corresponding to $z_{ijk} \leq 1$ from the minimal set of linearly independent tight inequalities. In other words, we do not have to count the inequality corresponding to $j$, for $j \in R_2$ and we can replace the term $|\mathcal{L}_+|$ by $|\mathcal{L}_+| - |R_2|$.

Further, by flow conservation we have $|\mathcal{L}^+| \geq \sum_{i \in \mathcal{A}', j \in \mathcal{L}, k \in V_i} z_{ijk} \geq \sum_{i \in \mathcal{A}'} r_i(S^i_{h_i}) \geq \sum_{i \in \mathcal{A}'} h_i$. Thus,

$$|\mathrm{supp}(z)| \leq |\mathcal{A}'| + 2|\mathcal{L}^+| - |R_1| - |R_2| .$$

□

**Corollary 4.4.8.** *Consider an optimal vertex solution $y$ of* (EG-NSW) *for Rado valuations. Then, $|\mathrm{supp}(y)| \leq |\mathcal{A}'| + 2|\mathcal{L}^+(y)| - |\mathcal{L}_1(y)|$, where*

$$\mathcal{L}^+(y) = \{j \in \mathcal{L} : \sum_{i \in \mathcal{A}'} y_{ij} > 0\},$$
$$\mathcal{L}_1(y) = \{j \in \mathcal{L} : \exists! i \in \mathcal{A}' \text{ such that } y_{ij} > 0\}.$$

*Proof.* The optimal vertex solution $y$ can be written as $y_{ij} = \sum_{(i,k) \in E_i} z_{ijk}$ for a vertex solution $z$ of $P^*$. We have $|\mathrm{supp}\,(z)| \leq |\mathcal{A}'| + 2|\mathcal{L}^+| - |R_1| - |R_2|$. The first condition holds by definition of $y$. By construction we also have $\mathcal{L}^+(y) = \mathcal{L}^+(z) =: \mathcal{L}^+$. Moreover, $R_1, R_2 \subseteq \mathcal{L}_1$.

By definition of $\mathcal{L}_1$, $R_1$ and $R_2$; we have $j \in \mathcal{L}_1 \setminus (R_1 \cup R_2)$ if and only if $j$ is allocated fully to a unique agent $i$ and there exist different $k_1, k_2 \in V_i$ with $z_{ijk_1} > 0$ and $z_{ijk_2} > 0$. Both variables $z_{ijk_1}$ and $z_{ijk_2}$ contribute that $y_{ij} > 0$ for the same $i, j$. Thus,

$$|\mathrm{supp}(y)| \leq |\mathcal{A}'| + 2|\mathcal{L}^+| - |R_1| - |R_2| - |\mathcal{L}_1 \setminus (R_1 \cup R_2)| = |\mathcal{A}'| + 2|\mathcal{L}^+| - |\mathcal{L}_1|.$$

□

**Further sparsification** We showed that any basic optimal solution to (EG-NSW) under Rado valuations has support of size $|\mathcal{A}'| + 2|\mathcal{L}_+| - |\mathcal{L}_1|$. Next, we show that any such sparse solution can be further sparsified by losing a fraction of valuation of each agent. The main observation is that given a feasible allocation for a Rado valuation function, all "sub-allocations" behave in a "locally subadditive" way, as explained next.

Let $y'$ be a feasible allocation and $z'$ its corresponding representation in (EG-Rado). Our argument will scale down $y_{ij} = q_{ij} y'_{ij}$ for some $q_{ij} \in [0, 1]$. We have $v_i(y'_i) = \sum_{j \in \mathcal{L}, k \in V_i} c_{ijk} z'_{ijk}$. Thus, we can write $v_i(y'_i) = \sum_{j \in \mathcal{L}} u(i, j)$ where $u(i, j) = \sum_{k \in V_i} c_{ijk} z'_{ijk}$ is the value agent $i$ gets from good $j$. Hence, we can represent $y_{ij} = q_{ij} y'_{ij}$ as $y_{ij} =$

$q_{ij} \sum_{k \in V_i} z'_{ijk}$. Assuming $q_{ij} \in [0,1]$ we have

$$v_i(y_i) \geq \sum_{j \in \mathcal{L}, k \in V_i} q_{ij} \cdot c_{ijk} z'_{ijk} = \sum_{j \in \mathcal{L}} q_{ij} \cdot u(i,j)\,,$$

where we use the fact that whenever $z'$ is feasible for (EG-Rado) then so is the allocation given by $q_{ij} z'_{ijk}$ for $j \in \mathcal{L}, k \in V_i$. In particular, this justifies the notation $y_{ij} = q_{ij} y'_{ij}$ for $q_{ij} \in [0,1]$ and it holds that $v_i(y_i) \geq \sum_{j \in \mathcal{L}} q_{ij} u(i,j)$. Such a property is used to prove the following lemma.

**Lemma 4.2.8.** *Suppose the functions $v_i$ are Rado valuations, and let $\hat{y}$ be a feasible solution to* (EG-NSW)*. Then, in polynomial time we can find a feasible solution $y$ such that*

*(i)* $v_i(y) \geq \frac{1}{2} v_i(\hat{y})$,

*(ii)* $|\mathrm{supp}(y)| \leq 2|\mathcal{A}'| + |\mathcal{L}^+|$ *where* $\mathcal{L}^+ := \mathcal{L}^+(y) = \{j \in \mathcal{G} \setminus \mathcal{H} : \sum_{i \in \mathcal{A}'} y_{ij} > 0\}$.

Given a $\hat{y}$, we can transform it to a vector $y'$ with $|\mathrm{supp}(y')| \leq |\mathcal{A}'| + 2|\mathcal{L}^+(y')| - |\mathcal{L}_1(y')|$ by Corollary 4.4.8. Then, the idea is to exhibit $q$ such that the vector $y$ defined as $y_{ij} = q_{ij} y'_{ij}$ satisfies the lemma. Such $q$ needs to preserve at least half of the value for each agent and should set at least $|\mathcal{L}^+| - |\mathcal{L}_1| - |\mathcal{A}'|$ values of $y'_{ij}$ to $0$. We can find such a $q$ as a basic feasible solution of a system of linear (in)equalities.

*Proof.* Let $y'$ be a solution of (EG-NSW) with $|\mathrm{supp}(y')| \leq |\mathcal{A}'| + 2|\mathcal{L}^+(y')| - |\mathcal{L}_1(y')|$, given by Corollary 4.4.8. Let $D = \{j \in \mathcal{L}^+(y') : \exists i, i', i \neq i' \text{ such that } y'_{ij} > 0 \text{ and } y'_{i'j} > 0\}$, i.e., $D$ is the set of items in $\mathcal{L}^+(y')$ allocated to at least two different agents by $y'$. Hence, $|D| = |\mathcal{L}^+(y')| - |\mathcal{L}_1(y')|$. For each $j \in D$, let $D(j)$ be a set containing two different agents $i, i'$ getting the item $j$ in $y'$. Such two agents are picked arbitrarily, but fixed throughout the proof for each $j$. Let $\mathcal{A}'' = \cup_{j \in D} D(j)$.

We consider the following linear system with variables $q$. The value $q_{ij}$ represents the fraction of $y'_{ij}$ agent $i$ keeps. By the above, if agent obtained $u(i,j)$ value from $y'_{ij}$ units of $j$ then agent receives $q_{ij} u(i,j)$ value from $q_{ij} y'_{ij}$ units of good $j$ whenever $q_{ij} \in [0,1]$.

$$\begin{aligned}
\sum_{j \in D} q_{ij} u(i,j) \geq \frac{1}{2} \sum_{j \in D} u(i,j) \quad & \forall i \in \mathcal{A}'' \\
q_{ij} + q_{i'j} = 1 \quad & \forall j \in D,\ \{i,i'\} = D(j) \\
q \geq 0\,. &
\end{aligned}$$

Let us define $y$: set $y_{ij} = 0$ if $q_{ij} = 0$ and $y_{ij} = y'_{ij}$ for all other values. Then for any feasible $q$ we have

- The second set of constraints together with non-negativity of $q$ guarantees $q_{ij} \in [0,1]$ and hence we can treat the values $v_i(y_i) \geq q_{ij} v_i(y'_i)$ as described before the statement of the lemma.

- By the first set of constraints and definition of $y$, we have

$$v_i(y_i) \geq \sum_{j\in D} q_{ij}u(i,j) + \sum_{j\in\mathcal{L}\setminus D} u(i,j) \geq \frac{1}{2}\sum_{j\in D} u(i,j) + \frac{1}{2}\sum_{j\in\mathcal{L}\setminus D} u(i,j) \geq \frac{1}{2}v_i(y')\,.$$

Therefore, any feasible solution of the linear system in $q$ gives an allocation that satisfies the first condition of the lemma. Let us show that the system is indeed feasible. Namely, setting $q_{ij} = \frac{1}{2}$ for all $i \in \mathcal{A}''$ and all $j \in D$ we see that the above system is feasible. Since, the system is feasible we can also find a basic feasible solution $q$. By counting the number of tight constraints we show that there are at least $|\mathcal{L}^+(y')| - |\mathcal{L}_1(y')| - |\mathcal{A}''|$ zeros in $q$. Thus, allocation $y$ defined as $y_{ij} = q_{ij}y'_{ij}$ will have support smaller by at least $|\mathcal{L}^+(y')| - |\mathcal{L}_1(y')| - |\mathcal{A}''|$.

The maximum number of (tight) constraints is obviously $|\mathcal{A}''| + |D|$. Thus, $|\mathrm{supp}(q)| \leq |\mathcal{A}''| + |D|$. Crucially, by the second constraint we have $\mathcal{L}^+(y) = \mathcal{L}^+(y')$. Hence, we only need to compare $|\mathrm{supp}(y')|$ and $|\mathrm{supp}(y)|$. The allocation $y'$ has exactly $2|D|$ positive variables when restricted on $D$ and $\mathcal{A}''$. On the other hand, $q$ and therefore $y$ take at most $|D| + |\mathcal{A}''|$ non-zero values on $D$ and $\mathcal{A}''$. It follows that $y$ has at least $|D| - |\mathcal{A}''|$ less positive variables than $y'$, i.e., $|\mathrm{supp}(y)| \leq |\mathrm{supp}(y')| - (|D| - |\mathcal{A}''|)$. By Corollary 4.4.8 and since $|\mathcal{A}''| \leq |\mathcal{A}'|$ we have $|\mathrm{supp}(y)| \leq 2|\mathcal{A}'| + 2|\mathcal{L}^+| - |\mathcal{L}_1(y')| - |D|$. By recalling that $|D| = |\mathcal{L}^+| - |\mathcal{L}_1(y')|$ we get $|\mathrm{supp}(y)| \leq 2|\mathcal{A}'| + |\mathcal{L}^+|$. □

## 4.5 Phase V: Rounding the mixed solution

We present the rounding for a sparse solution of (Mixed+matching). We recall that by *sparse* we mean a feasible solution $(y, \pi)$ of (Mixed+matching) satisfying:

$$\mathrm{supp}(y) \leq 2|\mathcal{A}| + |\mathcal{L}^+| \text{ where } \mathcal{L}^+ = \left\{ j \in \mathcal{G} \setminus \mathcal{H} : \sum_{i\in\mathcal{A}'} y_{ij} > 0 \right\}.$$

Such a sparse solution is rounded by setting $2|\mathcal{A}|$ positive variables in $y$ to $0$, i.e., a reduction of $(y, \pi)$ and allocating the items according to the support of the reduction. Formally, by a *reduction* of $(y, \pi)$ we mean a mixed integer solution $(y^r, \pi)$ obtained as follows (see Figure 4.1). For each item $j$ a fraction of which is allocated by $y$ (i.e., $j \in \mathcal{L}^+$), we pick an arbitrary agent $\kappa(j)$ getting the item (i.e., $y_{\kappa(j)j} > 0$). We set $y^r_{\kappa(j)j} = y_{\kappa(j)j}$, and set $y^r_{ij} = 0$ if $i \neq \kappa(j)$. In words, the agent $\kappa(j)$ keeps getting the same amount in reduction and no other agent receives any part of item $j$. By the bound on $\mathrm{supp}(y)$, this amounts to setting $\leq 2|\mathcal{A}|$ values $y_{ij}$ to 0. Looking at the *reduction* from the agents perspective: let $d_i$ be the number of items agent $i$ lost by reduction, i.e., the number of items $j$ for which $y_{ij} > 0$ and $y^r_{ij} = 0$. Then, $\sum_{i\in\mathcal{A}'} d_i \leq 2|\mathcal{A}|$.

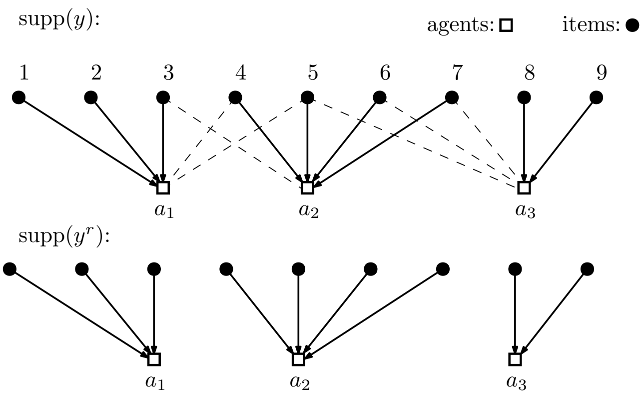


Figure 4.1: Support graph of an allocation $y$. Support graph of reduction $y^r$ obtained by $\kappa(1) = \kappa(2) = \kappa(3) = a_1$, $\kappa(4) = \kappa(5) = \kappa(6) = \kappa(7) = a_2$, and $\kappa(8) = \kappa(9) = a_3$. It follows that $d_{a_1} = 2$, $d_{a_2} = 1$ and $d_{a_3} = 3$.

The reduction $(y^r, \pi)$ might have an arbitrarily worse objective value than $(y, \pi)$ (e.g., if for agent $i$ we have $v_{i\pi(i)} = 0$ and reduction sets $y_i^r = 0$), but we show that we can find a different assignment $\rho$ such that $(y^r, \rho)$ is only worse by a constant factor than $(y, \pi)$, no matter how the reduction is carried out. The assignment $\rho$ is obtained as a combination of $\tau$ (the assignment obtained in Phase I) and $\pi$.

For a fixed reduction and the values $d_i$, $\rho$ and its properties are given by the following lemma.

**Lemma 4.5.1** (Key rounding lemma)**.** *Let $\mathcal{H}$ be the set of most preferred items, $(y, \pi)$ a feasible solution to* (Mixed+matching)*, and let $d_i \in \mathbb{N}$, $(d_i \geq 1)$ for each $i \in \mathcal{A}$. In $O(|\mathcal{A}|)$ time, we can find an assignment $\rho$ such that*

$$\overline{\mathrm{NSW}}(y, \rho) \geq \frac{1}{2} \left( \prod_{i \in \mathcal{A}} (d_i + 1)^{-w_i} \right)^{1/\sum_{i \in \mathcal{A}} w_i} \overline{\mathrm{NSW}}(y, \pi)$$

*and for each $i \in \mathcal{A}$ it holds either*

*(a) $v_{i\rho(i)} \geq \frac{1}{d_i} v_i(y_i)$, or*

*(b) for each $j \in \mathcal{L}$ it holds $v_{ij} \leq \frac{1}{d_i+1}(v_i(y_i) + v_{i\rho(i)})$.*

Intuitively, the above lemma states that starting with a feasible allocation $y$, we can find an assignment $\rho$ that might have smaller $\overline{\mathrm{NSW}}(y, \rho)$ than $\overline{\mathrm{NSW}}(y, \pi)$ but has the following nice property for each agent $i \in \mathcal{A}$:

- In case (a), $i$ values the item $\rho(i)$ at least as she values a $1/d_i$ fraction of $y_i$ (and thus at least a $1/(d_i+1)$ fraction of $v_i(y_i)+v_{i\rho(i)}$). Hence, agent $i$ keeps a $1/(d_i+1)$-fraction of her value just by keeping $\rho(i)$ even if we can take away all items $i$ gets from $\mathcal{L}$.

- In case (b), every item $\mathcal{L}$ has a small value for $i$ when compared to the combined value of $y_i$ and $\rho(i)$. That is, $i$ values $y_i$ and $\rho(i)$ significantly more than any $d_i$ items combined from $\mathcal{L}$. Looking at it from the other side, even if we were to take away any $d_i$ in $\mathcal{L}$ items from $i$ she will still keep a fraction of the value.

The essence of both cases is that the reduction will not hurt the agent too much. Before we present the proof of Lemma 4.5.1, we show that this is enough to prove Lemma 4.2.9.

**Lemma 4.2.9.** *Let $\mathcal{H}$ be the set of most preferred items, and let $(y, \pi)$ be a solution to (Mixed+matching) as in Theorem 4.2.5. Let $(y^r, \pi)$ be a reduction of $(y, \pi)$. Then in polynomial-time we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that*

$$\overline{\mathrm{NSW}}(y^r, \rho) \geq \frac{1}{32\gamma^2} \overline{\mathrm{NSW}}(y, \pi) \,.$$

*Further, if the valuations are linear, then we can find a matching $\rho : \mathcal{A} \to \mathcal{H}$ such that $\overline{\mathrm{NSW}}(y^r, \rho) \geq \frac{1}{8} \overline{\mathrm{NSW}}(y, \pi)$.*

*Proof of Lemma 4.2.9.* We first prove the lemma for the general case. Let $y^r$ be any reduction of $y$ and let $d_i$ be the number items agent $i$ lost in reduction. By sparsity in Theorem 4.2.5 we have $\sum_{i \in \mathcal{A}} d_i \leq 2|\mathcal{A}|$.

We use Lemma 4.5.1 to obtain $\rho$. Note that Lemma 4.5.1 requires $d_i \geq 1$ so we define $\overline{d}_i = \max\{1, d_i\}$. Thus, now we have the bound $\sum_{i \in \mathcal{A}} (\overline{d}_i + 1) \leq 4|\mathcal{A}|$. Let $\rho$ be the matching obtained by Lemma 4.5.1 given $\overline{d}_i$'s and $y$. By Lemma 4.1.2 we have

$$\left( \prod_{i \in \mathcal{A}} (\overline{d}_i + 1)^{-w_i} \right)^{1 / \sum_{i \in \mathcal{A}} w_i} \geq \frac{1}{4\gamma} \,.$$

Thus, $\overline{\mathrm{NSW}}(y, \rho) \geq \frac{1}{8\gamma} \overline{\mathrm{NSW}}(y, \pi)$. By the same inequality, it suffices to show that $\overline{\mathrm{NSW}}(y^r, \rho) \geq \left(\prod_{i \in A} (\overline{d}_i + 1)^{-w_i}\right)^{\sum_{i \in \mathcal{A}} w_i} \overline{\mathrm{NSW}}(y, \rho)$. We do so, by showing that for each $i \in \mathcal{A}$ it holds $v_i(y_i^r) + v_{i\rho(i)} \geq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$. By Lemma 4.5.1 for agent $i$ we have either (a) or (b).

(a) In this case we have $\overline{d}_i v_{i\rho(i)} \geq v_i(y_i)$. Thus, $v_{i\rho(i)} \geq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$. Consequently, $v_i(y_i^r) + v_{i\rho(i)} \geq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$.

(b) We have $v_{ij} \leq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$ for all $j \in \mathcal{L}$. Denote with $D_i$ the set of $d_i$ items $j$ for which $y_{ij} > 0$ and $y_{ij}^r = 0$. By subadditivity $v_i(D_i) \leq \sum_{j \in D_i} v_{ij}$. Therefore, $v_i(D_i) \leq \frac{d_i}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)}) \leq \frac{\overline{d}_i}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$. Hence, $v_i(y_i) - v_i(D_i) + v_{i\rho(i)} \geq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$. By subadditivity and monotonicity we have $v_i(y_i^r) \geq v_i(y_i) - v_i(D_i)$, proving in this case as well that $v_i(y_i^r) + v_{i\rho(i)} \geq \frac{1}{\overline{d}_i + 1}(v_i(y_i) + v_{i\rho(i)})$. The lemma follows.

For additive valuations, we recall Theorem 4.2.6. It gives an optimal solution of (EG-NSW) that is supported on a forest in which each tree contains an agent. In particular, this implies a nice property for the reductions of $y$. Namely, we can choose a reduction $y^r$ in which $d_i \leq 1$ for each agent $i \in \mathcal{A}$. Such a reduction is obtained by rooting each tree of the forest at an arbitrary agent and letting $\kappa(j)$ to be the parent agent of item $j$. Informally, each agent loses at most one item. Therefore, $\overline{d}_i = 1$ for all $i \in \mathcal{A}$. The lemma follows by Lemma 4.5.1. □

The proof of Lemma 4.5.1 is presented in the following section.

### 4.5.1 Constructing the new matching

Recall **Phase I** where we defined $\tau$ as an assignment maximizing $\left(\prod_{i\in\mathcal{A}} v_{i\tau(i)}^{w_i}\right)$ and $\mathcal{H}$ the set of items assigned by $\tau$. We number the agents $\mathcal{A} = \{1, 2, \dots, n\}$, and renumber the items $\mathcal{H} = \{1, 2, \dots, n\}$ such that $\tau = \{(i, i) : i \in A\}$. In other words, $\tau$ assigns item $i \in \mathcal{G}$ to agent $i \in \mathcal{A}$.

**Intuition** We are given a feasible solution $(y, \pi)$ of (Mixed+matching) and $\tau$. For the sake of illustration assume that by using the matching $\tau$ instead of $\pi$ we don't lose too much in the objective, i.e.,

$$\overline{\mathrm{NSW}}(y, \tau) \geq \left(\prod_{i\in\mathcal{A}} (d_i + 1)^{-w_i}\right)^{1/\sum_{i\in\mathcal{A}} w_i} \overline{\mathrm{NSW}}(y, \pi)\,.$$

In this case, each agent $i$ gets the item $i$ from $\mathcal{H}$. Let us show that under the above assumption we can set $\rho = \tau$, i.e., that for each agent $i$ either (a) or (b) holds.

**Claim 4.5.2.** *Let $i \in \mathcal{A}$. Then either $v_{ii} > \frac{1}{d_i} v_i(y_i)$ or for any $j \in \mathcal{L}$ it holds $v_{ij} \leq \frac{1}{d_i+1}(v_{ii} + v_i(y_i))$*

*Proof of Claim.* By the optimality of $\tau$ it then holds $v_{ii} \geq v_{ij}$ for all $j \in \mathcal{L}$. If $v_{ii} \geq \frac{1}{d_i} v_i(y_i)$ then (a) holds. Otherwise, we have that $d_i v_{ii} < v_i(y_i)$. Combining it with $v_{ij} < v_{ii}$, we have that

$$(d_i + 1)v_{ij} \leq (d_i + 1)v_{ii} < v_i(y_i) + v_{ii} = v_i(y_i) + v_{i\tau(i)}\,. \qquad \blacksquare$$

Therefore, our goal is to construct $\rho$ by "replacing" as much of $\pi$ with $\tau$ without losing too much in the objective. By Claim 4.5.2 for any agent for which $\rho(i) = \tau(i)$ we will have either (a) and (b). We formalize this idea below, and give a way of constructing $\rho$ such that even when $\rho(i) \neq \tau(i)$ still we have either (a) and (b).

**Algorithm** Let $(y, \pi)$ be a feasible solution of (Mixed+matching). We denote with $Y_i$ the value agent $i$ gets in $y$, i.e., $Y_i = v_i(y_i)$. We construct new assignment $\rho$ by combining $\pi$ and $\tau$. In particular, whenever $\pi(i) = \tau(i)$ then we set $\rho(i) := \pi(i) = \tau(i)$ and otherwise exactly one of the following will be the case: $\rho(i) = \tau(i)$, $\rho(i) = \pi(i)$ or $\rho(i) = \emptyset$. Notation $\rho(i) = \emptyset$ represents the case that $i$ is not allocated any item from $\mathcal{H}$. (Formally, we can allocate one item to each agent since $|\mathcal{H}| = |\mathcal{A}|$ but as some agents might value some items at $0$ it is simpler to say that agent is not allocated an item by $\rho$.)

Consider the symmetric difference of the two assignments $\pi\Delta\tau$. Each component is an alternating cycle; we consider the components one-by-one. Take any component $C$ of $\pi\Delta\tau$ with $c$ agents and $c$ items. Let the agents in the component be $a_1, a_2, \ldots, a_c$. The numbering is modulo $c$: $a_{c+k} = a_k$ for all $k \in \mathbb{Z}$. By the convention on the numbering, the corresponding items are also numbered $a_1, a_2, \ldots, a_c$, and $(a_k, a_k) \in \tau$ for all $k \in [c]$. We order the agents around the cycle such that $(a_k, a_{k-1}) \in \pi$ for all $k \in [c]$. Let $B := B(C) = \{t \in [c] : Y_{a_t} > d_{a_t} v_{a_t a_{t-1}}\}$. We consider two cases based on the size of $B$:

$|B| = 0$. In this case we set $\rho(a_t) = \pi(a_t) = a_{t-1}$ for all $t \in [c]$.

$|B| \geq 1$. First, we *trim* $\pi$ by setting $\pi(a_t) = \emptyset$ for each $t \in B$. We have $\frac{Y_{a_t} + v_{a_t a_{t-1}}}{Y_{a_t}} \leq 2$ for each $t \in B$ since $d_{a_t} \geq 1$. In words, each agent losses at most half of her value.

After trimming $\pi$, the connected component $C$ decomposes into several alternating paths, see Figure 4.2. Consider one such path, starting in agent $a_k$ and ending in item $a_r$. It follows that $k \in B$ and $t \notin B$ for all $k < t \leq r$. We consider the following ratio that measures the change in the objective value by augmenting $\pi$ over the previously mentioned path:

$$\varphi(C, k, r) := \left(\frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}}\right)^{w_{a_k}} \prod_{t=k+1}^{r} \left(\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}}\right)^{w_{a_t}} .$$

If it holds that $\varphi(C, k, r) \leq \prod_{t=k}^{r-1} (d_{a_t} + 1)^{w_{a_t}}$ then we say that the *interval* $[k, r]$ is *reversible*. Moreover, we set $\rho(a_t) = \tau(a_t) = a_t$ for all $k \leq t \leq r$. If $[k, r]$ is *not* reversible then we set $\rho(a_k) = \emptyset$ and $\rho(a_t) = \pi(a_t) = a_t$ for all $k < t \leq r$. We do the same for every augmenting path.

To prove Lemma 4.5.1, we first show that by changing the assignment from $\pi$ to $\rho$ the objective value of (Mixed+matching) cannot decrease by too much.

**Lemma 4.5.3.** *The assignment $\rho$ can be constructed in linear time (in $n$), and it holds*

$$\frac{\overline{\mathrm{NSW}}(y, \pi)}{\overline{\mathrm{NSW}}(y, \rho)} \leq 2 \cdot \left(\prod_{i \in \mathcal{A}} (d_i + 1)^{w_i}\right)^{1/\sum_{i \in \mathcal{A}} w_i} .$$

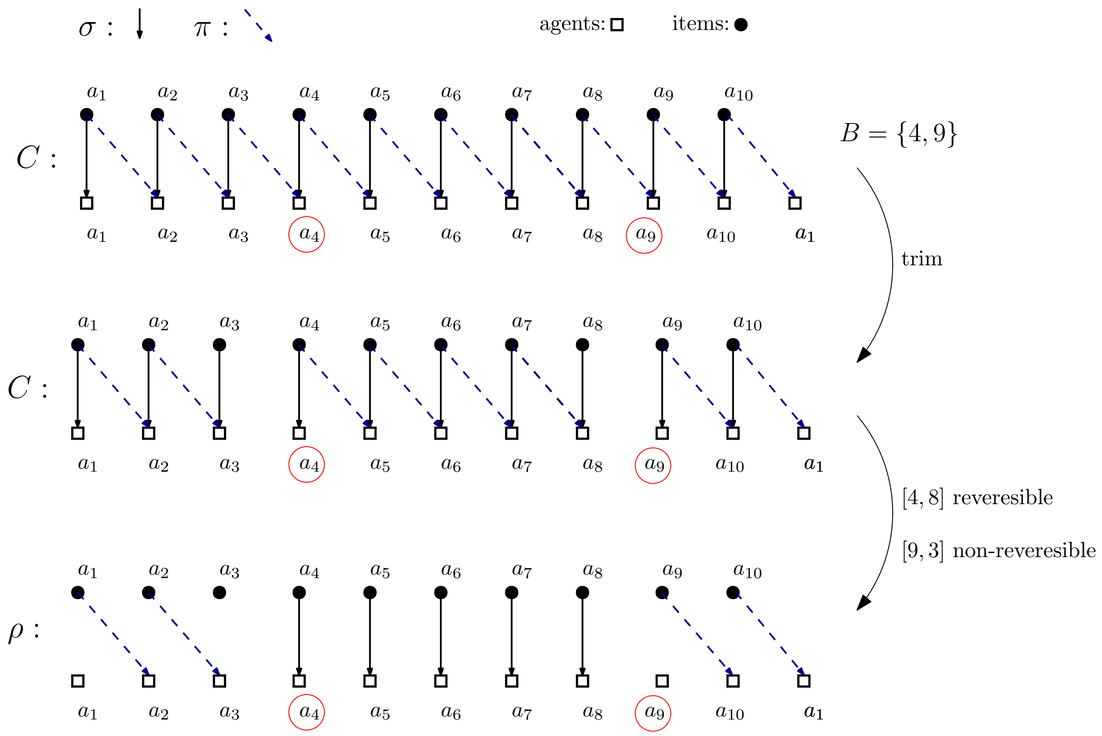


Figure 4.2: Assignments $\tau, \pi$, and $\rho$ resulting from $B = \{4, 9\}$ and reversible interval $[4, 8]$.

*Proof.* It suffices to prove the lemma for each of the connected components $C$ of $\pi \Delta \tau$. For $|B| = 0$ the lemma holds trivially. So assume that $|B| \geq 1$ for the rest of the proof.

The procedure terminates in linear time, as we only require one pass through the agents and items in $C$. To prove the bound on $\frac{\text{NSW}(y,\rho)}{\text{NSW}(y,\pi)}$, we show that for every interval $[k, r]$ the objective value "before averaging" decreases at most by factor $2^{w_{a_k}} \prod_{t=k}^{r} (d_{a_t} + 1)^{w_{a_t}}$.

If interval $[k, r]$ is not reversible, then the change in the objective function is captured by $\left(\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}}\right)^{w_{a_k}}$, as for every agent $a_t$ with $t \in [k+1, r]$, we have $\rho(a_t) = \pi(a_t)$, and $\rho(a_k) = \emptyset$. Since $k \in B$, it follows that $Y_{a_k} > d_{a_k} v_{a_k a_{k-1}} \geq v_{a_k a_{k-1}}$. Thus, $\left(\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}}\right)^{w_{a_k}} < 2^{w_{a_k}}$.

If, on the other hand, $[k, r]$ is reversible, then the difference in the objectives is exactly

$$\left(\frac{v_{a_k a_{k-1}} + Y_{a_k}}{v_{a_k a_k} + Y_{a_k}}\right)^{w_{a_k}} \prod_{t=k+1}^{r} \left(\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}}\right)^{w_{a_t}} =$$
$$\left(\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}} \cdot \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}}\right)^{w_{a_k}} \prod_{t=k+1}^{r} \left(\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}}\right)^{w_{a_t}}.$$

As $[k, r]$ is reversible $\varphi(C, k, r) = \left(\frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}}\right)^{w_{a_k}} \cdot \prod_{t=k+1}^{r} \left(\frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}}\right)^{w_{a_t}} < \prod_{t=k}^{r} (d_{a_t} + 1)^{w_{a_t}}$. Since $k \in B$ and $d_{a_k} \geq 1$ we again have $\frac{v_{a_k a_{k-1}} + Y_{a_k}}{Y_{a_k}} < 2$. Hence, the change in the objective value is bounded by $2^{w_{a_k}} \cdot \prod_{t=k}^{r} (d_{a_t} + 1)^{w_{a_t}}$. □

It is left to show that for each agent $i$ either (a) or (b) holds. Recall that $Y_i = v_i(y_i)$.

**Lemma 4.5.4.** *Let $i \in \mathcal{A}$. Then we either have*

*(a) $v_{i\rho(i)} \geq \frac{1}{d_i} v_i(y_i)$, or*

*(b) for each $j \in \mathcal{L}$ it holds $v_{ij} \leq \frac{1}{d_i+1}(v_i(y_i) + v_{i\rho(i)})$.*

To prove the lemma we use the following simple claim, which can applied to any agent $i \notin B$:

**Claim 4.5.5.** *For any agent $i \in \mathcal{A}$, if $Y_i \leq d_i v_{i\pi(i)}$, then $\dfrac{v_{i\pi(i)} + Y_i}{v_{ii} + Y_i} \leq \dfrac{(d_i+1)v_{i\pi(i)}}{v_{ii}}$.*

*Proof of Lemma 4.5.4.* If $\rho(i) = i$, that is, agent $i$ receives the same item in $\rho$ as in $\tau$ then the lemma follows by Claim 4.5.2. For the rest of the proof we assume $\rho(i) \neq i$. Hence, either $\rho(i) = \pi(i)$ or $\rho(i) = \emptyset$.

We consider the component $C$ of $\tau \Delta \pi$ containing an agent $i$. We use the notation as before, denoting the agents in $C$ by $a_1, a_2, \ldots, a_c$, and letting $i = a_k$.

If $\rho(a_k) = \pi(a_k) = a_{k-1}$ then for $i$ it holds (a). Namely, $\rho(a_k) = a_{k-1}$ implies that $k \notin B$ as otherwise this would be trimmed. Thus $Y_{a_k} \leq d_{a_k} v_{a_k a_{k-1}}$; or equivalently $v_{a_k a_{k-1}} \geq \frac{1}{d_{a_k}} Y_{a_k}$.

If on the other hand $\rho(a_k) = \emptyset$, we have that $k \in B$ and also that the interval $[k, r]$ with starting and $k$ and ending in $r$ that corresponds to some alternating path in $C$ is *not* reversible (otherwise, $\rho(a_k) = a_k$). Therefore, $\varphi(C, k, r) > \prod_{t=1}^{r} (d_{a_t} + 1)^{w_{a_t}}$. Recall that for each such considered interval we have $k \in B$ and $t \notin B$. Starting with $\prod_{t=k}^{r} (d_{a_t} + 1)^{w_{a_t}} < \varphi(C, k, r)$ and then by Claim 4.5.5 we obtain

$$
\begin{aligned}
1 &< \prod_{t=k}^{r-1} (d_{a_t} + 1)^{-w_{a_t}} \cdot \left( \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}} \right)^{w_{a_k}} \cdot \prod_{t=2}^{r} \left( \frac{v_{a_t a_{t-1}} + Y_{a_t}}{v_{a_t a_t} + Y_{a_t}} \right)^{w_{a_t}} \\
&\leq (d_{a_k} + 1)^{-w_{a_k}} \cdot \left( \frac{Y_{a_k}}{v_{a_k a_k} + Y_{a_k}} \right)^{w_{a_k}} \cdot \prod_{t=2}^{r} \left( \frac{v_{a_t a_{t-1}}}{v_{a_t a_t}} \right)^{w_{a_t}} .
\end{aligned}
$$

We further bound

$$
1 < (d_{a_k} + 1)^{-w_{a_k}} \cdot \left( \frac{Y_{a_k}}{v_{a_k j}} \cdot \frac{v_{a_k j}}{v_{a_k a_k}} \right)^{w_{a_k}} \cdot \prod_{t=2}^{r} \left( \frac{v_{a_t a_{t-1}}}{v_{a_t a_t}} \right)^{w_{a_t}} .
$$

By the optimal choice of $\tau$, for every $j \in \mathcal{L}$ we have

$$
1 \leq \left( \frac{v_{a_k a_k}}{v_{a_k j}} \right)^{w_{a_k}} \cdot \prod_{t=2}^{r} \left( \frac{v_{a_t a_t}}{v_{a_t a_{t-1}}} \right)^{w_{a_t}} .
$$

Combining the last two inequalities, we obtain $Y_{a_k} > (d_{a_k} + 1) v_{a_k j}$. Hence, in this case (b) holds, by recalling that $i = a_k$ and $\rho(a_k) = \emptyset$. $\square$

## 4.6 Connection to spending restricted equilibrium

The first constant factor approximation algorithm for the Nash social welfare problem was given by [35] using the spending restricted (SR) equilibrium; and we (Section 3.2) as well as other authors have since used this concept to desing approximation algorithms for the NSW problem [2, 33, 53, 56].

An important feature of the SR-equilibrium is that the items highly valued by the agents are recognized as items with price more than $1$ (*expensive*) in the equilibrium. Isolating such items is at the essence of the approximation algorithms in the literature. The main idea is that each of the expensive items must be allocated integrally to one agent only, thereby preventing the unbounded integrality gap arising when several agents share a very desirable good, see [35, Lemma 3.1].

In this section, we illustrate a connection between the approach we use and the SR-equilibrium. In that light, for the rest of the section we focus on the case of symmetric Nash social welfare problem where agents have additive valuations. We show that the *set of the most preferred items* $\mathcal{H}$ obtained in **Phase I** contains all the expensive items in an SR-equilibrium. Similarly to the algorithms relying on the SR-equilibrium where expensive items have special status during rounding, the items in $\mathcal{H}$ are allocated integrally throughout our algorithm. Intuitively, this is how we are overcoming the unbounded integrality gap.

**SR-equilibrium** From the definition, it follows that and $x$ and a price vector $p$ form an SR-equilibrium for additive valuations if and only if every agent spends all of her budget ($1$ unit) on her MBB items at prices $p$, and the total spending on each item is equal to $\min\{1, p_j\}$. By scaling the valuation of each agent we can assume that the maximum bang per buck is one for all agents. Under such a scaling, in an SR-equilibrium we also have that $v_{ij} = p_j$ whenever item $j$ is MBB for agent $i$ and $v_{ij} < p_j$ otherwise. We work with this assumption for the rest of this section.

**NSW and SR-equilibrium** Consider a NSW welfare instance with items $\mathcal{G}$ and agents $\mathcal{A}$ where each agent $i$ has additive valuation. For the NSW problem, the valuations are discrete function and the value of a subset of items $S$ for agent $i$ is given by $v_i(S) = \sum_{j\in S} v_{ij}$. The extension of an additive valuation $v_i$ to $\mathbb{R}_+^{\mathcal{G}}$ is naturally defined as $v_i(x_i) = \sum_{j\in\mathcal{G}} v_{ij}x_{ij}$ for all $x_i \in \mathbb{R}_+^{\mathcal{G}}$. We construct the market from the NSW instance from the same set of items $\mathcal{G}$ that are now declared divisible and the set of agents $\mathcal{A}$ each equipped with the extension of the discrete additive valuation and budget one.

Let $(x, p)$ be an SR-equilibrium in such a market. Define the set of *expensive goods* $\overline{H}$ as $\overline{H} := \{j \in \mathcal{G} : p_j > 1\}$. Cole nad Gkatzelis [35] proved that $\left(\prod_{j\in\overline{H}} p_j\right)^{1/|\mathcal{A}|}$ is an upper-bound on the optimal value of NSW, and gave a rounding algorithm that uses an SR-equilibrium as a starting point.

In the next lemma we show that $\overline{H} \subseteq \mathcal{H}$, where $\mathcal{H}$ is the set of the most preferred goods obtained in **Phase I** of our algorithm. Recall that $\tau$ is an assignment maximizing $\left(\prod_{i\in\mathcal{A}} v_{i\tau(i)}\right)^{1/|\mathcal{A}|}$ and that $\mathcal{H} := \tau(A)$. In words, $\tau$ maximizes the NSW welfare under the constraint that each agent gets exactly one item.

For the purposes of the proof recall that the the spending graph $(\mathcal{A}, \mathcal{G}; E_x)$ of an allocation $x$ is defined as $ij \in E_x$ if and only if $x_{ij} > 0$.

**Lemma 4.6.1.** *It holds $\overline{H} \subseteq \mathcal{H}$.*

*Proof.* Using a cycle canceling argument, we can assume that the spending graph of SR-equilibrium $(x, p)$ is a forest $F$. Moreover, since $x$ is an SR-equilibrium allocation, every tree contains at least one agent and one item. The next claim states that an expensive item is a leaf in some tree in $F$ only in a very special case.

**Claim 4.6.2.** *Let $T = (\mathcal{A}_1, \mathcal{G}_1; E_1)$ be a tree component of $F$ and $j \in \mathcal{G}_1$ an item in $T$. If $p_j > 1$ then either $|A_1| = |\mathcal{G}_1| = 1$ or $j$ is not a leaf of $T$.*

*Proof of Claim.* By definition of SR-equilibrium each agent spends all of her budget which is 1. If $j$ is a leaf, then there is unique agent $i$ buying $j$. Moreover, $i$ spends all 1 unit of her budget on $j$ and cannot buy any other item. Thus, $\mathcal{A}_1 = \{i\}$ and $\mathcal{G}_1 = \{j\}$. ■

Let $\kappa : \overline{H} \to \mathcal{A}$ such that $x_{\kappa(j)j} > 0$. Such an function $\kappa$ exists by definition of SR-equilibrium. Moreover, by Claim 4.6.2 we can choose $\kappa$ to be an assignment (root every tree of $F$ in an arbitrary item and assign the expensive item to any child agent). We are ready to prove the lemma.

For the sake of contradiction suppose that there is an item $j_1 \in \overline{H}$ such that $j_1 \notin \mathcal{H}$. In other words, $p_{j_1} > 1$ and $j_1$ is not allocated to any agent by $\tau$. By definition we have $\overline{H} \leq |\mathcal{A}| = |\mathcal{H}|$. Consider the component of the symmetric difference $\tau\Delta\kappa$ containing $j$. Since $j_1 \notin \mathcal{H}$ and $\mathcal{H} = \tau(\mathcal{A})$, this component forms a path starting in $j_1$ and ending in a vertex $j_{k+1}$ in $\mathcal{G} \setminus \overline{H}$; see Figure 4.3. Let us denote the path as $j_1, \kappa(j_1), j_2, \kappa(j_2), \ldots, \kappa(j_k), j_{k+1}$ where $j_{t+1} = \tau(\kappa(j_t))$ for $t \in [k]$, and $j_t \in \overline{H}$ for $t \leq k$.

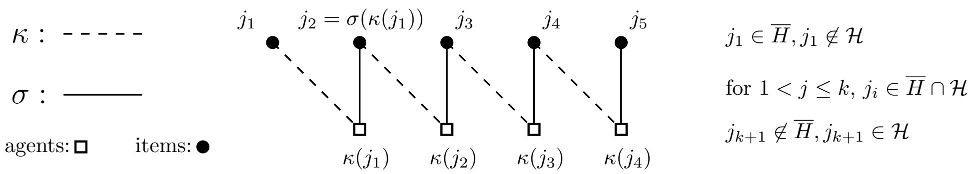


Figure 4.3: A component of $\kappa\Delta\tau$ containing $j_1$.

Recall, that MBB of each agent is one, therefore $v_{ij} = p_j$ for each $i, j$ with $x_{ij} > 0$. By definition of $\kappa$ we have that $v_{\kappa(j_t)j_t} \geq v_{\kappa(j_t)j_{t+1}}$ for $t \in [k-1]$. Moreover, we have

$v_{\kappa(j_1)j_1} = p_{j_1} > 1 \geq p_{j_{k+1}} \geq v_{\kappa(j_k),k+1}$. Since $j_{t+1} = \tau(\kappa(j_t))$, augmenting over the above path will contradict the optimality of $\tau$. □

## 4.7 Separating Rado and GS valuations

We show that Rado valuations form a strict subclass of GS valuations, thereby answering negatively the verions of Frank's question for the valuation functions. Lehmann, Lehmann, and Nisan [85, Example 1] gave an example that is GS valuations valuation but not OXS (assignment). We show that the same example is also not a Rado valuation; the proof is similar.

**Lemma 4.7.1.** *Consider the following valuation on the ground set $\mathcal{G} = \{1, 2, 3, 4\}$. We define $v(S) = 10$ if $|S| = 1$, and $v(S) = 19$ for all sets with $|S| \geq 2$ except $v(\{1, 3\}) = v(\{2, 4\}) = 15$. This is a GS valuation, but not a Rado valuation.*

*Proof.* The proof that $v$ is a gross substitutes valuation is given in [85, Claim 2]. Let us show that it is not a Rado valuation. For a contradiction, assume $v$ is a Rado valuation as in Definition 1.2.2. We can assume that the matroid on $V$ does not contain any loops (rank-0 elements), and any parallel elements, i.e., any set $S \subseteq V$ with $|S| \geq 2$ and $r(S) = 1$; we can contract any such set to a single element and obtain another representation.

Trivially, we can assume that no edge in the bipartite graph $(\mathcal{G}, V; E)$ has cost more than 10. By $v(\{1\}) = 10$ we have an element $u \in V$ with $c_{1u} = 10$. Since $v(\{2\}) = 10$, there is $u' \in V$ such that $c_{2u'} = 10$. Since $v(\{1, 2\}) < 20$ we have $u' = u$ as otherwise $(1, u), (2, u')$ would be an independent matching of cost 20, since $r(\{u, u'\}) = 2$ by the above assumption.

An analogous argument shows that $c_{ju} = 10$ for all $j \in \{1, 2, 3, 4\}$. We must have $c_{jk} \leq 5$ for any $j \in \{1, 2, 3, 4\}$ and any $k \in V \setminus \{v\}$, as otherwise we would have an independent matching of cost $> 15$ covering $\{1, 3\}$ or $\{2, 4\}$, again using the assumption of no parallel elements in $V$. Now, it is clear that we cannot realize $v(\{1, 2\}) = 19$. □

The reason why Rado valuations are not a rich enough class is that it is not closed under *endowment operations*. Given a valuation $v : 2^{\mathcal{G}} \to \mathbb{R}$ and a subset $T \subseteq \mathcal{G}$, we can define the valuation $v' : 2^{\mathcal{G}' \setminus T} \to \mathbb{R}_+$ as

$$v'(X) = v(X \cup T) - v(T) .$$

Using the definition of valuated generalized matroids, it is immediate that if $v$ is GS valuation than so is $v'$. It is not difficult to check that the example in Lemma 4.7.1 arises as the endowment of a Rado valuation, showing that Rado valuations are *not* closed under endowment operations. Note that endowement is combination of a contraction of the set $T$ and (additive) shift by $v(T)$. This fact is explained in more detail in Section 7.1.

# 5 Complete classes of valuated matroids

In this chapter, we study the classes of valuated matroids. A function $f : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ is a *valuated matroid* if and only if:

$$\forall X, Y \in \binom{V}{d} \text{ with } |X| = |Y| \text{ and } \forall i \in X \setminus Y :$$
$$f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X - i + j) + f(Y + i - j)\}.$$

We start by either recalling and introducing the valuated versions of the classical matroid operations (Section 5.1). These include deletion, contraction, dual, truncation, principal extension, induction by a bipartite graph, induction by a network, direct sum, and valuated matroid union. All of the previously mentioned operations turn out to preserve the property of being a valuated matroid.

In Section 5.2, we introduce complete classes of valuated matroids. A class of valuated matroids is complete if it is closed under deletion, contraction, dual, truncation and principal extension (although contraction could be dropped as it can be realized via deletion and dual, see Lemma 5.1.13). The main message here is that a complete class is also closed under all other of the above operations. We chose the five to define complete classes as they are the most basic ones.

The smallest complete class of valuated matroids are valuated gammoids – analogously to the matroids where the smallest complete class of matroids is the class of gammoids. We know that not all matroids are gammoids, so the next question arises: What is the smallest complete class of valuated matroids containing all trivially valuated matroids? The answer turns out to be the class of all R-minor valuated matroids (Section 5.3).

Chapter 6 is devoted to showing that not all valuated matroids are R-minor. Note that this is in contrast with complete classes of matroids where the above question has trivial answer.

**Preliminaries** We introduce the notation used in Chapters 5, 6 and 7. We denote a bipartite graph $G$ by $G = (V, U; E)$, where $V, U$ are the partitioned node sets and $E$ the edge set. If the bipartite graph is weighted, we denote the edges weights by $c \in \mathbb{R}^E$. For $Y \subseteq U$ or $Y \subseteq V$, we denote the set of neighbours of $Y$ by $\Gamma(Y) = \Gamma_G(Y)$. A network $N$ is a directed graph denoted as $(T, A)$ where $T$ is the node set and $A$ the arc set; if weighted then we denote the weights by $c \in \mathbb{R}^A$.

In these chapters the matroids are given as $\mathcal{M} = (U, r)$ where $U$ is the ground set and $r = r_{\mathcal{M}}$ is the rank of the matroid. The operations on matroids follows the notation of valuated matroids introduced in Section 5.1, as these are special cases of the valuated operations. Given a set $V$, we denote its set of subsets of cardinality $d$ by $\binom{V}{d}$.

## 5.1 Operations on valuated matroids

For a valuated matroid $f$, its *(effective) domain* $\mathrm{dom}(f)$ is formed by those sets $X$ on which $f(X) > -\infty$. The exchange property implies that it forms the set of bases of a matroid. The *rank* $\mathrm{rk}(f)$ of a valuated matroid $f$ is the rank of the underlying matroid $\mathrm{dom}(f)$.

**Definition 5.1.1.** *Let $f\colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ be a valuated matroid with $d = \mathrm{rk}(f)$, and $Y \subset V$ some subset of $V$.*

*(i) If $V - Y$ has full rank in $\mathrm{dom}(f)$ then the* deletion *of $f$ by $Y$ is the function $f \backslash Y\colon \binom{V-Y}{d} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f \setminus Y)(X) = f(X), \quad \forall X \in \binom{V-Y}{d}.$$

*This is also called the* restriction *to $V \setminus Y$ and denoted by $f|(V \setminus Y)$. If $V - Y$ does not have full rank in $\mathrm{dom}(f)$, the deletion is the function attaining only $-\infty$.*

*(ii) If $Y$ is independent in $\mathrm{dom}(f)$, then the* contraction *of $f$ by $Y$ is the function $f/Y\colon \binom{V-Y}{d-|Y|} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f/Y)(X) = f(X \cup Y), \quad \forall X \in \binom{V-Y}{d-|Y|}.$$

*If $Y$ is not independent in $\mathrm{dom}(f)$, the contraction is the function attaining only $-\infty$.*

*(iii) The* dual *of $f$ is the function $f^*\colon \binom{V}{|V|-d} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$f^*(X) = f(V - X), \quad \forall X \in \binom{V}{|V|-d}.$$

*(iv) The* truncation *of $f$ is the function $f^{(1)}\colon \binom{V}{d-1} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$f^{(1)}(X) = \max_{v \in V \setminus X} f(X \cup v), \quad \forall X \in \binom{V}{d-1}.$$

*The iterated truncation for $1 \leq r \leq d-1$ is given by $f^{(r+1)} = (f^{(r)})^{(1)}$.*

(v) *For $w \in (\mathbb{R} \cup \{-\infty\})^V$, the* principal extension *$f^w$ of $f$ with respect to $w$ is the valuated matroid on $V \cup p$ of rank $d$, for an additional element $p$, with $f^w|V = f$ and*

$$f^w(X \cup p) = \max_{v \in V \setminus X} (f(X \cup v) + w_v) \quad \text{for all} \quad X \in \binom{V}{d-1} .$$

**Remark 5.1.2.** *Our definition of deletion and contraction differs from the usual definition, e.g. in [42], in that we impose these rank conditions. The usual definition of deletion (and dually contraction) for matroids could equally be formulated by first performing a truncation (to the rank of the remaining set) and then a deletion. While for unvaluated matroids this is the same, for valuated matroids the naive deletion, where the remaining set does not have full rank, would result in a function only taking $-\infty$ as value. Our reason to be more restrictive with deletion and contraction is that these definitions allow for simple 'layer-wise' extensions to valuated generalized matroids in Section 7.1.*

**Example 5.1.3.** *The most basic examples of valuated matroids are those with* trivial valuation, *where only the values $0$ and $-\infty$ are attained (following naming as in [47]). Such valuated matroids can be identified with the underlying matroid. Observe that the operations listed in Definition 5.1.1 agree with the usual matroid operations for trivially valuated matroids.*

**Example 5.1.4.** *Valuated matroids corresponding to the layers of the assignment valuations are* transversally valuated matroids. *For a graph $G = (V, U; E)$ with edge weights $c \in \mathbb{R}^E$, we define transversally valuated matroid $f \colon \binom{V}{|U|} \to \mathbb{R} \cup \{-\infty\}$ for $X \in \binom{V}{d}$ as the maximum weight of a matching whose endpoints in $V$ are exactly $X$; if no such matching exists then we set $f(X) = -\infty$.*

*Let $V = [4]$ and consider the valuated matroid $f \colon \binom{V}{2} \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$f(12) = -\infty\ , \ f(13) = 0\ , \ f(14) = 0\ , \ f(23) = 1\ , \ f(24) = 1\ , \ f(34) = 1\ .$$

*This valuated matroid is transversally valuated as it can be realized via the weighted bipartite graph shown in Figure 5.1.*

**Example 5.1.5.** *One source of valuated matroids arises from matrices with polynomial entries. Let $A$ be a matrix with $d$ rows and columns labelled by $V$, whose entries are univariate polynomials over a field. For $J \subseteq V$, we denote by $A[J]$ the submatrix formed by the columns labelled by $J$. The valuated matroid induced by $A$ is defined to be*

$$f(J) = \deg \det A[J]\ ,$$

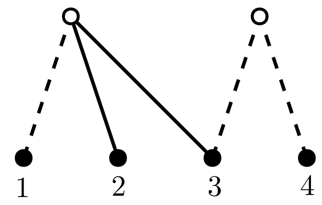

Figure 5.1: The bipartite graph realising the transversally valuated matroid from Example 5.1.4. The dashed edges have weight zero and the solid edges have weight one.

*where* $f(J) = -\infty$ *if* $\det A[J] = 0$ *or* $A[J]$ *is non-square, see [95, Section 2.4.2] for further details.*

*Recall the valuated matroid from Example 5.1.4. Observe that we can also represent this matrix via the polynomial matrix*

$$A = \begin{bmatrix} 1 & t & t & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

*e.g.* $f(23) = \deg(t) = 1$.

**Definition 5.1.6** (Direct sum, Valuated matroid union). *Let* $f_1$ *and* $f_2$ *be valuated matroids on ground sets* $V_1$ *and* $V_2$ *with ranks* $d_1$ *and* $d_2$.

- *For* $V_1 \cap V_2 = \emptyset$, *the* direct sum *of* $f_1$ *and* $f_2$ *is* $(f_1 \oplus f_2)\colon \binom{V_1 \cup V_2}{d_1+d_2} \to \mathbb{R} \cup \{-\infty\}$, *where*

$$(f_1 \oplus f_2)(X_1 \cup X_2) = f_1(X_1) + f_2(X_2) \text{ for all } X_1 \in \binom{V_1}{d_1}, X_2 \in \binom{V_2}{d_2}.$$

- *For* $V := V_1 \cup V_2$, *the* (valuated) matroid union *of* $f_1$ *and* $f_2$ *is* $(f_1 \vee f_2)\colon \binom{V}{d_1+d_2} \to \mathbb{R} \cup \{-\infty\}$, *where*

$$(f_1 \vee f_2)(X) = \max\left\{ f_1(Y) + f_2(X \setminus Y) : Y \subseteq X \,,\, Y \in \binom{V_1}{d_1} \,,\, X \setminus Y \in \binom{V_2}{d_2} \right\}.$$

*Undefined sets get the value* $-\infty$.

Actually, the direct sum can be considered as valuated matroid union by embedding both ground sets in a common bigger ground set. We give both definitions for sake of explicitness.

**Example 5.1.7.** *Given a matroid on some ground set, it is often useful to extend that ground set to a larger ground set by adding coloops, elements contained in all bases. The same construction can be generalized to valuated matroids in the following way.*

*Let* $f$ *be a valuated matroid on ground set* $V$, *and* $W$ *a disjoint set from* $V$. *We define the* free valuated matroid $fr_W$ *on* $W$ *to take the value 0 on* $W$ *and* $-\infty$ *everywhere else. Then the direct*

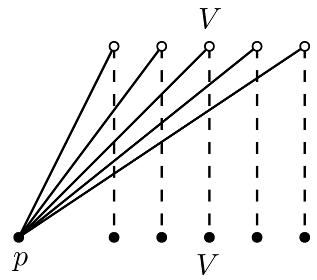


Figure 5.2: Given a valuated matroid $f$ on $V$ and $(w \in \mathbb{R} \cup \{-\infty\})^V$, the principal extension $f^w$ is realized as the induction of $f$ via the above bipartite graph. The dashed edges are weighted zero, while the solid edges $(p, v)$ are weighted $w_v$.

*sum of $f$ with $fr_W$ is given by*

$$(f \oplus fr_W)(X) = \begin{cases} f(Y) & X = Y \cup W \\ -\infty & \textit{otherwise} \end{cases}.$$

*In particular, note that $f = (f \oplus fr_W)/W$. This construction of adding coloops to a valuated matroid will be useful throughout.*

### 5.1.1 Induction by networks

The next operation is very powerful and can be seen as a vast generalization of Rado's theorem (Theorem 6.1.2). Somewhat surprisingly, we show that it can be modelled by the basic operations defined so far.

**Definition 5.1.8.** *Let $N = (T, A)$ be a directed network with a weight function $c \in \mathbb{R}^A$. Let $V, U \subseteq T$ be two non-empty subsets of nodes of $N$. Let $g$ be a valuated matroid on $U$ of rank $d$. Then the* induction of $g$ by $N$ *is a function $\Phi(N, g, c) \colon \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$. For $X \in \binom{V}{d}$, one sets $\Phi(N, g, c)(X)$ to*

$$\max \left\{ \sum_{a \in \mathcal{P}} c(a) + g(Y) : \textit{ node-disjoint paths } \mathcal{P} \textit{ in } N \colon \partial_V(\mathcal{P}) = X \wedge \partial_U(\mathcal{P}) = Y \right\}.$$

*Note that the maximization can also result in $-\infty$ if there exists no node-disjoint paths from $X$ to a set with finite value. It is even possible that $\mathrm{dom}(\Phi(N, g, c)) = \emptyset$.*

*In the special case that the directed network is bipartite with the edges directed from $V$ to $U$, we can also consider this as an undirected weighted bipartite graph and call the corresponding operation* induction by bipartite graphs*.*

**Theorem 5.1.9** (Special case of [95, Theorem 9.27])**.** *Let $N$, $g$ and $c$ as in Definition 5.1.8. Then if $\Phi(N, g, c) \not\equiv -\infty$ the induced function is a valuated matroid.*

While it is a special case of induction by networks, induction by bipartite graphs is an extremely powerful operation. Many of the operations introduced so far can be modelled using induction by bipartite graphs, which is a key observation in the proof of Theorem 5.1.12.

**Remark 5.1.10.** *One such example is principal extension, which is displayed in Figure 5.2. Explicitly, for a valuated matroid $g$ on ground set $U$ and weight vector $w \in \mathbb{R} \cup \{-\infty\}^U$, let $G = (U' \cup \{p\}, U; E)$ where $U'$ a copy of $U$, the edge set $E$ consists of $(u', u)$ and $(p, u)$ for all $u \in U$, and $c$ the weight function that takes the value zero on $(u', u)$ and $w_u$ on $(p, u)$. Then the principal extension $g^w$ of $g$ with respect to $w$ is precisely $\Phi(G, g, c)$. More details why this holds are provided in Lemma 5.1.15.*

Furthermore, the following lemma shows we can realize induction by a network as induction by a bipartite graph followed by a contraction. Given the power of this operation, it shall be a key construction throughout.

**Lemma 5.1.11.** *Let $N$ be a directed network with weight function $d$ and $g$ a valuated matroid such that $f = \Phi(N, g, d)$ is again a valuated matroid.*

*Then there is a bipartite graph $G$ with weight function $c$, a valuated matroid $h$ and a subset of the nodes of $G$ such that $f = (\Phi(G, h, c))/W$.*

*Proof.* Let $N = (T, A)$ be the weighted directed network such that the valuated matroid $f$ on the subset $V$ of $T$ is the induction of the valuated matroid $g$ on the subset $U$ of $T$ through $N$. Let $W = T \setminus (V \cup U)$ and $W'$ a disjoint copy of $W$. We define the bipartite graph $G = (V \cup W, U \cup W'; E)$ with weight function $c \in \mathbb{R}^E$ where for each arc $(a, b) \in A$, we add the edge $(a, b)$ if $b \in U$ or $(a, b')$ if $b \in W$ to $E$ with weight $d(a, b)$. Furthermore, we add the zero weight edges $(w, w')$ for all $w \in W$ with copy $w'$. An example of this construction is displayed in Figure 5.3.

Let $X \subseteq V$ and $Y \subseteq U$ be subsets of equal cardinality. We observe that node disjoint paths from $X$ to $Y$ in $N$ are in bijection with matchings from $X \cup W$ to $Y \cup W'$ in $G$, and furthermore preserve weights. Let $\mathcal{P}$ be a set of node disjoint paths in $N$, the edges of $G$ corresponding to arcs of $\mathcal{P}$ form a matching of equal weight on a subset of the nodes from $X \cup W$ to $Y \cup W'$. For any nodes $w \in W$ that are not used in $\mathcal{P}$, we add the corresponding zero weight edge $(w, w')$ to the matching: this gives a perfect matching from $X \cup W$ to $Y \cup W'$ of the same weight at $\mathcal{P}$. Conversely, any perfect matching $\mu$ from $X \cup W$ to $Y \cup W'$ gives rise to a set of node disjoint paths by contracting the $(w, w')$ in $G$ for all $w \in W$. This precisely recovers the network $N$ from $G$, and the matching $\mu$ becomes a set of node disjoint paths from $X$ to $Y$ in $N$.

We let $h$ be the valuated matroid $g \oplus \mathrm{fr}_{W'}$ as defined in Example 5.1.7. Consider $f(X)$ for some $X \subseteq V$. As node disjoint paths from $X$ in $N$ are bijection with matchings on

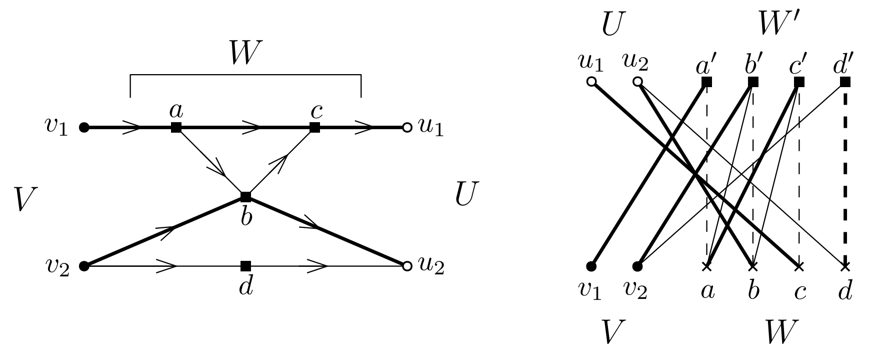


Figure 5.3: An example of the construction from Lemma 5.1.11: a network $N$ and the corresponding bipartite graph $G$. A set of node-disjoint paths in $N$ correspond to a matching in $G$, both displayed in bold.

$X \cup W$ in $G$, we can replace $N$ with $G$ in the definition of $f$:

$$f(X) = \max\left\{\sum_{e\in\mu} c(e) + g(Y) : \text{ matching } \mu \text{ in } G \wedge \partial_{V\cup W}(\mu) = X \cup W \wedge \partial_{U\cup W'}(\mu) = Y \cup W'\right\}.$$

Furthermore, by definition of $h$ we can replace $g(Y)$ with $h(Y \cup W')$ in the above equation. This implies that $f(X) = \Phi(G, h, c)(X \cup W)$; furthermore this holds for arbitrary $X$ and so $f = \Phi(G, h, c)/W$. □

We end this section by showing that valuated matroids are closed under all the operations introduced so far.

**Theorem 5.1.12.** *The class of valuated matroids is closed under the operations deletion, contraction, dualization, truncation, principal extension, direct sum, matroid union.*

To prove the lemma we use several lemmas.

**Lemma 5.1.13.** $f/Y = (f^* \setminus Y)^*$

*Proof.* At first, observe that the independence of $Y$ in $\operatorname{dom}(f)$ implies that it is contained in a basis. Hence, $V - Y$ has full rank in $\operatorname{dom}(f^*) = \operatorname{dom}(f)^*$ and we can actually apply the deletion operation.

Let $X \in \binom{V-Y}{d-|Y|}$. Then, as the codomain of $(f^* \setminus Y)$ is $V - Y$, we get $(f^* \setminus Y)^*(X) = (f^* \setminus Y)(V - (Y \cup X))$. Note that $X$ and $Y$ are disjoint by definition. Furthermore, from $V - (Y \cup X) \subseteq V - Y$ we obtain $(f^* \setminus Y)(V - (Y \cup X)) = f^*(V - (Y \cup X))$. Since the codomain of $f^*$ is $V$, this yields $f^*(V - (Y \cup X)) = f(X \cup Y)$. □

**Lemma 5.1.14.** $f^{(1)} = f^{\mathbf{0}}/\{p\}$, *where* $\mathbf{0}$ *is the zero vector and* $p$ *is the element added in the principal extension.*

*Proof.* As $p$ is not a loop of $\operatorname{dom}(f^{\mathbf{0}})$ one can form this contraction and $\operatorname{rk}(\{p\}) = 1$. Now the claim follows directly from the definition of contraction and truncation. □

The valuated truncation is further studied in [94]. It is shown that this actually gives rise to a valuation on all independent set such that this forms a generalized valuated matroid.

**Lemma 5.1.15.** *Let $G = (V, U; E)$ be a bipartite graph with weight function $c \in \mathbb{R}^E$ and $g$ be a valuated matroid on $U$. Then $\Phi(G, g, c) = ((\dots(g^{c_1})\dots)^{c_{|V|}}) \setminus U$, where $c_i \in (\mathbb{R} \cup \{-\infty\})^U$ is the function $c$ restricted to the edges incident with $i \in V$ extended with value $-\infty$ where it is not defined. Furthermore, these principal extensions commute.*

*Proof.* The claim follows by induction. We start with the bipartite graph $G_0 = (U', U; E_0)$ where $U'$ is a copy of $U$, and $E_0$ consists of edges $(u', u)$ between copies of elements. Furthermore the weight function $d_0$ takes the value zero on all elements of $E_0$.

We inductively define $G_i = (V_i, U; E_i)$ where $V_i = U' \cup \{1, \dots, i\}$ by adding the node $i \in V$ to $G_{i-1}$ with edges $(i, u)$ for all $u \in U$. Furthermore the weight function $d_i$ takes the value of $d_{i-1}$ for all edges in $E_{i-1}$, and the value $c_{iu}$ on the new edges $(i, u)$. These graphs are displayed in Figure 5.4. We claim that $\Phi(G_i, g, d_i) = (\dots(g^{c_1})\dots)^{c_i}$.

Note that for the base case, we have that $\Phi(G_0, g, d_0) = g$, as all edges in $G_0$ have weight zero.

For the general case, consider $\Phi(G_i, g, d_i)$ and let $X$ be a $d$-subset of $V_i = U \cup \{1, \dots, i\}$. If $i \notin X$, then $\Phi(G_i, g, d_i) = \Phi(G_{i-1}, g, d_{i-1})$ as the graphs $G_i$ and $G_{i-1}$ are the same outside of node $i$. If $X = i \cup Y$, then

$$\begin{aligned}
\Phi(G_i, g, d_i)(X) &= \max\left(\sum_{(k,v)\in\mathcal{P}} d_i(k, v) + g(Z) \;\middle|\; \partial_{V_i}(\mathcal{P}) = X\,,\ \partial_U(\mathcal{P}) = Z\right) \\
&= \max\left(c(i, u) + \sum_{(k,v)\in\mathcal{P}'} d_i(k, v) + g(Z' \cup u) \;\middle|\; \partial_{V_i}(\mathcal{P}') = Y\,,\ \partial_U(\mathcal{P}) = Z'\right) \\
&= \max_{u\in V_i\setminus Y} \left(c_{iu} + \Phi(G_{i-1}, g, d_i)(Y \cup u)\right)\,.
\end{aligned}$$

Note that for $u \notin U$, we define $c_{iu} = -\infty$, therefore this maximum will only be achieved for some $u \in U$ unless no matching $\mathcal{P}$ exists. This is precisely the principal extension of $\Phi(G_{i-1}, g, d_{i-1})$ with respect to $c_i$. By the inductive hypothesis, this implies $\Phi(G_i, g, d_i) = (\dots(g^{c_1})\dots)^{c_i}$.

The final observation is that $G$ is obtained from the graph $G_V$ by deleting the copy of $U$ that shares no edges with $V$. As they share no edges, deleting these nodes is equivalent to deletion on the level of valuated matroids, therefore $\Phi(G, g, c) = \Phi(G_V, g, d_V) \setminus U$.

Finally, we note that as elements of $V$ share no edges, we can inductively build the graph $G_V$ by adding nodes in any order. On the level of valuated matroids, this implies the principal extensions commute. □

Let $V_1$ and $V_2$ be the respective (not necessarily disjoint) ground sets for the valuated

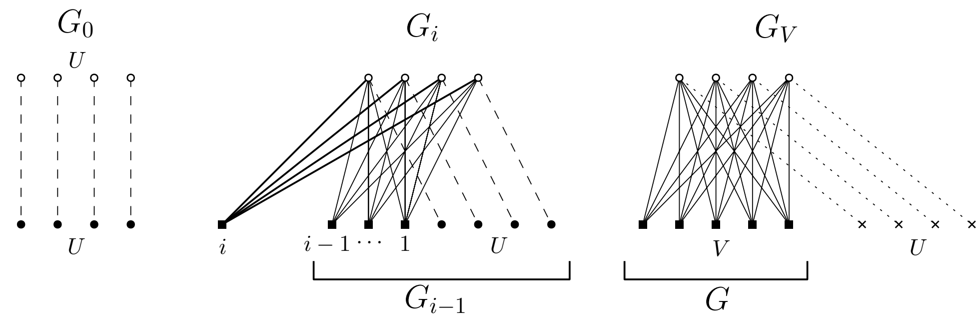


Figure 5.4: The inductive construction of graphs corresponding to principal extension from Lemma 5.1.15.

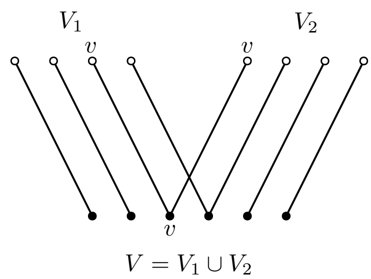


Figure 5.5: The graph $G$ that induces the union $f_1 \vee f_2$, as described before Lemma 5.1.16.

matroids $f_1$ and $f_2$ with ranks $d_1$ and $d_2$ and let $V = V_1 \cup V_2$. We define a bipartite graph $G = (V, V_1 \dot\cup V_2; E)$ where one colour class is $V$ and the other colour class is the disjoint union of copies of $V_1$ and $V_2$. The edge set $E$ consists of edges $(v, v)$ connecting a node to any of its copies, all weighted zero by weight function $c$; in particular, a node of $V$ has degree two if and only if it represents an element in $V_1 \cap V_2$. This graph is displayed in Figure 5.5.

**Lemma 5.1.16.** *The union $f_1 \vee f_2$ can be written as an induction $\Phi(G, f_1 \oplus f_2, c)$.*

*Proof.* Any matching $M$ such that $\partial_V(M) = X$ corresponds to a decomposition $X = X_1 \dot\cup X_2$ where $X_i \subseteq V_i$. Therefore

$$\Phi(G, f_1 \oplus f_2, c)(X) = \max\left\{(f_1 \oplus f_2)(X) : X_1 \in \binom{V_1}{d_1},\ X_2 = X \setminus X_1 \in \binom{V_2}{d_2}\right\},$$

which is precisely the definition of $f_1 \vee f_2$. □

*Proof of Theorem 5.1.12.* Deletion, dualization and direct sum are all covered by [95, Theorem 6.13], parts (6), (2) and (8) respectively. Lemma 5.1.13 implies closure under contraction. Lemma 5.1.16 and Remark 5.1.10 show matroid union and principal extension are special cases of induction by networks, which valuated matroids are closed under via Theorem 5.1.9. Finally, Lemma 5.1.14 implies closure under truncation. □

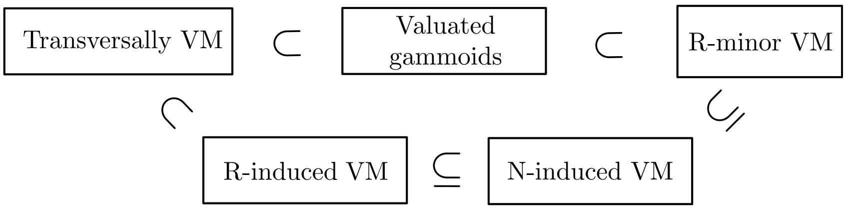


Figure 5.6: The inclusion relationship between classes of valuated matroids.

## 5.2 Classes of valuated matroids

In the following, we consider certain classes of valuated matroids that arise naturally in combinatorial optimization.

(i) The class of *transversally valuated matroids* are those valuated matroids arising from trivially valuated free matroids by induction via a bipartite graph.

(ii) The class of *valuated gammoids* are contractions of those valuated matroids arising from trivially valuated free matroids by induction via a bipartite graph.

(iii) The class of *R-induced valuated matroids* are those valuated matroids arising from trivially valuated matroids by induction via a bipartite graph.

(iv) The class of *N-induced valuated matroids* are those valuated matroids arising from trivially valuated matroids by induction via a network.

(v) The class of *R-minor valuated matroids* are those valuated matroids arising as contractions of R-induced valuated matroids.

Transversally valuated matroids are essentially the layers of assignment valuations. They were extensively studied in [47], which also considered the class of valuated strict gammoids, a subclass of valuated gammoids.

The inclusion relationship between these classes is shown in Figure 5.6. These are laid out in the following example and lemmas.

**Example 5.2.1.** *Consider the valuated matroid on six elements of rank two that takes the value* $-\infty$ *on* $\{12, 34, 56\}$*, and* $0$ *on all other pairs of elements. This valuated matroid, referred to as the "Snowflake": in particular it is not a transversally valuated matroid as shown in [47, Example 3.10]. However, it is both a valuated gammoid and an R-induced valuated matroid, as given by the representations in Figure 5.7.*

**Lemma 5.2.2.** *The class of valuated gammoids forms a strict subclass of R-minor valuated matroids.*

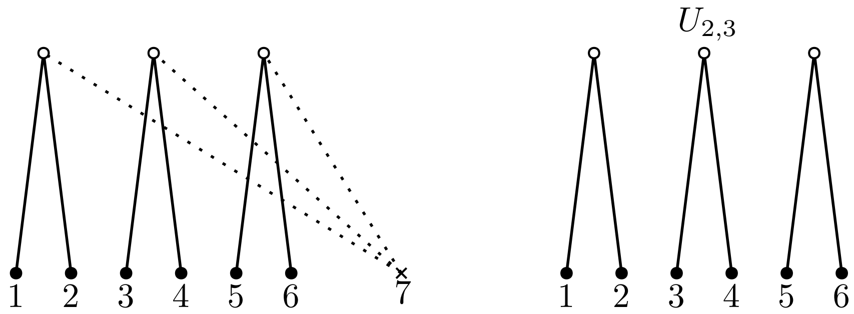


Figure 5.7: Two representations of the Snowflake, defined in Example 5.2.1. The left is a valuated gammoid representation, where the element 7 is contracted. The right is an R-induced representation with induced matroid $U_{2,3}$. All edges are weighted zero.

*Proof.* Containment is given by Theorem 5.2.6. By [104, Lemma 1], valuated gammoids are *strictly base-orderable*. However, any trivially valuated matroid that is not strictly base-orderable is an R-induced valuated matroid, giving strict containment. □

**Lemma 5.2.3.** *The class of R-induced valuated matroids forms a subclass of N-induced valuated matroids and a subclass of R-minor valuated matroids. Furthermore, N-induced valuated matroids form a subclass of R-minor valuated matroids.*

*Proof.* The inclusion of R-induced within N-induced and R-minor are immediate from definition. Furthermore, Lemma 5.1.11 shows how to represent an N-induced valuated matroid as an R-minor valuated matroid. □

The strictness of the inclusion between N-induced valuated matroids and R-minor valuated matroids remains unresolved. While this is reminiscent of the strict inclusion of transversal matroids within gammoids, the authors don't have a proof at hand for the valuated case. From an algorithmic point of view, it would be desirable for N-induced valuated matroids to exhibit concise representations in the spirit of the small representation of gammoids in [83]; see [110, Section 39.4a] for more on transversal matroids and their contractions, the gammoids.

**Conjecture 5.2.4.** *Let $N = (T, A)$ be a directed network with a weight function $c \in \mathbb{R}^A$. Let $V, U \subseteq T$ be two non-empty subsets of nodes of $N$. Let $g$ be a valuated matroid on $U$ of rank $d$.*

*Then there is a directed network $N'$ containing $U$ and $V$, and arc weights $c'$ such that $\Phi(N, g, c) = \Phi(N', g, c')$ and such that $|V(N')|$ is polynomial in $|V|$ and $|U|$.*

*Furthermore, N-induced valuated matroids form a strict subclass of R-minor valuated matroids.*

As we show in Section 6.4, R-induced valuated matroids have a polynomial size representation. However, the information-theoretic argument given does not extend to N-induced and R-minor valuated matroids as it cannot control the size of the contracted set. This suggests that several of the inclusions in Figure 5.6 should indeed be strict.

### 5.2.1 Complete Classes

**Definition 5.2.5** (Complete class)**.** *Let $\mathcal{V}$ be a subset of the set of valuated matroids. We call $\mathcal{V}$ a* complete class *if it is closed under taking restriction, duals, direct sum and principal extension.*

We extend several results in [19] from unvaluated to valuated matroids.

**Theorem 5.2.6.** *A complete class of valuated matroids is closed under taking contraction, truncation, induction by bipartite graphs, induction by directed graph and valuated union.*

*Furthermore, valuated gammoids forms the smallest complete class. Hence, they are contained in all complete classes.*

*Proof.* The points follow from Lemma 5.1.13, Lemma 5.1.14, Lemma 5.1.15, Lemma 5.1.16 and Lemma 5.1.11.

A non-empty complete class must contain the free matroid on one element. By taking iterated direct sum, this yields all free matroids. Then closure under induction by bipartite graphs and minors yields valuated gammoids. □

## 5.3 R-minor valuated matroids

The classes of valuated matroids discussed in the beginning of this section arising from induction through a network may only be induced by trivially valuated matroids. As discussed in Example 5.1.3, a trivially valuated matroid $g$ can be identified with its underlying matroid $\mathcal{M}$, where $g(X)$ takes the value zero on bases of $\mathcal{M}$ and $-\infty$ otherwise. Working with this underlying matroid shall be more convenient much of the time, therefore we extend the notation of Definition 5.1.8 to define $\Phi(N, \mathcal{M}, c) := \Phi(N, g, c)$.

Let $f$ be an R-minor valuated matroid on $V$. By definition, there exists an R-induced valuated matroid $\tilde{f}$ on $V \cup W$ such that $f = \tilde{f}/W$. By definition, there exists some bipartite graph $G = (V \cup W, U; E)$ with edge weights $c \in \mathbb{R}^E$ and matroid $\mathcal{M} = (U, r)$ such that $\tilde{f} = \Phi(G, \mathcal{M}, c)$; we say $\tilde{f}$ has an *R-induced representation* $(G, \mathcal{M}, c)$. As $f = \Phi(G, \mathcal{M}, c)/W$, we extend this notation to say that $f$ has an *R-minor representation* $(G, \mathcal{M}, c, W)$, where $W$ is the set to be contracted.

In the following, we show that R-minor valuated matroids are closed under deletion, principal extension, duality and direct sum, making them a complete class. In the following we shall assume $f$ is an R-minor matroid with representation $(G, \mathcal{M}, c, W)$ as above.

**Lemma 5.3.1.** *For a subset $X \subseteq V$, let $G \setminus X$ be the graph obtained from $G$ by deleting the nodes $X$ and all edges adjacent. The deletion $f \setminus X$ is represented by $(G \setminus X, \mathcal{M}, c, W)$.*

*Proof.* This follows by the definition of deletion. □

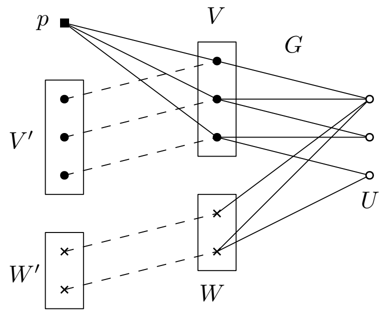


Figure 5.8: The network $N$ constructed from a graph $G$ inducing the principal extension of a R-minor valuated matroid, as described before and within Lemma 5.3.2.

Let $w \in (\mathbb{R} \cup \{-\infty\})^V$ and consider $f^w$. Let $V', W'$ denote copies of $V, W$, and define a network $N = (T, A)$ on the node set $T = (V' \cup W' \cup \{p\}) \cup (V \cup W) \cup U$, where $p$ is a new node. The arc set $A$ weighted by $c' \in \mathbb{R}^A$ consists of arcs $(v', v)$ with weight $0$, where $v' \in V' \cup W'$ denotes the copy of $v \in V \cup W$. We also add arcs $(p, v)$ with weight $w_v$ for all $v \in V$, and arcs $(v, u)$ for all edges $E$ of $G$ with weight inherited by $c$. The constructed network $N$ is displayed in Figure 5.8. This network can intuitively be thought of as the "concatenation" of $G$ with the graph from Remark 5.1.10.

**Lemma 5.3.2.** *The principal extension $f^w$ arises as the contraction of $\Phi(N, \mathcal{M}, c)$ by $W'$. In particular, it can be represented as an R-minor valuated matroid.*

*Proof.* Consider a subset $X \subseteq V \cup \{p\}$, the principal extension $f^w$ is defined as

$$f^w(X) = (\tilde{f}/W)^w(X) = \begin{cases} \max_{v \in V \setminus Y} \left( \tilde{f}(Y \cup v \cup W) + w_v \right) & X = Y \cup \{p\} \\ \tilde{f}(X \cup W) & p \notin X \end{cases} .$$

We claim that $\Phi(N, \mathcal{M}, c')(X' \cup W') = f^w(X)$ for $X' \subseteq V' \cup \{p\}$.
If $p \notin X'$, then the value of $\Phi(N, \mathcal{M}, c')(X' \cup W')$ is simply the maximal independent matching in $G$ to $X \cup W$ with no contribution from the zero edges, i.e. $\Phi(N, \mathcal{M}, c)(X') = \tilde{f}(X \cup W)$. If $X' = Y' \cup \{p\}$, then the value of $\Phi(N, \mathcal{M}, c')(X' \cup W')$ is the maximal independent matching in $G$ to $Y \cup v \cup W$ for some $v \in V \setminus Y$, plus $w_v$ picked up from the arc $(p, v)$, i.e.

$$\Phi(N, \mathcal{M}, c')(X' \cup W') = \max_{v \in V \setminus Y} \left( \tilde{f}(Y \cup v \cup W) + w_v \right) ,$$

which is precisely the value of $f^w$. Therefore $f^w = \Phi(N, \mathcal{M}, c')/W'$. Applying Lemma 5.1.11, we can represent $\Phi(N, \mathcal{M}, c')$ as an R-minor valuated matroid, and therefore also $f^w$. $\square$

Consider the dual valuated matroid $f^*$, we claim it can be represented in the following way. Let $U', V', W'$ be copies of $U, V, W$ respectively. Let $G' = (U \cup V \cup W, U' \cup V' \cup W', E')$

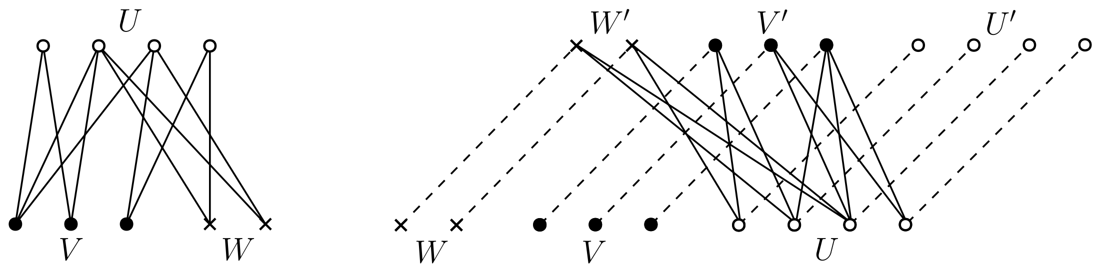


Figure 5.9: Construction of R-minor representation for $f^*$.

whose edge set $E'$ consists of edges

$$E' = \{(v, v') : v \in U \cup V \cup W\} \cup \{(u, v') : (v, u) \in E\} \ .$$

The edge weights are given by $c' \in \mathbb{R}^{E'}$ where $c'(v, v') = 0$ and $c'(u, v') = c(v, u)$. This graph is displayed in Figure 5.9. We also use in our representation the matroid $\mathcal{M}' = \mathcal{M}^* \oplus \mathrm{fr}_{V' \cup W'}$, the direct sum of the dual matroid $\mathcal{M}^* = (U', r^*)$ and the free matroid on $V' \cup W'$.

**Lemma 5.3.3.** *The dual $f^*$ is a R-minor valuated matroid.*

*Proof.* Let $f = \tilde{f}/W$, then its dual is $f^* = (\tilde{f}/W)^* = (\tilde{f})^* \setminus W$ by Lemma 5.1.13. As R-minor valuated matroids are closed under deletion by Lemma 5.3.1, we are done if we can show $(\tilde{f})^*$ is an R-minor valuated matroid. We claim that $(\tilde{f})^*$ is represented by $(G', \mathcal{M}', c', U)$.

Fix some $X \subseteq V$, we shall compute $\Phi(G', \mathcal{M}', c')(X \cup U)$. First observe that $v \in X$ can only be matched to $v' \in X'$ with weight zero, and that there are no matroid constraints on these edges. Therefore the rest of the matching is an independent matching from $U$ to $(U \cup V' \cup W') \setminus X'$. For any independent matching, $Y \subseteq U$ matches to $(V' \cup W') \setminus X'$ if and only if $U' \setminus Y'$ is independent in $\mathcal{M}*$, which by matroid duality only occurs when $Y$ is independent in $\mathcal{M}$. Therefore all independent matchings are of the form

$$\{(u, v') : (v, u) \in \mu\} \cup \{(v, v') : v \in X \cup (U \setminus Y)\}$$

where $\mu$ is an independent matching in $G$ from $(V \cup W) \setminus X$ to $Y \subseteq U$. As the weights of these edges are either $0$ or inherited from $G$, we have

$$\Phi(G', \mathcal{M}', c')(X \cup U) = \tilde{f}((V \cup W) \setminus X) = (\tilde{f})^*(X) \,,$$

implying that $(\tilde{f})^* = \Phi(G', \mathcal{M}', c')/U$ as claimed. As $U, W$ are disjoint, contracting and/or deleting them commute and so $f^*$ has the representation $(G' \setminus W, \mathcal{M}', c', U)$; the same representation as $(\tilde{f})^*$, but with $W$ deleted from $G'$. $\square$

**Lemma 5.3.4.** *Let $f_1$ and $f_2$ be two R-minor valuated matroids represented by $(G_1, \mathcal{M}_1, c_1, W_1)$ and $(G_2, \mathcal{M}_2, c_2, W_2)$. Then $f_1 \oplus f_2$ is represented by $(G', \mathcal{M}_1 \oplus \mathcal{M}_2, c', W_1 \cup W_2)$, where $G'$ and its weight function $c'$ arises by taking the union of the weighted graphs $G_1$ and $G_2$.*

*Proof.* This just follows from the definitions. □

**Theorem 5.3.5.** *The set of R-minor valuated matroids forms a complete class of valuated matroids.*

*Proof.* This follows directly from Lemmas 5.3.1, 5.3.2, 5.3.3 and 5.3.4. □

# 6 Non R-minor valuated matroids

Our goal is to show that there are valuated matroids that are not R-minor. Recall that a function $f : \binom{V}{d} \to \mathbb{R}$ is an R-minor valuated matroid valuated matroid if there exists a bipartite graph $G = (V \cup W, U; E)$ with edge weights $c \in \mathbb{R}^E$, and a matroid $\mathcal{M}$ on $U$ of rank $d + |W|$, such that: the value $f(X)$ is the maximum weight of a matching in $G$ whose endpoints in $V \cup W$ are $X \cup W$; and the endpoints in $U$ form a basis in $\mathcal{M}$.

In particular, we will show that none of the valuated matroids in $\mathcal{F}_n$ are not R-minor (Theorem 1.3.3). We recall the definition of $\mathcal{F}_n$.

**Definition 1.3.2.** *For $n \geq 2$, we define $\mathcal{F}_n$ as the following family of functions $\binom{[2n]}{4} \to \mathbb{R}$. Let $V = [2n]$, $P_i = \{2i-1, 2i\}$ for $i \in [n]$, and let*

$$\mathcal{H} = \{P_i \cup P_j : ij \equiv 0 \mod 2\} \qquad (\mathcal{H}\text{-def})$$

*i.e. we take pairs such that at least one of $i, j$ is even. Let $X^* = P_1 \cup P_2 = \{1, 2, 3, 4\}$. A function $h : \binom{V}{4} \to \mathbb{R} \cup \{-\infty\}$ is in the family $\mathcal{F}_n$ if and only if the following hold:*

- *$h(X) = 0$ if $X \in \binom{V}{4} \setminus \mathcal{H}$,*
- *$h(X) < 0$ if $X \in \mathcal{H}$, and*
- *$h(X^*)$ is the unique largest nonzero value of the function.*

The proof that functions in $\mathcal{F}_n$ are valuated matroids follows from simple case analysis given in Appendix 6.3.1. We prove that no function in $\mathcal{F}_n$ arises as an R-induced minor function in Section 6.3; the proof uses several lemmas on Rado representations of matroids given in Section 6.1, and lemmas on the LP representation of R-minor valuated matroids given in Section 6.2. An overview of this proof is given below.

## A guide for the core proof

We now give an overview of a complex technical argument in Section 6.3 showing that functions in $\mathcal{F}_n$ (Definition 1.3.2) are not R-minor. Recall that the domain of each of these functions contains $B_0 := \binom{V}{4} \setminus \mathcal{H}$. We reduce the study of the family to the combinatorial structure and Rado representations of the matroids $B_0$ and the domain of the function. To achieve this, we use a canonical linear programming formulation and the submodularity of the rank of the neighbourhood function arising in the bipartite graph of an R-minor representation. Finally, we impose several extremality assumptions on a potential representation which we exploit by applying local modifications. Now, we elaborate on these steps.

The first main ingredient is the linear programming dual of the R-minor representation (Section 6.2). Let $h$ be a function of rank $d$ on $V = [2n]$ represented by a bipartite graph $G = (V \cup W, U; E)$, matroid $\mathcal{M} = (U, r)$, and weights $c \in \mathbb{R}^E$, such that $r(\mathcal{M}) = |W| + d$. (In our construction, $d = 4$.) The maximum weight independent matching problem of size $|W| + d$ can be formulated as a linear program. The dual program has variables $\pi \in \mathbb{R}^{V \cup W}$ and $\tau \in \mathbb{R}^U$ that form a vertex cover, i.e. $\pi_i + \tau_j \geq c_{ij}$ for every edge $(i, j) \in E$. The objective can be equivalently written as $\min \pi(V \cup W) + \hat{r}(\tau)$, where $\hat{r}$ is the Lovász-extension of $r$, i.e., $\hat{r}(\tau)$ is the maximum $\tau$-weight of any basis.

Note that for $h \in \mathcal{F}_n$ the maximum is 0 and the set of maximizers equals $B_0$. The optimality criteria of the LP are as follows: let $E_0 \subseteq E$ denote the tight edges ($\pi_i + \tau_j = c_{ij}$) and $\mathcal{M}_\tau$ the matroid formed by the maximum $\tau$-weight bases. Then, for $X \subseteq V$ with $|X| = 4$, we have $X \in B_0$ if and only if $W \cup X$ has an independent matching to a base in $\mathcal{M}_\tau$ using edges in $E_0$ only. We also let $E^* \subseteq E$ denote the union of all maximum weight independent matchings. By complementary slackness, $E^* \subseteq E_0$ for any dual optimal solution.

A key proof strategy is to work with the purely combinatorial structure of Rado-minor representations of two matroids: the one with bases $B_0$ and the larger one with bases $B_1$, where $B_1 := \mathrm{dom}(h)$ is the *effective domain*, i.e., where $h(X) > -\infty$. For $B_0$, this means that $X \in B_0$ if and only if there is a matching between $X \cup W$ and a basis of $\mathcal{M}_\tau$ using edges from $E_0$; for $B_1$, we use the matroid $\mathcal{M}$ and edge set $E$ instead. Note that the edge weights $c$ are not used in these representations. We review the necessary concepts and results in Section 6.1.

We fix $n \geq 16$, and prove by contradiction that no function in $h \in \mathcal{F}_n$ can be represented. We carefully select a counterexample that satisfies certain minimality criteria. Most importantly, we require that *(a)* $B_1$ is minimal; subject to this, that *(b)* the contracted set $|W|$ is minimal, and finally, that *(c)* $|E \setminus E^*|$ is minimal. From these, we can easily deduce that one of two main cases (Lemma 6.3.4):

(CI) There exist a dual optimal solution $(\pi, \tau)$ such that $E = E_0 \cup \{(i', j')\}$ for an edge $(i', j')$, $\mathcal{M}_\tau = \mathcal{M}$, and $B_1 = B_0 \cup \{X^*\}$.

(CII) $E = E^*$ and $\mathcal{M}_\tau \neq \mathcal{M}$ for any dual optimal solution $(\pi, \tau)$.

Thus, in case (CI), all bases in $\mathcal{M}$ have the same $\tau$-weight, and there is a single non-tight edge. Further, $h(X^*)$ is the only finite value outside $B_0$. In contrast, in case (CII), all edges are tight, but we need to work with two different matroids on $U$.

We now explain the proof for the base case $W = \emptyset$, i.e., that $h$ is not R-induced (Section 6.3.3). (This can be alternatively proved by an information-theoretic argument as in Section 6.4.) We show that case (CI) must apply. Otherwise, $\mathcal{M}_\tau \neq \mathcal{M}$, in which case one can show that $B_0$ is *fully reducible*, that is, it can be written as a full-rank matroid union of smaller matroids (Lemma 6.1.8). In Lemma 6.3.7, we show that this is not the case for $B_0$, exploiting the combinatorics of the pairs $P_i$ in the construction.

To complete the proof of the base case $W = \emptyset$, we note that the set $X^* = P_1 \cup P_2$ does not have an independent matching in $E_0$ but has one in $E_1 = E_0 \cup \{(i', j')\}$. Hence, $(i', j')$ is incident to $X^*$; say, $i' \in P_1$. With an uncrossing argument using the submodularity of the rank of the neighbourhood function, we show that $(i', j')$ should create an independent matching also for another set $X = P_1 \cup P_k \notin B_0$. Since $\mathcal{M} = \mathcal{M}_\tau$ and this is the single non-tight edge, it follows that $0 > h(Z) \geq h(X^*)$, a contradiction that $h(X^*)$ is the unique largest negative function value.

In Section 6.3.4 we analyze Rado-representation of *robust* matroids: a common generalization of $B_0$ and $B_0 \cup \{X^*\}$, sparse paving matroids with elements arranged in pairs $P_i$. It turns out that the structure of the pairs $P_i$ forces itself on the full representation; in particular, for each pair $P_i$ there exists a unique largest 'extension set' $Z_i \subseteq V \cup W$ such that $Z_i \cap V = P_i$, and these are tight with respect to Rado's condition. Moreover, the $Z_i$'s are pairwise disjoint, and and encode all relevant information of the robust matroid, $B_0$ or $B_1$. The structural analysis is based on careful uncrossing arguments of the rank of the neighbourhood function in the Rado-representation.

In Section 6.3.5, we apply this structure to first show that $B_1 = B_0 \cup \{X^*\}$, that is, in the first selection criterion, $\mathrm{dom}(h) = B_1$ is as small as it can be. We also show that the sets $Z_i^0$ and $Z_i^1$—obtained for each pair $P_i$ from the robust matroid analysis for $B_0$ and $B_1$—are closely related: $Z_i^0 = Z_i^1 \cup Q_0$ for a certain set $Q_0$. Both cases (CI) (Section 6.3.6) and (CII) (Section 6.3.7) can be derived by exposing the discrepancy between two near-identical representations of two near-identical (yet different) matroids.

## 6.1 Rado representation of matroids

In this section, we specialize R-induced and R-minor representation to matroids without valuation. This means that the bipartite graph has only zero weights and the starting valuated matroid has only trivial valuation. Then the construction boils down to well-known results in matroid theory. This allows us to deduce strong structural statements on these representations in Section 6.1.1.

**Definition 6.1.1** (Rado representation)**.** *Let $G = (V, U; E)$ be a bipartite graph and $\mathcal{M} = (U, r_{\mathcal{M}})$ be a matroid. We define a matroid $\mathcal{N}$ on $V$ as follows. A set $X \subseteq V$ is independent in $\mathcal{N}$*

*if there exists $S \subseteq U$ such that there is a perfect matching in the subgraph induced by $(X, S)$ and $S$ is independent in $\mathcal{M}$. We say that $(G, \mathcal{M})$ is* Rado representation *of $\mathcal{N}$.*

The following theorem verifies that this construction indeed defines a matroid, and characterizes its rank function.

**Theorem 6.1.2** (Rado's theorem [107, 105])**.** *Let $\mathcal{N}$ be as in Definition 6.1.1. Then $\mathcal{N}$ is a matroid. Moreover, a set $X \subseteq V$ is independent in $\mathcal{N}$ if and only if $r_{\mathcal{M}}(\Gamma(Y)) \geq |Y|$ for all $Y \subseteq X$. If a set $X \subseteq V$ is a circuit in $\mathcal{N}$, then $r_{\mathcal{M}}(\Gamma(Y)) = |Y| - 1$.*

A more general representation can be obtained as minors of the above.

**Definition 6.1.3** (Rado-minor representation)**.** *Let $G = (V \cup W, U; E)$ be a bipartite graph and $\mathcal{M} = (U, r_{\mathcal{M}})$ be a matroid. We define a matroid $\mathcal{N}$ on $V$ as follows. A set $X \subseteq V$ is independent in $\mathcal{N}$ if there exists $S \subseteq U$ such that there is a perfect matching in the subgraph induced by $(X \cup W, S)$ and $S$ is independent in $\mathcal{M}$. We say that $(G, \mathcal{M}, W)$ is* Rado-minor representation *of $\mathcal{N}$.*

**Proposition 6.1.4.** *Let $\mathcal{N}$ be as in Definition 6.1.3. Then $\mathcal{N}$ is a matroid. Moreover, $X \subset V$ is independent in $\mathcal{N}$ if and only if for all $Z \subseteq X \cup W$ it holds $r_{\mathcal{M}}(\Gamma(Z)) \geq |Z|$.*

*Proof.* Consider $G, W$ and $\mathcal{M}$ as in Definition 6.1.4. Then, let $\mathcal{N}'$ be the matroid on $V \cup W$ with Rado representation $(G, \mathcal{M})$. It is easy to see that $\mathcal{N}$ can be obtained by contracting $W$ in $\mathcal{N}'$. The proposition follows. $\square$

Any matroid that has no independent sets other than the empty set is said to be an *empty matroid*. We next introduce some basic matroidal notions, and present their properties in the context of Rado representations. Note that definitions of *matroid sum* and *matroid union* are specializations of the operation *direct sum* and *valuated matroid union* to trivially valuated matroids, see Definition 5.1.6. We state them here for clarity.

**Definition 6.1.5** (Matroid sum, Disconnected)**.** *Let $U_1, \ldots, U_k$ be disjoint sets. For $i \in [k]$ let $B_i$ be the bases of matroid $\mathcal{M}_i$ on $U_i$. We define the* matroid sum *$\mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ as a matroid $\mathcal{M}$ on $U = \dot\bigcup_{i=1}^{k} U_i$ with bases $B = \{\dot\bigcup_{i=1}^{k} B_i : B_i \in B_k\}$. We say that a matroid $\mathcal{M}$ is* disconnected *if and only if it is a* matroid sum *of at least two non-empty matroids. A matroid is* connected *if it is not* disconnected*.*

**Definition 6.1.6** (Matroid union, Fully reducible)**.** *For $i \in [k]$ let $B_i$ be the bases of matroid $\mathcal{M}_i$ on $U$. We define the* matroid union *$\mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ as a matroid $\mathcal{M}$ on $U$ with bases $B = \{\cup_{i=1}^{k} B_i : B_i \in B_k\}$.*

*We say that a matroid $\mathcal{M}$ is* reducible *if and only if it is a* matroid union *of at least two non-empty matroids. Further, $\mathcal{M}$ is* fully reducible*, if $\mathcal{M} = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ for non-empty matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$, $k \geq 2$, and $r(\mathcal{M}) = \sum_{i=1}^{k} r(\mathcal{M}_i)$. Given the latter condition, this is a* full-rank matroid union *[29].*

We will use the rank formula of matroid union, see e.g. [49, Theorem 13.3.1].

**Theorem 6.1.7** (Edmonds and Fulkerson, 1965)**.** *Consider the matroid union $\mathcal{M} = \mathcal{M}_1 \vee \cdots \vee \mathcal{M}_k$ for matroids $\mathcal{M}_1, \ldots, \mathcal{M}_k$, $k \geq 2$ on the ground set $U$, and let $r_i$ denote the rank function of the $i$-th matroid. Then for any $X \subseteq U$, the rank $r(X)$ in $\mathcal{M}$ equals*

$$r(X) = \min \left\{ \sum_{i=1}^{k} r_i(Z) + |X \setminus Z| : Z \subseteq X \right\} .$$

*Consequently, if $X$ is a circuit in $\mathcal{M}$ then $\sum_{i=1}^{k} r_i(X) = |X| - 1$.*

**Lemma 6.1.8.** *Let $\mathcal{N}$ be a matroid with a Rado-representation $(G, \mathcal{M})$, where $G = (V, U; E)$ and $\mathcal{M} = (U, r)$. Assume that $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_k$ for matroids $\mathcal{M}_i = (U_i, r_i)$, and $\Gamma(V) \cap U_i \neq \emptyset$ for every component of $\mathcal{M}$. Then, $\mathcal{N}$ is reducible. Further, if $r_{\mathcal{N}}(V) = r_{\mathcal{M}}(U)$, then $\mathcal{N}$ is fully reducible.*

*Proof.* Let $\mathcal{N}_i$ be the matroid with Rado representation $(G_i, \mathcal{M}_i)$, where $G_i = (V, U_i; E_i)$ and $E_i$ is the set of edges between $V$ and $U_i$. It follows from definitions that $\mathcal{N} = \mathcal{N}_1 \vee \ldots \mathcal{N}_k$. By the assumption, $\Gamma(V) \cap U_i \neq \emptyset$ for each component, hence each $\mathcal{N}_i$ is a non-empty matroid. The second part is immediate. □

### 6.1.1 Uncrossing properties for Rado-minor representation

We now present some technical statements for Rado-minor representations that will be used in the proof of Theorem 1.3.3. Consider a matroid $\mathcal{N}$ on ground set $V$ with Rado-minor representation $(G, \mathcal{M}, W)$ where $G = (V \cup W, U; E)$ and $\mathcal{M} = (U, r)$.

For a subset $X$ of the ground set $V$ of $\mathcal{N}$, we say that $Z \subseteq V \cup W$ is an $X$-set if $Z \cap V = X$. For $Z \subseteq V \cup W$, let

$$\rho(Z) := r(\Gamma(Z)) - |Z| .$$

For an $X$-set $Z$, we give lower bounds on $\rho(Z)$ depending on the independence of $X$ in $\mathcal{N}$. Throughout, we will use $X, Y$ for subsets of $V$; and $Z, I, J$ for subsets of $V \cup W$, i.e., $X$-sets for some $X \subseteq V$ are denoted with letters $Z, I, J$.

**Lemma 6.1.9.** *The function $\rho : 2^{V \cup W} \to \mathbb{Z}$ defined above is submodular. Let $X \subseteq V$ and consider any $X$-set $Z$.*

*(ind)* *If $X$ is independent in $\mathcal{N}$, then $\rho(Z) \geq 0$.*

*(cir)* *If $X$ is a circuit in $\mathcal{N}$, then $\rho(Z) \geq -1$. Moreover, in this case there is an $X$-set $Z$ such that $\rho(Z) = -1$.*

*(dep)* *If $X$ is dependent in $\mathcal{N}$ and contains a basis, then $\rho(Z) \geq r(\mathcal{N}) - |X|$.*

*Proof.* Function $\rho$ is the difference of a submodular function $r(\Gamma(.))$ and a modular function $|.|$, and thus submodular. (Function $r(\Gamma(.))$ is submodular, see [105, Lemma 11.2.13].)

(ind) follows immediately from Proposition 6.1.4. Let us show (cir). Using previous, we have $\rho(Z \setminus \{i\}) \geq 0$ for $i \in Z \cap V$. Since the marginal value of any element with respect to $\rho$ is at least $-1$, it follows that $\rho(Z) \geq -1$ by submodularity. As $Z \cap V = X$ is dependent, it must be the case that $\rho(Z') < 0$ for some $Z' \subseteq V \cup W$ with $Z' \cap V = X$. For such $Z'$ we have $\rho(Z') = -1$.

For (dep), let $B \subset Z \cap V$ be a basis. Then, using the monotonicity of $r(\Gamma(.))$ we have

$$\begin{aligned}
\rho(Z) = r(\Gamma(Z)) - |Z| &= r(\Gamma(Z)) - |B \cup (Z \cap W)| - |Z \cap V \setminus B| \\
&\geq r(\Gamma(B \cup (Z \cap W))) - |B \cup (Z \cap W)| - |Z \cap V \setminus B| \\
&= \rho(B \cup (Z \cap W)) - |Z \cap V \setminus B| \,.
\end{aligned}$$

Using (ind) for the basis $B$ and the fact that $B$ has cardinality $r(\mathcal{N})$ yields

$$\rho(Z) \geq 0 - (|Z \cap V| - |B|) = r(\mathcal{N}) - |Z \cap V| \,. \qquad \square$$

**Lemma 6.1.10.** *If* $\rho(I) + \rho(J) = \rho(I \cup J) + \rho(I \cap J)$, *then* $\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)] = \mathrm{cl}[\Gamma(I \cap J)]$.

*Proof.* As $\rho(I) + \rho(J) = \rho(I \cup J) + \rho(I \cap J)$, we have $r(\,\Gamma(I)\,) + r(\,\Gamma(J)\,) = r(\,\Gamma(I \cup J)\,) + r(\,\Gamma(I \cap J)\,)$. Then trivially,

$$r(\,\mathrm{cl}[\Gamma(I)]\,) + r(\,\mathrm{cl}[\Gamma(J)]\,) = r(\,\mathrm{cl}[\Gamma(I \cup J)]\,) + r(\,\mathrm{cl}[\Gamma(I \cap J)]\,)\,. \tag{6.1}$$

On the other hand, we have

$$\begin{aligned}
r(\,\mathrm{cl}[\Gamma(I)]\,) + r(\,\mathrm{cl}[\Gamma(J)]\,) &\geq r(\,\mathrm{cl}[\Gamma(I)] \cup \mathrm{cl}[\Gamma(J)]\,) + r(\,\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)]\,) \\
&\geq r(\,\mathrm{cl}[\Gamma(I) \cup \Gamma(J)]\,) + r(\,\mathrm{cl}[\Gamma(I) \cap \Gamma(J)]\,) \\
&\geq r(\,\mathrm{cl}[\Gamma(I \cup J)]\,) + r(\,\mathrm{cl}[\Gamma(I \cap J)]\,)\,.
\end{aligned}$$

The first inequality follows by submodularity of $r$. The second inequality follows since $r(\,\mathrm{cl}[\Gamma(I)] \cup \mathrm{cl}[\Gamma(J)]\,) = r(\,\mathrm{cl}[\Gamma(I) \cup \Gamma(J)]\,)$ (the previous follows form $\mathrm{cl}[\,\mathrm{cl}[\Gamma(I)] \cup \mathrm{cl}[\Gamma(J)]\,] = \mathrm{cl}[\Gamma(I) \cup \Gamma(J)]$ ) and since $\mathrm{cl}[\Gamma(I) \cap \Gamma(J)] \subseteq \mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)]$. The third inequality follows from $\Gamma(I) \cup \Gamma(J) = \Gamma(I \cup J)$ and since $\Gamma(I \cap J) \subseteq \Gamma(I) \cap \Gamma(J)$.

Thus, by (6.1), we have $r(\,\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)]\,) = r(\,\mathrm{cl}[\Gamma(I \cap J)]\,)$. Now, $\mathrm{cl}[\Gamma(I \cap J)]$ is a closed set that is subset of closed set $\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)]$, and both $\mathrm{cl}[\Gamma(I \cap J)]$ and $\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)]$ have the same rank. Thus, $\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)] = \mathrm{cl}[\Gamma(I \cap J)]$. $\square$

Throughout we shall refer to the following uncrossing lemmas liberally.

**Lemma 6.1.11** (Uncrossing I). *For $X, Y \subseteq V$ let $I, J \subseteq V \cup W$ be any $X$-set and any $Y$-set respectively, and assume $\rho(I) = \rho(J) = 0$. If $X \cup Y$ is independent in $\mathcal{N}$ then,*

$$\rho(I \cap J), \rho(I \cup J) = 0\,.$$

*In particular, if $X = Y$ for an independent set $X$ in $\mathcal{N}$, and $\rho(I) = 0$ for some $X$-set $I$, then there exists a unique largest maximal set $I$ with $\rho(I) = 0$.*

*Proof.* By submodularity, we have $0 = \rho(I) + \rho(J) \geq \rho(I \cap J) + \rho(I \cup J)$. Trivially, $I \cap J$ is an $(X \cap Y)$-set and $I \cap J$ is an $(X \cup Y)$-set. Since both $X \cap Y$ and $X \cup Y$ are independent, we have $\rho(I \cap J), \rho(I \cup J) \geq 0$ by Lemma 6.1.10. The first part follows.

By the first part the family of sets $I$ that are $X$-sets with $\rho(I) = 0$ is is closed under intersection and union. If this family is non-empty then there exists unique largest $X$-set $I$ with $\rho(I) = 0$. □

In other words, the above lemma states that the set of $X$-sets $I$ where $X$ is independent in $\mathcal{N}$ and with $\rho$-value 0 is a lattice over $V \cup W$ with respect to the union and intersection.

By the uncrossing lemma for $X = Y = \emptyset$ and since $\rho(\emptyset) = 0$, we have the following corollary.

**Corollary 6.1.12.** *There exists a unique largest set $Q \subseteq W$ such that $\rho(Q) = 0$.*

**Lemma 6.1.13** (Uncrossing II). *Let $X, Y \subseteq V$ be two different circuits in matroid $\mathcal{N}$ whose union contains a basis. Consider an $X$-set $I$ and a $Y$-set $J$ with $\rho(I), \rho(J) = -1$. Then, we have $\rho(I \cap J) = 0$ and $\rho(I \cup J) = -2$.*

*Proof.* Since $I \cap J$ is a $(X \cap Y)$-set and since $X \cap Y$ is an independent set we have $\rho(I \cap J) \geq 0$. Since $I \cup J$ is a $(X \cap Y)$-set and since $X \cup Y$ contains a basis we have $\rho(I \cup J) \geq -2$. By submodularity we get $-2 = \rho(I) + \rho(J) \geq \rho(I \cap J) + \rho(I \cup J) \geq 0 - 2$. Hence, the equalities $\rho(I \cap J) = 0$ and $\rho(I \cup J) = -2$ hold. □

### 6.1.2 Lovász extension and the matroid of maximum weight bases

**Definition 6.1.14** (Lovász extension). *Let $\mathcal{M} = (U, r)$ be a matroid. The* Lovász-extension *$\hat{r} : \mathbb{R}^U \to \mathbb{R}$ of the rank function $r$ is defined for $\tau \in \mathbb{R}^U$ as the maximum $\tau$-weight of a basis of $\mathcal{M}$.*

For a given $\tau \in \mathbb{R}^U$, the value $\hat{r}(\tau)$ can be calculated by the following well-known characterization, see e.g., [49, Theorem 5.5.5].

Lovász extension is often used more general for any submodular function, and in fact can be used to characterize submodularity. Namely, Lovász extension of a function is convex if and only if the function is submodular. It is a basic building block for many submodular optimization algorithms [90].

**Lemma 6.1.15.** *Let $\mathcal{M} = (U, r)$ be a matroid. For $\tau \in \mathbb{R}^U$, the* Lovász-extension $\hat{r}(\tau)$ *equals*

$$\hat{r}(\tau) = r(U)\tau_{u_n} + \sum_{i=1}^{n-1} r(U_i)(\tau_{u_i} - \tau_{u_{i-1}}),$$

*where we reordered $U = \{u_1, u_2, \ldots, u_n\}$ such that $\tau_{u_1} \geq \tau_{u_2} \geq \ldots \geq \tau_{u_n}$, and $U_i = \{u_1, \ldots, u_i\}$ for all $i \in [n]$.*

In this context, we say that $S \subseteq U$ is a *level set* of $\tau$ if $S = \emptyset$, $S = U$, or $S = U_i$ for some $i \in [n]$ with $\tau_{v_i} > \tau_{v_{i+1}}$. Thus, the level sets of $\tau$ form a chain. Using these level sets we can nicely capture all maximum weight bases in a matroid. The following lemma follows from the greedy algorithm for finding maximum weight bases in a matroid.

**Lemma 6.1.16.** *For a matroid $\mathcal{M} = (U, r)$ and $\tau \in \mathbb{R}^U$, let $\emptyset = S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \ldots S_t \subsetneq S_{t+1} = U$ denote the level sets of $\tau$. Let us define the matroid*

$$\mathcal{M}_\tau := \bigoplus_{\ell=1}^{t+1} \left(\mathcal{M}\big|_{S_\ell}\right)/S_{\ell-1}.$$

*This is the matroid formed by the maximum $\tau$-weight bases of $\mathcal{M}$. That is, a basis $B$ in $\mathcal{M}$ maximizes $\sum_{i\in B} \tau_i$ if and only if $B$ is a basis in $\mathcal{M}_\tau$.*

## 6.2 Linear Programming representation of R-minor functions

R-induced valuated matroids are defined via independent matchings. This can be naturally captured by the following linear program.

Throughout this section, unless stated otherwise, $f$ is an R-minor valuated matroid with representation $(G, \mathcal{M}, c, W)$ given by a bipartite graph $G = (V \cup W, U; E)$, edge weights $c \in \mathbb{R}^E$ and a matroid $\mathcal{M} = (U, r)$.

**Lemma 6.2.1.** *For $X \subseteq V$, $f(X)$ is the objective value of the linear program*

$$\begin{aligned} \max \quad & \textstyle\sum_{(i,j)\in E} c_{ij} x_{ij} \\ \textit{s.t.:} \quad & \textstyle\sum_{j\in U} x_{ij} = \mathbb{1}_{i\in X\cup W} && \forall i \in V \cup W \\ & \textstyle\sum_{i\in V\cup W, j\in S} x_{ij} \leq r(S) && \forall S \subset U \\ & \textstyle\sum_{i\in V\cup W, j\in U} x_{ij} = r(U) \\ & x_{ij} \geq 0 && \forall i \in V\cup W, \forall j \in U\,. \end{aligned} \tag{6.2}$$

*Here, $\mathbb{1}_{i\in Z}$ is the indicator function of the set $Z$, taking value 1 if $i \in Z$ and 0 otherwise.*

*Proof.* The formulation clearly gives a relaxation of the integer program defining the value of $f(X)$. Using the total-dual integrality of polymatroid intersection, see [110, Theorem 46.1 and Corollary 41.12b], the existence of an integer optimal solution $x \in \mathbb{Z}^E$ is guaranteed; see the proof of Lemma 6.2.2 for more detail. By the first set of constraints and since $\sum_{i\in V} x_{ij} \le r(\{j\}) \le 1$ for all $j \in U$, it is clear that $x = \chi_\mu$ for a matching $\mu$. Moreover, it holds $\partial_{V\cup W}(\mu) = X \cup W$ and $\partial_U(\mu)$ is a basis in $\mathcal{M}$. The lemma follows. $\square$

We next characterize the set of maximizers of an R-minor valuated matroid.

**Lemma 6.2.2.** *Let $B$ be the set of maximizers of $f$. Then $B$ corresponds to the set of integral optimal solutions of*

$$\begin{array}{lrll} \max & \sum_{(i,j)\in E} c_{ij}x_{ij} & & \\ \text{s.t.:} & \sum_{j\in U} x_{ij} \le 1 & \forall i \in V & \\ & \sum_{j\in U} x_{ij} = 1 & \forall i \in W & \\ & \sum_{i\in V\cup W, j\in S} x_{ij} \le r(S) & \forall S \subset U & \\ & \sum_{i\in V\cup W, j\in U} x_{ij} = r(U) & & \\ & x_{ij} \ge 0 & \forall i \in V\cup W, \forall j \in U\,. & \end{array} \tag{6.3}$$

*The dual of (6.3) is then*

$$\begin{array}{lll} \min & \pi(V) + \pi(W) + \hat{r}(\tau) & \\ \text{s.t.:} & \pi_i + \tau_j \ge c_{ij} & \forall (i,j) \in E \\ & \pi_i \ge 0 & \forall i \in V \\ & \pi_i - \textit{free} & \forall i \in W \\ & \tau - \textit{free}. & \end{array} \tag{6.4}$$

*Above, $\hat{r}$ is the Lovász extension of the matroid rank function $r$. Let $E_0 = \{(i,j) \in E : \pi_i + \tau_j = c_{ij}\}$ denote the set of tight edges, and $G_0 = (V \cup W, U; E_0)$ the tight subgraph. Let $\emptyset = S_0 \subsetneq S_1 \subsetneq S_2 \subsetneq \dots S_t \subsetneq S_{t+1} = U$ be the level sets of $\tau$ in $U$, and denote with $\mathcal{M}_\tau$ the matroid of maximum weight bases. Let $\mathcal{N}$ be the matroid on $V \cup W$ with bases $\{B \cup W : B \in B\}$. Then, $(G_0, \mathcal{M}_\tau)$ is a Rado representation of $\mathcal{N}$. We have $\pi_i = 0$ for all $i \in V$ for which there is a maximizer set $X \in B$ with $i \notin X$.*

*Further, the optimal solution $(\pi, \tau)$ can be chosen with the following additional properties:*

- *Every level set $S_\ell$, $\ell \in [t+1]$ is a flat in $\mathcal{M}$.*
- *For every $\ell \in [t+1]$, $(S_\ell \setminus S_{\ell-1}) \cap \Gamma_{E_0}(V) \neq \emptyset$.*

*Proof.* Observe that the problem is a special case of matroid intersection. We can define two matroids on the edge set $E$: a partition matroid enforcing that only one edge can be selected incident to every node in $V \cup W$, and a second matroid enforcing that the set of endpoints in $U$ must be independent in $\mathcal{M}$; this can be obtained from $\mathcal{M}$ by replacing every node $u \in U$ by parallel copies corresponding to the edges incident to $u$. By the integrality of polymatroid intersection [110, Theorem 46.1 and Corollary 41.12b], the set $\arg\max\{f(X) : X \subseteq V\}$ corresponds to the set of integral solutions of (6.3).

The dual LP formulation can be easily derived from Frank's weight splitting theorem [49, Theorem 13.2.4], interpreted in this bipartite setting. The Rado representation of $\mathcal{N}$ and the condition on the $\pi_i = 0$ values follow by complementary slackness.

Let us now show that the additional properties can be ensured. Consider the smallest level set $S_\ell$ that is not a flat. Thus, $S_\ell = \{i \in U : \tau_i \geq \lambda\}$ for some $\lambda \in \mathbb{R}$. Let us increase $\tau_j$ to $\lambda$ for every $j \in \mathrm{cl}(S_\ell) \setminus S_\ell$. By definition of the Lovász extension, this does not change the value $\hat{r}(\tau)$; and since we only increase $\tau$, the solution remains feasible. After the change, $\mathrm{cl}(S_\ell)$ replaces $S_\ell$ as a level set. Thus, after at most $|U|$ such changes, we can guarantee that all level sets are flats.

We show that this also implies the final property, i.e. that for every $i \in [t+1]$, there exists a tight edge $(i,j) \in E_0$ with $j \in S_\ell \setminus S_{\ell-1}$. Indeed, if no such edge exists, then we can decrease $\tau_k$ by some positive $\varepsilon > 0$ for every $k \in S_\ell \setminus S_{\ell-1}$ such that $(\pi, \tau)$ remains feasible, and $S_\ell$ remains a level set, i.e. $\tau_k > \tau_{k'}$ for any $k \in S_\ell$, $k' \in S_{\ell+1}$. This decreases $\hat{r}(\tau)$ by $\varepsilon\,(r(S_\ell) - r(S_{\ell-1})) > 0$, a contradiction to optimality. $\square$

Note that as an immediate corollary, the set of maximizers $B$ is a matroid with Rado-minor representation $(G_0, \mathcal{M}_\tau, W)$.

**Lemma 6.2.3.** *Let $f$ be an $R$-*induced *represented by $(G, \mathcal{M}, c)$ and $B$ be the set of maximizers of $f$. Consider a dual optimal solution $(\pi, \tau)$ as in Lemma 6.2.2. If $\tau_i \neq \tau_j$ for some $i, j \in U$, then the matroid on $V$ defined by the bases $B$ is fully reducible.*

*Proof.* By Lemma 6.2.2, $(V, U; E_0)$ and $\mathcal{M}_\tau$ gives a Rado representation of the matroid with bases $B$ (since $W = \emptyset$). For the flats $S_\ell$, $\mathcal{M}_\tau$ is the direct sum of the matroids $\left(\mathcal{M}|_{S_\ell}\right)/S_{\ell-1}$ (Lemma 6.1.16). Since all level sets $S_\ell$ are flats, each matroid $\left(\mathcal{M}|_{S_\ell}\right)/S_{\ell-1}$ is non-empty. If there are more than two terms, then Lemma 6.1.8 implies that $B$ corresponds to a fully reducible matroid. Otherwise, the only flats can be $S_0 = \emptyset$ and $S_1 = U$; consequently, $\tau_i$ is the same for all $i \in U$. $\square$

## 6.3 R-minor functions do not cover valuated matroids

In this section we prove that no function in $\mathcal{F}_n$ arises as R-minor valuated matroid, that is, we prove Theorem 1.3.3. Recall that $\mathcal{F}_n$ (Definition 1.3.2) is a family of valuated matroids defined over ground set $V = [2n]$, and using *pairs* $P_i = \{2i-1, 2i\}$ for $i \in [n]$. We let $\mathcal{H}$ be the set of pairs such that at least one of $i, j$ is even and we let $X^* = P_1 \cup P_2 = \{1,2,3,4\}$. A function $h : \binom{V}{4} \to \mathbb{R} \cup \{-\infty\}$ is in $\mathcal{F}_n$ if and only if the following hold:

- $h(X) = 0$ if $X \in \binom{V}{4} \setminus \mathcal{H}$,
- $h(X) < 0$ if $X \in \mathcal{H}$, and
- $h(X^*)$ is the unique largest nonzero value of the function.

First, we prove that all functions in $\mathcal{F}_n$ are valuated matroids. Then we proceed to the main proof that no function in $\mathcal{F}_n$ is a an R-minor valuated matroid. This is proved by showing that a carefully chosen minimal counterexample does not exist.

### 6.3.1 All functions in $\mathcal{F}_n$ are valuated matroids

**Lemma 6.3.1.** *Let $B_0 = \binom{V}{4} \setminus \mathcal{H}$ and $B_1 = \mathrm{dom}(h)$. Then $B_0$ and $B_1$ are sparse paving matroids.*

*Proof.* Let $J$ be the Johnson graph with nodes $\binom{V}{4}$ with edges $(X, Y)$ if and only if $|X \cap Y| = 3$. By [12, Lemma 8], a set system $U$ forms an independent set of $J$ if and only if $\binom{V}{4} \setminus U$ forms the bases of a sparse paving matroid. As elements of $\mathcal{H}$ can intersect in at most two elements, they form an independent set of $J$ and so $B_0$ is a sparse paving matroid. As $B_1$ is obtained by removing elements of $\mathcal{H}$, it is also a sparse paving matroid. □

**Lemma 6.3.2.** *For every $n \geq 2$, all functions in $\mathcal{F}_n$ are valuated matroids.*

*Proof.* Required to show each $h \in \mathcal{F}_n$ satisfies (1.1b). We consider three cases:

- Let $X, Y \in B_0$ and $i \in X \setminus Y$. By Lemma 6.3.1, the basis exchange axiom holds within $B_0$. Therefore we can find $j \in Y \setminus i$ such that $X \setminus i \cup j, Y \cup i \setminus j$ are both in $B_0$, taking the value zero and satisfying (1.1b).
- Let $X \in B_0, Y \in \mathcal{H}$ without loss of generality. If there exists $j \in Y \setminus X$ such that $X \setminus i \cup j \in B_0$, then $Y \cup i \setminus j$ is also in $B_0$ and we satisfy (1.1b). If such a $j$ does not exist, there cannot be distinct $j_1, j_2 \in Y \setminus X$, else $X \setminus i \cup j_1, X \setminus i \cup j_2$ are both elements of $\mathcal{H}$ and have intersection of cardinality 3, something elements of $\mathcal{H}$ cannot have. Therefore $Y = X \setminus i \cup j$ and so (1.1b) is satisfied with equality.
- Let $X, Y \in \mathcal{H}$ and $i \in X \setminus Y$. As elements of $\mathcal{H}$ can intersect in at most two elements, picking any $j \in Y \setminus X$ to exchange yields two sets in $B_0$ with value zero, satisfying (1.1b). □

**Remark 6.3.3.** *We can extend the above construction of the valuated matroid $h$ to any sparse paving matroid $B$, where $\mathcal{H} = \binom{V}{4} \setminus B$ is the set of circuits of rank 4. The proof of Lemma 6.3.2 generalizes as it only uses the property that elements of $\mathcal{H}$ cannot intersect in three elements, as stated in [106, Lemma 19].*

### 6.3.2 A minimal counterexample

Let us fix a value $n \geq 16$. For a contradiction, let us assume there exists a valuated matroid $h \in \mathcal{F}_n$ that is $R$-minor arising via a bipartite graph $G = (V \cup W, U; E)$, matroid $\mathcal{M} = (U, r)$, and weights $c \in \mathbb{R}^E$. Define

$$B_0 := \binom{V}{4} \setminus \mathcal{H}\,, \qquad B_1 := \mathrm{dom}(h)\,.$$

By Lemma 6.3.1 both $B_0$ and $B_1$ are (sparse) paving matroids. From the definition of $\mathcal{F}_n$, we have $B_0 \cup \{X^*\} \subseteq B_1$. Define

$$E^* = \{(i,j) :\ (i,j) \in \mu \text{ for some independent matching with } c(\mu) = 0\}$$

as the union of all maximum weight independent matchings in $G$.

**Selection criteria for $h$.** Let us select a valuated matroid $h \in \mathcal{F}_n$ that admits an R-minor representation $(G, \mathcal{M}, c, W)$ according to the following criteria:

(S1) The function $h$ has minimal effective domain, that is, $|B_1|$ is minimal.

(S2) Subject to this, $|W|$ is minimal.

(S3) Subject to this, $|E \setminus E^*|$ is minimal.

Note that (S1) only depends on $h$, whereas (S2) and (S3) on also on the representation.

We will refer to this choice as the *minimal counterexample*. This choice is well-defined, since all criteria minimize over non-negative integers. For (S1), note that the extreme case is $B_1 = B_0 \cup \{X^*\}$; a key step in the proof is to show that this must always be the case.

**Dual solutions and the two main cases** We will also select an optimal dual solution $(\pi, \tau)$ to (6.4) in Lemma 6.2.2. Let us introduce some notation; the choice of the particular solution will be specified in Lemma 6.3.4.

Let $E_0 = \{(i,j) \in E :\ \pi_i + \tau_j = c_{ij}\}$ denote the set of tight edges. By complementarity, $E^* \subseteq E_0$ must hold for any optimal dual $(\pi, \tau)$. Recall that $\mathcal{M}_\tau$ denotes the matroid of the maximum $\tau$-weight bases as in Lemma 6.1.16. The bipartite graph $G = (V \cup W, U; E)$ and matroid $\mathcal{M} = (U, r)$ and $W$ give a Rado-minor representation of $B_1$, while $G_0 = (V \cup W, U; E_0)$ and $\mathcal{M}_\tau = (U, r_\tau)$ and $W$ give a Rado-minor representation of $B_0$.

For $Z \subseteq V \cup W$, we let $\Gamma(Z)$ and $\Gamma_0(Z)$ denote the set of neighbours of $Z$ in $U$ in the edge sets $E$ and $E_0$, respectively. Furthermore, for $Z \subseteq V \cup W$ we define

$$\begin{aligned} \rho_0(Z) &:= r_\tau(\Gamma_0(Z)) - |Z|\,, \\ \rho_1(Z) &:= r(\Gamma(Z)) - |Z|\,. \end{aligned}$$

Note that $\rho_1(Z) \geq \rho_0(Z)$ for every $Z \subseteq V \cup W$. Finally, let $Q_0$ denote the unique largest subset of $W$ with $\rho_0(Q_0) = 0$ as in Corollary 6.1.12.

Further, for every $X \in B_1$, select a maximum weight independent matching $\mu^X$ with $\partial_{V \cup W}(\mu^X) = X \cup W$; let $\mathcal{L}$ be the set of all these matchings.

**Lemma 6.3.4.** *The minimal counterexample can be selected to satisfy one of the following properties:*

*(CI) We can choose dual optimal solution* $(\pi, \tau)$ *such that* $E = E_0 \cup \{(i', j')\}$ *for an edge* $(i', j')$ *where* $i' \in X^* \cup W$*,* $\mathcal{M}_\tau = \mathcal{M}$*, and* $B_1 = B_0 \cup \{X^*\}$*.*

*(CII)* $E = E_0 = E^*$ *and* $\mathcal{M}_\tau \neq \mathcal{M}$ *for any dual optimal* $(\pi, \tau)$*.*

Intuitively, the above lemma states that the difference between Rado-minor representations of $B_0$ and $B_1$ is either in the edge set only, or in the matroid on $U$ only. In case (CII) we can select an arbitrary $(\pi, \tau)$; in case (CI) we will use the dual asserted in the lemma.

*Proof of Lemma 6.3.4.* Let $\mu^{X^*} \in \mathcal{L}$ denote a maximum weight independent matching covering $X^* \cup W$. First, we show that $E = E^* \cup \mu^{X^*}$. Indeed, removing an edge in $E \setminus (E^* \cup \mu^{X^*})$ does not affect $h(X)$ for $X \in B_0 \cup \{X^*\}$ as all matchings $\mu^X$ for $X \in B_0$ lie in $E^*$. For any other set, $h(X)$ may decrease (possibly to $-\infty$); but this would yield another function in $\mathcal{F}_n$ that is the same or better on criterion (S1), the same on (S2), and strictly better on (S3). Hence, $E = E^* \cup \mu^{X^*}$.

Now, assume that $E \setminus E^* = \mu^{X^*} \setminus E^* \neq \emptyset$. Let $(i', j')$ be an arbitrary edge in $\mu^{X^*} \setminus E^*$, i.e., $i' \in X^* \cup W$. We start increasing $c$ to $c'$ for $\varepsilon \geq 0$ as follows

$$c_{ij} = \begin{cases} c_{ij} + \varepsilon & \text{for } (i,j) = (i', j') \\ c_{ij} & \text{otherwise.} \end{cases}$$

Pick the largest $\varepsilon \geq 0$ such that the maximum weight of an independent matching in $G$, $\mathcal{M}$, $c$ remains $0$, i.e., such that the optimum value of the LP (6.3) does not change.

**Claim 6.3.5.** $\varepsilon = -h(X^*)$.

*Proof.* Suppose that $\varepsilon < -h(X^*)$. By definition of $\mathcal{F}_n$, we have stopped increasing $\varepsilon$ as the edge $(i', j')$ has now entered $E^*$ and increasing the value further would increase the optimal value via a set $X \in B_0$. This is a contradiction on (S3). □

Next, we note that $B_0 \cup \{X^*\}$ is the set of maximizers of LP (6.3) under the increased weights $c'$. Indeed, by the choice of $\varepsilon$ all previous maximizers $B_0$ remain maximizers and now $\mu^{X^*}$ achieves the same value thereby becoming a maximizer as well. Moreover, for $X \in \mathcal{H} \setminus \{X^*\}$, we have $c'(\mu^X) \leq c(\mu^X) + \varepsilon < c(\mu^{X^*}) + \varepsilon = 0$.

Let us pick an optimal dual solution $(\pi, \tau)$ to (6.4) under $c'$. Recall that $E = E^* \cup \mu^{X^*}$ and therefore all edges $E$ are tight with respect to $c'$. Since $c' \geq c$ and the optimum value is the same for the two cost functions, it follows that $(\pi, \tau)$ is also optimal to (6.4) with the original weights $c$.

Since $c$ and $c'$ differ only on $(i', j')$, all edges $E \setminus \{(i', j')\}$ are tight under $(\pi, \tau)$ for $c$; thus, $E_0 = E \setminus \{(i', j')\}$.

As $\partial_U(\mu^{X^*})$ is a maximum $\tau$-weight basis in $\mathcal{M}$, it follows that we can replace $\mathcal{M}$ by $\mathcal{M}_\tau$. This is because all $\mu^X \in \mathcal{L}$ for $X \in B_0 \cup \{X^*\}$ remain independent matchings. The function value $h(X)$ might decrease for $X \notin B_0 \cup \{X^*\}$, but this may only lead to improvement in (S1), or otherwise we get another solution that is equally good on the selection criteria.

It is left to show $B_1 = B_0 \cup \{X^*\}$. Take any $X \in B_1$. Since every basis in $\mathcal{M}$ has maximum $\tau$-weight, the value of $c(\mu)$ is the optimum minus the sum of the slack values on the edges, that is, $h(X) = c(\mu^X) = -\sum_{(i,j)\in\mu}(\pi_i + \tau_j - c_{ij})$. Since $(i', j')$ is the only edge with positive slack, this means that $h(X) = 0$ if $(i', j') \notin \mu^X$ and $h(X) = h(X^*)$ if $(i', j') \in \mu^X$. Since $X^*$ is the unique set with the largest negative function value, this implies $B_1 = B_0 \cup \{X^*\}$.

Finally assume $E = E^*$ . Then, $\mu^{X^*} \subseteq E^* \subseteq E_0$. Thus, $\partial_U(\mu)$ cannot be independent in $\mathcal{M}_\tau$, as otherwise $h(X^*) = 0$ would follow by complementary slackness. Hence, $\mathcal{M} \neq \mathcal{M}_\tau$, giving case (CII). $\square$

**Lemma 6.3.6.** *In the minimal representation for each $Z \subseteq W$, $Z \neq \emptyset$ it holds $\rho_1(Z) > 0$.*

*Proof.* For a contradiction let $Z \subseteq W$ be a non-empty set with $\rho_1(Z) \geq 1$. Let $T = \mathrm{cl}(\Gamma(Z))$. Since $r(T) = |Z|$, for every independent matching $\mu$, we must have $|\partial_U(\mu) \cap T| = r(T)$. This implies that the weight of the edges covering $Z$ must be the same value $\delta$ for any independent matching. This follows since for any two independent matchings $\mu, \mu'$, we can replace by the set of edges covering $Z$ in $\mu$ by the set of edges covering $Z$ in $\mu'$ and obtain another independent matching covering $Z \cup X$.

Let $\mathcal{M}'$ denote the contraction of $T$ in $U$, and $U' = U \setminus T$. Then, we obtain a smaller $R$-minor representation by restricting to $W' = W \setminus Z$, and using $\mathcal{M}'$ on $U'$. Moreover, we define the new weight function on the edges as $c'(i, j) = c(i, j) + \delta / r(\mathcal{M})$ for each edge $(i, j)$ with $i \in (V \cup W) \setminus Z$ and $j \in U \setminus T$ to obtain the same $h(X)$ values. This contradicts criterion (S2) whenever $Z \neq \emptyset$. $\square$

### 6.3.3 $h$ is not R-induced

We start by showing that $W = \emptyset$ is not possible; in other words, $h$ cannot have an $R$-induced representation. (An alternative proof is given in Section 6.4.) We start with a structural claim on $\mathcal{B}_0$.

**Lemma 6.3.7.** *The matroid on $[2n]$ defined by bases $\mathcal{B}_0$ is not fully reducible for $n \geq 10$.*

*Proof.* For a contradiction, assume $\mathcal{B}_0$ is obtained as the union of two matroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on $V = [2n]$ with rank functions $r_1$ and $r_2$, such that $r_1(V)+r_2(V) = 4$; let $r(.)$ denote the rank function of $\mathcal{B}_0$. W.l.o.g. $r_1(V) \leq r_2(V)$. We distinguish two cases.

**Case I:** $r_1(V) = 1$**,** $r_2(V) = 3$**.** Let $T = \{v \in V : r_1(\{v\}) = 0\}$ denote the set of loops in $\mathcal{M}_1$. We claim that $T$ may intersect at most three different pairs $P_i$. Indeed, every $X \subseteq V$ with $|X| = 4$ intersects four different pairs is in $B_0$, and therefore $X \subseteq T$ cannot be the case. Let us select four pairs $P_i$, $P_j$, $P_k$, $P_\ell$ that do not intersect $T$, $i$ and $j$ are odd, $k$ and $\ell$ are even; such selection is possible for $n \geq 10$. Since $P_i \cup P_j \in B_0$, we must have $r_2(P_i \cup P_j) = 3$; w.l.o.g. assume $r_2(P_i) = 2$.

Consider $P_i \cup P_k$ and recall that it forms a circuit in $\mathcal{M}$. By Theorem 6.1.7, $r_1(P_i \cup P_k) + r_2(P_i \cup P_k) = 3$, implying $r_2(P_i \cup P_k) = 2$. Similarly, $r_2(P_i \cup P_\ell) = 2$. By submodularity, we have $r_2(P_i \cup P_k \cup P_\ell) = 2$, and thus $r(P_i \cup P_k \cup P_\ell) = 3$, a contradiction as the union of any three pairs contains a basis.

**Case II:** $r_1(V) = r_2(V) = 2$**.** Note that there can be at most one pair $P_t$ such that $r_1(P_t) = 0$, and at most one pair $P_{t'}$ with $r_2(P_{t'}) = 0$. Otherwise, if there existed $P_a, P_b$ such that $r_1(P_a) = r_1(P_b) = 0$, then $r_1(P_a \cup P_b) = 0$, contradicting that the union of any two pairs has rank at least 3 in $B_0$.

Let us select $P_i$, $P_j$, $P_k$, $P_\ell$ such that $i$ is even, $j$, $k$, and $\ell$ are odd, and all these pairs have rank $\geq 1$ in both matroids; again such sets can be selected for $n \geq 10$. Since $P_i \cup P_j$ is a circuit, $r_1(P_i \cup P_j) + r_2(P_i \cup P_j) = 3$. Similarly, $r_1(P_i \cup P_k) + r_2(P_i \cup P_k) = 3$ and $r_1(P_i \cup P_\ell) + r_2(P_i \cup P_\ell) = 3$. W.l.o.g. $r_1(P_i \cup P_j) = r_1(P_i \cup P_k) = 1$. By the assumption $r_1(P_i) \geq 1$, submodularity gives $r_1(P_i \cup P_j \cup P_k) = 1$. This again contradicts the fact that $r(P_i \cup P_j \cup P_k) = 4$. □

**Lemma 6.3.8.** *If $W = \emptyset$, then we must have $\pi \equiv 0$ and $\tau \equiv 0$ for the optimal dual $(\pi, \tau)$ in* (6.4).

*Proof.* By definition of $h \in \mathcal{F}_n$, the optimum value of the LP (6.3) is 0. Since for any $i \in V$ there is an $X \in B_0$ not containing $i$, it follows that $\pi_i = 0$ for all $i \in V$. From Lemma 6.2.3, it follows that $\tau_i$ has the same value for all $i \in U$; let $\alpha$ be this common value. Then, the objective value of the dual program (6.4) is $0 = \alpha \cdot r(\mathcal{M})$. Consequently, $\alpha = 0$, and therefore $\tau = 0$. □

Therefore $\mathcal{M}_\tau = \mathcal{M}$, implying case (CI) of Lemma 6.3.4: $E = E_0 \cup \{(i^*, j^*)\}$ and $B_1 = B_0 \cup \{X^*\}$. The rest of the analysis is covered by the argument in Section 6.3.6 for (CI). We include a simpler direct proof that also illustrates some key ideas of the more complex subsequent arguments.

Let $\ell \in \{1, 2\}$ such that $i^* \in P_\ell$. We note that the cases $\ell = 1$ and $\ell = 2$ are not symmetric, because of different parity.

**Claim 6.3.9.** *We have* $r(\Gamma(P_\ell)) = 3$.

*Proof of Claim.* Note first that $j^* \notin \mathrm{cl}[\Gamma_0(X^*)]$ as otherwise there is no independent matching $\mu^{X^*}$ covering $X^*$. Trivially $j^* \notin \mathrm{cl}[\Gamma_0(P_\ell)]$. Since $P_\ell$ is a subset of a basis in $B_0$ we have $r(\mathrm{cl}[\Gamma_0(P_\ell))]) \geq 2$. Thus, $r(\Gamma(P_\ell)) = r(\Gamma_0(P_\ell) \cup \{j^*\}) = r(\mathrm{cl}[\Gamma_0(P_\ell))] \cup \{j^*\}) \geq 3$.

For any $i \in [n]$ it holds $r(\Gamma(P_i)) \geq r(\Gamma_0(P_i)) \geq 2$ since each $P_i$ is a subset of a basis in $B_0$; and it particular for $i = 4$. Recall that $P_\ell \cup P_4 \in \mathcal{H} \setminus \{X^*\}$, i.e., $P_\ell \cup P_4 \notin B_0 \cup X^*$. Thus we have $r(\Gamma(P_\ell)) \leq 3$ as otherwise there is an independent matching $\mu^{P_\ell \cup P_4} \in \mathcal{L}$. The claim follows. ■

**Proposition 6.3.10.** *$h$ is not an R-induced valuated matroid.*

*Proof.* Let $X, Y \in \mathcal{H} \setminus \{X^*\}$ be two sets whose intersection is $P_\ell$. If $\ell = 1$, we can select $X = P_1 \cup P_4 = \{1, 2, 7, 8\}$ and $Y = P_1 \cup P_6 = \{1, 2, 11, 12\}$, and if $\ell = 2$, we can select $X, Y \in \mathcal{H} \setminus \{X^*\}$ intersecting in $P_2$, such as $X = P_2 \cup P_3 = \{3, 4, 5, 6\}$ and $Y = P_2 \cup P_4 = \{3, 4, 7, 8\}$.

Let $X, Y \in \mathcal{H} \setminus \{X^*\}$ be two sets whose intersection is $P_\ell$. If $\ell = 1$, we can select $X = \{1, 2, 7, 8\}$ and $Y = \{1, 2, 11, 12\}$, and if $\ell = 2$, we can select $X, Y \in \mathcal{H} \setminus \{X^*\}$ intersecting in $P_2$, such as $X = \{3, 4, 5, 6\}$ and $Y = \{3, 4, 7, 8\}$.

Since $h(X), h(Y) = -\infty$ by $B_1 = B_0 \cup \{X^*\}$, there is no independent matching in $E$ covering $X$ or $Y$. By Theorem 6.1.2, we have $r(\Gamma(X)) = r(\Gamma(Y)) = 3$. By Claim 6.3.9, it follows that $\Gamma(X), \Gamma(Y) \subseteq \mathrm{cl}(\Gamma(P_\ell))$. This further implies that $r(\Gamma(X \cup Y)) \leq r(\Gamma(P_\ell)) = 3$, a contradiction since $X \cup Y$ contains a set in $B$. (For $\ell = 1$, one such set is $\{1, 2, 7, 11\}$, and for $\ell = 2$, we can select $\{3, 4, 5, 7\}$.) □

### 6.3.4 Robust matroids and their Rado-minor representations

In this section we study some additional properties of Rado-minor representations of the matroid $B_0$. We formulate the properties more generally, so that we can also use them whenever $B_1 = B_0 \cup \{X^*\}$. This always holds in case (CI), and we will later show that it must also be true in case (CII).

**Definition 6.3.11** (Robust matroid). *Let $V = [2n]$, and let $P_i = \{2i - 1, 2i\}$ for $i \in [n]$; these are called* pairs. *We define a matroid by its set of bases $\mathcal{B} \subseteq \binom{V}{4}$ and let $\mathcal{H} = \binom{V}{4} \setminus \mathcal{B}$. We say that $\mathcal{B}$ forms the bases of a* robust *matroid if*

*(D1) Every circuit in $\mathcal{H}$ is the union of two pairs $P_i \cup P_j$,*

*(D2) Consider a graph $([n], H)$ where $\{i, j\} \in H$ if and only if $P_i \cup P_j \in \mathcal{H}$. Then, we can partition $[n]$ into two sets $S$ and $K$ such that $|S| \geq 2$, $K$ is a clique in $H$ with $|K| \geq 3$,*

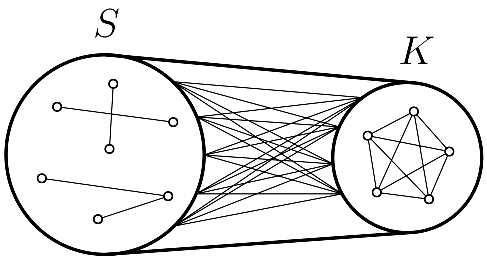


Figure 6.1: The graph $H$ of a robust matroid defined in (D2).

*and every node in $S$ is adjacent to every node in $K$. Moreover, for each $i \in S$ there is $j \in S$ such that $i$ is non-adjacent to $j$ in $H$. (A schematic view of $H$ is given in Figure 6.1.)*

Note that this defines a sparse paving matroid.

**Lemma 6.3.12.** *Both $B_0$ and $B_0 \cup \{X^*\}$ are robust matroids for $n \geq 8$.*

*Proof.* The first property is immediate. For (D2), in $B_0$ (respectively $B_0 \cup \{X^*\}$), it suffices to choose $K$ as the set of even indices (respectively the set of even indices different from 2). In both cases, $S = [n] \setminus K$. □

Let $B$ be a robust matroid on $V$. Consider a Rado-minor representation $(G, \mathcal{M})$ with bipartite graph $G = (V \cup W, U; E)$ and $\mathcal{M} = (U, r)$. We now derive strong structural properties for such a representation of the matroid $B$.

Recall that for $Z \subseteq V \cup W$, $\rho(Z) := r(\Gamma(Z)) - |Z|$. In the following proofs, we make heavy use Lemma 6.1.9. Note that the rank of a robust matroid is 4. Thus, in this section we use the following assumption. Recall that $Q$ is the unique maximal subset of $W$ such that $\rho(Q) = 0$ by Corollary 6.1.12.

**Lemma 6.3.13.** *For each pair $P_k$, there exists a unique largest $P_k$-set $Z_k$ with $\rho(Z_k) = 0$; and $Q \subset Z_k$.*

*Proof.* Let $k \in [n]$. By (D2), there exists different indices $i, j \in [n] \setminus \{k\}$ such that $P_i \cup P_j$ and $P_i \cup P_k$ are circuits in $\mathcal{H}$. By (cir), there exists a $(P_i \cup P_k)$-set $I$ and a $(P_j \cup P_k)$-set $J$ with $\rho(I) = \rho(J) = -1$. By the second uncrossing lemma, $I \cap J$ is a $P_k$-set with $\rho(I \cap J) = 0$. This shows existence of a $P_k$-set with $\rho$-value 0. The existence of a unique largest such set follows by the first uncrossing lemma by choosing $X = Y = P_k$ there.

To see that $Q \subseteq Z_k$, we apply the first uncrossing lemma for $X = \emptyset, I = W_0$ and $Y = P_k, J = Z_k$. Namely, $Q \cap Z_k$ is $\emptyset$-set and $Q \cup Z_k$ is $P_k$-set. Thus, $\rho(Q \cup Z_k) = 0$ and $Q \subseteq Z_\kappa$. □

Let us interpret the above lemma. It states that for any pair $P_k$ there exists unique largest set $Z_k$ containing exactly $P_k$ in $V$ with $\rho(Z_k) = 0$. Having $\rho(Z_k) = 0$ means that any independent matching $\mu$ in the Rado-minor representation with $\partial_{V \cup W}(\mu) = P_k \cup W$, must match the nodes in $Z_k$ to $\mathrm{cl}[\Gamma(Z_i)]$ and no other node is matched to a node in $\mathrm{cl}[\Gamma(Z_i)]$.

Next we describe how the sets $Z_k$, given by Lemma 6.3.13, interact with each other.

**Lemma 6.3.14.** *For any $i, j \in [n]$, $i \neq j$, we have*

- *If $P_i \cup P_j \in \mathcal{B}$ then $\rho(Z_i \cup Z_j) = 0$;*
- *if $P_i \cup P_j \in \mathcal{H}$ then $\rho(Z_i \cup Z_j) = -1$.*
- *For all $i, j \in [n]$, $i \neq j$ we have $Z_i \cap Z_j = Q$ and $\mathrm{cl}[\Gamma(Z_i)] \cap \mathrm{cl}[\Gamma(Z_j)] = \mathrm{cl}[\Gamma(Q)]$.*

*Proof.* First, we show the lemma for pairs $P_i$ and $P_j$ such that $P_i \cup P_j$ is a basis in $B$. We have that $Z_i \cap Z_j$ is $\emptyset$-set and $Z_i \cup Z_j$ is $(P_i \cup P_j)$-set. By the first uncrossing lemma, as $P_i \cup P_j$ is an independent set, we have $\rho(Z_i \cap Z_j), \rho(Z_i \cup Z_j) = 0$. By the maximality of $Q$ and since $Q \subseteq Z_i, Z_j$, we have $Z_i \cap Z_j = Q$. Finally, Lemma 6.1.10 implies $\mathrm{cl}(\Gamma(Z_i)) \cap \mathrm{cl}(\Gamma(Z_j)) = \mathrm{cl}(\Gamma(W_0))$. This proves the lemma for $i, j \in [n]$ with $P_i \cup P_j \in B$.

For the rest of the proof consider pairs $P_i$ and $P_j$ such that $P_i \cup P_j$ is a circuit in $\mathcal{H}$. We show that $\rho(Z_i \cup Z_j) = -1$. By (cir), there is a $(P_i \cup P_j)$-set $A$ with $\rho(A) = -1$. Let $k \in K \setminus \{i, j\}$ be such that $P_i \cup P_k$ and $P_j \cup P_k$ are circuits $\mathcal{H}$; such $k$ is guaranteed by (D2). Again by (cir), there exist a $(P_i \cup P_k)$-set $I$ and a $(P_j \cup P_k)$-set $J$ such that $\rho(I) = \rho(J) = -1$. By the second uncrossing lemma, we have $\rho(I \cup J) = -2$.

Using $\rho(A) = -1$ and $\rho(I \cup J) = -2$, we uncross $A$ and $I \cup J$:

$$-3 = \rho(A) + \rho(I \cup J) \geq \rho(A \cap (I \cup J)) + \rho(A \cup I \cup J) \geq -1 - 2\,,$$

by (cir) and (dep), since $C = A \cap (I \cup J)$ is a $(P_i \cup P_j)$-set and $A \cup I \cup J$ is a $(P_i \cup P_j \cup P_k)$-set. Thus, $\rho(C) = -1$. We can write $C = (A \cap I) \cup (A \cap J)$. By the maximality of $Z_i$ and $Z_j$ we have $A \cap I \subseteq Z_i$, $A \cap J \subseteq Z_j$. Consequently, $C \subseteq Z_i \cup Z_j$. Finally, we uncross $C$ with $Z_i$ (resp. with $Z_j$), and then uncross $C \cup Z_i$ and $C \cup Z_j$ to see that $\rho(Z_i \cup Z_j) = -1$.

Next, we show that $Z_i \cap Z_j = Q$. For a contradiction, assume there exists $w \in Z_i \cap Z_j \setminus Q \subseteq W$. Consider $k \in K \setminus \{i, j\}$ as before, i.e., $k \in K \setminus \{i, j\}$ such that $\{i, j, k\}$ is a triangle in graph $H$. By the second uncrossing lemma for $I = Z_i \cup Z_k$ and $J = Z_j \cup Z_k$, we see that $\rho(I \cap J) = 0$. Since $Z_k \subseteq I \cap J$ and $Z_k$ is the largest $P_k$-set with $\rho(Z_k) = 0$, it follows that $I \cap J = Z_k$. Consequently, $Z_i \cap Z_j \subseteq Z_k$ and $w \in Z_k$ for all $k \in K$.

Let $k, k' \in K$, and consider any $\ell \in S$. These three indices again from a triangle in the graph $([n], H)$. By the same argument as in the previous paragraph, we conclude $w \in Z_\ell$ for all $\ell \in S$. Hence, $w \in Z_\ell$ for all $\ell \in [n]$. This is contradiction as we have already showed that $Z_a \cap Z_b = Q$ whenever $P_a \cup P_b$ is a basis in $B$.

Finally, we show that $\mathrm{cl}[\Gamma(Z_i)] \cap \mathrm{cl}[\Gamma(Z_j)] = \mathrm{cl}[\Gamma(Q)]$. Similarly to the previous argument, we assume for the contradiction that there exists $u \in \mathrm{cl}[\Gamma(Z_i)] \cap \mathrm{cl}[\Gamma(Z_j)] \setminus \mathrm{cl}[\Gamma(Q)]$. Again, by the second uncrossing lemma for $I = Z_i \cup Z_k$ and $J = Z_j \cup Z_k$, we have $\rho(I \cap J) = 0$ and $I \cap J = Z_k$. Moreover, it holds $\rho(I) + \rho(J) = \rho(I \cap J) + \rho(I \cup J)$. Lemma 6.1.10

implies that $\mathrm{cl}[\Gamma(I)] \cap \mathrm{cl}[\Gamma(J)] = \mathrm{cl}[\Gamma(Z_k)]$; consequently, $\mathrm{cl}[\Gamma(Z_i)] \cap \mathrm{cl}[\Gamma(Z_j)] \subseteq \mathrm{cl}[\Gamma(Z_k)]$. As before, this implies that $u \in \mathrm{cl}[\Gamma(Z_\ell)]$ for all $\ell \in [n]$. This is contradiction as we have already shown that $\mathrm{cl}[\Gamma(Z_a)] \cap \mathrm{cl}[\Gamma(Z_b)] = \mathrm{cl}[\Gamma(Q)]$ whenever $P_a \cup P_b$ is a basis in $B$. □

**Lemma 6.3.15.** *We have $\rho(\cup_{i=1}^n Z_i) = 4-2n$ and and $\rho(\cup_{i\in[n]\setminus\{j\}} Z_i) = 2-2n$ for every $j \in [n]$.*

*Proof.* We rely on the following two claims.

**Claim 6.3.16.** *Consider three different indices $i, j, k \in [n]$ such that at least two out of $\{i, j\}$, $\{i, k\}$, and $\{j, k\}$ are edges in $H$. Then, $\rho(Z_i \cup Z_j \cup Z_k) = -2$.*

*Proof of Claim.* Consider the pairs $P_i$, $P_j$, and $P_k$ with indices as in the claim. Without loss of generality assume that $\{i, k\}, \{j, k\}$ are edges in $H$. Thus, $P_i \cup P_k$ and $P_j \cup P_k$ are circuits in $\mathcal{H}$. Then, we have $\rho(Z_i \cup Z_k) = \rho(Z_j \cup Z_k) = -1$ by the second part of Lemma 6.3.14. Let us uncross these two sets. By submodularity and Lemma 6.1.9, we have

$$-2 = \rho(Z_i \cup Z_k) + \rho(Z_j \cup Z_k) \geq \rho(Z_k) + \rho(Z_i \cup Z_j \cup Z_k) \geq 0 - 2\,.$$

Hence, $\rho(Z_i \cup Z_j \cup Z_k) = -2$. ■

**Claim 6.3.17.** *Let $L \subseteq [n]$ such that $|L \cap K| \geq 3$ and $L \cap S$ contains two non-adjacent indices $i$ and $j$. (Recall that $K$ and $S$ are the sets given by (D2).) Then, $\rho(\cup_{i\in L} Z_i) = 4 - 2|L|$.*

*Proof of Claim.* As $\{i, j\} \notin \mathcal{H}$ then $P_i \cup P_j \in B$ and thus $\rho(Z_i \cup Z_j) = 0$ by the first part of Lemma 6.3.14. Consider any index $k \in K$. By Claim 6.3.16, $\rho(Z_k \cup Z_i \cup Z_j) = -2$. Therefore, adding $Z_k$ to $Z_i \cup Z_j$ decreases the $\rho$ value by 2. In other words, for any $k \in K$ we have

$$\Delta_\rho(Z_k | Z_i \cup Z_j) := \rho(Z_k \cup Z_i \cup Z_j) - \rho(Z_i \cup Z_j) = -2 - 0 = -2\,. \tag{6.5}$$

By submodularity, adding $\ell$ different sets $Z_k$ with $k \in K$ to $Z_i \cup Z_j$ decreases $\rho$ by at least $2\ell$. We proceed to prove a similar statement for sets $Z_k$ with $k \in S$.

Next, consider three different indices $a, b, c \in K \cap L$. Let $Y = Z_a \cup Z_b \cup Z_c$. We then have, $\rho(Y \cup Z_i \cup Z_j) \leq 4 - 2\cdot 5$. By Claim 6.3.16, we have $\rho(Y) = -2$. By Lemma 6.1.9 (dep), we also have $\rho(Y \cup Z_i \cup Z_j) \geq 4 - 2\cdot 5$ and consequently $\rho(Y \cup Z_i \cup Z_j) = 4 - 2\cdot 5$. (Which proves the claim if $L = \{a, b, c, i, j\}$.) Similarly, $\rho(Y \cup Z_i) = 4 - 2\cdot 4$.

Rearranging the above we conclude that whenever $\{i, j\} \notin H$, we have

$$\Delta_\rho(Z_i \cup Z_j | Y) = -4\,. \tag{6.6}$$

In other words, adding $Z_i \cup Z_j$ to $Y$ leads to a decrease of 4 in the $\rho$ value. By Lemma 6.1.9 (dep) we also have $\rho(Y \cup Z_i) \geq 4 - 2\cdot 4 = -4$, and $\rho(Y \cup Z_i \cup Z_j) \geq$

$4-2\cdot 5=-6$. Combining it with the previous paragraph, we have $\Delta_\rho(Z_i|Y)\geq -2$, and $\Delta_\rho(Z_j|Y\cup Z_i)\geq -2$. Using (6.6) and submodularity we conclude that the inequalities hold with equality. That is, we have

$$\Delta_\rho(Z_i|Y)=-2 \tag{6.7}$$

for every $i$ such that $\{i,j\}\notin H$ for some $j\in S$, i.e., by (D2), for every $i\in S$. By submodularity, adding $\ell$ different sets $Z_i$ with $i\in S$ to $Y$ decreases the $\rho$ value by at least $2\cdot\ell$.

Thus, for our set $L$, by submodularity and combing (6.5) and (6.7) we have $\rho(\cup_{i\in L}Z_i)\leq 4-2\cdot|L|$. The equality holds by Lemma 6.1.9 (dep). ■

The lemma follows by applying the last claim for $L=[n]$ and $L=[n]\setminus\{i\}$. □

### 6.3.5 Bounding the support of $h$

Since $B_0$ is always a robust matroid, we can use the results in the previous section for $B=B_0$. Let $Z_i^0$ denote the $Z_i$-sets, and $Q_0$ the unique largest subset of $W$ with $\rho_0(Q)=0$.

Our first goal is to show Lemma 6.3.20 below, namely, that in both case (CI) and (CII), we have that $\mathrm{dom}(h)=B_1=B_0\cup(X^*)$. Thus, we get the smallest possible size according to the main selection criterion (S1). This will enable us to also use the robust matroid analysis on $B=B_1$. The proof will rely on the following 'compression' of the matroid $\mathcal{M}$.

**Compressing $\mathcal{M}$** We replace $\mathcal{M}$ on $U$ by the following matroid $\overline{\mathcal{M}}$: a set $T\in\binom{U}{|W|+4}$ is a basis in $\overline{\mathcal{M}}$ if and only if there is a matching in $E$ between $T$ and a basis in

$$\overline{B}:=\{X\cup W:\ X\in B_0\cup\{X^*\}\}\,.$$

These sets $T$ form the bases of a matroid by Rado's theorem. Since $h(X)$ is finite for all $X\in B_0\cup\{X^*\}$, this will be a submatroid of $\mathcal{M}$, i.e., all bases of $\overline{\mathcal{M}}$ are bases in $\mathcal{M}$. Let $\bar{h}(X)$ be the function corresponding to the modified representation $(G,\overline{\mathcal{M}},c,W)$. Clearly, $\bar{h}(X)=h(X)$ for every $X\in B_0\cup\{X^*\}$ and $\bar{h}(X)\leq h(X)$ otherwise. As $\bar{h}$ has the same or better criteria (S1)–(S3) than $h$, we assume that $h=\bar{h}$ and $\overline{\mathcal{M}}=\mathcal{M}$.

Using this construction, we first show that $Q_0=\emptyset$ in (CII). However, $Q_0\neq\emptyset$ may still be possible in case (CI).

**Lemma 6.3.18.** *In case (CII), i.e., if $E=E^*$, then $Q_0=\emptyset$ must hold. Thus, $\rho_1(Z_i^0)\geq\rho_0(Z_i^0)\geq 1$ for all $Z\subseteq W, Z\neq\emptyset$ in this case.*

*Proof.* Denote with $T_0=\Gamma(Q_0)$. By definition of $\rho_0$, $r_\tau(T_0)=|Q_0|$. We claim that also $r(T_0)=|Q_0|$. The next claim will be needed for this proof.

**Claim 6.3.19.** *There is no edge $(i, j) \in E$ with $i \in (V \cup W) \setminus Q_0$ and $j \in T_0$.*

*Proof of Claim.* Suppose there is such an edge. By definition of $E^*$ ($= E$), there exists an independent matching $\mu$ containing $(i, j)$ with weight $0$. Trivially, this matching also covers $Q_0$ as $Q_0 \subseteq W$. Thus, $\mu$ matches $Q_0$ and $i$ to the set $T_0$ in $U$. By optimality criteria the endpoints of $\mu$ in $U$ must form a basis in $\mathcal{M}_\tau$. This is a contradiction, since $|Q_0 \cup \{i\}| > r_\tau(T_0) = |Q_0|$. ∎

Suppose that $r(T_0) > |Q_0|$. Then there is a basis $S$ of $\mathcal{M}$ such that $|S \cap T_0| > |Q_0|$. As $\overline{\mathcal{M}} = \mathcal{M}$ there is an independent matching, matching $S \cap T_0$ to a subset of size $> |Q_0|$ in $V \cup W$. This is impossible as the neighbourhood of $T_0$ in $V$ is $Q_0$ by Claim 6.3.19. Hence, $r(T_0) = |Q_0|$. This contradicts Lemma 6.3.6. □

**Lemma 6.3.20.** *$B_1 = B_0 \cup \{X^*\}$ must hold.*

*Proof.* There is nothing to prove in (CI), so let us assume we are in case (CII); thus, $E = E^*$. According to the previous lemma, we also have $Q_0 = \emptyset$. Let $Z^* = \cup_{i=1}^n Z_i^0$; in particular $V \subseteq Z^*$.

**Claim 6.3.21.** *There are no edges between $W \setminus Z^*$ and $\Gamma(Z^*)$.*

*Proof.* Let $F$ denote the edge set in the claim. Let $T^* = \Gamma(Z^*)$. By Lemma 6.3.15 and Lemma 6.3.18, we have that $\rho_0(Z^*) = 4 - 2n$. As $\rho_0(Z^*) = r_\tau(\Gamma_0(Z^*)) - |Z^*| = r_\tau(T^*) - |Z^*|$ we have $r_\tau(T^*) = 4 + |Z^* \cap W|$. Consequently, an independent matching $\mu$ of weight $0$ cannot use any of the edges in $F$, since $|\partial_{Z^*}(\mu^X)| = 4 + |Z^* \cap W|$ and thus $\partial_{Z^*}(\mu^X)$ must be matched to a maximal independent set in $T^*$. Hence, $E^* \cap F = \emptyset$. Then $F = \emptyset$ as $E = E^*$. □

Consider any $X \in B_1 \setminus (B_0 \cup \{X^*\})$. We have $X = P_i \cup P_j$ for some $i, j \in [n]$, $\{i, j\} \neq \{1, 2\}$ by the definition of $\mathcal{F}_n$. Let $S = Z_i^0 \cup Z_j^0$ and $T = \Gamma(S)$. The next claim shows that $r(T) < |S| - 1$.

**Claim 6.3.22.** *$r(T) < |S|$.*

*Proof of Claim.* By Lemma 6.3.14 and Lemma 6.3.18 ($Q_0 = \emptyset$), there are no edges connecting $T$ and any $Z_k^0$, $k \notin \{i, j\}$. As $F = \emptyset$ there are no edges between $T \subseteq \Gamma(Z^*)$ and $W \setminus Z^*$. We conclude that $\Gamma(T) = S$ (the direction $\Gamma(T) \supseteq S$ follows by definition as $T = \Gamma(S)$). Therefore, $T = \Gamma(S)$ and $S = \Gamma(T)$.

Since $\mathcal{M} = \overline{\mathcal{M}}$ and using Rado's theorem, if we have $r(T) \geq |S|$ then $B^*$ has a basis intersecting $S$ in at least $|S|$ elements. As $S = Z_i^0 \cup Z_j^0$ for some $X = P_i \cup P_j \notin B_0 \cup \{X^*\}$ this means that, $P_i \cup P_j \cup W$ is a basis of $\overline{B}$. ∎

By the above claim and Rado's theorem, there cannot be any independent matching in $G, \mathcal{M}$ covering $X \cup W$; thus, $h(X) = -\infty$ proving the lemma. □

In light of the above Lemma, we can apply the techniques in Section 6.3.4 to the robust matroid $B_1 = B_0 \cup \{X^*\}$. Let $Z_i^1$ denote the corresponding sets in Lemma 6.3.14, and recall that $\rho_1(Z) = r(\Gamma(Z)) - |Z|$. By Lemma 6.3.6, the largest subset $Q_1$ of $W$ with $\rho_1(Q_1) = 0$ is $Q_1 = \emptyset$.

**Lemma 6.3.23.** *In the minimal counterexample we have* $\cup_{i=1}^n Z_i^1 = V \cup W$.

*Proof.* We use the following claim stating that, in the minimal counterexample, for any $V$-set $Z$ with sufficiently large $\rho_1$-value it holds $V \cup W = Z$.

**Claim 6.3.24.** *Let* $Z = V \cup W'$ *for* $W' \subseteq W$ *such that* $\rho_1(Z) = 4 - |V|$. *Then, in a minimal counterexample we must have* $W' = W$ *or equivalently* $Z = V \cup W$.

*Proof of Claim.* For a contradiction assume that $W' \neq W$. Let $T = \mathrm{cl}[\Gamma(V \cup W')]$. By definition of $\rho_1$, having $\rho_1(V \cup W') = 4 - |V|$ means $r(T) = |V \cup W'| + 4 - |V| = |W'| + 4$. Thus, for any $X \in B_1$ ($|X| = 4$) the corresponding matching $\mu^X \in \mathcal{L}$ matches exactly $r(T)$ nodes in $T$ to the nodes in $X \cup W'$. In other words, any matching $\mu^X \in \mathcal{L}$ matches nodes $W \setminus W'$ to $|W \setminus W'|$ nodes in $U \setminus T$.

Similarly to the proof of Lemma 6.3.6, it follows that in any $\mu^X \in \mathcal{L}$, the cost of the edges covering $W \setminus W'$ is the same. Hence, we can get a smaller representation by restricting $W$ to $W'$ and $U$ to $U'$. ■

Lemma 6.3.15 for $B_1$ gives $\rho_1(\cup_{i=1}^n Z_i^1) = 4 - 2n$. Also noting that $V \subseteq \cup_{i=1}^n Z_i^1$, the statement follows by Claim 6.3.24. □

**Lemma 6.3.25.** *In a minimal counterexample we have* $Z_i^0 = Z_i^1 \cup Q_0$ *(in particular,* $Z_i^0 = Z_i^1$ *in case (CII)) for every* $i \in [n]$.

*Proof.* Let us first show $Z_i^1 \cup Q_0 \subseteq Z_i^0$. By Lemma 6.3.13, $Q_0 \subseteq Z_i^0$. Let us show $Z_i^1 \subseteq Z_i^0$. We have $\rho_0(Z_i^1) \geq 0$ by (ind) since $Z_i^1$ is a $P_i$-set, and also $\rho_0(Z_i^1) \leq \rho_1(Z_i^1) = 0$. Thus, $\rho_0(Z_i^1) = 0$. By the maximality of $Z_i^0$ (Lemma 6.3.13), it follows that $Z_i^1 \subseteq Z_i^0$.

We next show that equality holds. For the sake of contradiction, assume that we have $w \in Z_i^0 \setminus (Z_i^1 \cup Q_0)$ for some $i \in [n]$. Lemma 6.3.23 shows that $\cup_{i=1}^n Z_i^1 = V \cup W$, and hence we must have $w \in (Z_i^0 \cap Z_j^1) \setminus Q_0$ for some $i \neq j$. By the third part of Lemma 6.3.14 we then have $Q_0 = Z_i^0 \cap Z_j^0 \supseteq Z_i^1 \cap Z_j^0 \supseteq \{w\}$, a contradiction. □

### 6.3.6 The case (CI)

We are ready to show that case (CI) cannot occur. In this case, we have $\mathcal{M}_\tau = \mathcal{M}$, $E = E_0 \cup \{(i^*, j^*)\}$, and $B_1 = B_0 \cup \{X^*\}$.

**Lemma 6.3.26.** *Either* $Q_0 = \emptyset$ *or there exists unique* $q \in [n]$ *such that* $Q_0 \subseteq Z_q^1$.

*Proof.* The sets $Z_i^1$ are pairwise disjoint by Lemmas 6.3.6 and 6.3.14. Suppose $Q_0 \cap Z_q^1 \neq \emptyset$ for some $q \in [n]$. Let us uncross these two sets. Trivially $\rho_1(Z_q^1) = 0$, by Lemma 6.3.6 we have $\rho_1(Q_0) \geq 1$ and $\rho_1(Z_q^1 \cap Q_0) \geq 1$. Further $\rho_1(Z_q^1 \cup Q_0) \geq 0$ holds since $Z_q^1 \cup Q_0$ is a $P_q$-set. By submodularity it follows

$$0 + 1 = \rho_1(Z_q^1) + \rho_1(Q_0) \geq \rho_1(Z_q^1 \cap Q_0) + \rho_1(Z_q^1 \cup Q_0) \geq 1 + 0\,,$$

implying $\rho_1(Z_q^1 \cup Q_0) = 0$. By maximality of $Z_q^1$ we have $Q_0 \subseteq Z_q^1$. □

**Lemma 6.3.27.** *We have $\rho_0(Z_1^0 \cup Z_2^0) = -1$ and $\rho_0(Z_1^1 \cup Z_2^1) = 0$. Consequently, $Q_0 \neq \emptyset$ and $q \notin \{1,2\}$ for $q$ as in Lemma 6.3.26.*

*Proof.* Recall that $\rho_0(Z_1^0 \cup Z_2^0) = -1$ by Lemma 6.3.14 as $h(X^*) < 0$. We claim that $\rho_0(Z_1^1 \cup Z_2^1) = 0$.

Recall that $\mathcal{M} = \mathcal{M}_\tau$. For a contradiction, assume that $\rho_0(Z_1^1 \cup Z_2^1) = 0$. It can only be that $\rho_0(Z_1^1 \cup Z_2^1) < \rho_1(Z_1^1 \cup Z_2^1) = 0$. In particular, $\rho_0(Z_1^1 \cup Z_2^1) = -1$ as every three element set is independent in $B_0$. Hence, $\rho_0(Z_1^1 \cup Z_2^1) = -1 < 0 = \rho_1(Z_1^1 \cup Z_2^1)$. This means that $r(\Gamma(Z_1^1 \cup Z_2^1)) > r(\Gamma_0(Z_1^1 \cup Z_2^1))$. Thus, the single edge $(i', j') \in E \setminus E_0$ is incident to $Z_1^1 \cup Z_2^1$. Let $\ell \in \{1,2\}$ such that $i' \in Z_\ell^1$. Now, we must have $0 \leq \rho_0(Z_\ell^1) < \rho_1(Z_\ell^1) = 0$, a contradiction.

The last statements follow since if $Q_0 = \emptyset$ or $q \in \{1,2\}$, then $Z_1^0 \cup Z_2^0 = Z_1^1 \cup Z_2^1$ by Lemma 6.3.25. □

**Lemma 6.3.28.** *Let $q \in [n]$ such that $Q_0 \subseteq Z_q^1$, and let $Y = \cup_{i \in [n] \setminus \{q\}} Z_i^1$. Then, $\rho_0(Y) = 2 - 2n$.*

*Proof.* By the second part of Lemma 6.3.15 for $\rho_1$, we have $\rho_1(Y) = 2 - 2n$. We show that the same holds for $\rho_0$.

By Lemma 6.3.26 (and Lemma 6.3.27) we know that $Q_0 \subseteq Z_q^1$ for a unique $q \in [n]$. As all $Z_i^1$ are disjoint (Lemma 6.3.14 and $Q_1 = \emptyset$) we have $Q_0 \cap Z_i^1 = \emptyset$ for all $i \in [n] \setminus \{q\}$. Then by Lemma 6.3.25 we have that $Z_i^1 = Z_i^0 \setminus Q_0$. We use this below at the second line to show $\rho_0(Y) \geq 2 - 2n$:

$$\begin{aligned}
\rho_0(Y) &= \rho_0(\cup_{i \in [n] \setminus \{q\}} Z_i^1) \\
&= \rho_0(\cup_{i \in [n] \setminus \{q\}} (Z_i^0 \setminus Q_0)) \\
&= \rho_0\left((\cup_{i \in [n] \setminus \{q\}} Z_i^0) \setminus Q_0\right) + \rho_0(Q_0) && (\rho_0(Q_0) = 0) \\
&\geq \rho_0(\cup_{i \in [n] \setminus \{q\}} Z_i^0) + \rho_0(\emptyset) && \text{(submodularity)} \\
&= 2 - 2n\,. && \text{(Lemma 6.3.15 for } \rho_0\text{)}
\end{aligned}$$

Since $\rho_0(Y) \leq \rho_1(Y)$ we conclude $\rho_0(Y) = 2 - 2n$. □

Let us now derive the final contradiction for (CI). As $Q_0$ only intersects $Z_q^1$, by submod-

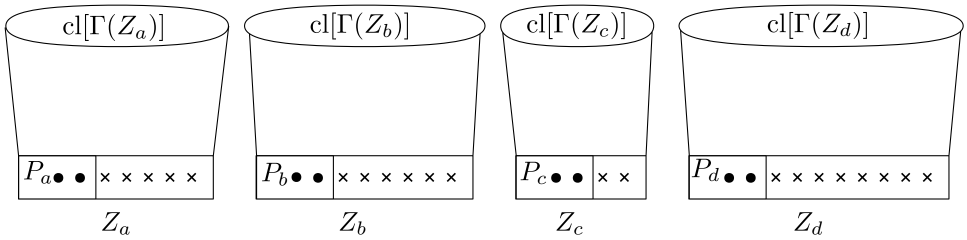


Figure 6.2: Schematic example of matroid $B_1$ with its Rado-minor representation $(G, \mathcal{M}, W)$. Here, the neighbourhoods if taken in the edge set $E$, and the closure in the matroid $\mathcal{M}$. The black dots represent set $V$ and the white dots represent $W$. Similarly, a Rado-minor representation holds for $B_0$ once we replace $\mathcal{M}$ (and closure) by $\mathcal{M}_\tau$.

ularity

$$\rho_0(Y \cup Q_0) + \rho_0(Z_1^1 \cup Z_2^1) \leq \rho_0(Z_1^1 \cup Z_2^1 \cup Q_0) + \rho_0(Y)\,.$$

Then, by Lemma 6.3.27 we further have

$$\rho_0(Y \cup Q_0) - \rho_0(Y) \leq \rho_0(Z_1^1 \cup Z_2^1 \cup Q_0) - \rho_0(Z_1^1 \cup Z_2^1) = \rho_0(Z_1^0 \cup Z_2^0) - \rho_0(Z_1^1 \cup Z_2^1) = -1\,.$$

Hence, $\rho_0(Y \cup Q_0) \leq 1 - 2n$. On the other hand $\rho_0(Y \cup Q_0) = \rho_0(\cup_{i \in [n] \setminus \{q\}} Z_i^1 \cup Q_0) = \rho_0(\cup_{i \in [n] \setminus \{q\}} Z_i^0) = 2 - 2n$. A contradiction.

### 6.3.7 The case (CII)

In the remaining case (CII), we have $E = E_0 = E^*$ but $\mathcal{M}_\tau \neq \mathcal{M}$. In Section 6.3.5, we have already showed some strong properties for this case: $Q_0 = \emptyset$ (Lemma 6.3.18), $B_1 = B_0 \cup \{X^*\}$ (Lemma 6.3.20), and $Z_i^0 = Z_i^1$ for all $i \in [n]$ (Lemma 6.3.25). In light of this, we can simplify the notation to $Z_i = Z_i^0 = Z_i^1$.

Let $D_i := \mathrm{cl}[\Gamma_E(Z_i)]$; see Figure 6.2. By Lemma 6.3.14, there are no edges with one end point in $Z_i$ and the other in $D_j$ whenever $i \neq j$.

Let us additionally modify the bipartite graph in the representations: we may assume that $E = E_0 = E^*$ is a complete bipartite graph between $Z_i$ and $D_i$ for any $i \in [n]$. Indeed, recall that any independent matching covering $Z_i$ has to match $Z_i$ to $D_i$ and no node outside of $Z_i$ can be matched to a node in $D_i$. Thus, adding new edges between these sets cannot add a new basis to either the set of all bases $B_1$ or to the set of maximum weight bases $B_0$.

Introducing these new edges allows us to describe the representations of $B_0$ and $B_1$ in purely set-theoretic and matroidal terms. We introduce definition that under the representation constructed above captures the matroids $B_0$ and $B_1$.

**Definition 6.3.29.** *For a set $X \in \binom{V}{4}$, we say that a set $S \subseteq U$, $|S| = |W| + 4$* conforms *$X$ if*

$|S \cap D_i| = |X \cap P_i| + |Z_i| - 2$ *for all* $i \in [n]$.

The requirements on our matroids $\mathcal{M}$ and $\mathcal{M}_\tau$ can be stated as follows:

- For any $X \in \binom{V}{4}$, there exists a basis $S$ in $\mathcal{M}$ conforming $X$ if and only if $X \in B_1$.
- For any $X \in \binom{V}{4}$, there exists a basis $S$ in $\mathcal{M}_\tau$ conforming $X$ if and only if $X \in B_0$.

The next lemma concludes the proof of Theorem 1.3.3, by showing that $W = \emptyset$ in a minimal representation. Thus, the existence of an R-minor representation would imply the existence of an R-induced representation, which we have already shown cannot exist.

Recall from Lemma 6.3.15 (applied to both $\rho_0$ and $\rho_1$) that $\rho_0(\cup_{i=1}^n Z_i) = \rho_1(\cup_{i=1}^n Z_i) = 4 - 2n$ and and $\rho_0(\cup_{i\in[n]\setminus\{j\}} Z_i) = \rho_1(\cup_{i\in[n]\setminus\{j\}} Z_i) = 2 - 2n$ for every $j \in [n]$.

**Lemma 6.3.30.** *In a minimal representation we must have* $W = \emptyset$.

*Proof.* For a contradiction, assume $W \neq \emptyset$; pick $i \in [n]$ such that $|Z_i| > 2$. Now, every basis in $\mathcal{M}$ (and thus in $\mathcal{M}_\tau$) must intersect $D_i$ in at least $|Z_i| - 2$ elements (due to the modified representation above, or the second part of Lemma 6.3.15). This guarantees the existence of a $u \in D_i$ such that $u \notin \mathrm{cl}_{\mathcal{M}_\tau}(U \setminus D_i)$. We claim that a smaller representation can be obtained by contracting $u$ in $D_i$ and deleting a node from $W \cap Z_i$.

To see this, it suffices to prove that for every $X \in B_0$ there exists a basis $S$ in $\mathcal{M}_\tau$ conforming $X$ with $u \in S$, and there exists a basis $S_1$ in $\mathcal{M}$ conforming $X_1$ with $u \in S_1$. Then, the requirements listed above remain true in the smaller instance. Note that we do not require that $S_1$ has the largest possible $\tau$-weight; as long as we can guarantee the existence of a basis in $\mathcal{M}$ but not in $\mathcal{M}_\tau$ that conforms $X_1$, we get a function in $\mathcal{F}_n$ that is the same on (S1), but better on (S2) (with possibly different negative value $h(X^*)$.)

Consider any $X \in B_0$ and a basis $S$ in $\mathcal{M}_\tau$ conforming $X$ but $u \notin S$. Let $C \subseteq S \cup \{u\}$ be the fundamental circuit of $u$ with respect to $S$. Then, $(C \setminus u) \cap D_i \neq \emptyset$: otherwise, $C \setminus u \subseteq U \setminus D_i$ would yield $u \in \mathrm{cl}_{\mathcal{M}_\tau}(U \setminus D_i)$, a contradiction to the choice of $u$. Hence, we can exchange $u$ with an element of $S \cap D_i$ and thereby obtain another basis $S'$ conforming $X$ with $u \in S'$.

The same argument applies for the basis $S_1$ in $\mathcal{M}$ conforming $X_1$, noting that $\mathrm{cl}_{\mathcal{M}}(U \setminus D_i) \subseteq \mathrm{cl}_{\mathcal{M}_\tau}(U \setminus D_i)$. □

## 6.4 The size of R-induced representations

We show that any R-induced valuated matroid has an R-induced representation where the bipartite graph has size $O(|V| \cdot d)$, where $d$ is its rank. A corollary is that not all valuated matroids are R-induced.

**Lemma 6.4.1.** *Let $f : \binom{V}{d} \to \mathbb{R} \cup \{-\infty\}$ be an R-induced valuated matroid with representation $G = (V, U; E)$, $\mathcal{M} = (U, r)$ and $c \in \mathbb{R}^E$. Then, there is an R-induced representation of $f$ with $G' = (V, U'; E')$, $\mathcal{M}' = (U', r')$ and $c' \in \mathbb{R}^{E'}$ such that $|\Gamma_{G'}(v)| \le d$ for all $v \in V$. In particular, $|E'| + |U'| + |V| \in O(|V| \cdot d)$.*

*Proof.* Consider an arbitrary node $v \in V$, and the set of its neighbours $\Gamma_G(v)$ in $U$. Let us define a weight function $\omega$ over $\Gamma_G(v)$ as $\omega(u) = c_{vu}$ for $u \in \Gamma_G(v)$. Let $S$ be a maximum weight basis in the matroid $\mathcal{M}$ restricted to $\Gamma_G(v)$ with respect to the weights $\omega$. As $\mathcal{M}$ has rank $d$ it follows that $|S| := s \le d$.

To prove the lemma, it suffices to show that for any set $X \in \mathrm{dom}(f)$ with $v \in X$, in any maximum weight independent matching $\mu^X$ defining $f(X)$ the edge incident to $v$ can be switched to have the other end point in $S$.

Let $\mu^X$ be an independent matching covering $X$ where $X \in \mathrm{dom}(f)$ and $v \in X$. Denote with $u$ the node in $U$ matched to $v$ by $\mu^X$. If $u \in S$, there is nothing to show. So, assume $u \notin S$. Let $T$ be the set of all other endpoints of $\mu^X$ in $U$. That is, the set of endpoints of $\mu^X$ in $U$ is exactly $T \cup \{u\}$, where $u \notin T$ and $|T| = |X| - 1$. We show that we can swap $(v, u)$ by an edge $(v, u')$ for $u' \in S$ without decreasing the weight of the matching.

Denote the elements of the neighbourhood $\Gamma_G(v)$ by $u_1, \ldots, u_s$ such that $\omega(u_1) \ge \cdots \ge \omega(u_s)$. Since $S$ is a maximum weight basis, there is a $k \in [s]$ such that $\omega(u_1) = c_{vu_1} \ge \cdots \ge \omega(u_k) = c_{vu_k} \ge \omega(u) = c_{vu}$ and $u \in \mathrm{cl}(\{u_1, \ldots, u_k\})$ (by the greedy algorithm for finding a maximum weight basis in a matroid).

If we can replace $(v, u)$ by an edge $(v, u_t)$ for $t \in [k]$ in $\mu^X$, we get a new independent matching with weight at least as much as the weight of $\mu^X$. On the other hand, suppose that for any $t \in [k]$ the set $\mu^X \cup \{(v, u_t)\} \setminus \{(v, u)\}$ is not an independent matching. Then, it must be the case that $\{u_1, \ldots, u_k\} \subseteq \mathrm{cl}(T)$. Since, $u \in \mathrm{cl}(\{u_1, \ldots, u_k\})$ it follows that $u \in \mathrm{cl}(T)$. A contradiction. It follows that we can always swap $(v, u) \in \mu^X$ for an edge $(v, u')$ where $u' \in S$, to obtain a matching with weight at least the weight of $\mu^X$. The lemma follows. □

**Information-theoretic separation** We use the above lemma to give an alternative proof that not all valuated matroids are R-induced. Note that this is also proved in Proposition 6.3.10.

Let $f : \binom{V}{4} \to \mathbb{R} \cup \{-\infty\}$ be an R-induced valuated matroid and consider its R-induced representation $(G, \mathcal{M}, c)$ given by Lemma 6.4.1; in particular, $G = (V, U; E)$ where $|E| \le |V| \cdot \mathrm{rk}(f) = 4|V|$. Let $C = \{c_{ij} : (i, j) \in E\}$ be the set of weights appearing on the edges; note that we trivially have $|C| \le 4|V|$. For any set $X \in \binom{V}{4}$, the value $f(X)$ is either $-\infty$ or a sum of precisely four numbers in $C$. This implies the set of function values is contained in the $\mathbb{Q}$-vector space generated by $C$. In particular, the dimension of this vector space is bounded above by $|C|$.

We now exhibit a family of valuated matroids for which the $\mathbb{Q}$-vector space generated by its attained values has dimension greater than $4|V|$. Recall from Definition 1.3.2 and Appendix 6.3.1 the sparse paving matroid with bases $\binom{V}{4} \setminus \mathcal{H}$, where $\mathcal{H}$ the set of pairs $P_i \cup P_j$ where at least one of $i, j$ are even. We define a valuated matroid by

$$h(X) = \begin{cases} 0 & X \in \binom{V}{4} \setminus \mathcal{H} \\ \alpha_X & X \in \mathcal{H} \end{cases}, \ \alpha_X < 0\,.$$

In particular, the values $\alpha_X$ for $X \in \mathcal{H}$ can be assigned freely. Consider such a function for which the set $A = \{\alpha_X : X \in \mathcal{H}\}$ is a set of linearly independent real numbers over $\mathbb{Q}$. Therefore the $\mathbb{Q}$-vector space generated by values of $h$ has dimension at least $|A|$. By definition of $\mathcal{H}$ we have, $|A| = \binom{n}{2} - \binom{\lfloor n/2 \rfloor}{2}$; in particular, this grows quadratically as opposed to $|C|$ which grows linearly. For $n \geq 23$, we have that $|A| > 4 \cdot 2n = 4|V|$. Hence, such a function $h$ is not an R-induced valuated matroid.

Finally we mention that with a similar proof, it is easy to show an analogous lemma for R$^\natural$-induced valuated generalized matroids.

**Lemma 6.4.2.** *Let $f : 2^V \to \mathbb{R} \cup \{-\infty\}$ be an R$^\natural$-induced valuated matroid with representation $G = (V, U; E)$, $\mathcal{M} = (U, r)$ and $c \in \mathbb{R}^E$. Then, there is an R$^\natural$-induced representation of $f$ with $G' = (V, U'; E')$, $\mathcal{M}' = (U', r')$ and $c' \in \mathbb{R}^{E'}$ such that $|\Gamma_{G'}(v)| \leq \min\{n, r(\mathcal{M})\}$ for all $v \in V$. In particular, $|E'| + |U'| + |V| \in O(|V|^2)$.*

# 7 Refuting the MBV conjecture

In this chapter, we build on the family $\mathcal{F}_n$ of counterexamples in Theorem 1.3.3 to refute the MBV conjecture. Theorem 1.3.3 states that all functions in $\mathcal{F}_n$ are valuated matroids but not R-minor valuated matroids.

To refute the MBV conjecture, we extend the class of R-minor valuated matroids to R$^\natural$-minor valuated generalized matroids, and show this contains matroid based valuations as a subclass. Furthermore, we extend our main counterexample to a valuated generalized matroid that is not R$^\natural$-minor and therefore not a matroid based valuation, refuting the MBV conjecture.

## 7.1 Valuated generalized matroids

Recall that a function $f : 2^V \to \mathbb{R} \cup \{-\infty\}$ is a valuated generalized matroid if and only two properties (1.1a) and (1.1b) hold:

$$\begin{aligned} &\forall X, Y \subseteq V \text{ with } |X| < |Y| : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X + j) + f(Y - j)\} \end{aligned}$$

$$\begin{aligned} &\forall X, Y \subseteq V \text{ with } |X| = |Y| \text{ and } \forall i \in X \setminus Y : \\ &f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{f(X - i + j) + f(Y + i - j)\}. \end{aligned}$$

In Section 7.2, we demonstrate a construction which allows one to consider valuated generalized matroids as special cases of valuated matroids on a larger ground set. On the other hand, we already saw valuated matroids as special class of valuated generalized matroids. Another important class are the trivially valuated generalized matroids, those taking only values $0$ and $-\infty$. This includes the characteristic functions of the family of independent sets of a matroid. Indeed, if $g(\emptyset) > -\infty$ for a valuated generalized matroid, then $\operatorname{dom}(g)$ is the family of independent sets of a matroid [99, Corollary 1.4].

We defined several constructions for valuated matroids which are defined on the layers of those subsets with fixed size in Section 5.1. It turns out that these operations extend essentially layer-wise to valuated generalized matroids. In the following we denote the restriction of a valuated generalized matroid $g$ on $V$ to $\binom{V}{k}$ by $\ell^k(g)$.

**Definition 7.1.1.** *Let $N = (T, A)$ be a directed network with a weight function $c \in \mathbb{R}^A$. Let $V, U \subseteq T$ be two non-empty subsets of nodes of $N$. Let $g$ be a valuated generalized matroid on $U$. Then the* induction of $g$ by $N$ *is the function $\Phi(N, g, c)\colon 2^V \to \mathbb{R} \cup \{-\infty\}$ such that*

$$\ell^k\left(\Phi(N, g, c)\right) = \Phi(N, \ell^k\left(g\right), c),$$

*where $\Phi(N, g, c)(\emptyset) = g(\emptyset)$.*

*In the special case that the directed network is bipartite with the edges directed from $V$ to $U$, we can also consider this as an undirected weighted bipartite graph and call the corresponding operation* induction by bipartite graphs*.*

Analogous to Theorem 5.1.9 this is just a special case of transformation by networks.

**Theorem 7.1.2** (Special case of [95, Theorem 9.27])**.** *Let $N, g, c$ as in Definition 7.1.1. Then if $\Phi(N, g, c) \not\equiv -\infty$ the induced function is a valuated generalized matroid.*

As with induction of valuated matroids, we shall often be most interested in the induction of trivially valuated generalized matroids. A trivially valuated generalized matroid $g$ can be identified with its underlying domain $\mathcal{I}$, where $g(I) = 0$ if $I \in \mathcal{I}$ and $-\infty$ otherwise. As stated previously, if $\emptyset \in \mathcal{I}$ then $\mathcal{I}$ forms the set of independent sets of a matroid; however this does not have to be the case, $\mathcal{I}$ only has to satisfy the independent set exchange axiom (the unvaluated equivalent of (1.1a)). We call such an $\mathcal{I}$ a *generalized matroid*. As working with $\mathcal{I}$ directly will be convenient in some situations, we extend the notation of Definition 7.1.1 to define $\Phi(N, \mathcal{I}, c) := \Phi(N, g, c)$.

The following example shows why induction of trivially valuated generalized matroids is a natural construction to consider.

**Example 7.1.3.** *Let $\mathcal{I}$ be the independent sets of a matroid $\mathcal{M}$ on ground set $V$. A* weighted rank function *$r^w : 2^V \to \mathbb{R}_{\geq 0}$ with weight $w \in \mathbb{R}^n_{\geq 0}$ is*

$$r^w(X) = \max\left\{\sum_{i \in I} w_i : I \subseteq X\,,\, I \in \mathcal{I}\right\} \quad .$$

*Note that if $w$ is the vector of all ones, then $r^w$ is precisely the rank function of $\mathcal{M}$.*

*Let $V'$ and $V''$ be copies of $V$ and let $\overline{\mathcal{I}}$ be the independent sets of the matroid $\overline{\mathcal{M}} = \mathcal{M} \oplus fr_{V''}$ on $V' \cup V''$. Furthermore, we define the bipartite graph $G = (V, V' \cup V''; E)$ where $E$ consists of the edges $(v, v')$ and $(v, v'')$ connected each node in $V$ its copies in $V'$ and $V''$. We attach weights $c \in \mathbb{R}^E$ where the edge $(v, v')$ gets the weight $w_v$ and the edge $(v, v'')$ gets the weight $0$.*

*Let $I \subseteq X$ be the max weight independent set contained in $X$. The value of $\Phi(G, \overline{\mathcal{I}}, c)(X)$ is obtained by connecting elements of $I$ to $I' \subseteq V'$ via edges of weight $w_i$, and then connecting elements of $X \setminus I$ to their copy in $V''$ by edges of weight zero. In this way $r^w = \Phi(G, \overline{\mathcal{I}}, c)$ arises from a trivially valuated generalized matroid by induction via a bipartite graph.*

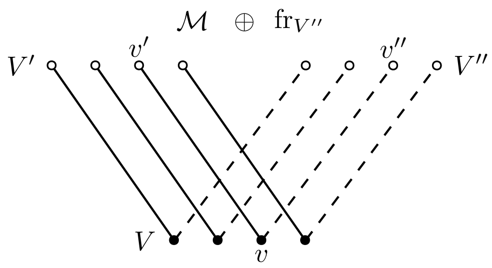


Figure 7.1: The graph $G = (V, V' \cup V''; E)$ realising the weighted matroid rank function from Example 7.1.3. Edges of weight $w_v$ are solid while edges of weight zero are dashed.

Many of the operations on valuated matroids extend to valuated generalized matroids by acting layerwise.

**Definition 7.1.4.** *Let $f\colon 2^V \to \mathbb{R} \cup \{-\infty\}$ be a valuated generalized matroid and $Y \subset V$ some subset of $V$. The operations* deletion (restriction), contraction, dualization, truncation, principal extension *are defined by the respective operations on the layers from Definition 5.1.1.*

Note that direct sum and valuated matroid union do not extend layerwise to valuated generalized matroids. Intuitively, this is because the $k$-th layer of the union must take information from multiple layers of the constituent valuated generalized matroids, all $i$-th and $j$-th layers such that $k = i + j$. The analogue of direct sum and valuated matroid union for valuated generalized matroids is the following operation.

**Definition 7.1.5.** *Let $f, g\colon 2^V \to \mathbb{R} \cup \{-\infty\}$. The* merge *of $f$ and $g$ is the function $f * g\colon 2^V \to \mathbb{R} \cup \{-\infty\}$ defined as*

$$(f * g)(X) = \max\{f(Y) + g(X \setminus Y) : Y \subseteq X\}, \quad \forall X \subseteq V.$$

With these operations, we get an analogue of Theorem 5.1.12.

**Theorem 7.1.6.** *The class of valuated generalized matroids is closed under the operations deletion, contraction, dualization, truncation, principal extension, merge.*

*Proof.* Deletion, dualization and merge are covered by [95, Theorem 6.15]; the latter is integer infimal convolution restricted to the interval $[0, 1]$, parts (8) and (5) respectively. Lemma 5.1.13 implies layerwise closure under contraction and therefore globally closed contraction. Remark 5.1.10 shows principal extension are special cases of induction by networks, which valuated generalized matroids are closed under via Theorem 7.1.2. Finally, Lemma 5.1.14 implies layerwise closure under truncation and therefore globally closed under truncation. □

It was shown in [10] that valuated generalized matroids are not covered by the cone of matroid rank functions; note that not even all non-negative combinations of matroid rank functions are valuated generalized matroids. In particular, not every valuated generalized matroid can be represented as a weighted matroid rank function [113, Theorem 4].

However, it was conjectured that allowing two operations, merge and endowment, would suffice to construct all. Here, the endowment by $T \subseteq V$ of a function $f\colon 2^V \to \mathbb{R} \cup \{-\infty\}$ is the function $\Delta_T(f)\colon 2^{V\setminus T} \to \mathbb{R} \cup \{-\infty\}$ with $\Delta_T(f)(X) = f(X \cup T) - f(T)$ for all $X \subseteq V \setminus T$.

With this, the class of matroid based valuations are those functions arising from the class of weighted matroid rank functions by arbitrary application of merge and endowment.

**Conjecture 7.1.7** (MBV conjecture [104])**.** *The class of matroid based valuations is equal to the class of monotone valuated generalized matroids taking value zero on the empty set.*

We study a subclass of valuated generalized matroids which is an extension of the class of R-minor valuated matroids. This allows us to use the results from Section 6.3.

**Definition 7.1.8.** *The class of* R$^\natural$-induced functions *are valuated generalized matroids arising from trivially valuated generalized matroids via induction by bipartite graph.*

*The class of* R$^\natural$-minor functions *are valuated generalized matroids arising from contractions of R$^\natural$-induced functions.*

Throughout the proofs in this section, we use the same notation as introduced in Section 5.3. Let $f$ be an R$^\natural$-minor function on $V$; by definition, there exists an R$^\natural$-induced function $\tilde{f}$ on $V \cup W$ such that $f = \tilde{f}/W$. By definition, there exists some bipartite graph $G = (V \cup W, U; E)$ with edge weights $c \in \mathbb{R}^E$ and generalized matroid $\mathcal{I}$ on $U$ such that $\tilde{f} = \Phi(G, \mathcal{I}, c)$; we say $\tilde{f}$ has an *R$^\natural$-induced representation* $(G, \mathcal{I}, c)$. As $f = \Phi(G, \mathcal{I}, c)/W$, we extend this notation to say that $f$ has an *R$^\natural$-minor representation* $(G, \mathcal{I}, c, W)$, where $W$ is the set to be contracted.

**Lemma 7.1.9.** *The class of R$^\natural$-minor functions is closed under endowment.*

*Proof.* Given $f$ as above, we show we can represent $\Delta_T(f)$ as an R$^\natural$-minor function for some $T \subseteq V$. As $f$ is a contraction of $\tilde{f}$ by $W$, the endowment by $T$ can be written as

$$\Delta_T(f) = f(X \cup T) - f(T) = \tilde{f}(X \cup T \cup W) - \tilde{f}(T \cup W) = \Delta_{T \cup W}(\tilde{f})\,.$$

Let $\delta = \tilde{f}(T \cup W)/|T \cup W|$ and consider a new edge weight function $c'(e)$ that takes the value $c(e) - \delta$ on all edges adjacent to $T \cup W$, and $c(e)$ otherwise. Then the induction of $\mathcal{I}$ through the graph $G$ with altered weight function $c'$ is

$$(\Phi(G, \mathcal{I}, c'))(Z) = \tilde{f}(Z) - \delta \cdot |Z \cap (T \cup W)|\,.$$

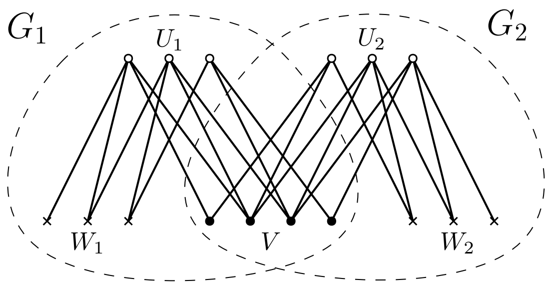


Figure 7.2: The graph $G'$ constructed in the proof of Lemma 7.1.10, obtained by gluing $G_1$ and $G_2$ at their common node set $V$.

Taking the contraction of $\Phi(G, \mathcal{I}, c')$ by $T \cup W$ yields

$$(\Phi(G, \mathcal{I}, c')/(T \cup W))(X) = \tilde{f}(X \cup T \cup W) - \delta \cdot |T \cup W| = \Delta_T(f)(X)\,. \qquad \square$$

**Lemma 7.1.10.** *The class of $R^\natural$-minor functions is closed under merge.*

*Proof.* Let $f_1, f_2$ be $\mathrm{R}^\natural$-minor functions on a common ground set $V$ with representation $(G_i, \mathcal{I}_i, c_i, W_i)$ where $G_i = (V \cup W_i, U_i, E_i)$ for $i = 1, 2$. In particular, we can choose the contracted sets to be disjoint i.e., $W_1 \cap W_2 = \emptyset$. This last assertion is particularly important as it allows merge and contraction to commute. By extending $\tilde{f}_1$ and $\tilde{f}_2$ to the ground set $V \cup W_1 \cup W_2$, taking the value $-\infty$ where previously undefined, we see that for any $X \subseteq V$,

$$\begin{aligned}
(f_1 * f_2)(X) &= (\tilde{f}_1/W_1 * \tilde{f}_2/W_2)(X) \\
&= \max\{\tilde{f}_1(Y \cup W_1) + \tilde{f}_2((X \setminus Y) \cup W_2) : Y \subseteq X\} \\
&= \max\{\tilde{f}_1(Z) + \tilde{f}_2((X \cup W_1 \cup W_2) \setminus Z) : Z \subseteq X \cup W_1 \cup W_2\} \\
&= (\tilde{f}_1 * \tilde{f}_2)(X \cup W_1 \cup W_2) \\
&= ((\tilde{f}_1 * \tilde{f}_2)/(W_1 \cup W_2))(X)\,.
\end{aligned}$$

Therefore if we can represent $(\tilde{f}_1 * \tilde{f}_2)$ via induction by bipartite graph, contracting $W_1 \cup W_2$ completes the proof.

Let $G'$ be a graph obtained by "gluing" $G_1$ and $G_2$ along their common ground set. Explicitly, $G' = (V \cup W_1 \cup W_2, U_1 \cup U_2; E_1 \cup E_2)$ whose weight function $c'$ inherits the same weights from $c_1$ and $c_2$. The graph is given in Figure 7.2. We consider the trivially valuated generalized matroid $\mathcal{I}' = \mathcal{I}_1 \oplus \mathcal{I}_2$. Then the value of $\Phi(G', \mathcal{I}', c')(Z)$ is the maximum over all matchings from $Y \subset Z$ to $U_1$ and matchings $Z \setminus Y$ to $U_2$, ranging over subsets $Y \subset Z$, precisely realizing $(\tilde{f}_1 * \tilde{f}_2)$ as an $\mathrm{R}^\natural$-induced function. $\square$

Example 7.1.3 showed that weighted matroid rank functions are special cases of $\mathrm{R}^\natural$-induced functions. Combining this with Lemmas 7.1.9 and 7.1.10, we see that matroid based valuations are a subclass of R$^\natural$-minor functions.

**Corollary 7.1.11.** *Matroid based valuations form a subclass of R$^\natural$-minor functions with the properties that they are monotone, real-valued and have value* $0$ *on the empty set.*

### 7.1.1 A valuated generalized matroid extending a robust matroid

Let $h$ be an arbitrary function in the class $\mathcal{F}_n$ in Definition 1.3.2 which takes only values in $(-1, 0]$.

We define a function $h^\natural : 2^V \to \mathbb{R}$ by

$$h^\natural(X) = \begin{cases} |X| & \text{for } |X| \leq 3 \\ 4 + h(X) & \text{for } |X| = 4 \\ 4 & \text{for } |X| \geq 5 \end{cases}$$

Note that $h^\natural$ is a perturbed rank function of the uniform matroid on $V$ of rank $4$.

**Lemma 7.1.12.** *The function $h^\natural$ is a valuated generalized matroid.*

*Proof.* We first show $h^\natural$ satisfies (1.1b), where $|X| = |Y| = k$. When $k \neq 4$, all sets of that cardinality $k$ have the same value and so $h^\natural$ satisfies (1.1b). The case when $k = 4$ follows from Lemma 6.3.2 and all sets being scaled by the same value.

We next show $h^\natural$ satisfies (1.1a), where without loss of generality $|X| < |Y|$.

- If $|X| \geq 5$, then all sets take the value 4, and therefore trivially satisfy (1.1a).
- If $|X| = 4$, then $h^\natural(X) + h^\natural(Y) \leq 8$. If we can pick $i \in Y \setminus X$ such that $Y \setminus i \notin \mathcal{H}$, then $h^\natural(X + i) + h^\natural(Y - i) = 8$ and this case holds. If no such $i$ exists, then $|Y| = 5$. Furthermore, there cannot be two distinct elements $i, j \in Y \setminus X$, else $Y - i, Y - j \in \mathcal{H}$ intersect in three elements, which no pairs in $\mathcal{H}$ do. Therefore $Y = X \cup i$, and so (1.1a) holds with equality.
- If $|Y| = 4$, then $h^\natural(X) + h^\natural(Y) \leq |X| + 4$. If we can pick $i \in Y \setminus X$ such that $X \cup i \notin \mathcal{H}$, then $h^\natural(X + i) + h^\natural(Y - i) = |X| + 4$ and this case holds. By a similar argument as above, if no such $i$ exists then $Y = X \cup i$, and so (1.1a) holds with equality.
- If $|Y| \leq 3$, then all sets take the value of their cardinality, and therefore trivially satisfy (1.1a). □

**Lemma 7.1.13.** *For $n \geq 16$, the function $h^\natural$ is not an R$^\natural$-minor function.*

*Proof.* Suppose $h^\natural$ is R$^\natural$-minor, therefore it has representation $(G, \mathcal{I}, c, W)$ for some graph $G = (V \cup W, U; E)$. We claim we can find an R-minor representation for $h$.

First note that

$$\begin{aligned} h(X) &= \ell^4\left(h^\natural\right)(X) - 4 \\ &= \ell^{|W|+4}\left(\Phi(G,\mathcal{I},c)\right)(X\cup W) - 4 \\ &= \Phi(G,\ell^{|W|+4}\left(\mathcal{I}\right),c)(X\cup W) - 4\,. \end{aligned}$$

By introducing the altered weight function $c'(e) = c(e) - 4/(|W|+4)$, we get

$$\Phi(G,\ell^{|W|+4}\left(\mathcal{I}\right),c')(X\cup W) = \Phi(G,\ell^{|W|+4}\left(\mathcal{I}\right),c)(X\cup W)) - \frac{4|X\cup W|}{|W|+4} = h(X)\,.$$

Therefore, $h$ has the R-minor representation $(G,\ell^{|W|+4}\left(\mathcal{I}\right),c',W)$, contradicting Theorem 1.3.3. □

**Theorem 7.1.14.** *The class of R$^\natural$-minor functions is not equal to the class of valuated generalized matroids. In particular, Conjecture 7.1.7 is false.*

*Proof.* The first claim follows immediately from Lemmas 7.1.12 and 7.1.13. For the second claim, we observe that $h^\natural$ is a monotone and only takes finite values. However, by Corollary 7.1.11 it is not a matroid based valuation, providing a counterexample to Conjecture 7.1.7. □

## 7.2 From valuated generalized matroids to valuated matroids

By definition, valuated matroids are defined only on a layer of the ground set, but it is easy to check that each valuated matroid is also a valuated generalized matroid if we set the function value to $-\infty$ outside of the layer. Another way to obtain a valuated generalized matroid from a valuated matroid is by truncation (introduced in Section 5.1) and elongation. The interested reader is referred to [94], in particular Theorem 3.2.

Here, we demonstrate how to go in the other direction, i.e., how to represent a valuated generalized matroid as a valuated matroid. Then we show an explicit construction for the case of R$^\natural$-minor valuated generalized matroids.

Let $f : 2^{V_1} \to \mathbb{R}\cup\{-\infty\}$ be a valuated generalized matroid. Denote with $n$ the size of $V_1$ and let $V_2$ be a copy of $V_1$. We define a function $g_f : \binom{V_1\cup V_2}{n} \to \mathbb{R}\cup\{-\infty\}$ for $X \in \binom{V_1\cup V_2}{n}$ as

$$g_f(X) := f(X\cap V_1)\,.$$

Then, it is a straightforward check via the valuated (generalized) matroid axioms that the function $g_f$ is a valuated matroid. Note that given such a function $g_f$, we can recover $f$ as

$f(X) = g_f(X \cup Y)$ for any $Y \subseteq V_2$ of size $n - |X|$.

Starting with an R$^\natural$-induced or an R$^\natural$-minor valuated generalized matroid, a similar construction gives rise to an R-minor valuated matroid. Let $f : 2^{V_1} \to \mathbb{R} \cup \{-\infty\}$ be an R$^\natural$-minor valuated generalized matroid represented by $(G_1, \mathcal{M}_1, c, W)$ where $G_1 = (V_1 \cup W, U_1; E)$. For $n = |V_1|$, let $V_2, U_2$ be two disjoint sets, each with $n$ elements, and disjoint from $V_1 \cup W \cup U_1$. Let $\mathcal{M}_2$ be the free matroid on $U_2$. Consider the R-minor valuated matroid $g$ defined by the bipartite graph $G = (\,(V_1 \cup V_2) \cup W,\ U_1 \cup U_2;\ E')$, matroid $\mathcal{M}$ on $U_1 \cup U_2$, $c' \in \mathbb{R}^{E'}$, and $W$; where

- $\mathcal{M}$ is obtained by truncating $\mathcal{M}_1 \oplus \mathcal{M}_2$ to the size $|W| + n$, and
- $E'$ is obtained from $E$ by adding all possible edges $(i, j)$, for $i \in V_2$, $j \in U_1 \cup U_2$,
- $c'$ extends $c$ to $E'$ by weighting all edges in $E' \setminus E_0$ by zero.

Then, a maximal independent matching in $G$ on $X \cup W$ must come from a maximal independent matching in $G_1$ with additional zero weight edges adjacent to all nodes in $X \cap V_2$, verifying that $g$ is the same valuated matroid as $g_f$ defined in the previous paragraph.

# 8 Conclusion and future directions

We presented three main results in this thesis. We recall these results, and propose several questions and open problems.

We gave an auction algorithm for finding an approximate market equilibrium in Arrow-Debreu exchange markets when agents have weak gross substitutes (WGS) demands. We believe that this class of demands is a maximal class of demands for which auction-type algorithms converge to an $\epsilon$-equilibrium as in the case of markets with indivisible items. An interesting direction is to give a compelling argument on when and why an equilibrium problem with divisible goods admits an auction-type algorithm, i.e., an algorithm with increasing prices. As we have seen, in some cases the equilibria are captured as the optimal solutions of a convex program, where the Lagrangian multipliers of the constraints correspond to the prices. In such cases the (non-) existence of an auction-type algorithm could be explained from a convex programming perspective. This line of work was suggested by Chandra Chekuri during the PhD viva.

The auction algorithm framework is a robust one. As we have shown, our auction algorithm is easily modified for finding spending-restricted market equilibria in Fisher markets and assuming agent have WGS demands. Previous auction algorithms [59] have been extended to the markets where agents satisfy the WGS property only approximately [78]. In this case the auction algorithm converges to an approximate equilibrium where the approximation factor additionally depends on how close the demands are to being WGS, e.g., if demands satisfy $\delta$-approximate WGS property, then there is an auction algorithm that converges to a $(\delta + \epsilon)$-approximate equilibrium. We expect that our auction algorithm also extends to the setting where agents' demands satisfy the WGS property only approximately.

Our second main contribution is a constant-factor approximation algorithm for the symmetric Nash social welfare (NSW) problem under Rado valuations. The algorithm also works for the asymmetric NSW problem under Rado valuations and produces an $O(\gamma^3)$-approximation algorithm, where $\gamma$ is such that the weights of all agents fall in the interval $[1, \gamma - 1]$ for $\gamma \geq 2$. In a subsequent work, Li and Vondrák [89] obtained a 320-approximation algorithm for the symmetric NSW problem under arbitrary submodular valuations. They extended and strengthened our approach. Their algorithm likely extends also to the asymmetric NSW under submodular valuations with approximation guarantee depending on $\gamma$. Hence, two obvious open questions remain:

- Design a constant-factor approximation algorithm for the asymmetric NSW problem under additive valuations (or more general valuations). Recall that our approximation algorithm loses $\gamma$ only in **Phase II**.
- Improve the approximation factor for the NSW problem under submodular valuations.

The Nash social welfare problem asks for an allocation of the items to the agents while maximizing the geometric mean of agents' valuations. The geometric mean is a $p$-mean for $p = 0$; and one could define the same problem for any other $p$-mean for $p \in [-\infty, 1]$. When $p = -\infty$ and $p = 1$ we recover the max-min welfare (Santa Claus) problem, and the social welfare problem, respectively. It is increasingly popular to investigate the general problem of allocating goods to the agents in order to maximize a $p$-mean of their valuation, see e.g. [25, 14] and references therein. An interesting problem is to find a constant-factor approximation algorithm that works for any given $p$-mean even for additive valuations. Note that finding such an algorithm is quite challenging as the special case of $p = -\infty$ is the Santa Claus problem [8] – a significant open problem.

As an encouragement, we point out that the complementary problem of minimum-norm load balancing admits a $(2 + \varepsilon)$-approximation algorithm that works for any symmetric norm $f$ [70]. In the minimum-norm load balancing problem, the goal is to allocate the items (jobs) to the agents (machines) in order minimize the $f$-norm of the agents' valuations (machine loads).

For our third main contribution we exhibited a family of valuated matroids that are not R-minor valuated matroids. As a corollary we showed that the Matroid Based Valuation conjecture does not hold. Hence, the quest for a constructive characterization of gross substitutes valuations and valuated matroids remains open. From the perspective of complete classes there are two natural next options. The first option is to find a necessary conditions for a minimal class of valuations $\mathcal{C}$ such that the complete class containing $\mathcal{C}$ covers all GS valuations. We showed that $\mathcal{C}$ cannot be the class of matroid rank functions. The second option is to add additional operations and define complete class to be closed under the additional operations as well. A related possibility is that all GS valuations arise as sums of R-minor valuated matroids. Note that it has been shown that there are GS valuations that are not sums of weighted matroid rank functions [11].

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