AN APPLICATION OF THE TECHNIQUES OF LINEAR PROGRAMMING TO THE TRANSPORTATION OF COAL.

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Abstract of the Thesis

The object of this study was to find the minimum transport cost routing pattern for coal delivered to coke ovens. The quantities available at each colliery and required at each coke oven were assumed to be the same as the quantities actually available and required during one particular month.

A model is developed from the Koopmans transport model of linear programming for minimizing rail ton-mileage travelled by a homogeneous commodity. The method of finding the solution to this model, which makes it possible to compute a larger problem by hand than would be possible using the standard transport model and standard computing methods, is described in detail.

The problems peculiar to applying the model to coal transport are considered and the methods of modifying the model to take account of these peculiarities--particularly the non-homogeneity of coal--are discussed.

The optimum solution is compared to the actual transport pattern, and it is found that a reduction of 10% in the total ton-mileage could have been made by the application of linear programming. It is estimated that this is equivalent to a saving of the order of £500 thousand per annum in the transport cost

of coking coal.

The field of problems which remain to be studied is briefly surveyed. This includes applications of the model to all forms of coal transport and to other commodities, and the further research required into computing methods.

The "dual" solution of the linear programming model can be interpreted as the prices for coal in the different geographical regions. The relevance of these prices to the pricing policy and the development policy of the National Coal Board is discussed.

Contents of the Thesis

Introduction

I

This study is the report on an application of linear programming to the transport of coal for coking. The main interest is in the development of the model and the method of obtaining the solution. The application to coal transport is in the nature of a test of the model, by comparing an optimum solution with an actual distribution of coking coal within one month.

There is a growing interest in the field of study known alternatively as "linear programming" or "activity analysis", which may be regarded either as a technique of operations research or as a branch of economics. Very briefly, it is a method of listing all the resources available in a system together with all the different activities in which these resources might be employed, and then computing by a process of successive approximation the best combination of the activities to maximize (or minimize) some function of them, such as profit (or cost).

Linear programming belongs to operations research because it can be used in the planning of actual operations or activities such as production, transport, or militrry operations. In that use the interest is focussed on the minimization of some loss (money cost or time or men, for instance) or the maximization of some gain (money profit or the production of some commodity, for instance) or on the rate of substitution between two or more such desirable ends.[1]

Linear programming belongs also to economics, because the best use of limited resources is the central problem of economics. Every maximum or minimum solution of a linear programming problem has also a "price" solution which, as it were, evaluates all the commodities in the model in terms of the "commodity" which has been maximized or minimized. This "price" solution is called the <u>dual solùtion</u>. Usually there is a single set of such evaluations uniquely associated with the optimum use of the commodities in a model. A large part of micro-economics

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1. Called in economics terminology the "transformation curve" between two or more desirable ends.

centres on the concept of equilibrium prices, thought of primarily as the set of prices which would be established by market forces. Here, again, a dual concept exists in that (under certain rigidly defined conditions) equilibrium prices can be shown to be associated with the best use of resources.

In other words, given a set of commodities and the knowledge of various possible ways in which they might be used and produced, the practical production controller would focus his attention on the best way of using and producing the commodities, and the economist would think in terms of the price set--the marginal costs and marginal revenues of the commodities. Linear programming provides a link between these dual approaches. On the one hand it shows that solving the production problem is impossible without also solving the other, or dual, aspect, the pricing problem (as economists have always claimed). But on the other hand it stresses what economists have not always realized, that to evaluate the prices (i.e., marginal costs and revenues)--as distinct from leaving them to be determined by market forces--is also impossible without at the same time solving the dual aspect of the problem, the actual direction of the production process.

Thus business men who apply linear programming techniques to specific management problems, are controlling those segments of their organizations by the very principles which economists have always claimed would maximize their profits. But whereas economics as such had very little to say on the actual method of applying these principles, except in the very simplest industrial situations (e.g., the single product firm), linear programming provides the technique for obtaining both aspects of the dual solution in actual situations.

While it is true that the dual aspects of the solution of a linear programming problem are indissolubly linked, so that knowledge of the equilibrium price set implies also knowledge of the optimum pattern of control of activities, it may still be that in practice the control is better carried out via the price set than by direct orders. That would depend entirely on the individual situation, including the cost of recomputation as the data changed.

II

The only drawback to the wholesale application of linear programming is that there are costs involved in its use. In the first place there is the cost of constructing the model. Secondly, there is the cost of communication in the sense of the regular collection of data and despatch of instructions. Thirdly there is the cost of computation.

There is a standard form of linear programming which is in principle applicable to every problem of minimization or maximization of a linear function (cost or profit) subject to linear constraints (limited resources). But in practice there has to be selection and compression of data in order to construct a manageable model. Also, some simplifying assumptions, which may or may not be reasonable in a particular instance, enable a simpler form of linear programming to be used--that which is known as the "transport problem", associated with the name of T.C. Koopmans.[1]

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1. T.C. Koopmans & S. Reiter "A Model of Transportation" Chapter XIV of "Activity Analysis of Production & Allocation" Cowles Commission Monograph No. 13. ed. T.C. Koopmans.

Thus model construction depends on the nature of the particular problem for which it is being designed. However, it is normally a capital cost, in that once it is designed it would usually be applied regularly to data which were changing through time.

The cost of communication, again, depends closely on the nature of the problem: it may be trivial in a case of machine shop management or very heavy in a transport problem where by the very nature of the problem data have to be collected from and instructions issued to individuals in widely separated places. It is in this type of situation where a case might exist for decentralizing detailed decision taking and exercising overall control via the dual price solution.

The cost of computation is bound to be heavy. For instance, a program may have to be computed electronically in order to get an answer soon enough to be useful. Here again it may be possible to effect savings by taking advantage of particular features of the problem. As the cost of calculation rises with the complexity of the problem rather than with the magnitudes involved it seems probable that linear programming will be most useful on large scale problems, where even a small percentage improvement over previous methods is sufficient in absolute amount to make the computation worth while. This means that the most likely field for its application at the present time is in the large industrial organizations, including the nationalized industries.

III

As has been mentioned above, there is one group of problems very much simpler than the general linear programming model. The transport model was independently developed and only later fitted into the general linear programming mould. Basically this problem is to allocate a homogeneous commodity available in known quantities at several different places to various other places, in such a way as to minimize the cost of transport. It is not strictly limited to transport (through space and time) since some simple production allocation jobs can be handled in this form with more or less sacrifice in accuracy, but with great gains in simplicity and computational speed. Nevertheless, the possible field of application of the transport model in the transport of goods and location of industry would seem to be sufficiently wide to justify a presentation in those terms alone, without reference to the general linear programming model. It is felt that the principles of the transport problem can be convincingly, if not rigorously, demonstrated by an appeal to the visual sense, and thereby be made intelligible to a larger number of possible users than if they are subsumed in the general principles of linear programming. This is the intention of Chapter 1, "The Model". To make the discussion more concrete, it has been framed in terms of the movement of coking coal, and indeed in terms of an actual numerical problem--the problem discussed in the later chapters. But this is purely an expositional device: it could be any homogeneous commodity available and required anywhere in the world.

In Chapter 2, the applicability of the model to the actual problems of coal transport is discussed. The particular choice of coking coal is determined by the availability of data which enable a comparison of the theoretical optimum with the actual solution to be made.

In Chapter 3, the method of calculation used in this project is explained, Chapter 4 discusses the results of the calculation, and Chapter 5 some possible other applications of the same or similar models.

Chapter 1 The Model

I

A common problem of large industrial organisations, both public and private, which have plants located in different parts of the country is to move goods from one place to another as cheaply as possible. For instance, a firm may have factories in a dozen towns in widely separated parts of the country, and sell its products over the whole country: how should the market be divided between the different factories? A large part of the answer will often seem to be obvious; the difficulties arise over where to draw the boundary lines. There are also the allied questions of which factories to expand and which to contract; the location of new factories, or of warehouses to store goods for future distribution.

The economist might suppose that the price mechanism would provide an answer to this type of problem by a variation in the local prices. But in practice, standardized products are often sold at an equal price in all markets if the seller bears the transport cost, or at all factories if the consumer bears the transport cost. If the product is one which does not enter the market, the only price mechanism is that of cost accounting. This might, of course, be so constructed as to reproduce a sort of competitive price structure, in which case it would provide a valid guide to transport decisions as it would to other decisions internal to the organisation. As will be seen, the methods presented here could be used as the basis of such a costing system.

The complete answer to all types of transport problem will not be found here, or even the complete theoretical answer. But it is claimed that the "best" answer to any such problem will rest on some of the basic concepts which follow. The procedure for arriving at the "best" answer, even for the cases which exactly fit the theoretical models, is not itself costless. Whether it is worth carrying out must depend, in any particular case, on the magnitude of the saving to be made, relative to the cost of reaching the "best" solution rather than a "rule-of-thumb" solution.

The particular model which will be developed here assumes an organisation with many plants producing a single homogeneous commodity and supplying it to many different consumers. It assumes, furthermore, that the only method of transport is the railway, and that the rail charge is constant per ton, and strictly proportional to the shortest rail distance connecting any pair of points.[1] If it is also assumed that the amounts of the commodity both available and required are known and fixed for the period under consideration, the allocation problem is simply that of determining which plant shall supply which consumer in such a way as to minimise the ton-mileage travelled.

For the sake of concreteness and to lead on to the application of the model in later chapters, let us say that we are trying to find the best allocation of coal from 160 pits to 60 coke ovens during one month, after the amount to be supplied by each pit and received by each coke oven has been determined. In Chapter 2 we

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1. But it can be modified if the charge is proportional to working-distance. See Appendix B to Chapter 3.

shall investigate how far our model is applicable to the actual transport of coal--at present we are concerned only with the development and exposition of the model itself.

The specific numerical problem is represented pictorially on Map 1.[1] The lines are part of the railway network of England and Scotland; the black balls are tonnage available and the red balls tonnage required. The amount available is equal to the amount required, but this is not an essential feature of the problem because building up or running down of stocks can be allowed in the short period.[2] But in the long run this is a necessary equality like that of supply and demand.

The solution of this allocation problem is by no means trivial. Rule of thumb methods, such as allocating the coal first to thos coke ovens nearest a pit and then those further away, etc., inevitably involve long cross

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1. The actual numerical data are to be found in Appendix A to Chapter 2.

2. See Appendix C to Chapter 1.

hauls on the last remaining consignments.

Map 2 represents the solution of this problem (calculated by the method described in Chapter 3). The heavy black lines represent the "flows" of coal, in the directions indicated by the small adjacent arrows. A comparison has been made with the way in which the coal was actually allocated.

The proof that Map 2 shows the cheapest possible solution of the problem will be found elsewhere.[1] We shall discuss here some easily demonstrated and convincing characteristics of the solution.

II

Two related terms basic to the whole subject of optimal commodity transportation are "tree" and "loop".[2] Regarding the routes used in the transport of the commodity in Map 2 simply as the lines of a graph, without taking account of

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1. Appendix C to this chapter and "A Model of Transportation", T.C. Koopmans and S. Reiter, Chapter XIV of "Activity Analysis of Production & Allocation" ed. T.C. Koopmans, Cowles Commission Monograph No. 13.

2. Koopmans & Reiter, l.c., use the term "circuit" where "loop" is used here.

their direction or the tonnage they carry it will be seen that, with the exception of two small areas (just south of Leeds, and south west of Colne)the lines form a connected graph. In other words, one could travel from any point on that graph to any other point using only the railway lines used in the solution.

Less obvious is the fact that there is only one such connection between any pair of points: or, to say the same thing in a different way, there are no loops in the graph. By a "loop" is meant the existence of a complete circuit through some part of the solution graph so that it is possible to take a round trip from some point and back to that point using only the routes along which coal is being moved. This is difficult to appreciate on a map of this scale, as lines crossing without any connection cannot be distinguished from junctions. For instance, there appears to be at least one loop between Barnsley and Sheffield and another just west of Doncaster. But in fact they are all due to lines crossing with no junctions. A connected graph which contains no loops is called a "tree".

There always exists a minimum cost solution to this type of problem which contains no loops. In other words, if there is an allocation pattern claimed to be a minimum cost allocation, which contains loops in it, then all the loops can be removed and the pattern reduced to a tree, without increasing the total cost of the operation. Thus one is not restricted in seeking the minimum total cost by confining oneself to patterns of allocation which contain no loops. The pattern containing all possible minimum cost allocations[1] may have loops in it, but it can be readily obtained, if it should be required, from a minimum cost tree. This superfluousness or actual disadvantage of a loop can be illustrated as follows.

The very simplest form of loop would be one pit supplying a certain coke oven by two different routes:-

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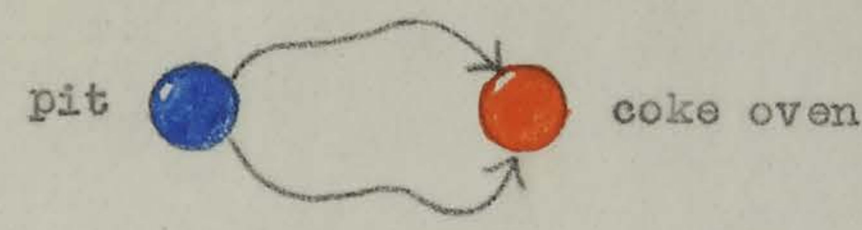
1. Known as the maximal efficient graph.

Diagram 1



It is clear that such a loop would not form part of the minimum cost solution except in the special case where both routes were the same distance: and that even in the special case a solution could be found using only one arm of the loop. Another very simple loop which is intuitively obviously inefficient is a movement of coal in opposite directions along a stretch of line.

The loop remains basically simple if we consider three points, say, two sending and one receiving, or one sending and two receiving:-

Diagram 2

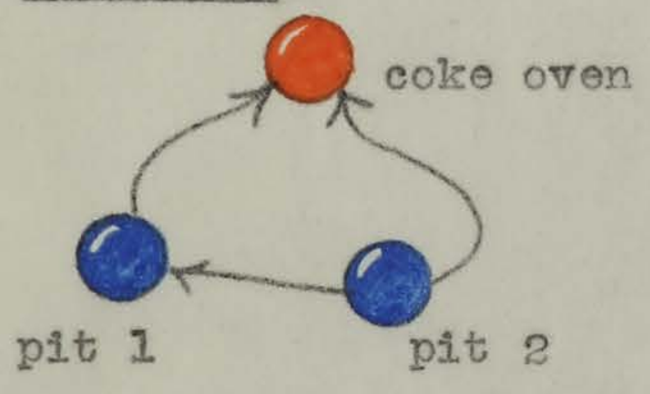
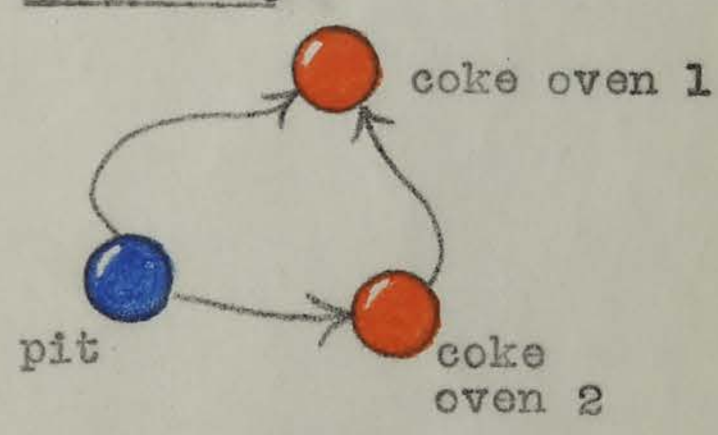


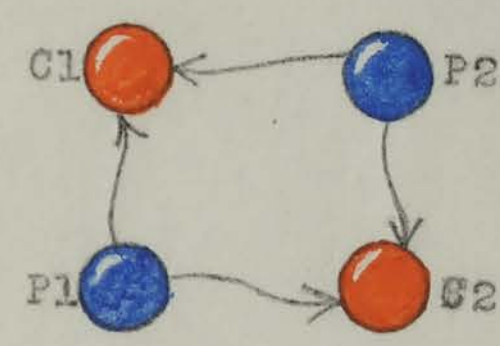
Diagram 3



Either of these cases can be interpreted as a pit delivering coal to a coke oven by two different routes, one of which will, in general, be shorter than the other (not necessarily the one which does not pass through another point).

To have a more complicated form of loop, we must consider at least four points, for instance, two sending and two receiving:-

Diagram 4

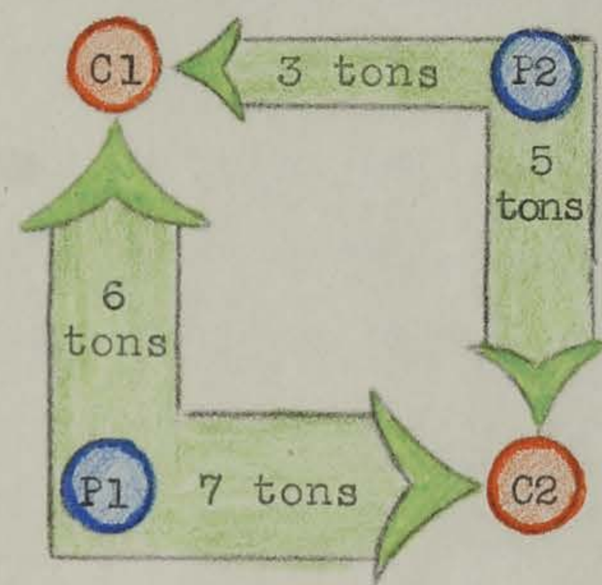


This is certainly a loop for, disregarding the direction of the routes, one can make a tour of the whole figure by the lines of the graph, in the counter clockwise direction, from P2 to C1 to C2 and back to P2. But now it is no longer possible to distinguish two sides of the loop, one of which would, in general, be shorter than the other. The two pits may both be supplying both coke ovens by the shortest possible routes. But this loop can be shown to be inefficient (except in the special case of an exact equality[1]) in the same way as the simple loop of Diagram 1. Suppose, for example, that the tonnages on each route were as shown by the circled figures in

1. Koopmans & Reiter, l.c., call the special case a "neutral circuit".

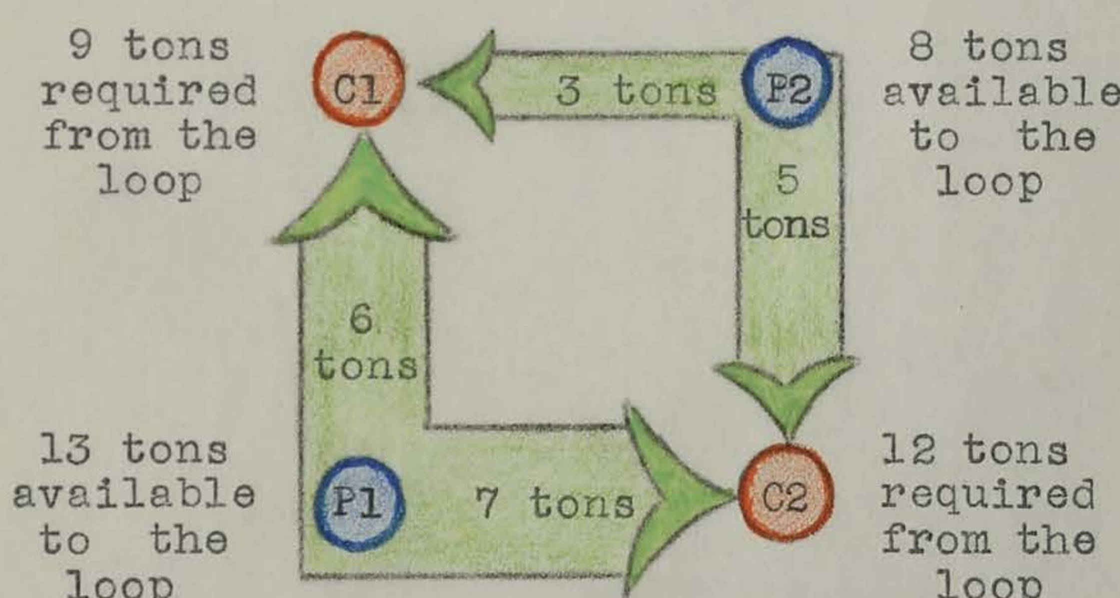
Diagram 5, and represented by the width of the arrows[1]. The lengths of the arrows are not intended to represent the lengths of the routes:-

Diagram 5



From Diagram 5 we can derive the amounts each point sends to or receives from other points within the loop, see Diagram 6:-

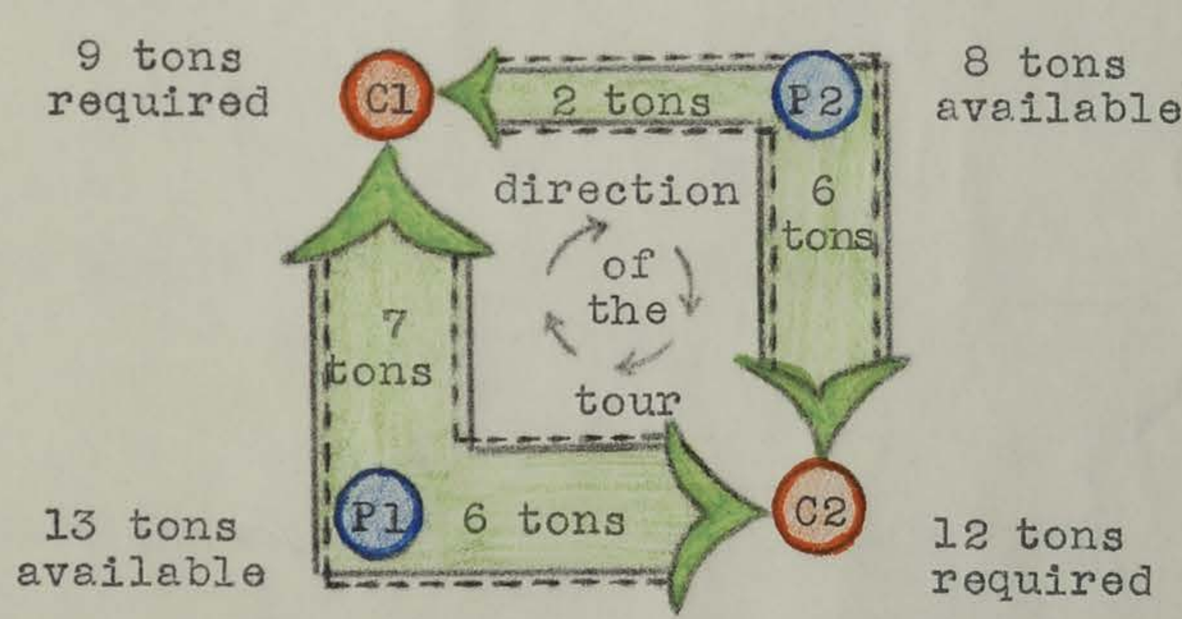
Diagram 6



1. This does not correspond with the representation on Map 2 where the width of the lines is proportional to the square root of the tonnage. I.e., in Map 2 the flows are intended to be visualized as pipes or tubes of varying thickness, whereas in Diagram 5 they are streams of equal depth but varying width.

Suppose C1 took one less ton from P2. This would leave P2 free to send an extra ton to C2, which could therefore take one less from P1, leaving P1 free to send an extra ton to C1 and thus maintain its total receipt of coal at nine tons, despite the reduction of one ton from P2. These adjustments involve a complete clockwise tour of the loop, adding one ton where the tour coincides in direction with the coal movement, and subtracting one ton where the tour moves in the opposite direction to the coal.

Diagram 7



On Diagram 7 the dotted lines represent the flows of coal before the adjustment, i.e., they correspond to Diagrams 5 and 6. From the way in which the adjustment round the loop was made, subtracting one ton when the tour moves counter to the movement of coal, and adding one ton when the tour moves with the movement of coal, the total tonnage received by each coke oven and sent by each pit is unchanged.[1]

Such a tour of adjustments will make certain changes to the total cost of the routing pattern, adding to it where the tonnage is increased and subtracting from it where the tonnage is reduced. In the clockwise tour of adjustments one ton was added to routes:-

P1-C1

P2-C2

and one ton subtracted from routes:-

P2-C1

P1-C2

therefore the change in total ton-mileage will depend on whether the sum of the distances of the first two routes is greater than or less than the sum of the distances of the second two routes.

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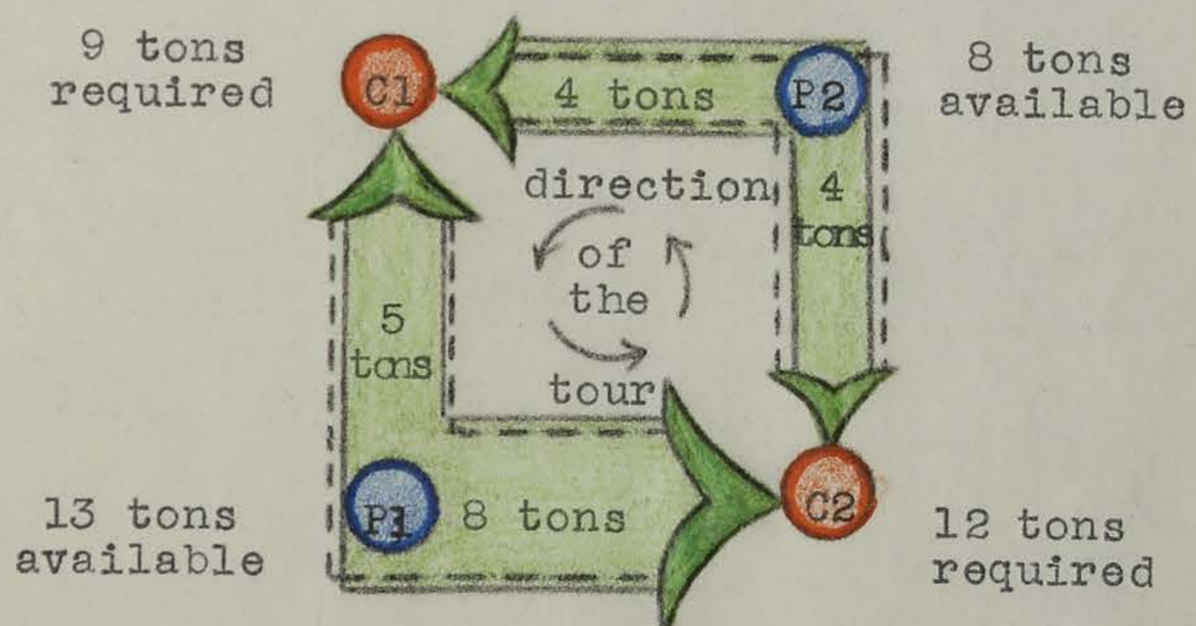
1. See Appendix A to this chapter.

If P1-C1 + P2-C2 is > P2-C1 + P1-C2 we should add to the total cost of transport and the adjustment is clearly better not carried out. The tour around the loop proceeded over a longer total distance in the same direction as the coal (the routes to which tonnage was added) than counter to the movement of coal (the routes from which tonnage was subtracted). Not only should this tour of adjustment not be carried out--the above inequality would imply that the reverse tour of adjustment ought to be carried out, with a consequent saving in total cost. This would result in:-

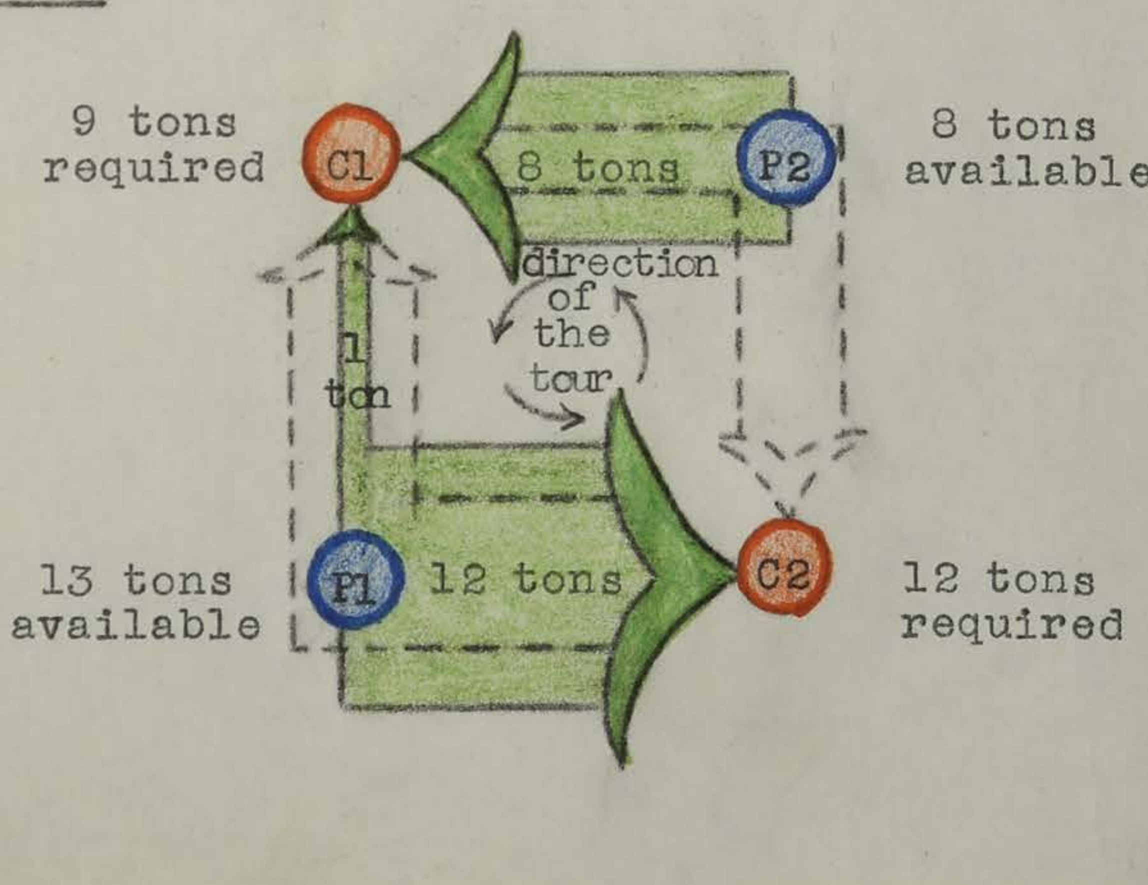
Diagram 8



Hence we see that unless the distances P1-C1 + P1-C2 happen to equal the distances P2-C1 + P1-C2 either the pattern of Diagram 7 or that of Diagram 8 must represent a lower total cost than the original pattern of Diagram 6.

Suppose it is the change represented by Diagram 8, the anti-clockwise tour, which yields a saving in cost, then it is clear that a further saving in cost can be achieved by carrying out the tour of adjustments again, with a further saving of (P1-C1 + P2-C2) - (P1-C2 + P2-C1). The limit of such adjustment is reached (see Diagram 9) when there is no tonnage left on route P2-C2, so

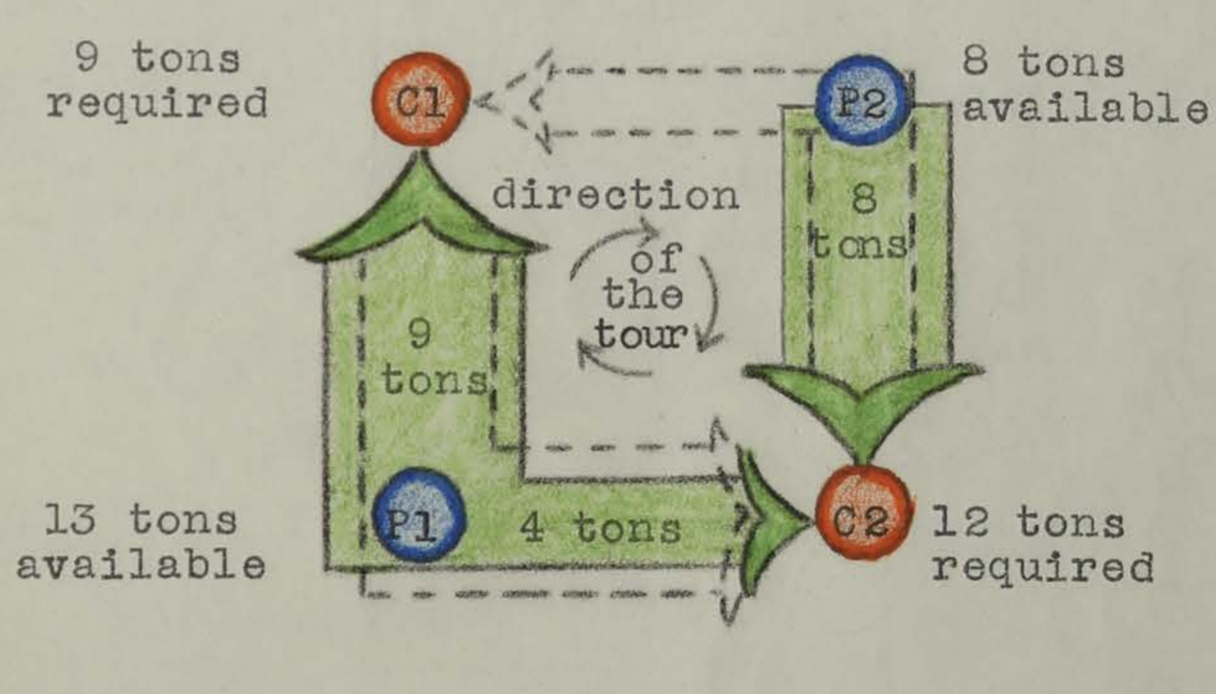
Diagram 9



that further subtraction from it is not possible.[1] The other route from which tonnage is being subtracted, P1-C1, still bears a positive tonnage of one ton--but the limit is set by the <u>smallest</u> tonnage travelling in the direction counter to the tour.

If it is the adjustment represented by Diagram 7, the clockwise tour, which yields the saving in total cost, the limit will be set by the route P2-C1 which carries only three tons, and the resultant pattern is:-

<u>Diagram 10</u>



1. Even if negative tonnage is interpreted as a reversal of the route, the saving in cost from the situation where P2-C2 carries 0 tons to the case where it carries -1 (C2-P2 carries one ton) will not be (P1-C1 + P2-C2) - (P1-C2 + P2-C1) but rather (P1-C1) - (P2-C2 + P1-C2 + P2-C1) and therefore only worth carrying out if P1-C1 is longer than the other three routes combined.

If the two pairs of routes have the same total cost, no saving is effected by breaking the loop either as in Diagram 9 or as in Diagram 10: but equally no addition to cost is incurred. Therefore we can say, at least, that if there is a loop in a proposed routing pattern one can break the loop by making adjustments in tonnage, with, in general, an improvement in total cost, or (in the special case of an exact equality) at least without an increase in total cost.

More complicated loops than that illustrated in Diagrams 4 to 8 involve no more complicated analysis. An actual routing pattern might contain very many loops, and even loops within loops, but the foregoing discussion applies to any one path from a point P, through any number of other points (once only) and back to point P. The argument is not affected by the fact that there are many such possible paths out from and back to P--it applies to each such single loop considered separately.

The sort of loop which might be encountered in practice is illustrated in Appendix B to this chapter.

III

Faced with a problem of the type which forms the subject of this report, a transport manager could do worse than solve it in the accustomed fashion, or by some rule of thumb method, and then go through it and systematically eliminate all loops. But he could do better than this. We can say of the pattern of Map 2 not only that it contains no loops but also that there is no unused stretch of railway line[1], connecting any two points of the solution graph, which could be used to form a loop and enable a tour of adjustments to be made resulting in a reduction of total cost.

Let us consider an unused route from A to B, where A and B are points on the solution graph. In so far as the solution forms a connected tree[2] there is a chain of routes belonging to the solution pattern (probably partly

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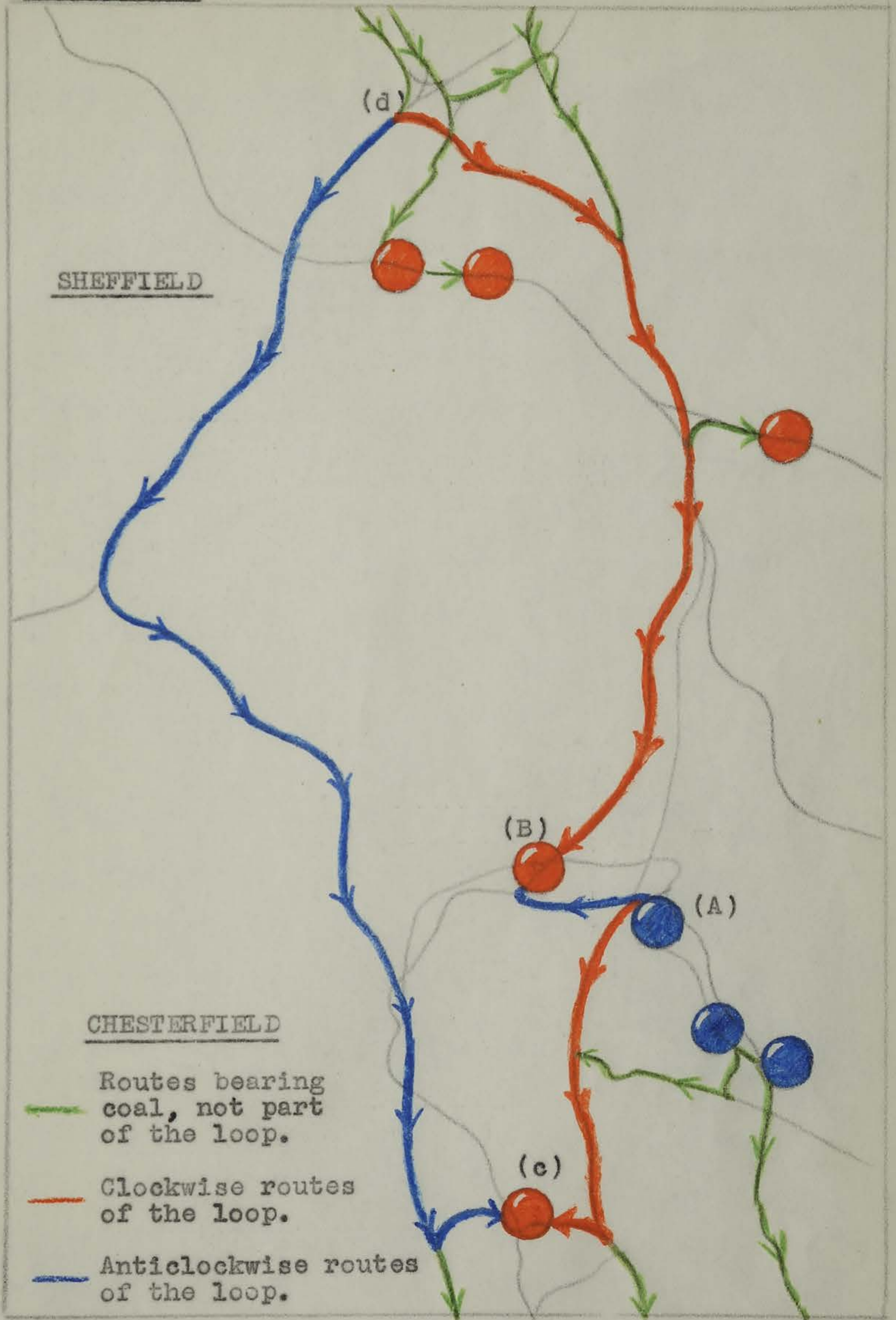
1. The term "unused stretch of railway line" means not only parts of the railway network not used at all in the solution, but also the reverse direction of those routes which are used.

2. The question of the unconnected parts of the solution graph will be discussed at a later stage.

with the direction of coal movement, and partly counter to it) between A and B. To transfer one ton of coal from A to B, one could either use the unused route AB or make a series of adjustments along the chain of routes connecting A to B in the solution tree, adding one ton on to each route on which coal moves in the same direction as the chain, and subtracting one ton from the routes on which coal moves in the direction opposite to the chain. The net result of such a series of adjustments would be an indirect transfer of one ton of coal from A to B, just as effective as the direct transfer. For instance, consider the loop of Diagram 11. This loop was formed by adding to the solution tree of Map 2 the route joining the point A to the point B. One ton of coal could be transferred from A to B either directly or by sending one ton extra from A to the coke oven at c, one less ton from d to that coke oven, and hence one more from d to B.

In general, either the direct shipment or the indirect tour of adjustments may be the cheaper. The answer, in any particular

Dia gram 11


(d)
SHEFFIELD
(B)
(A)
CHESTERFIELD
(c)
Routes bearing coal, not part of the loop.
Clockwise routes of the loop.
Anticlockwise routes of the loop.

case, will be found to depend on whether the loop which would be formed by sending tonnage on AB ought to be broken at the route AB itself, or by the opposite tour of adjustments which involves increasing tonnage on AB. For instance, on Diagram 11, it will be cheaper to transport coal directly from A to B if and only if the distance AB is less than Ac - dc - dB. In other words, if the distance of the anticlockwise route AB is less than the sum of the distances of the clockwise routes of the loop, minus the sum of the distances of the other anti-clockwise routes. This will be seen to be equivalent to the condition that all the anti-clockwise routes (including AB) sum to less than the clockwise routes, with the established consequence that the tonnage on the anti-clockwise routes (including the "new" route AB) should be increased and the tonnage on the clockwide routes decreased until one of the clockwise routes is reduced to carrying zero tons, and the loop is broken.

Thus the statement that there is no unused stretch of line which could be used to form a loop in the routes of the solution graph with a consequent reduction in total cost, is seen to be equivalent to saying that every unused direct route has a cheaper indirect substitute amongst the routes of the solution. That this condition implies that the irreducible minimum cost has been reached, is proved elsewhere:[1] here, all that is claimed for it is a certain plausibility.

IV

We might leave this discussion of the characteristics of the solution of the transport problem at this point, except that the procedure which starts out as being merely the mechanics of checking that there are no cost-saving direct routes to be incorporated, turns out to yield magnitudes which can be identified with price differentials for coal at different places. The economic significance of these price differentials is briefly discussed in section V of this chapter.

No doubt it would be possible to select each unused route and work out for each one whether the loop it formed could be more

1. Appendix C to this chapter, and Koopmans & Reiter, l.c.

profitably broken at the smallest-tonnage route going in the direction opposite to the proposed new route. But this would involve wasteful repetition of computing effort by going over the same stretches of the solution graph many different times. Instead we can go over the graph once and assign to each point a number, such that subtracting the number assigned to A from the number assigned to B would yield the sum of the distances of the routes going towards B from A (on the chain connecting the two points), minus the sum of the distances of the routes going in the opposite direction. Then to determine whether it would be profitable to utilize any particular unused route, it is only necessary to compare its distance with this difference.

In order to see how these figures are obtained, it will be helpful to enlarge to some extent on the nature of "tree" graphs. Because every point on such a graph is connected to every other by a unique chain of routes, without any loops, it is possible to select any one point and construct a chart showing the connections of every point to that origin. In fact in this model the chart will be a formalized representation of the actual routing graph on the railway network. One way in which such a chart can be arranged is like an inverted tree, with branches spreading outwards and downwards from the origin point. The directions of some of the routes will be downwards coinciding with the direction of the chain from the origin to every point, and some of the routes will be upwards, counter to the direction of the chain. The chart formalizing the solution of Map 2 is illustrated by the photograph (Diagram 1 of Chapter 3) on page 3a, which is basically as described above except that it has been cut into six segments because of its disproportionate vertical and horizontal measurements.

Starting with zero at the origin[1], a number can be assigned to every point on the tree by adding the distances of every down-pointing route and subtracting the distance of every upward-pointing route along every branch.

Let us first see how these numbers

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1. Or an arbitrary figure--some figure large enough to make all the subsequent ones positive is convenient.

can be used to check whether any particular unused route ought to be used. The loop formed by the route may or may not pass through the zero-valued origin point. Let us take the former possibility, where the loop passes through the origin point. We can illustrate the case with a simple chart:-

Diagram 12

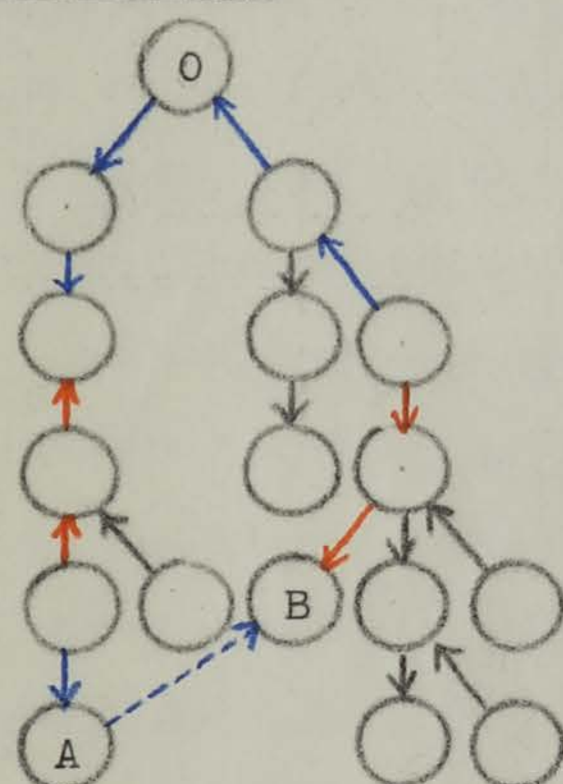
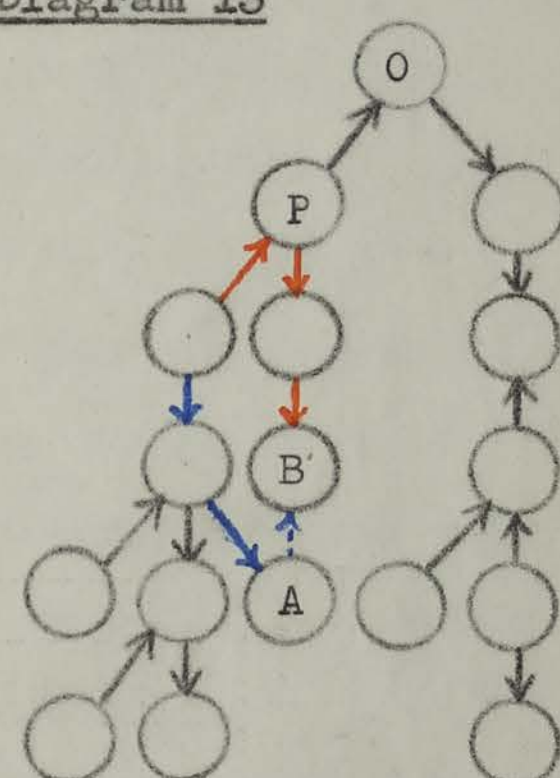


Diagram 13



The proposed new route is AB, and the loop completed by it passes through the origin. The number assigned to B is the sum of the distances of the clockwise routes, minus the sum of the anticbckwise routes on the stretch of the loop, OB. The number assigned to A, on the other hand, is the sum of the distances of the anticlockwise routes, minus the sum of the clockwise routes. If, therefore, the number assigned to A is subtracted from the number assigned to B, the resultant number is the sum of the clockwise routes between A and B, minus the sum of the anticlockwise routes. If from that number is subtracted the distance of the anticlockwise new route AB, we have a figure which tells us the net effect on total ton-mileage of making a clockwise tour of adjustments to the tonnages on the routes of the loop. If it is positive, therefore, an anticlockwise tour involving putting tonnage on the proposed new route AB is indicated. If it is negative, a clockwise tour is indicated, or in other words, the new route continues to bear zero tonnage.

If the loop does not pass through the origin, then it must pass through some point, say P, which is common to the chain from the origin to A and to the chain from the origin to B, as on Diagram 13. In this case, when the number assigned to A is subtracted from that assigned to B, the number at P (reflecting the stretch from the origin to P which occurs in both the B and the A figures) cancels out, and the difference again represents the sum of the clockwise routes minus the sum of the anticlockwise routes.

We shall call the figure obtained by adding the distances of the downward routes and subtracting the distances of the upward routes from the origin to each point, the "price of coal" at that point. The distance of each route we shall call the "cost of the route". The difference between the price at the receiving end of a route and that at the sending end, whether the route is used or not, we shall call the "revenue on the route". The" revenue" minus the "cost" gives the "profit on the route".

Thus the check whether the route AB should be used or not, can be described in the new vocabularly as "subtract the price at A from the price at B to obtain the revenue on the route AB. Subtract from the revenue the cost of the route BB. If the resulting profit is positive, the route AB could be incorporated into the solution with a saving in total cost. If the profit is negative, the route AB should remain unused."

V

If there is a fixed price per ton mile for transporting coal, so that the length of each route can be strictly equated with the cost of operating it, we can think of the terms "price", "cost", "revenue" and "profit" as being expressed in money terms.

Let us suppose it is decreed that the real price of coal is to reflect the cost of transporting it, in the sense that if coal is moved from A to B then the price at B is to be higher than at A by the cost of the route. Then the price differences must be the same as the numbers we have labelled "price" above. In the description of obtaining the price for each point from an arbitrary origin, price was increased by the cost of the route when the receiving point of a route was being priced from the sending point (a down pointing route on the chart of the tree) and decreased when it was the sending point of the route which was obtaining its price from the receiving point (an upward pointing route). This corresponds to the condition that the price at B should be higher than at A by the cost of the route if coal is sent from A to B. The price differences could not be determined at all unless the routing pattern formed a tree, as the presence of a loop (except one where the sum of the clockwise distances equals the sum of the anti-clockwise distances) would mean that there were two different ways of obtaining the price at A from the price at B (where A and B are two points on such a loop).

Suppose there were free entry to the transporting business in the sense that any entrepreneur is free to purchase coal at the ruling price at one place, transport it at the given ton-mileage rate to another place, and sell it at the ruling price there. Then the centrally planned pricing pattern would not be the equilibrium pattern unless there were no profits on the unused routes, i.e., unless the planned pattern fulfilled the conditions of the solution of the minimum cost transport program as set out in this chapter.

This equivalence between the solution to a minimization problem and the equilibrium price pattern of economic theory, is characteristic of all forms of linear programming, or activity analysis, of which the transport problem is a specialized form[1].

VI

We have seen that the solution to the transport problem must form a "tree" graph--that is to say a graph containing no loops--such that no unused route could be profitably substituted for some series of routes within the solution. What has so far not been discussed is the method of obtaining this solution. Various possible methods of computation are possible--the one used in this study is described in Chapter 3--but most are basically the simplex procedure, an iterative procedure of starting with any solution which forms a tree; finding which unused routes could be profitably incorporated and taking them one at a time (starting with the largest profit); forming the consequent loop; and breaking it at the smallest tonnage route in the opposite direction.

1. See Appendix C to this chapter for the relationship between this problem and the standard form of linear programming. Also Koopmans and Reiter, l.c.

This ensures that there is a steady reduction in total cost (except that some loops contain routes which are only notionally in the solution and bear zero tons, and hence enable a zero improvement in total cost to be made). It is a somewhat unsatisfactory method of computation, in that it is slow and there is no way of knowing how far from the solution one is at any stage.

VII

The possibility of disconnected independent subgraphs has been mentioned, and indeed two such areas are to be seen on Maps 2 and 3. This can only happen if the tonnages available match the tonnages required in this group of pits and coke ovens. The price differences of the points in the sub-area are determinate amongst themselves, but only determinate within a range vis-à-vis the rest of the graph. In fact, the group of prices is free to range between a level so low that it becomes profitable to export coal from the area, to a level so high that it attracts tonnage from the rest of the graph. From the point of view of calculation, the whole graph is notionally made into a connected one by assuming that some route between the main tree and the subarea is part of the solution but happens to carry zero tons, so that the calculation procedure outlined in section VI of this chapter can be followed without substantial modification.

Chapter 2. Application of the Model to Coal Transport.

I

In Chapter 1 we discussed a highly simplified problem where known quantities of "coal" were available at points scattered over a railway network, and were required at other points on the railway network (Map 1). The problem was to move the coal from the pits to the coke ovens in such a way as to minimize the total ton-mileage travelled by the coal. The model both allocated the coal available at each pit between the different coke ovens (the conventional linear programming transport model) and planned the actual routing of the coal in such a way as to ensure that the shortest route between any pair of points was used.

This model fails to correspond in many respects with the actual situation. Coking coal is not a homogeneous commodity, with certain standardized properties. The properties of coal --size, chemical composition, dirt content, etc.-- vary greatly from one pit, (and indeed seam) to another, and coke ovens have been designed to use different types, and different combinations of types, of coal. Then again, marketing of coal is a matter for local arrangement and the data for such a nationwide planning of allocation are not normally available centrally. The amount of coal from each pit which is to go to coke ovens is not, in practice, determined entirely independently of the allocation between different coke ovens, as we have assumed in our model. Furthermore, the allocation and the planning of the routing are not carried out by the same authority. The decision is taken to deliver coal from Pit A to coke oven B, and the railway authority instructed to that effect. In planning the actual routing of that and all the other coal, the railways take into account not merely the shortest distance, but timetables, convenient marshalling yards, and problems of congestion. But they cannot change the allocation pattern, even if they are called upon to carry the same type of coal from A to B and from B to A again. Other complications which have been neglected are the availability of other forms of transport, and the question of the frequency with which the plan should be changed.

This chapter will be concerned with the possible modifications in the simple model of Chapter 1 which could be made in order to use it for the actual planning of the allocation or the routing of coal. But before proceeding to these questions of detail, let us briefly examine the existing method of allocation, and a possible alternative method also claimed to minimize transport costs, namely, a pricing system.

II

The actual method of allocating coal is by individual contract between coke ovens and area marketing officers, with only occasional co-ordination of supplies at the regional and national levels. The principle of the allocation is apparently "fair shares" as between the different qualities of coal.

The price of coal is not used for allocation, either geographical or between different uses, but is regarded principally as a method of raising revenue. There have been arguments elsewhere[1] as to the desirability of equating the

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1. E.g., I.M.D. Little "The Price of Fuel".

over all demand for and supply of coal by means of its price (the present price is well below such an equilibrium valüe). Differential prices could also be used to carry out the allocation of coal of different types and at different places. Indeed one can say that for any set of physical data and of demand and supply elasticities, there exists a set of prices for each type of coal at each place which will exactly clear the market during the period for which those data remain unchanged. Furthermore, the cost of transport for each type of coal will be minimized by the price pattern if arbitrage between different areas is possible.[1] But those who advocate the institution of a price mechanism have not usually in mind the complete solution of all the equations relating to each time period in order to obtain such a unique price set, but rather a system of approximation with prices adjusted in successive time periods, according to the errors of the preceeding period. Such a system would eventually reach the equilibrium set if the data

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1. See P.A. Samuelson"Spatial Price Equilibrium and Linear Programming". American Economic Review, June, 1952.

were unchanging in successive time periods, and should work reasonably well as a practical method of control so long as the data do not change too violently. But there are considerable practical difficulties in the way of instituting such a price system, particularly the great number of separate price quotations to be brought into some sort of relationship with one another. Admittedly it would not be expected that the initial price set would be the equilibrium one, but it would certainly be a relevant consideration that no price set is a possible equilibrium one unless it conforms to the conditions of the dual solution of the linear programming problem. In fact the only practicable way of instituting a pricing system, as opposed to working one already in operation, might be by solving the appropriate linear programs. It is of interest in this context that it has not proved possible to institute fixed delivered prices, as opposed to pithead plus cost of transport prices, for industrial coal as has been done for household coal. Either the house coal retailers have been given sufficient inducement to accept the scheme, or the optimum price pattern has been found for this grade of coal[1]. Only if pithead and consuming point prices were those of a minimum transport cost pattern would each consumer find that he was unable to improve his position by purchasing his coal at pithead (or at some other consuming region) and paying his own transport costs, rather than accept the delivered price quoted for his area.

The application of the model to coal transport can be regarded quite equivalently either as a method of direct control or as part of the problem of finding the appropriate prices. But note that transport cost can be minimized for each group of coal considered in isolation if the program is used for direct control, whilst a pricing system of control can be used only if it is complete. That is to say, not only must each grade of coal vary in price appropriately from one place to another, but also the grades must be appropriately priced with respect to one another, and the price of all coal must be such as to just clear the market. Unless this total pricing problem

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1. An interesting investigation of the zone delivered prices could be made on the basis of the quantities available and supplied for each grade of household coal in each area.

is solved, the cost of transport alone cannot be minimized by a pricing system, but only by direct controls.

III

The distinction between the allocation and the routing of the coal has been briefly mentioned. By "allocation" is meant the decision as to which pit is to supply which coke oven. The usual form of the Koopmans transport model is concerned with taking these decisions in such a way as to minimize transport costs when the cost of transport is known for every possible connection between a consuming region and a supplying region. If these costs are based on the shortest route between the two points, then this shortest distance will have to be computed first, for each possible connection.

From the point of view of the authority making the allocation plan, the best plan is that which minimizes the money cost of transport. If the transport authority is following the recommended pricing policy for public monopolies and charging according to marginal real cost, minimizing money cost will also minimize the real cost of transport, and there is no conflict between the two. In this instance the money costs of transport do not reflect the real costs accurately. Rail charges are based on the shortest rail distance between any pair of points, without regard to whether that distance is the one actually travelled by the goods, or even if it is a possible route. In fact the shortest rail distance would not usually coincide with the shortest practicable working distance, which has to take account of the limited manoeuvrability of trains. There is no reason in principle why charges should not be based on minimum working distance instead of on minimum rail distance as at present. Still closer approximation to marginal real cost would be to allow charges on congested routes to rise sufficiently to cut the volume of traffic on them down to the manageable level. This would mean charging according to the distance of the alternative route, bypassing the congested segment, travelled by the "marginal" ton[1].

Although this marginal real cost charging is a possibility for the future, there remains the present question as to whether the allocation of coal ought to be based on money costs or real costs of transport. Minimization of

1. See Appendix B to Chapter 3.

real cost implies co-operation between the National Coal Board and the transport authorities in the joint planning of the allocation and the routing of the coal. It means that the Coal Board should not pursue single-mindedly money profit criteria. If it were costless to compute minimum transport cost programs, it would pay the transport authorities to co-operate with the Coal Board in allocating coal according to real transport cost even though the Coal Board were charged a total sum of money based on a minimum money transport cost allocation, computed but not used.

It is inefficient from the national point of view that the allocation should be carried out on the basis of transport costs related to minimum rail distance and then the routing of each individual allocation planned to minimize its real cost: the best routing plan can only be made if the allocation of the coal between the different pits and coke ovens is also free to vary. The model of Chapter 1 is such a combined allocation and routing plan, but its basic cost data, the adjacent point to point distances, are such that the routing is being planned according to the shortest rail distance, instead of the feasible working distances. As the rail charges are based on these distances, the allocation part of the solution of this model is that which will minimize the money cost of transport. The model is very simply modified to carry out the real cost allocation and routing plan[1], if this should be required additionally or alternatively. The remainder of this chapter, however, will be concerned with the minimization of money cost of transport.

If British Railways quotes special rates for some pit to coke oven routes, they could be incorporated in the model as though they were new stretches of railway line, and present no computational difficulty[1]. Analagous to these special rate routes is the possibility of using road or water transport for some shipments. Here it is not enough to know the distance by road or water; some conversion factor reflecting the different costs per ton-mile must also be incorporated. But here again the difficulties are not ones of principle. In fact, there is no reason why one should not deal with three networks, rail, road and water, perhaps with a suitable handling charge to move goods from one network to another

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1. See Appendix B to Chapter 3.

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1. See Appendix B to Chapter 3.

at the appropriate points. This model would be sufficiently flexible to take the best advantage of the seasonal fluctuations in coastwise shipping rates, for instance. These other forms of transport have been neglected in this study in the belief that a reasonable first approximation can be obtained from a consideration of rail traffic only.

IV

The data for this study are the results of a survey into the sources of coal for each coke oven during one particular month. It was the availability of these data which determined that coking coal, rather than, for instance, house coal, should be the object of the calculation. For each coke oven the survey yields the tonnage of coal it received from each pit, the pithead price, the delivered price, and the size of the coal.

Not by any means all the collieries in the country contributed coal to coking ovens during that month, nor did all thos collieries which entered the survey contribute their total output. Thus one respect in which Map 1 is misleading is in assuming that definite amounts of coal are available at each pit, and that the allocation to the different coke ovens is a subsequent task. However, it should prove feasible to incorporate this primary decision into the linear programming framework, if it were being applied to all forms of coal traffic, not simply to coking coal. For the present purpose it seems reasonable to say that this coal was in fact delivered to coke ovens during the month of the survey and that that same coal could have been allocated according to our model with a certain saving in transport costs.

A much more serious objection is that coking coal is not a homogeneous commodity. The model rests on the assumption that it is a matter of indifference to each coke oven manager which pit supplies him. This may not be the case. For this reason, household coal, classified into groups which the householder is expected to regard as homogeneous, would have been a more satisfactory project, if the data had been available. Besides a cost comparison one would have had a comparison of price patterns betwee n the computed solution and the actual one of the zone-delivered price system.

No basis of classification of the coal in the coke ovens survey could be obtained and it was therefore decided to carry out the calculation on the assumption of homogeneity as a first approximation. To object that this calculation overstates the possibility of saving in transport cost because some coke ovens require particular grades of coal which they cannot obtain from the pits to which they are connected by this solution, implies that one can obtain a basis for a classification of coking coal into smaller groups which can then be regarded as homogeneous for this purpose.

Once the coal has been so classified, the calculation can be carried out on each group in turn. The total minimum cost must necessarily be greater than if the coal is freely substitutable. On the other hand, the different groups of coal would presumably be partially substitutable and the solution to the transport program for each group is relevant to the initial allocation of the different types. For each grade of coal one has the "dual solution" or set of locational price differentials. It will be re-called that these are computed on an arbitrary origin, so that the whole set of prices for each grade can move up or down, but must preserve their absolute differences from one place to another. If these price sets are fixed, say, at one coke oven, to reflect the valuation the given quantities of each group have in the production of coke at that point (e.g. by means of a linear program, or by more orthodox methods) then the prices of each group are fixed at all points. Only if these prices coincide with the valuations of each group of coal at each point, is the initial allocation of the different grades justifiable. This is not to say that doing a transport program for each grade provides the whole answer to the allocation problem--for instance, prices could be brought to equality with valuations at other coke ovens either by changing the arbitrary origins of the pricing sets or by changing the quantities received of the different grades--but it would provide marginal transport cost data for each group of coal which ought to form part of the allocation procedure.

V

Our problem has been to determine how the coal which supplied coke ovens during one month might have been allocated compared with the way in which it was in fact allocated. This involved taking a whole month's supply of and demand for coal as static quantities, and carrying out a once-for-all allocation of those quantities. But clearly no matter how great an improvement could be made in this theoretical case, it would not be worth doing unless it had some possibility of being used currently on the same problem. The object of this study is not to point out how much has been wasted, but to see if any saving can be made. Two new considerations now become relevant --the length of the planning period, and the problems of communication arising out of the centralization of planning.

So long as there is no discount for large quantities in railway transport, it makes no difference to our calculation or its solution whether we regard the quantities of coal used in that one month as stocks or as a flow of production and transport, planned on a monthly basis. If this form of planning were to be used in current practice, the best length of the planning period would depend on such considerations as the variability in supply and demand; the capacity at pits and coke ovens for holding stocks; the reliability of estimates of supply and demand; and on the cost of communication of data and of carrying out the calculation. If, in a more representative period than one month, many more pits than those represented in the survey would provide consignments of coal to coking ovens, then one month is probably too short a planning period. In general, unless the coal becomes available at all the relevant pits in a fairly smooth flow, the longer the planning period the more efficient can be the allocation. But against this must be set the cost of holding stocks for those coke ovens scheduled to receive coal at longer intervals.

It has been mentioned already that this calculation has only been possible because a survey into the sources of coal for coke ovens was made. In general, the data for such a program would not be available centrally--and certainly not before the allocation has to be made. Therefore to the cost of the calculation would have to be added the cost of establishing a two-way flow of information, i.e., estimates of availabilities and requirements to the central organisation, and instructions for the disposal of the coal to the pits. This might possibly be worth establishing for all forms of coal, even if it were not worth doing for coking coal alone.

No consideration has been given to these problems of organisation. The first step to be taken when considering the usefulness of linear programming in any application is to determine whether the difference between the optimum and the actual is sufficiently large to warrant an investigation into the possibility of instituting it as a regular method of control. That first step is the main object of this study.

Chapter 3 Computational Method

I

In the original development of the transport problem by T.C. Koopmans (l.c.) the cost data were the sailing time in days between the major ports of the world, and the problem was to minimize the time ships spent in ballast in getting from the ports which had a surplus of empty ships to those which required empty ships. The calculation was apparently carried out by drawing routes on a map of the world to form a tree, and making successive improvements.

Dantzig[1] developed a method of carrying out basically the same operations in a matrix representation. All the sending ports are listed down the left of the matrix and the receiving ports along the top of the matrix. Thus each cell of the matrix stands for one port-to-port connection, and the same matrix can be used to represent the cost of the routes, the tonnage on each route, the revenues, and the profits. His method starts by arbitrarily selecting a number of routes

1. G.B. Dantzig, "Application of the Simplex Method to a Transportation Problem", Chapter XXIII of "Activity Analysis of Production and Allocation", ed. T.C. Koopmans, Cowles Commission Monograph, No. 13.

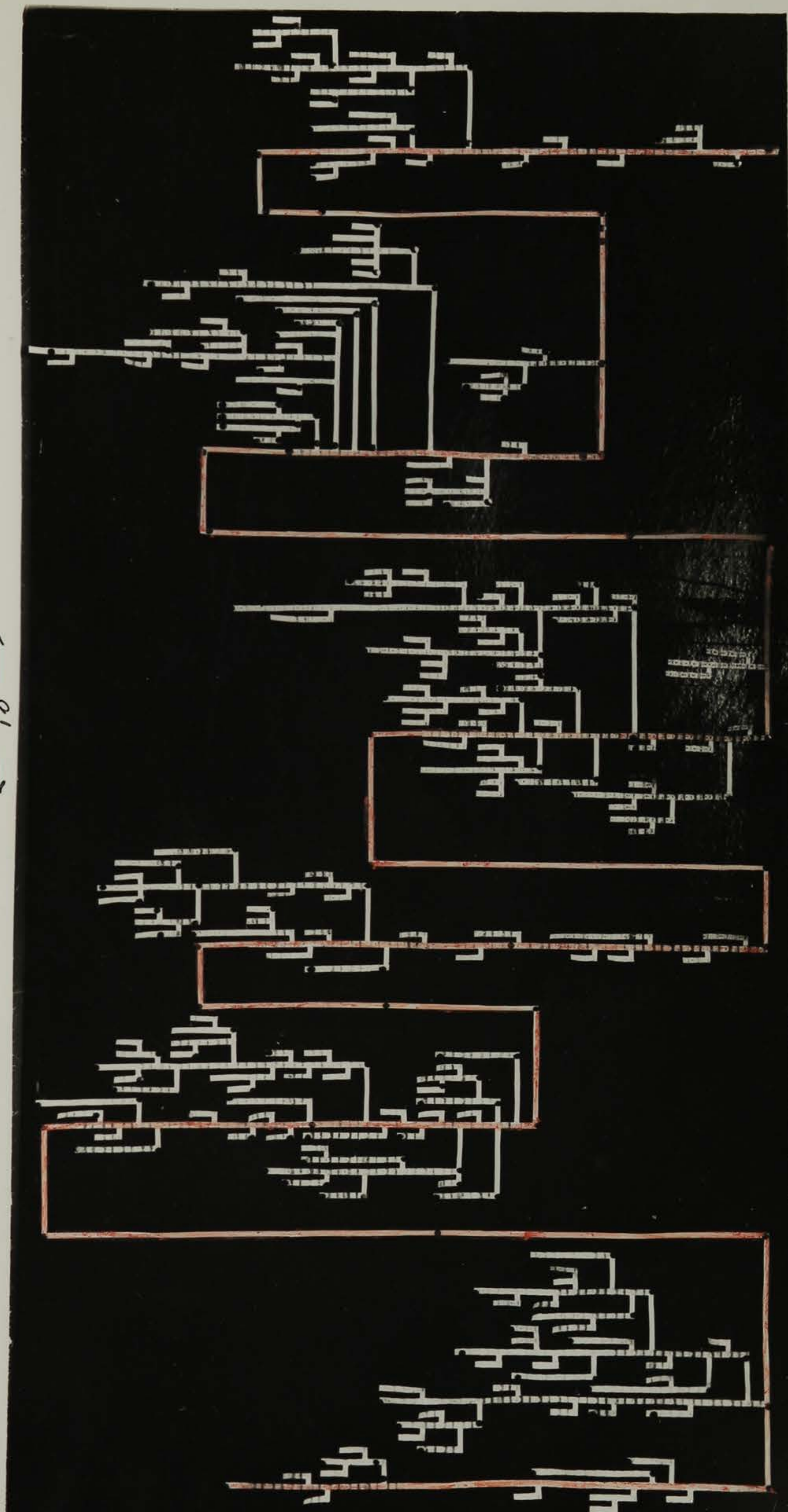
which represent a very simple form of tree, and making successive improvements from there. This works well on small problems (e.g. 20 x 20) for hand computation, particularly if it is adapted to make use of the fact that only some of the prices change at each iteration[1]. It is possibly also satisfactory for electronic computation of large problems, but is not feasible for a hand computation of the magnitude of the pit to coke ovens problem.

For this problem a method of calculation has been devised which is to some extent a reversion to the earlier graphical method. But instead of using a map, the routing pattern is represented formally by a chart in the form of a tree, as in Diagram 1. All the other information, both that which is constant and that which varies during the course of the computation, is contained in the "Profits Table", one page of which is reproduced here as Diagram 2.

The basic cost data are the adjacent point-to-point distances of the railway network. That is to say, not all of the 9,600 shortest rail

1. See Appendix A to Chapter 3.

← 18' →



← 4' →

DIAGRAM 1.

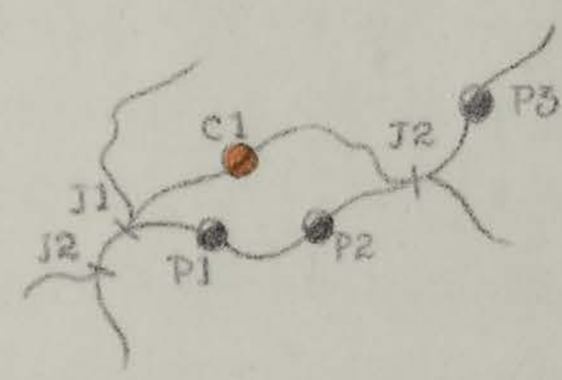
Diagram 2[1] PROFITS TABLE

Point	Map Ref.	Order	Tonnage Available	Price	Sending to	Cost of Route	Profit	Tons
P1	EM. 7	1	2,694	10.00	J1	1.00	in	
					P2	1.25	- .25	
P2	EM.11	6	10,678	11.00	P1	1.25	-2.25	
					J3	2.00	in	
C1	EM.11	4	-1,868	12.25	J2	.50	-1.00	
					J3	.75	in	
C2	EM.10	6	-11,504	14.00	J3	1.00	-2.00	
					J4	.50	-1.00	
J1	EM. 6	2	0	11.00	P1	1.00	-2.00	
					J4	3.00	-1.50	
					J2	.75	in	
J2	EM 7	3	0	11.75	C1	.50	in	
					J1	.75	-1.50	
					J4	1.50	.25	
J3	EM.10	5	0	13.00	C2	1.00	in	
					C1	.75	-1.50	
					P2	2.00	-4.00	
J4	EM.10	7	0	13.50	C2	.50	in	
					J1	3.00	-4.50	
					J2	1.50	-3.25	

1. The table represents the fictitious example of Diagram 8 of this chapter. The tonnage column is not used until the total cost of the final solution is being computed.

distances from each pit to each coke oven were obtainable as they would have to be in order to use the matrix method of calculation[1]. All that is known is the distance from each pit to the nearest adjacent junction on either side, and from that junction to all its possible immediate connections, and so on. The solution not only allocates the coal between coke ovens, but also determines the shortest rail distances of those routes used in the solution.

Diagram 3



To take a very simple case from the actual map (north of Manchester), there are pits at the points labelled P1, P2 and P3 (and at other points not shown here) and a coke oven at C1.

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1. It may still be worth using the chart method of computation even if all the matrix distances are known--see Appendix A to Chapter 3.

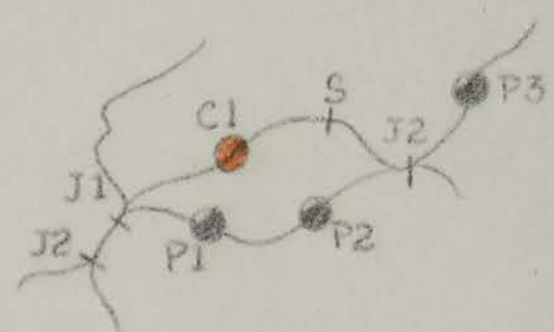
The only distances known are the segments between points, including those points labelled J1, J2, and J3 at junctions. For instance, the distance P2→C1 is either P2→J2 plus J2→C1, or P2→P1 plus P1→J1 plus J1→C1, whichever is the shorter. But it is not known at the start of the calculation which is the shorter route. This is a particularly simplified part of the network; in reality each of these junctions has possible connections to other junctions not shown on Diagram 3. In general, there is a very large number of possible routes for each of the 9,600 pit-to-coke-oven connections. Even those routes amongst which the eye cannot discern the shortest may be very numerous. As a matter of interest, British Railways are at present engaged on a massive calculation on electronic equipment to obtain the shortest distance from every railway station to every other station, using the basic point-to-point distances. This pair-wise tabulation will then be the basis of their charging system and will save computing the shortest distance for each new connection as it arises for the first time. These data which British Railways are using for their calculation would have been the most satisfactory starting point of this calculation, and should certainly be obtained if possible for any future similar application[1]. In fact the point-to-point distances were obtained for the coal calculation from $2\frac{1}{2}$ inch to the mile Ordnance Survey Maps, after a sample check had revealed a close correspondence to the correct distances as obtained from railway working timetables. Not the whole of the railway network was taken into account. It was clearly reasonable to limit the southern and northern extensions of the network, but besides this some other routes have been eliminated as not useful to the problem. The lines actually used provide the background to both the maps, but they are not very clear on so small a scale.

By a "route" is meant from here onwards a single segment of line between two adjacent points of the railway network. By a "point" is meant a pit, a coke-oven, or a junction.

1. The shortest distance from each pit to each coke oven--the matrix distances--will, of course, be made available by British Railways' computation. But this is not necessarily the best starting point of the transport computation--see Section II of Chapter 5.

In general, stations are not points in this calculation as they merely multiply the number of points and the number of routes with no compensating advantage. Thus on Diagram 3 there would be no advantage in taking a station at S, say, (Diagram 4) as a point on the network.

Diagram 4



The single route J2→C1 would merely be replaced by the two routes J2→S and S→C1. On the other hand, there is no difficulty about insterting such superfluous points. This might be done, for instance, if it were desired to use the British Railways point-to-point distances in future computations, without first eliminating superfluous points. This flexibility of the model means, for instance, that new sources of supply could be incorporated without difficulty, and indeed that the same basic cost data could be used for many entirely different commodity transport problems.

Quite a few collieries have coke ovens attached. In this case the net tonnage available or required has been computed, and those points which have a surplus to send out are labelled pits, and those with a net deficiency of coal are labelled coke ovens. Thus Map 1 cannot be taken to show the distribution of mining of coking coal, but only the distribution of coal which has to be moved.

Every pit, coke oven and junction has been given a number, and a key to its position on the map. Then the routes, or segments of the railway network, have been listed under the point numbers. In principle, there is nothing to determine whether a route should be listed under the point at one end or the point at the other end of its length. In practice, it was found quite convenient to duplicate the entire list of routes by recording them under both points. Then the two different directions of the segment can be thought of as two different routes, each listed under its sending point. E.g. (Diagram 4) the route P2→J2 would be listed under P2, and the route J2→P2 under J2.

At this stage the Profits Table (Diagram 2) would have been filled in in the columns labelled "Point", "Map reference" "sending to", "cost of route". The column "tonnage available" would be zero for junctions and would vary from one planning period to the next for pits and coke ovens (for this study, of course, the tonnage figures for pits and coke ovens are known and fixed.)

There are approximately 1,000 points and 2,000 routes (1,000 segments of railway line). If the shortest distance from each pit to each coke oven had been known, there would have been approximately 160 + 60 = 220 points, and 160 x 60 = 9,600 routes.

II

The first step in the actual calculation is to make a tree chart of all the points, connected by a selection of the routes. In practice, it is probably best to work from maps, marking in stretches of lines as they are used in the chart and ticking off the points in the Profits Table list as they are connected to the tree. Note that we are not in the least concerned with tonnage at this stage, though it is as well to see that the more obviously useful continuous stretches of railway lines are used.

If this method were being used as a continuous method of control, and if the changes in the amounts available and required were relatively small in each new planning period, the best starting tree would be that of the solution for the previous period[1].

The tree chart is formed of paper tape printed[2] as in Diagram 5.

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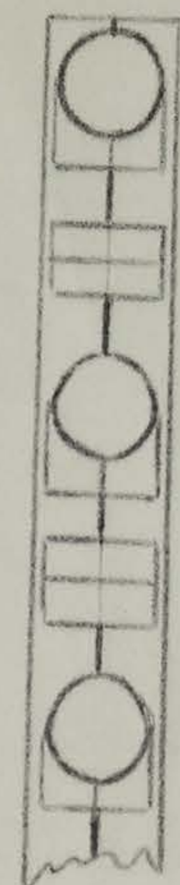
1. If some method of storing solution trees (perhaps suboptimal trees as well) were being used, then the best starting tree from all those stored might be selected by obtaining the total cost of the new program from each of the different locational price sets. This is analagous to the method described in a forthcoming article "International Comparison of Retail Food Prices" by G. Morton, for selecting the best first basis for a computation out of a store of different inverse matrices.

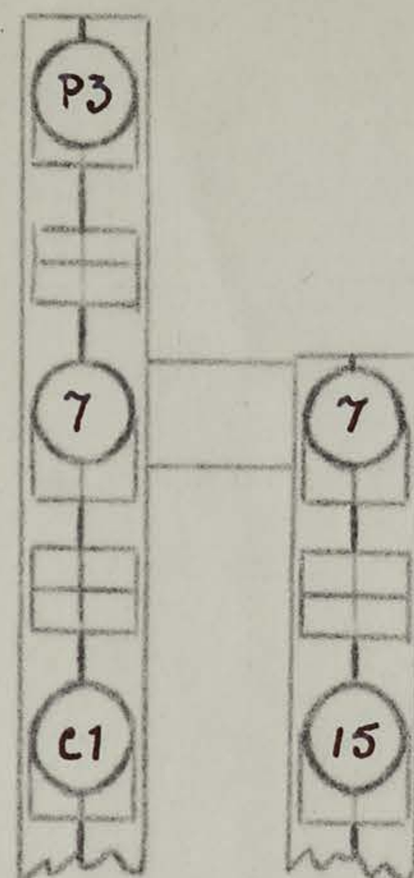
2. In fact, the paper tape was manufactured from sheets of foocscap, stencilled in this pattern.

Diagram 5



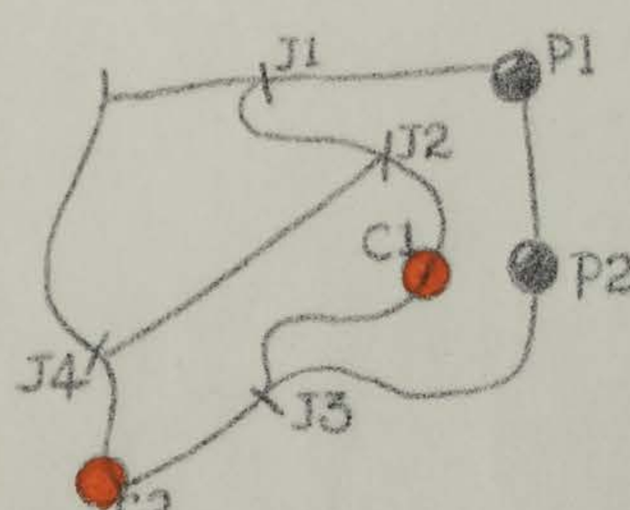
The circles represent the points, the lines, routes. On each route there is a label for writing in the distance and the tonnage--but this is not used at this stage. As each point is added to the tree it is ticked off on the Profits Table list. Where the tree branches, the point at which it branches is written in again at the top of the new strip, and the two strips attached by a cross piece glued to both of them, as at the point 7 in Diagram 6.

Diagram 6



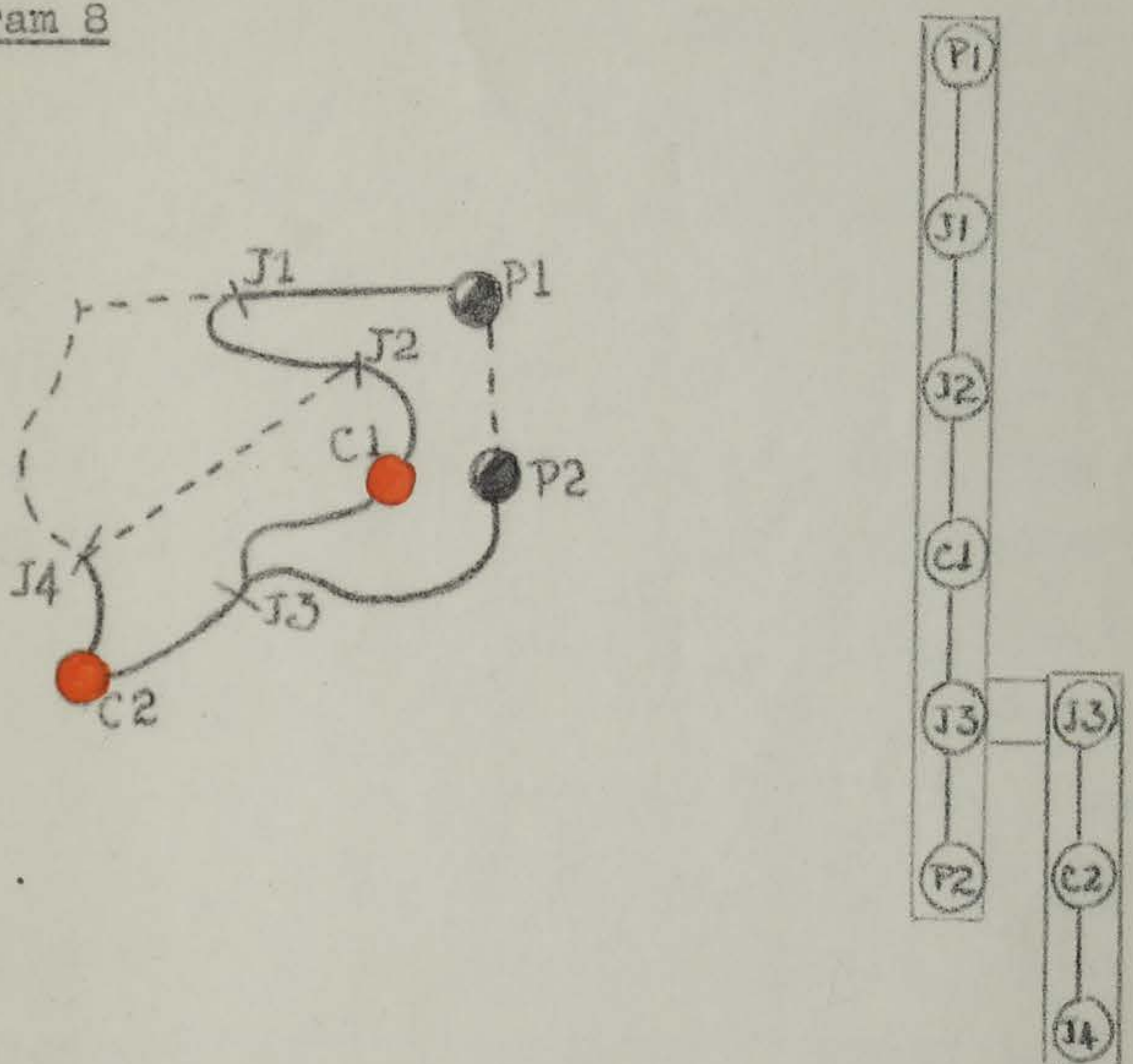
Let us take a very simple example:-

Diagram 7



An initial tree chart, called a <u>first basis</u> might be constructed, starting at P1 and using some of the most useful-looking routes as follows:-

<u>Diagram 8</u>



The routes used in the chart are marked in heavy lines on the map.

One can start the tree at any point on the map and move from it in any direction, but it is probably best on a large scale problem to make a main stem first, by using the most likely looking connections between the areas. In the case of Map 1, a reasonable start would be to take the chain from the extreme northerly point to the furthest south point on the map, with subsidiary stems to right and left. The chart is complete when every point has been ticked off as it was attached to the tree. If the number of points is n, n-1 segments of railway line are needed to attach all the points to the tree, and any more would result in the formation of loops.

As the chart is completed it is pinned to a board with drawing pins. The board is marked with numbered horizontal lines (not visible on the photograph, Diagram 1) at the same spacing as the points on the printed strips. These lines are used to provide a key to the position on the board of each point. Thus each one, as it is attached to the board, receives an "order" number, which is recorded on the Profits Table (see Diagram 2) in the appropriate column. To find any particular point thereafter it is only necessary to run along the horizontal line on the board corresponding to its order number. The chart pictured on Diagram 1 has, for convenience, been divided into 6 vertical sections, and arranged horizontally. The bottom of each section is attached to the top of the following one by a strip of plain paper tape. The "main stem" of the tree (including the plain strips) is coloured red on the photograph.

III

The next step is to fill in the tonnages on these routes which have been selected for the first basis. Each point is labelled with its net tonnage available--that is to say, pits labelled with a positive figure showing their tonnage available, coke ovens with a negative figure showing the tonnage required, and junctions with zero. These figures are known as the "boundary conditions" and they necessarily total to zero, because the total tonnage available equals the total tonnage required[1].

An important characteristic of a set of routes forming a tree is that any set of tonnages can be filled in on any tree, and filled in in one wa y only. Every point on the tree is connected by a unique set of routes to every other point. Thus it must be possible to dispose of all

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1. Even if there is an ~~###~~ inequality in fact, for the purpose of calculation there will formally be an equality--see Appendix C to Chapter 1.

the coal available at all the pits to some coke oven or other, though that allocation may be inefficient. It is perhaps less intuitively obvious that once the tree is determined there is no longer any freedom of choice about the tonnage which each route shall bear.

To enter the tonnages we start at the end point of one of the branches. The distinguishing feature of an end point is that it is connected by only one route to the rest of the tree (e.g. the points P2 and J4 on the chart of Diagram 8). Therefore, if that point has a positive boundary condition (i.e., if it is a pit) it must send out the whole of that amount along the one available route--if it is negative (a coke oven) it must receive all of its tonnage along the one available route. If the boundary of the end point is zero, the tonnage on the route must be zero, and the direction is indeterminate.

The tonnage is written on the label of the route, and its direction shown by an arrow.

We proceed next to the point immediately above the end point, which assuming that it is not a point at which the tree branches (e.g., C2 on Diagram 8, but not J3), it has two routes attached to it, one of which (the one below it on the chart) has already had its tonnage and direction determined. Therefore the tonnage and direction of the route above it is also determinate, in such a way that the total amount coming into the point from both routes equals the total amount going out, if the point is a junction; or so that the net amount sent on the two routes together equals the total amount available plus (possibly) the total amount received, if the point is a pit; and so on.

In this manner, one can continue up from the end point of a branch, filling in the only possible directions and tonnages of the routes which will make each point send or receive the correct net amounts. This can continue so long as there is only one route whose tonnage is undetermined as one comes to each new point--in other words, until one comes to a branching point. For instance, suppose that tonnage and direction of each route have been determined from J4 up to J3 on Diagram 8. Now J3 has 3 separate routes attached to it, only one of which has been determined. Therefore the other two are indeterminate.

When it is no longer possible to proceed further up the tree, another end point is selected and followed up until again a branching point is reached. It may be, as this procedure is repeated, that when the branching point is reached, two (or more) of the the routes have already been determined so that only the one route above the point has not been determined. This would be the case, for instance, if the branch from J4 up to J3 were determined, and then the branch from P2 to J3, so that of the three routes at J3, two are now determined. Then it is possible to continue upwards above the branching point, determining the direction and tonnage of the one remaining route belonging to the point--the one immediately above it.

In this way, every branch of the tree can be determined in such a way as to make every point send and receive the net amount indicated by its boundary condition, until one reaches the last route at the top of the chart (J1-P1 in Diagram 8) which is again determined by the chain below it, and must also match the boundary condition of the topmost point. It must so match because every other point is in balance--pits putting coal into the network, coke ovens taking coal out, and junctions merely passing coal through--and the total amount of coal put into the network by the pits has been made equal initially to the total amount of coal taken out of the network by the coke ovens.

Note that at no point in this process was there any choice about the tonnage to put on any route. The only choice was when a route was to bear zero tonnage, in which case it could be considered as going in either direction. It seems likely that a more efficient method of computation would leave that choice unsettled, but under the present method, the direction of zero tonnage routes was simply settled arbitrarily.

It will be noticed that the tree as described here includes every point on the railway network, whereas the solution tree, as illustrated on Map 2, does not include all the points. But this is because only the part of the tree bearing actual tonnage is illustrated on the map, not the parts bearing zero tonnage.

Where zero tonnage routes come between a group of actual tonnage routes and the rest of the tree, it appears on Map 2 as an independent region.

IV

A "first basis" has now been found--in other words, some method of allocating the coal between coke ovens and of routing it, such that the routing pattern forms a tree. In general, one would expect that this first basis neither made the best allocation of the coal nor used the shortest path between pits and coke ovens. The rest of the calculation is to make successive improvements until the minimum cost solution is reached. It will be recalled from Chapter 1 that it is possible to economize on computation in checking each route for its potential usefulness by assigning "prices" to each point. This is the next step.

The price at some one point must be fixed arbitrarily. It is useful, but by no means essential, to fix the arbitrary price sufficiently high to ensure that no prices are negative. The origin of the pricing system need not be at the top-most point of the chart, but it helps to avoid errors if it is so fixed.

Suppose the topmost point is assumed to have a price per ton for coal of 10 "miles"[1]. If the point (or points) immediately below the origin receive coal from it--including a notional receipt of zero tons--then its price must be greater than 10 by the distance of the route. If the lower point sends coal to the origin point then its price must be less than 10 by the distance of the route. It will be recalled that the routes have been listed under their origin points in the Profits Table (Diagram 2). The prices of the points, as they are obtained, are recorded against the point numbers in this list. The price of every point is obtained from the one above it in the same way--adding the cost of the route if the coal movement is down the chart and subtracting the cost of the route if the coal movement is up the chart. This step is complete when every point on the list has been assigned a price.

1. As we are minimizing mileage travelled (representing money cost) prices are also fixed in miles.

It will be recalled that an unused route should be brought into the solution if the price at the receiving end minus the price at the sending end minus the cost of the route--called the "profit"--is positive. To obtain these profits is the next step. It will be seen that by definition the profit on those routes which are in the solution is zero. These are labelled "in" on the Profit Table.

We have distinguished as two separate routes the two directions of each segment of railway line. Calling the prices at the two end points of a segment A and B, and its distance, x, the two profits are (A-B-x) and (B-A-x). The sum of the two profits is $-2x$. This condition always holds--that the sum of the profits of the two different directions of a segment is equal to twice the cost of the route. In particular, where one of the two directions forms part of the solution tree and hence has zero profit, the profit on the other direction is simply minus twice the cost of the route. These relationships, when compared with the profit as obtained from the prices, provide a check on the accuracy of the pricing system.

Suppose that at this or some other stage an error were found in the pricing system, and that it had been traced to the particular route where it occurred, then it would not be necessary to recompute all the succeeding prices. If the error on the one route has to be corrected by adding a certain amount to the incorrect price, then that same amount should be added to the price of every point which follows it in the pricing tree. Furthermore, if the error is found to have occurred in one of the uppermost parts of the main stem, so that the greater part of the points would have to be corrected, the best procedure is to alter the prices of the other points in the opposite sense. If the points whose prices have been altered in this fashion are listed, the profits can be altered in the same way as for a change of basis (see below).

V

The first basis has now been completed and all the routes, used and unused, have been evaluated on that basis. If none of the unused routes has a positive profit, there is no improvement to be made and the first basis is the minimum cost solution. If, as one would expect to be the case, there are positive profits, the largest of these should be selected, the resultant loop formed and rebroken, the appropria te changes in profits made, and the existence of positive profits again checked. This is the unit of calculation from here on, and it is known as the "change of basis". Note that prices are not recomputed each time, and only some of the profits changed.

To avoid having to scan the whole list of routes at the end of each change of basis it is worth keeping a list of those routes with a positive profit.

The first step in the change of basis is to find, by means of the order numbers, the two end points of the most profitable route, and mark them, say, with red pins on the chart. The next step is to check the profit by adding and subtracting the costs of the routes along the chain connecting the two points. Taking in the proposed new route must complete a loop (see Chapter 1), and starting a tour of the loop with that route, those routes where the direction of coal movement coincides with the direction of the tour will be added to the total cost, and those where it moves in the opposite direction will be subtracted. Prices and profits have been so defined that a certain positive profit equals a reduction by that amount in the total cost, and must check with the amount computed by making a tour of the loop.

Assuming that the profit from the Profits Table checks with the board, the next step is to determine which route is to be removed from the loop. It will be recalled that it will be one of the routes on which coal moves in the direction round the loop opposite to the new route. Of all such routes, there may be one or more bearing less tonnage than the others. To choose between two or more routes all bearing the same minimum tonnage, the most useful rule has been found to be to select the one with the greatest cost (i.e., the longest)[1].

1. Dantzig, l.c., presents a rule which guarantees that there is no "cycling" in the solution (i.e., returning to the same basis) but the above rule has not caused any cycling. Presumably the rule could be applied if cycling were suspected.

If the route selected as above were to be removed from the solution, the new route not having been brought in yet, the whole tree would be cut into two parts. The next step is to make a list of all the points in the smaller of the two parts of the tree--we can call this the "price change list for change of basis No. ...." This list may comprise only one point, or, in the worst case, one half of all the points[1].

Now one must adjust the tonnage around the loop, removing the route cutting the tree into two parts, and reconnecting the tree by writing in the new route. The amount of the adjustment equals the tonnage on the route being removed, and may be zero. That tonnage must be added to the new route and to all the routes going in the same direction round the loop, and subtracted from those going in the opposite direction. As the adjustment is made, it should be checked that the tonnage into and out of each point of the loop corresponds to its net availability or boundary condition.

The strip of paper representing the route being removed must be cut out, leaving

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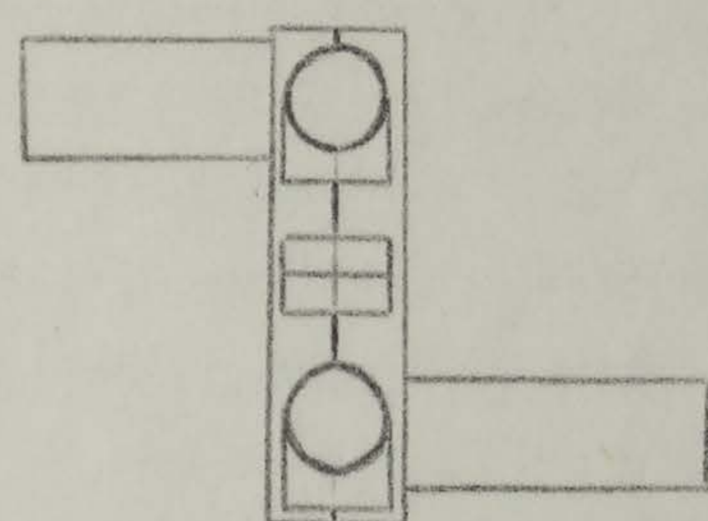
1. It is not worth counting the points if the two parts are approximatbly equal--in that case, either part can be listed.

the two end points of the route, one in each part of the old tree. If one part of the tree is very must smaller than the other it can be lifted and placed so that the two end points of the new route are adjacent to one another, and the new route written on a strip with tabs either side as in Diagram 9, for instance. The two tabs are then

<u>Diagram 9</u>



glued to the existing representations of the two end points and the tree thus reconnected. If the tree has been cut into two nearly equal parts, it may be simplest not to move either part, but to represent the new route by a long strip, if necessary winding down to the edge of the board and up again. In practice, it was found that it was simple to tidy up sections of the tree by redrawing them occasionally when there were too many such long connections.

If one part of the old tree has been bodily moved, it will most likely have moved up or down the board as well as laterally. In this case, the order numbers of all the points which have been listed on the Price Change List will have been raised or lowered by a certain amount, and this should be recorded at the top of the list.

The remaining operation to complete the change of basis, is to alter the profits. It is assumed that the prices are changed of those points which have been listed, and that those of the rest of the tree are unchanged. Calling the new route A→B, then A is in one part of the tree and B is in the other. Suppose A is in the smaller part, which has been listed, A→B was brought into the solution because it had a positive profit--say, of x miles per ton. In other words, the price at B minus the price at A minus the cost of route A→B e quals x. After the change of basis, the route A→B being now part of the solution tree, must bear zero profit. If the price at B is to remain unchanged, the price at A must rise by x miles to make the profit zero. But all the points listed with A, as belonging to the smaller part of the tree, are connected by an unchanged sequence of routes to A, and their price must therefore also rise by x miles along with the price at A. Therefore, at the top of the Price Change List, with the change in the order number (if any) should be recorded the rise (by the amount of the profit on the new route) in the prices of all the points on the list. If B, the receiving point of the new route, is on the list of changing prices and A on the unchanged part of the tree, the profit on A→B is brought to zero by the price at B falling by the amount of the profit, and hence by all the points on the list falling in price to that extent.

It is not necessary, however, actually to change the prices on the main list, so long as the profits are changed. The profit is unchanged on any route which either has both end points unchanged, or which has both end points changed by the same amount. It is changed on any route which has one end price changed and one end price unchanged. In practice, one goes through the list of points of which the prices have changed, looking them up in the full list of routes on the Profits Table (Diagram 2). For each route listed under such a point, check whether the point at the at the other end is also on the list and so also has its price changed. If not, alter the profit up or down by the amount of the profit on the new route according as the prices at the two ends have moved further apart or have approached closer toward each other. Next the route in the opposite direction must be changed in the opposite way. Any profits changed from negative to positive must be added to the short list of positive profits, and those changed from positive to negative must be removed from the short list.

The change of basis is now complete. A record should be kept of the route in and the route out and the tonnage of the adjustment, together with the list of points of which the price is changed, in case an error has to be traced. Cumulative error in the pricing system cannot arise because the profit is checked on each new route before it is incorporated. At the worst, the largest-profit route is not selected for each change , which may increase the number of changes of basis needed to the reach the solution.

In any case, no matter what state of chaos the calculation has got into (and all the automatic checks invented cannot prevent the inexperienced computor from creating chaos), loops can be removed and points restored to the chart so as to reform a tree. Then it is always possible to start again at Section III of this chapter, without losing the results of all the prece#ding work.

VI

At each change of basis, some profits are raised and some reduced so that there is no guarantee that the greatest profit is lower at each stage than in the prece#ding one, nor that the number of routes with positive profits will be lower. The amount by which the total cost falls at each change of basis depends not only on the profit of the new route but also on the tonnage which can be sent along it, which may be zero. The main determinant of the time taken for any particular change of basis is the length of the list of prices to be be changed, and this tends to be relatively short for zero tonnage changes.

Note that the only arithmetical operations in the calculation are addition and subtraction. A desk listing machine was used.

By way of summary, the different information stores are:

(1) The tree chart (Diagram 1) showing the current solution including tonnage and cost of routes in use.

(2) The list of all points and all routes on the Profits Table (Diagram 2), including both those routes used and those routes unused in the current solution, showing the prices associated with the first basis, the order numbers of the points, the costs of the routes, and the current profit of each route.

(3) Short list of positive profits

(4) Record of changes of basis, including for each one the Price Change List of points.

When there are no positive profits, the points are repriced and the profits recomputed as a check that the solution has actually been reached.

The solution tree can be translated into pit-to-coke-oven allocations (see Chapter 4).

The total cost can then be obtained in three ways:

(1) Multiplying the tonnage and cost of each route and totalling, from the tree chart itself.

(2) Obtaining the cost of the pit-to-coke-oven routes actually used from the difference of the prices at pit and coke oven, and multiplying these by the tonnages of the allocations.

(3) Obtaining the total "value" of coal at coke ovens by multiplying the amount required by the price at each coke oven, and subtracting from this the total "value" of coal at pits, obtained by multiplying the quantity available by the price.

These three figures are identically equal.

Chapter 4 Results

The simple transport problem represented by Map 1 has been solved to minimize total ton-mileage travelled by the coal, by the method described in Chapter 3. The data and the solution are described in Appendices A and B to this chapter.

Appendix A shows both the actual distribution of the coal and the optimum pattern. The pits and coke ovens are referred to by numbers in this Appendix, and can be identified from the list in Appendix B. The entire name and postal address of each coke oven has been listed, but in the interest of brevity only the names of the collieries as printed in the Mines Index[1] have been reproduced.

The matrix of all the pit-to-coke-oven routes is too large to reproduce conveniently. Furthermore, owing to the particular form of this computation, there are no data for the routes of that matrix which did not form part of either the actual solution or the optimum solution, so that the bulk of the matrix is necessarily blank. There are two ways in which the matrix could be compressed, either row-wise or column-wise, i.e., the routes classified either under pits or under coke ovens. In fact, only one of these is reproduced in Appendix A. Thus the source of every consignment of coal received by each coke oven under

1. See for example "Guide to the Coalfields" published annually by the Colliery Guardian.

either method of allocation can be seen immediately, but it would be necessary to reclassify the data of Appendix A in order to compare the destination of coal from each pit under the two solutions.

The first column of Appendix A shows the coke oven number. The second lists (in numerical order) all the pits which send coal to that coke oven under either solution. The third column shows the shortest rail distance from the pit to the coke oven. For the routes used in the optimum solution, the distance is obtained as part of the solution. It is the dual price (expressed in miles) at the coke oven minus the dual price at the pit. For the routes of the actual solution which are not also routes of the optimum solution, the shortest rail distances have been separately computed in order to have a basis of comparison between the two allocations.

The fourth column in Appendix A shows the way in which the coal was actually allocated during the month of the survey. The fifth column is the product of the tonnage and the mileage on each route. The tonnage and the ton-mileage of the actual solution are totalled for each coke oven, and for each page of the table. The cumulative total for each page is also shown.

The optimum solution is similarly shown

in the sixth and seventh columns. There are many more admissable pit-to-coke-oven routes (576) than the minimum necessary to fulfill the program (218)[1]. In order to make the comparison between the optimum and the actual solutions, one of the possible patterns of tonnages (using the minimum number of routes) has been selected from amongst the admissable routes. The other admissable routes are shown by <u>zero's</u> in the tonnage and ton-mileage columns of the optimum solution. Those routes which are used in the actual solution and are not admissable in the optimum solution are left <u>blank</u> in the optimum columns.

If linear programming were being used as a current method of control, the full list of admissable routes could be issued to the marketing officers who could work within it to satisfy as far as possible the particular wishes of the consumers. The drawback to the latter method is that it would probably result in more than the minimum number of different routes being used, and therefore possibly smaller consignments.

Again the tonnage and the ton-mileage

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1. It is never necessary to use more routes than n-1 (where n is the number of origins plus the number of destinations) in a transport problem. But where two or more routes go through a single point, the solution will be indeterminate between such routes (cp. Koopmans, op. cit, page 253). This is the case for very many routes when the connections are restricted to the railway network.

of the optimum solution are totalled for each coke oven. The total tonnage is, of course, the same as under the actual allocation for each coke oven. It will be noticed that the optimum ton-mileage is by no means lower for every coke oven than the actual ton-mileage. This is as one would expect from the principle that the coke ovens have been regarded as merely units of a national economy for which the cost of transport is to be minimized, rather than as individual cost-minimizing units in an imperfect market.

For the optimum solution, as for the actual solution, the tonnage and the ton-mileage have been totalled for each page, and the cumulative total also recorded.

The explanation of the two remaining columns of Appendix A will be found in Section V (below) of this chapter.

The grand total tonnage (as shown on the last page of Appendix A) is 1,592,873. The total actual ton-mileage is 31,949,212.18, and the optimum, 28,726,056.38, a reduction of 10.09%. Or, to look at it in another way, the average distance travelled by each ton of coal was 20.06 miles, whereas under the linear programming solution it would have been 18.03 miles.

II

The dual solution of the linear programming problem is shown in Appendix B, in the "price" column associated with each coke oven and colliery. It will be recalled that these "prices" are not absolute but rather the price differences which would have to be established between the different geographical points it it were desired to allocate optimally the given quantities of coal available and required by means of a price mechanism. Thus the price at Pit 2 would have to be the money equivalent of two miles higher than the price at Pit 1. But there is no direct significance in the prices themselves, particularly as there is no colliery or coke oven priced zero so that all the other prices could be regarded as the excess price as compared to the zero point. In fact, the origin has been so chosen that one colliery (No. 84) has a negative price. This merely means that Pit 84 would have the lowest price in the above price system.

If the prices themselves in Appendix B have no direct significance, the figures in the "value" column have even less. They are simply the product of the price and the tonnage available or required at each pit or coke oven. They cannot be compared one with another, because a shift in the origin of the pricing system would shift the value figures by varying amounts, according to the tonnages. What would not vary under such a shift in the pricing system, however, is the difference between the total value of the coal required at the coke ovens and the total value of the coal available at the pits (since the total tonnage is the same in both cases). It will be seen from Appendix B that the total value of coal at coke ovens is 153,939,655.10 ton miles, and the total value at collieries is 125,213,598.72 tons miles. The difference, 28,726,056.38 ton miles, is equal to the total transport cost of the optimum solution, as it must be (see equation 25 of Appendix C to Chapter 1). Thus this calculation constitutes a check on the optimum ton-mileage.

III

The calculations have been performed entirely in terms of ton-mileage. This can be regarded (see Chapter 3) either as an approximation to minimizing the money cost of transport or as a better approximation than money cost to minimizing the real cost of the transport. In either case, however, the money equivalent is of some interest.

The 154 pits actually supplied the 65 coke ovens by 355 different connections (or 322, omitting those where the coke oven is located on the colliery) as compared to 216 (or 189 omitting those where the coke oven is located on the colliery) of the linear programming solution. This reduction in the number of pit-to-coke-oven connections may be itself an additional reduction in the real cost of transport not reflected in the simple reduction of ton-mileage as it may enable larger consignments to be made up. On the other hand, it might lead to additional congestion on some routes, a factor which has not been taken into consideration in this computation, though there is no difficulty in principle in so doing. No allowance is made for this factor in the following estimates of the money saving.

In principle, railway freight rates per ton are based on shortest rail distances. But in practice there are many special rates. British Railways are not prepared to quote any one rate per ton-mile, although they will quote for particular routes. However, it was not possible to obtain all the 65 x 154 = 10,010 quotations.

It is conceivable that the special rates are such as to make the actual solution cheaper than the optimum one. But in fact there seems no reason to doubt that a significant reduction in ton-mileage would be associated with some substantial reduction in money cost.

What was available in the original data was the transport cost (actually, the difference between pithead and delivered price) of those routes used in the actual solution. Regarding these as a sample set of cost quotations from which to estimate the ton mileage rate, the following regression equation for cost upon mileage was obtained:-

$$x = 3s.1.483d. + 1.327d.y$$

where x = the transport cost per ton, and y = the shortest rail mileage. The correlation coefficient between x and y is 0.905. Thus the transport cost is made up, on the average, of 3/1 per ton handling charges, plus 1.3d per ton-mile travelled. The reduction in ton-mileage would affect only the latter item.

Thus the variable part of the total transport cost due to mileage was actually 1.327d. x 31,949,212.18 = £176,653, whereas the optimum would have been 1.327d. x 28,726,056.38 = £158,831; a saving of £17,822 for the one month, or £213,864 per annum. (South Wales has been excluded altogether from both the optimum and the actual solutions).

The data for these calculations are incomplete (by excluding South Wales) and out of date. To make a very rough estimate of the potential current annual saving in transport cost, we can assume that the average saving in mileage per ton is applicable to the to the whole country and to the present time.

The total quantity of coal delivered to coke ovens in the year up to August 1956 was 28,747,000 tons[1]. Assuming that on the average each ton travelled 20.06 miles, total ton-mileage was 576,665,000. During the period since the survey was taken, railway freight rates for coal traffic have risen by a uniform 84.04%[2]. Therefore the average cost per ton mile (on the basis of the regression equation) is now 1.327 x 184.04% = 2.442d. per ton-mile.

Thus that part of total transport cost with is variable with mileage should now be of the order of 2.442d. x 576,665 = £5,867,000 per annum, and the potential saving from using linear programming to plan the transport of coking coal, of the order of £5,867,000 x 10.09% = £590,000 per annum.

The foregoing estimate of the current potential saving from the use of linear programming in planning the distribution of coking coal is based on some very crude assumptions. To make some estimate of the order of magnitude of the possible saving from applying linear programming to all forms of coal, an even cruder assumption has to be made--that all forms of

1. Monthly Digest of Statistics, September 1956.

2. May, 1950, 16 2/3%; April,1951, 10%; December, 1951, 10%; December,1952, 5%; March, 1954, 10%; June, 1955, 7½%; April, 1956, 5%.

coal travel the same average distance as coking coal, and that the same reduction in average distance could be achieved by linear programming. As coal consumed by coke ovens in the year up to August 1956 formed about 1/7 of all coal going to inland consumption[1], we can obtain an estimate of the potential saving of £4,470,000 per annum.

IV

Whilst it would be unreasonable to expect British Railways to quote for 10,010 hypothetical routes, it might be possible to obtain them for the 410 (non-zero mileage) admissable routes of the optimum solution[2] (not also part of the actual solution). This would give a genuine money cost comparison between the actual and the optimum solutions, and would, furthermore, enable a move towards the true minimum money cost solution if it were the case that some of the actual routes not used in the optimum now became profitable. This might still not be the true money cost solution, but it would at least have the virtue that it would take into account any special rates taken advantage of in the actual solution.

1. Monthly Digest of Statistics, September 1956.

2. or 110 actually used non-zero routes of the optimum solution (not also part of the actual solution).

There may be other cases when either the inaccessibility of data or the magnitude of a straightforward transport problem forces one to use a ton-mileage approximation to the true money cost solution. In general, cost data will be known for the routes which have been used in the past, and can be combined with the limited number of new routes suggested by the minimum ton-mileage problem to obtain a manageable money cost matrix on which a standard transport calculation can be performed.

V

The size of the potential saving from the use of linear programming is sufficiently great for the solution to be worth a closer study.

Every route used or available for use has associated with it a "profit" figure in the solution. The criterion that the solution had been reached was that no unused route show a positive profit. By definition, those which are used have zero profit. In the calculations carried out in this study, the routes are simply the segments of railway line. But the condition on the profits applies also to the pit-to-coke-oven connections. One most of these routes the mileage is unknown and the (negative) profit is therefore also unknown. But as the mileage has been computed for the routes used in the actual solution and

and not in the optimum, the profit on them can be computed. These negative figures have been recorded as the positive "loss per ton" against each route of the actual solution in Appendix A of Chapter 4.

The loss per ton on any route is the excess of the mileage of the route over the price difference between the end points. It can be interpreted as the marginal cost per ton of using that route directly, as opposed to using the (unique)series of indirect adjustments within the solution tree which would also result in transferring one ton from the origin point to the destination point of the route.

It should be noted that the "loss per ton" column, relating to absolute differences between prices, is independent of the origin of the pricing tree.

The column "total loss" is the product of the loss per ton and the tonnage on each route of the actual solution. It is a measure of the share of the total excess cost of the actual solution over the optimum solution contributed by each consignment. Where the route is one also used by the optimum solution, the loss per ton is, of course, zero.

The sum of all the entries in the total loss column is equal to 3,223,155.80, the difference between the actual ton-mileage and the optimum[1]. Again, this column is totalled for each coke oven, and for each page, and accumulated for each page. Note that the total loss for each coke oven bears no relationship to the difference between the ton-mileage of the two solutions for each coke oven: the equality holds only over the whole "economy".

The loss per ton on the different routes varies from a fraction of a mile to over 20

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1. That it must be so follows from the general theory of linear programming. In this application it can be demonstrated as follows:-
Let us consider the route R_{ij} from the point i to the point j ($i = 1,2,\ldots,m$ where m is the number of pits; $j = 1,2,\ldots,n$ where n is the number of coke ovens). The dual price at i is p_i; at j is p_j. The cost of the route is c_{ij}. The loss per ton, $l_{ij} = c_{ij} - (p_j - p_i)$. If x_{ij} is the actual tonnage on the route, the total loss, $L_{ij} = x_{ij}\left[c_{ij} - (p_j - p_i)\right]$. Summing over all routes,

$$\sum_{i=1}^{m}\sum_{j=1}^{n} L_{ij} = \sum_i \sum_j x_{ij}c_{ij} - \left[\sum_i \sum_j x_{ij}p_j - \sum_i \sum_j x_{ij}p_i\right].$$

But $\sum_i \sum_j x_{ij}c_{ij}$ is the total actual ton-mileage; $\sum_i \sum_j x_{ij}p_i = \sum_i x_i p_i$ (where x_i is the availability of coal at pit i) which is the total value of coal at pits; $\sum_i \sum_j x_{ij}p_j = \sum_j x_j p_j$ (where x_j is the requirement for coal at coke oven j) which is the total value of coal at coke ovens. Therefore $\left[\sum_i \sum_j x_{ij}p_j - \sum_i \sum_j x_{ij}p_i\right]$ is an expression for the optimum ton mileage, and the whole expression $\sum_i \sum_j L_{ij}$ is the difference between the actual ton-mileage and the optimum.

miles. One objection to the use of linear programming may be the very restricted pattern of allocation of the coal indicated by the solution. It is apparent from Map 2 that many pits in the solution are prohibited from sending coal to nearby coke ovens and instead send it over long distances. The "loss per ton" figure for each route gives us a measure of the cost of relaxing the conditions of the solution by permitting non-optimal routes to be used. The loss per ton on any route cannot exceed twice the mileage of the route, so that allowing routes to be used on which the loss per ton is small, allows (amongst others) routes between nearby pits and coke ovens to be used. Table 1 shows the effect of allowing routes of the actual solution to be used if the loss per ton does not exceed a certain value.

Table 1

Loss per ton	Total loss	Cumulative loss	Cumulative loss as a percentage of difference between optimum solution and actual solution.
Under 0.25	12,395.02	12,395.02	0.38
0.50	16,167.06	29,562.08	0.91
0.75	9,150.00	38,712.08	1.20
1.00	24,839.00	63,551.08	1.97
2.00	95,291.40	158,842.48	4.93
3.00	262,948.78	421,791.26	13.09
4.00	142,469.44	564,260.70	17.51
5.00	154,806.98	719,067.68	22.31
10.00	709,260.64	1,428,328.32	44.31
20.00	1,097,702.38	2,526,030.70	78,37
Over 20.00	697,125.10	3,223,155.80	100.00

Thus if the routes of the actual solution were permitted for which the loss per ton does not exceed 0.25 ton-miles, and if those routes were used with the same tonnage as in the actual solution, the total cost would rise by 12,395.02 ton-miles, or 0.38% of the total difference between the optimum solution and the actual solution. If the routes on which the loss per ton does not exceed 0.50 ton-miles were permitted, the total cost would rise by a further 17,167.06 ton-miles, or a total of 0.91% of the difference between the two solutions. And so on.

Thus the cost of each degree of relaxation in the conditions of the solution can be seen. It might be considered that the extra freedom of manoeuvre by the sales department from issuing a list of permissable routes including not only those on which the loss per ton is zero, but also those for which the loss per ton is less than (say) two miles might be worth the sacrifice of 5% of the possible saving from using linear programming. Of course, this implies that the non-optimal routes would not be used more intensively than before the institution of linear programming. It might prove more costly than 5% of the saving, and the limit might have to be lowered. On the strict solution, 165 of the 355 actual routes are also optimal. The relaxation of the limit to a loss not greater than two miles would leave 215 of the actual routes still usable.

There would, of course, be other routes for which the loss per ton does not exceed two miles, for which no cost information is known in this program. If the cost is obtained on any route the loss per ton can also be immediately computed, so that the list of permissable routes could be extended as new routes were proposed and examined.

VI

It should be repeated that the most serious objection to the application of linear programming to the distribution of coking coal lies in the fact that the coal is not a strictly homogeneous commodity. The computations of the loss per ton provide an easy way of checking the addition to the total cost which would follow from insisting that route A-B must form part of the solution because A's coal is of a special type which is required in the same quantities by the coke ovens which received it in the actual solution.[1]

It will be apparent from Map 2 that Durham coal is to be sent out to all the other regions,

1. It is understood that in fact Durham and South Wales coal is particularly suitable for coking, and that in some cases it is required in order to blend it with less satisfactory coal.

but not to every coke oven in each region. In particular, we see that nearly all the Durham coal going to the Yorkshire and Midlands area is in fact diverted eastwards towards Scunthorpe. It will also be seen, however, that the general direction of coal movement in that area is southward, and that the southward moving Durham coal comes within a very short distance of the rest of the graph for that area. The loss on that short stretch of line between the Durham coal and the rest of the Yorkshire graph cannot be more than twice the distance of the segment. One could, therefore, take that loss, and the loss on any other segments which would have to be added to the graph to connect Durham to each particular coke oven, as the measure of the additional cost which would be incurred by distinguishing Durham coal as a separate commodity.

A cruder measure has been taken, as illustrated in Table 2.

Table 2 Total coal from Durham to coke ovens not scheduled to receive Durham coal in the solution.

CO	Pit	Miles	Tons	Loss per ton	Total Loss
4	83	164.96	529	2.75	1,454.75
57	82	150.41	3,136	7.00	21,952.00
	84	159.89	2,557	7.00	17,899.00
59	35	187.40	4,163	12.43	51,746.09
	38	185.95	2,872	8.50	24,412.00
	60	202.14	4,289	8.50	36,456.50
	61	201.39	2,919	8.50	24,811.50
	69	208.65	3,728	25.52	95,138.56
60	42	127.77	2,697	7.50	20,227.50
					294,097.90

The total loss on these consignments is 294.097.90 ton miles, or 9.12% of the total saving from using linear programming. This is really the loss from obtaining coal from those particular pits, rather than just any Durham coal, and therefore it overstates the additional cost which would be incurred if the program were split into two parts, Durham coal and other coal.

Chapter 5 Conclusions

I
Further applications of the model to coal transport.

It has been claimed in Chapter 4 that appreciable savings in transport costs could be made by using linear programming in planning the distribution of coking coal. Coking coal was chosen for this study because the data were available, not because it seemed the most likely place to find an improvement. It is reasonable to assume that similar savings could be made on other classes of coal. But further research on the application of linear programming to the transport of coal can only be carried out within the National Coal Board and the various coal-using organisations.

The interaction between the programs for different types of coals with their varying degrees of substitutability is the most obvious line of future development arising out of this study. It seems probable that fairly simple adaptations of the calculation procedure would enable the transport model to be used not only to allocate coal between different users of the same general type (e.g., coking, household, etc.) but also to assist in the assignment of coal to the appropriate type of user. The result might be only an approximation to the theoretical optimum which would require a general model linear program. But such an adaptation might be manageable where the general model might be impossibly large. The adaptation of the transport model should at least yield a lower total cost than the sum of the minimum transport costs for each group of coal taken separately.

This type of investigation is one which could only be done with access to a constant flow of detailed information as to the qualities and quantities of the coals available and required in each period. It would arise most naturally from the successive practical application of the transport model to an ever greater range of types of coal.

II

Applications to other commodities

The distance of every "point" (junction or station) to all its adjacent points[1]

1. The best source of these data for future applications would the data being used by British Railways in their computation of the shortest distance of every point to every other point on the railway network.

could be used as the basis for a variety of other transport problems for commodities where the cost of transport is proportional to the rail ton-mileage. This type of problem is faced by many large industrial organisations with factories or warehouses located in different parts of the country, supplying widely scattered consumers. Normally one would also require cost data for the other forms of inland transport and the cost relationship between the different forms (which might vary for different commodities).

One possible application is the return of empty railway waggons from the consuming regions to the producing regions of the country. This is similar to the original Koopmans problem of the routing of empty ships. It is probable, however, that this should be based on working distance rather than on rail distance (see Chapter 2). It may be that the best procedure would be to make an initial computation on the rail distance basis and then eliminate any sections which are not conceivable working distance routes by putting a very high cost on the use of certain junctions in the "wrong" direction.

A very direct application of the British Railways data of adjacent point-to-point distances would be in the computation of the shortest distance of every point on the railway network to every other point. This would be, as it were, a succession of zero tonnage problems[1].

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1. One would start by constructing a tree with all the routes directed outwards from an origin point: compute prices for all the points and profits for all the adjacent point-to-point routes; change basis until there are no positive profits. From the resulting tree one can obtain the shortest distance (including, if desired, the actual route) between all pairs of points connected by a chain of routes all pointing in the same direction, by subtracting the price at one end from the price at the other. This will complete the distance from the origin to every other point and all intermediate distances on routes from the origin. Then one would take another starting point (probably one near the previous origin) and change directions of the routes so that again the new origin sends out to every other point. The prices do not need to be recomputed, and the profits would be changed by constant amounts where-ever a route has been reversed in direction (see Chapter 3). Most of the tree would be unchanged, but it might be necessary to make one or two changes of basis until there are again no positive profits. This tree would yield more shortest distances. And so one could proceed, taking origins near the previous one where necessary to fill gaps in the matrix of point-to-point distances. In each new tree the only basis changes would be in the region of the new origin, leaving the bulk of the tree unchanged.

This computation of the shortest distance of every point to every other point is at present being carried out by other methods, so that this procedure is of somewhat academic interest, although it might find a use in other countries or for other forms of transport[1]. It is difficult to estimate the cost of such an operation. The necessary manipulations of the tree for a problem of, say, 5,000 points could probably be carried out by two people with a desk adding machine in two or three years. But the actual subtractions and printing of the 25,000,000 point-to-point distances would probably be best handled by electronic equipment.

III

Relative advantages of standard form & modified form

Once the shortest distance from every point to every other point is known, the question arises whether this matrix of distances is not a better form of basic data for future commodity transport problems than the adjacent point-to-point distances. The answer will vary with the problem. The size of the tree is determined by the number of points used, so that it will be much larger if junctions as well as origins and destinations are included. But the size of tree is much less

1. c.f. A. Orden "The Transhipment Problem" Management Science, April 1956, Vol II No. 3 p.276.

important than the size of the profits table which has to be modified at each change of basis, and this is determined by the number of routes. Even supposing an entire railway network of 5,000 points to be relevant to the problem the number of (directed) routes will be not very much more than 10,000. This number would not change significantly from one problem to another, whereas the number of routes in an ordinary matrix formulation of the transport problem is the product of the number of origins and the number of destinations. Thus a problem greater than 100x100 (or 50x200 or 20x500) would be more conveniently dealt with by the adjacent point routes. In practice (for instance the coal-to-coke oven problem) the relevant part of the railway network may contain considerably fewer than 10,000 routes. Furthermore a given change of basis would cause more profit alterations in a profits matrix than in a profits table of segments of railway line. For instance, consider a basis change for which the prices of one half of the tree are to be altered relatively to the other half. In the matrix form the profits would have to be altered for every connection from an origin in the one half to a destination in the other half.

But in the railway network form, only those stretches of line potentially connecting the two halves of the tree have to be altered. Nevertheless, the decision cannot be made a priori.

IV

Computational Method

The main contribution of this study to the field of linear programming has been the development of a transport model suitable for hand computation. Whether or not this is a useful contribution depends on how it compares in both speed and cost with existing electronic methods of computation, and also on whether the new method can be developed to use special or general purpose mechanical or electronic devices with any saving in time or cost.

It is understood that there is virtually no limit to the size of problems which can be handled electronically by existing programs in the United States, but the cost and time are unknown. In the United Kingdom the only electronic computer known to have a program for the transport problem is the Ferranti Pegasus. This is designed to deal with up to 128 points and up to about 1,000 routes. The only information as to the cost of this program is that the solution of a problem with 8 sources and 56 destinations has been estimated to cost between £20 and £25.

To do by hand a similar problem to the one in this study, using the railway network adjacent point distances (assuming maps to be already available) should take one experienced person two or three months using a desk adding and listing machine. To recompute the program, however, as the quantities available and required changed (say, by not more than 10 or 15%, either way, at each point) should take only about a week or a fortnight.

The part of the computational procedure which would yield the greatest returns from some form of mechanisation is the modification of the profits table at each change of basis. It will be recalled that the profits table consists of a list of routes (2,000 in the present study) with their associated profits. At each change of basis there is a list of points of which the prices are raised by a constant x (which may be positive or negative). For each route there are four possibilities: (1) neither end point of the route is on the list of points--in which case the profit is unchanged (2) both end points of the route are on the list--in which case the profit is unchanged (3) the origin point of the route is on the list and the destination point is not--in which case the profit is reduced by x (4) the destination point of the route is on the list and the origin point is not--in which case the profit is raised by x. One also requires an indication at each change of basis of the largest profit.

This operation could be readily carried out on a standard electronic digital computer, using, say, magnetic tape to record the profits table. But this implies the regular but intermittent availability of a computer for perhaps 10 minutes at a time at, say, half hour intervals. This would not normally be practicable (unless of course there were several such programs running concurrently). Furthermore, the facilities of such a computer would be wasted on so simple a task.

It is possible that a special purpose computer to operate on the profits table would be worth constructing. It could be used in conjunction with a somewhat improved form of the chart which could probably also be mechanized to some extent.

It is not possible to decide which, if any, of these methods is worth further consideration without considering how often the model is likely to be used.

One other possibility is to consider whether the methods used in this study for hand computation could usefully be incorporated in a complete program for a general purpose electronic digital computer. There is no real reason why its great advantage in hand computation should carry over into electronic computation as it has been devised to exploit to the full the visual sense (in tracing the chain between two points and in deciding which is the smaller "half" of the tree) whereas the electronic computer would have to "grope" for the chain. At an earlier stage in the development of the model, the tracing of the chain and the building up of the price change list were done purely numerically (at a cost, of course, of carrying extra information). It was the lack of a means of finding the smaller "half" of the tree which led to the chart method being developed. But this is a deficiency in the numerical method which might conceivably be remedied (again by carrying extra information). The whole point of changing the standard simplex method at all, however, was to make use of the fact that not all the prices and profits change at each basis change. If, in fact, the electronic computer has to scan every entry in the profits table in any case, it probably makes little difference if it also recomputes each one.

Nevertheless, it is considered that the hand computational advantage of this method is sufficiently great to warrant planning an electronic program on these lines and making an estimate of the time required to change a basis, and of the storage space required.

One completely different method under consideration is an analogue computer which would yield an instantaneous answer to a transport problem. Such an analogue appears to be possible in theory, but the investigation is at too early a stage to determine whether it will prove to be practicable.

V

Relevance of the transport model to pricing

This study has been concerned with linear programming principally as a form of operations research, and hence the dual solution has been only touched upon. It will be recalled (Chapter 1) that the solution of a transport problem consists not only of the minimum cost allocation pattern but also of a set of equilibrium price differences for the commodity in the different geographical locations. These are equilibrium prices in the limited sense that only if this were the equilibrium price pattern would the quantities demanded and supplied be the same as those which form the initial data of the transport problem. It is quite possible that they are not the prices which would be established in the long run by a free market when the quantities available and required could also vary. However, such a long run equilibrium set of prices would itself be the dual solution set for the quantities established by the demand and supply conditions at each place.

The setting of prices is a continuing problem for all large industrial organizations, and particularly for the nationalized industries which are by their nature unusually isolated from competitive considerations, and which do not generally fix their prices on simple profit maximisation principles. The question arises as to whether linear programming is relevant to this type of problem as well as to the physical direction of operations. In more concrete terms, can linear programming and specifically the transport problem throw light on the pricing policy of the National Coal Board.

The theoretical problem of finding the equilibrium prices and quantities for a situation in which there are n places, separated by known transport costs, each of which has a known demand and supply schedule for a homogeneous commodity, has been shown to be soluble by an electrical analogue, by S. Enke.[1] Samuelson[2] has shown that the solution to this problem contains within it the solution of the minimum transport cost problem for the quantities actually traded in the

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1. S.Enke "Equilibrium among Spatially Separated Markets: Solution by Electric Analogue". Econometrica, Jan.1951.
2. P.A.Samuelson "Spatial Price Equilibrium & Linear Programming". American Economic Review. June 1952.

equilibrium situation. Oddly, he queries whether it also solves the dual solution of the transport problem. In fact the conditions of the Enke equilibrium prices are precisely the dual price conditions for the minimum transport cost linear programming problem.

In practice, no analogue computer has been constructed to solve such a problem. But it should be possible to devize a modification of the simplex procedure to solve this problem. For instance, one could start by selecting any consistent set of quantities to be imported and exported at each point, and solve the minimum transport cost problem for those quantities. The quantities to be exported and imported for the next iteration would be those associated with the dual prices of the first iteration, the zero position of the dual set being selected to make the total volume of imports equal to the total volume of exports for the second iteration. Then the new transport problem could be solved, the dual prices again used to determine a third set of quantities. The solution would be reached when the quantities for the final iteration can be disposed along the routes of the previous tree without any reversal of direction.

This modification of the transport problem amounts to a form of non-linear programming where the boundary conditions are functions of the y (dual price) values. It would have to be proved that the procedure outlined above (or some modification of it) is in fact convergent to the unique solution. But the close analogy to the economic procedure of adjustment to equilibrium gives grounds for confidence in its convergence.

There remains the serious objection that demand and supply schedules are in fact unknown. It may even be the case that a nationalized industry is in the habit of thinking in terms of quantities demanded almost as independent of prices, partly because of their monopoly position and partly because, in the case of coal at least, prices have been consistently below the equilibrium level since nationalization. Indeed, in the short run, the demand for coal may be very price inelastic, as it is determined largely by installed equipment. The demand for coking coal, in particular, is not price sensitive, as there is no alternative for the production of coke, and as the cost of coal can be readily passed on to the coke and ultimately to the steel, both of which are currently sold at prices below the equilibrium level[1].

In the long run, however, the price at which coal is sold in each market must have some effect

1. L. Foldes "Iron and Steel Prices" <u>Economica</u>, Nov.1956

on the quantities purchased. For most uses of coal there are alternatives available. The change-over cost for existing plant may be high, but the initial cost when new plant is being built may be much the same, so that the longer the period considered, the more sensitive would demand be to the relative prices of coal and other fuels. In so far as the cost of coal is a significant item in determing the location of new industrial development there is another source of price elasticity of the demand for coal at different places.

Perhaps the most important source of price elasticity of demand, however, is in the substitution of one grade of coal for another. It is in this respect that the pricing policy of the National Coal Board is most open to criticism. So far a system of zone delivered prices has not been instituted for any coal except household grades, but it is the eventual aim of pricing policy to extend such a system to all coal. If it were intended that (as in the case of household coal) each grade of coal should bear the same price relationship to other grades in every zone, such a pricing system would have a very distorting effect on the demand for the different types. To take the very simplest case, it would be in accord with the best distribution of national coal resources if a plant situated on a coal field producing poor quality coal had a relatively larger price incentive to install equipment capable of using that coal, than a plant situated on a coal field producing good quality coal. This incentive would be lost if a uniform price relationship were imposed for every zone.

So much for demand elasticity. The price elasticity of supply of coal at each pit may be zero if the National Coal Board chooses to make it so. The development plans can be determined completely without regard for the selling price of coal, and a uniform price charged regardless of costs. There is no obligation on the National Coal Board to ensure that each individual colliery covers its costs[1]. In fact, one assumes that in order to keep down the

1. The only requirement as to pricing policy in the Coal Industry Nationalization Act, 1946, is that coal should be made available "at such prices as may seem to them best calculated to further the public interest in all respects, including the avoidance of undue or unreasonable preference or advantage" (Section 1, Subsection (1)): and "that the revenues of the Board shall not be less than sufficient for meeting all their outgoings properly chargeable to revenue account...on an average of good and bad years" (Section 1, Subsection (4),(c)).

average cost of production as far as possible, development effort is concentrated on the lowest production cost places.

It will be recalled that the dual solution assigns prices to coal not only at each consuming point but also at each pit. These prices are the locational values (with respect to the actual quantities available and required) of the pits. In a theoretical perfect competition model the equilibrium locational values would be equal to the marginal production costs[1]. In practice it is reasonable to claim that pits with a high locational value should be expanded even if at a higher cost of production than pits less favourably located with respect to the market. Probably this consideration is not neglected in development plans, if only in that the costs at the different pits would be considered relative to the cost of the coal field rather than to the national average. Nevertheless, the further the pattern of production and consumption moves away from the pre-war pattern which, however imperfectly, underlies the present price structure, the greater is the need for a redetermination of the locational values, even if these

1. The greater the production at any point, the lower would its locational value fall, and, in general, the greater the production the higher would rise the marginal costs of production. Thus the equilibrium equality would arise from the movement of both factors.

are regarded as purely notional prices and not reflected in market prices.

It may be objected that determination of coal prices along the lines of linear programming would lead to continual fluctuations in prices owing to purely accidental variations in quantities demanded and supplied, or to the maturing of development plans changing the sources of supply; and that such instability would make planning impossible, both by the producers of coal and by the users of coal. Certainly fluctuating prices would cause considerable protest from coal users. But some degree of price flexibility is necessary in a changing economy and the period for which prices are fixed could be as short or as long as required. It would be reasonable to make estimates of demand and supply for the period (which may be as long as five or ten years hence, but must surely be foreseeable) when it is expected that the price of coal should no longer be chronically below the equilibrium price, and carry out the price determination problem as described above, taking also into account the effects of variation in the price of one type on the demand for another. It is not claimed that this calculation would be easy, but it would be feasible. It would not, for instance, call for the construction of continuous demand and supply schedules for each type of coal at each place, but only for estimates of some points on such schedules--which estimates must underlie any development and pricing plans, even if only implicitly. It might be assumed initially, for instance, that all price elasticities and cross-elasticities were zero, and the resulting price-quantity pattern checked for reasonableness at each point.

This price pattern could then be the basis of the market price structure, reduced throughout by the amount which it was intended to hold the price of coal below the equilibrium value. The pattern could be subject to annual revision in the light of changing estimates of demand and supply. As estimates for the future change more gradually than the actual situation, this would ensure as great a degree of price stability as is consistent with a changing economy. Such a price pattern would be a valid guide to development both for coal users and for the coal producers.

Note that the distinction between pithead and delivered pricing is irrelevant if the absolute price differences between the different locations are in accordance with the dual solutions for quantities actually coming on to the market. If the prices are fixed at the pits and announced for the consuming points, no purchaser could do better by purchasing at the pithead and arranging for the transport of the coal himself than by accepting the announced price. (This is another way of expressing the conditions of the linear programming solution).

This outline of a pricing policy is unlikely to find favour with the National Coal Board or with coal users in their corporate form of the Federation of British Industries. Both these bodies[1] are committed to the doctrine of the "fair" price. A complicated assessment of coals has been made by calorific content, ash content, etc., and the price of each coal based on this analysis. As however it is impossible to legislate for prices without regard to quantities, it will not surprise economists to find in the description of coal pricing (l.c.) the following heads:-

"Coalfield adjustment

"The price structure herein described is entirely based on pit head prices and the evaluation up to this point has merely been related to the quality and characteristics of the coal irrespective of the district in which it is produced.

"Throughout the history of the industry there has been competition between districts based on costs of production and transport advantages to the chief markets. There have in fact been wide differences in the pit head prices of coal of equal value to the consumer in the various coal-fields. If these differences are ignored the existing pattern of distribution would be largely changed. There would probably be great confusion, and also grave economic effects in some districts.

"With the object therefore of maintaining approximately the present relations between the main producing districts and their various markets there must be some difference in the pit head prices of similar coals produced in different areas. To effect this the Board have introduced a coalfield adjustment which is added to the pit head price as determined above." (There follows

1. See, for example, p.2 of "Coal: the Price Structure" second edition, published by the Federation of British Industries.

a table showing the amounts, but not the method of computation.)

"Selective increases

"As a result of experience since the scheme was introduced the National Coal Board have found it necessary to take into account in fixing prices the relative scarcity of particular kinds of coal, the increasing demand both at home and abroad for certain qualities of coal, and the unavoidably high cost of production incurred, and likely to be incurred, in producing some coals in particular coalfields. They have done so by applying selective increases to certain coals,...."

These modifications to the "fair" price open the door to the demand and supply type of price determination. The exact determination of the coal price structure goes only half-way. The methods suggested in this chapter might be developed to provice an objective formula for these otherwise arbitrary or historically determined adjustments.

VI

The theory of location

The theory of location has remained a relatively under-developed part of economics, largely, one suspects, because of the discontinuous steps introduced into the functions by transport costs, which prohibit the use of the usual calculus concepts. It may be that linear programming, which deals with such discontinuities explicitly can provide alternative formulations of the theorems of interregional trade, which incorporate transport costs, and so lead to a useful theory of location.

It was claimed above that even the simple application of the transport model to the allocation of a homgeneous commodity had a direct bearing on the problem of the location of new industrial development. But it was not claimed, for instance, that the price of coal was the only factor determining the location of coke ovens. If it were, and if the location of coal production (as distinct from the location of the coal seams) were itself completely unvariable, the theoretical optimum location of the coke ovens would be on the collieries.

If one were attempting to develop a theory of the location of coke ovens, the next degree of approximation would be to consider the pull of the market for coke. This is largely provided by the iron and steel industry. Assuming that they were fixed by other considerations, it should be possible to show that coke ovens should be located either on the collieries or at the iron and steel plants, depending on the relative cost of transport of coal and coke. In fact, coke ovens fall into two groups, those located at collieries and those located at the market. It would be interesting to find the conditions which would predict such a pattern, and to see whether the conditions are still fulfilled, or whether the trend were towards one type of location or the other.

VII

Conclusions

Although this report would have been more complete if it had been possible to follow up all the lines of research which have been suggested in this Chapter, from another point of view the most satisfactory aspect of this study has been that very multiplication of possibilities. That the transport model has so many possible applications and extensions must be regarded as corroboratory evidence for its potential value in the simple problem on which it has been tested.

In so far as this study has "conclusions" they are (1) that a prima facie case has been made out for applying linear programming to the distribution of coking coal: (2) that it is possible to handle very large transport problems by hand computation: (3) that a large class of other problems, both of transport and of pricing, are also likely to benefit from the application of linear programming: (4) that there are considerable returns to be expected from further research into methods of computation and into the adaptation of the transport model to specific problems.

Appendix A to Chapter 1 - Adjustment of tonnage round a loop

If there is a loop in the routing pattern, in which there is some positive tonnage on every route in the loop, it must be possible to make a tour or the loop subtracting x tons from those routes where the tour moves counter to the direction of coal movement, and adding x tons to those routes where the tour moves with the direction of coal movement (where x is no more than the minimum tonnage of the routes where the coal movement is counter to the direction of the tour); and this tour will leave every point in the loop with its net receipt or despatch of coal[1] unchanged, all tonnages in the loop remaining non-negative.

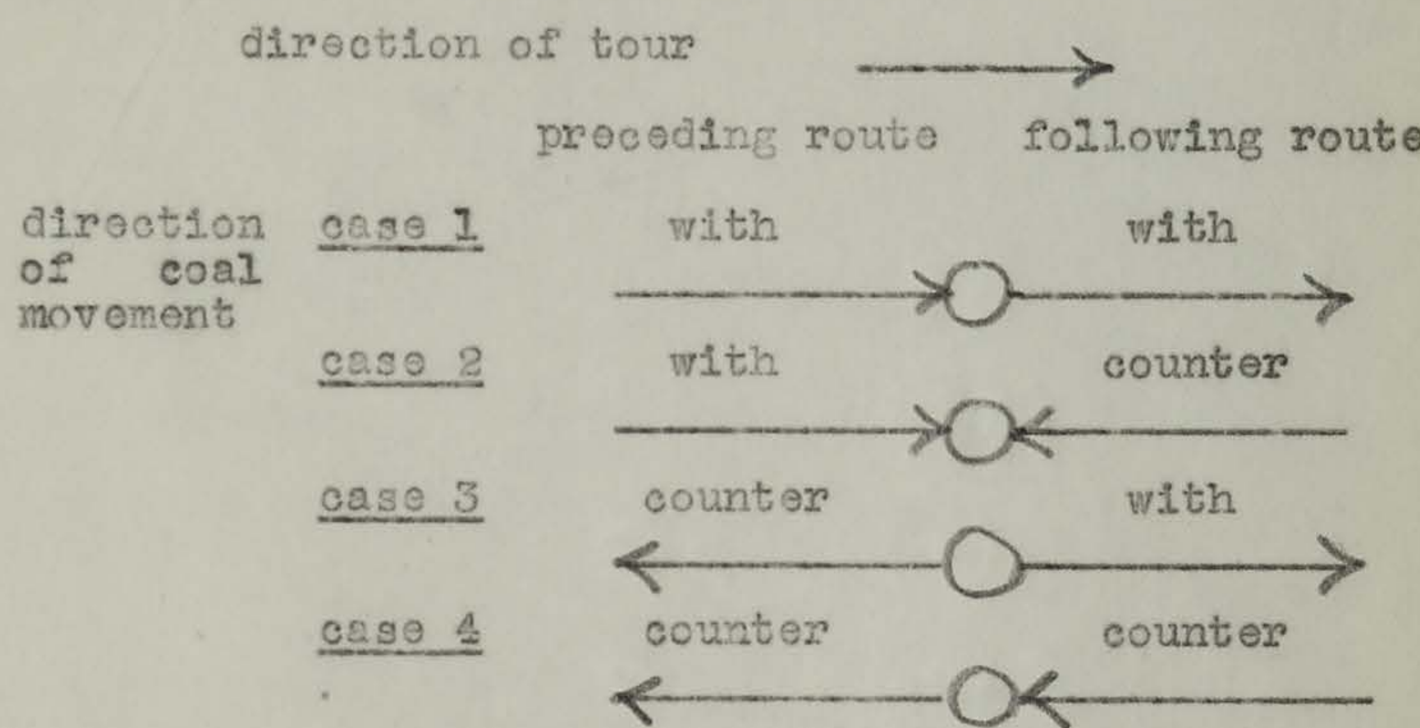
Proof (i) Every point in the loop is connected by one route to the point preceding it in the tour, and by one route to the point following it in the tour.

(ii) The preceding route either sends in coal to the point, or it sends coal out of the point. If it sends coal in, the coal is moving with the direction of the tour, and if it sends coal out, the coal is moving counter to the direction of the tour.

1. "Net receipt or despatch of coal" rather than "boundary condition" because we mean here net of possible change within the loop. A net receiver of coal within a loop may be merely a point passing on a certain tonnage of coal to destinations outside the loop, rather than a final destination.

The reverse is true for the route which leads to the following point of the tour, giving us four possible combinations for any point on the loop, as in Diagram 1.

Diagram 1



In case 1, where the point receives coal and passes it on, in the same direction as the tour, both routes have x tons of coal added to them, and the net receipt or despatch of coal by the point is unchanged.

In case 4, the point receives coal and passes it on in the direction counter to that of the tour, both routes have x tons subtracted from them, and again the net receipt and despatch of coal by the point is unchanged.

In case 2, the preceding route sends in x tons more than before, and the following route x tons less, leaving the net receipt unchanged.

In case 3, the preceding route sends out x tons less than before, and the following route sends out x tons more, leaving the net despatch unchanged.

The only limitation on this adjustment of tonnage around a loop is that x cannot be greater than the smallest tonnage on a route which is to have x subtracted from it by the tour. In other words, x must not be so great as to result in some route carrying negative tonnage.[1]

If the tour of adjustments is taken in the opposite direction, the limiting amount, x, will, in general, be different, because now the routes from which tonnage is to be subtracted are the ones

---

1. For some purposes we shall find it convenient to regard negative tonnage as indicating reversal of the route. If we allow this possibility there is no limit to the extent to which the adjustment can be carried out in either direction. The loop then corresponds with the physical railway loop and a very large x can be interpreted as routing tonnage round and round the loop as well as getting coal from the net despatching points to the net receiving points by carrying it all in one direction round the loop. Taking the tour of the loop in the opposite direction with a very large x leads to a similar picture with the coal moving in the opposite direction.

to which tonnage was to be added in the previous tour, and hence the maximum tonnage which can be subtracted will occur elsewhere.

A loop, therefore, can be thought of as a series of routes on which a range of different self-consistent[1] sets of tonnage can be send, limited by the smallest tonnage encountered when traversing the loop in the two different directions.

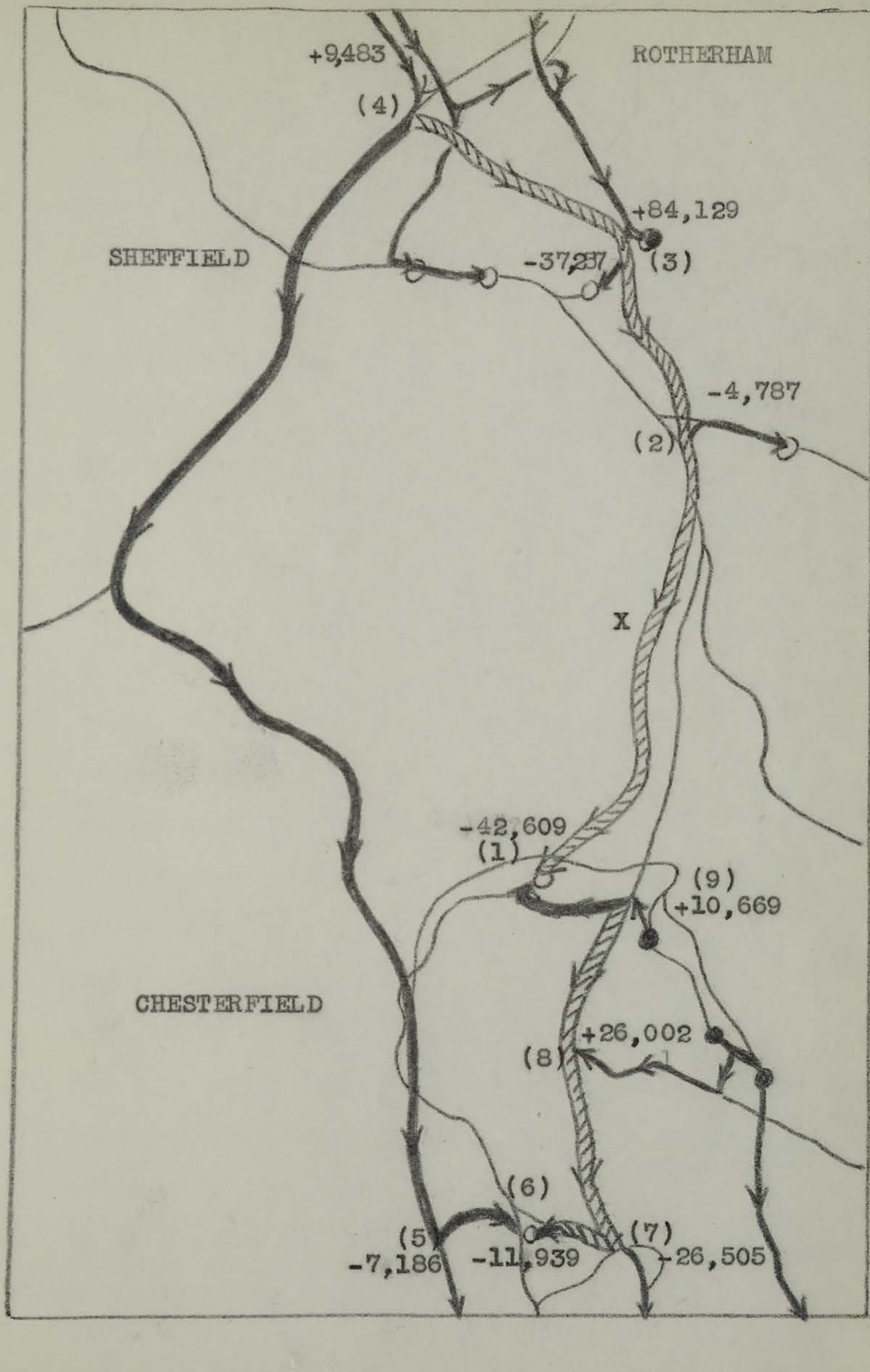
1. "Self-consistent" in the sense that the absolute difference between the tonnage on route P and that on route Q (P and Q any two routes of the loop) is the same for every possible set of tonnages.

Appendix B to Chapter 1 - Elimination of a loop

Diagram 1 is an enlargement from Map 2 with the addition of a route linking a pit and a coke oven at Chesterfield (the route from the point labelled (9) to the point (1) on Diagram 1). Those lines which carry coal are shown by the darker lines, and those which form the loop are in thick lines. Of the routes forming the loop, those which flow in a clockwise direction are indicated by hatched lines, and those which flow anti-clockwise are indicated by solid lines.[1] Those points of the loop where there is either a net input of coal to the loop or a net output from the loop are numbered from (1) to (9), with the tonnage in or out indicated by a negative sign for coal leaving the loop, either to a coke oven or to other parts of the network, and by a positive sign for coal entering the loop. As the tonnage available for the whole program must equal the tonnage required, so the amount flowing into a loop must equal the amount flowing out.

1. There is, of course, no significance in the terms "clockwise" and "anticlockwise"--they are used merely to distinguish the two opposite directions in which a tour of the loop could be made. The terms could be reversed without affecting the argument--only their antithesis is important. In some cases a loop may cross itself to form a figure of eight, in which case some other terms than "clockwise" and "anticlockwise" would have to be found.

Diagram 1



+9,483
ROTHERHAM
(4)
+84,129
SHEFFIELD
(3)
-4,787
(2)
X
-42,609
(1)
(9)
+10,669
CHESTERFIELD
+26,002
(8)
(6)
(5)
(7)
-7,186
-26,505

There is a whole range of possible tonnage patterns on this loop, from putting as great a tonnage as possible on the clockwise routes, to putting as great a tonnage as possible on the anti-clockwise routes. The limit to the range is set in the first case by one of the anti-clockwise routes being reduced to carrying zero tons, and in the second place by one of clockwise routes being reduced to carrying zero tons.

Once the amounts entering and leaving the loop at each point are known, the differences between the tonnages on each route are determined. In other words, if the tonnage on any one route is fixed, the others are also determinate. We can express the tonnage on all the routes in terms of the tonnage on any one. Every point on the loop is visited only once[1], therefore each point has a route preceding it in the tour, and one following it. If the tonnage is determined on the preceding route, then it is also determined on the following route. But that route itself is the preceding route for the next point on the tour, and so on, round the entire loop.

1. If a point were visited twice we should be dealing with two loops not one. All the discussion about loops would apply to each loop taken singly.

By way of example, let us call the tonnage on the route from the point (2) to the point (1) on Diagram 1, "x" and determine all the other tonnages by a clockwise tour of the loop:-

Route		tonnage	
(2)→(1)	=	x	The coke oven at point (1) which receives 42,609 tons altogether, receives x tons from (2), and therefore 42,609 - x tons from (9):-
(9)→(1)	=	42,609 - x	The point (9) receives 10,669 tons from outside the loop. It sends (42,609 - x) to (1), therefore it must send 10,669 - (42,609 - x) to (8):-
(9)→(8)	=	x - 31,940	The point (8) receives 26,002 from outside the loop and (x - 31,940) from (8), therefore it must send 26,002 + (x - 31,940) to (7):-
(8)→(7)	=	x - 5,938	The point (7) receives (x - 5938) and sends 26,505 outside the loop, therefore it must send the remaining (x - 5,938) - 26,505 to (6):-
(7)→(6)	=	x - 32,443	The point (6) is to receive 11,939 altogether, therefore from (5) it must receive 11,939 - (x - 32,443):-
(5)→(6)	=	44,382 - x	(5) sends out 7,186 and (44,382 - x), therefore it must receive the same:-
(4)→(5)	=	51,568 - x	(4) receives 9,483, and sends to (5) (51,568 - x), therefore it must send to (3) 9,483 - (51,568 - x):-
(4)→(3)	=	x - 42,085	(3) receives (x - 42,085) from (4), 84,129 from outside the loop, and sends 37,257 to a coke oven outside the loop, therefore it must send 84,129 + (x - 42,085) - 37,257 to (2):-
(3)→(2)	=	x + 4,787	which leaves the point (2) in balance, receiving (4,787 x) from (3) and sending 4,787 outside the loop, plus x to the point (1).

It will be seen that the clockwise routes ((9)→(8), (8)→(7), (7)→(6), (4)→(3), (3)→(2)) involve positive terms in x, x being the tonnage on the clockwise route (2)→(1), and the anti-clockwise routes ((9)→(1), (5)→(6), (4)→(5)) involve negative terms in x. This is as we should expect, for as we increase x we must also increase the tonnage on other clockwise routes, and decrease the tonnage on the anti-clockwise routes. That increase cannot be taken so far that any of the anti-clockwise routes are made to carry negative tonnage. The route (9)→(1) would be made negative if x were $>$ 42,609; (5)→(6), if x $>$ 44,382; (4)→(5) if x $>$ 51,568. Therefore x must not be $>$ 42,609.

At the other end of the range, x must not be reduced to the point of making any of the other clockwise routes negative. From the point of view of route (9)→(8), x must be $\geqq$ 31,940; (8)→(7), x $\geqq$ 5,938; (7)→(6), x $\geqq$ 32,443; (4)→(3), x $\geqq$ 42,085; (3)→(2), x $\geqq$ -4,787. The greatest of these is 42,085, therefore x must be at least 42,085.

This gives us a maximum range of x = 42,085 to x = 42,609, and the alternative possibilities of eliminating either the route (4)→(3) or the route (9)→(1), according to which is shorter, the sum of the distances of the clockwise routes or that of the anti-clockwise routes.

If the maximum x were less than the minimum x it would mean that the directions of the routes were inconsistent with the tonnage availabilities and requirements. A negative tonnage on a route can be interpreted as a reversal of direction. The maximum and minimum figures for x were such as to ensure that no route bore negative tonnage, i.e., the maximum and minimum associated with the selected directions for each route of the loop. By reversing the directions of the routes which effectively constrain either the maximum or the minimum, the difference (minimum - maximum) can be made successively smaller until the maximum exceeds the minimum and the directions of the routes are feasible having regard to the tonnage requirements and availabilities. The directions of the routes in the example must be feasible because the loop is formed by the addition of a route to the solution pattern: therefore the minimum x cannot exceed the maximum x.

Appendix C to Chapter 1--Relationship of the model to the standard transport problem and to general linear programming.

I

An attempt has been made in Chapter 1 to treat the transport problem in isolation as a technique of management control, in non-mathematical terms. This appendix is concerned with placing the transport problem in the wider context of linear programming.

In Section II of this appendix there is a brief mathematical statement of the Linear Programming problem. In Chapter 1 it was asserted that the irreducible minimum cost had been found if there were no positive profit on an unused route. This condition is expressed for the general form of linear programming as equations (11) and (21) in section II of this appendix, and the assertion proved. The dual (pricing) aspect of the solution is introduced, and the equivalence of the total value of the commodities with the total cost of the optimum solution is demonstrated. But the "dual theorem" of linear programming is stated without proof, and is in fact not strictly applicable to the formulation of the Linear Programming problem in this Appendix. Thus Section II is intended to provide an introduction to the concepts of general linear programming to a mathematician interest in application. For a full mathematical exposition see "Activity Analysis of Production and Allocation", Cowles Commission Monograph No. 13., ed., T.C. Koopmans; "An Introduction to Linear Programming" by A. Charnes, W.W. Cooper, and A. Henderson; "An Application of Linear Programming to the Theory of the Firm" by Dorfman.

Section III of this appendix is a translation of section II into non-mathematical terms, using only simultaneous equations.

Section IV relates the transport problem to the general linear programming form, and section V describes the modifications to the usual form of the transport model used in this study.

II

The formal statement of linear programming can be expressed briefly:-

(1) Minimize $c'x = \sum c_j x_j \quad (j = 1,2,\ldots,m)$

(2) Subject to: $x_j \geqq 0$ for all $j = 1,2,\ldots,m$

(3) and subject to: $Ax = b$ or $\sum_{j=1}^{m} a_{ij} x_j = b_i$ for all $i = 1,2,\ldots,n$

$(n \leqq m)$

If there is a finite solution it can be demonstrated that there exists a solution x^*

containing not more than n positive elements x_j, the remaining (m-n) elements x_j being zero. Re-ordering the elements of x^* so that the last m-n elements are zero, and re-ordering the matrix A and the vector c in the same way, the two vectors and the matrix can be partitioned:-

(4) $A = (P;Q)$ — P has elements a_{ip}; Q has elements a_{iq}

(5) $c = \begin{pmatrix} c_P \\ \cdots \\ c_Q \end{pmatrix}$ — c_P has elements c_p; c_Q has elements c_q

(6) $x^* = \begin{pmatrix} x_P^* \\ \cdots \\ x_Q^* \end{pmatrix}$ — x_P^* has elements x_p; x_Q^* has elements x_q

$i,p = 1,2,\ldots,n$; $q = n+1,n+2,\ldots,m$

(7) $x_q = 0$ for all $q = n+1, n+2,\ldots,m$

$x_p \geqq 0$

Then (3) becomes:-

(8) $Px_P^* + Qx_Q^* = b$

and, because of (7):-

(9) $Px_P^* = b$

(10) $x_P^* = P^{-1}b$ (provided P is non-singular. It is always possible to construct the program so that if there is a solution then one of this form exists, viz., with P non-singular).

A sufficient condition[1] that x^* should be the solution is:-

(11) $c_P' P^{-1} Q \leqq c_Q'$

1. Although in some cases there may be minimum solutions where this condition does not hold, it is always possible to find one where it does. Therefore it can be treated as a necessary as well as a sufficient condition.

For let $\bar{x}$ be an alternative solution such that:-

(12) $A\bar{x} = b$

(13) $\bar{x} \geqq 0$

Partitioning $\bar{x}$ to correspond to the ordering and partitioning of A into (P:Q):-

(14) $\bar{x} = \begin{pmatrix} \bar{x}_P \\ \cdot\cdot \\ \bar{x}_Q \end{pmatrix}$

Then (12) becomes:-

(15) $P\bar{x}_P + Q\bar{x}_Q = b$

(16) $c'x^* = c_P'x_P^*$ ($\because x_Q^* = 0$), therefore

(17) $c'x^* = c_P'P^{-1}b$ (from (10)), therefore

(18) $c'x^* = c_P'P^{-1}(P\bar{x}_P + Q\bar{x}_Q)$ (from (15)), therefore

(19) $c'x^* = c_P'\bar{x}_P + c_P'P^{-1}Q\bar{x}_Q \leqq c_P'\bar{x}_P + c_Q'\bar{x}_Q = c'\bar{x}$

since $c_P'P^{-1}Q \leqq c_Q'$ and $\bar{x}_Q \geqq 0$. Q.E.D.

(20) Let $y^{*\prime} = c_P'P^{-1}$

Then (11) becomes:-

(21) $y^{*\prime}Q \leqq c_Q'$

Also

(22) $y^{*\prime}P = c_P'P^{-1}P = c_P'$, therefore

(23) $y^{*\prime}A = y^{*\prime}(P:Q) \leqq (c_P':c_Q') \leqq c'$

Also

(24) $y^{*\prime}b = c_P'P^{-1}b = c_P'x_P^* = c'x^*$

The "dual theorem" of linear programming is:-

$$(25)\qquad \text{Minimum } c'x \;=\; \text{Maximum } y'b$$

$$\text{subject to } Ax \geqq b \;\&\; x \geqq 0 \quad\Big|\quad \text{subject to } y'A \leqq c' \;\&\; y \geqq 0$$

III

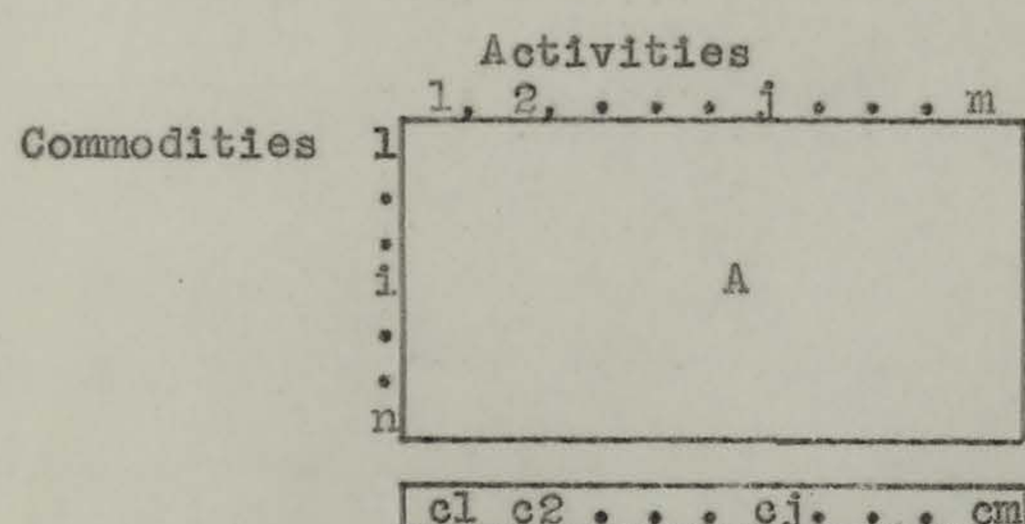
Section II of this appendix can be expressed verbally in a nomenclature that owes much to economics.

The n x m matrix A is known as the "technology matrix", each column of which represents and "activity" or (in a more limited context) a "productive process".

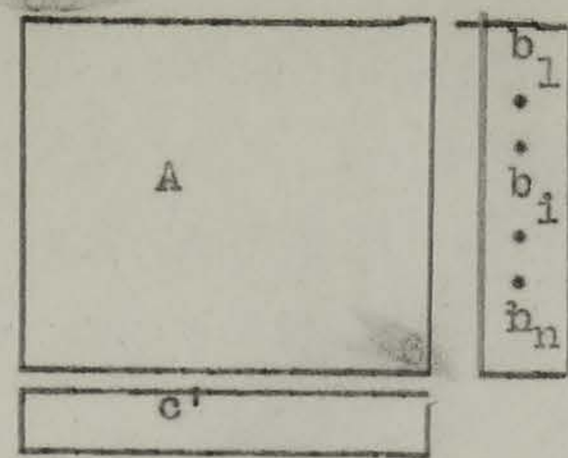
$$\begin{bmatrix} a_{11} & a_{12}\cdots & a_{1j}\cdots & a_{1m} \\ a_{21} & a_{22}\cdots & a_{2j}\cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2}\cdots & a_{ij}\cdots & a_{im} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2}\cdots & a_{nj}\cdots & a_{nm} \end{bmatrix} = A$$

Each row of A refers to a single homogeneous commodity expressed in any convenient units, in the usual economic sense of the term which includes services as well as material commodities. A negative figure in the matrix represents an input of so many units of a commodity, and a positive figure an output. Thus any one column will list the amounts of each commodity in the system which are inputs or outputs for any one activity.

Associated with each activity is the cost of carrying it out at the unit level. In other contexts it may be a revenue from carrying it out at the unit level. But in all cases of linear programming there must be a single "commodity" (which may be money) of which it is desired to minimize the input or maximize the output. If there are two or more such desired ends, then the calculation procedure can be used to trace out the transformation function between them, but it cannot yield a unique optimum. The row of these costs, one associated with each activity column, is the vector c' of Section II.



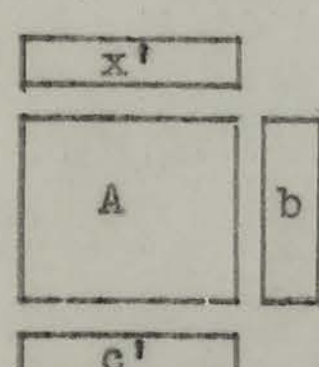
Each commodity is assumed to be either limited in initial availability or required in some minimum net amount (which might be zero). These limitations, or constraints, are known as the boundary conditions, and are represented by the vector b in Section II.



The linear programming problem as set out in equations (1), (2) and (3) of Section II is to find the levels at which each activity should be carried out, assuming that there are strictly constant returns to scale within each activity. If activity j is carried out at intensity x_j then every input and output in the j^{th} column will be multiplied by x_j, and the total cost of the j^{th} activity will be c_jx_j. The total cost of the whole program will be

$$c_1x_1 + c_2x_2 + \dots + c_jx_j + \dots + c_mx_m$$

or, in other words, $\sum_{j=1}^{m} c_jx_j (j=1,2,\dots,m)$ or, still more briefly, c'x (see equation (1), section II). This is the quantity which it is required to minimize.



The constraint (2) is the fairly obvious one that no activity should be carried out at a negative level.

The other constraint, equation (3), is that the total use and production of commodities should equal the boundary conditions, b. It is, in fact, a whole set of equations:-

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + \cdots + & a_{1j}x_j & + \cdots + & a_{1m}x_m & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + \cdots + & a_{2j}x_j & + \cdots + & a_{2m}x_m & = & b_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{i1}x_1 & + & a_{i2}x_2 & + \cdots + & a_{ij}x_j & + \cdots + & a_{im}x_m & = & b_i \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + \cdots + & a_{nj}x_j & + \cdots + & a_{nm}x_m & = & b_n
\end{array}
$$

Note that there are more unknowns (m) than equations (n) so that finding the levels at which the activities are to be used is not simply a question of solving a set of equations.

Some of the commodity constraints may not be equalities but rather minimum or maximum conditions. If a certain machine has available 48 machine-hours per week, it might be over-restrictive to insist that the whole 48 hours be fully used up. But it is useful to have the constraints in the form of equalities, and this can be done by the device of "disposal activities" or "slack vectors" which for each commodity "throw away" any surplus amounts at no cost. If, in fact, some of the constraints are true equalities, it is still useful to include disposal activities on those commodities, but at a very high cost. This high cost can even be left undefined, but simply assumed to be so great at every stage of the calculation that it will never become profitable to use it. Or, in some cases, it may be that there is a genuine disposal activity with a known cost which is appropriate. But, whether the cost of the activity is zero, finite, or indefinitely great, every one of the n commodities has a disposal activity associated with it--a column containing all zeros except in the relevant row, where there is a minus one.

The existence of n disposal activities within the A matrix is the reason that m can positively be asserted to be greater than n.

We have n equations in m unknowns. The solution can be thought of in terms of selecting n activities from the m available to be carried out at positive levels, the other (m-n) being at zero level. Once the n activities are selected, the determination of the individual levels (the x's) is a matter of the solution of n equations in n unknowns: the linear programming problem arises in the selection of the appropriate set of n activities to minimize cost.

Suppose that the solution is known, and that the A matrix has been re-arranged so that the n active activities are first, and the m-n zero level activities follow. The A matrix can be regarded as split into the active P, and non-active Q:-

$$\boxed{\quad A \quad} = \boxed{P}\boxed{\quad Q \quad}$$

Some of the disposal activities may be in P, indicating that some initially available commodity is not fully used up, or that some minimum production requirement is overfulfilled by the optimum program.

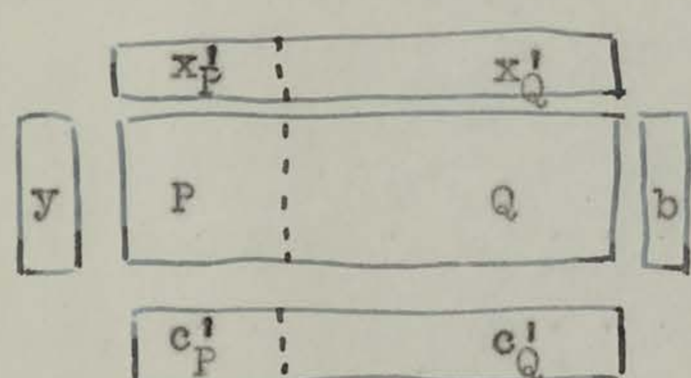
The first n x's are such as to satisfy the equations:-

$$\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdot \cdot \cdot + a_{1p}x_p + \cdot \cdot \cdot + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdot \cdot \cdot + a_{2p}x_p + \cdot \cdot \cdot + a_{2n}x_n = b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdot \cdot \cdot + a_{ip}x_p + \cdot \cdot \cdot + a_{in}x_n = b_i \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdot \cdot \cdot + a_{np}x_p + \cdot \cdot \cdot + a_{nn}x_n = b_n
\end{array}$$

Or, briefly, $Px_p = b$ (Equation (9) Section II). The remaining (m-n) x's are, of course, zero.

The same set of technical coefficients in the P matrix can be regarded as the coefficients of n equations reading vertically. This can be thought of as imputing the total cost amongst the commodities in the system. Reading the equations vertically:-

$$
\begin{array}{ccccc}
a_{11}y_1 & a_{12}y_1 \cdot\cdot\cdot & a_{1p}y_1 \cdot\cdot\cdot & a_{1n}y_1 \\
+ & + & + & + \\
a_{21}y_2 & a_{22}y_2 \cdot\cdot\cdot & a_{2p}y_2 \cdot\cdot\cdot & a_{2n}y_2 \\
+ & + & + & + \\
\vdots & \vdots & \vdots & \vdots \\
+ & + & + & + \\
a_{i1}y_i & a_{i2}y_i \cdot\cdot\cdot & a_{ip}y_i \cdot\cdot\cdot & a_{in}y_i \\
+ & + & + & + \\
\vdots & \vdots & \vdots & \vdots \\
+ & + & + & + \\
a_{n1}y_n & a_{n2}y_n \cdot\cdot\cdot & a_{np}y_n \cdot\cdot\cdot & a_{nn}y_n \\
= & = & = & = \\
c_1 & c_2 \; \ldots\ldots & c_p \; \ldots\ldots & c_n
\end{array}
$$



These equations can be interpreted as imputing those "prices" for each commodity which exactly distribute the total cost of each of the used activities. The y's can also be identified with the marginal values in economics, in that to produce of an output, i, would add y_i to the total cost, etc. They also distribute the total cost of the program between all the commodities of the boundary conditions, since:-

$$y_1b_1+y_2b_2+\cdot\ \cdot+y_ib_i+\cdot\ \cdot+y_nb_n = c_1x_1+c_2x_2+\cdot\ \cdot+c_jx_j+\cdot\ \cdot+c_mx_m$$

Total value of the commodities in the system — Total cost of the program[1]

In a more limited context, where all the commodities are factors contributing to the production of one single commodity (the maximand) we have a proposition familiar in economics: That under constant returns to scale, payments to the factors of production at rates equal to their marginal physical products, will exactly exhaust the total product.

These prices are derived from the activities used in the optimum solution, and express the value which each commodity has in that pattern of production. Those activites not used can be costed on the imputed prices and that cost called the "indirect saving" of the activity. The economic interpretation of the indirect saving on a certain

1. See equation (24) of Section II. The equality can be seen by multiplying every element of the A matrix by its appropriate x and y:-

$$\begin{matrix} x_1a_{11}y_1 & x_2a_{12}y_1 & \cdot\ \cdot & x_ja_{1j}y_1 & \cdot\ \cdot & x_ma_{1m}y_1 \\ x_1a_{21}y_2 & x_2a_{22}y_2 & \cdot\ \cdot & x_ja_{2j}y_2 & \cdot\ \cdot & x_ma_{2m}y_2 \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1a_{i1}y_i & x_2a_{i2}y_i & \cdot\ \cdot & x_ja_{ij}y_i & \cdot\ \cdot & x_ma_{im}y_i \\ \vdots & \vdots & & \vdots & & \vdots \\ x_1a_{n1}y_n & x_2a_{n2}y_n & \cdot\ \cdot & x_ja_{nj}y_n & \cdot\ \cdot & x_ma_{nm}y_n \end{matrix}$$

The left hand (total value) side of the equality is the sum of the row sums, and the right hand (total cost) side is the sum of the column sums--these are necessarily equal.

activity is the amount which could be saved on the existing program by carrying out that activity at the unit level, and readjusting the rest of the programm optimally. Each activity has also its direct cost, given by the c' row. The net effect of using the activity at unit level is the direct cost minus the indirect saving. If the program is truly optimal, the direct cost on any unused activity will exceed the indirect saving. This is the significance of equations (11) and (21) in section II.

IV

The Koopmans transport problem is characterised by a particularly simple form of the technology matrix.

(1) Every commodity is always either an input or an output, so that the A matrix can be divided vertically into two sections, one section containing only negative figures and zeros, and the other only positive figures and zeros.

A	inputs (negative)	b_1
		:
	outputs (positive)	b_n

(2) Every activity has only two non-zero entries, a minus one and a plus one: one input and one output.

The <u>commodities</u> are a single homogeneous commodity differentiated by geographical location. The boundary column shows the quantities of that commodity available initially at some of the places and required at others.

The activities each have an input of one unit of the commodity at one place and an output of one unit at another place, at a certain cost shown in the c' row.

It is normally assumed that the A matrix contains only real transport activities, not disposal activities. However, it is then impossible to pick out a set of n activities which will form a non-singular matrix P. This corresponds to the statement in Chapter 1 that if more than n-1 routes are used (n being the number of points) there must be a loop in the graph.

This problem is avoided by allowing each constraint to be an inequality, and inserting disposal activities. If if fact the total amount required, the P matrix will contain one disposal activity, used with zero intensity. The associated commodity will have zero price--all the others being positive. If there is more of the commodity available than there is required, a disposal activity will be used with a positive weight. If there is more required than available the constraints are inconsistent and the problem cannot be solved unless it is assumed that there is a possibility of running down stocks to an amount just sufficient to fill the gap, or the possibility of production at a cost, both possibilities being expressed as additional activities.

The disposal activity used in the solution will be the one on the lowest price origin point, which will therefore have zero price. A negative price on any point would violate the conditions of the solution.

Because there is no overlap between input commodities and output commodities, and because each activity involves only one of each, the program can be condensed into a pair-wise matrix of costs

Inputs \ Outputs	b_{k+1}	. . .	b_j	. . .	b_n
b_1	$c_{1,k+1}$	. . .	c_{1j}	. . .	c_{1n}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
b_i	$c_{i,k+1}$	. . .	c_{ij}	. . .	c_{in}
$\vdots$	$\vdots$		$\vdots$		$\vdots$
b_k	$c_{k,k+1}$	. . .	c_{kj}	. . .	c_{kn}

and the problem is written:

Minimize $\sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} x_{ij}$

subject to $\sum_{i=1}^{k} x_{ij} = b_j$ for all $j = k+1, \ldots, n$

and $\sum_{j=k+1}^{n} x_{ij} = b_i$ for all $i = 1, 2, \ldots, k$

and $x_{ij} \geqq 0$ for all i, j.

In other words, listing sending points down the left hand side and receiving points along the top, each entry in the table represents one route, i.e., activity. The problem is to allocate the tonnage to the different routes in such a way as to ensure that the tonnage on each row totals to the amount available and the tonnage on each column to the amount required.

This formulation is only possible if the row sum total is equal to the column sum total. But disposal activities can be introduced if necessary by adding a destination (n+1) to which each origin can send at zero cost ($c_{i,n+1} = 0$ for all $i = 1,2,\ldots,k$) and which is to receive the surplus amount ($b_{n+1} = \sum_{i=1}^{k} b_i - \sum_{j=k+1}^{n} b_j$)

This "cost matrix" form of the transport problem is only a convenient method of organizing the data and the calculation procedure. At any stage the information exists for translating the data into the standard form. The inverse of P can be obtained from the "tree".

V

The modification to the form of the standard transport program used in this study is the introduction of intermediate commodities which represent the geographical locations of "coal" where it is neither initially available nor finally required[1]. The intermediate commodities are constrained to be $\geqq 0$ indicating that not more can be sent out from any point than it receives. Another modification is that each route appears as two activities, reversing the input and output coefficients, but both having positive costs. E.g., the possibility of using a segment of line A-B, distance 3 miles, would be represented by

	Activity 1	Activity 2
"coal" at A	-1	+1
"coal" at B	+1	-1
Cost	3	3

Only one direction could enter the solution[2].

The transport problem so modified remains an ordinary linear program. It loses one of the important characteristics of the Koopmans problem--the separation of the input commodities from the output commodities throughout the table. The

1. A different modification with a similar effect is described in "The Transhipment Problem" by A. Orden, Management Science, April 1956, Vol. II No. 3.

2. The solution matrix, P, cannot contain both directions as it would then be singular.

same commodity will now be an input for one activity and an output for another. This appears to mean that it cannot be conveniently condensed into the cost matrix form. However, it retains the other important characteristic--that each column has only two entries, a plus one and a minus one. This, in fact, is the essential characteristic of a transport problem, which leads to the tree form of the solution.

The cost matrix representation was not used in this study, but rather the method described in Chapter 3. However, if it should be required, it is readily achieved, although it may be very cumbersome. All points which may be sending points are listed down the left hand side, and all those which may be receiving points are listed along the top of the matrix. In the coal-to-coke-ovens study this would mean almost the entire list of 1,000 points in both dimensions of the matrix, as most pits and coke ovens were also treated as being intermediate points. It might, however, be a useful method of handling a case where a transport problem naturally breaks down into a few areas, linked together by a few intermediate points.

The column under an intermediate point shows the costs of routes along which it can receive. Some of these costs will be "M" (greater than any indirect saving from using the route). The row associated with the intermediate point shows the costs of the routes along which it can send out the commodity. The common element of the row and column can be regarded as the activity which transforms the commodity received into the commodity available for send out. It is restricted to bearing negative or zero tonnage . It will be readily seen that this is the condition which enables a positive amount to be received in the column and sent out in the row, whilst keeping the net total of both equal to zero.

Such an apparent contradition of the most fundamental rule of linear programming can be simply justified by re-casting the standard linear programming form of the problem. Instead of considering a commodity "goods at J" we can distinguish "goods received at J" and "goods available for despatch from J". Then we can again split the A matrix vertically into sending points and receiving points, adding an activity to transform "goods received at J" into "goods available for despatch from J" at zero cost, for each intermediate point:-

Commodities \ Activities		"real activities	"transforming" act.
goods available for despatch at	1 2 . J . .	elements -1 or 0 only	elements -1 or 0 only
goods received at	1 2 . J . .	elements -1 or 0 only	elements -1 or 0 only
cost			0 0 . . 0 . . . 0

Note that this program could be converted into one where there is only the commodity "goods available for despatch at J" and not "goods received at J" by adding the free "transforming" activity to every real activity with an entry in the latter row. The the program is back in the form which underlies the model used in this study.

The program in the above expanded form requires all activities to be used at levels $\geqq 0$. But exactly equivalent would be to multiply each "transforming activity" throughout by -1 and require that it take place at levels $\leqq 0$. Then all commodities would be either negative throughout or positive throughout, and the cost matrix condensation would be applicable with a "non-positivity" constraint on the elements common to the row and column of an intermediate commodity[1].

VI

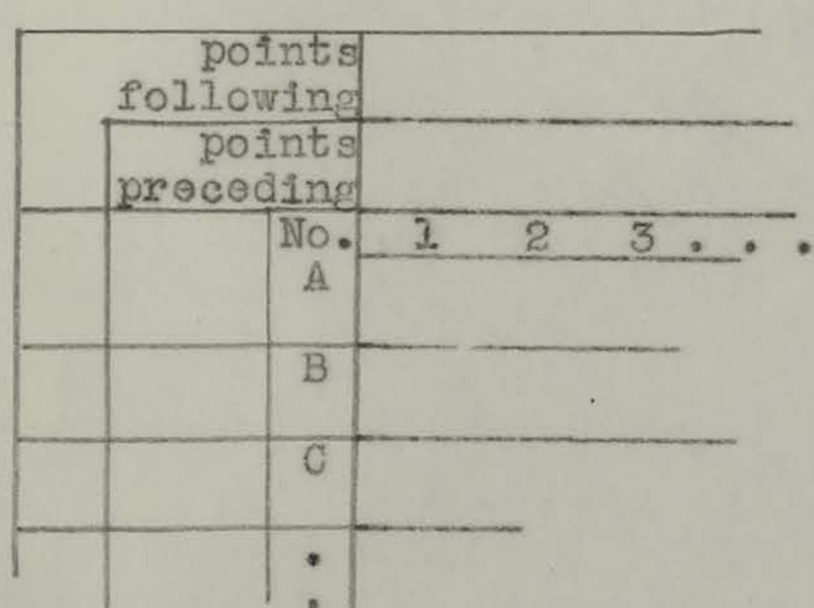
Incidentally, it can also be shown that a negative tonnage is permissable anywhere in any cost-matrix representation of a transport problem, interpreted as a reversal of direction of the route, so long as the direction of the pricing is also reversed. The practical application of this arises when one has a pattern of routes known to form a tree (perhaps the solution pattern for the previous period) and the tonnages available and required can only be fitted on the routing pattern with some negative tonnages. The pricing must not be taken over unchanged from the existing tree, but reversed wherever the direction of the route is reversed. Once the profits have been recomputed, the computation proceeds normally, except that the limit on the extent of a change of basis may be when one of the negative routes is raised to zero tons, instead of the normal limit of reducing a positive tonnage to zero. If the costs of the matrix are based on the true shortest distances between the points, the computation procedure will eliminate all negative tonnage routes. But if, in fact, the cost A→B→C→D is less than the cost A→D as shown in the matrix, the negative tonnage on the route C→B may persist.

Appendix A to Chapter 3 Modifications to the standard form of the transport model for hand computation.

I

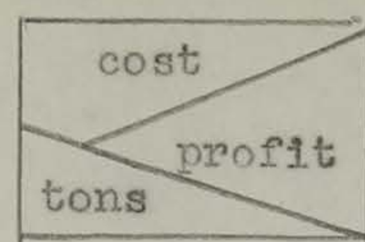
Without using the tree chart it is possible to take advantage in hand computation of the fact that not all prices and profits change at any given change of basis in the standard form of the transport problem. To the information normally carried is added for each point an indication of the one point preceding it in the tree (proceeding from an arbitrary origin) and the list of points (if any) which follow it in the tree.

Diagram 1



Each cell of the matrix can contain three pieces of information for each route, as in Diagram 2.

Diagram 2



The cost figure is unchanging and should therefore be printed indelibly: the profit and tons figures are liable to alteration and should be written lightly in pencil so that the same table can serve for all iterations. ("Overwriting" preceding steps is not open to the usual criticism, as it is never necessary to trace back the source of an error in this computation. It is always possible to correct an error by adjusting to a feasible solution, if necessary admitting negative tonnages--see Section VI of Appendix C to Chapter 1.)

The computation starts with any feasible solution, but preferrably with a "good" one[1]. When computing the prices, enter for the

1. E.g., either using maps or some such procedure as picking out the smallest cost route in the matrix, putting on it whichever is the smaller, the row total tonnage or the column total tonnage and eliminating that row or column from further consideration: fill in the other row or column in which the first tonnage has been entered using the smallest cost routes available, and so on.

origin point "points preceding, nil", and list in the "points following" cell the other end of each route used in the origin row or column. Follow up the pricing tree along each of these branches in turn, ticking it off in the "points following" cell. As each new point is priced from the one preceding it in the tree (either by adding or by subtracting the cost of the connecting route), the preceding point is listed in the appropriate cell.

Once the prices and profits have been computed, the calculation proceeds by basis changes until all profits are negative or zero. To find the tonnage alterations at each basis change, one can use the standard procedure[1] or make a list of the chain back from each end point of the new route until a common point on the two chains is reached. As the prices and profits are not recomputed at each basis change it is important to check that the sum of the routes in the direction of the new route round the loop are less by the amount of the profit than the sum of the routes in

1. G.B. Dantzig, "Application of the Simplex Method to a Transportation Problem", Chapter XXIII of "ACtivity Analysis of Production and Allocation" ed. T.C. Koopmans, Cowles Commission Monograph, No.13.

the opposite direction[1].

The alterations in the tonnages can be made and checked (by seeing that the net receipt or despatch of tonnage within the loop for each point of the loop is the same before and after the change) on the separate list and then copied into the main matrix.

The list of points whose prices change is started from the route removed from the loop, each branch being followed up in the same way as in the determination of the prices. It is not necessary to change the prices, but only the profits on the routes for which one end point of the route is on the list and the other end point not on the list. The list may or may not contain the smaller "half" of the tree.

Perhaps the biggest disadvantage of this method as opposed to the chart method is that where a section of the tree is reversed in direction it is necessary to switch the entry in the "preceding point" cell with one of the points in the "following points" cell for each point in the

1. In the standard form of the transport problem with no negative tonnages, the routes are in alternate directions, as each destination point is priced from an origin point and each origin point from a destination point (except the point taken as the origin of the pricing tree.)

reversed stretch of the tree[1].

II

The chart method of computation is not dependent on the transport network form of the cost data. If the entire cost matrix is known, the tree chart would consist of all the origin and destination points connected by the selection of routes which form the basis. The tree would be smaller and would have more multiple branching poihts. The profits table would now be a profits matrix and at each basis change a profit would be unchanged if both its row and its column point were unchanged, or if both its row and its column point were changed, and the profit would be changed if one of the row or the column point were changed and not the other.

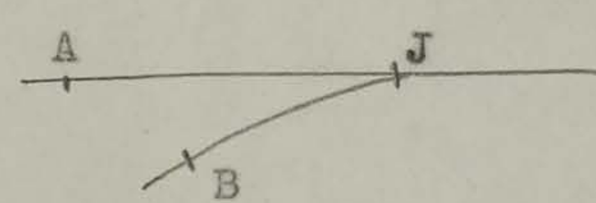
1. This is not necessary in the chart method, as the chain connecting each point to the origin is visually obvious.

Appendix B to Chapter 3 Modifications to the Computational Procedure to use Working Distance rather than Rail Distance, etc.

I

If working distance rather than rail distance is to be minimized, the problem is to exclude from the solution routes which, although they are the shortest rail distance between two points, are not feasible working routes. For instance:

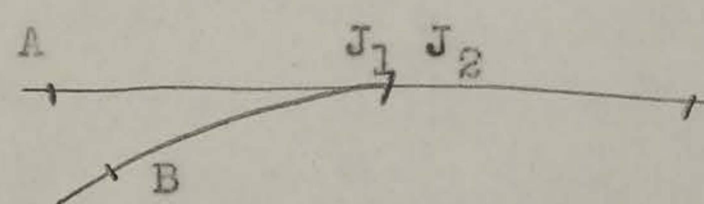
Diagram 1



The shortest rail distance from A to B is via J, but in general this is not an allowable working route because it involves reversing a train along the stretch from J to B.

This can be dealt with by distinguishing two points at J:

Diagram 2



J_1 can receive tonnage from A or from B and pass it on towards C; J_2 can receive tonnage from C and pass it on towards A or B. There is no route between J_1 and J_2.

It would burden the computation with almost a double number of points to split every junction point in this way to start with, so the best procedure would be to start with the computation to minimize rail distance and then systematically eliminate any routes which are not feasible working routes by inserting an extra point as above. Initially tonnage would travel from J_1 to J_2, but this can be thought of as costing M[1].

The resulting solution may involve A sending tonnage to B by some other route, or it may distribute the tonnage in a way which excludes A to B altogether.

II

If most routes are charged according to shortest rail mileage but there are some specially cheap rates, then the transport network can still be used with the addition of the direct pit-to-coke-oven routes converted to a mileage equivalent. Care must be taken that tonnage not actually originating at that origin or not actually destined for that coke oven cannot use the special route. This can always be ensured by means of irreversible zero cost

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1. It is not, of course, necessary to add M to the profits, but only to seek the smallest negative profit from a point in the "half" of the tree containing J_1 to the "half" containing J_2 until the tonnage on the route $J_1 \rightarrow J_2$ is reduced to zero by one or more basis changes.

routes out of the pit to a point on the network and from a point on the network into the coke oven. These points may very well have different dual prices in the solution from the pit and coke oven themselves, indicating that if it were possible to divert other coal along the special route it would be profitable.

III

An analagous problem is congestion on some route. There the problem might present itself as an agreement with British Railways not to exceed a certain fixed tonnage of coal on a certain route. If the tonnage of coking coal alone is limited to a certain figure which is exceeded by the solution pattern, it would be possible to experimentally increase the cost of the congested segment until the tonnage is reduced to the allowable figure.

The tonnage on any segment is a step function of the cost of the segment, and the function can be obtained over as great a range as required from the solution pattern. If the transport model were being used on other forms of coal then the distribution of the allowable tonnage between the different forms of coal would have to be such that the cost of the congested route was the same in each program.[1]

1. The step functions for the different programs could be added horizontally to obtain the correct cost.

Appendix A to Chapter IV The numbrical solution

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
1	12	1.50	5,102	7,653.00	5,330	7,995.00	0	0
	13	17.69	228	4,033.32			10.52	2,398.56
			5,330	11,686.32	5,330	7,995.00		2,398.56
2	4	0	9,242	0	9,242	0	0	0
	5	3.50	1,787	6,254.50			5.00	8,935.00
	6	1.00			1,787	1,787.00		
			11,029	6,254.50	11,029	1787.00		8,935.00
3	12	0	3,564	0	3,336	0	0	0
	13	16.19	2,616	42,353.04			10.52	27,520.32
	15	3.00			2,844	8,532.00		
			6,180	42,353.04	6,180	8,532.00		27,520.32
4	1	4.25			2,681	11,394.25		
	2	2.25	4,543	10,221.75	4,610	10,372.50	0	0
	3	3.50	1,981	6,933.50			2.50	4,952.50
	6	9.50	5,471	51,974.50			0.50	2,735.50
	7	16.75	9,866	165,255.50	9,958	166,796.50	0	0
	8	11.25	926	10,417.50			2.75	2,546.50
	13	24.53			5,715	140,188.95		
	15	21.86			352	7,694.72		
	83	164.96	529	87,263.84			2.75	1,454.75
			23,316	332,066.59	23,316	336,446.92		11,689.25
5	1	12.00	7,270	87,240.00	4,589	55,068.00	0	0
	2	10.00	67	670.00	0	0	0	0
	3	10.00	5,797	57,970.00			1.25	7,246.25
	7	24.50	92	2,254.00	0	0	0	0
	8	16.25	9,715	157,868.75	10,641	172,916.25	0	0
	9	35.25	6,685	235,646.25	6,685	235,646.25	0	0
	10	21.00	1,117	23,457.00	14,366	301,686.00	0	0
	13	32.28	2,871	92,675.88	0	0	0	0
	15	29.61	3,196	94,633.56	0	0	0	0
	51	168.46	2,852	480,447.92			9.50	27,094.00
	57	168.04			0	0		
	59	166.96			0	0		
	60	165.71	934	154,773.14	0	0	0	0
	61	164.96			0	0		
	62	167.71			0	0		
	66	164.36			0	0		
	67	152.90			3,582	547,687.80		
	68	162.36			0	0		
	69	155.20			2,836	440,147.20		
	74	165.36			0	0		
	75	165.61			0	0		
	79	163.86			0	0		
	83	169.96			0	0		
	84	170.46			0	0		
	86	171.53	2,103	360,727.59			5.52	11,608.56
	89	162.11			0	0		
			42,699	1,748,364.09	42,699	1,753,151.50		45,948.81
Total			88,554	2,140,724.54	88,554	2,107,912.42		96,491.94
Brought fd			0	0	0	0		0
Carried fd			88,554	2,140,724.54	88,554	2,107,912.42		96,491.94

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
6	3	10.50			7,778	81,669.00		
	5	16.00			1,787	28,592.00		
	6	18.50			3,684	68,154.00		
	8	18.00			0	0		
	9	37.00			0	0		
	10	22.75	20,414	464,418.50	7,165	163,003.75	0	0
	57	169.79			0	0		
	59	168.71			0	0		
	61	166.71			0	0		
	62	169.46			0	0		
	66	166.11			0	0		
	67	154.65			0	0		
	68	164.11			0	0		
	69	156.95			0	0		
	74	167.11			0	0		
	75	167.36			0	0		
	79	165.61			0	0		
	83	171.71			0	0		
	84	172.21			0	0		
	89	163.86			0	0		
			20,414	464,418.50	20,414	341,418.75		0
7	16	6.00	2,142	12,852.00	2,142	12,852.00	0	0
	17	3.50	1,602	5,607.00	1,602	5,607.00	0	0
	18	0	13,100	0	13,100	0	0	0
	19	6.50	6,376	41,444.00	6,376	41,444.00	0	0
	31	113.94	9,512	1,083,797.28			29.47	280,318.64
	57	102.59			565	57,963.35		
	59	101.51			0	0		
	61	99.51			0	0		
	62	102.26	2,544	257,605.44	0	0	0	0
	63	91.91			6,428	590,797.48		
	66	98.91			0	0		
	67	87.45			0	0		
	68	96.91			0	0		
	69	89.75			3,886	348,768.50		
	74	99.91			0	0		
	75	100.16			0	0		
	79	98.41			0	0		
	83	104.51			0	0		
	84	105.01	94	9,776.94	0	0	0	0
	89	96.66			1,271	122,854.86		
			35,370	1,413,720.66	35,370	1,180,287.19		280,318.64
Total			55,784	1,878,139.16	55,784	1,521,705.94		280,318.64
Brought fd			88,554	2,140,724.54	88,554	2,107,912.42		96,491.94
Carried fd			144,338	4,018,863.70	144,338	3,629,618.36		376,810.58

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
8	28	20.23	9,727	196,777.21	0	0	0	0
	30	18.99	5,860	111,281.40	0	0	0	0
	31	23.08			0	0		
	32	27.73	1,159	32,139.07	10,693	296,516.89	0	0
	34	22.59	991	22,386.69	0	0	0	0
	35	20.20			0	0		
	37	20.20	719	14,523.80	0	0	0	0
	38	22.68	683	15,490.44	0	0	0	0
	39	18.59	1,500	27,885.00	0	0	0	0
	43	29.15	269	7,841.35			2.08	559.52
	46	29.93	610	18,257.30			5.00	3,050.00
	47	34.37	58	1,993.46			5.00	290.00
	50	31.45			0	0		
	51	32.12			0	0		
	53	32.61			0	0		
	55	30.41			0	0		
	56	32.64			0	0		
	57	41.14			0	0		
	59	40.12			0	0		
	61	38.12			0	0		
	62	40.87	3,205	130,988.35	14,088	575,776.56	0	0
	82	34.14			0	0		
	83	43.12			0	0		
	84	43.62			0	0		
			24,781	579,564.07	24,781	872,293.45		3,899.52
9	28	18.23			0	0		
	30	16.99			0	0		
	31	21.08	4,776	100,678.08	5,996	126,395.68	0	0
	32	25.73	1,380	35,507.40	0	0	0	0
	34	20.59	1,656	34,097.04	0	0	0	0
	35	18.20			0	0		
	37	18.20	3,096	56,347.20	0	0	0	0
	38	20.68	6,817	140,975.56	0	0	0	0
	39	16.59	3,355	55,659.45	0	0	0	0
	40	23.92	100	2,392.00			5.20	520.00
	50	29.45			0	0		
	51	30.12			0	0		
	53	30.61			0	0		
	55	28.41			0	0		
	56	30.64			0	0		
	57	39.14			0	0		
	59	38.12			23	876.76		
	61	36.12			0	0		
	62	38.87			0	0		
	82	32.14			0	0		
	83	41.12			529	21,752.48		
	84	41.62			14,632	608,983.84		
			21,180	425,656.73	21,180	758,008.76		520.00
Total			45,961	1,005,220.80	45,961	1,630,302.21		4,419.52
Brought fd			144,338	4,018,863.70	144,338	3,629,618.36		376,810.58
Carried fd			190,299	5,024,084.50	190,299	5,259,920.57		381,230.10

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
10	21	38.05	2,282	86,830.10			0.10	228.20
	28	22.40			0	0		
	30	21.16			0	0		
	31	25.25	16,569	418,367.25	12,092	305,323.00	0	0
	32	29.90	6,208	185,619.20	0	0	0	0
	34	24.76	6,493	160,766.68	0	0	0	0
	35	22.37			0	0		
	37	22.37	15,457	345,773.09	19,272	431,114.64	0	0
	38	24.85	26,730	664,240.50	26,016	646,497.60	0	0
	39	20.76	14,301	296,888.76	21,103	438,098.28	0	0
	40	28.09	100	2,809.00			5.20	520.00
	41	29.33	48	1,407.84			7.68	368.64
	50	33.62			0	0		
	51	34.29			3,929	134,725.41		
	53	34.78			5,930	206,245.40		
	55	32.58			0	0		
	56	34.81			3,349	116,578.69		
	57	43.31			0	0		
	59	42.29			0	0		
	60	41.04	475	19,494.00	0	0	0	0
	61	40.29			0	0		
	62	43.04			0	0		
	75	55.79	3,028	168,932.12			14.85	44,965.80
	82	36.31			0	0		
	83	45.29			0	0		
	84	45.79			0	0		
			91,691	2,351,128.54	91,691	2,278,583.02		46,082.64
11	46	0	13,326	0	21,921	0	0	0
	47	4.44			1,260	5,594.40		
	50	19.20	5,245	100,704.00			12.68	66,506.60
	56	20.39	4,610	93,997.90			12.68	58,454.80
			23,181	194,701.90	23,181	5,594.40		124,961.40
12	43	2.82	256	721.92			3.91	1,000.96
	44	4.82	1,728	8,328.96			3.91	6,756.48
	49	0	10,191	0	12,581	0	0	0
	50	5.12	1,980	10,137.60			1.83	3,623.40
	53	4.45			1,574	7,004.30		
			14,155	19,188.48	14,155	7,004.30		11,380.84
13	42	0	16,409	0	2[illegible],674	0	0	0
	43	3.32	4,852	16,108.64	6,056	20,105.92	0	0
	44	5.32	11,469	61,015.08	0	0	0	0
			32,730	77,123.72	32,730	20,105.92		0
Total			161,757	2,642,142.64	161,757	2,311,287.64		182,424.88
Brought fd			190,299	5,024,084.50	190,299	5,259,920.57		381,230.10
Carried fd			352,056	7,666,227.14	352,056	7,571,208.21		563,654.98

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
14	57	10.68	5,716	61,046.88	0	0	0	0
	59	9.60			0	0		
	61	7.60			7,103	53,982.80		
	62	10.35	11,119	115,081.65	14,561	150,706.35	0	0
	63	0	822	0	3,595	0	0	0
	66	7.00	22,470	157,290.00	22,470	157,290.00	0	0
	67	4.46	3,582	15,975.72			8.92	31,951.44
	68	5.00	2,218	11,090.00	2,218	11,090.00	0	0
	74	8.00			0	0		
	75	8.25	3,000	24,750.00	0	0	0	0
	79	6.50	155	1,007.50	1,969	12,798.50	0	0
	83	12.60			0	0		
	84	13.10	5,100	66,810.00	0	0	0	0
	89	4.75	1,272	6,042.00	3,538	16,805.50	0	0
			55,454	459,093.75	55,454	402,673.15		31,951.44
15	45	18.38	85	1,562.30			24.23	2,059.55
	46	18.88	1,113	21,013.44			24.23	26,967.99
	50	7.24	145	1,049.80			6.07	880.15
	52	0	13,964	0	14,078	0	0	0
	53	2.33			1,656	3,858.48		
	56	8.43	427	3,599.61			6.07	2,591.89
			15,734	27,225.15	15,734	3,858.48		32,499.58
16	21	9.02			2,282	20,583.64		
	22	6.07			2,009	12,194.63		
	57	14.44			12,777	184,499.88		
	74	12.00	15,239	182,868.00			0.24	3,657.36
	75	12.25	15	183.75			0.24	3.60
	77	8.25	3,038	25,063.50	3,038	25,063.50	0	0
	79	10.50	1,814	19,047.00			0.24	435.36
			20,106	227,162.25	20,106	242,341.65		4,096.32
17	44	0	7,591	0			0.40	3,036.40
	49	3.14	2,390	7,504.60			4.45	10,635.50
	50	1.98			0	0		
	55	.94			5,473	5,144.62		
	56	3.17			4,508	14,290.36		
	82	4.67			0	0		
			9,981	7,504.60	9,981	19,434.98		13,671.90
18	62	12.60	7,642	96,289.20			4.50	34,389.00
	63	2.25	391	879.75			4.50	1,759.50
	74	5.75	2,182	12,546.50	17,839	102,574.25	0	0
	75	6.00			0	0		
	79	4.25			0	0		
	84	15.35	10,468	160,683.80			4.50	47,106.00
	89	2.50	9,322	23,305.00	12,166	30,415.00	0	0
			30,005	293,704.25	30,005	132,989.25		83,254.50
		Total	131,280	1,014,690.00	131,280	801,297.51		165,473.74
		Brought fd	352,056	7,666,227.14	352,056	7,571,208.21		563,654.98
		Carried fd	483,336	8,680,917.14	483,336	8,372,505.72		729,128.72

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
19	57	12.01	4,145	49,781.45	0	0	0	0
	59	10.93			0	0		
	61	8.93			0	0		
	62	11.68			11,147	130,196.96		
	63	1.33	8,810	11,717.30	0	0	0	0
	66	8.33			0	0		
	68	6.33			0	0		
	70	4.55	1,176	5,350.80			5.02	5,903.52
	74	9.33			0	0		
	75	9.58			6,043	57,891.94		
	79	7.83			0	0		
	83	13.93			0	0		
	84	14.43	207	2,987.01	0	0	0	0
	89	6.08	2,852	17,340.16	0	0	0	0
			17,190	87,176.72	17,190	188,088.90		5,903.52
20	52	14.66	114	1,671.24			23.25	2,650.50
	59	1.25	6,231	7,788.75	6,208	7,760.00	0	0
	60	0			10,332	0		
	61	0.75	10,195	7,646.25			1.50	15,292.50
	62	2.00			0	0		
	83	4.25			0	0		
	84	4.75			0	0		
			16,540	17,106.24	16,540	7,760.00		17,943.00
21	34	1.55			40	62.00		
	39	2.45	40	98.00			4.90	196.00
			40	98.00	40	62.00		196.00
22	38	0.50			12,848	6,424.00		
	42	10.49	3,472	36,421.28			8.92	30,970.24
	45	10.97	978	10,728.66			8.72	8,528.16
	46	11.47	2,866	32,873.02			8.72	24,991.52
	47	15.91	1,388	22,083.08			8.72	12,103.36
	53	16.83	2,511	42,260.13			6.40	16,070.40
	82	18.36	1,633	29,981.88			6.40	10,451.20
			12,848	174,348.05	12,848	6,424.00		103,114.88
23	57	13.23			0	0		
	70	0.75	289	216.75	1,465	1,098.75	0	0
	71	1.00	4,222	4,222.00	4,222	4,222.00	0	0
	73	2.35	822	1,931.70	822	1,931.70	0	0
	74	12.63	418	5,279.34			2.08	869.44
	84	15.65			2,771	43,366.15		
	86	11.20			0	0		
	87	12.35			0	0		
	89	9.38	3,529	33,102.02			2.08	7,340.32
			9,280	44,751.81	9,280	50,618.60		8,209.76
24	46	27.73	4,006	111,086.38			41.09	164,606.54
	60	10.08	2,053	20,694.24			9.50	19,503.50
	69	14.00	249	3,486.00			23.93	5,958.57
	84	5.33	389	2,073.37	4,594	24,486.02	0	0
	86	0.88	52,860	46,516.80	54,963	48,367.44	0	0
	87	2.03	12,765	25,912.95	12,765	25,912.95	0	0
			72,322	209,769.74	72,322	98,766.41		190,068.61
Total			128,220	533,250.56	128,220	351,719.91		325,435.77
Brought fd			483,336	8,680,917.14	483,336	8,372,505.72		729,128.72
Carried fd			611,556	9,214,167.70	611,556	8,724,225.63		1,054,564.49

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
25	28	14.79			9,727	143,862.33		
	30	13.55			7,246	98,183.30		
	31	17.64			0	0		
	32	22.29			0	0		
	34	17.15			9,100	156,065.00		
	35	14.76			4,163	61,445.88		
	37	14.76			0	0		
	38	17.24			0	0		
	39	13.15	19	249.85	0	0	0	0
	42	23.51	4,096	96,296.96			5.20	21,299.20
	43	26.83	3,429	92,000.07			5.20	17,830.80
	50	26.01	2,409	62,658.09	9,779	254,351.79	0	0
	51	26.68	1,077	28,734.36	0	0	0	0
	53	27.17	6,649	180,653.33	0	0	0	0
	55	24.97	5,473	136,660.81	0	0	0	0
	56	27.20	2,820	76,704.00	0	0	0	0
	57	35.76	3,481	124,480.56	0	0	0	0
	59	34.68			0	0		
	60	33.43	2,581	86,282.83	0	0	0	0
	61	32.68	1,697	55,457.96	3,589	117,288.52	0	0
	62	35.43	9,873	349,800.39	0	0	0	0
	82	28.70			0	0		
	83	37.68			0	0		
	84	38.18			0	0		
	92	22.73			0	0		
			43,604	1,289,979.21	43,604	831,196.82		39,130.00
26	40	1.24	3,200	3,968.00	3,152	3,908.48	0	0
	41	0	8,468	0	8,516	0	0	0
			11,668	3,968.00	11,668	3,908.48		0
27	23	17.10	1,404	24,008.40	1,404	24,008.40	0	0
	24	12.92	3,663	47,325.96	3,663	47,325.96	0	0
	26	7.02			8,156	57,255.12		
	29	11.03	2,026	22,346.78	2,026	22,367.04	0	0
	30	9.25	4,697	43,447.25	3,311	30,626.75	0	0
	32	22.62	1,946	44,018.52			4.63	9,009.98
	38	20.68	770	15,923.60			7.74	5,959.80
	39	18.37	1,888	34,682.56			9.52	17,973.76
	62	31.39	2,166	67,990.74			0.26	563.16
			18,560	299,743.81	18,560	181,583.27		33,506.70
28	110	0	10,593	0	10,593	0	0	0
	130	2.00			0	0		
	142	8.75			0	0		
	144	9.75			0	0		
			10,593	0	10,593	0		0
29	139	0	11,208	0	11,208	0	0	0
			11,208	0	11,208	0		0
Total			95,633	1,593,711.28	95,633	1,016,688.57		72,636.70
Brought fd			611,556	9,214,167.70	611,556	8,724,225.63		1,054,564.49
Carried fd			707,189	10,807,878.98	707,189	9,740,914.20		1,127,201.19

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
30	109	3.50	359	1,256.50			3.00	1,077.00
	114	0	8,958	0	8,958	0	0	0
	117	7.25	211	1,529.75			3.25	685.75
	119	2.00			6,702	13,404.00		
	120	5.50	1,933	10,631.50			0.75	1,449.75
	121	5.75	906	5,209.50			0.50	453.00
	129	10.25	3,040	31,160.00			1.50	4,560.00
	134	13.00	253	3,289.00			5.50	1,391.50
			15,660	53,076.25	15,660	13,404.00		9,617.00
31	93	6.00			0	0		
	104	0	19,514	0	19,514	0	0	0
	105	13.25			0	0		
			19,514	0	19,514	0		0
32	31	83.10			0	0		
	32	87.75			0	0		
	34	82.61			0	0		
	38	82.70			0	0		
	40	80.74			0	0		
	43	87.09			2,505	218,160.45		
	44	89.09			0	0		
	45	84.45			0	0		
	47	89.39			0	0		
	50	91.47			0	0		
	51	92.14			0	0		
	53	92.63			0	0		
	55	90.43			0	0		
	56	92.66			0	0		
	57	101.22			0	0		
	59	100.14			0	0		
	61	98.14			0	0		
	62	100.89			0	0		
	82	94.16			0	0		
	83	103.14			0	0		
	84	103.64			0	0		
	140	32.75	63	2,063.25			29.50	1,858.50
	143	15.75	64	1,008.00			13.50	864.00
	145	10.50	4,477	47,008.50			5.75	25,742.75
	146	5.75	2,014	11,580.50			3.00	6,042.00
	151	1.75			8,204	14,357.00		
	152	5.00	4,091	20,455.00			1.00	4,091.00
	153	0	12,029	0	12,029	0	0	0
	154	1.00	9,234	9,234.00	9,234	9,234.00	0	0
			31,972	91,349.25	31,972	241,751.45		38,598.25
Total			67,146	144,425.50	67,146	255,155.45		48,215.25
Brought fd			707,189	10,807,878.98	707,189	9,740,914.20		1,127,201.19
Carried fd			774,335	10,952,304.48	774,335	9,996,069.65		1,175,416.44

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
33	107	3.75			16,072	60,270.00		
	110	1.75			10,322	18,063.50		
	116	5.50	10,211	56,160.50			1.25	12,763.75
	120	0	20,303	0			0.25	5,075.75
	121	0.25			4,120	1,030.00		
	130	3.75			0	0		
	142	10.50			0	0		
	144	11.50			0	0		
			30,514	56,160.50	30,514	79,363.50		17,839.50
34	121	3.50	1,809	6,331.50			7.00	12,663.00
	128	4.75	187	888.25			3.25	607.75
	130	0	10,575	0	10,997	0	0	0
	138	5.00	26	130.00			5.00	130.00
	142	6.75			0	0		
	144	7.75			1,600	12,400.00		
			12,597	7,349.75	12,597	12,400.00		13,400.75
35	98	18.25	160	2,920.00			11.50	1,840.00
	99	1.50			1,960	2,940.00		
	100	0	302	0	6,870	0	0	0
	108	18.75	3,943	73,931.25			10.50	41,401.50
	118	12.00	201	2,412.00			3.00	603.00
	120	11.25	2,138	24,052.50			1.75	3,741.50
	129	15.25	514	7,838.50			1.75	899.50
	134	12.25	1,572	19,257.00	0	0	0	0
	137	11.00			0	0		
			8,830	130,411.25	8,830	2,940.00		48,485.50
36	99	0	10,698	0	8,738	0	0	0
	121	11.50	9	103.50			3.00	27.00
	129	16.00	1,423	22,768.00			4.00	5,692.00
	134	10.75	1	10.75	0	0	0	0
	136	7.25			3,768	27,318.00		
	137	9.50			0	0		
	138	13.50	375	5,062.50			1.50	562.50
			12,506	27,944.75	12,506	27,318.00		6,281.50
37	132	0	22,484	0	22,484	0	0	0
	142	5.75			0	0		
	144	6.75			0	0		
			22,484	0	22,484	0		0
38	148	4.25	4,408	18,734.00	4,408	18,734.00	0	0
	150	2.75	7,116	19,569.00	7,116	19,569.00	0	0
			11,524	38,303.00	11,524	38,303.00		0
Total			98,455	260,169.25	98,455	160,324.50		86,007.25
Brought fd			774,335	10,952,304.48	774,335	9,996,069.65		1,175,416.44
Carried fd			872,790	11,212,473.73	872,790	10,156,394.15		1,261,423.69

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
39	107	11.75			0	0		
	109	3.50	7,129	24,951.50	0	0	0	0
	110	9.75			0	0		
	112	2.50			0	0		
	113	0	2,464	0	2,464	0	0	0
	116	12.25			0	0		
	119	5.00			0	0		
	120	7.75	1,820	14,105.00	0	0	0	0
	121	8.25	194	1,600.50	0	0	0	0
	128	13.25			0	0		
	129	11.75			0	0		
	130	11.75			0	0		
	133	15.25			902	13,755.50		
	134	10.50	2,383	25,021.50	0	0	0	0
	136	7.00			3,443	24,101.00		
	137	9.25			0	0		
	139	21.25			1,062	22,567.50		
	142	18.50			0	0		
	143	16.75			6,119	102,493.25		
	144	19.50			0	0		
			13,990	65,678.50	13,990	162,917.25		0
40	133	10.75	902	9,696.50			1.00	902.00
	134	5.00	7,631	38,155.00	19,129	95,645.00	0	0
	135	0	17,183	0	17,183	0	0	0
	136	1.50	12,874	19,311.00	2,278	3,417.00	0	0
	137	3.75			0	0		
			38,590	67,162.50	38,590	9,062.00		902.00
Total			52,580	132,841.00	52,580	261,979.25		902.00
Brought fd			872,790	11,212,473.73	872,790	10,156,394.15		1,261,423.69
Carried fd			925,370	11,345,314.73	925,370	10,418,373.40		1,262,325.69

CO	Pit	Miles	Actual		Optimum		Loss per Ton	Total Loss
			Tons	Ton miles	Tons	Ton miles		
41	97	1.25	310	387.50	5,566	6,957.50	0	0
	98	13.25	3,015	39,948.75			1.75	5,276.25
	100	5.00	6,568	32,840.00			0.25	1,642.00
	103	5.25			0	0		
	107	18.25			0	0		
	109	10.00			0	0		
	110	16.25			0	0		
	112	9.00			0	0		
	113	6.50			0	0		
	116	18.75			0	0		
	119	11.50			0	0		
	120	14.25			0	0		
	121	14.75			0	0		
	128	19.75			4,787	94,543.25		
	129	18.25			0	0		
	130	18.25			0	0		
	133	21.75			0	0		
	134	17.00	460	7,820.00	0	0	0	0
	136	13.50			0	0		
	137	15.75			0	0		
	138	18.25			0	0		
	139	27.75			0	0		
	142	25.00			0	0		
	143	23.25			0	0		
	144	26.00			0	0		
			10,353	80,996.25	10,353	101,500.75		6,918.25
42	117	4.50	103	463.50			6.00	618.00
	118	3.25	97	315.25			4.50	436.50
	120	0.25	2,958	739.50			1.00	2,958.00
	121	0	5,411	0			0.25	1,352.75
	128	4.75			0	0		
	129	3.25			8,569	27,849.25		
	133	6.75			0	0		
	139	12.75			0	0		
	143	8.25			0	0		
			8,569	1,518.25	8,569	27,849.25		5,365.25
43	137	0	6,912	0	6,912	0	0	0
			6,912	0	6,912	0		0
44	143	0	8,494	0	8,494	0	0	0
			8,494	0	8,494	0		0
45	131	0	27,907	0	27,907	0	0	0
	137	2.75	710	1,952.50			5.00	3,550.00
	138	1.25	8,048	10,060.00			1.00	8,048.00
	142	7.00			1,870	13,090.00		
	144	8.00			6,888	55,104.00		
			36,665	12,012.50	36,665	68,194.00		11,598.00
		Total	70,993	94,527.00	70,993	197,544.00		23,881.50
		Brought fd	925,370	11,345,314.73	925,370	10,418,373.40		1,262,325.69
		Carried fd	996,363	11,439,841.73	996,363	10,615,917.40		1,286,207.19

Co	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
46	123	0	7,518	0	7,518	0	0	0
	125	2.75	9,367	25,759.25			5.50	51,518.50
	145	9.00			4,477	40,293.00		
	146	7.00			3,545	24,815.00		
	149	4.00			0	0		
	152	8.25			0	0		
	155	8.50			1,345	11,432.50		
			16,885	25,759.25	16,885	76,540.50		51,518.50
47	124	0	17,973	0	17,973	0	0	0
	126	9.75	467	4,553.25			12.75	5,954.25
	128	5.00	879	4,395.00			5.50	4,834.50
	129	5.50	76	418.00			7.50	570.00
	130	8.00	143	1,144.00			10.00	1,430.00
	143	3.00			2,800	8,400.00		
	144	10.75	1,235	13,276.25			5.00	6,175.00
			20,773	23,786.50	20,773	8,400.00		18,963.75
48	107	9.25	2,322	21,478.50	0	0	0	0
	108	15.50	2,105	32,627.50			11.50	24,207.50
	109	1.00			0	0		
	110	7.25	7,312	53,012.00	17,476	126,701.00	0	0
	112	0	10,409	0	10,409	0	0	0
	116	9.75			0	0		
	118	6.00	4,115	24,690.00			1.25	5,143.75
	119	2.50			0	0		
	120	5.25			0	0		
	121	5.75	1,477	8,492.75	0	0	0	0
	122	10.50	1,507	15,823.50			4.00	6,028.00
	128	10.75			0	0		
	129	9.25	2,624	24,272.00	0	0	0	0
	130	9.25	279	2,580.75	0	0	0	0
	133	12.75			0	0		
	134	8.00			0	0		
	136	4.50			3,385	15,232.50		
	137	6.75			880	5,940.00		
	139	18.75			0	0		
	142	16.00			0	0		
	143	14.25			0	0		
	144	17.00			0	0		
			32,150	182,977.00	32,150	147,873.50		35,379.25
49	97	0	11,916	0	13,750	0	0	0
	105	21.75	1,834	39,889.50			4.25	7,794.50
			13,750	39,889.50	13,750	0		7,794.50
Total			83,558	272,412.25	83,558	232,814.00		113,656.00
Brought fd			996,363	11,439,841.73	996,363	10,615,917.40		1,286,207.19
Carried fd			1,079,921	11,712,253.98	1,079,921	10,848,731.40		1,399,863.19

			Actual		Optimum		Loss per	
CO	Pit	Miles	Tons	Ton miles	Tons	Ton miles	Ton	Total Loss
50	97	3.50	2,695	9,432.50			6.50	17,517.50
	98	18.00	396	7,128.00			10.75	4,257.00
	101	0	16,435	0	16,435	0	0	0
	102	14.25	6,633	94,520.25			10.75	11,304.75
	103	1.00	8,667	8,667.00	8,667	8,667.00	0	0
	104	12.00	93	1,116.00			10.75	999.75
	105	25.25	5,180	130795.00			10.75	55,685.00
	107	14.00	334	4,676.00	0	0	0	0
	108	17.50	2,068	36,190.00			8.75	18,095.00
	109	5.75			7,488	43,056.00		
	110	12.00			0	0		
	112	4.75			0	0		
	113	2.25			0	0		
	116	14.50			2,898	42,021.00		
	119	7.25			2,401	17,407.25		
	120	10.00			0	0		
	121	10.50			1,238	12,999.00		
	122	15.25	123	1,875.75			4.00	492.00
	125	16.75	869	14,555.75			7.00	6,083.00
	128	15.50			0	0		
	129	14.00	101	1,414.00	2,714	37,996.00	0	0
	130	14.00			0	0		
	132	20.50	3,954	81,057.00			5.50	21,747.00
	133	17.50			0	0		
	134	12.75	5,374	68,518.50	0	0	0	0
	136	9.25			0	0		
	137	11.50			0	0		
	138	14.00	573	8,022.00	2,297	32,158.00	0	0
	139	23.50			0	0		
	142	20.75	197	4,087.75	0	0	0	0
	143	19.00			0	0		
	144	21.75			9,554	207,799.50		
			53,692	472,055.50	53,692	402,103.75		196,181.00
Total			53,692	472,055.50	53,692	402,103.75		196,181.00
Brought fd			1,079,921	11,712,253.98	1,079,921	10,848,731.40		1,399,863.19
Carried fd			1,133,613	12,184,309.48	1,133,613	11,250,835.15		1,596,044.19

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
51	22	134.85	2,009	270,913.65			4.00	8,036.00
	26	128.79	500	64,395.00			14.01	7,005.00
	31	121.10			12,769	1,546,325.90		
	32	125.75			0	0		
	34	120.61			0	0		
	38	120.70	992	119,734.40	0	0	0	0
	40	118.74			248	29,447.52		
	43	125.09			245	30,647.05		
	44	127.09			20,788	2,641,946.92		
	45	122.45			1,063	130,164.35		
	47	127.39			186	23,694.54		
	50	129.47			0	0		
	51	130.14			0	0		
	53	130.63			0	0		
	55	128.43			0	0		
	56	130.66			0	0		
	57	139.22			0	0		
	59	138.14			0	0		
	61	136.14			2,532	344,706.48		
	62	138.89			0	0		
	69	143.40	2,745	393,633.00			17.02	46,719.90
	82	132.16			4,769	630,271.04		
	83	141.14			0	0		
	84	141.64			0	0		
	97	45.00	4,395	197,775.00			26.75	117,566.25
	102	38.50	4,336	166,936.00			13.75	59,620.00
	105	35.75			0	0		
	106	35.75			0	0		
	107	37.50	3,257	122,137.50			2.25	7,328.25
	108	30.00	1,734	52,020.00	0	0	0	0
	110	35.50	3,565	126,557.50			2.25	8,021.25
	116	35.75			0	0		
	117	30.50			0	0		
	118	30.75			0	0		
	122	32.50	904	29,380.00	0	0	0	0
	123	33.75			0	0		
	125	31.00			0	0		
	126	34.25			0	0		
	128	38.50	3,085	118,772.50			1.75	5,398.75
	134	41.25	248	10,230.00			7.25	1,798.00
	139	46.50	1,062	49,383.00			1.75	1,858.50
	142	44.25	3,014	133,369.50			2.25	6,781.50
	143	42.00	3,441	144,522.00			1.75	6,021.75
	144	45.25	7,313	330,913.25			2.25	16,454.25
	145	42.75			0	0		
	146	40.75			0	0		
	149	37.75			0	0		
	152	42.00			0	0		
	155	42.25			0	0		
			42,600	2,330,672.30	42,600	5,377,203.80		292,609.40
Total			42,600	2,330,672.30	42,600	5,377,203.80		292,609.40
Brought fd			1,133,613	12,184,309.48	1,133,613	11,250,835.15		1,596,044.19
Carried fd			1,176,213	14,514,981.78	1,176,213	16,628,038.95		1,888,653.59

CO	Pit	Miles	Actual		Optimum		Loss per Ton	Total Loss
			Tons	Ton miles	Tons	Ton miles		
52	26	127.79	7,656	978,360.24			14.01	107,260.56
	31	120.10			0	0		
	32	124.75			0	0		
	34	119.61			0	0		
	38	119.70			0	0		
	40	117.74			0	0		
	43	124.09			0	0		
	44	126.09			0	0		
	45	121.45			0	0		
	47	126.39			0	0		
	50	128.47			0	0		
	51	129.14			0	0		
	53	129.63			0	0		
	55	127.43			0	0		
	56	129.66			0	0		
	57	138.22			0	0		
	59	137.14			0	0		
	61	135.14			1,587	214,467.18		
	62	137.89			0	0		
	82	131.16			0	0		
	83	140.14			0	0		
	84	140.64			0	0		
	105	34.75			4,089	142,092.75		
	106	34.75			11,202	389,269.50		
	107	36.50	2,134	77,891.00			2.25	4,801.50
	108	29.00	5,753	166,837.00	0	0	0	0
	110	34.50	2,748	94,806.00			2.25	6,183.00
	116	34.75			0	0		
	117	29.50			314	9,263.00		
	118	29.75	5,818	173,085.50	3,500	104,125.00	0	0
	121	33.00	816	26,928.00			2.25	1,836.00
	122	31.50	998	31,437.00	3,532	111,258.00	0	0
	123	32.75			0	0		
	125	30.00			10,236	307,080.00		
	126	33.25			2,868	95,361.00		
	128	37.50	1,846	69,225.00			1.75	3,230.50
	129	37.50	2,723	102,112.50			3.25	8,849.75
	132	43.00	2,799	120,357.00			7.75	21,692.25
	134	40.25	1,207	48,581.75			7.25	8,750.75
	137	41.50	170	7,055.00			9.75	1,657.50
	138	41.50	461	19,131.50			7.25	3,342.25
	140	56.50	1,955	110,457.50			16.25	31,768.75
	142	43.25	4,763	205,999.75			2.25	10,716.75
	143	41.00	2,464	101,024.00			1.75	4,312.00
	145	41.75			0	0		
	146	39.75			758	30,130.50		
	149	36.75			10,338	379,921.50		
	151	42.25	8,204	346,619.00			3.50	28,714.00
	152	41.00			4,091	167,731.00		
	155	41.25			0	0		
			52,515	2,679,907.74	52,515	1,950,699.43		243,115.56
Total			52,515	2,679,907.74	52,515	1,950,699.43		243,115.56
Brought fd			1,176,213	14,514,981.78	1,176,213	16,628,038.95		1,888,653.59
Carried fd			1,228,728	17,194,889.52	1,228,728	18,578,738.38		2,131,769.15

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
53	31	120.85			0	0		
	32	125.50			0	0		
	34	120.36			0	0		
	38	120.45			0	0		
	40	118.49			0	0		
	43	124.84			0	0		
	44	126.84			0	0		
	45	122.20			0	0		
	47	127.14			0	0		
	50	129.22			0	0		
	51	129.89			0	0		
	53	130.38			0	0		
	55	128.18			0	0		
	56	130.41			0	0		
	57	138.97			0	0		
	59	137.89			0	0		
	61	135.89			0	0		
	62	138.64	3,247	450,164.08	0	0	0	0
	82	131.91			0	0		
	83	140.89			0	0		
	84	141.39	3,182	449,902.98	0	0	0	0
	98	32.25	1,848	59,598.00			4.00	7,392.00
	105	35.50	798	28,329.00	0	0	0	0
	106	35.50			0	0		
	108	29.75	2,047	60,898.25	20,075	597,231.25	0	0
	116	35.50			7,313	259,611.50		
	117	30.25			0	0		
	118	30.50			10,331	315,095.50		
	121	33.75	936	31,590.00			2.25	2,106.00
	122	32.25			0	0		
	123	33.50			0	0		
	125	30.75			0	0		
	126	34.00	2,401	81,634.00	0	0	0	0
	128	38.25	2,062	78,871.50			1.75	3,603.50
	129	38.25	782	29,911.50			3.25	2,541.50
	142	44.00	3,557	156,508.00			2.25	8,003.25
	143	41.75	130	5,427.50			1.75	227.50
	144	45.00	3,497	157,365.00			2.25	7,868.25
	145	42.50			0	0		
	146	40.50	1,549	62,734.50	0	0	0	0
	149	37.50	10,338	387,675.00	0	0	0	0
	152	41.75			0	0		
	155	42.00	1,345	56,490.00	0	0	0	0
			37,719	2,097,099.31	37,719	1,171,938.25		31,747.00
		Total	37,719	2,097,099.31	37,719	1,171,938.25		31,747.00
		Brought fd	1,228,728	17,194,889.52	1,228,728	18,578,738.38		2,131,769.15
		Carried fd	1,266,447	19,291,988.83	1,266,447	19,750,676.63		2,163,516.15

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
54	163	8.00	109	872.00	109	872.00	0	0
	164	2.75	3,916	10,769.00	3,916	10,769.00	0	0
	165	7.00	891	6,237.00	891	6,237.00	0	0
	166	6.25	7,047	44,043.75	7,047	44,043.75	0	0
	167	8.00	9,229	73,832.00	9,229	73,832.00	0	0
	168	7.25	2,110	15,297.50	2,110	15,297.50	0	0
			23,302	151,051.25	23,302	151,051.25		0
55	107	55.00	1,805	99,275.00			15.00	27,075.00
	110	49.50	6,106	302,247.00			11.50	70,219.00
	132	41.00			1,243	50,963.00		
	140	46.00	3,674	169,004.00	0	0	0	0
	142	46.75			14,908	696,949.00		
	143	50.00	2,820	141,000.00			5.00	14,100.00
	144	47.75	1,746	83,371.50	0	0	0	0
	159	12.50	5,377	67,212.50	5,377	67,212.50	0	0
	160	8.50	3,527	29,979.50	3,527	29,979.50	0	0
	161	11.50	3,294	37,881.00	3,294	37,881.00	0	0
	162	12.50	3,201	40,012.50	3,201	40,012.50	0	0
			31,550	969,983.00	31,550	922,997.50		111,394.00
56	171	16.25			4,934	80,177.50		
	172	16.25			0	0		
	174	9.50			0	0		
	175	7.50			1,450	10,875.00		
	176	12.75	3,687	47,009.25			0.50	1,843.50
	178	3.00	2,650	7,950.00	2,650	7,950.00	0	0
	179	4.50			6,143	27,643.50		
	180	4.50	8,840	39,780.00			1.00	8,840.00
	181	1.25	123	153.75	123	153.75	0	0
			15,300	94,893.00	15,300	126,799.75		10,683.50
57	82	150.41	3,136	471,685.76			7.00	21,952.00
	84	159.89	2,557	408,838.73			7.00	17,899.00
	142	53.75	5,247	282,026.25			0.50	2,623.50
	144	54.75	4,251	232,742.25			0.50	2,125.50
	171	25.75			5,735	147,676.25		
	172	25.75			15,836	407,777.00		
	174	19.00			6,242	118,598.00		
	175	17.00			0	0		
	178	12.50			0	0		
	179	14.00			0	0		
	182	21.50	5,049	108,553.50			3.50	17,671.50
	183	18.00	7,573	136,314.00			0.50	3,786.50
			27,813	1,640,160.49	27,813	674,051.25		66,058.00
Total			97,965	2,856,087.74	97,965	1,874,899.75		188,135.50
Brought fd			1,266,447	19,291,988.83	1,266,447	19,750,676.63		2,163,516.15
Carried fd			1,364,412	22,148,076.57	1,364,412	21,625,576.38		2,351,651.65

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
58	103	9.00			0	0		
	107	22.00			0	0		
	109	13.75			0	0		
	110	20.00			0	0		
	112	12.75			0	0		
	113	10.25			0	0		
	116	22.50			0	0		
	119	15.25			0	0		
	120	18.00			29,887	537,966.00		
	121	18.50			6,200	114,700.00		
	128	23.50			6,522	153,267.00		
	129	22.00			0	0		
	130	22.00			0	0		
	133	25.50			0	0		
	134	20.75			0	0		
	136	17.25			0	0		
	137	19.50			0	0		
	138	22.00			0	0		
	139	31.50			0	0		
	142	28.75			0	0		
	143	27.00			0	0		
	144	29.75			0	0		
	171	2.00	10,267	20,534.00			0.75	7,700.25
	172	4.25	24,821	105,489.25			3.00	74,463.00
	176	7.50	7,521	56,407.50			10.25	77,090.25
			42,609	182,430.75	42,609	805,933.00		159,253.50
59	35	187.40	4,163	780,146.20			12.43	51,746.09
	38	185.95	2,872	534,048.40			8.50	24,412.00
	60	202.14	4,289	866,978.46			8.50	36,456.50
	61	201.39	2,919	587,857.41			8.50	24,811.50
	69	208.65	3,728	777,847.20			25.52	95,138.56
	98	85.25	3,020	257,455.00	8,439	719,424.75	0	0
	102	81.50			10,969	893,973.50		
	104	79.25			93	7,370.25		
	105	92.50	1,398	129,315.00	5,121	473,692.50	0	0
	107	94.75	6,220	589,345.00			2.75	17,105.00
	108	90.00	2,425	218,250.00			3.25	7,881.25
	110	93.00	8,067	750,231.00			3.00	24,201.00
	118	91.50	3,600	329,400.00			4.00	14,400.00
	119	87.50	9,103	796,512.50			2.25	20,481.75
	120	91.25	735	67,068.75			3.25	2,388.75
	128	97.25	3,250	316,062.50			3.75	12,187.50
	146	107.00	740	79,180.00			9.50	7,030.00
	170	70.25	625	43,906.25	625	43,906.25	0	0
	171	71.25			0	0		
	172	71.25			1,524	108,585.00		
	174	66.00	1,919	126,654.00			1.50	2,878.50
	176	67.25			19,680	1,323,480.00		
	182	63.50	3,079	195,516.50	8,128	516,128.00	0	0
	183	63.00	9,328	587,664.00	16,901	1,064,763.00	0	0
			71,480	8,033,438.17	71,480	5,151,328.25		341,118.40
Total			114,089	8,215,868.92	114,089	5,957,256.25		500,371.90
Brought fd			1,364,412	22,148,076.57	1,364,412	21,625,576.38		2,351,651.65
Carried fd			1,478,501	30,363,945.49	1,478,501	27,582,832.63		2,852,023.55

CO	Pit	Miles	Actual Tons	Actual Ton miles	Optimum Tons	Optimum Ton miles	Loss per Ton	Total Loss
60	42	127.77	2,697	344,595.69			7.50	20,227.50
	179	1.25	8,291	10,363.75	2,148	2,685.00	0	0
	180	0.25			8,840	2,210.00		
			10,988	354,959.44	10,988	4,895.00		20,227.50
61	138	28.75			7,186	206,597.50		
	172	11.25	3,429	38,576.25			3.25	11,144.25
	175	4.25	1,450	6,162.50			5.00	7,250.00
	176	12.50	2,307	28,837.50			8.50	19,609.50
			7,186	73,576.25	7,186	206,597.50		38,003.75
62	171	7.50	402	3,015.00	0	0	0	0
	172	7.50	1,049	7,867.50	11,939	89,542.50	0	0
	173	0	2,927	0	2,927	0	0	0
	174	2.75	4,323	11,888.25			2.00	8,646.00
	176	10.50	6,165	64,732.50			7.00	43,155.00
			14,866	87,503.25	14,866	89,542.50		51,801.00
63	171	6.75			0	0		
	172	6.75			0	0		
	174	0	14,387	0	14,387	0	0	0
			14,387	0	14,387	0		0
64	132	58.00			5,510	319,580.00		
	140	63.00			5,692	358,596.00		
	142	63.75			0	0		
	144	64.75			0	0		
	184	3.25	7,042	22,886.50			3.00	21,126.00
	186	4.75	3,392	16,112.00			8.00	27,136.00
	191	1.75	11,944	20,902.00	11,176	19,558.00	0	0
			22,378	59,900.50	22,378	697,734.00		48,262.00
65	106	81.25	11,202	910,162.50			19.00	212,838.00
	132	62.75			0	0		
	140	67.75			0	0		
	142	68.50			0	0		
	144	69.50			0	0		
	184	5.00			7,042	35,210.00		
	185	0.75	12,547	9,410.25	12,547	9,410.25	0	0
	186	1.50	2,802	4,203.00	6,194	9,291.00	0	0
	188	3.75	9,166	34,372.50	9,166	34,372.50	0	0
	189	7.50	5,051	37,882.50	5,051	37,882.50	0	0
	190	3.50	3,799	13,296.50	3,799	13,296.50	0	0
	191	6.50			768	4,992.00		
			44,567	1,009,327.25	44,567	144,454.75		212,838.00
Total			114,372	1,585,266.69	114,372	1,143,233.75		371,132.25
Brought fd			1,478,501	30,363,945.49	1,478,501	27,582,832.63		2,852,023.55
GRAND TOTAL			1,592,873	31,949,212.18	1,592,873	28,726,056.38		3,223,155.80

Appendix B to Chapter 4 Key to Pits and Coke Ovens

Pit No.	"Price"	Tons available	Total "value"	Name
1	157.97	7,270	1,148,441.90	Auchengeich
2	159.97	4,610	737,461.70	Bedlay
3	161.22	7,778	1,253,969.16	Cardowan
4	154.22	9,242	1,425,301.24	Dumbreck
5	155.72	1,787	278,271.64	Gartshore
6	153.22	5,471	838,266.62	Twechar
7	145.47	9,958	1,448,590.26	Easton
8	153.72	10,641	1,635,734.52	Kingshill No.1.
9	134.72	6,685	900,603.20	Kingshill No.2.
10	148.97	21,531	3,207,473.07	Polkemmet
12	143.36	8,666	1,242,357.76	Bannockburn & Plean
13	137.69	5,715	786,898.35	Kinneil
15	140.36	3,196	448,590.56	Polmaise
16	98.52	2,142	211,029.84	Risehow
17	101.02	1,602	161,834.04	St. Helens
18	104.52	13,100	1,369,212.00	Harrington
19	98.02	6,376	624,975.52	Walkmill
21	7.35	2,282	16,772.70	Washington
22	10.30	2,009	20,692.70	Usworth
23	16.29	1,404	22,871.16	Ryhope
24	20.47	3,663	74,981.61	Dawdon
26	26.37	8,156	215,073.72	Blackhall
28	22.90	9,727	222,748.30	Deaf Hill
29	22.35	2,026	45,281.10	Shotton
30	24.14	10,557	254,845.98	Wingate
31	20.05	30,857	618,682.85	Bowburn
32	15.40	10,693	164,672.20	Sherburn Hill
34	20.54	9,140	187,735.60	East Hetton
35	22.93	4,163	95,457.59	Fishburn
37	22.93	19,272	441,906.96	Chilton
38	20.45	38,864	794,768.80	Dean & Chapter
39	24.54	21,103	517,867.62	Mainsforth
40	22.41	3,400	76,194.00	Ramshaw
41	23.65	8,516	201,403.40	Randolph
42	19.38	26,674	516,942.12	Brancepeth
43	16.06	8,806	141,424.36	Brandon
44	14.06	20,788	292,279.28	Brandon, Pit House
45	18.70	1,063	19,878.10	Hole-in-the-Wall
46	18.20	21,921	398,962.20	Wooley & Roddymoor
47	13.76	1,446	19,896.96	West Thornley Harvey Drift
49	14.97	12,581	188,337.57	Bearpark
50	11.68	9,779	114,218.72	Esh
51	11.01	3,929	43,258.29	Kimblesworth
52	12.85	14,078	180,902.30	Langley Park
53	10.52	9,160	96,363.20	Malton Amy
55	12.72	5,473	69,616.56	Ushaw Moor
56	10.49	7,857	82,419.93	Waterhouses
57	1.93	13,342	25,750.06	Beamish Mary & Second
59	3.01	6,231	18,755.31	Pelton
60	4.26	10,332	44,014.32	Sacriston
61	5.01	14,811	74,203.11	South Pelaw
62	2.26	39,796	89,938.96	Craghead "Busty" & "Oswald" & Twizell Burn
	Total	539,669	24,108,129.02	
	Brought fd	0	0	
	Carried fd	539,669	24,108,129.02	

Pit No.	"Price"	Tons available	Total "value"	Name
63	12.61	10,023	126,390.03	Axwell Park & Blaydon Burn
66	5.61	22,470	126,056.70	Chopwell
67	17.07	3,582	61,144.74	Clara Vale
68	7.61	2,218	16,878.98	Garesfield
69	14.77	6,722	99,283.94	Greenside
70	14.41	1,465	21,110.65	Lilley Drift
71	14.16	4,222	59,783.52	Victoria Garesfield
73	12.81	822	10,529.82	Burnopfield
74	4.61	17,839	82,237.79	Byermoor
75	4.36	6,043	26,347.48	East Tanfield & Tanfield Lea
77	8.12	3,038	24,668.56	Kibblesworth
79	6.11	1,969	12,030.59	Marley Hill & Blackburn Fell Drift
82	8.99	4,769	42,873.31	Hedley
83	.01	529	5.29	Louisa
84	-.49	21,997	-10,778.53	Morrison Busty
86	3.96	54,963	217,653.48	Derwent & Crookhall "Victory"
87	2.81	12,765	35,869.65	Eden & South Medomsley
89	7.86	16,975	133,423.50	Watergate
97	122.90	19,316	2,373,936.40	Brookhouse
98	112.65	8,439	950,653.35	Maltby Main
99	117.90	10,698	1,261,294.20	Nunnery
100	119.40	6,870	820,278.00	Handsworth
101	119.90	16,435	1,970,556.50	Orgreave
102	116.40	10,969	1,276,791.60	Thurcroft Main
103	118.90	8,667	1,030,506.30	Treeton
104	118.65	19,607	2,326,370.55	Dinnington Main
105	105.40	9,210	970,734.00	Firbeck Main
106	105.40	11,202	1,180,690.80	Brodsworth Main
107	105.90	16,072	1,702,024.80	Hickleton Main
108	111.15	20,075	2,231,336.25	Yorkshire Main
109	114.15	7,488	854,755.20	Aldwarke Main
110	107.90	38,391	4,142,388.90	Cortonwood & Elsecar Main & Wentworth
112	115.15	10,409	1,198,596.35	New Stubbin
113	117.65	2,464	289,889.60	Rotherham Main
114	114.65	8,958	1,027,034.70	Silverwood
116	105.40	10,211	1,076,239.40	Barborough Main
117	110.65	314	34,744.10	Cadeby Main
118	110.40	13,831	1,526,942.40	Denaby Main
119	112.65	9,103	1,025,452.95	Kilnhurst
120	109.90	29,887	3,284,581.30	Manvers Main
121	109.40	11,558	1,264,445.20	Wath Main
122	108.65	3,532	383,751.80	Frickley
123	107.40	7,518	807,433.20	Hemsworth
124	103.90	17,973	1,867,394.70	New Monckton Nos. 1 & 2
125	110.15	10,236	1,127,495.40	South Kirkby
126	106.90	2,868	306,589.20	Upton
128	104.40	11,309	1,180,659.60	Grimethorpe
129	105.90	11,283	1,194,869.70	Houghton Main
130	105.90	10,997	1,164,582.30	Mitchell Main & Darfield Main
	Total	568,301	42,938,528.25	
	Brought fd	539,669	24,108,129.02	
	Carried fd	1,107,970	67,046,657.27	

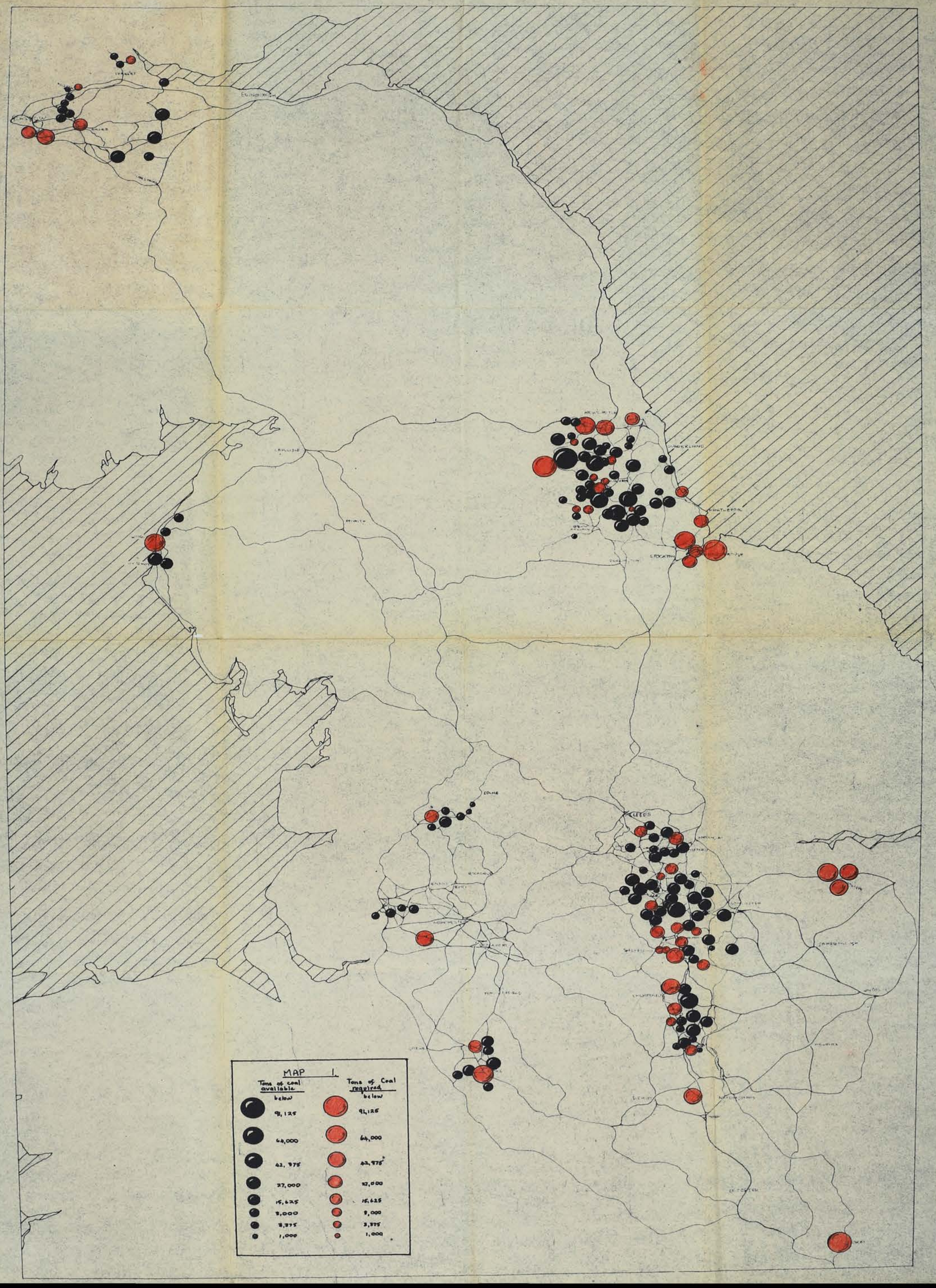
Pit No.	"Price"	Tons available	Total "value"	Name
131	106.15	27,907	2,962,328.05	Barrow
132	104.90	29,237	3,066,961.30	Dodworth
133	102.40	902	92,364.80	Monk Bretton
134	107.15	19,129	2,049,672.35	Rockingham
135	112.15	17,183	1,927,073.45	Smithy Wood
136	110.65	12,874	1,424,508.10	Thorncliffe
137	108.40	7,792	844,652.80	Wharncliffe Silkstone
138	105.90	9,483	1,004,249.70	Wombwell Main
139	96.40	12,270	1,182,828.00	Crigglestone
140	99.90	5,692	568,630.80	Park Mill & Emley Moor
142	99.15	16,778	1,663,538.70	North Gawber
143	100.90	17,413	1,756,971.70	Wharncliffe Woodmoor
144	98.15	18,042	1,770,822.30	Woolley
145	98.40	4,477	440,536.80	Old Roundwood
146	100.40	4,303	432,021.20	Sharlston
148	97.90	4,408	431,543.20	Rothwell
149	103.40	10,338	1,068,949.20	Sharlston West
150	99.40	7,116	707,330.40	Newmarket Silkstone
151	101.40	8,204	831,885.60	Whitwood
152	99.15	4,091	405,622.65	Ackton Hall
153	103.15	12,029	1,240,791.35	Glass Houghton
154	102.15	9,234	943,253.10	Prince of Wales
155	98.90	1,345	133,020.50	St. Johns
159	133.40	5,377	717,291.80	Nook, Gin & Astley Green
160	137.40	3,527	484,609.80	Maypole
161	134.40	3,294	442,713.60	Parsonage
162	133.40	3,201	427,013.40	Garswood Hall
163	119.65	109	13,041.85	Bank Hall
164	124.90	3,916	489,108.40	Calder
165	120.65	891	107,499.15	Clifton
166	121.40	7,047	855,505.80	Hapton Valley
167	119.65	9,229	1,104,249.85	Huncoat
168	120.40	2,110	254,044.00	Scaitcliffe
170	127.65	625	79,781.25	Bolsover
171	126.65	10,669	1,351,228.85	Ireland
172	126.65	29,299	3,710,718.35	Markham
173	134.15	2,927	392,657.05	Grassmoor
174	133.40	20,629	2,751,908.60	Holmewood
175	135.40	1,450	196,330.00	Pilsley
176	130.65	19,680	2,571,192.00	Glapwell
178	139.90	2,650	370,735.00	Alfreton
179	138.40	8,291	1,147,474.40	Blackwell "A" Winning
180	139.40	8,840	1,232,296.00	Blackwell "B" Winning
181	141.65	123	17,422.95	Langton
182	134.40	8,128	1,092,403.20	Pleasley
183	134.90	16,901	2,279,944.90	Silverhill
184	162.65	7,042	1,145,381.30	Chatterley Whitfield
185	166.90	12,547	2,094,094.30	Deep Pit
186	166.15	6,194	1,029,133.10	Sneyd
188	163.90	9,166	1,502,307.40	Apedale & Holditch
189	160.15	5,051	808,917.65	Madeley
190	164.15	3,799	623,605.85	Stafford
191	161.15	11,944	1,924,775.60	Victoria
	Total	484,903	58,166,941.45	
	Brought fd	1,107,970	67,046,657.27	
	Grand Total	1,592,873	125,213,598.72	

CO No	"Price"	Tons required	Total "value"	Name
1	144.86	5,330	772,103.80	Carnock Coking Plant, Falkirk
2	154.22	11,029	1,700,892.38	Dumbreck Coking Plant, Twechar, nr. Glasgow
3	143.36	6,180	885,964.80	Plean Coking Plant, Plean, Stirlingshire
4	162.22	23,316	3,782,321.52	Bairds & Scottish Steel Ltd., Gartsherrie Iron Works, Coatbridge
5	169.97	42,699	7,257,549.03	Colvilles Ltd., Clyde Ironworks, Tollcross, nr. Glasgow
6	171.72	20,414	3,505,492.08	Dixons' Ironworks Ltd., Govan Ironworks, Crown Street, Glasgow
7	104.52	35,370	3,696,872.40	United Steel Companies Ltd., (Workington Iron & Steel Branch) Moss Bay, Workington, Cumb.
8	43.13	24,781	1,068,804.53	Cargo Fleet Iron Co. Ltd., Cargo Fleet Iron Works, Middlesborough
9	41.13	21,180	871,133.40	Dorman Long & Co. Ltd., Acklam Iron & Steel Works, Middlesborough
10	45.30	91,691	4,153,602.30	Dorman Long & Co. Ltd., Cleveland Works, South Bank, Yorks.
11	18.20	23,181	421,894.20	Bankfoot Coking Plant, Bankfoot, Crook, Co. Durham
12	14.97	14,155	211,900.35	Bearpark Coking Plant, Bearpark Colliery, Co. Durham
13	19.38	32,730	634,307.40	Brancepeth Gas & Coke (Strakers & Love) Ltd., Brancepeth Coke Works, Willington, Co. Durham
14	12.61	55,454	699,274.94	Derwenthaugh Coking Plant, Blaydon-on-Tyne
15	12.85	15,734	202,181.90	Langley Park Coking Plant, Langley Park Co. Durham
16	16.37	20,106	329,135.22	Monckton Coke & Gas Works, Gateshead 10, Co. Durham
17	13.66	9,981	136,340.46	New Brancepeth Coking Plant, New Brancepeth Co. Durham
18	10.36	30,005	310,851.80	Norwood Coking Plant, Dunston, Gateshead on Tyne
	Total	483,336	30,640,622.51	
	Brought fd	0	0	
	Carried fd	483,336	30,640,622.51	

CO No	"Price"	Tons required	Total "value"	Name
19	13.94	17,190	239,628.60	Ottovale Coking Plant, Blaydon-on-Tyne
20	4.26	16,540	70,460.40	Stella Gill Coking Plant, Pelton Fell, Chester-le-Street, Co. Durham
21	22.09	40	883.60	Thristlington Coking Plant, West Cornforth, Ferryhill
22	20.95	12,848	269,165.60	Tudhoe Coking Plant, Spennymoor, Co. Durham
23	15.16	9,280	140,684.80	Victoria Coking Plant, Victoria Garesfield Colliery, Rowlands Gill, Co. Durham
24	4.84	72,322	350,038.48	Consett Iron Co. Ltd., Coke & Brick Works Office, Templetown, Consett
25	37.69	43,604	1,643,434.76	I.C.I. Ltd., Billingham Division, Billingham, Co. Durham
26	23.65	11,668	275,948.20	Sadler & Co. Ltd., Middlesbrough, Randolph Coke Ovens, Evenwood, nr. Bishop Auckland
27	33.39	18,560	619,718.40	South Durham Steel & Iron Co. Ltd., Cargo Fleet Ironworks, Middlesbrough
28	107.90	10,593	1,142,984.70	Cortonwood Coking Plant, Wombwell, nr. Barnsley
29	96.40	11,208	1,080,451.20	Crigglestone Coking Plant, Crigglestone, nr. Wakefield
30	114.65	15,660	1,795,419.00	Dalton Main Coking Plant, Silverwood Colliery, Thrybergh, nr. Rotherham
31	118.65	19,514	2,315,336.10	Dinnington Coking Plant, Dinnington, nr. Sheffield
32	103.15	31,972	3,297,911.80	Glasshoughton Coking Plant, Castleford, Yorks.
33	109.65	30,514	3,345,860.10	Manvers Main Coking Plant, Wath-on-Dearne, nr. Rotherham
34	105.90	12,597	1,334,022.30	Mitchell Main Coking Plant, Wombwell, nr. Barnsley
35	119.40	8,830	1,054,302.00	Nunnery (Handsworth) Coking Plant, High Hazels, Handsworth, Sheffield
Total		342,940	18,976,250.04	
Brought fd		483,336	30,640,622.51	
Carried fd		826,276	49,616,872.55	

CO No	"Price"	Tons required	Total "value"	Name
36	117.90	12,506	1,474,457.40	Nunnery (Sheffield) Coking Plant, Attercliffe, Sheffield
37	104.90	22,484	2,358,571.60	Old Silkstone Coking Plant, Dodworth, nr. Barnsley
38	102.15	11,524	1,177,176.60	Robin Hood Coking Plant, Robin Hood, nr. Wakefield
39	117.65	13,990	1,645,923.50	Rotherham Main Coking Plant, Canklow, nr. Rotherham
40	112.15	38,590	4,327,868.50	Smithy Wood Coking Plant, Chapeltown, nr. Sheffield
41	124.15	10,353	1,285,324.95	Waleswood Coking Plant, Waleswood, nr. Sheffield
42	109.15	8,569	935,306.35	Wath Main Coking Plant, Wath-on-Dearne, nr. Rotherham
43	108.40	6,912	749,260.80	Wharncliffe Silkstone Coking Plant, Tankersley, nr. Barnsley
44	100.90	8,494	857,044.60	Wharncliffe Woodmoor Coke Oven Plant, Carlton, nr. Barnsley
45	106.15	36,665	3,891,989.75	The Barnsley District Coking Co. Ltd., Warsborough, nr Barnsley
46	107.40	16,885	1,813,449.00	Hemsworth & United Kingdom Coke Oven Co. Ltd., Fitzwilliam, nr. Pontefract
47	103.90	20,773	2,158,314.70	The Monckton Coke & Chemical Co. Ltd., Monckton, nr. Barnsley
48	115.15	32,150	3,702,072.50	South Yorkshire Chemical Works Ltd., Parkgate, nr. Rotherham
49	122.90	13,750	1,689,875.00	United Coke & Chemicals Co. Ltd., Brookhouse Coke Ovens, Beighton, nr. Sheffield
50	119.90	53,692	6,437,670.80	United Steel Companies Ltd., Orgreave, Handsworth, Sheffield
51	141.15	42,600	6,012,990.00	Appleby-Frodingham Steel Co. Ltd., Scunthorpe, Lincs.
52	140.15	52,515	7,359,977.25	John Lysaght's Scunthorpe Works Ltd., Normanby Park Steel Works, Scunthorpe, Lincs.
Total		402,452	47,877,273.30	
Brought fd		826,276	49,616,872.55	
Carried fd		1,228,728	97,494,145.85	

CO No	"Price"	Tons required	Total "value"	Name
53	140.90	37,719	5,314,607.10	Richard Thomas & Baldwin Ltd., Redbourn Works, Scunthorpe
54	127.65	23,302	2,974,500.30	Altham Coking Plant, Altham, nr. Accrington
55	145.90	31,550	4,603,145.00	The Lancashire Steel Corporation Ltd., Irlam Works, Manchester
56	142.90	15,300	2,186,370.00	The Notts. & Derby Coke & By-Product Co. Ltd., Pinxton, Nottingham
57	152.40	27,813	4,238,701.20	The Stanton Ironworks Co. Ltd., Stanton, nr. Nottingham
58	127.90	42,609	5,449,691.10	The Stavely Iron & Chemical Co. Ltd., Hollingwood, nr. Chesterfield
59	197.90	71,480	14,145,892.00	Stewarts & Lloyds Ltd., Corby, nr. Kettering Northants
60	139.65	10,988	1,534,474.20	Blackwell Coking Plant, Huthwaite, Mansfield, Notts
61	134.65	7,186	967,594.90	Clay Cross Coking Plant, Clay Cross, nr. Chesterfield
62	134.15	14,866	1,994,273.90	Grassmoor Coking Plant, Grassmoor Collieries, nr. Chesterfield
63	133.40	14,387	1,919,225.80	Hardwick Coking Plant, Holmewood, nr. Chesterfield
64	162.90	22,378	3,645,376.20	Birchenwood Coal & Coke Co. Ltd., Kidsgrove, nr. Stoke-on-Trent
65	167.65	44,567	7,471,657.55	The Shelton Iron, Steel, & Coal Co. Ltd., Stoke-on-Trent
	Total	364,145	56,445,509.25	
	Brought fd	1,228,728	97,494,145.85	
	Grand Total	1,592,873	153,939,655.10	


MAP 1.
Tons of coal available below
Tons of Coal required below
91,125
64,000
42,875
27,000
15,625
8,000
3,375
1,000
EDINBURGH
CARLISLE
NEWCASTLE
SUNDERLAND
STOCKTON
LEEDS
COLNE


MAP 2.
Route carrying below 400 tons of coal
- 1,600 " " "
- 6,400 " " "
- 14,400 " " "
- 25,600 " " "
- 40,000 " " "
- 57,600 " " "
- 78,400 " " "
- 102,400 " " "
- 129,600 " " "
- 160,000 " " "
- 193,600 " " "
Colliery
Coke Oven
direction of coal movement.