Interrupted Brownian Motion and Other Related Processes

A thesis presented for the degree of Doctor of Philosophy

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it). The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgement is made. This thesis may not be reproduced without my prior written consent. I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

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Abstract

In this thesis, we study the distributional properties of functionals of the Brownian motion. The thesis starts with an analysis of the occupation time process of Brownian motion in which the joint Laplace transforms of the occupation time processes in different regions and their driving Brownian motion are computed for different starting points using martingale methodology. The corresponding joint density functions are also derived. A version of the Brownian motion, called the interrupted Brownian motion is introduced in the next chapter where the paths of the Brownian motion within a certain band are eliminated. Some distributional properties of the interrupted Brownian motion are derived using the perturbation method. The study of the local time at a certain level of the Brownian motion is then investigated using the Feynman-Kac formulas to derive the joint Laplace transforms of the local time evaluated at the first inverse time local time of the Brownian motion. We repeat the procedure for a compound Poisson process with drift. This thesis is concluded with a discussion on hitting and exit times of other diffusion using symmetry methods. In particular, we look at a diffusion related to Nicholson's integral and another diffusion by conditioning on the Brownian motion.

Contents

Chapter 1

Introduction

In this thesis, as the title suggested, we investigate the distributional properties of a version of the Brownian motion, which we call the interrupted Brownian motion. Before going into the details, let us consider a continuous, adapted process $W = \{W_t,\ t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and taking values in the space $(\mathbb{R}, \mathcal{B})$ where $\mathbb{F} = \{\mathcal{F}_t,\ t \geq 0\}$ is the filtration generated by W and $\mathcal{B} = \mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra on $\mathbb{R}$.

Definition 1.0.1. ***Standard Brownian Motion.***
The process W starting from 0 is called a standard one-dimensional Brownian motion if it satisfies one of the following equivalent properties:

I. *For any $0 = t_0 < t_1 < \ldots < t_n < \infty$ and $n \geq 1$, the increments of W, defined as $\{W_{t_i} - W_{t_{i-1}}\}_{i=1}^n$ are independent and $W_{t_i} - W_{t_{i-1}}$ is distributed as $\mathcal{N}(0, t_i - t_{i-1})$.*

II. *W is a Gaussian process such that $\mathbb{E}(W_t) = 0$ and cov$(W_s, W_t) = \mathbb{E}(W_s W_t)$=min(s,t) for $s, t \in \mathbb{R}_+$.*

III. *The processes $\{W_t,\ \mathcal{F}_t;\ t \geq 0\}$ and $\{X_t^2 - t,\ \mathcal{F}_t;\ t \geq 0\}$ are local martingales.*

IV. *The process $\{e^{\lambda W_t - \frac{\lambda^2}{2}t},\ \mathcal{F}_t;\ t \geq 0\}$ is a local martingale for any fixed $\lambda \in \mathbb{R}$.*

V. *The process $\{e^{i\lambda W_t + \frac{\lambda^2}{2}t},\ \mathcal{F}_t;\ t \geq 0\}$ is a local martingale for any fixed $\lambda \in \mathbb{R}$ and $i \in \mathbb{C}$ denotes the square root of -1.*

Definition 1.0.2. ***Brownian Motion with Drift.***

An adapted, continuous process $X = \{X_t = \mu t + \sigma W_t;\ t \geq 0\}$ is called a Brownian motion with drift $\mu \in \mathbb{R}$ and diffusion coefficient $\sigma \in \mathbb{R}_+$.

For more details, see Karatzas and Shreve (1991), Revuz and Yor (1991), Borodin and Salminen (2002) and Jeanblanc et al. (2009). Louis Bachelier presented the Bachelier model in his PhD thesis "Théorie de la Spéculation" which uses an arithmetic Brownian motion to model the dynamics of stock prices to study the theory of the valuation of financial options. Bachelier (1900) assumes that stock price $S = \{S_t,\ t \geq 0\}$ follow an arithmetic Brownian motion defined as

$$S_t = x + \mu t + \sigma W_t,$$

where $x \in \mathbb{R}$ is the starting point, $\mu \in \mathbb{R}$ is the drift (or growth rate), $\sigma \in \mathbb{R}_+$ is the volatility and W is the Brownian motion as defined in Definition 1.0.1. In the paper Samuelson (2015) in 1965, building on the model introduced by Louis Bachelier, Paul Samuelson introduced the geometric Brownian motion (or economic Brownian motion) with the following stochastic differential equation (SDE):

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.1}$$

in place of the arithmetic Brownian motion on the grounds that stock prices should only take non-negative values.

Definition 1.0.3. ***Black & Scholes Model.***
In 1973, Fischer Black and Myron Scholes developed the famous Black & Scholes model

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$
$$dB_t = r B_t dt,$$

with the following assumptions:

I. The risk free interest rate $r \in \mathbb{R}$ is constant.

II. The stock price process $S = \{S_t,\ t \geq 0\}$ follows the geometric Brownian motion with SDE (1.1).

III. Short positions are allowed.

IV. The stock pays no dividends.

V. Only European options are considered.

VI. Market is frictionless.

VII. Unlimited credit is allowed.

where $B = \{B_t,\ B \geq 0\}$ *is the money market account.*

See Black and Scholes (1973) and Merton (1973) for more details. The pricing of derivative securities which hinges around the Black-Scholes model is closely related to the studies concerning the problems of finding the distributions of measurable functionals of Brownian motion such as the maximum and minimum functionals. The distribution of the running maximum Brownian motion is well explored by Karatzas and Shreve (1991) which gives the results

$$\mathbb{P}_0\left[W_t \in da,\ M_t \in db\right] = \frac{2(2b-a)}{\sqrt{2\pi t^3}} e^{-\frac{(2b-a)^2}{2t}} dadb,$$

for $M = \{M_t,\ t \geq 0\}$ the running maximum of the Brownian motion defined as

$$M_t = \sup_{0\leq s\leq t} W_s.$$

The law of the running minimum $m = \{m_t,\ t \geq 0\}$ of the Brownian motion can be deduced by

$$m_t := \inf_{0\leq s\leq t} W_s = -\sup_{0\leq s\leq t}(-W_s) = -\sup_{0\leq s\leq t} B_s,$$

where $B = \{B_t = -W_t,\ t \geq 0\}$ is again a Brownian motion. See Jeanblanc et al. (2009) for more details. The distribution of the running maximum and minimum of the Brownian motion is widely used in the pricing of some path dependent options, such as the barrier option and the lookback option. The barrier options are a type of path dependent options which can be divided in to two classes:

- Knock-out barrier option: The option is eliminated, or "knocked-out" when the a predetermined barrier is reached.
- Knock-in barrier option: The option comes into play, or "knocked-in" when the predetermined barrier is reached.

For more details regarding the pricing of Barrier option in the discrete time setting, see P Wilmott and Howison (1993), Chesney et al. (1995), Pliska (1997), Zhang (1997), Wilmott (1998) and Musiela and Rutkowski (2006). For pricing of this option in continuous time setting, see Rubinstein (1991), Rich (1994), Heynen and Kat (1995), Carr and Chou (1997), Baldi et al. (1999), Andersen et al. (2000), Linetsky (2004a), Suchanecki (2004), Jeanblanc et al. (2009).

A lookback option is a type of option with path dependency feature whose payoff depends not only on the value of the underlying stock price at maturity but on the optimal value over the life of the option. More details on the pricing of a lookback option can be found in Goldman et al. (1979a), Goldman et al. (1979b), Conze (1991), He et al. (1998), Shreve et al. (2004), Musiela and Rutkowski (2006) and Jeanblanc et al. (2009).

Let us look at a type of option with path dependency structure, the α-quantile option for $\alpha \in [0, 1]$, first introduced by Miura (1992) which motivated the main part of the studies in this thesis. For a fixed strike price $K > 0$, the α-quantile option can be seen as an extension of the barrier option (see Broadie and Detemple (2004)) where as an α-quantile option with floating strike can serve as the extension of a lookback option (see Cai (2011)). The payoff of this option depends on the α-quantile of the underlying process $X = \{X_t,\ t \geq 0\}$ as defined in Definition 1.0.2 is given as

$$M(\alpha, t) := \inf\{x \ : \ \int_0^t \mathbb{1}_{\{X_s \leq x\}} ds > \alpha t\}, \tag{1.2}$$

which is the smallest level, or barrier such that the underlying process spends a fraction of time exceeding α below that level. The quantity

$$\int_0^t \mathbb{1}_{\{B\}}(X_s) ds = \text{meas}\{0 \leq s \leq t \ : \ X_s \in B\}, \qquad t \in \mathbb{R}_+,$$

is known as the occupation time of the process X, a Brownian motion with drift for every fixed Borel set $B \in \mathcal{B}(\mathbb{R})$ and meas denotes the Lebesgue measure. The occupation time process has received much attention in the study of ruin problems for general diffusion processes where the ruin probabilities can be expressed in terms of the occupation time processes. The event of ruin was first introduced as the first time the surplus process of an insurance company dips below zero for the first time. However, many have observed that this definition of ruin can be a bit conservative as even when the surplus drops below zero, most companies can still endure a small period of negative surplus and cam quickly recover to a positive surplus level. This leads to the discussion which encourages the distinction between ruin and bankruptcy as in Dassios and Wu (2008) who introduced the concept of Parisian ruin, which is a type of ruin that occurs only when the surplus process stays below the pre-determined ruin level for a continuous time interval of some length. For a surplus process in continuous time $X = \{X_t,\ t \geq 0\}$

$$X_t = x + ct - \sum_{i=0}^{N_t} Y_i,$$

for $x \in \mathbb{R}_+$, c a constant rate for premium payment, $N_t \sim \text{Poisson}(\lambda)$ for $\lambda \in \mathbb{R}_+$ and $Y_i,\ i = 1, 2, \ldots$ are independent and identically distributed claim sizes which are independent of N_t, the authors defined the excursions

$$\begin{aligned} g_t^X &= \sup\{s < t \ : \ \text{sign}(X_s)\text{sign}(X_t) \leq 0\}, \\ d_t^X &= \inf\{s > t \ : \ \text{sign}(X_s)\text{sign}(X_t) \leq 0\}, \end{aligned}$$

where $\text{sign}(x)$ is the sign function. The Parisian ruin in the finite horizon is defined as the event $\{\tau_d^X \leq t\}$ where for $d \in \mathbb{R}_+$, τ_d^X is given as

$$\tau_d^X := \inf\{t \geq 0 \ : \ \mathbb{1}_{\{X_t < 0\}}(t - g_t^X) \geq d\}.$$

Landriault et al. (2010) extended the study of Parisian ruin in an insurance risk model with underlying spectrally negative Lévy process of bounded variation. The authors proved that the Parisian ruin with exponential implementation clock with mean $\frac{1}{d}$ can be obtained in terms of the occupation time process of X, given as

$$\mathbb{P}\left[\tau_d^X < \infty\right] = 1 - \mathbb{E}\left[e^{-d\int_0^\infty \mathbb{1}_{\{X_s \leq 0\}} ds}\right],$$

where X here is a spectrally negative Lévy process. For more details on Parisian deterministic delays, see Landriault et al. (2011), Loeffen et al. (2013), Czarna et al. (2014), Wong and Cheung (2015), Czarna et al. (2017) and Loeffen et al. (2018). The extension of Parisian deterministic delay to Parisian stochastic delay is explored by Landriault et al. (2014), Baurdoux et al. (2016), Albrecher and Ivanovs (2017) and Frostig and Keren-Pinhasik (2020). In a model analysed by Kyprianou and Loeffen (2010), the occupation time process is considered for $X = \{X_t,\ t \geq 0\}$, a spectrally negative Lévy process and $U = \{U_t,\ t \geq 0\}$ a refracted Lévy process in (b, ∞) described by

$$U_t = X_t - \delta \int_0^t \mathbb{1}_{\{U_s > b\}} ds, \qquad t \geq 0.$$

Then, X can be thought of as a Lévy insurance risk process with dividend policy at rate $\delta > 0$ whenever the process goes above the barrier level $b > 0$. In Chapter 2 of this thesis, we take $X = \{X_t,\ t \geq 0\}$ to be a Brownian motion as defined in Definition 1.0.1 and we modify this model to study the process

$$X_\tau - \alpha_1 V_\tau - \alpha_2 Z_\tau^{(2)} = X_\tau - \alpha_1 \int_0^\tau \mathbb{1}_{\{-a < X_s < a\}} ds - \alpha_2 \int_0^\tau \mathbb{1}_{\{X_s < -a\}} ds, \qquad 0 < \alpha_1 < \alpha_2 < \infty,$$

where we have the following occupation time processes of the Brownian motion X with different barriers for $a > 0$:

$$\begin{aligned} Z_t^{(1)} &= \int_0^t \mathbb{1}_{\{X_s > a\}} ds \\ V_t &= \int_0^t \mathbb{1}_{\{-a < X_s < a\}} ds \\ Z_t^{(2)} &= \int_0^t \mathbb{1}_{\{X_s < -a\}} ds, \end{aligned} \tag{1.3}$$

and τ is defined as the right-continuous inverse of the occupation time $Z_t^{(1)}$:

$$\tau := \inf\{t : \ Z_t^{(1)} = z\}. \tag{1.4}$$

This model can be considered as an insurance risk process with penalising rates α_1 and α_2 for $0 < \alpha_1 < \alpha_2 < \infty$ such that when the process is penalised at the lower rate α_1 when it is in the region between $-a$ and a, and at a higher rate α_2 when the process falls beyond the barrier $-a$, until the process stabalizes in the sense that the process spends a predetermined proportion of time above the level a (this is when τ comes into play). For negative rates α_1 and α_2, the company is forced to have an influx at rates α_1 and α_2 instead. We can then consider a flip of the process $-X$ where we study

$$X_\tau - \delta_1 V_{\tau^*} - \beta_2 Z^{(1)}_{\tau^*} = Y_{\tau^*} - \delta_1 \int_0^{\tau^*} \mathbb{1}_{\{-a<X_s<a\}} ds - \delta_2 \int_0^{\tau^*} \mathbb{1}_{\{X>a\}} ds, \qquad 0<\delta_1<\delta_2,$$

where

$$\tau^* := \inf\{t \geq 0 \ : \ Z^{(2)}_t = z\}.$$

The intuition remains the same as the case for X but now we have the company paying dividends or investing at rates δ_1 and δ_2 according to the region the process is in, until the first time the process stays below the level $-a$ for an amount z of time.

With this in mind, we derive the distributional properties of the occupation time processes defined in (1.3) evaluated at the stopping time τ given in (1.4) by extending the martingale methodology as seen in Dassios and Embrechts (1989), Dassios and Jang (2003) and Dassios and Jang (2005). We derive the joint Laplace transforms for these quantities at different starting points and proceed to obtain the respective density functions.

Our results of the occupation time process also find application in structural credit risk modelling. The study of credit risk is the investigation surrounding the potential loss arising from possible default of counterparty. There are two main classes of credit risk models, which are the structural model and the reduced form method. The structural credit models were first developed by Merton (1974) following the argument of Black and Scholes (1973) in option pricing in order to study default behaviour. The Black & Cox model proposed in Black and Cox (1976) is based on a default time of the form :

$$\tau = \inf\{t \geq 0 \ : \ V_t \leq B(t)\}$$

where $V = \{V_t, \ t \geq 0\}$ is the value process of the firm and $B(t)$ is a predetermined time-dependent default barrier. In Mukhopadhyay and Makarov (2019), a structural credit risk model based on the occupation time of the underlying process is studied. The default of the firm happens at time τ where τ is given as

$$\tau = \inf\{t \geq 0 \ : \ \int_0^t \mathbb{1}_{\{V_s \leq B(s)\}} ds \geq v\},$$

for V and $B(s)$ as defined in Black and Cox (1976) and $v > 0$ is some predetermined threshold level.

As mentioned, a large part of this thesis is motivated by the path dependent option called the α-quantile option introduced by Miura (1992). For an underlying process $X = \{X_t,\ t \geq 0\}$, its α-quantile $M(\alpha, t)$ is given in (1.2) for $0 < \alpha < 1$. It follows that

$$\lim_{\alpha \to 0} M(\alpha, t) = \inf_{0 \leq s \leq t} X_s \qquad a.s.$$

$$\lim_{\alpha \to 1} M(\alpha, t) = \sup_{0 \leq s \leq t} X_s \qquad a.s.$$

The distribution of the α-quantile of a process has been studied by many, including Yor (1995) who investigated the connection between the arc-sine law and the distribution of the α-quantile of a a Brownian motion. Dassios (1995) proved the remarkable identity in law for $X = \{X_t,\ t \geq 0\}$ using Feynman-Kac method detailed in Kac (1951):

$$M(\alpha, t) \stackrel{(d)}{=} \sup_{0 \leq s \leq \alpha t} X_s^{(1)} + \inf_{0 \leq s \leq (1-\alpha) t} X_s^{(2)}, \tag{1.5}$$

where $\stackrel{(d)}{=}$ denotes equality in distribution and $X_s^{(2)}$ is an independent copy of $X_s^{(1)}$. Dassios (1996b) then obtained a similar representation of (1.5) for a joint distribution of α-quantile and the process X, a process with stationary and independent increments with paths in $D[0, \infty)$. Dassios (1996a) later investigated the identity (1.5) for a renewal reward process $X = \{X_t,\ t \geq 0\}$ given as

$$X_t = \begin{cases} \sum_{i=1}^{N_t} Y_i, & N_t = 1, 2, \ldots, \\ 0, & N_t = 0, \end{cases}$$

for the renewal process $N = \{N_t,\ t \geq 0\}$ defined as

$$N_t = \sup_{n \in \mathbb{Z}_0^+} \{n\ :\ \sum_{i=1}^{n} T_i \leq t\},$$

where the sequence of pairs of independent and identically distributed random variables $\{(T_i, Y_i),\ i = 1, 2, \ldots\}$ taking vales in $\mathbb{R}_+ \times \mathbb{R}$. Embrechts et al. (1995) provided two different proofs of the identity (1.5) for a Brownian motion with drift. The first proof uses the following identity in law:

$$\int_0^t \mathbb{1}_{\{X_s \geq 0\}} ds \overset{(d)}{=} \sup_{0<s<t} \{s \ : \ \sup_{u \leq s} X_u = X_s\},$$

where as the second proof follows an extension of Bertoin (1991) rearrangement of the positive and negative excursions of a Brownian motion with drift. Bertoin et al. (1997) extended the identity (1.5) for renewal reward processes using a predictable transformation and for chains with exchangable increments using an optional transformation.

The pricing of this option in the Black-Scholes model as detailed in Definition 1.0.3 is explored by Akahori (1995) and Dassios (1995) who derived the explicit expression of the price of α-quantile call option by applying the risk neutral precedure as detailed in Harrison and Pliska (1981). For more literature on the pricing of an α-quantile option, see Linetsky (1999), Hugonnier (1999), Pechtl (1999), Fusai (2000), Fusai and Tagliani (2001) and Davydov and Linetsky (2002). For pricing approaches beyond the Black-Scholes model, see Sun Leung and Kwok (2007) who obtained the pricing formula for α-quantile option from the distribution functions of occupation times under the constant elasticity of variance process, Cai et al. (2010) who derived the solutions to the pricing problem under Kou's double exponential jump diffusion model and Cai (2011) for a Laplace transform based pricing solution under a hyperexponential jump diffusion model.

Inspired by the α-quantile option, the study in Chapter 3 is motivated by the double quantile option of the underlying process which is a path dependent option that is the smallest level, or barrier such that the fraction of time spent by the process above that level or below the negative part of that level in the time period $[0, t]$ exceeds the amount α for $0 < \alpha < 1$. More formally, we define

$$M(\alpha, t) := \inf\{x \in \mathbb{R}_+ \ : \ \int_0^t \mathbb{1}_{\{W_s \geq x\}} ds + \int_0^t \mathbb{1}_{\{W_s \leq -x\}} ds > \alpha t\},$$

where $W = \{W_t, \ t \geq 0\}$ is the Brownian motion as detailed in Definition 1.0.1. The structure of this option has prompted the study of the joint distribution of the occupation time process of W above a predetermined barrier a and below the barrier $-a$ for any $a \in \mathbb{R}_+$. To do this, we introduce a version of the Brownian motion which we only consider the paths of the Brownian motion above a and below $-a$ by joining them. We call this process the Interrupted Brownian Motion. As we will see later, the interrupted Brownian motion is a continuous version of the Brownian motion by construction. The main body of Chapter 3 focuses on the construction and derivation of the distributional properties of the interrupted Brownian motion. We derive the stochastic differential equation of the interrupted Brownian motion with the help of Doob's h-transform and obtain some distributional properties of the interrupted Brownian motion using the extended martingale methodology as seen in Dassios and Embrechts (1989), Dassios and Jang (2003) and Dassios (2005). In the last part of this chapter, we derive the Laplace transform of the maximum height of the excursion of an interrupted Brownian motion with exponential time using the perturbed Brownian motion introduced by Dassios and Wu (2008).

In Chapter 4, we look at the stochastic process $L^x = \{L_t^x,\ (t,x) \in [0,\infty) \times \mathbb{R}\}$ taking values in $[0,\infty)$ which describes the amount of time spent by a continuous time stochastic process $X = \{X_t,\ t \geq 0\}$ in the neighbourhood of a point $x \in E$ for E the state space of the stochastic process X. The Lebesgue measure of the time spent at the level x can be derived using

$$L_t^x = \int_0^t \mathbb{1}_{\{X_s = x\}} ds.$$

However, this does not make sense for when X is a Brownian motion, for example, as we have

$$\text{meas}\{0 \leq t < \infty \ :\ W_t(\omega) = x\} = 0, \qquad \text{for } \mathbb{P} - \text{a.e., } \omega \in \Omega,$$

With $E = \mathbb{R}$, this is not helpful as it does not tell us how much time the Brownian motion has spent in the neighbourhood of the point $x \in \mathbb{R}$. In order to provide a meaningful interpretation for this measure of time, Paul Lévy in Lévy (1940) first introduced the notion of local time of Brownian motion by defining the following random field for $t \in [0,\infty)$ and $x \in \mathbb{R}$,

$$L_t^{x,\epsilon} = \frac{1}{2\epsilon}\text{meas}\{0 \leq s \leq t \ :\ |W_s - x| \leq \epsilon\}.$$

He showed that the limit below almost surely exists for all $t > 0$

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon}\text{meas}\{0 \leq s \leq t \ :\ |W_s - x| \leq \epsilon\}, \tag{1.6}$$

and this is called the *mesure du voisinage* or "measure of the time spent by the Brownian path in the vicinity of the point x". L_t^x shall be referred to as the *local time* from here onwards. Interested readers can refer to Lévy (1940), Itō and McKean (1974a), Balkema (1991), Balkema and Chung (1991), Karatzas and Shreve (1991), Chung and Durrett (2008), and Jeanblanc et al. (2009). The introduction of the Brownian local time finds many applications in terms of the development of the theory of stochastic calculus. In particular, the local time process is crucial in the generalisation of the celebrated Itô rule for convex functions.

Definition 1.0.4. ***Itô Process.***

Let $\mu = \{\mu_t,\ t \geq 0\}$ and $\sigma = \{\sigma_t,\ t \geq 0\}$ be two predictable processes. If the following integrability conditions hold for all $t \geq 0$:

$$\mathbb{P}\left[\int_0^t |\mu_s| ds < \infty\right] = 1,$$
$$\mathbb{P}\left[\int_0^t \sigma_s^2 ds < \infty\right] = 1.$$

Then $X = \{X_t,\ t \geq 0\}$ is called an Itô process which satisfies

$$X_t = x + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s, \qquad t \geq 0, x \in \mathbb{R},$$

where W is as defined in Definition 1.0.1.

Theorem 1.0.5. *For a function $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ and X an Itô process as in Definition 1.0.4, the process $\{f(t, X_t),\ t \geq 0\}$ is a continuous semi-martingale given as*

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial f(x, X_s)}{\partial t} ds + \int_0^t \frac{\partial f(s, X_s)}{\partial x} dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 f(s, X_s)}{\partial x^2} d\left\langle X \right\rangle_s,$$

where $\langle X \rangle$ is the predictable quadratic variation of X. See Revuz and Yor (1991) for a proof on the invariance of semi-martingale under "smooth" transformation.

Remark 1.0.6. *For a function $f \in C^{1,2}(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$ satisfying the partial differential equation:*

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0,$$

the process $\{f(t, W_t),\ t \geq 0\}$ is a local martingale for W a Brownian motion.

See for example Itô (1944) and Kunita and Watanabe (1967). The Itô rule plays an important role as it acts as the foundation to stochastic calculus, but as we can see from the definition above, Itô rule requires the existence of the second derivative for the formula to make sense. The local time process finds application in the generalisation of the Itô rule for convex function $f : \mathbb{R} \mapsto \mathbb{R}$ which are not necessarily twice differentiable. The generalised Itô rule is given as

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_s) dX_s + \int_{-\infty}^{\infty} L_t^x \mu(dx), \qquad 0 \leq t < \infty,$$

for every $X_0 \in \mathbb{R}$, D_f^- as the left derivative of f, μ to be taken as the second derivative measure in the distribution sense and L^x is the local time process as given in (1.6). For more details, see Chung et al. (1990), Revuz and Yor (1991), Karatzas and Shreve (1991), Kallenberg (1997), Rogers and Williams (2000b) and Borodin and Salminen (2002).

Besides contributing to the generalised Itô rule, the local time process also plays a significant part in mathematical finance. Leblanc (1997) made use of the definition of local time to derive the Dupire's formula for local volatility which states that

$$\frac{1}{2} K^2 \sigma^2(T, K) = \frac{\partial_T C(K,T) + rK \partial_K C(K,T)}{\partial_{KK}^2 C(K,T)},$$

where ∂_T (resp. ∂_K) is the partial derivative operator with respect to the maturity (resp. the strike), $C(K,T) = \mathbb{E}\left[e^{-rT}(S_T - K)_+\right]$ is the price of the European call for any maturity $T \in [0, \infty)$ and any strike price $K \in \mathbb{R}_+$ for $S = \{S_t,\ t \geq 0\}$ the price process, $r \in \mathbb{R}_+$ the instantaneous risk free rate and $\sigma(T, K)$ the dispersion of S. In the Black-Scholes model defined in Definition 1.0.3, the instant volatility σ_t is assumed to be deterministic. Here, the local volatility term has a dependence on time is therefore a function of time represented as $\sigma^2(T, K)$ and ∂ is the partial derivative operator. The Dupire's formula is well used as it serves as a direct method to deduce the local volatility function from the prices of call options in the market. For early works of local volatility model, see more details regarding the derivation of the Dupire formula, see Dupire et al. (1994) for the continuous case and Derman and Kani (1994) for the discrete case. Further details of the derivations of the Dupire's formula can be found in Buraschi and Dumas (2001), Esser and Schlag (2002), Gatheral (2004), Derman and Miller (2016) and Itkin (2020).

Another established application of the local time process is the pricing of a knock-out BOOST option studied by Leblanc (1997) which is an option that pays at maturity, for the amount of time when the underlying price process stays above a level $b \in \mathbb{R}_+$ until the time when the price process touches level $a \in \mathbb{R}_+$ for the first time, for positive levels a and b such that $b < a$. The local time process is also being employed in the form of the Ito-Tanaka formula for the pricing of a special type of contingent claim which is known as a passport option. This option gives its holder the right to engage in an optimal trading strategy of choice. For a finite horizon model, let $S = \{S_t,\ 0 \leq t \leq T\}$ be the price process, $\{q_t, ; 0 \leq t \leq T\}$ be the predictable strategy, $r \in \mathbb{R}_+$ be the deterministic interest rate and $\psi^{(q)} = \{\psi_t^{(q)},\ 0 \leq t \leq T\}$ be the gains from the trade process. The payoff of a passport option with expiry time $T \in \mathbb{R}_+$ is defined as

$$\max\{\psi_T^{(q)}, 0\}.$$

Following the arguments in Harrison and Pliska (1981), the price of the passport option can be determined using

$$\max_{|q_t| \leq K} e^{-r(T-t)} \mathbb{E}_t \left[\max\{\psi_T^{(q)}, 0\} \right].$$

For more details on pricing a passport option, see Hyer et al. (1997) who applied the pricing partial differential approach to derive the pricing formula for passport option, Andersen et al. (1998) who employed a change of measure method for the derivation of a pricing formula for a passport option, Shreve and Vecer (1998) who made use of probabilistic methods to price this option and Andersen et al. (1998) who utilized the concepts of local time process and Skorokhod lemma in deriving the price of a passport option.

In Chapter 4, we employ the methods in Borodin (1994) and Karatzas and Shreve (1991) to investigate the joint distribution of the local times of a Brownian motion with drift at 2 distinct levels evaluated at the first hitting time of level 0 of the driving Brownian motion. Using the results obtained, we proceed to compute the joint distribution of the local times at the 2 levels evaluated at right inverse of the local time of the driving process at 0. We then continue to compute the same quantity for a compound Poisson process with drift at different starting points. Our results find applications in counterparty credit risk management concerning the Accumulator option. We compute the expected exposure of this derivative using the density function derived. The accumulator option is a derivative with high path dependency which is popular among investors with appetite for high risk. The accumulator option is settled periodically thought its term, and this option with a knock-out feature, vanishes if the underlying price process reaches above a pre-determined barrier. If the underlying price process lies inside the knock-out barrier and the strike price, then the buyer of this option "accumulates" the stock at the strike price. Otherwise, the buyer is met with the obligation to purchase the stock at the strike price with some gearing ratio g (this is usually set to 2). The payoff (P_i) on observation day t_i, $i \in \mathbb{N}_+$ of the accumulator derivative is then:

$$P_i = \begin{cases} 0, & \text{if } \max_{0\le s\le t_i} S_s \ge b, \\ Q(S_{t_i} - K), & \text{if } \max_{0\le s\le t_i} S_s < b,\ S_s \ge K, \\ gQ(S_{t_i} - K), & \text{if } \max_{0\le s\le t_i} S_s < b,\ S_s < K, \end{cases}$$

where b is the knock-out barrier level, K is the strike price, Q is the purchase quantity and g the gearing ratio. See Lam et al. (2009) and Bonollo et al. (2017) for more details. We conclude the chapter by following the argument of Bonollo et al. (2017) to use our local time results to estimate the counterparty credit risk relating to the accumulator option.

In Chapter 5, we look at the role of symmetry methods in obtaining the solutions to some differential equations. There are many tools for obtaining solutions to differential equations, for example if the differential equation is separable:

$$g(y)\frac{dy}{dx} = f(x),$$

then the differential equation can be easily solved by separating the dependent and independent variables to the form:

$$g(y)dy = f(x)dx. \tag{1.7}$$

A closer inspection reveals that the underlying method allowing the separation technique to be possible is the presence of a Lie group symmetry. Indeed, the solution of the separable differential equation involves integrating both sides of equation (1.7) which gives us the

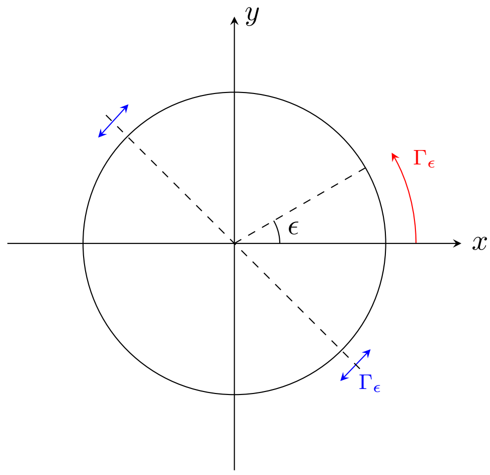

Figure 1.1: Symmetries of the unit circle.

solution of the form

$$y = h(x, c), \tag{1.8}$$

where $c \in \mathbb{R}$ is the integration constant. This constant of integration is exactly the adjustable parameter of a continuous transformation that takes each solution curve (1.8) into another. The theory of Lie group symmetry, is a topic developed by Marius Sophus Lie in Lie (1970). The theory of Lie group symmetry has then received a lot of attention and has been extensively studied by many, see Bluman and Kumei (1989), Stephani (1989), Olver (1993), Hydon and Hydon (2000), Starrett (2007) to name a few. The development of Lie group symmetry has profound influence in, but not limited to the fields of pure and applied Mathematics, Physics and Engineering. In particular, the Lie group symmetry can be applied in vast areas of studies, such as algebraic topology, differential geometry, invariant theory, bifurcation theory, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and many others. Before moving on to the application of Lie group symmetry, we provide some important definitions surrounding this topic. According to Martin (2012), a symmetry of a geometrical object involves a transformation i for i in the set of transformation, such that i maps the object to itself. The object is said to be invariant under transformation. For example, a circle has two different types of symmetries: rotation symmetry about the origin and reflection symmetry in the diagonals. See Figure 1.1 for more details.

Extending this notion of symmetry to the symmetry of ordinary differential equation, we list some important definitions for the one-parameter Lie group of transformation.

Definition 1.0.7. ***One-Parameter Lie Group of Transformation.***
On the Euclidean plane, let $\boldsymbol{x} = (x, y)$ *and* $\hat{\boldsymbol{x}} = (\hat{x}, \hat{y})$ *be some points on the plane. For* $\epsilon \in \mathbb{C}$, *the transformation*

$$\Gamma_\epsilon : \boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x}, \epsilon),$$

such that

$$\hat{x} = f(x, y, \epsilon); \qquad \hat{y} = g(x, y, \epsilon),$$

is the one-parameter group of transformation with parameter ϵ, *if the following hold:*

I. Γ_0 *gives the identity transformation, i.e.*

$$f(x, y, 0) = x; \qquad g(x, y, 0) = y.$$

II. Γ *is closed under composition. This means that for* $\epsilon_1, \epsilon_2 \in \mathbb{C}$,

$$\Gamma_{\epsilon_2}\Gamma_{\epsilon_1} = \Gamma_{\epsilon_2+\epsilon_1},$$

i.e.

$$x^* = f(\hat{x}, \hat{y}, \delta) = f(x, y, \epsilon_1 + \epsilon_2); \qquad y^* = f(\hat{x}, \hat{y}, \delta) = f(x, y, \epsilon_1 + \epsilon_2).$$

III. Γ_ϵ^{-1} *gives the inverse transformation:*

$$\Gamma_\epsilon^{-1} = \Gamma_{-\epsilon}.$$

IV. Each $\hat{\boldsymbol{x}}$ *can be represented as a Taylor series in* ϵ.

Example 1.0.8. *For the following ordinary differential equation:*

$$\frac{dy}{dx} = 0,$$

we know the solution is of the form $y = C$ *for* $C \in \mathbb{R}$. *There are a few valid symmetries available for this differential equation. One possible symmetry for this is translational symmetry* *in the x-direction, giving us*

$$(\hat{x}, \hat{y}) = (x + \epsilon, y).$$

We can spot that this is a trivial symmetry since every solution is mapped to itself. Another symmetry is to perform the transformation in the y-direction. This gives us

$$(\hat{x}, \hat{y}) = (x, y + \epsilon).$$

This is no longer trivial as under this transformation, the solution curve is mapped from $y = C$ to $y = C + \epsilon$, and we can easily see that this solution curve satisfies the original differential equation.

As we discussed, it is straightforward to solve a differential equation which is separable, i.e. of the form in (1.7). However, this is not the case in many differential equations that we encounter, for example an ordinary differential equation of the form

$$\frac{dy}{dx} = \omega(x, y),$$

can be quite complicated to solve depending on the form of ω. This is when canonical coordinates come in handy.

Definition 1.0.9. ***Canonical Coordinates.***
Any coordinates $(r(x, y), s(x, y))$ satisfying

$$\xi(x, y) r_x + \eta(x, y) r_y = 0,$$
$$\xi(x, y) s_x + \eta(x, y) s_y = 1,$$

and

$$r_x s_y - r_y s_x \neq 0,$$

is called the canonical coordinates, where (ξ, η) is the tangent vector at (x, y) to the curve, $r_x = \frac{\partial}{\partial x} r(x, y)$, $r_y = \frac{\partial}{\partial y} r(x, y)$ and similarly defined for s_x and s_y. The canonical coordinates can be obtained using the method of characteristics such that

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)} = ds.$$

The ODE of the form $\frac{dy}{dx} = \omega(x, y)$ *can then be transformed to its canonical coordinates:*

$$\frac{ds}{dr} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y}.$$

In canonical coordinates $r(x, y), s(x, y)$, a differential equation becomes separable, which we can manage. We summarise this paragraph by giving the steps to solve a first-order ordinary differential equation using symmetry methods:

I. When the given differential equation

$$\frac{dy}{dx} = \omega(x, y),$$

is not separable, identity the form of Lie symmetry of the solutions using ansatz which satisfies the linearized symmetry condition:

$$\eta_x + (\eta_y - \xi_x)\omega(x, y) - \xi_y\omega^2(x, y) = \xi\omega_x(x, y) + \eta\omega_y(x, y).$$

II. Make use of the canonical coordinates to transform the original differential equation to a separable differential equation. The canonical coordinates $r(x, y), s(x, y)$ can be identified from $\xi(x, y)$ and $\eta(x, y)$, the solutions to the linearized symmetry condition using method of characteristics.

III. Substitute the canonical coordinates obtained in the previous step into the differential equation for (r, s):

$$\frac{ds}{dr} = \frac{s_x + \omega(x, y)s_y}{r_x + \omega(x, y)r_y}.$$

IV. Solve this differential equation in terms of the canonical coordinates.

V. Obtain the solution of the original differential equation by using the inverse relation $x = (x(r, s))$ and $y = y(r, s)$.

We use the symmetry method for differential equation to obtain a diffusion that links to the Nicholson's integral. The Nicholson's integral, introduced by J.W.Nicholson in Nicholson (1910) and Nicholson (1911) is the generalized version for the case of Bessel functions of the well known relationship between cosine and sine:

$$\cos^2(x) + \sin^2(x) = 1.$$

For $J_n(x)$ and $Y_n(x)$ Bessel functions of order n of the first and second kinds, respectively, the Nicholson's integral is given as

$$J_n^2(x) + Y_n^2(x) = \frac{8}{\pi^2} \int_0^\infty K_0(2x \sinh(t)) \cosh(2nt) dt,$$

where $K_0(x)$ is the modified Bessel function of the second kind, $\sinh(x)$ and $\cosh(x)$ are the hyperbolic sine and cosine functions respectively. The Nicholson's integral is proved Watson (1922) for $x \in \mathbb{C}_+$ using Hardy's theory of generalised integrals developed by Hardy (1903) and Cauchy's integral theorem which states that

$$\int_C f(z) dz = 0,$$

for $f : U \to \mathbb{C}$ a holomorphic function with $U \subseteq \mathbb{C}$ and C is a smooth closed curve in U. Dixon and Ferrar (1930) proved the Nicholson's integral using a transformation following the method in Nicholson (1910) for all values of n if $\Re(x) > 0$ and when x is purely imaginary for $\Re(n) < \frac{3}{4}$. Durand (1975) extended the result for Nicholson's integral for general Gegenbauer and Legendre functions. The functions, $C_\lambda^{(\alpha)}(x)$ and $D_\lambda^{(\alpha)}(x)$, the Gegenbauer functions of the first and second kind solves the differential equation of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - (2\alpha + 1)x\frac{dy}{dx} + \lambda(\lambda + 2\alpha)y = 0,$$

for arbitrary α and λ and $x \in \mathbb{C}$. The generalisation for Gegenbauer functions obtained by Durand (1975) reads

$$\begin{aligned}
&(1 - x^2)^n \left([C_{\lambda-n}^{(\alpha+n)}(x)]^2 + [D_{\lambda-n}^{(\alpha+n)}(x)]^2 \right) \\
&= 2^{-2\alpha-2n+3} e^{-i\pi\alpha} \frac{\Gamma(2\alpha - 1)\Gamma(n + 1)\Gamma(\lambda + 2\alpha + n)}{[\Gamma(\alpha + n)]^2 \Gamma(n + 2\alpha - 1)\Gamma(\lambda - n + 1)} \\
&\quad \cdot \int_1^\infty D_\lambda^{(\alpha)}(x^2 + (1 - x^2)z) C_n^{(\alpha-\frac{1}{2})}(z)(z^2 - 1)^{\alpha-1} dz,
\end{aligned}$$

for $\Re(\lambda - n + 1) > 0$ and $\Re(\alpha) > 0$. For $P_n(x)$ and $Q_n(x)$, the Legendre function of the first and second kind, the Legendre differential equation is

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0,$$

where n is the degree of the Legendre function. The generalisations for ordinary Legendre functions derived by Durand (1975) for $-1 < x < 1$ is

$$P_n^2(x) + \frac{4}{\pi^2}Q_n^2(x) = \frac{4}{\pi^2}\int_1^{\infty} Q_n(x^2 + (1-x^2)z)\frac{1}{\sqrt{z^2-1}}dz.$$

We end the chapter by constructing a conditioned Brownian motion and we proceed to find the Laplace transform of its first exit time as well as density function with the help of the symmetry methods for differential equation established.

Chapter 2

Occupation Times of Brownian Motion

2.1 Introduction

The Bachelier model presented by Louis Bachelier in his PhD thesis is an option pricing model which marks the birth of modern mathematical finance. His results received attention after Chicago Mercantile Exchange (CME) and Intercontinental Exchange (ICE) decided to adopt the Bachelier model to model oil and natural gas options after the price of the oil future dropped to negative for the first time in history due to the lack of demand owing to the spread of Covid-19. In 1973, Black and Scholes developed the much celebrated Black & Scholes model (see Black and Scholes (1973), Merton (1973)) for the problems of pricing derivative securities which are closely related to the the problems of finding the distributions of measurable functional of Brownian motion such as the first passage time, maximum and minimum functionals. The distribution of the maximum and minimum of Brownian motion is well explored and widely used in the pricing of certain path dependent options such as the barrier option and lookback option. For more details regarding the pricing of Barrier option, see Rich (1994), Kunitomo and Ikeda (1992), and Goldman et al. (1979a), Goldman et al. (1979b) and Conze (1991) for lookback options.

Consider $X = \{X_t,\ t \geq 0\}$ a Brownian motion as given in Definition 1.0.1. The occupation time of X is a well studied functional defined as

$$\Gamma_t(B) := \int_0^t \mathbb{1}_B(X_s)ds = \text{meas}\{0 \le s \le t : X_s \in B\}, \qquad 0 \le t \le \infty,$$

for every fixed Borel set $B \in \mathcal{B}(\mathbb{R})$ and meas denotes Lebesgue measure. The occupation time of Brownian motion measures the amount of time up till a deterministic time t that the Brownian motion stays in the set B. The resulting process $\Gamma(B) = \{\Gamma_t(B),\ t \ge 0\}$ is adapted and continuous. For more details regarding the Brownian occupation time, see Karatzas and Shreve (1991) and Borodin and Salminen (2002). For occupation time for more general diffusion processes, see Pitman and Yor (2003). The study of the occupation time process finds applications in the pricing of the α-quantile option first introduced by Miura (1992) and the pricing of this option is investigated using Feynman-Kac formula by Akahori (1995) who derived the explicit form of the distribution function of the occupation time of a Brownian motion and Dassios (1995) who showed the identity in law between the sum of max and min of independent Brownian motions and the Brownian quantiles.

The occupation time process also received attention in the study of in the study of ruin problems for general diffusion processes where the ruin probabilities can be expressed in terms of the occupation times. The idea of Parisian ruin introduced by Dassios and Wu (2008), is a special type of ruin that takes into consideration that companies may be able to withstand some periods of negative surplus before experiencing bankruptcy. This concept is extended to an insurance risk model with underlying spectrally negative Lévy process of bounded variation in Landriault et al. (2010) and to an omega risk process in Li and Zhou (2013).

In this chapter, we study the occupation time of the Brownian path over different regions. We define the following occupation times for a pre-determined level $a \in \mathbb{R}_+$:

$$\begin{aligned} Z_t^{(1)} &= \int_0^t \mathbb{1}_{\{X_s > a\}} ds, \\ V_t &= \int_0^t \mathbb{1}_{\{-a < X_s < a\}} ds, \\ Z_t^{(2)} &= \int_0^t \mathbb{1}_{\{X_s < -a\}} ds, \end{aligned} \tag{2.1}$$

and let us define τ as the right-continuous inverse of the occupation time $Z_t^{(1)}$ for $z \in \mathbb{R}_+$:

$$\tau := \inf\{t : \; Z_t^{(1)} = z\}, \tag{2.2}$$

using the usual convention $\inf(\emptyset) = \infty$. We continue to derive the joint Laplace transform and joint density function of the occupation time processes in the region above the level a, between the levels $-a$ and a and their driving Brownian motion evaluated at τ. Our results can be applied to the study of insurance risk models which will be discussed later.

2.2 Preliminaries

Let an adapted stochastic process $X = \{X_t : t \geq 0\}$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a Markov process (with continuous paths) taking values in measurable space $(E, \mathcal{E})$. For simplicity, we take $E = \mathbb{R}$ and $\mathcal{E} = \mathcal{B}$ the Borel σ-algebra on $\mathbb{R}$. Then, the infinitesimal generator of X applied to a function $f : \; \mathbb{R} \to \mathbb{R}$ is defined as the operator $\mathcal{A}$ such that:

$$\mathcal{A}f(x) := \lim_{t \downarrow 0} \frac{\mathbb{E}_x\left[f(X_t)\right] - f(x)}{t}, \qquad \forall x \in \mathbb{R},$$

where $\mathbb{E}_x\left[\cdot\right] = \mathbb{E}\left[\cdot|\mathcal{F}_0\right] = \mathbb{E}\left[\cdot|X_0 = x\right]$ by the Markov property. This relation holds for every f in a suitable subclass of the space $\mathcal{C}^2(\mathbb{R})$ of real-valued, twice continuously differentiable functions on $\mathbb{R}$. From the theory of Markov processes in Dynkin (1965). The infinitesimal generator $\mathcal{A}$ applied to function f is a second order differential operator given by

$$\mathcal{A}f(x) = b(x)\frac{\partial f(x)}{\partial x} + \frac{1}{2}a(x)\frac{\partial^2 f(x)}{\partial x^2},$$

for suitable Borel-measureable functions $b, a : \mathbb{R} \to \mathbb{R}$. For X a Brownian motion with $b(x) = 0$ and $a(x) = 1$, we see that X is a standard one-dimensional Brownian motion introduced in Definition 1.0.1. This process can be defined as a linear diffusion on $\mathbb{R}$ and for any $f \in \mathcal{C}_b^2(\mathbb{R})$, the infinitesimal generator of X is given by

$$\mathcal{A}f(x) := \frac{1}{2}\partial_{xx} f(x).$$

Remark 2.2.1. *A simple application of Itô formula implies that*

$$f(X_t) - \int_0^t \mathcal{A}f(X_s)ds,$$

is a martingale. This gives us an approach to identify martingale candidates of the form $f(X_t)$ by solving

$$\mathcal{A}f = 0,$$

subject to some boundary conditions.

2.3 Construction

In order to find a martingale of the form

$$f(X_t, V_t, Z_t^{(1)}, Z_t^{(2)}),$$

for $X, V, Z^{(1)}, Z^{(2)}$ as defined in (2.1), we use the martingale approach discussed in Remark 2.2.1. Consider a function $f : \mathbb{R}^4 \to \mathbb{R}$ such that its domain X is a subset of $\mathbb{R}^4$ that contains a nonempty open set. The infinitesimal generator $\mathcal{A}$ acting on the function f in its domain is such that

$$\begin{aligned}\mathcal{A}f(x, v, z_1, z_2) &= \mathbb{1}_{\{x>a\}}\frac{\partial f(x, v, z_1, z_2)}{\partial z_1} + \mathbb{1}_{\{-a<x<a\}}\frac{\partial f(x, v, z_1, z_2)}{\partial v} + \mathbb{1}_{\{x<-a\}}\frac{\partial f(x, v, z_1, z_2)}{\partial z_2} \\ &\quad + \frac{1}{2}\frac{\partial^2 f(x, v, z_1, z_2)}{\partial x^2} \\ &= \begin{cases} \frac{\partial f_1(x,v,z_1,z_2)}{\partial z_1} + \frac{1}{2}\frac{\partial^2 f_1(x,v,z_1,z_2)}{\partial x^2}, & x > a, \\ \frac{\partial f_2(x,v,z_1,z_2)}{\partial v} + \frac{1}{2}\frac{\partial^2 f_2(x,v,z_1,z_2)}{\partial x^2}, & -a < x < a, \\ \frac{\partial f_3(x,v,z_1,z_2)}{\partial z_2} + \frac{1}{2}\frac{\partial^2 f_3(x,v,z_1,z_2)}{\partial x^2}, & x < -a. \end{cases}\end{aligned} \tag{2.3}$$

Trying a solution of the following form

$$f_i(x, v, z_1, z_2) = e^{-\beta_1 v}\, e^{-\beta_2 z_2}\, e^{\gamma z_1} f_i(x); \qquad i = 1, 2, 3 \tag{2.4}$$

for $\beta_1, \beta_2, \gamma \in \mathbb{C}_+$. We solve $\mathcal{A}f = 0$, which gives us the following ordinary differential equations:

$$\begin{cases} 0 = \gamma f_1(x) + \frac{1}{2} f_1''(x), & x > a, \\ 0 = -\beta_1 f_2(x) + \frac{1}{2} f_2''(x), & -a < x < a, \\ 0 = -\beta_2 f_3(x) + \frac{1}{2} f_3''(x), & x < -a. \end{cases} \tag{2.5}$$

Solving the differential equations gives

$$\begin{aligned} f_1(x) &= A_1 \cos(\sqrt{2\gamma} x) + A_2 \sin(\sqrt{2\gamma} x), \\ f_2(x) &= B_1 e^{\sqrt{2\beta_1} x} + B_2 e^{-\sqrt{2\beta_1} x}, \\ f_3(x) &= C e^{\sqrt{2\beta_2} x}, \end{aligned} \tag{2.6}$$

where $A_1, A_2, B_1, B_2, C \in \mathbb{R}$ are some constants to be determined subject the boundary conditions for the functions $f_i \in \mathcal{C}^2(\mathbb{R}), i = 1, 2, 3$:

$$\begin{cases} f_1(a) &= f_2(a), \\ f_1'(a) &= f_2'(a), \\ f_2(-a) &= f_3(-a), \\ f_2'(-a) &= f_3'(-a). \end{cases} \tag{2.7}$$

Following Remark 2.2.1, we obtain martingales of the form

$$f_i(X_t, V_t, Z_t^{(1)}, Z_t^{(2)}) = e^{-\beta_1 V_t} e^{-\beta_2 Z_t^{(2)}} e^{\gamma Z_t^{(1)}} f_i(X_t), \qquad i = 1, 2, 3 \tag{2.8}$$

where the functions $f_i(z)$ are derived as

$$\begin{aligned} f_1(x) &= \left(e^{\sqrt{2\beta_1} a} + e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) B_1 \cos(\sqrt{2\gamma}[x - a]) \\ &\quad + \left(e^{\sqrt{2\beta_1} a} - e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) \frac{\sqrt{2\beta_1}}{\sqrt{2\gamma}} B_1 \sin(\sqrt{2\gamma}[x - a]), \\ f_2(x) &= B_1 e^{\sqrt{2\beta_1} x} + \left(B_1 e^{-2\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) e^{-\sqrt{2\beta_1} x}, \\ f_3(x) &= B_1 e^{-\sqrt{2\beta_1} a} e^{\sqrt{2\beta_2}(a + x)} \left(\frac{2\sqrt{2\beta_1}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right). \end{aligned} \tag{2.9}$$

Note that we could have obtained the same equations more easily by having

$$
\begin{aligned}
f_1(x) &= A_1 \cos(\sqrt{2\gamma}[x-a]) + A_2 \sin(\sqrt{2\gamma}[x-a]), \\
f_2(x) &= B_1 e^{\sqrt{2\beta_1}x} + B_2 e^{-\sqrt{2\beta_1}x}, \\
f_3(x) &= Ce^{\sqrt{2\beta_2}[x+a]},
\end{aligned}
$$

with the same boundary conditions as above.

Remark 2.3.1. *Substituting (2.6) into (2.7), we obtain an under-determined system of equations with 5 parameters and 4 equations. Setting the free variable to an arbitrary value (say 1) as we have in Appendix 2.9.1, we can obtain the rest of the required parameters.*

2.4 Distributional Properties

In this section, we explore the relationship between V, $Z^{(2)}$ and X evaluated at time τ by deriving first the joint density of V and $Z^{(2)}$ from the inversion of the double Laplace transform. We then derive the joint density of V, $Z^{(2)}$ and X at time τ by inverting the triple Laplace transform. We look at the three different cases when we have the starting point $X_0 = x$ to be below the level $-a$, above the level a and finally between the levels $-a$ and a.

2.5 Case 1 : $X_0 = x < -a$

We look at the first case where the starting point X_0 is below the level $-a$.

2.5.1 Joint distribution of V_τ and $Z_\tau^{(2)}$

Lemma 2.5.1. *For $\beta_1, \beta_2 \in \mathbb{C}_+$ and the first hitting time τ defined as*

$$\tau := \inf\{t : \; Z_t^{(1)} = z\}, \tag{2.10}$$

for $z \in \mathbb{R}_+$, the joint Laplace transform of V_τ and $Z_\tau^{(2)}$ can be derived as

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= 2\sqrt{\frac{2}{\pi}} \frac{e^{-2\sqrt{2\beta_1}a} e^{\sqrt{2\beta_2}(a+x)}}{\left(1-e^{-4\sqrt{2\beta_1}a}\right)} \int_0^\infty e^{-\frac{1}{2}s^2} \exp\left(-s\frac{\sqrt{2z}\beta_1(1-h^2(\beta_1))}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right) \frac{e^{-s\sqrt{2z\beta_1}h(\beta_1)}\sqrt{\beta_1}}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}} ds,
\end{aligned}
\tag{2.11}
$$

where the function $h(\beta_1)$ is

$$
h(\beta_1) = \frac{1+e^{-4\sqrt{2\beta_1}a}}{1-e^{-4\sqrt{2\beta_1}a}} = \frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}, \tag{2.12}
$$

and $\cosh(x) = \frac{e^x+e^{-x}}{2}$ *is the hyperbolic cosine function and* $\sinh(x) = \frac{e^x-e^{-x}}{2}$ *is the hyperbolic sine function.*

Proof. The proof is in the appendix (2.9.1). $\square$

Theorem 2.5.2. *The joint density of V and $Z^{(2)}$ evaluated at τ is*

$$
\begin{aligned}
&\mathbb{P}\left[V_\tau \in dm, Z_\tau^{(2)} \in dy\right] \\
&= \frac{1}{2\pi\sqrt{y^3}} \int_0^\infty \int_{\sqrt{2}(-a-x)}^\infty e^{-\frac{1}{2}s^2} t e^{-\frac{t^2}{4y}} \sum_{k=0}^\infty \sum_{l=0}^\infty \frac{\left(s\frac{\sqrt{z}}{\sqrt{2}}\left[t+\sqrt{2}(a+x)\right]\right)^k}{\Gamma(k+1)k!} \frac{(-1)^l}{l!} 2^{1+2k+l} \\
&\quad \cdot \sum_{j=0}^\infty \frac{\Gamma(1+2k+l+j)}{\sqrt{2\pi}m^{1+\frac{1+2k+l}{2}}\Gamma(1+2k+l)j!} \exp\left(-\frac{[(1+k+l+j)(4a)+s\sqrt{z}+x+\frac{t}{\sqrt{2}}-a]^2}{4m}\right) \\
&\quad \cdot D_{2+2k+l}\left(\frac{(1+k+l+j)4a+s\sqrt{z}+x+\frac{t}{\sqrt{2}}-a}{\sqrt{m}}\right) dtds,
\end{aligned}
$$

with the conditions

$$
2a > 0, \qquad 2a + \frac{\sqrt{2}(s\sqrt{z}+a+x)+t}{\sqrt{2}} > 0,
$$

where $D_n(x)$ is a parabolic cylinder function.

Proof. There are two steps to this proof. The first step is to consider inverting the Laplace transform in (2.11) with respect to the parameter β_2. Letting $\mathcal{L}_{\beta_2}^{-1}$ denote the inverse Laplace transform operator with respect to β_2, we want to compute

$$
\mathcal{L}_{\beta_2}^{-1}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right]\right)(y).
$$

For any $\beta > 0$, the following inverse Laplace transform holds

$$\mathcal{L}_{\beta}^{-1}\left(\frac{1}{\beta}e^{-\frac{s\sqrt{2z}\beta_1(1-h^2(\beta_1))}{\beta}}\right)(y) = I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1))y}\right). \tag{2.13}$$

Using a change of variable of the form $\beta + \sqrt{\beta_1}h(\beta_1)$ as the argument of the inverse Laplace gives

$$\mathcal{L}_{\beta}^{-1}\left(F_1\left[\beta + \sqrt{\beta_1}h(\beta_1)\right]\right)(y) = e^{-\sqrt{\beta_1}h(\beta_1)y}I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1))y}\right), \tag{2.14}$$

where the function F_1 is defined as

$$F_1(x) = \frac{1}{x}e^{-\frac{s\sqrt{2z}\beta_1(1-h^2(\beta_1))}{x}}.$$

We can add the term $e^{\sqrt{\beta_2}\sqrt{2}(a+x)} = e^{-\sqrt{\beta_2}\sqrt{2}(-a-x)}$ to the inversion and proceed as follows:

$$\begin{aligned}
&\mathcal{L}_{\beta}^{-1}\left(e^{-\sqrt{2}(-a-x)\beta}F_2(\beta)\right)(y)\\
&= e^{-\sqrt{\beta_1}h(\beta_1)[y-\sqrt{2}(-a-x)]}I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1)[y-\sqrt{2}(-a-x)])}\right) y > \sqrt{2}(-a-x)\\
&= e^{-\sqrt{\beta_1}h(\beta_1)[y+\sqrt{2}(a+x)]}I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1)[y+\sqrt{2}(a+x)])}\right)\mathbb{1}_{\{y>\sqrt{2}(-a-x)\}},
\end{aligned} \tag{2.15}$$

where the function F_2 is defined as

$$F_2(x) = F_1(x + \sqrt{2\beta_1}h(\beta_1)) = \frac{1}{x+\sqrt{2\beta_1}h(\beta_1)}e^{-\frac{s\sqrt{2z}\beta_1(1-h^2(\beta_1))}{x+\sqrt{2\beta_1}h(\beta_1)}},$$

with the condition $\sqrt{2}(-a-x) > 0$. We then take into account the square root term by considering:

$$\begin{aligned}
&\mathcal{L}_{\beta_2}^{-1}\left(F_3(\sqrt{\beta})\right)(y)\\
&= \frac{1}{2\sqrt{\pi y^3}}\int_0^{\infty} te^{-\frac{t^2}{4y}}e^{-\sqrt{\beta_1}h(\beta_1)[t+\sqrt{2}(a+x)]}\mathbb{1}_{\{t>\sqrt{2}(-a-x)\}}\\
&\quad\cdot I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1)[t+\sqrt{2}(a+x)])}\right)dt,
\end{aligned} \tag{2.16}$$

where the function F_3 is defined as

$$F_3(x) = e^{-\sqrt{2}(-a-x)x}F_2(x).$$

Therefore, we have that

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_2}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z^{(2)}_\tau}\right]\right)(y)\\
&=\frac{4}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\frac{1}{2}s^2}e^{-s\sqrt{2z\beta_1}h(\beta_1)}e^{-\sqrt{2\beta_1}a}\sqrt{\beta_1}}{e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a}}\\
&\quad\cdot\mathcal{L}^{-1}_{\beta_2}\left(\exp\left(-\frac{s\sqrt{2z}\beta_1(1-h^2(\beta_1))}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right)\frac{e^{\sqrt{2\beta_2}(a+x)}}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right)(y)ds\\
&=\frac{\sqrt{2}}{\pi\sqrt{y^3}}\int_0^\infty\int_0^\infty \frac{e^{-\frac{1}{2}s^2}e^{-s\sqrt{2z\beta_1}h(\beta_1)}e^{-\sqrt{2\beta_1}a}\sqrt{\beta_1}}{e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a}}\;te^{-\frac{t^2}{4y}}e^{-\sqrt{\beta_1}h(\beta_1)[t+\sqrt{2}(a+x)]}\\
&\quad\cdot I_0\left(2i\sqrt{s\sqrt{2z}\beta_1(1-h^2(\beta_1)[t+\sqrt{2}(a+x)])}\right)\mathbb{1}_{\{t>\sqrt{2}(-a-x)\}}dtds,
\end{aligned}\tag{2.17}$$

where in the first equality we exchange the order of inverse Laplace transform and integration. Recall that the inverse Laplace transform of a function F is given as the Bromwich integral:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{st}F(s)ds,$$

where $c\in\mathbb{R}$ is chosen such that it is greater than the real parts of all the possible singularities of $F(s)$ on the complex plane.

The next step is to invert expression (2.17) with respect to β_1 in order to obtain the joint density function. Using the following inversion

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_1}\left(\frac{1}{2\sqrt{2}}e^{-\frac{\sqrt{2}(s\sqrt{z}+a+x)+t}{\sqrt{2}}\frac{\sqrt{2\beta_1}\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}}\frac{\sqrt{2\beta_1}}{\sinh(2a\sqrt{2\beta_1})}\right.\\
&\qquad\left.\cdot I_0\left(-2\sqrt{s\sqrt{\frac{z}{2}}[t+\sqrt{2}(a+x)]\frac{\sqrt{2\beta_1}}{\sinh(2a\sqrt{2\beta_1})}}\right)\right)(m)\\
&=\frac{1}{2\sqrt{2}}\sum_{l=0}^\infty\frac{\left(s\frac{\sqrt{z}}{\sqrt{2}}[t+\sqrt{2}(a+x)]\right)^l}{\Gamma(l+1)l!}\mathrm{es}_m\left(1+2l,1+2l,2a,0,\frac{\sqrt{2}(s\sqrt{z}+a+x)+t}{\sqrt{2}}\right),
\end{aligned}\tag{2.18}$$

with the conditions

$$2a>0,\qquad 2a+\frac{\sqrt{2}(s\sqrt{z}+a+x)+t}{\sqrt{2}}>0,$$

where the function es_m is such that for $t > 0, \nu \geq 0, \nu t + z > 0$ and $\nu t + x + z > 0$,

- $\mathrm{es}_y(\mu, \nu, t, x, z) = \displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \, s_y(\mu + k, \nu + k, t, x + z + kt),$
- $s_t(\mu, \nu, t, z) \quad = 2^{\nu} \displaystyle\sum_{k=0}^{\infty} \frac{\Gamma(\nu + k) \, e^{-\frac{(\nu t + z + 2kt)^2}{4y}}}{\sqrt{2\pi} y^{1+\frac{\mu}{2}} \Gamma(\nu) k!} \, D_{\mu+1} \left(\frac{\nu t + z + 2kt}{\sqrt{y}} \right),$
- $D_n(x) \quad = 2^{-\frac{n}{2}} \, e^{-\frac{x^2}{4}} \, H_n \left(\frac{x}{\sqrt{2}} \right) = e^{-\frac{x^2}{4}} \mathrm{He}_n(x),$

where He is the modified Hermite function. We can invert $\mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \right]$ twice with respect to β_2 and β_1 respectively to compute the joint density. □

2.5.2 Joint distribution of V_τ, $Z_\tau^{(2)}$ and X_τ

Lemma 2.5.3. *For $\beta_1, \beta_2, \xi \in \mathbb{C}_+$ and the first hitting time τ as defined in (2.10), the joint Laplace transform of V_τ, $Z_\tau^{(2)}$ and X_τ can be derived as*

$$
\begin{aligned}
&\mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi [X_\tau - a]} \right] \\
&= \frac{4\xi e^{-\sqrt{2\beta_1} a} e^{\sqrt{2\beta_2}(a+x)} \sqrt{2\beta_1} \; e^{\xi^2 \frac{z}{2}} \Phi(-\xi \sqrt{z})}{H(\xi, \beta_1, \beta_2)} \\
&\quad - \frac{\sqrt{2\beta_1} [\tilde{H}_2(\beta_1, \beta_2)]}{H(\xi, \beta_1, \beta_2)} \frac{4}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} \exp\left(-s\sqrt{z} \frac{2\beta_1(1 - h^2(\beta_1))}{\sqrt{2\beta_1} h(\beta_1) + \sqrt{2\beta_2}} \right) \\
&\quad \cdot \frac{e^{-s\sqrt{2z\beta_1} h(\beta_1)} e^{-\sqrt{2\beta_1} a} e^{\sqrt{2\beta_2}(a+x)} \sqrt{2\beta_1}}{\tilde{H}_1(\beta_1, \beta_2)} ds,
\end{aligned}
\tag{2.19}
$$

where the functions $h, H, \tilde{H}_1$ and $\tilde{H}_2$ are such that

$$
\begin{aligned}
h(\beta_1) \quad &= \frac{1 + e^{-4\sqrt{2\beta_1} a}}{1 - e^{-4\sqrt{2\beta_1} a}}, \\
H(\xi, \beta_1, \beta_2) &= \xi \tilde{H}_1(\beta_1, \beta_2) - \sqrt{2\beta_1} \tilde{H}_2(\beta_1, \beta_2), \\
\tilde{H}_1(\beta_1, \beta_2) \quad &= e^{\sqrt{2\beta_1} a} (\sqrt{2\beta_1} + \sqrt{2\beta_2}) + e^{-3\sqrt{2\beta_1} a} (\sqrt{2\beta_1} - \sqrt{2\beta_2}), \\
\tilde{H}_2(\beta_1, \beta_2) \quad &= e^{\sqrt{2\beta_1} a} (\sqrt{2\beta_1} + \sqrt{2\beta_2}) - e^{-3\sqrt{2\beta_1} a} (\sqrt{2\beta_1} - \sqrt{2\beta_2}).
\end{aligned}
$$

Proof. From (2.37), we can see that the expression for the triple Laplace transform is derived in terms of the double Laplace transform (2.11):

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}e^{-\xi[X_\tau - a]}\right]\\
&= \frac{4\xi e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}\, e^{\xi^2\frac{z}{2}}\Phi(-\xi\sqrt{z})}{H(\xi,\beta_1,\beta_2)}\\
&\quad - \frac{\sqrt{2\beta_1}\left[e^{\sqrt{2\beta_1}a}(\sqrt{2\beta_1}+\sqrt{2\beta_2}) - e^{-3\sqrt{2\beta_1}a}(\sqrt{2\beta_1}-\sqrt{2\beta_2})\right]\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}\right]}{H(\xi,\beta_1,\beta_2)}.
\end{aligned}
$$

Substituting the expression derived for the double Laplace transform (2.11), we can derive the expression for the triple Laplace transform. □

Theorem 2.5.4. *The joint density of V_τ, $Z_\tau^{(2)}$ and X_τ can be obtained as*

$$
\begin{aligned}
&\mathbb{P}\left[V_\tau \in dp, Z_\tau^{(2)} \in dm, X_\tau \in dy\right]\\
&= \int_0^\infty \frac{2e^{-\frac{1}{2}s^2}}{\sqrt{2m\pi^3 p^5}}\delta(y - s\sqrt{2z})e^{-\frac{(a+x)^2}{2m}}\sum_{k=0}^{\infty}[(2k+1)^2(2a)^2 - p]e^{-\frac{(2k+1)^2(2a)^2}{2p}}ds\\
&\quad - \int_0^\infty \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}}\delta(y - s\sqrt{z})\int_0^\infty e^{-\frac{(a+x)^2}{2m}}e^{-r^2}e^{\frac{\sqrt{2}r(a+x)}{\sqrt{m}}}\sum_{k=0}^{\infty}\frac{(-\sqrt{2m}r)^k}{\sqrt{\pi}k!}\\
&\quad \cdot\Big[\tilde{S}_p(2+k, 2+k, 2a, -2a+\sqrt{2m}s+2ka)\\
&\qquad\quad +\tilde{S}_p(2+k, 2+k, 2a, 2a+\sqrt{2m}s+2ka)\Big]\,drds,
\end{aligned}
$$

where the function $\tilde{S}$ is such that for $t>0, \nu\geq 0, \nu t + z > 0$:

$$
\tilde{S}_y(\mu,\nu,t,z) = 2^\nu\sum_{k=0}^{\infty}\frac{\Gamma(\nu+k)e^{-\frac{(\nu t+z+2kt)^2}{4y}}}{\sqrt{2\pi}y^{1+\frac{\mu}{2}}\Gamma(\nu)k!}D_{\mu+1}\left(\frac{\nu t + z + 2kt}{\sqrt{y}}\right).
$$

Proof. We focus on the first term of (2.19), which we define as $g_1(\xi,\beta_1,\beta_2)$.

$$
\begin{aligned}
&g_1(\xi,\beta_1,\beta_2)\\
&= \frac{4\xi e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}\, e^{\xi^2\frac{z}{2}}\Phi(-\xi\sqrt{z})}{H(\xi,\beta_1,\beta_2)}\\
&= \frac{4\xi e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}\, e^{\xi^2\frac{z}{2}}\Phi(-\xi\sqrt{z})}{\tilde{H}_1(\beta_1,\beta_2)}\cdot\frac{1}{\xi - \sqrt{2\beta_1}\left(\frac{\tilde{H}_2(\beta_1,\beta_2)}{\tilde{H}_1(\beta_1,\beta_2)}\right)} \qquad (2.20)\\
&= \int_0^\infty \frac{4\xi e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}e^{-\frac{1}{2}s^2}e^{-\xi s\sqrt{z}}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)}\cdot\frac{1}{\xi - \sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)}ds.
\end{aligned}
$$

We will proceed to invert $g(\xi, \beta_1, \beta_2)$ with respect to ξ, β_2 and finally β_1. The inversion with respect to ξ gives us

$$\begin{aligned}
&\mathcal{L}_\xi^{-1}\left(\int_0^\infty \frac{4e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)} \cdot \frac{\xi e^{-\xi s\sqrt{z}}}{\xi - \sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)}ds\right)(y) \\
&= \mathcal{L}_\xi^{-1}\left(\int_0^\infty \frac{4e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)} e^{-\xi s\sqrt{z}}ds\right)(y) \\
&\quad + \mathcal{L}_\xi^{-1}\left(\int_0^\infty \frac{4e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)} \frac{e^{-\xi s\sqrt{z}}\sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)}{\xi - \sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)}ds\right)(y) \\
&= \int_0^\infty \frac{4e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)}\delta(y - s\sqrt{z})ds \\
&\quad + \int_0^\infty \frac{4e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}(2\beta_1)e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)} \frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}} \\
&\quad \cdot e^{\sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)(y-s\sqrt{z})}\mathbb{1}_{\{y>s\sqrt{z}\}}ds,
\end{aligned} \tag{2.21}$$

where $\delta(x)$ is the Dirac Delta function. We then proceed with the inversion with respect to β_2:

$$\begin{aligned}
&\mathcal{L}_{\beta_2}^{-1}\left(\int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)}\delta(y - s\sqrt{z})ds\right)(m) \\
&\quad + \mathcal{L}_{\beta_2}^{-1}\left(\int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}(2\beta_1)\mathbb{1}_{\{y>s\sqrt{z}\}}}{\sqrt{2\pi}\tilde{H}_1(\beta_1,\beta_2)} \frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right. \\
&\quad \left.\cdot e^{\sqrt{2\beta_1}\left(\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right)[y-s\sqrt{z}]}ds\right)(m).
\end{aligned}$$

Exchanging the order of integration and inversion gives:

$$\begin{aligned}
&\int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}\sqrt{2\beta_1}}{\sqrt{2\pi}} \frac{\delta(y - s\sqrt{z})}{\sqrt{2}(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a})}\mathcal{L}_{\beta_2}^{-1}\left(\frac{e^{\sqrt{\beta_2}\sqrt{2}(a+x)}}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right)(m)ds \\
&\quad + \int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}(2\beta_1)}{\sqrt{2\pi}} \frac{\mathbb{1}_{\{y>s\sqrt{z}\}}h(\beta_1)}{\sqrt{2}(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a})}e^{\sqrt{2\beta_1}h(\beta_1)[y-s\sqrt{z}]} \\
&\quad \cdot \mathcal{L}_{\beta_2}^{-1}\left(\frac{e^{\sqrt{2\beta_2}(a+x)}}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}e^{\frac{\sqrt{2}\beta_1[1-h^2(\beta_1)](y-s\sqrt{z})}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}}\right)(m)ds \\
&\quad + \int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}(2\beta_1)}{\sqrt{2\pi}}\mathbb{1}_{\{y>s\sqrt{z}\}} \frac{\sqrt{\beta_1}[1-h^2(\beta_1)]}{\sqrt{2}(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a})}e^{\sqrt{2\beta_1}h(\beta_1)[y-s\sqrt{z}]} \\
&\quad \cdot \mathcal{L}_{\beta_2}^{-1}\left(\frac{e^{\sqrt{2\beta_2}(a+x)}}{(\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2})^2}e^{\frac{\sqrt{2}\beta_1[1-h^2(\beta_1)](y-s\sqrt{z})}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}}\right)(m)\,ds.
\end{aligned}$$

Finally, we can obtain the inversion with respect to β_2 is obtained as:

$$\int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}\sqrt{2\beta_1}}{\sqrt{2\pi}} \frac{\delta(y-s\sqrt{z})}{\sqrt{2}(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a})} \frac{e^{-\frac{(a+x)^2}{2m}}}{\sqrt{\pi m}} ds$$
$$- \int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}\sqrt{2\beta_1}}{\sqrt{2\pi}} \frac{\delta(y-s\sqrt{z})}{\sqrt{2}(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a})} \sqrt{\beta_1}h(\beta_1)$$
$$\cdot e^{-\sqrt{2\beta_1}h(\beta_1)(a+x)}e^{\beta_1 h^2(\beta_1)m}\mathrm{erfc}\left(h(\beta_1)\sqrt{\beta_1 m}-\frac{(a+x)}{\sqrt{2m}}\right) ds$$
$$+ \int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}(2\beta_1)}{\sqrt{2\pi}} \frac{h(\beta_1)e^{\sqrt{2\beta_1}h(\beta_1)[y-s\sqrt{z}]}}{\sqrt{2}(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a})} \int_0^\infty \frac{te^{-\frac{t^2}{4m}}e^{-\sqrt{\beta_1}h(\beta_1)[t+\sqrt{2}(a+x)]}}{2\sqrt{\pi m^3}}$$
$$\cdot I_0\left(2i\sqrt{\sqrt{2}[s\sqrt{z}-y]\beta_1(1-h^2(\beta_1))[t+\sqrt{2}(a+x)]}\right) \mathbb{1}_{\{t>-\sqrt{2}(a+x)\}}\mathbb{1}_{\{y>s\sqrt{z}\}} dtds$$
$$+ \int_0^\infty \frac{4e^{-\frac{1}{2}s^2}e^{-\sqrt{2\beta_1}a}(2\beta_1)}{\sqrt{2\pi}} \frac{\sqrt{\beta_1}[1-h^2(\beta_1)]e^{\sqrt{2\beta_1}h(\beta_1)[y-s\sqrt{z}]}}{\sqrt{2}(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a})} \int_0^\infty \frac{te^{-\frac{t^2}{4m}}e^{-\sqrt{\beta_1}h(\beta_1)[t+\sqrt{2}(a+x)]}}{2\sqrt{\pi m^3}}$$
$$\cdot \left(\frac{t+\sqrt{2}(a+x)}{-\sqrt{2}[s\sqrt{z}-y]\beta_1(1-h^2(\beta_1))}\right)^{\frac{1}{2}} I_1\left(2i\sqrt{\sqrt{2}[s\sqrt{z}-y]\beta_1(1-h^2(\beta_1))[t+\sqrt{2}(a+x)]}\right)$$
$$\cdot \mathbb{1}_{\{t>-\sqrt{2}(a+x)\}}\mathbb{1}_{\{y>s\sqrt{z}\}} dtds.$$

We then proceed to invert with respect to β_1 using the following relationships for $a>0$:

$$\mathcal{L}^{-1}_{\beta_1}\left(\frac{\sqrt{2\beta_1}}{2\sinh(2a\sqrt{2\beta_1})}\right)(p) = \frac{\sqrt{2}}{2\sqrt{\pi p^5}}\sum_{k=0}^\infty [(2k+1)^2(2a)^2-p]e^{-\frac{(2k+1)^2(2a)^2}{2p}},$$
$$\mathcal{L}^{-1}_{\beta_1}\left(\frac{\beta_1 h(\beta_1)e^{-2s\sqrt{\beta_1 m}h(\beta_1)}}{\sinh(2a\sqrt{2\beta_1})}\right)(p) = \sum_{k=0}^\infty \frac{(-\sqrt{2m}s)^k}{4k!}\tilde{S}_p(2+k,2+k,2a,-2a+\sqrt{2m}s+2ka)$$
$$+ \sum_{k=0}^\infty \frac{(-\sqrt{2m}s)^k}{4k!}\tilde{S}_p(2+k,2+k,2a,2a+\sqrt{2m}s+2ka),$$

where the function $\tilde{S}$ is such that for $t>0, \nu\geq 0, \nu t+z>0$:

$$\tilde{S}_y(\mu,\nu,t,z) = 2^\nu \sum_{k=0}^\infty \frac{\Gamma(\nu+k)e^{-\frac{(\nu t+z+2kt)^2}{4y}}}{\sqrt{2\pi}y^{1+\frac{\mu}{2}}\Gamma(\nu)k!} D_{\mu+1}\left(\frac{\nu t+z+2kt}{\sqrt{y}}\right),$$

and we use the following relationship:

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_1}\left(\frac{\sqrt{2\beta_1}e^{2\sqrt{2\beta_1}a}}{\sinh(2\sqrt{2\beta_1}a)}e^{-\frac{\sqrt{\beta_1}[t+\sqrt{2}(a+x+s\sqrt{z}-y)]}{\tanh(2\sqrt{2\beta_1}a)}}\right.\\
&\qquad\left.\cdot I_0\left(\frac{-2\sqrt{2\beta_1}}{\sinh(\sqrt{2\beta_1}2a)}\left[\frac{\sqrt{\sqrt{2}[s\sqrt{z}-y][t+\sqrt{2}(a+x)]}}{\sqrt{2}}\right]\right)\right)(p)\\
&=\sum_{k=0}^{\infty}\left(-\frac{\sqrt{\sqrt{2}[s\sqrt{z}-y][t+\sqrt{2}(a+x)]}}{\sqrt{2}}\right)^{2k}\frac{1}{\Gamma(k+1)k!}\\
&\qquad\cdot \mathrm{es}_p\left(1+2k,1+2k,2a,-2a,\frac{t+\sqrt{2}(a+x+s\sqrt{z}-y)}{\sqrt{2}}\right).
\end{aligned}\tag{2.22}$$

We can then finally obtain that

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_1}\left(\mathcal{L}^{-1}_{\beta_2}\left(\mathcal{L}^{-1}_{\xi}\left(g_1(\xi,\beta_1,\beta_2)\right)(y)\right)(m)\right)(p)\\
&=\int_0^\infty \frac{2e^{-\frac{1}{2}s^2}}{\sqrt{2m\pi^3p^5}}\delta(y-s\sqrt{2z})e^{-\frac{(a+x)^2}{2m}}\sum_{k=0}^{\infty}[(2k+1)^2(2a)^2-p]e^{-\frac{(2k+1)^2(2a)^2}{2p}}ds\\
&\quad-\int_0^\infty \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}}\delta(y-s\sqrt{z})\int_0^\infty e^{-\frac{(a+x)^2}{2m}}e^{-r^2}e^{\frac{\sqrt{2}r(a+x)}{\sqrt{m}}}\sum_{k=0}^{\infty}\frac{(-\sqrt{2m}r)^k}{\sqrt{\pi}k!}\\
&\qquad\cdot\tilde{S}_p(2+k,2+k,2a,-2a+\sqrt{2m}s+2ka)drds\\
&\quad-\int_0^\infty \frac{e^{-\frac{1}{2}s^2}}{\sqrt{2\pi}}\delta(y-s\sqrt{z})\int_0^\infty e^{-\frac{(a+x)^2}{2m}}e^{-r^2}e^{\frac{\sqrt{2}r(a+x)}{\sqrt{m}}}\sum_{k=0}^{\infty}\frac{(-\sqrt{2m}r)^k}{\sqrt{\pi}k!}\\
&\qquad\cdot\tilde{S}_p(2+k,2+k,2a,2a+\sqrt{2m}s+2ka)drds\\
&\quad+\int_0^\infty\int_0^\infty \frac{2e^{-\frac{1}{2}s^2}}{\sqrt{\pi^2m^3}}te^{-\frac{t^2}{4m}}\mathbb{1}_{\{t>-\sqrt{2}(a+x)\}}(2.23)dtds\\
&\quad+\int_0^\infty\int_0^\infty \frac{2e^{-\frac{1}{2}s^2}}{\sqrt{\pi^2m^3}}te^{-\frac{t^2}{4m}}\mathbb{1}_{\{t>-\sqrt{2}(a+x)\}}(2.24)dtds,
\end{aligned}$$

where (2.23) is such that

$$\begin{aligned}
&\frac{1}{8}\frac{\sqrt{2}}{\sqrt{\pi p^5}}\sum_{k=0}^{\infty}\left((2k+1)^2(2a)^2-p\right)e^{-\frac{(2k+1)^2(2a)^2}{2p}} \\
&\quad *\sum_{k=0}^{\infty}\frac{\left(-\frac{\sqrt{\sqrt{2}[s\sqrt{z}-y][t+\sqrt{2}(a+x)]}}{\sqrt{2}}\right)^{2k}}{\Gamma(k+1)k!} \\
&\quad\cdot\left[\mathrm{es}_p\left(1+2k,1+2k,2a,-2a,\frac{t+\sqrt{2}(a+x+s\sqrt{z}-y)}{\sqrt{2}}\right)\right. \\
&\qquad\left.+\mathrm{es}_p\left(1+2k,1+2k,2a,2a,\frac{t+\sqrt{2}(a+x+s\sqrt{z}-y)}{\sqrt{2}}\right)\right],
\end{aligned} \tag{2.23}$$

and (2.24) is such that

$$\begin{aligned}
&\frac{(-1)}{4}\left(\frac{t+\sqrt{2}(a+x)}{\sqrt{2}[s\sqrt{z}-y]}\right)^{\frac{1}{2}}\cdot\frac{\sqrt{2}}{\sqrt{\pi p^5}}\sum_{k=0}^{\infty}((2k+1)^2(2a)^2-p)e^{-\frac{(2k+1)^2(2a)^2}{2p}} \\
&\quad *\sum_{k=0}^{\infty}\frac{\left(-\frac{\sqrt{\sqrt{2}[s\sqrt{z}-y][t+\sqrt{2}(a+x)]}}{\sqrt{2}}\right)^{1+2k}}{\Gamma(2+k)k!} \\
&\quad\cdot\mathrm{es}_p\left(2+2k,2+2k,2a,0,\frac{t+\sqrt{2}(a+x+s\sqrt{z}-y)}{\sqrt{2}}\right),
\end{aligned} \tag{2.24}$$

with $*$ denoting convolution. We repeat the same steps to compute the inversion of the second term of (2.19) to finally obtain the joint density. □

2.6 Case 2: $X_0 = x > a$

We have looked at the case when we start in state 3 by setting $X_0 = x < -a$. In this section, we will look at the case when we start in state 1 by having $X_0 = x > a$.

2.6.1 Joint distribution of V_τ and $Z_\tau^{(2)}$

Lemma 2.6.1. *For $\beta_1, \beta_2 \in \mathbb{C}_+$ and the first hitting time τ as defined in (2.10), the joint Laplace transform of V_τ and $Z_\tau^{(2)}$ can be derived as*

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= \frac{2}{\sqrt{\pi}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-2s\frac{x-a}{\sqrt{2z}}} e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)} \exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right) ds \\
&\quad + erf\left(\frac{x-a}{\sqrt{2z}}\right),
\end{aligned} \tag{2.25}$$

where erf is the error function defined as

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt,$$

and $h(\beta_1)$ is as previously defined (2.12).

Proof. The proof of this Lemma is included in the Appendix (2.9.2). □

Theorem 2.6.2. *The joint density of V and $Z^{(2)}$ at τ can be derived as*

$$\begin{aligned}
&\mathbb{P}\left[V_\tau \in dm, Z_\tau^{(2)} \in dy\right] \\
&= \frac{i}{\sqrt{\pi^2 y^3}} \int_0^\infty \int_0^\infty e^{-\frac{(x-a)^2}{2z}} e^{-s^2} e^{-2s\frac{x-a}{\sqrt{2z}}} t e^{-\frac{t^2}{4y}} i\sqrt{\frac{s\sqrt{z}}{t}}\, is_m\left(1, 2a, 0, \frac{t}{\sqrt{2}} + s\sqrt{2z}, -\sqrt{st\sqrt{z}}\right) dtds \\
&\quad + \frac{1}{\sqrt{\pi^2 y^3}} \int_0^\infty \int_0^\infty e^{-\frac{(x-a)^2}{2z}} e^{-s^2} e^{-2s\frac{x-a}{\sqrt{2z}}} t e^{-\frac{t^2}{4y}} \delta(t) es_m\left(0, 0, 2a, 0, \frac{t}{\sqrt{2}} + s\sqrt{2z}\right) dtds \\
&\quad + erf\left(\frac{x-a}{\sqrt{2z}}\right)\delta(y)\delta(m),
\end{aligned}$$

where the functions is_m and es_m are defined as

$$\begin{aligned}
is_y(v,t,r,z,x) &= \sum_{l=0}^\infty \frac{x^{v+2l}}{\Gamma(v+l+1)l!} es_y(1+v+2l, 1+v+2l, t, r, z), \\
es_y(u,v,t,x,z) &= \sum_{k=0}^\infty \frac{(-z)^k}{k!} s_y(u+k, v+k, t, x+z+kt), \\
s_y(u,v,t,z) &= 2^v \sum_{k=0}^\infty \frac{\Gamma(v+k) e^{-\frac{(vt+z+2kt)^2}{4y}}}{\sqrt{2\pi} y^{1+\frac{u}{2}} \Gamma(v) k!} D_{u+1}\left(\frac{vt+z+2kt}{\sqrt{y}}\right),
\end{aligned}$$

with the conditions

$$2a > 0, \qquad \frac{t}{\sqrt{2}} + s\sqrt{2z} > 0, \qquad 4a + \frac{t}{\sqrt{2}} + s\sqrt{2z} > 0.$$

Proof. The proof of this theorem can be divided into two steps. Inverting (2.25) with respect to β_2 and using Fubini's theorem gives

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_2}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z^{(2)}_\tau}\right]\right)(y)\\
&= \mathcal{L}^{-1}_{\beta_2}\Bigg(\frac{2}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)}\\
&\qquad\cdot\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)ds\Bigg)(y) + \mathcal{L}^{-1}_{\beta_2}\left(\operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\right)(y)\\
&= \frac{2}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)}\\
&\qquad\cdot\mathcal{L}^{-1}_{\beta_2}\left(\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)\right)(y)ds + \operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\mathcal{L}^{-1}_{\beta_2}(1)(y).
\end{aligned}$$

The inversion in the first term can be derived by first noticing that

$$\begin{aligned}
&\mathcal{L}^{-1}_{\beta_2}\left(\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)\right)(y)\\
&= \mathcal{L}^{-1}_{\beta-2}\left(\exp\left(-s\sqrt{z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]\right)\right)(y).
\end{aligned}$$

We then use the following steps:

- Considering a change of variable of the form $\beta = \sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}$, the inversion becomes

  $$\begin{aligned}
  &\mathcal{L}^{-1}_{\beta}\left(\exp\left(-s\sqrt{z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\beta}\right]\right)\right)(y)\\
  &= \mathcal{L}^{-1}_{\beta}\left(\exp\left(-s\sqrt{z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\beta}\right]\right)-1\right)(y) + \mathcal{L}^{-1}_{\beta}(1)(y)\\
  &= i\sqrt{\frac{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]}{y}}I_1\left(2i\sqrt{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]y}\right) + \delta(y).
  \end{aligned}$$

  where $I_1(x)$ is the modified Bessel function of the first kind and $\delta(x)$ is the Dirac delta function.

  In the second equality, we used the theory of residues and the concept of a distribution.

The function $\delta(y)$ is an infinite spike centered at $y=0$, such that its total mass is 1 and the inverse Laplace transform of 1 which can be written in the integral form as

$$\mathcal{L}_{\beta}^{-1}\left(1\right)(y)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\beta y}d\beta=\lim_{p\to\infty}\frac{\sin(py)}{\pi y}=\delta(y),$$

using complex analysis, since the function 1 has no poles we can take $c=0$ for convenience.

- Using the following relationship:

$$\begin{aligned}&\text{If } \mathcal{L}_{\gamma}^{-1}\left(F(\gamma)\right)(y)=:f(y), \text{ where } F(\gamma)=\int_0^{\infty}e^{-\gamma y}f(y)dy, \quad \Re(\gamma)\geq 0,\\ &\text{then } \mathcal{L}_{\gamma}^{-1}\left(F(a\gamma+\beta)\right)=\frac{1}{\alpha}e^{-\frac{\beta y}{\alpha}}f\left(\frac{y}{\alpha}\right), \quad \alpha>0.\end{aligned}$$

We can add the terms with β_1 in

$$\begin{aligned}&=\mathcal{L}_{\beta}^{-1}\left(\exp\left(-s\sqrt{z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{\beta_1}h(\beta_1)+\beta}\right]\right)\right)(y)\\ &=\mathcal{L}_{\beta}^{-1}\left(F(\sqrt{\beta_1}h(\beta_1)+\beta)\right)(y)\\ &=e^{-\sqrt{\beta_1}h(\beta_1)y}\left[i\sqrt{\frac{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]}{y}}I_1\left(2i\sqrt{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]y}\right)+\delta(y)\right].\end{aligned}$$

- Using the following relationship,

$$\begin{aligned}&\text{If } \mathcal{L}_{\gamma}^{-1}\left(F(\gamma)\right)(y)=:f(y), \text{ where } F(\gamma)=\int_0^{\infty}e^{-\gamma y}f(y)dy, \quad \Re(\gamma)\geq 0,\\ &\text{then } \mathcal{L}_{\gamma}^{-1}\left(F(\sqrt{\gamma})\right)(y)=\frac{1}{2\sqrt{\pi y^3}}\int_0^{\infty}xe^{-\frac{x^2}{4y}}f(x)dx.\end{aligned}$$

We can then add the square root in to obtained the desired inverse Laplace transformation with respect to β_2

$$\mathcal{L}_{\beta_2}^{-1}\left(\exp\left(-s\sqrt{z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]\right)\right)(y)$$
$$=\frac{1}{2\sqrt{\pi y^3}}\int_0^\infty te^{-\frac{t^2}{4y}}e^{-\sqrt{\beta_1}h(\beta_1)t}$$
$$\cdot\left[i\sqrt{\frac{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]}{t}}I_1\left(2i\sqrt{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]t}\right)+\delta(t)\right]dt.$$

Therefore, the inversion with respect to β_2 for the double Laplace transform is given as

$$\mathcal{L}_{\beta_2}^{-1}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}\right]\right)(y)$$
$$=\frac{2}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)}$$
$$\cdot\mathcal{L}_{\beta_2}^{-1}\left(\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)\right)(y)ds+\operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\mathcal{L}_{\beta_2}^{-1}(1)(y)$$
$$=\frac{i}{\sqrt{\pi^2y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}}te^{-\frac{t^2}{4y}}e^{-\sqrt{\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}t}$$
$$\cdot\sqrt{\frac{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]}{t}}I_1\left(2i\sqrt{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]t}\right)dtds$$
$$+\frac{1}{\sqrt{\pi^2y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}te^{-\frac{t^2}{4y}}e^{-\sqrt{2\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}\left[\frac{t}{\sqrt{2}}+s\sqrt{2z}\right]}\delta(t)dtds$$
$$+\operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\delta(y),$$

where the function $h(\beta_1)$ is as defined earlier. We can now proceed to the second part of the proof where we invert the expression with respect to β_1. Therefore, using Fubini's theorem, we have that the joint density can be obtained as

$$
\begin{aligned}
&\mathcal{L}^{-1}_{\beta_1}\left(\mathcal{L}^{-1}_{\beta_2}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z^{(2)}_\tau}\right]\right)(y)\right)(m)\\
&=\frac{i}{\sqrt{\pi^2 y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}te^{-\frac{t^2}{4y}}\mathcal{L}^{-1}_{\beta_1}\left(e^{-s\sqrt{2z}\sqrt{2\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}}e^{-\sqrt{\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}t}\right.\\
&\qquad\left.\cdot\sqrt{\frac{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]}{t}}I_1\left(2i\sqrt{s\sqrt{z}(2\beta_1)[1-h^2(\beta_1)]t}\right)\right)(m)dtds\\
&\quad+\frac{1}{\sqrt{\pi^2 y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}te^{-\frac{t^2}{4y}}\delta(t)\mathcal{L}^{-1}_{\beta_1}\left(e^{-\sqrt{2\beta_1}\frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}\left[\frac{t}{\sqrt{2}}+s\sqrt{2z}\right]}\right)(m)dtds\\
&\quad+\operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\delta(y)\mathcal{L}^{-1}_{\beta_1}(1)(m)\\
&=\frac{i}{\sqrt{\pi^2 y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}te^{-\frac{t^2}{4y}}i\sqrt{\frac{s\sqrt{z}}{t}}\,is_m\left(1,2a,0,\frac{t}{\sqrt{2}}+s\sqrt{2z},-\sqrt{st\sqrt{z}}\right)dtds\\
&\quad+\frac{1}{\sqrt{\pi^2 y^3}}\int_0^\infty\int_0^\infty e^{-\frac{(x-a)^2}{2z}}e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}te^{-\frac{t^2}{4y}}\delta(t)es_m\left(0,0,2a,0,\frac{t}{\sqrt{2}}+s\sqrt{2z}\right)dtds\\
&\quad+\operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right)\delta(y)\delta(m),
\end{aligned}
$$

where the functions is_m and es_m are defined as

$$
\begin{aligned}
is_y(v,t,r,z,x)&=\sum_{l=0}^{\infty}\frac{x^{v+2l}}{\Gamma(v+l+1)l!}es_y(1+v+2l,1+v+2l,t,r,z),\\
es_y(u,v,t,x,z)&=\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}s_y(u+k,v+k,t,x+z+kt),\\
s_y(u,v,t,z)&=2^v\sum_{k=0}^{\infty}\frac{\Gamma(v+k)e^{-\frac{(vt+z+2kt)^2}{4y}}}{\sqrt{2\pi}y^{1+\frac{u}{2}}\Gamma(v)k!}D_{u+1}\left(\frac{vt+z+2kt}{\sqrt{y}}\right),
\end{aligned}
$$

with the conditions

$$
2a>0,\qquad \frac{t}{\sqrt{2}}+s\sqrt{2z}>0,\qquad 4a+\frac{t}{\sqrt{2}}+s\sqrt{2z}>0.
$$

□

2.6.2 Joint distribution of V_τ, $Z_\tau^{(2)}$ and X_τ

Lemma 2.6.3. *For $\beta_1, \beta_2, \xi \in \mathbb{R}_+$ and the first hitting time τ as defined in (2.10), the joint Laplace transform of $V_\tau, Z_\tau^{(2)}$ and X_τ can be derived as*

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]}\right] \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-2s\sqrt{\frac{z}{2}}\xi} e^{2s\frac{x-a}{\sqrt{2z}}} ds + \frac{1}{\sqrt{\pi}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-2s\sqrt{\frac{z}{2}}\xi} e^{-2s\frac{x-a}{\sqrt{2z}}} ds \\
&\quad + \frac{2}{\sqrt{\pi}} \frac{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}{\xi - \sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-s\sqrt{2z}\xi} e^{-2s\frac{x-a}{\sqrt{2z}}} ds \\
&\quad - \frac{2}{\sqrt{\pi}} \frac{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}{\xi - \sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-2s\frac{x-a}{\sqrt{2z}}} e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)} \\
&\qquad \cdot \exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right) ds,
\end{aligned}
\tag{2.26}
$$

where $h(\beta_1)$ is as defined in (2.12).

Proof. We have from (2.25), that the double Laplace transform is

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= \frac{2}{\sqrt{\pi}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} e^{-2s\frac{x-a}{\sqrt{2z}}} e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)} \exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right) ds \\
&\quad + \operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right).
\end{aligned}
$$

Substituting this into (2.39), we can easily obtain the desired expression for the triple Laplace transform. $\square$

Theorem 2.6.4. *The joint density of $V_\tau, Z_\tau^{(2)}$ and X_τ is given as*

$$
\begin{aligned}
&\mathbb{P}\left[V_\tau \in dp, Z_\tau^{(2)} \in dm, X_\tau \in dy\right] \\
&= \frac{1}{\sqrt{\pi}} e^{-\frac{(x-a)^2}{2z}} \int_0^\infty e^{-s^2} \delta(y - s\sqrt{2z}) \delta(m) \left[e^{2s\frac{x-a}{\sqrt{2z}}} + e^{-2s\frac{x-a}{\sqrt{2z}}}\right] \delta(p) ds \\
&\quad + \frac{\sqrt{2}}{\sqrt{\pi^2 m^3}} \int_0^\infty \int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2} t e^{-\frac{t^2}{4m}} \left[\theta(y - s\sqrt{2z}) - 1\right] \left\{-\frac{1}{2}\, s_p\,(2a)\right. \\
&\qquad \left. * is_p \left(0, 2a, 0, \frac{t}{\sqrt{2}} + s\sqrt{2z} - y, -\sqrt{\left(s\sqrt{2z} - y\right) \frac{t}{\sqrt{2}}}\right)\right\} dtds \\
&\quad + \frac{1}{\sqrt{\pi^2 m^3}} \int_0^\infty \int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2} t e^{-\frac{t^2}{4y}} \left[\theta(y - s\sqrt{2z}) - 1\right] \left\{-\sqrt{2}\sqrt{\frac{\sqrt{2}\,(s\sqrt{2z} - y)}{t}}\right. \\
&\qquad \left. \cdot \left[f(p) * is_p \left(1, 2a, 0, \frac{t}{\sqrt{2}} + s\sqrt{2z} - y, -\sqrt{\left(s\sqrt{2z} - y\right) \frac{t}{\sqrt{2}}}\right)\right]\right\} dtds \\
&\quad + \frac{1}{\sqrt{\pi^2 m^3}} \int_0^\infty \int_0^\infty dtds\; e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2} t e^{-\frac{t^2}{4y}} \left[\theta(y - s\sqrt{2z}) - 1\right] \delta(t) \cdot \sum_{k=0}^\infty \frac{\left(-2\left[\frac{t}{\sqrt{2}} + s\sqrt{2z} - y\right]\right)^k}{k!} \\
&\qquad \cdot \left\{\sum_{l=0}^\infty \frac{\Gamma(k+l+2) e^{-\frac{\left[(k+l)(4a) + \frac{t}{\sqrt{2}} + s\sqrt{2z} - y\right]^2}{2p}}}{\sqrt{2\pi} p^{1+\frac{k+1}{2}} \Gamma(k+2) l!} He_{k+2}\left(\frac{(k+l)(4a) + \frac{t}{\sqrt{2}} + s\sqrt{2z} - y}{\sqrt{p}}\right)\right. \\
&\qquad \left. - \sum_{l=0}^\infty \frac{\Gamma(k+l+2) e^{-\frac{\left[(k+l+2)(4a) + \frac{t}{\sqrt{2}} + s\sqrt{2z} - y\right]^2}{2p}}}{\sqrt{2\pi} p^{1+\frac{k+1}{2}} \Gamma(k+2) l!} He_{k+2}\left(\frac{(k+l+2)(4a) + \frac{t}{\sqrt{2}} + s\sqrt{2z} - y}{\sqrt{p}}\right)\right\},
\end{aligned}
$$

with the conditions

$$2a > 0, \qquad \frac{t}{\sqrt{2}} + s\sqrt{2z} - y > 0,$$

where He is the modified Hermite function, $\delta(t)$ the Dirac delta function and $\theta(x)$ the Heaviside step function.

Proof. There are three steps to this proof. First, we look at the inversion of the triple Laplace transform with respect to ξ.

$$
\begin{aligned}
&\mathcal{L}_\xi^{-1}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}e^{-\xi[X_\tau-a]}\right]\right)(y)\\
&=\mathcal{L}_\xi^{-1}\left(\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\sqrt{\frac{z}{2}}\xi}e^{2s\frac{x-a}{\sqrt{2z}}}ds\right)(y)\\
&\quad+\mathcal{L}_\xi^{-1}\left(\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\sqrt{\frac{z}{2}}\xi}e^{-2s\frac{x-a}{\sqrt{2z}}}ds\right)(y)\\
&\quad+\mathcal{L}_\xi^{-1}\left(\frac{2}{\sqrt{\pi}}\frac{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}{\xi-\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-s\sqrt{2z}\,\xi}e^{-2s\frac{x-a}{\sqrt{2z}}}ds\right)(y)\\
&\quad-\mathcal{L}_\xi^{-1}\left(\frac{2}{\sqrt{\pi}}\frac{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}{\xi-\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)}\right.\\
&\qquad\left.\cdot\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)ds\right)(y)\\
&=\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{2s\frac{x-a}{\sqrt{2z}}}\delta(y-s\sqrt{2z})ds\\
&\quad+\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}\delta(y-s\sqrt{2z})ds\\
&\quad+\frac{2}{\sqrt{\pi}}\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}e^{-\frac{(x-a)^2}{2z}}\\
&\qquad\cdot\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}(y-s\sqrt{2z})}\theta(y-s\sqrt{2z})ds\\
&\quad-\frac{2}{\sqrt{\pi}}\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{-s\sqrt{2z}\sqrt{2\beta_1}h(\beta_1)}\\
&\qquad\cdot e^{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}y}\exp\left(-s\sqrt{2z}\left[\frac{2\beta_1\left[1-h^2(\beta_1)\right]}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\right]\right)ds,
\end{aligned}
\tag{2.27}
$$

where we have used the following inversions:

$$
\begin{aligned}
\mathcal{L}_\xi^{-1}\left(e^{-2s\sqrt{\frac{z}{2}}\xi}\right)(y)&=\delta\left(y-2s\sqrt{\frac{z}{2}}\right)=\delta(y-s\sqrt{2z}),\\
\mathcal{L}_\xi^{-1}\left(\frac{e^{-2s\sqrt{\frac{z}{2}}\xi}}{\xi-\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}\right)(y)&=e^{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}(y-s\sqrt{2z})}\theta(y-s\sqrt{2z}),\\
\mathcal{L}_\xi^{-1}\left(\frac{1}{\xi-\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}}\right)(y)&=e^{\sqrt{2\beta_1}\frac{\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}y},
\end{aligned}
$$

for $\delta(x)$ is the Dirac delta function and $\theta(x)$ is the Heaviside step function defined as

$$\theta(x) = \frac{1}{2}\left[1 + \mathrm{sgn}(x)\right],$$

where $\mathrm{sgn}(x)$ is the sign function defined as

$$\mathrm{sign(x)} = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

We can then proceed to invert the expression in (2.27) with respect to β_2. We consider a change of variable of the form $\beta = \sqrt{\beta_1} h(\beta_1) + \sqrt{\beta_2}$ and define

$$\mathcal{L}_\gamma^{-1}\left(F(\gamma)\right)(y) =: f(y),$$

where for $\Re(\gamma) \geq 0$,

$$F(\gamma) = \int_0^\infty e^{-\gamma y} f(y) dy.$$

Using the following inversions,

$$\mathcal{L}_\gamma^{-1}\left(F(a\gamma + \beta)\right)(y) = \frac{1}{\alpha} e^{-\frac{\beta y}{\alpha}} f\left(\frac{y}{\alpha}\right), \quad \alpha > 0,$$
$$\mathcal{L}_\gamma^{-1}\left(F(\sqrt{\gamma})\right)(y) = \frac{1}{2\sqrt{\pi y^3}} \int_0^\infty x e^{-\frac{x^2}{4y}} f(x) dx,$$

and Fubini's thereom, we can obtain the inversion with respect to β_2 for the triple Laplace transform:

$$
\begin{aligned}
&\mathcal{L}^{-1}_{\beta_2}\left(\mathcal{L}^{-1}_{\xi}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z^{(2)}_\tau}e^{-\xi[X_\tau-a]}\right]\right)(y)\right)(m)\\
&=\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}\delta(y-s\sqrt{2z})\left[e^{2s\frac{x-a}{\sqrt{2z}}}+e^{-2s\frac{x-a}{\sqrt{2z}}}\right]\mathcal{L}^{-1}_{\beta_2}(1)(m)ds\\
&\quad+\frac{\sqrt{2}}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{(y-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}2\beta_1[1-h^2(\beta_1)]\left[\theta(y-s\sqrt{2z})-1\right]\\
&\qquad\cdot\mathcal{L}^{-1}_{\beta_2}\left(\left(\frac{1}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right)e^{\left(\frac{y-s\sqrt{2z}}{\sqrt{2}}\right)\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}}\right)(m)ds\\
&\quad+\frac{2}{\sqrt{\pi}}\sqrt{2\beta_1}h(\beta_1)e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\frac{x-a}{\sqrt{2z}}}e^{(y-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\left[\theta(y-s\sqrt{2z})-1\right]\\
&\qquad\cdot\mathcal{L}^{-1}_{\beta_2}\left(e^{\left(\frac{y-s\sqrt{2z}}{\sqrt{2}}\right)\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}}\right)(m)ds\\
&=\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}\delta(y-s\sqrt{2z})\delta(m)\left[e^{2s\frac{x-a}{\sqrt{2z}}}+e^{-2s\frac{x-a}{\sqrt{2z}}}\right]ds\\
&\quad+\frac{\sqrt{2}}{\sqrt{\pi^2m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4m}}e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\beta_1[1-h^2(\beta_1)]\\
&\qquad\cdot\left[\theta(y-s\sqrt{2z})-1\right]I_0\left(2i\sqrt{\sqrt{2}\left(s\sqrt{2z}-y\right)\beta_1[1-h^2(\beta_1)]t}\right)dtds\\
&\quad+\frac{1}{\sqrt{\pi^2m^3}}\sqrt{2\beta_1}h(\beta_1)\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\left[\theta(y-s\sqrt{2z})-1\right]\\
&\qquad\cdot\sqrt{\frac{\sqrt{2}\left(y-s\sqrt{2z}\right)\beta_1[1-h^2(\beta_1)]}{t}}I_1\left(2i\sqrt{\sqrt{2}\left(s\sqrt{2z}-y\right)\beta_1[1-h^2(\beta_1)]t}\right)dtds\\
&\quad+\frac{\sqrt{2\beta_1}h(\beta_1)}{\sqrt{\pi^2m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\left[\theta(y-s\sqrt{2z})-1\right]\delta(t)dtds.
\end{aligned}
\tag{2.28}
$$

Finally, the joint density can be obtained by inverting (2.28) with respect to β_1.

$$\mathcal{L}^{-1}_{\beta_1}\left(\mathcal{L}^{-1}_{\beta_2}\left(\mathcal{L}^{-1}_{\xi}\left(\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z^{(2)}_\tau}e^{-\xi[X_\tau-a]}\right]\right)(y)\right)(m)\right)(p)$$
$$= \frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}\delta(y-s\sqrt{2z})\delta(m)\left[e^{2s\frac{x-a}{\sqrt{2z}}}+e^{-2s\frac{x-a}{\sqrt{2z}}}\right]\mathcal{L}^{-1}_{\beta_1}(1)(p)ds$$
$$+\frac{\sqrt{2}}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4m}}\left[\theta(y-s\sqrt{2z})-1\right]\mathcal{L}^{-1}_{\beta_1}\left(e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\right.$$
$$\left.\cdot\,\beta_1[1-h^2(\beta_1)]I_0\left(2i\sqrt{\sqrt{2}\left(s\sqrt{2z}-y\right)\beta_1[1-h^2(\beta_1)]t}\right)\right)(p)dtds$$
$$+\frac{1}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty dtds\; e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}\left[\theta(y-s\sqrt{2z})-1\right]$$
$$\cdot\,\mathcal{L}^{-1}_{\beta_1}\left(\sqrt{2\beta_1}h(\beta_1)e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\right.$$
$$\left.\cdot\sqrt{\frac{\sqrt{2}\left(y-s\sqrt{2z}\right)\beta_1[1-h^2(\beta_1)]}{t}}I_1\left(2i\sqrt{\sqrt{2}\left(s\sqrt{2z}-y\right)\beta_1[1-h^2(\beta_1)]t}\right)\right)(p)$$
$$+\frac{1}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}\left[\theta(y-s\sqrt{2z})-1\right]\delta(t)$$
$$\cdot\,\mathcal{L}^{-1}_{\beta_1}\left(\sqrt{2\beta_1}h(\beta_1)e^{(y-\frac{t}{\sqrt{2}}-s\sqrt{2z})\sqrt{2\beta_1}h(\beta_1)}\right)(p)dtds.$$

This can be computed as:

$$\frac{1}{\sqrt{\pi}}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}\delta(y-s\sqrt{2z})\delta(m)\left[e^{2s\frac{x-a}{\sqrt{2z}}}+e^{-2s\frac{x-a}{\sqrt{2z}}}\right]\mathcal{L}^{-1}_{\beta_1}(1)(p)ds$$
$$-\frac{\sqrt{2}}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4m}}\left[\theta(y-s\sqrt{2z})-1\right]$$
$$\cdot\left(\frac{1}{2}s_p\left(2a\right)\right)*is_p\left(0,2a,0,\frac{t}{\sqrt{2}}+s\sqrt{2z}-y,-\sqrt{\left(s\sqrt{2z}-y\right)\frac{t}{\sqrt{2}}}\right)dtds$$
$$-\frac{1}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty dtds\; e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}\left[\theta(y-s\sqrt{2z})-1\right]\sqrt{2}\sqrt{\frac{\sqrt{2}\left(s\sqrt{2z}-y\right)}{t}}$$
$$\cdot\left[\frac{d}{dp}\left(\frac{1}{\sqrt{2\pi p}}\sum_{k=-\infty}^{\infty}e^{-\frac{[4a(k+1)]^2}{2p}}\right)*is_p\left(1,2a,0,\frac{t}{\sqrt{2}}+s\sqrt{2z}-y,-\sqrt{\left(s\sqrt{2z}-y\right)\frac{t}{\sqrt{2}}}\right)\right]$$
$$+\frac{1}{\sqrt{\pi^2 m^3}}\int_0^\infty\int_0^\infty e^{-\left[s+\frac{x-a}{\sqrt{2z}}\right]^2}te^{-\frac{t^2}{4y}}\left[\theta(y-s\sqrt{2z})-1\right]\delta(t)$$
$$\cdot\,escs_p\left(2a,2a,2a,\frac{t}{\sqrt{2}}+s\sqrt{2z}-y\right)dtds,$$

where the functions s, is and $escs$ are such that

$$
\begin{aligned}
s_y(t) &= \frac{\sqrt{2}}{\sqrt{\pi}y^{\frac{5}{2}}}\sum_{k=0}^{\infty}((2k+1)^2t^2-y)e^{-\frac{(2k+1)^2t^2}{2y}},\\
is_y(v,t,r,z,x) &= \sum_{l=0}^{\infty}\frac{x^{v+2l}}{\Gamma(v+l+1)l!}es_y(1+v+2l,1+v+2l,t,r,z),\\
es_y(u,v,t,x,z) &= \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}s_y(u+k,v+k,t,x+z+kt),\\
s_y(u,v,t,z) &= 2^v\sum_{k=0}^{\infty}\frac{\Gamma(v+k)e^{-\frac{(vt+z+2kt)^2}{4y}}}{\sqrt{2\pi}y^{1+\frac{u}{2}}\Gamma(v)k!}D_{u+1}\left(\frac{vt+z+2kt}{\sqrt{y}}\right),\\
escs_y(u,v,t,z) &= \frac{1}{4}es_y(1,2,t,-u-v,z)-\frac{1}{4}es_y(1,2,t,u+v,z),\\
&+\frac{1}{4}es_y(1,2,t,-u+v,z)-\frac{1}{4}es_y(1,2,t,u-v,z),
\end{aligned}
$$

with the conditions

$$2a>0, \qquad \frac{t}{\sqrt{2}}+s\sqrt{2z}-y>0.$$

□

2.7 Case 3: $-a < X_0 = x < a$

We have looked at the cases where $X_0 = x < -a$ (we start in state 3) and $X_0 = x > a$ (we start in state 1). We now look at the case where $-a < X_0 = x < a$ when we start in state 2. This is the most complicated case out of the three.

2.7.1 Joint distribution of V_τ and $Z_\tau^{(2)}$

Lemma 2.7.1. *For $\beta_1, \beta_2 \in \mathbb{C}_+$ and the first hitting time τ as defined in (2.10), the joint Laplace transform of V_τ and $Z_\tau^{(2)}$ can be derived as*

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]} \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})}\left[\frac{2\sqrt{2\beta_1}\cosh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]}\right] ds \\
&\quad - \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]} \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})}\left[\frac{\sqrt{2\beta_1}h(\beta_1)2\sinh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]}\right] ds \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]} \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})}\left(2\sinh([a+x]\sqrt{2\beta_1})\right) ds,
\end{aligned} \tag{2.29}$$

where the functions $\sinh(x)$ *and* $\cosh(x)$ *are the hyperbolic sine and cosine functions respectively and the function $h(\beta_1)$ is as defined in (2.12).*

Proof. With the starting point $-a < X_0 = x < a$, applying the optional stopping theorem to the martingales $f_i(X_t, V_t, Z_t^{(1)}, Z_t^{(2)})$ with a bounded stopping time $\tau \wedge t$ gives

$$\mathbb{E}\left[e^{-\beta_1 V_{\tau\wedge t}} e^{-\beta_2 Z_{\tau\wedge t}^{(2)}} e^{\gamma Z_{\tau\wedge t}^{(1)}} f_1(X_{\tau\wedge t})\middle| X_0 = x\right] = e^{-\beta_1 V_0} e^{-\beta_2 Z_0^{(2)}} e^{\gamma Z_0^{(1)}} f_2(X_0).$$

Taking limits for $t \to \infty$ gives us

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{\gamma z} f_1(X_\tau)\right] &= f_2(x). \\
\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau)\right] &= e^{-\gamma z} f_2(x).
\end{aligned}$$

We proceed using the same steps as in the previous section. With the change of variable of the form $\omega = \sqrt{2\gamma}$, multiplying both sides with $\frac{1}{\omega^2+\xi^2}$ and integrating with respect to ω over the range of ω, the right hand side of the expressions gives

$$\begin{aligned}
&\int_0^\infty e^{-\gamma z} f_2(x) \frac{1}{\omega^2+\xi^2} d\omega \\
&= \int_0^\infty e^{-\frac{\omega^2}{2}z}\left[B_1 e^{\sqrt{2\beta_1}x} + \left(B_1 e^{-2\sqrt{2\beta_1}a}\frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) e^{-\sqrt{2\beta_1}x}\right]\frac{1}{\omega^2+\xi^2} d\omega \\
&= B_1\left[e^{\sqrt{2\beta_1}x} + e^{-\sqrt{2\beta_1}(2a+x)}\frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right]\frac{\pi}{\xi} e^{\xi^2\frac{z}{2}}\Phi\left(-\xi\sqrt{z}\right).
\end{aligned}$$

Now, computing the left hand side of the expression with $\omega = \sqrt{2\gamma}$ gives

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau) \right] \frac{1}{\omega^2 + \xi^2} d\omega \\
&= B_1 \frac{\pi}{2\xi} \left(e^{\sqrt{2\beta_1} a} + e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]} \right] \\
&\quad + B_1 \frac{\pi}{2\xi^2} \sqrt{2\beta_1} \left(e^{\sqrt{2\beta_1} a} - e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \right] \\
&\quad - B_1 \frac{\pi}{2\xi^2} \sqrt{2\beta_1} \left(e^{\sqrt{2\beta_1} a} - e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \right) \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]} \right].
\end{aligned}
$$

Equating the 2 expressions and making the triple Laplace transform the subject gives

$$
\begin{aligned}
&\mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]} \right] \\
&= \frac{1}{\xi - \sqrt{2\beta_1} \frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)}} \left\{ \frac{2}{g_1(\beta_1,\beta_2)} \xi e^{\sqrt{2\beta_1} x} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-s\xi\sqrt{z}} ds \right. \\
&\quad + \frac{2\xi e^{-\sqrt{2\beta_1}(2a+x)}}{g_1(\beta_1,\beta_2)} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-s\xi\sqrt{z}} ds \\
&\quad \left. - \sqrt{2\beta_1} \frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)} \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \right] \right\}.
\end{aligned} \tag{2.30}
$$

where the functions g_1 and g_2 are defined as

$$
\begin{aligned}
g_1(\beta_1,\beta_2) &:= e^{\sqrt{2\beta_1} a} + e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}}, \\
g_2(\beta_2,\beta_2) &:= e^{\sqrt{2\beta_1} a} - e^{-3\sqrt{2\beta_1} a} \frac{\sqrt{2\beta_1} - \sqrt{2\beta_2}}{\sqrt{2\beta_1} + \sqrt{2\beta_2}}.
\end{aligned}
$$

Looking at the expression in (2.30), we can say that there exists a $\xi^* > 0$ such that when the denominator goes to 0 at ξ^*, the numerator goes to 0 at the same point for the triple Laplace transform exists and not to be identically 0, when the denominator goes to 0 at $\xi = \xi^*$, we should have that the numerator goes to 0 at the same point. The value of ξ^* can be determined by setting

$$
\xi^* = \sqrt{2\beta_1} \frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)} = \frac{2\beta_1[1 - h^2(\beta_1)]}{\sqrt{2\beta_1} h(\beta_1) + \sqrt{2\beta_2}} + \sqrt{2\beta_1} h(\beta_1).
$$

The double Laplace transform can be obtained directly by setting the numerator to 0 at ξ^*. □

2.7.2 Joint distribution of V_τ, $Z_\tau^{(2)}$ and X_τ

Lemma 2.7.2. *For $\beta_1, \beta_2, \xi \in \mathbb{R}_+$ and the first hitting time τ as defined in (2.10), the joint Laplace transform of $V_\tau, Z_\tau^{(2)}$ and X_τ can be derived as*

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]}\right] \\
&= \frac{1}{\xi - \sqrt{2\beta_1}\frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)}} \frac{2}{g_1(\beta_1,\beta_2)} \xi e^{\sqrt{2\beta_1}x} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2}\, e^{-s\xi\sqrt{z}} ds \\
&\quad + \frac{1}{\xi - \sqrt{2\beta_1}\frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)}} \frac{2}{g_1(\beta_1,\beta_2)} \xi e^{-\sqrt{2\beta_1}(2a+x)}\, \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2}\, e^{-s\xi\sqrt{z}} ds \\
&\quad - \frac{1}{\xi - \sqrt{2\beta_1}\frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)}} \sqrt{2\beta_1} \frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)} \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]} \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})} \\
&\quad \cdot \left[\frac{2\sqrt{2\beta_1}\cosh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]} - \frac{\sqrt{2\beta_1}h(\beta_1)2\sinh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]} + 2\sinh([a+x]\sqrt{2\beta_1})\right] ds.
\end{aligned}
$$

Proof. From (2.29), we have

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]}\, \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})} \left[\frac{2\sqrt{2\beta_1}\cosh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]}\right] ds \\
&\quad - \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]}\, \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})} \left[\frac{\sqrt{2\beta_1}h(\beta_1)2\sinh([a+x]\sqrt{2\beta_1})}{\sqrt{2}[\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}]}\right] ds \\
&\quad + \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2}s^2} e^{-\frac{s\sqrt{z}}{\sqrt{2}}\left[\frac{2\beta_1[1-h^2(\beta_1)]}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}\right]}\, \frac{e^{-\sqrt{2\beta_1}\left[s\sqrt{z}h(\beta_1)\right]}}{\sinh(2a\sqrt{2\beta_1})} \left(2\sinh([a+x]\sqrt{2\beta_1})\right) ds.
\end{aligned}
\tag{2.31}
$$

Substituting this into (2.30) gives us the desired triple Laplace transform. □

Remark 2.7.3. *We can derive the joint densities using the same steps as in the previous cases.*

2.8 Concluding Remarks

In this chapter, we derived and discussed in details the joint Laplace transform and the joint density functions of the Brownian occupation time processes under three different cases. Our results find applications in the study of insurance risk models. This is discussed in Chapter 1 .

2.9 Appendix

2.9.1 Proof of Lemma 2.5.1

We provide some steps to obtain the expression in Lemma 2.5.1.

Proof. With the starting point $X_0 = x < -a$ (ie starting in state 3), applying the optional stopping theorem to the martingales $f_i(X_t, V_t, Z_t^{(1)}, Z_t^{(2)})$, $i = 1, 2, 3$ derived with a bounded stopping time $\tau \wedge t$ gives

$$\mathbb{E}\left[e^{-\beta_1 V_{\tau\wedge t}}\, e^{-\beta_2 Z^{(2)}_{\tau\wedge t}}\, e^{\gamma Z^{(1)}_{\tau\wedge t}} f_1(X_{\tau\wedge t}) | X_0 = x\right] = e^{\beta_1 V_0} e^{-\beta_2 Z_0^{(2)}} e^{\gamma Z_0^{(1)}} f_3(X_0), \tag{2.32}$$

where $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot|X_0 = x]$. Taking limits for $t \to \infty$ and recalling from the definition of τ that $Z_\tau^{(1)} = z$ gives

$$\begin{aligned} \mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{\gamma Z_\tau^{(1)}} f_1(X_\tau)\right] &= f_3(x), \\ \mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau)\right] &= e^{-\gamma z} f_3(x). \end{aligned} \tag{2.33}$$

We proceed by using a change of variable of the form $\omega = \sqrt{2\gamma}$ and multiplying both sides with $\frac{1}{\omega^2+\xi^2}$ and integrating with respect to ω from 0 to ∞. Using Schwinger parametrisation, the right hand side of the expression gives

$$\int_0^\infty e^{-\gamma z} f_3(x) \frac{1}{\omega^2+\xi^2} d\omega \quad = B_1 e^{-\sqrt{2\beta_1}a} e^{\sqrt{2\beta_2}(a+x)} \frac{2\sqrt{2\beta_1}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \int_0^\infty e^{-\xi^2 u} \frac{\sqrt{\pi}}{2\sqrt{\frac{z}{2}+u}} du. \tag{2.34}$$

Setting $t = \sqrt{\frac{z}{2}+u}$ and $\frac{1}{2}s^2 = \xi^2 t^2$ gives

$$\int_0^\infty e^{-\gamma z} f_3(x) \frac{1}{\omega^2+\xi^2} d\omega \quad = B_1 e^{-\sqrt{2\beta_1}a} e^{\sqrt{2\beta_2}(a+x)} \frac{2\sqrt{2\beta_1}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \frac{\pi}{\xi}\, e^{\xi^2 \frac{z}{2}} \Phi(-\xi\sqrt{z}), \tag{2.35}$$

where Φ is the CDF of a standard Normal distribution. Letting $\omega = \sqrt{2\gamma}$ and applying Fubini's theorem on the left hand side gives

$$\begin{aligned} &\int_0^\infty \mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau)\right] \frac{1}{\omega^2+\xi^2} d\omega \\ &= \mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]}\right] \left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) B_1 \frac{\pi}{2\xi} \\ &\quad + \mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \left(1 - e^{-\xi[X_\tau - a]}\right)\right] \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) \sqrt{2\beta_1} B_1 \frac{\pi}{2\xi^2}. \end{aligned} \tag{2.36}$$

Equating the expressions on the LHS and RHS then gives

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}e^{-\xi[X_\tau - a]}\right] \\
&= \frac{4\xi e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}\, e^{\xi^2\frac{z}{2}}\Phi(-\xi\sqrt{z})}{H(\xi,\beta_1,\beta_2)} \\
&\quad - \frac{\sqrt{2\beta_1}\left[e^{\sqrt{2\beta_1}a}(\sqrt{2\beta_1}+\sqrt{2\beta_2}) - e^{-3\sqrt{2\beta_1}a}(\sqrt{2\beta_1}-\sqrt{2\beta_2})\right]\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}\right]}{H(\xi,\beta_1,\beta_2)},
\end{aligned} \tag{2.37}$$

where the function H is such that

$$\begin{aligned}
H(\xi,\beta_1,\beta_2) = \xi&\left[e^{\sqrt{2\beta_1}a}(\sqrt{2\beta_1}+\sqrt{2\beta_2}) + e^{-3\sqrt{2\beta_1}a}(\sqrt{2\beta_1}-\sqrt{2\beta_2})\right] \\
&- \sqrt{2\beta_1}\left[e^{\sqrt{2\beta_1}a}(\sqrt{2\beta_1}+\sqrt{2\beta_2}) - e^{-3\sqrt{2\beta_1}a}(\sqrt{2\beta_1}-\sqrt{2\beta_2})\right].
\end{aligned}$$

Looking at the expression in (2.37), we can say that there exists a $\xi^* > 0$ such that when the denominator to go to zero at ξ^*, the numerator goes to 0 at the same point for the triple Laplace transform to exist and not be identically 0. The value of ξ^* can be determined from the denominator by setting $H(\xi,\beta_1,\beta_2) = 0$ and this gives

$$\xi^* = \frac{2\beta_1}{\sqrt{2\beta_1}\,h(\beta_1) + \sqrt{2\beta_2}}\left(1 - h^2(\beta_1)\right) + \sqrt{2\beta_1}\,h(\beta_1),$$

where the function $h(\beta_1)$ is as defined in (2.12) Setting the numerator to 0 at ξ^* and notice that with $s = t - \xi\sqrt{z}$, we have

$$e^{\xi^2\frac{z}{2}}\Phi(-\xi\sqrt{z}) = e^{\xi^2\frac{z}{2}}\int_{\xi\sqrt{z}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}dt = \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2}\, e^{-s\xi\sqrt{z}}ds, \tag{2.38}$$

gives the joint Laplace transform

$$\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}\right] \\
&= \frac{4\xi^* e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{2\beta_1}\,\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2}\, e^{-s\xi^*\sqrt{z}}ds}{\sqrt{2\beta_1}\left[e^{\sqrt{2\beta_1}a}(\sqrt{2\beta_1}+\sqrt{2\beta_2}) - e^{-3\sqrt{2\beta_1}a}(\sqrt{2\beta_1}-\sqrt{2\beta_2})\right]} \\
&= \frac{4}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{1}{2}s^2}\exp\left(-s\sqrt{z}\frac{\sqrt{2}\beta_1(1-h^2(\beta_1))}{\sqrt{\beta_1}\,h(\beta_1)+\sqrt{\beta_2}}\right)\frac{1}{e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a}} \\
&\qquad \cdot \frac{e^{-s\sqrt{2z\beta_1}h(\beta_1)}e^{-\sqrt{2\beta_1}a}e^{\sqrt{2\beta_2}(a+x)}\sqrt{\beta_1}}{\sqrt{\beta_1}h(\beta_1)+\sqrt{\beta_2}}ds.
\end{aligned}$$

$\square$

2.9.2 Proof of Lemma 2.6.1

We now provide the steps to obtain the expression for the double Laplace transform in Lemma 2.6.1.

Proof. With the starting point $X_0 = x > a$, applying the optional stopping theorem to the martingales $f_i(X_t, V_t, Z_t^{(1)}, Z_t^{(2)}),\ i = 1, 2, 3$ with a bounded stopping time $\tau \wedge t$ gives

$$\mathbb{E}\left[e^{-\beta_1 V_{\tau\wedge t}} e^{-\beta_2 Z^{(2)}_{\tau\wedge t}} e^{\gamma Z^{(1)}_{\tau\wedge t}} f_1(X_{\tau\wedge t}) \middle| X_0 = x\right] = e^{-\beta_1 V_0} e^{-\beta_2 Z_0^{(2)}} e^{\gamma Z_0^{(1)}} f_1(X_0).$$

Taking limits for $t \to \infty$ gives us

$$\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{\gamma z} f_1(X_\tau)\right] = f_1(x),$$

$$\mathbb{E}_x\left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau)\right] = e^{-\gamma z} f_1(x).$$

We proceed using the same steps as in the previous section. With the change of variable of the form $\omega = \sqrt{2\gamma}$, multiplying both sides with $\frac{1}{\omega^2+\xi^2}$ and integrating with respect to ω over the range of ω, the right hand side of the expression gives

$$\begin{aligned}
&\int_0^\infty e^{-\gamma z} f_1(x) \frac{1}{\omega^2+\xi^2} d\omega \\
&= \int_0^\infty e^{-\frac{\omega^2}{2}z} \frac{1}{\omega^2+\xi^2}\left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a}\frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) B_1\cos(\omega[x-a])d\omega \\
&\quad + \int_0^\infty e^{-\frac{\omega^2}{2}z} \frac{1}{\omega^2+\xi^2}\left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a}\frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right)\frac{\sqrt{2\beta_1}}{\sqrt{2\gamma}} B_1\sin(\omega[x-a])d\omega.
\end{aligned}$$

The integrals can be computed as follows:

- The first integral with cosine function

$$\begin{aligned}
&\int_0^\infty e^{-\frac{\omega^2}{2}z}\frac{1}{\omega^2+\xi^2}\cos(\omega[x-a])d\omega \\
&= \frac{\pi}{4\xi} e^{\frac{z}{2}\xi^2}\left[e^{-\xi(x-a)}\mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi - \frac{x-a}{\sqrt{2z}}\right) + e^{\xi(x-a)}\mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi + \frac{x-a}{\sqrt{2z}}\right)\right],
\end{aligned}$$

  with the conditions

$$\Re\left(\frac{z}{2}\right) > 0, \qquad \Re(\xi) > 0,$$

where $\operatorname{erfc}(z)$ is the complementary error function defined as

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z).$$

- Using partial fraction, the second integral with sine function

$$\begin{aligned}
&\int_0^\infty e^{-\frac{\omega^2}{2}z} \frac{1}{\omega^2+\xi^2} \frac{1}{\sqrt{2\gamma}} \sin(\omega[x-a]) d\omega \\
&= \int_0^\infty e^{-\frac{\omega^2}{2}z} \frac{1}{\omega(\omega^2+\xi^2)} \sin(\omega[x-a]) d\omega \\
&= \frac{\pi}{2\xi^2} \operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right) - \frac{\pi}{4\xi^2} e^{\frac{z}{2}\xi^2} e^{-\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi - \frac{x-a}{\sqrt{2z}}\right) \\
&\quad + \frac{\pi}{4\xi^2} e^{\frac{z}{2}\xi^2} e^{\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi + \frac{x-a}{\sqrt{2z}}\right),
\end{aligned}$$

with the conditions

$$\arg\left(\frac{z}{2}\right) < \frac{\pi}{2}, \qquad \Re\left(\sqrt{\frac{z}{2}}\right) > 0, \qquad \Re(\xi) > 0.$$

Therefore, RHS of the expression becomes

$$\begin{aligned}
&\int_0^\infty e^{-\gamma z} f_1(x) \frac{1}{\omega^2+\xi^2} d\omega \\
&= \left[e^{-\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi - \frac{x-a}{\sqrt{2z}}\right) + e^{\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi + \frac{x-a}{\sqrt{2z}}\right)\right] \\
&\quad \cdot \left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) B_1 \frac{\pi}{4\xi} e^{\frac{z}{2}\xi^2} \\
&\quad + \left[\frac{\pi}{2\xi^2} \operatorname{erf}\left(\frac{x-a}{\sqrt{2z}}\right) - \frac{\pi}{4\xi^2} e^{\frac{z}{2}\xi^2} e^{-\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi - \frac{x-a}{\sqrt{2z}}\right)\right] \\
&\quad \cdot \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) \sqrt{2\beta_1} B_1 \\
&\quad + \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\right) \sqrt{2\beta_1} B_1 \frac{\pi}{4\xi^2} e^{\frac{z}{2}\xi^2} e^{\xi(x-a)} \operatorname{erfc}\left(\sqrt{\frac{z}{2}}\xi + \frac{x-a}{\sqrt{2z}}\right).
\end{aligned}$$

Now, computing the left hand side of the expression with $\omega = \sqrt{2\gamma}$ and Fubini's theorem, we have

$$
\begin{aligned}
&\int_0^{\infty} \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} f_1(X_\tau) \right] \frac{1}{\omega^2+\xi^2} d\omega \\
&= \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \int_0^{\infty} \left\{ \left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) B_1 \cos(\omega[X_\tau - a]) \right.\right. \\
&\qquad \left.\left. + \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) \frac{\sqrt{2\beta_1}}{\omega} B_1 \sin(\omega[X_\tau - a]) \right\} \frac{1}{\omega^2+\xi^2} d\omega \right] \\
&= \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) B_1 \frac{\pi}{2\xi} e^{-\xi[X_\tau - a]} \right] \\
&\qquad + \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) \sqrt{2\beta_1} B_1 \left(1 - e^{-\xi[X_\tau - a]}\right) \frac{\pi}{2\xi^2} \right].
\end{aligned}
$$

Let us define the following functions to simplify our calculations:

$$
\begin{aligned}
h(\beta_1) &= \frac{1+e^{-4\sqrt{2\beta_1}a}}{1-e^{-4\sqrt{2\beta_1}a}} = \frac{\cosh(2a\sqrt{2\beta_1})}{\sinh(2a\sqrt{2\beta_1})}, \\
g_1(\beta_1,\beta_2) &:= \left(e^{\sqrt{2\beta_1}a} + e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) \\
&= \frac{1}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \left[\sqrt{2\beta_1} h(\beta_1) + \sqrt{2\beta_2} \right] \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \right), \\
g_2(\beta_1,\beta_2) &:= \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \frac{\sqrt{2\beta_1}-\sqrt{2\beta_2}}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \right) \\
&= \frac{1}{\sqrt{2\beta_1}+\sqrt{2\beta_2}} \left[\sqrt{2\beta_1} + \sqrt{2\beta_2} h(\beta_1) \right] \left(e^{\sqrt{2\beta_1}a} - e^{-3\sqrt{2\beta_1}a} \right).
\end{aligned}
$$

Equating the expressions for LHS and RHS gives

$$
\begin{aligned}
&\mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} e^{-\xi[X_\tau - a]} \right] \left(g_1(\beta_1,\beta_2) - \frac{\sqrt{2\beta_1}}{\xi} g_2(\beta_1,\beta_2) \right) \\
&\qquad + \mathbb{E}_x \left[e^{-\beta_1 V_\tau} e^{-\beta_2 Z_\tau^{(2)}} \right] \frac{\sqrt{2\beta_1}}{\xi} g_2(\beta_1,\beta_2) \\
&= \frac{1}{2} e^{\frac{z}{2}\xi^2} e^{-\xi(x-a)} \mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi - \frac{x-a}{\sqrt{2z}} \right) \left[g_1(\beta_1,\beta_2) - \frac{\sqrt{2\beta_1}}{\xi} g_2(\beta_1,\beta_2) \right] \\
&\qquad + \frac{\sqrt{2\beta_1}}{\xi} g_2(\beta_1,\beta_2) \mathrm{erf}\left(\frac{x-a}{\sqrt{2z}} \right) \\
&\qquad + \frac{1}{2} e^{\frac{z}{2}\xi^2} e^{\xi(x-a)} \mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi + \frac{x-a}{\sqrt{2z}} \right) \left[g_1(\beta_1,\beta_2) + \frac{\sqrt{2\beta_1}}{\xi} g_2(\beta_1,\beta_2) \right].
\end{aligned}
$$

Making the triple Laplace transform the subject gives

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}e^{-\xi[X_\tau-a]}\right]\\
&=\frac{1}{\xi g_1(\beta_1,\beta_2)-\sqrt{2\beta_1}g_2(\beta_1,\beta_2)}\\
&\quad\cdot\left[\frac{1}{2}e^{\frac{z}{2}\xi^2}e^{-\xi(x-a)}\mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi-\frac{x-a}{\sqrt{2z}}\right)\left[\xi g_1(\beta_1,\beta_2)-\sqrt{2\beta_1}g_2(\beta_1,\beta_2)\right]\right.\\
&\qquad+\frac{1}{2}e^{\frac{z}{2}\xi^2}e^{\xi(x-a)}\mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi+\frac{x-a}{\sqrt{2z}}\right)\left[\xi g_1(\beta_1,\beta_2)+\sqrt{2\beta_1}g_2(\beta_1,\beta_2)\right]\\
&\qquad\left.+\sqrt{2\beta_1}g_2(\beta_1,\beta_2)\mathrm{erf}\left(\frac{x-a}{\sqrt{2z}}\right)-\mathbb{E}_x\left[e^{-\beta_1 V_\tau}e^{-\beta_2 Z_\tau^{(2)}}\right]\sqrt{2\beta_1}g_2(\beta_1,\beta_2)\right].
\end{aligned}
\tag{2.39}
$$

Looking at the expression in (2.39), we can say that there exists a $\xi^*>0$ such that when the denominator goes to 0 at ξ^*, the numerator goes to 0 at the same point for the triple Laplace transform to exist and not to be identically 0. The value of ξ^* can be determined by setting the denominator to 0.

$$
\begin{aligned}
0&=\xi^* g_1(\beta_1,\beta_2)-\sqrt{2\beta_1}g_2(\beta_1,\beta_2),\\
\xi^*&=\sqrt{2\beta_1}\frac{g_2(\beta_1,\beta_2)}{g_1(\beta_1,\beta_2)}\\
&=\sqrt{2\beta_1}\;\frac{\frac{1}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\left[\sqrt{2\beta_1}+\sqrt{2\beta_2}h(\beta_1)\right]\left(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a}\right)}{\frac{1}{\sqrt{2\beta_1}+\sqrt{2\beta_2}}\left[\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}\right]\left(e^{\sqrt{2\beta_1}a}-e^{-3\sqrt{2\beta_1}a}\right)}\\
&=\frac{2\beta_1}{\sqrt{2\beta_1}h(\beta_1)+\sqrt{2\beta_2}}\left[1-h^2(\beta_1)\right]+\sqrt{2\beta_1}h(\beta_1).
\end{aligned}
$$

Notice that with $s=t-\left[\sqrt{\frac{z}{2}}\xi+\frac{x-a}{\sqrt{2z}}\right]$, we can derive

$$
\mathrm{erfc}\left(\sqrt{\frac{z}{2}}\xi+\frac{x-a}{\sqrt{2z}}\right)=\frac{2}{\sqrt{\pi}}e^{-\frac{z}{2}\xi^2}e^{-(x-a)\xi}e^{-\frac{(x-a)^2}{2z}}\int_0^\infty e^{-s^2}e^{-2s\left(\sqrt{\frac{z}{2}}\xi+\frac{x-a}{\sqrt{2z}}\right)}ds.
$$

The double Laplace transform can then be obtained by setting the numerator to be 0 at ξ^*. $\square$

Chapter 3

Interrupted Brownian Motion

3.1 Introduction

A path dependent option is an option, as the name suggests, whose payoff is determined by the path history of the underlying asset price either throughout the whole or part of the life of the option. Path dependent options have received much attention in the recent years due to their innovative structure which can be designed to allow for different payoff outcomes in order to accommodate different risk profiles. Barrier option is a path dependent option whose payoff depends on the price of the underlying asset breaching a predetermined barrier level. The option then either comes into existence or vanishes prior to expiration, depending on the type or structure of the barrier option. Barrier options are widely used because they allow the buyers to incorporate their views on the movement of the asset price in the structure of the options. For example, a buyer who is concerned about possible sharp increase in asset price can invest in a knock-out barrier option as it can protect the buyer from unlimited liabilities when there is a rise in the underlying asset price. For more results on barrier options in discrete time setting, see P Wilmott and Howison (1993), Chesney et al. (1995), Pliska (1997), Zhang (1997), Wilmott (1998) and Musiela and Rutkowski (2006). For continuous time setting, see Rubinstein (1991), Rich (1994), Heynen and Kat (1995), Carr and Chou (1997), Baldi et al. (1999), Andersen et al. (2000), Linetsky (2004a) and Suchanecki (2004) and Jeanblanc et al. (2009).

Another type of option with path dependency structure is the lookback option whose payoff is related to the optimal value of the price of the underlying asset (either the maximum or the minimum) over certain period or the life of the option. Lookback options are attractive because they help to minimize regrets by lowering the uncertainty on the optimal timing on entering into or exiting from the market. This option is therefore particularly popular among buyers with some market movements anticipation during the life of the option but not knowing the exact time of these occurrences. Early works on lookback options can be found in Goldman et al. (1979a), Goldman et al. (1979b) and Conze (1991). More results can be found in He et al. (1998), Shreve et al. (2004), Musiela and Rutkowski (2006) and Jeanblanc et al. (2009).

An Asian option is a type of path dependent option in which the average of the underlying asset price over some duration of the life of the option is used to determine its payoff. This option is desired by many as the Asian option is usually cheaper than European or American option and its averaging structure helps to reduce price manipulation of the underlying asset close to the maturity period. For more results on Asian options, see Geman and Yor (1992), Yor (1995), Dufresne (2005), Schröder (2000), Donati-Martin et al. (2001), Geman and Yor (2001), Schröder (2001), Carr and Schröder (2004), Dufresne (2000), Linetsky (2004b) and Jeanblanc et al. (2009).

We now look at a special option with path dependency structure, the α-quantile option for $0 < \alpha < 1$. The study of the α-quantile option revolves around the α-quantile of the process $W = \{W_s,\ 0 \leq s \leq t\}$ which is defined as

$$M(\alpha, t) = \inf\{x : \int_0^t \mathbb{1}_{\{X_s \leq x\}} ds > \alpha t\},$$

for $\sigma \in \mathbb{R}^+, \mu \in \mathbb{R}$ and $W_t = \mu t + \sigma B_t$ where $B = \{B_t,\ t \geq 0\}$ is a one-dimensional Brownian motion starting from 0 as in Definition 1.0.1. The α-quantile option first introduced by Miura (1992) and the pricing of this option is investigated using Feynman-Kac formula by Akahori (1995) who derived the explicit form of the distribution function of the occupation time of a Brownian motion and Dassios (1995) who showed the identity in law between the sum of maximum and minimum of independent Brownian motions and the Brownian quantiles:

$$M(\alpha,t) \stackrel{(d)}{=} \sup_{0\le s\le \alpha t} W_s^{(1)} + \inf_{0\le s\le (1-\alpha)t} W_s^{(2)},$$

where $\stackrel{(d)}{=}$ denotes equality in distribution and W_s^2 is an independent copy of $W_1^{(1)}$. More results on the alpha-quantile of a Brownian motion can be found in Embrechts et al. (1995), Yor (1995), Takács (1996), Fusai (2000), Dassios (2005), Detemple (2005) and Jeanblanc et al. (2009).

In this chapter, we will focus on the double-quantile option which we have to consider for $0<\alpha<1$:

$$M(\alpha,t) = \inf\{x : \int_0^t \mathbb{1}_{\{W_s\ge x\}} ds + \int_0^t \mathbb{1}_{\{W_s\le -x\}} ds > \alpha t\}.$$

This leads us to the study of the distribution of occupation times of the Brownian motion above a predetermined level x and below the level $-x$ for any $x\in\mathbb{R}^+$. We proceed by introducing a new version of the Brownian motion which we call the Interrupted Brownian motion. It is the continuous version of a Brownian motion where we eliminate the paths of the Brownian motion within the band from $-x$ to x and join the remaining paths.

The main focus of this chapter is the construction of the interrupted Brownian motion using an reflected Brownian motion. We derive the stochastic differential equation (SDE) of the interrupted Brownian motion and compare this to the SDEs of some well-known processes. We then obtain some distributional properties of the interrupted Brownian motion such as the joint Laplace transform at a deterministic time t of the reflected Brownian motion and the number of interruption as well as the probability generating function using the definition of the infinitesimal generator and the extension of the martingale methodology developed by Dassios and Embrechts (1989) and Dassios and Jang (2003). We also look at the some distributional properties of the interrupted Brownian at the first passage time. In the last part of this chapter, we employ a new process, called the perturbed Brownian motion introduced by Dassios and Wu (2011) in order to study the excursion of the interrupted Brownian motion. We then derive the Laplace transform of the maximum height of the excursion of an interrupted Brownian motion with exponential time.

3.2 Connection

In this section, we look at the connection of the interrupted Brownian motion with other variants of Brownian motion by deriving the stochastic differential equations, (SDE) of the processes. The main ingredient for this section is the well known Doob's h-transform discussed in Doob (1957).

Theorem 3.2.1. *Let $X = \{X_t,\ t \geq 0\}$ be a 1-dimensional diffusion starting at $X_0 = x$ with SDE of the form*

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

with infinitesimal generator applied to a function $f \in \mathcal{D}(\mathcal{A})$

$$\mathcal{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \mu(x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(t,x), \tag{3.1}$$

and the function $h(t,x)$ is a positive harmonic function. Then, referring to Williams (1974) and Doob (1957), we can define the probability measure $\mathbb{P}^*$ as follows:

$$\mathbb{E}^*_x(Z) = \frac{1}{h(X_0)}\mathbb{E}_{X_0}(h(X_t)Z) = \frac{1}{h(x)}\mathbb{E}_x(h(X_t)Z),$$

for Z a $\mathcal{F}_t$-measurable random variable. The new measure $\mathbb{P}^*$ is the Doob's h-transform of $\mathbb{P}$. Under this measure $\mathbb{P}^*$, the process $\{X_t,\ t \geq 0\}$ is a Markov process with infinitesimal generator $\mathcal{A}^*$ for a bounded and measurable function f of the form:

$$\mathcal{A}^*f = h^{-1}\mathcal{A}(hf).$$

From (3.1), it follows that the infinitesimal generator under $\mathbb{P}^*$ applied to a function $f \in \mathcal{D}(\mathcal{A}^*)$ can be derived as:

$$\mathcal{A}^*f(t,x) = \frac{\partial f}{\partial t}(t,x) + \left(\mu(x) + \frac{\sigma^2(x)}{h(t,x)}\frac{\partial h}{\partial x}(t,x)\right)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\sigma^2(x)\frac{\partial^2 f}{\partial x^2}(t,x). \tag{3.2}$$

3.2.1 SDE: Brownian Bridge

In order to construct a Brownian bridge, $W^{(Br)} = \{W_t^{(Br)}),\ 0 \leq t \leq T\}$ from x to y for every $y \in \mathbb{R}$, we have to condition on its starting and ending points. We consider the transition probability density for a Brownian motion such that given $W_0 = x$, the probability density for W_{T-t}, where T is fixed, is given by:

$$h(t,x,y) = \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}}.$$

Therefore, using (3.2) with $\mu = 0$ and $\sigma = 1$, we can deduce that the infinitesimal generator of the conditioned process is

$$\begin{aligned}
&\mathcal{A}^* f(t,x) \\
&= \frac{\mathcal{A}(fh)(t,x)}{h_1(t,x)} \\
&\overset{(3.2)}{=} \frac{\partial f}{\partial t} + \left(\mu(x) + \frac{\sigma(x)}{h_1}\frac{\partial h_1}{\partial x}\right)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2 f}{\partial x^2} \\
&= \frac{\partial f}{\partial t} + \left(\mu(x) + \frac{\sigma(x)}{h_1}\frac{y-x}{(T-t)\sqrt{2\pi(T-t)}} e^{-\frac{x^2}{2(T-t)}}\right)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma(x)\frac{\partial^2 f}{\partial x^2} \\
&\overset{(\mu=0,\sigma=1)}{=} \frac{\partial f}{\partial t} + \left(\frac{y-x}{T-t}\right)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}.
\end{aligned}$$

The dynamics of the Brownian bridge for $t \in [0,T]$ is then of the form

$$dW_t^{(Br)} = \frac{y - W_T^{(Br)}}{T-t} dt + dW_t. \tag{3.3}$$

For more properties of Brownian bridge, see for example Pitman (1999) and Vervaat (1979).

3.2.2 SDE: Brownian Meander

For a Brownian meander, $W^{(Me)} = \{W_t^{(Me)}), \ 0 \le t \le T\}$, we condition on a Brownian motion to stay positive until T. This is equivalent to conditioning on the event that the first time the Brownian motion reaches zero is after the period $[0,T]$. Therefore, we use the function $h(t,x)$ for Brownian meander of the following form:

$$\begin{aligned}
h(t,x) &= \int_{T-t}^{\infty} f_{\tau_0}(x) ds \\
&= \int_{T-t}^{\infty} \frac{x}{\sqrt{2\pi s^3}} e^{-\frac{x^2}{2s}} ds \\
&= \operatorname{erf}\left(\frac{x}{\sqrt{2(T-t)}}\right).
\end{aligned}$$

Using the Doob's h-transform in Theorem 3.2.1, the conditioned process has the infinitesimal generator as follwos:

$$\begin{aligned}\mathcal{A}^* f(t,x) &= \frac{\mathcal{A}f(t,x)}{h(t,x)} \\ &= \frac{\partial f}{\partial t}(t,x) + \frac{1}{\operatorname{erf}\left(\frac{x}{\sqrt{2(T-t)}}\right)} \frac{2\, e^{-\frac{x^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}} \frac{\partial f}{\partial x}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x),\end{aligned}$$

and the dynamics of a Brownian meander is:

$$dW_t^{(Me)} = \left(\frac{2}{\sqrt{2\pi(T-t)}} \frac{1}{\exp\left(\left[\frac{W_t^{(Me)}}{\sqrt{2(T-t)}}\right]^2\right) \operatorname{erf}\left(\frac{W_t^{(Me)}}{\sqrt{2(T-t)}}\right)} \right) dt + dW_t. \tag{3.4}$$

For more properties of Brownian meander, see for example Durrett and Iglehart (1977), Pitman (1999).

3.2.3 SDE: Brownian Excursion

For the Brownian excursion $W^{(Ex)} = \{W_t^{(Ex)},\ 0 \le t \le T\}$, we condition on the event of hitting 0 at time T and not before. We choose the function h to be the first hitting time density such that

$$h(t,x) = \frac{x}{\sqrt{2\pi(T-t)^3}} e^{-\frac{x^2}{2(T-t)}}.$$

The infinitesimal generator of the conditioned process is:

$$\begin{aligned}&\mathcal{A}^* f(t,x) \\ &= \frac{\mathcal{A}(fh_1)(t,x)}{h_1(t,x)} \\ &\overset{(\mu=0,\sigma=1)}{=} \frac{\partial f}{\partial t} + \left(\frac{1}{x} - \frac{x}{T-t}\right)\frac{\partial f}{\partial x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}.\end{aligned} \tag{3.5}$$

Therefore, we can obtain the SDE for a Brownian excursion,

$$dW_t^{(Ex)} = \left(\frac{1}{W_t^{(Ex)}} - \frac{W_t^{(Ex)}}{T-t}\right) dt + dW_t. \tag{3.6}$$

For more properties of Brownian excursion, see for example Durrett and Iglehart (1977), Pitman (1999) and Vervaat (1979).

3.3 Construction of Interrupted Brownian Motion

3.3.1 Definition

We present the formal definition of an interrupted Brownian motion.

Definition 3.3.1. ***Interrupted Brownian Motion***
The interrupted Brownian motion can be constructed by defining it as

$$(-1)^{N_t} X_t, \tag{3.7}$$

where

- $\{N_t,\ t \geq 0\}$, *number of interruption is a renewal process defined by the sequence of arrival time process* $(T_0, T_1, \ldots)$ *such that the time of the n-th arrival is*

  $$T_n = \sum_{i=1}^{n} L_i^{(\tau)},$$

  where $L_i^{(\tau)}$ *is the occupation time process of a Brownian motion defined as*

  $$L_i^{(\tau)} = \int_0^{\tau} \mathbb{1}_{\{B_s^{(i)} > 0\}} ds,$$

  for $\tau = \inf\{s \leq t :\ B_s = -2a\}$ *for* B *a standard Brownian motion.*

- $\{X_t,\ t \geq 0\}$ *is a reflected Brownian motion with reflecting barrier at* 0.

From this definition, it is clear that the **interrupted Brownian motion** is a continuous version of the Brownian motion whose paths within a specified interval have been eliminated. This is further presented by Figures 3.1 and 3.2, which show the construction of an interrupted Brownian motion with the standard Brownian motion. This is done by discarding the negative negative excursions and adjusting the time scale to provide appropriate representation of the bahviour of the interrupted Brownian motion.

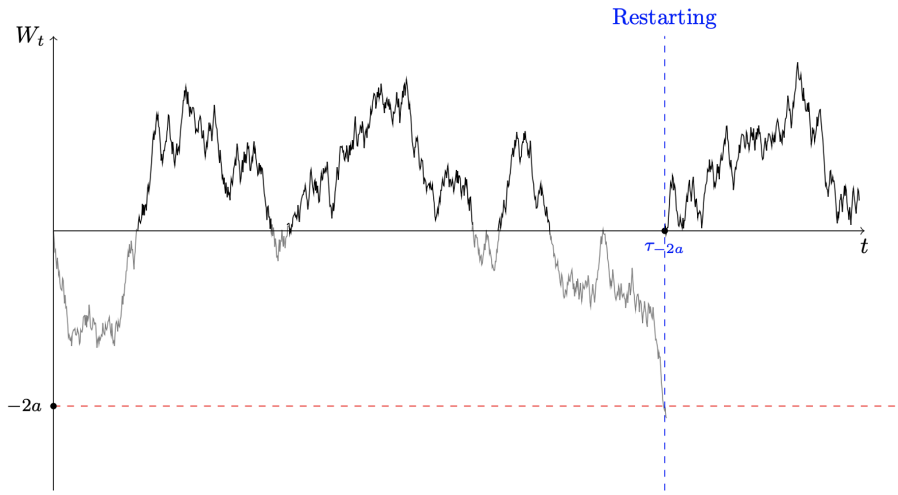

Figure 3.1: Standard Brownian Motion with Renewal.

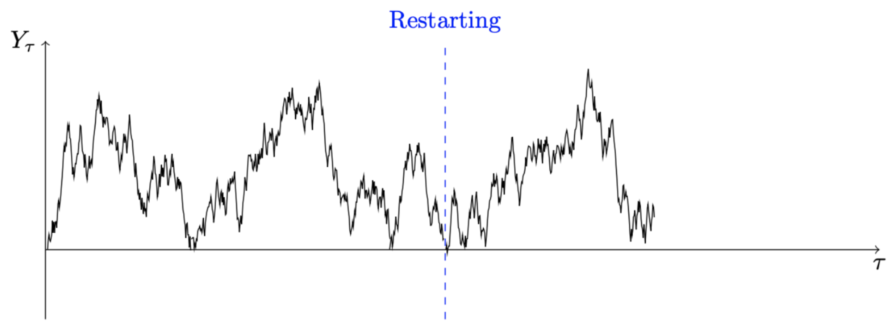

Figure 3.2: Interrupted Brownian Motion.

3.3.2 SDE: Interrupted Brownian Motion

In this subsection, we provide, in details, the derivation of the SDE of an interrupted Brownian motion. Let us define the occupation time process as follows:

$$\begin{aligned} \Gamma_t &:= \text{meas}\{0 \le s \le t; W_s > 0\}, \\ &= \int_0^t \mathbb{1}_{\{W_s>0\}} ds. \end{aligned} \tag{3.8}$$

Let us now introduce the right-continuous inverses of the occupation time $\{\Gamma_+(t),\ t \geq 0\}$ as

$$\Gamma^{-1}(\tau) = \inf\{t \geq 0;\ \Gamma_t > \tau\}; \qquad 0 \leq \tau < \infty. \tag{3.9}$$

The interrupted Brownian motion is then given as

$$W_{\Gamma^{-1}(\tau)}\Big| \tau < \Gamma(\tau_{-2a}). \tag{3.10}$$

From Karatzas and Shreve (1991), it is known that the time changed process

$$W_{\Gamma^{-1}(t)}; \qquad t \geq 0, \tag{3.11}$$

is a continuous Markov process. If f is a locally integrable function such that the support of f is an interval which contains the origin, then from Bertoin (1999), we have that the generator of the process $W_{\Gamma_\tau^{-1}}$ is such that

$$\mathcal{A}f(t,x) = \frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x); \qquad x > 0, \tag{3.12}$$

such that

$$\lim_{x \downarrow 0} f'(t,x) = 0.$$

For further reference, we direct readers to McKean (1963), Itō and McKean (1974b) and Jeanblanc et al. (2009). In order to take into account the condition for the interrupted Brownian motion, we use Doob's h-transform where h is the function such that

$$h(\tau,x) = 1 + e^{\frac{x}{2a}}\ e^{\frac{\tau}{2(2a)^2}}\ \mathrm{erfc}\left(\frac{x}{\sqrt{2\tau}} + \frac{\sqrt{\tau}}{\sqrt{2}(2a)}\right) - \mathrm{erfc}\left(\frac{x}{\sqrt{2\tau}}\right). \tag{3.13}$$

Therefore, we have that

$$\frac{\partial h}{\partial x}(\tau,x) = \frac{1}{2a}e^{\frac{x}{2a}}\ e^{\frac{\tau}{2(2a)^2}}\mathrm{erfc}\left(\frac{x}{\sqrt{2\tau}} + \frac{\sqrt{\tau}}{\sqrt{2}(2a)}\right). \tag{3.14}$$

With $\tau = T - t$, the infinitesimal generator $\mathcal{A}^*$ for the conditioned process is then given as

$$\begin{aligned}
&\mathcal{A}^* f(t,x) \\
&= \frac{\partial f}{\partial t}(t,x) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) \\
&\quad + \left[\frac{\frac{1}{2a}\, e^{\frac{x}{2a}}\, e^{\frac{T-t}{2(2a)^2}} \operatorname{erfc}\left(\frac{x}{\sqrt{2(T-t)}} + \frac{\sqrt{T-t}}{\sqrt{2}(2a)} \right)}{1 + e^{\frac{x}{2a}}\, e^{\frac{T-t}{2(2a)^2}} \operatorname{erfc}\left(\frac{x}{\sqrt{2(T-t)}} + \frac{\sqrt{T-t}}{\sqrt{2}(2a)} \right) - \operatorname{erfc}\left(\frac{x}{\sqrt{2(T-t)}} \right)} \right] \frac{\partial f}{\partial x}(t,x),
\end{aligned} \tag{3.15}$$

for $x \in \mathbb{R}_+$ and with the condition

$$\lim_{x \downarrow 0} f'(x) = 0.$$

The dynamics of the interrupted Brownian motion, $Y = \{Y_t,\ 0 \leq t \leq T\}$ is then given as

$$dY_t = \left[\frac{\frac{1}{2a}\, e^{\frac{Y_t}{2a}}\, e^{\frac{T-t}{2(2a)^2}} \operatorname{erfc}\left(\frac{Y_t}{\sqrt{2(T-t)}} + \frac{\sqrt{T-t}}{\sqrt{2}(2a)} \right)}{1 + e^{\frac{Y_t}{2a}}\, e^{\frac{T-t}{2a^2}} \operatorname{erfc}\left(\frac{Y_t}{\sqrt{2(T-t)}} + \frac{\sqrt{T-t}}{\sqrt{2}(2a)} \right) - \operatorname{erfc}\left(\frac{Y_t}{\sqrt{2(T-t)}} \right)} \right] dt + dW_t. \tag{3.16}$$

Long Term Behavior

In this part, we investigate the long term behaviour of the interrupted Brownian motion. Using the SDE derived in (3.16), we see that as $T \to \infty$, the drift term becomes

$$\begin{aligned}
&\lim_{\tau \to \infty} \frac{\frac{1}{2a} e^{\frac{x}{2a}}\, e^{\frac{\tau}{2(2a)^2}} \operatorname{erfc}\left(\frac{x}{\sqrt{2\tau}} + \frac{\sqrt{\tau}}{\sqrt{2}(2a)} \right)}{1 + e^{\frac{x}{2a}}\, e^{\frac{\tau}{2(2a)^2}} \operatorname{erfc}\left(\frac{x}{\sqrt{2\tau}} + \frac{\sqrt{\tau}}{\sqrt{2}(2a)} \right) - \operatorname{erfc}\left(\frac{x}{\sqrt{2\tau}} \right)} \\
&= \frac{1}{2a + x}.
\end{aligned}$$

The long term dynamics of the interrupted Brownian motion is

$$dY_t = \frac{1}{2a + Y_t} dt + dW_t; \qquad Y_t > 0.$$

We can see that as $T \to \infty$, the process behaves like a Bessel process staying above the level $2a$.

3.3.3 Infinitesimal Generator

We use the martingale approach discussed in Remark 2.2.1 to find a martingale of the form

$$f(X_t, W_t, t),$$

where $\{X_t,\ t \geq 0\}$ is a reflected Brownian motion and the process $W = \{W_t,\ t \geq 0\}$ is given as

$$W_t = \int_0^t e^{-\beta s} \theta^{N_s} e^{-\gamma X_s} ds.$$

The infinitesimal generator of the process $\{(X_t, t),\ t \geq 0\}$ acting on a function $f_n : \mathbb{R}_+^2 \to \mathbb{R}$ belonging to its domain is given by:

$$\mathcal{A} f_n(x, t) = \frac{\partial f_n}{\partial t}(x, t) + \frac{1}{2} \frac{\partial^2 f_n}{\partial x^2}(x, t); \qquad x > 0.$$

We then extend this process by adding one other component, the process $\{W_t,\ t \geq 0\}$ so that the infinitesimal generator of the process $\{(X_t, W_t, t),\ t \geq 0\}$ acting on a function $f_n : \mathbb{R}_+^3 \to \mathbb{R}$ belonging to its domain is given by:

$$\mathcal{A} f_n(x, w, t) = \frac{\partial f_n}{\partial t}(x, w, t) + e^{-\beta t} \theta^n e^{-\gamma x} \frac{\partial f_n}{\partial w}(x, w, t) + \frac{1}{2} \frac{\partial^2 f_n}{\partial x^2}(x, w, t). \tag{3.17}$$

Let us assume that f_n takes the following form:

$$f_n(x, w, t) = w + e^{-\beta t} f_n(x). \tag{3.18}$$

Substituting (3.18) into (3.17) and setting it to 0, we have to solve the second-order linear non-homogeneous differential equation of the form:

$$-\beta f_n(x) + \theta^n e^{-\gamma x} + \frac{1}{2} f_n''(x) = 0, \tag{3.19}$$

where $f_n''(x)$ is the second derivative with respect to x of $f_n(x)$. The solution to (3.19) is the sum of the particular solution of the form $e^{-\gamma x}$ and the complementary solution which can be obtained by solving the homogeneous differntial equation:

$$\frac{1}{2} f_n''(x) - \beta f_n(x) = 0.$$

We can obtain that $f_n(x)$ as:

$$f_n(x) = \frac{2\theta^2 e^{-\gamma x}}{2\beta - \gamma^2} + C_n e^{-\sqrt{2\beta}x}, \tag{3.20}$$

where we have also ensured that f_n as defined above is bounded. In order to derive a condition satisfied by the function $f_n(x)$ for the interrupted Brownian motion, we start by restricting a standard Brownian motion to only run above level $-2a$ where $a > 0$ with interruption levels at 0 and $-2a$. By imposing some differentiability conditions and by forcing this Brownian motion to start afresh at the level 0 when it hits the level $-2a$, we can establish a condition for the function $f_n(x)$. We then include N_t by recording the number of times the process restarts. With this, we can deduce that the condition that $f_n(x)$ has to satisfy is

$$f_{n+1}(0) = -2af_n'(0) + f_n(0). \tag{3.21}$$

Solving the function (3.20) subject to the condition (3.21), we can obtain $f_n(x)$:

$$f_n(x) = \frac{2\theta^n}{(\sqrt{2\beta}-\gamma)(\sqrt{2\beta}+\gamma)} \left[e^{-\gamma x} + \frac{1+2a\gamma-\theta}{\theta - 1 - 2a\sqrt{2\beta}} e^{-\sqrt{2\beta}x} \right]. \tag{3.22}$$

Therefore, from (3.18), we see that $f_{N_t}(X_t, W_t, t)$ is a martingale:

$$\begin{aligned} & f_{N_t}(X_t, W_t, t) \\ &= W_t + e^{-\beta t} f_{N_t}(X_t) \\ &= W_t + e^{-\beta t} \left(\frac{2\theta^{N_t}}{(\sqrt{2\beta}-\gamma)(\sqrt{2\beta}+\gamma)} \left[e^{-\gamma X_t} + \frac{1+2a\gamma-\theta}{\theta-1-2a\sqrt{2\beta}} e^{-\sqrt{2\beta}X_t} \right] \right). \end{aligned} \tag{3.23}$$

3.4 Distributional Properties

In this section, we derive the joint Laplace transform of (N_t, X_t), the probability generating function (N_t, X_t) and the probability generating function of N_t.

3.4.1 Joint Laplace Transform of $\mathbb{E}(\theta^{N_t}e^{-\gamma X_t})$

Theorem 3.4.1. *For $|\theta| \leq 1$ and $\gamma \in \mathbb{C}_+$, the joint Laplace transform of $\mathbb{E}\left(\theta^{N_t}e^{-\gamma X_t}\right)$ is obtained as:*

$$\int_0^\infty e^{-\beta s}\mathbb{E}\left(\theta^{N_s}e^{-\gamma X_s}\right) ds = \frac{2\theta^{2N_0}e^{-\gamma x}}{2\beta-\gamma^2} + \frac{2\theta^{2N_0}(1+2a\gamma-\theta)}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}. \tag{3.24}$$

Proof. From (3.23), we can easily deduce that $W_t + e^{-\beta t}\theta^{N_t} f_{N_t}(X_t)$ is a martingale, therefore, using property of martingales, we have that:

$$\mathbb{E}(W_t) + \mathbb{E}\left(e^{-\beta t}\theta^{N_t} f_{N_t}(X_t)\right) = \theta^{N_0} f_{N_0}(X_0).$$

Taking limits $t \to \infty$ on both sides and using Dominated Convergence Theorem (DCT), we have

$$\lim_{t\to\infty}\mathbb{E}(W_t) + \lim_{t\to\infty}\mathbb{E}\left(e^{-\beta t}\theta^{N_t} f_{N_t}(X_t)\right) = \lim_{t\to\infty}\theta^{N_0} f_{N_0}(X_0)$$

$$\lim_{t\to\infty}\mathbb{E}(W_t) = \theta^{N_0} f_{N_0}(X_0).$$

The second term goes to 0 as $t \to \infty$ because $|\theta| \leq 1$ and the function f_n is bounded by construction. Following from the previous equation and with the aid of Fubini's theorem, (3.24) follows.

□

3.4.2 Probability Generating Function of (N_t, X_t)

Theorem 3.4.2. *For $|\theta| \leq 1$ and $\gamma \in \mathbb{C}_+$, we can obtain the joint expectation of N_t and X_t as:*

$$\begin{aligned}
&\mathbb{E}\left(\theta^{N_t}e^{-\gamma X_t}\right) \\
&= 2\theta^{2N_0}e^{\frac{(\theta-1)^2 t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}\frac{\theta-1}{2a\gamma+\theta-1} + \frac{\theta^{2N_0}}{2}e^{\frac{t\gamma^2}{2}}e^{-\gamma x}\,erfc\left(\gamma\sqrt{\frac{t}{2}} - \frac{x}{\sqrt{2t}}\right) \\
&\quad + \theta^{2N_0}e^{\frac{t\gamma^2}{2}}e^{\gamma x}\,erfc\left(\gamma\sqrt{\frac{t}{2}} + \frac{x}{\sqrt{2t}}\right)\left[\frac{1}{2} - \frac{\theta-1}{\theta-1+2a\gamma}\right] \\
&\quad - \frac{\theta^{2N_0}(\theta-1)}{\theta-1+2a\gamma}e^{\frac{(\theta-1)^2 t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}\,erfc\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}} - \frac{x}{\sqrt{2t}}\right),
\end{aligned} \tag{3.25}$$

where erfc(x) is the complementary error function defined as

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt.$$

Proof. From (3.24), the joint Laplace transform of (N_t, X_t), we can invert with respect to β to obtain $\mathbb{E}\left(\theta^{N_s} e^{-\gamma X_s}\right)$. We can do this term by term:

- For the first term:

$$\mathcal{L}_\beta^{-1}\left(\frac{2\theta^{2N_0}e^{-\gamma x}}{2\beta - \gamma^2}\right)(t) = \theta^{2N_0}e^{-\gamma x + \frac{1}{2}\gamma^2 t}. \tag{3.26}$$

- For the second term, we invert the Laplace transform to get:

$$\begin{aligned}
&\mathcal{L}_\beta^{-1}\left(\frac{2\theta^{2N_0}(1+2a\gamma-\theta)}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}\right)(t) \\
&= -e^{\frac{\gamma^2 t}{2}}e^{-\gamma x}\theta^{2N_0} + 2\theta^{2N_0}e^{\frac{(\theta-1)^2 t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}(\theta-1)\frac{1}{2a\gamma+\theta-1} \\
&\quad + \frac{\theta^{2N_0}}{\theta-1+2a\gamma}e^{\frac{t\gamma^2}{2}}e^{-\gamma x}\text{erfc}\left(\gamma\sqrt{\frac{t}{2}} - \frac{x}{\sqrt{2t}}\right)\left[\frac{\theta-1}{2}+a\gamma\right] \\
&\quad + \frac{\theta^{2N_0}}{\theta-1+2a\gamma}e^{\frac{t\gamma^2}{2}}e^{\gamma x}\text{erfc}\left(\gamma\sqrt{\frac{t}{2}} + \frac{x}{\sqrt{2t}}\right)\left[a\gamma - \frac{\theta-1}{2}\right] \\
&\quad - \frac{\theta^{2N_0}(\theta-1)}{\theta-1+2a\gamma}e^{\frac{(\theta-1)^2 t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}\text{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}} - \frac{x}{\sqrt{2t}}\right).
\end{aligned} \tag{3.27}$$

The details on inverting this Laplace transform are included in the Appendix 3.6. Combining (3.26) and (3.27), we have that the inversion with respect to β of (3.24) is (3.25).

□

3.4.3 Joint Distribution of N_t and X_t

Theorem 3.4.3. *For $|\theta| \leq 1$ and $\mathbb{1}_{\{A\}}$ be the indicator function, we have:*

$$
\begin{aligned}
&\mathbb{E}\left[\theta^{N_t}\mathbb{1}_{\{X_t\in dy\}}\right] \\
&= \frac{\theta^{2N_0}(\theta-1)}{2a} e^{\frac{(\theta-1)^2t}{8a^2}} e^{-\frac{(\theta-1)}{2a}(x+y)} \left[2 - erfc\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}} - \frac{x}{\sqrt{2t}}\right)\right] \\
&+ \frac{\theta^{2N_0}}{\sqrt{2\pi t}}\left[e^{-\left(\frac{y}{\sqrt{2t}}-\frac{x}{\sqrt{2t}}\right)^2} + e^{-\left(\frac{y}{\sqrt{2t}}+\frac{x}{\sqrt{2t}}\right)^2}\right] \\
&- \frac{\theta^{2N_0}(\theta-1)}{2a} e^{-\frac{x^2}{2t}} e^{-\frac{\theta-1}{2a}y}\, e^{\frac{\left(\frac{\theta-1}{2a}-\frac{x}{t}\right)^2}{2}t}\left[erf\left(\frac{y}{\sqrt{2t}} - \frac{\left(\frac{\theta-1}{2a}-\frac{x}{t}\right)\sqrt{t}}{\sqrt{2}}\right)\right. \\
&\qquad\left. + erf\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}} - \frac{x}{\sqrt{2t}}\right)\right],
\end{aligned}
\tag{3.28}
$$

where $erfc(x)$ is as defined earlier and erf, the error function is given as:

$$
erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.
$$

Proof. With the expectation as defined in (3.25), we can derive the joint distribution of N_t, the number of interruptions and X_t, the reflected Brownian motion by inverting the expectation with respect to γ. We can do this term by term as before but with respect to γ this time to obtain the joint distribution of N_t and X_t:

- For the first term:

$$
\begin{aligned}
&\mathcal{L}_\gamma^{-1}\left(2\theta^{2N_0} e^{\frac{(\theta-1)^2t}{8a^2}} e^{-\frac{(\theta-1)x}{2a}}(\theta-1)\frac{1}{2a\gamma+\theta-1}\right)(y) \\
&= \frac{\theta-1}{a}\,\theta^{2N_0} e^{\frac{(\theta-1)^2t}{8a^2}} e^{-\frac{(\theta-1)}{2a}(x+y)}.
\end{aligned}
\tag{3.29}
$$

- For the second term:

$$
\begin{aligned}
&\mathcal{L}_\gamma^{-1}\left(\frac{\theta^{2N_0}}{2} e^{\frac{t\gamma^2}{2}} e^{-\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}} - \frac{x}{\sqrt{2t}}\right)\right)(y) \\
&= \frac{\theta^{2N_0}}{\sqrt{2\pi t}}\, e^{-\left(\frac{y}{\sqrt{2t}}-\frac{x}{\sqrt{2t}}\right)^2}.
\end{aligned}
\tag{3.30}
$$

- For the third term

$$
\begin{aligned}
&\mathcal{L}_{\gamma}^{-1}\left(\theta^{2N_0}e^{\frac{t\gamma^2}{2}}e^{\gamma x}\text{erfc}\left(\gamma\sqrt{\frac{t}{2}}+\frac{x}{\sqrt{2t}}\right)\left[\frac{1}{2}-\frac{\theta-1}{\theta-1+2a\gamma}\right]\right)(y)\\
&=\frac{\theta^{2N_0}}{\sqrt{2\pi t}}e^{-\left(\frac{y}{\sqrt{2t}}+\frac{x}{\sqrt{2t}}\right)^2}\\
&-\frac{\theta^{2N_0}(\theta-1)}{2a}e^{-\frac{x^2}{2t}}e^{-\frac{\theta-1}{2a}y}\,e^{\frac{\left(\frac{\theta-1}{2a}-\frac{x}{t}\right)^2}{2}t}\left[\text{erf}\left(\frac{y}{\sqrt{2t}}-\frac{\left(\frac{\theta-1}{2a}-\frac{x}{t}\right)\sqrt{t}}{\sqrt{2}}\right)\right.\\
&\left.+\,\text{erf}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right)\right].
\end{aligned}
\tag{3.31}
$$

- For the forth term

$$
\begin{aligned}
&\mathcal{L}_{\gamma}^{-1}\left(-\frac{\theta^{2N_0}(\theta-1)}{\theta-1+2a\gamma}e^{\frac{(\theta-1)^2t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}\text{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right)\right)(y)\\
&=-\frac{\theta^{2N_0}(\theta-1)}{2a}e^{\frac{(\theta-1)^2t}{8a^2}}e^{-\frac{(\theta-1)}{2a}(x+y)}\text{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right).
\end{aligned}
\tag{3.32}
$$

Combining (3.29), (3.30), (3.31) and (3.32), we can obtain (3.28).

□

3.4.4 Probability Generating Function of N_t

Theorem 3.4.4. *For $|\theta|\le 1$, the probability generating function of N can be obtain as*

$$
\begin{aligned}
&\mathbb{E}(\theta^{N_t})=:G_{N_t}(\theta)\\
&=\theta^{2N_0}\ \mathit{erf}\left(\frac{x}{\sqrt{2t}}\right)+\theta^{2N_0}e^{\frac{t(\theta-1)^2}{8a^2}}e^{-\frac{x(\theta-1)}{2a}}\left[2-\mathit{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right)\right],
\end{aligned}
\tag{3.33}
$$

where erf(x) and erfc(x) are as defined earlier.

Proof. From equation (3.24), we have that

$$
\int_0^{\infty}e^{-\beta s}\mathbb{E}\left(\theta^{N_s}e^{-\gamma X_s}\right)ds=\frac{2\theta^{2N_0}e^{-\gamma x}}{2\beta-\gamma^2}+\frac{2\theta^{2N_0}(1+2a\gamma-\theta)}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}.
$$

If we take limits $\gamma \to 0$, we can obtain the following expression:

$$\int_0^\infty e^{-\beta s}\mathbb{E}\left(\theta^{N_s}\right) ds = \frac{2\theta^{2N_0}}{2\beta} + \frac{2\theta^{2N_0}(1-\theta)}{(2\beta)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}.$$

This is the Laplace transform of the $\mathbb{E}(\theta^{N_t})$. We can then obtain the probability generating function by inverting the Laplace transform with respect to β to obtain (3.33).

□

Remark 3.4.5. *We can also obtain the probability generating function for N_t in (3.33) from (3.28) by integrating over the range of y.*

Corollary 3.4.6. *Given the probability generating function of N_t as above, we can derive the $\mathbb{P}(N_t = n)$ using the following formula:*

$$\mathbb{P}(N_t = n) = \frac{G_{N_t}^{(n)}(0)}{n!},$$

where $G_{N_t}^{(n)}(\theta)$ is the n-th derivative of $G_{N_t}(\theta)$. This gives us probability of the n numbers of interruptions that the process had undergone.

Corollary 3.4.7. *Using the definition of a probability generating function, we have that*

$$G_{N_t}(\theta) = \mathbb{E}(\theta^{N_t}) = \sum_{n=0}^{\infty} P_{N_t}(n)\theta^n,$$

where P_{N_t} is the probability mass function of N_t. By setting $\theta = -1$, we observe that

$$\begin{aligned}\mathbb{E}\left(-1^{N_t}\right) &= \mathbb{P}(N_t = 0) - \mathbb{P}(N_t = 1) + \mathbb{P}(N_t = 2) - \mathbb{P}(N_t = 3) + \ldots \\ &= \mathbb{P}(N_t = even) - \mathbb{P}(N_t = odd).\end{aligned}$$

This shows us that when we choose the value of θ to be $\theta = -1$, we obtain a sequence with alternating signs and this gives us information about the sign of the interrupted Brownian motion.

3.4.5 Distribution of First Passage Time of Interrupted Brownian Motion

We want to find the first hitting time of the level $b \in \mathbb{R}_+$ of the interrupted Brownian motion before the first interruption occurs. Let us define the following hitting times:

$$\begin{aligned} \tilde{\tau}_b^* &:= \inf\{t \geq 0 \ : \ Y_t \geq b\}, \\ \tau_b &:= \inf\{t \geq 0 \ : \ W_t \geq b\}, \\ \tau_{-a,b} &:= \inf\{t \geq 0 \ : \ W_t \notin (-a,b)\}, \end{aligned} \tag{3.34}$$

where $\{Y_t, \ t \geq 0\}$ is the interrupted Brownian motion and $\{W_t, \ t \geq 0\}$ is the standard Brownian motion. Let us also define the occupation times of the standard Brownian motion to be

$$\begin{aligned} \Gamma_t^+ &:= \int_0^t \mathbb{1}_{\{W_s \geq 0\}} ds, \\ \tilde{\Gamma}_t^{(c_1,c_2)} &:= \int_0^t \mathbb{1}_{\{c_1 \leq W_s \leq c_2\}} ds. \end{aligned}$$

where c_1 and c_2 are non-negative constants.

Theorem 3.4.8. *The density of the first passage time of the level $b > 0$ of the interrupted Brownian motion can be derived as*

$$\begin{aligned} &\mathbb{P}_x[\tilde{\tau}_b^* \in dy] \\ &= \begin{cases} (x+a)\tilde{rc}_y(1,b,0,a)dy; & -a \leq x \leq 0, \\ ss_y(x,b)dy + a \cdot ss_y(b-x,b) * \tilde{rc}_y(1,b,0,a)dy; & 0 \leq x \leq b, \end{cases} \end{aligned} \tag{3.35}$$

where

- $f(t) * g(t) \quad := \displaystyle\int_0^t f(s)g(t-s)ds$ *is the convolution of functions f and g.*
- $ss_y(x,b) \quad := \displaystyle\sum_{k=-\infty}^{\infty} \frac{b-x+2kb}{\sqrt{2\pi y^3}}\, e^{-\frac{(b-x+2kb)^2}{2y}}.$
- $\tilde{rc}_y(1,b,0,a) := \displaystyle\sum_{k=0}^{\infty} \frac{2(-1)^k}{a^{k+1}} \sum_{l=0}^{k} (-1)^l \binom{k}{l} \sum_{j=0}^{\infty} \frac{(-1)^j\, e^{-\frac{(2b(k+j-l)+b)^2}{4y}}}{\sqrt{2\pi}\, y^{1-\frac{k}{2}}} \binom{k+j}{j}$

  $\displaystyle\cdot D_{1-k}\left(\frac{2b(k+j-l)+b}{\sqrt{y}}\right),$

where $D_n(x)$ is the parabolic cylinder function of order n.

Proof. Since the events $\{\tau_b < \tau_{-a}\}$ and $\{W_{\tau_{-a,b}} = b\}$ are equivalent, we then have

$$\begin{aligned}
&\mathbb{P}_x[\tilde{\tau}_b^* > t] \\
&= \mathbb{P}_x[\Gamma^+_{\tau_b} > t \ , \ \tau_b < \tau_{-a}] \\
&= \mathbb{P}_x[\tilde{\Gamma}^{(0,b)}_{\tau_{-a,b}} > t \ , \ W_{\tau_{-a,b}} = b].
\end{aligned}$$

From Borodin and Salminen (2002), it is given that

$$\begin{aligned}
&\mathbb{P}_x[\tilde{\Gamma}^{(r,b)}_{\tau_{-a,b}} > t \ , \ W_{\tau_{-a,b}} = b] \\
&= \begin{cases} (x+a)\tilde{\mathrm{rc}}_y(1, b-r, 0, r+a)dy, & -a \le x \le r, \\ \mathrm{ss}_y(x-r, b-r)dy + (r+a) \cdot \mathrm{ss}_y(b-x, b-r) * \tilde{\mathrm{rc}}_y(1, b-r, 0, r+a)dy, & r \le x \le b. \end{cases}
\end{aligned}$$

Setting $r = 0$ gives the desired result. □

3.4.6 Joint Distribution of First Passage Time and Counter

We look at another approach to compute the first passage time of the interrupted Brownian motion in this subsection. This approach gives us an additional piece of information, the distribution of the counter N. Let us define the first hitting time τ to be:

$$\tau = \inf\{t \ : \ Y_t = \gamma\}, \tag{3.36}$$

where γ is a positive, attainable level.

Theorem 3.4.9. *The joint distribution of the first passage time τ and the counter N is given* *as:*

$$
\begin{aligned}
&\mathbb{E}_x\left[\theta^{N_\tau}\mathbb{1}_{\{\tau\in dy\}}\right]\\
&=\theta^{N_0}\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{2a}{1-\theta}\right)^k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{j=0}^{\infty}\frac{(-1)^j\, e^{-\frac{[2\gamma(k+j-l)+\gamma-x]^2}{4y}}}{\sqrt{2\pi}y^{1-\frac{k}{2}}}\binom{k+j}{j}\\
&\quad\cdot D_{1-k}\left(\frac{2\gamma(k+j-l)+\gamma-x}{\sqrt{y}}\right)\\
&\quad+\theta^{N_0}\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{2a}{1-\theta}\right)^k}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{j=0}^{\infty}\frac{(-1)^j e^{-\frac{[2\gamma(k+j-l)+\gamma+x]^2}{4y}}}{\sqrt{2\pi}y^{1-\frac{k}{2}}}\binom{k+j}{j}\\
&\quad\cdot D_{1-k}\left(\frac{2\gamma(k+j-l)+\gamma+x}{\sqrt{y}}\right)\\
&\quad+\theta^{N_0}\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{2a}{1-\theta}\right)^{k+1}}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{j=0}^{\infty}\frac{(-1)^j e^{-\frac{[2\gamma(k+j-l)+\gamma-x]^2}{4y}}}{\sqrt{2\pi}y^{1-\frac{k+1}{2}}}\binom{k+j}{j}\\
&\quad\cdot D_{-k}\left(\frac{2\gamma(k+j-l)+\gamma-x}{\sqrt{y}}\right)\\
&\quad-\theta^{N_0}\sum_{k=0}^{\infty}\frac{(-1)^k}{\left(\frac{2a}{1-\theta}\right)^{k+1}}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{j=0}^{\infty}\frac{(-1)^j e^{-\frac{[2\gamma(k+j-l)+\gamma+x]^2}{4y}}}{\sqrt{2\pi}y^{1-\frac{k+1}{2}}}\binom{k+j}{j}\\
&\quad\cdot D_{-k}\left(\frac{2\gamma(k+j-l)+\gamma+x}{\sqrt{y}}\right),
\end{aligned}
\tag{3.37}
$$

where $D_n(x)$ is the parabolic cylinder function of order n.

Proof. We have from (3.12) that the infinitesimal generator of an interrupted Brownian motion has the following form:

$$\mathcal{A}f_n(x,t)=\frac{\partial f_n(x,t)}{\partial t}+\frac{1}{2}\frac{\partial^2 f_n(x,t)}{\partial x^2};\qquad x>0,$$

with the following condition:

$$f_{n+1}(0)=-2af_n'(0)+f_n(0). \tag{3.38}$$

Using a solution of the form

$$f_n(x,t)=e^{-\beta t}f_n(x)=e^{-\beta t}\theta^n f(x),$$

we have that

$$f(x) = C_1 e^{\sqrt{2\beta}x} + C_2 e^{-\sqrt{2\beta}x}.$$

Substituting the $f(x)$ obtained into condition (3.38), we have

$$C_1 = \frac{\frac{2a\sqrt{2\beta}}{\theta-1} - 1}{\frac{2a\sqrt{2\beta}}{\theta-1} + 1} C_2.$$

We see that for boundedness of C_1 and C_2, we need $\theta \neq 1$. Therefore, $f(x)$ is such that

$$f(x) = \left(\frac{\frac{2a\sqrt{2\beta}}{\theta-1} - 1}{\frac{2a\sqrt{2\beta}}{\theta-1} + 1} \right) C_2 e^{\sqrt{2\beta}x} + C_2 e^{-\sqrt{2\beta}x}.$$

We then conclude that

$$e^{-\beta t}\theta^{N_t} f(X_t),$$

is a martingale. Using Doob's optional stopping theorem, we have that

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta\tau}\theta^{N_\tau} f(X_\tau)\mathbb{1}_{\{\tau<t\}}\right] + \mathbb{E}_x\left[e^{-\beta t}\theta^{N_t} f(X_t)\mathbb{1}_{\{t<\tau\}}\right] &= e^{-\beta(0)}\theta^{N_0} f(X_0) \\
\mathbb{E}_x\left[e^{-\beta\tau}\theta^{N_\tau}\right] &= \theta^{N_0}\frac{f(x)}{f(\gamma)} \\
\theta^{N_0}\left[\frac{\left(\frac{\frac{2a\sqrt{2\beta}}{\theta-1}-1}{\frac{2a\sqrt{2\beta}}{\theta-1}+1}\right) C_2 e^{\sqrt{2\beta}x} + C_2 e^{-\sqrt{2\beta}x}}{\left(\frac{\frac{2a\sqrt{2\beta}}{\theta-1}-1}{\frac{2a\sqrt{2\beta}}{\theta-1}+1}\right) C_2 e^{\sqrt{2\beta}\gamma} + C_2 e^{-\sqrt{2\beta}\gamma}}\right] &\overset{(3.4.6)}{=} \mathbb{E}_x\left[e^{-\beta\tau}\theta^{N_\tau}\right] \\
\theta^{N_0}\left[\frac{\frac{2a\sqrt{2\beta}}{\theta-1}\cosh\left(\sqrt{2\beta}x\right) - \sinh\left(\sqrt{2\beta}x\right)}{\frac{2a\sqrt{2\beta}}{\theta-1}\cosh\left(\sqrt{2\beta}\gamma\right) - \sinh\left(\sqrt{2\beta}\gamma\right)}\right] &= \mathbb{E}_x\left[e^{-\beta\tau}\theta^{N_\tau}\right].
\end{aligned}$$

We have included in the condition $|\theta| < 1$ in our calculations. We can then obtain the desired result (3.37) using Borodin and Salminen (2002) with the following conditions:

$$\frac{2a}{1-\theta} \neq 0; \qquad \gamma > 0; \qquad x < \gamma; \qquad x > -\gamma.$$

$\square$

3.5 Perturbation Method

3.5.1 Formulation of Problem

We would like to study the excursions of the interrupted Brownian motion by first studying the excursions of a Brownian motion. However, due to the properties of paths of Brownian motion, the Brownian motion has an infinite number of very small excursions which make studying the excursion results very difficult. Therefore, in an attempt to overcome this issue, we introduce a new process, the perturbed Brownian motion, $X^{(\epsilon)}$, where $\epsilon > 0$. Let X be a standard Brownian motion starting from $X_0 = 0$. Define a sequence of stopping times as follows:

$$
\begin{aligned}
\sigma^-_{1,\epsilon} &= 0 \\
\sigma^+_{1,\epsilon} &= \inf\{t \geq \sigma^-_{1,\epsilon} \ : \ X_t = \epsilon\} \\
\sigma^-_{2,\epsilon} &= \inf\{t \geq \sigma^+_{1,\epsilon} \ : \ X_t = 0\} \\
\sigma^+_{n,\epsilon} &= \inf\{t \geq \sigma^-_{n,\epsilon} \ : \ X_t = \epsilon\} \\
\sigma^-_{n+1,\epsilon} &= \inf\{t \geq \sigma^-_{n,\epsilon} \ : \ X_t = 0\},
\end{aligned}
$$

where $n = 1, 2, \ldots$. Let us now define the perturbed Brownian motion as follows

$$
X^{(\epsilon)}_t = \begin{cases} X_t - \epsilon; & \sigma^-_{k,\epsilon} \leq t \leq \sigma^+_{k,\epsilon} \\ X_t; & \sigma^+_{k,\epsilon} \leq t \leq \sigma^-_{k+1,\epsilon}. \end{cases}
$$

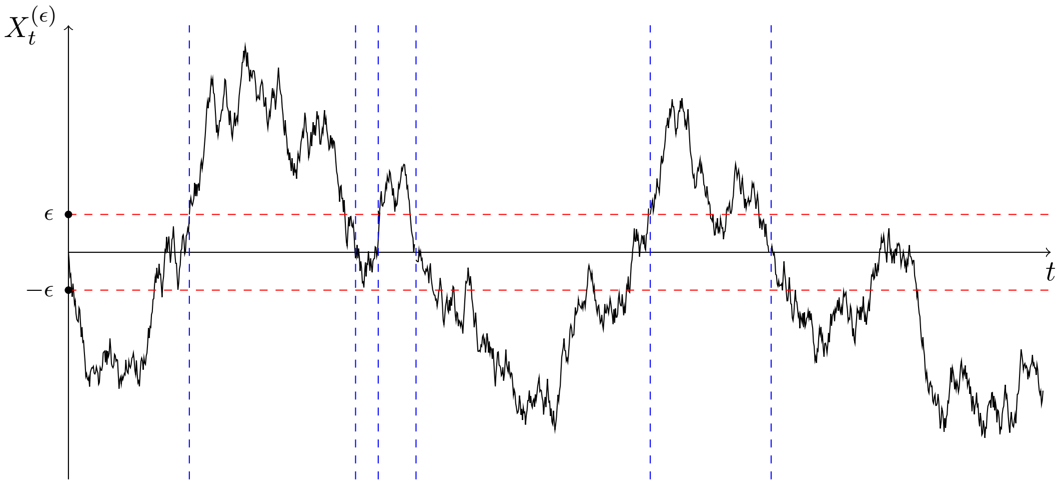

Figure 3.3: Brownian motion X_t (before perturbation).

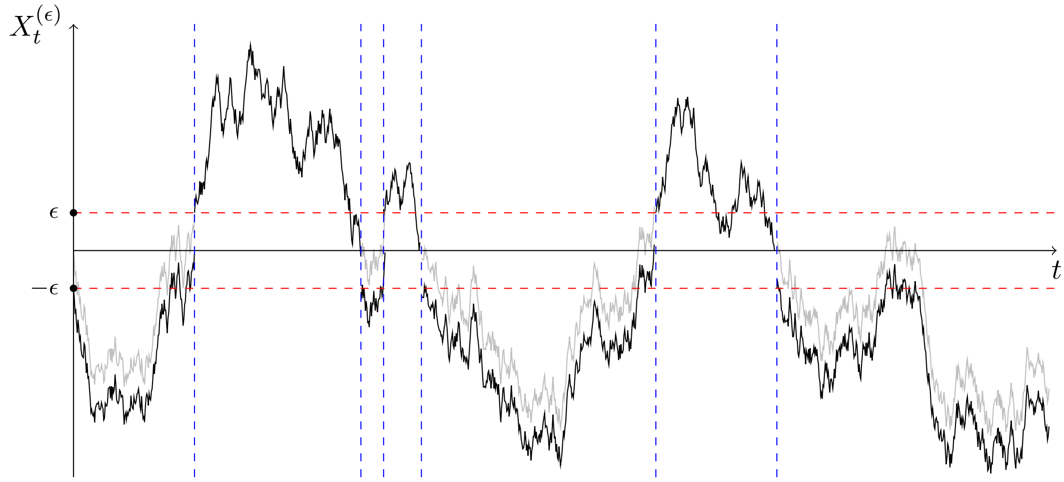

Figure 3.4: Perturbed Brownian motion $X_t^{(\epsilon)}$

The process $X_t^{(\epsilon)}$ can be thought of as a Brownian motion with the level 0 as a boundary and the process jumps to the other side of the boundary whenever the boundary is hit. We can see that the process $X_t^{(\epsilon)}$ has alternating positive and negative excursions with length of each excursion being greater than 0, making 0 an irregular point.

Let t be the new clock for the interrupted Brownian motion. This clock only accumulates time when the path of the Brownian motion is above the level 0 before the event of hitting the level $-a$ happens. We define the following quantities:

- $Y_t^{(\epsilon)}$ be the perturbed interrupted Brownian motion
- $\Gamma_{t,\epsilon} := \text{meas}\{0 \leq s \leq t \ : \ Y_t^{(\epsilon)} > 0\}$ be the occupation time of $Y_t^{(\epsilon)}$ in the interval $(0, \infty)$.
- $\Gamma_{t,\epsilon}^{-1}$ be the left inverse of $\Gamma_{t,\epsilon}$ such that the jumps as $\{\sigma_{k,\epsilon}^{-} - \sigma_{k,\epsilon}^{+}, \ k \in \mathbb{N}\}$
- $\tau_{0,k}^{(\epsilon)} = \sigma_{k+1,\epsilon}^{-} - \sigma_{k,\epsilon}^{+}$
- $\tau_{-a-\epsilon} := \inf\{t \geq 0 \ : \ X_t^{\epsilon} = -a - \epsilon\}$

3.5.2 Laplace Transform of Maximum Height of Excursion

Theorem 3.5.1. *The Laplace transform of the maximum height of excursion of the interrupted Brownian motion with exponential time $\mathbb{e}_\theta$ where $\mathbb{e}_\theta \sim \exp(\theta)$ for $\theta > 0$ and $\mathbb{e}_\theta$ is* *independent of the Brownian motion can be derived as*

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\mathbb{e}_\theta]}Y_s\right)\right]\\&=\frac{q}{\sqrt{2\theta}+\frac{1}{a}}\left[\log\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)-\frac{\sqrt{2\theta}}{q}-\psi\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)\right],\end{aligned}\tag{3.39}$$

where the function $\psi(x)$ is the logarithmic derivative of the Gamma function known as the Digamma function. It is defined as

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-xt}}{1-e^{-t}}\right)dt.$$

Proof. Using the definition of interrupted Brownian motion, we can rewrite the expression in terms of the perturbed Brownian motion.

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\mathbb{e}_\theta]}Y_s\right)\right]\\&=\lim_{\epsilon\to0}\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}-\Gamma^{-1}_{\mathbb{e}_\theta-,\epsilon}]}X_{s+\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}}\right)\sum_{n=1}^\infty\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau^{(\epsilon)}_{0,k}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right].\end{aligned}$$

With tower property, we have

$$\begin{aligned}&\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}-\Gamma^{-1}_{\mathbb{e}_\theta-,\epsilon}]}X_{s+\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}}\right)\sum_{n=1}^\infty\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau^{(\epsilon)}_{0,k}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right]\\&=\mathbb{E}\left[\sum_{n=1}^\infty\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\sigma^-_{n+1,\epsilon}-\sigma^+_{n,\epsilon}]}B_{s+\sigma^+_{n,\epsilon}}\right)\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau^{(\epsilon)}_{0,k}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right]\middle|\mathcal{F}_{\sigma^+_{n,\epsilon}}\right]\\&=\sum_{n=1}^\infty\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^+_{n,\epsilon}\}}f_1\left(\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k},\epsilon\right)\right],\end{aligned}\tag{3.40}$$

where the function f_1 can be derived as using the strong Markov property of Brownian motion to be:

$$\begin{aligned}f_1(t,\epsilon)&=\mathbb{E}_\epsilon\left[\exp\left(-q\max_{s\in[0,\tau_0^-]}B_s\right)\mathbb{1}_{\{\mathbb{e}_\theta<t+\tau_0^-\}}\mathbb{1}_{\{t<\mathbb{e}_\theta\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right]\\&=e^{-\theta t}\,\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\mathbb{1}_{\{\underline{B}_{\mathbb{e}_\theta}>0\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right],\end{aligned}$$

where $\bar{B}_t = \max_{s\in[0,t]} B_s$. Substituting the expression for f_1 into (3.40), we can derive

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-q \max_{s\in[0,\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}-\Gamma^{-1}_{\mathbb{e}_\theta-,\epsilon}]} X_{s+\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}}\right)\sum_{n=1}^{\infty}\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau^{(\epsilon)}_{0,k}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right]\\
&=\sum_{n=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^+_{n,\epsilon}\}}\, f_1\left(\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k},\epsilon\right)\right]\\
&=\sum_{n=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^+_{n,\epsilon}\}}\cdot e^{-\theta\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}}\,\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\,\mathbb{1}_{\{\underline{B}_{\mathbb{e}_\theta}>0\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right]\right].
\end{aligned}$$

Using tower property again, we derive

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(-q \max_{s\in[0,\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}-\Gamma^{-1}_{\mathbb{e}_\theta-,\epsilon}]} X_{s+\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}}\right)\sum_{n=1}^{\infty}\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau^{(\epsilon)}_{0,k}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right]\\
&=\sum_{n=1}^{\infty}\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^+_{n,\epsilon}\}}e^{-\theta\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}}\right]\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\,\mathbb{1}_{\{\underline{B}_{\mathbb{e}_\theta}>0\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right]\\
&=\sum_{n=1}^{\infty}\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\,\mathbb{1}_{\{\underline{B}_{\mathbb{e}_\theta}>0\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right]\\
&\qquad\cdot\mathbb{E}\left[\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}-\sigma^-_{n,\epsilon}>\sigma^+_{n,\epsilon}-\sigma^-_{n,\epsilon}\}}\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^-_{n,\epsilon}\}}e^{-\theta\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}}\right]\middle|\mathcal{F}_{\sigma^-_{n,\epsilon}}\right]\\
&=\sum_{n=1}^{\infty}\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\,\mathbb{1}_{\{\underline{B}_{\mathbb{e}_\theta}>0\}}\mathbb{1}_{\{\tau_0^-<\tau_{-a-\epsilon}\}}\right]\mathbb{E}\left[\mathbb{1}_{\{\tau_{-a-\epsilon}>\sigma^-_{n,\epsilon}\}}e^{-\theta\sum_{k=1}^{n-1}\tau^{(\epsilon)}_{0,k}}\right]\mathbb{P}\left[\tau_{-a-\epsilon}>\tau^+_\epsilon\right]\\
&=\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\right]\frac{1}{\frac{1}{\mathbb{P}\left[\tau_{-a-\epsilon}>\tau^+_\epsilon\right]}-\mathbb{E}_\epsilon\left(e^{-\theta\tau_0^-}\right)}-\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}e^{-\theta\tau_0^-}\right]\frac{1}{\frac{1}{\mathbb{P}\left[\tau_{-a-\epsilon}>\tau^+_\epsilon\right]}-\mathbb{E}_\epsilon\left(e^{-\theta\tau_0^-}\right)}.
\end{aligned}\tag{3.41}$$

We can compute the following:

$$\begin{aligned}
\mathbb{E}_\epsilon\left[e^{-\theta\tau_0^-}\right]&=e^{-\epsilon\sqrt{2\theta}},\\
\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}\right]&=e^{-q\epsilon}-q\epsilon\int_\epsilon^\infty\frac{1}{y}e^{-qy}dy,\\
\mathbb{P}\left[\tau_{-a-\epsilon}>\tau^+_\epsilon\right]=\mathbb{P}\left[W_{\tau_{-a-\epsilon},\epsilon}=\epsilon\right]&=\frac{a+\epsilon}{a+2\epsilon}.
\end{aligned}$$

Finally, the double Laplace transform can be obtained as

$$\mathbb{E}_\epsilon\left[e^{-q\bar{B}_{\tau_0^-}}e^{-\theta\tau_0^-}\right]=\int_{x\in[\epsilon,\infty)}qe^{-qx}\frac{\sinh([x-\epsilon]\sqrt{2\theta})}{\sinh(x\sqrt{2\theta})}dx.$$

Substituting these quantities into the expectation (3.41) gives us the

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\mathbb{e}_\theta]} Y_s\right)\right] \\
&= \lim_{\epsilon\to 0}\mathbb{E}\left[\exp\left(-q\max_{s\in[0,\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}-\Gamma^{-1}_{\mathbb{e}_\theta-,\epsilon}]} X_{s+\Gamma^{-1}_{\mathbb{e}_\theta,\epsilon}}\right)\sum_{n=1}^{\infty}\mathbb{1}_{\{\sum_{k=1}^{n-1}\tau_{0,k}^{(\epsilon)}<\mathbb{e}_\theta<\sum_{k=1}^{n}\tau_{0,k}^{(\epsilon)}\}}\mathbb{1}_{\{\sigma^-_{n+1,\epsilon}<\tau_{-a-\epsilon}\}}\right] \\
&= \lim_{\epsilon\to 0}\frac{e^{-q\epsilon}-q\epsilon\int_\epsilon^\infty\frac{1}{y}e^{-qy}dy-\int_{x\in[\epsilon,\infty)}qe^{-qx}\frac{\sinh([x-\epsilon]\sqrt{2\theta})}{\sinh(x\sqrt{2\theta})}dx}{\frac{a+2\epsilon}{a+\epsilon}-e^{-\epsilon\sqrt{2\theta}}} \\
&= \frac{q}{\sqrt{2\theta}+\frac{1}{a}}\left[\log\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)-\frac{\sqrt{2\theta}}{q}-\psi\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)\right],
\end{aligned}
$$

where the function $\psi(x)$ is the logarithmic derivative of the Gamma function known as the Digamma function. $\square$

3.5.3 Density of Maximum height of Excursion

Remark 3.5.2. *We can obtain the density of the maximum height of excursion of the interrupted Brownian motion by inverting the Laplace transform in (3.39) with respect to q. We need to invert the following Laplace transform with respect to q:*

$$
\mathcal{L}_q^{-1}\left(\frac{q}{\sqrt{2\theta}+\frac{1}{a}}\left[\log\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)-\frac{\sqrt{2\theta}}{q}-\psi\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)\right]\right)(s).
$$

Using results from integral transform, the inverse Laplace transform with respect to q can be obtained as

$$
\begin{aligned}
&\mathbb{P}\left[\max_{u\in[0,\mathbb{e}_\theta]} Y_u\in ds\right] \\
&= \mathcal{L}_q^{-1}\left(\frac{q}{\sqrt{2\theta}+\frac{1}{a}}\left[\log\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)-\frac{\sqrt{2\theta}}{q}-\psi\left(\frac{1}{2}\frac{q}{\sqrt{2\theta}}\right)\right]\right)(s) \\
&= \frac{1}{\sqrt{2\theta}+\frac{1}{a}}\left[\frac{1}{s^2}-\left(\frac{2\sqrt{2\theta}}{1-e^{-2s\sqrt{2\theta}}}\right)^2 e^{-2s\sqrt{2\theta}}\right]ds \\
&= \frac{1}{s^2}\frac{1}{\sqrt{2\theta}+\frac{1}{a}}-\frac{1}{\sqrt{2\theta}+\frac{1}{a}}\left(\frac{2e^{-2s\sqrt{2\theta}}}{1-e^{-2s\sqrt{2\theta}}}\right)^2\cdot 2\theta\, e^{2s\sqrt{2\theta}}ds.
\end{aligned}
$$

Remark 3.5.3. *We can further invert with respect to θ to recover the original clock.*

Therefore, we have that

$$\mathbb{P}\left[\max_{u\in[0,s]} Y_u \in dp\right]$$
$$\mathcal{L}_\theta^{-1}\left(\frac{1}{s^2}\frac{1}{\sqrt{2\theta}+\frac{1}{a}} - \frac{1}{\sqrt{2\theta}+\frac{1}{a}}\left(\frac{2e^{-2s\sqrt{2\theta}}}{1-e^{-2s\sqrt{2\theta}}}\right)^2 \cdot 2\theta\, e^{2s\sqrt{2\theta}}\right)(p)$$
$$= \frac{1}{\sqrt{2}s^2}\cdot\frac{1}{2a^2}\left[\frac{1}{\sqrt{\pi p}} - \frac{e^{\frac{p}{2a^2}}\ erfc\left(\frac{\sqrt{p}}{a\sqrt{2}}\right)}{a\sqrt{2}}\right]dp$$
$$- \frac{1}{2\sqrt{2}}\left(\left[\frac{1}{\sqrt{\pi p}} - \frac{1}{a\sqrt{2}}\, e^{\frac{p}{2a^2}}\ erfc\left(\frac{\sqrt{p}}{a\sqrt{2}}\right)\right] * s_p\left(4,2,2s,-2s\right)\right)dp,$$

where the function $s_p(4,2,2s,-2s)$ *is defined as*

$$s_p\left(4,2,2s,-2s\right)$$
$$= 2^2\sum_{k=0}^{\infty}\frac{\Gamma(2+k)e^{-\frac{(2s+4ks)^2}{4p}}}{\sqrt{2\pi p^3}\Gamma(2)k!}D_5\left(\frac{2s+4ks}{\sqrt{p}}\right).$$

3.6 Appendix

We provide now the rest of the proof for in Theorem 3.4.2. We want to invert the Laplace transform of (3.27) with respect to β. Considering only the required terms, we have:

$$\begin{aligned}&\mathcal{L}_\beta^{-1}\left(\frac{2\theta^{2N_0}(1+2a\gamma-\theta)}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}\right)(t)\\ &= 2\theta^{2N_0}(1+2a\gamma-\theta)\mathcal{L}_\beta^{-1}\left(\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)(t).\end{aligned}\tag{3.42}$$

We then follow the steps:

- Using the definition of inverse Laplace transform, the desired inversion can be expressed as
$$f(t) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\beta t}\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}d\beta,\tag{3.43}$$
where c is a constant to be determined.

- We determine the poles $\beta_n = (\beta_1,\beta_2)$ present and establish the contour, C to be used for the expression in (3.42). Here, it is easy to see that we require a keyhole contour due to the presence of branch cut.

- With the singularities, we choose the constant c in the limits of the integration such that the contour C contains all the poles. We therefore need $c > \max(\beta_1, \beta_2)$.

- Define the function $g(\beta)$ to be

$$g(\beta) := \frac{e^{\beta t} e^{-\sqrt{2\beta}x}}{(2\beta - \gamma^2)(\theta - 1 - 2a\sqrt{2\beta})}.$$

Since the function $g(\beta)$ is analytic everywhere except at the poles, according to Residue Theorem, we have that

$$\frac{1}{2\pi i} \oint_C \left(\frac{e^{\beta t} e^{-\sqrt{2\beta}x}}{(2\beta - \gamma^2)(\theta - 1 - 2a\sqrt{2\beta})} \right) d\beta = \sum_{i=1}^{2} \operatorname*{Res}_{\beta=\beta_i} g(\beta), \tag{3.44}$$

where C is a keyhole contour and β_i for $i = 1, 2$ are the poles such that

$$\beta_1 = \frac{\gamma^2}{2},$$
$$\beta_2 = \frac{\theta^2 - 2\theta + 1}{8a^2}.$$

Computing the residues at the poles enables us to express (3.44) as

$$\begin{aligned} &\frac{1}{2\pi i} \oint_C \left(\frac{e^{\beta t} e^{-\sqrt{2\beta}x}}{(2\beta - \gamma^2)(\theta - 1 - 2a\sqrt{2\beta})} \right) d\beta \\ &= e^{\frac{\gamma^2}{2}t} e^{-\gamma x} \frac{1}{2(\theta - 1 - 2a\gamma)} + e^{\frac{\theta^2 - 2\theta + 1}{8a^2}t} e^{-\frac{\theta - 1}{2a}x} \frac{\theta - 1}{4a^2\gamma^2 - (\theta - 1)^2}. \end{aligned} \tag{3.45}$$

- By Cauchy's Theorem, the contour integral on the LHS of (3.45) can be broken down to

$$\oint_C g(\beta) d\beta = \left(\oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \oint_{C_4} + \oint_{C_5} + \oint_{C_6} \right) g(\beta) d\beta,$$

where $C_i, i = 1, \ldots, 6$ form a closed path of C.

- Then, by the Bromwich Inversion formula, we have that the inverse Laplace transform,

$f(t)$ can be evaluated using

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\oint_C g(\beta)d\beta \\ &- \lim_{R\to\infty,\, r\to 0}\sum_{j=2}^{6}\frac{1}{2\pi i}\oint_{C_j}\left(e^{\beta t}\,\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta, \end{aligned} \tag{3.46}$$

where R is the radius of the big circular contours C_2 and C_6 centred around $(0,0)$, whereas r is the radius of the small circular contour C_4 centered around $(0,0)$. What is left to do is to compute the contour integrals in (3.46) for contours $C_j, j=2,\ldots,6$.

- Using the parametrisation $\beta = Re^{iz}$ for different values of z accordingly, the Estimation Lemma tells us that

$$\begin{aligned} \lim_{R\to\infty}\oint_{C_2}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta = 0, \\ \lim_{R\to\infty}\oint_{C_6}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta = 0. \end{aligned} \tag{3.47}$$

- Computing contour C_4 using the parametrisation $\beta = re^{iz}$ gives

$$\lim_{r\to 0}\oint_{C_4}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta = 0. \tag{3.48}$$

- We now compute the contour C_3 where β takes values from $-R$ to $-r$.

$$\oint_{C_3}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta = \int_R^r\left(\frac{e^{-yt}e^{-ix\sqrt{2y}}}{(2y+\gamma^2)(\theta-1-2ai\sqrt{2y})}\right)dy. \tag{3.49}$$

- Computing contour C_5 where β takes values from $-r$ to $-R$ gives us

$$\oint_{C_5}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta = \int_r^R\left(\frac{e^{-yt}e^{ix\sqrt{2y}}}{(2y+\gamma^2)(\theta-1+2ai\sqrt{2y})}\right)dy. \tag{3.50}$$

- Adding $\oint_{C_3}$ in (3.49) and $\oint_{C_5}$ in (3.50) gives

$$\begin{aligned}
&\oint_{C_3}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta+\oint_{C_5}\left(\frac{e^{\beta t}e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta\\
&=\int_0^\infty\frac{2ie^{-yt}}{2y+\gamma^2}\cdot\frac{\theta-1}{(\theta-1)^2+8a^2y}\sin\left(x\sqrt{2y}\right)dy\\
&\quad-\int_0^\infty\frac{2e^{-yt}}{2y+\gamma^2}\cdot\frac{2ai\sqrt{2y}}{(\theta-1)^2+8a^2y}\cos\left(x\sqrt{2y}\right)dy.
\end{aligned}$$

The integrals can be computed with the help of Weierstrass test and a change of variable.

- From (3.46), we know that the inverse Laplace transform is

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{\beta t}\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}d\beta\\
&=\frac{1}{2\pi i}\oint_C g(\beta)d\beta-\lim_{R\to\infty,\,r\to 0}\sum_{j=2}^{6}\frac{1}{2\pi i}\oint_{C_j}\left(e^{\beta t}\,\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta\\
&=\frac{1}{2\pi i}\oint_C g(\beta)d\beta-\frac{1}{2\pi i}\left(\oint_{C_3}+\oint_{C_5}\right)\left(e^{\beta t}\,\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)d\beta\\
&=e^{\frac{\gamma^2}{2}t}e^{-\gamma x}\,\frac{1}{2(\theta-1-2a\gamma)}+e^{\frac{\theta^2-2\theta+1}{8a^2}t}e^{-\frac{\theta-1}{2a}x}\frac{\theta-1}{4a^2\gamma^2-(\theta-1)^2}\\
&\quad-\psi\frac{\theta-1}{4}e^{\frac{t\gamma^2}{2}}\left[e^{-\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}-\frac{x}{\sqrt{2t}}\right)-e^{\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}+\frac{x}{\sqrt{2t}}\right)\right]\\
&\quad+\psi\frac{\theta-1}{4}e^{\frac{(\theta-1)^2t}{8a^2}}\left[e^{-\frac{(\theta-1)x}{2a}}\mathrm{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right)\right.\\
&\quad\quad\left.-e^{\frac{(\theta-1)x}{2a}}\mathrm{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}+\frac{x}{\sqrt{2t}}\right)\right]\\
&\quad-\psi\frac{a}{2}e^{\frac{t\gamma^2}{2}}\left[\gamma e^{-\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}-\frac{x}{\sqrt{2t}}\right)+\gamma e^{\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}+\frac{x}{\sqrt{2t}}\right)\right]\\
&\quad+\psi\frac{a}{2}e^{\frac{(\theta-1)^2t}{8a^2}}\left[\frac{\theta-1}{2a}e^{\frac{(\theta-1)x}{2a}}\mathrm{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}+\frac{x}{\sqrt{2t}}\right)\right.\\
&\quad\quad\left.+\frac{\theta-1}{2a}e^{-\frac{(\theta-1)x}{2a}}\mathrm{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right)\right],
\end{aligned}\tag{3.51}$$

where $\psi=\frac{1}{(\theta-1)^2-4a^2\gamma^2}=\frac{1}{(\theta-1-2a\gamma)(\theta-1+2a\gamma)}$.

- From (3.42), we can finally express the inverse Laplace transform as

$$
\begin{aligned}
&\mathcal{L}_\beta^{-1}\left(\frac{2\theta^{2N_0}(1+2a\gamma-\theta)}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}e^{-\sqrt{2\beta}x}\right)(t)\\
&=2\theta^{2N_0}(1+2a\gamma-\theta)\mathcal{L}_\beta^{-1}\left(\frac{e^{-\sqrt{2\beta}x}}{(2\beta-\gamma^2)(\theta-1-2a\sqrt{2\beta})}\right)(t)\\
&=-e^{\frac{\gamma^2 t}{2}}e^{-\gamma x}\theta^{2N_0}+2\theta^{2N_0}e^{\frac{(\theta-1)^2t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}(\theta-1)\frac{1}{2a\gamma+\theta-1}\\
&\quad+\frac{\theta^{2N_0}}{\theta-1+2a\gamma}e^{\frac{t\gamma^2}{2}}e^{-\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}-\frac{x}{\sqrt{2t}}\right)\left[\frac{\theta-1}{2}+a\gamma\right]\\
&\quad+\frac{\theta^{2N_0}}{\theta-1+2a\gamma}e^{\frac{t\gamma^2}{2}}e^{\gamma x}\mathrm{erfc}\left(\gamma\sqrt{\frac{t}{2}}+\frac{x}{\sqrt{2t}}\right)\left[a\gamma-\frac{\theta-1}{2}\right]\\
&\quad-\frac{\theta^{2N_0}(\theta-1)}{\theta-1+2a\gamma}e^{\frac{(\theta-1)^2t}{8a^2}}e^{-\frac{(\theta-1)x}{2a}}\mathrm{erfc}\left(\frac{(\theta-1)\sqrt{t}}{2a\sqrt{2}}-\frac{x}{\sqrt{2t}}\right).
\end{aligned}
\tag{3.52}
$$

Chapter 4

Local Time Process

4.1 Introduction

In this chapter, we study the stochastic process $\{\Gamma_t^x : a \in \mathbb{R}, t \geq 0\}$ which characterises the amount of time spent by a continuous time stochastic process $X = \{X_t \ : \ t \geq 0\}$ in the neighbourhood of a point $x \in E$ where E is the state space of the stochastic process. The Lebesgue measure of the time spent at the level x can be derived using

$$\Gamma_t^x = \int_0^t \mathbb{1}_{\{X_s = x\}} ds.$$

However, this does not make sense for X a Brownian motion as we have

$$\text{meas}\{0 \leq t < \infty \ ; \ W_t(\omega) = x\} = 0, \qquad \text{for } \mathbb{P} - \text{a.e.}, \ \omega \in \Omega.$$

This is not helpful as it does not tell us how much time the Brownian motion has spent in the neighbourhood of the point $x \in \mathbb{R}$. In order to provide a meaningful interpretation for this measure of time, Paul Lévy introduced the random field for $t \in [0, \infty)$ and $x \in \mathbb{R}$,

$$L_t^x = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \text{meas}\{0 \leq s \leq t; |W_s - x| \leq \epsilon\},$$

which is called the *mesure du voisinage* or "measure of the time spent by the Brownian path in the vicinity of the point x". L_t^x shall be referred to as the *local time* from here onwards. Interested readers can refer to Lévy (1940), Itō and McKean (1974a), Karatzas and Shreve

(1991), Chung and Durrett (2008), and Jeanblanc et al. (2009).

The introduction of the local time process gained a lot of attention, with rapid development in both theory and applications in many fields such as stochastic integration, excursion and stochastic differential equation. In particular, the local time process is also taken as a tool to generalise Itô rule for convex functions. It is well known that the celebrated Itô rule

$$f(X_t) = f(X_0) = \int_0^t f'(X_s)dMs + \int_0^t f'(X_s)dB_s + \frac{1}{2}\int_0^t f''(X_s)d\langle M\rangle_s,$$

for $f : \mathbb{R} \to \mathbb{R}$ a function in C^2 and $X = \{X_t,\ 0 \le t < \infty\}$ a continuous semimartingale with decomposition

$$X_t = X_0 + M_t + B_t,$$

for M a continuous local martingale and B a continuous adapted process. See for example Itô (1944) and Kunita and Watanabe (1967). The Itô rule plays an important role as it is the key to the world of stochastic calculus but as we can see from the definition above, Itô rule requires the existence of the second derivative for the formula to make sense.
The local time process comes in for the generalisation of the Itô rule for convex function $f : \mathbb{R} \mapsto \mathbb{R}$ which are not necessarily twice differentiable. The generalised Itô rule is given as

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_s)dX_s + \int_{-\infty}^{\infty} L_t(x)\mu(dx); \qquad 0 \le t < \infty,$$

for every $X_0 \in \mathbb{R}$, D_f^- as the left derivative of f, μ to be taken as the second derivative measure in the distribution sense and L is the local time process as defined earlier. For more details, see Chung et al. (1990), Revuz and Yor (1991), Karatzas and Shreve (1991), Kallenberg (1997), Rogers and Williams (2000b) and Borodin and Salminen (2002).

Besides contributing to the generalised Itô rule, the local time process also plays a significant part in mathematical finance. Leblanc (1997) made use of the definition of local time to derive the Dupire's formula for local volatility which states that

$$\frac{1}{2}K^2\sigma^2(T,K) = \frac{\partial_T C(K,T) + rK\partial_K C(K,T)}{\partial^2_{KK} C(K,T)}$$

where $C(K,T) = \mathbb{E}\left[e^{-rT(S_T-K)_+}\right]$ is the price of the European call for any maturity $T \in [0,\infty)$ and any strike price $K \in \mathbb{R}_+$, $\sigma^2(T,K)$ is the local volatility and ∂ is the partial derivative operator. The Dupire's formula is well used as it serves as a direct method to deduce the local volatility function from the prices of call options in the market.

Another established application of the local time process is the pricing of a knock-out BOOST option studied by Leblanc (1997) which is an option that pays at maturity, for the amount of time when the underlying price process stays above a level $b \in \mathbb{R}_+$ until the time when the price process touches level $a \in \mathbb{R}_+$ for the first time, for positive levels a and b such that $b < a$. The local time process is also being employed in the form of the Itô-Tanaka formula which is an extention of the Tanaka's formula. The Itô-Tanaka formula develop for formula for $\{f(X_t),\ t \geq 0\}$ as a semi-martingale for the function f which is the difference of two convex functions and $\{X_t,\ t \geq 0\}$ is a continuous semi-martingale. Then,

$$f(X_t) = f(X_0) + \int_0^t D^- f(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da).$$

In particular, $\{f(X_t),\ t \geq 0\}$ is again a semi-martingale. The Itô-Tanaka formula is used for the pricing of a special type of contingent claim which is known as a passport option, see Shreve and Vecer (1998). This option gives its holder the right to engage in an optimal trading strategy of choice.

The study in this chapter is motivated by the the computation of the expected exposure of the Accumulator option which is a path-dependent option. We look at the distribution of the local times of a Brownian motion with drift evaluated at the first time its local time at 0 exceeds some amount $l \in \mathbb{R}_+$. We employ the techniques in Karatzas and Shreve (1991) to first compute the joint Laplace transform of local times at the first time the Brownian motion with drift hits 0. We then proceed to compute the same quantity for a compound Poisson process with drift at different starting points. We finish this chapter with some concluding remarks on the application of our results in the setting of counterparty credit risk management.

4.2 Brownian Motion

In this section, we focus on the local time of the Brownian motion at some level $a \in \mathbb{R}$.

4.2.1 Construction

We follow the same construction as in Karatzas and Shreve (1991) to derive the distributions of Brownian local time at one or several points. The procedure is outlined as follows: in the interest of studying the Brownian local time, we first consider the elastic Brownian motion $W^* = \{W_t^*,\ 0 \leq t < \infty\}$ which is defined as

$$W^* := \begin{cases} W_t; & \text{if } t < \xi, \\ \partial; & \text{if } t > \xi. \end{cases}$$

where ∂ denotes the cemetery state and ξ is called the lifetime of the elastic Brownian motion such that there exists $1 \leq i \leq n$ where

$$\xi := \inf\{t \geq 0 \ : \ L_t^{(a_i)} > R_i\},$$

and $R_1, R_2, \dots, R_n$ are independent and exponentially distributed random variables with parameters $\gamma_1, \gamma_2, \dots, \gamma_n$ respectively on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and ξ and W^* are defined on the enlarged probability space $(\Omega^*, \mathcal{F}^*, \mathbb{P}^*) = (\Omega \times \Omega', \mathcal{F} \otimes \mathcal{F}', \mathbb{P} \times \mathbb{P}')$. The elastic Brownian motion can then be seen as the original Brownian motion conditioned to stop at time ξ, the first time the local times at any of the n distinct levels $a_1, a_2, \dots, a_n \in \mathbb{R}$ exceeds the corresponding level R_i. Using the description of the elastic Brownian motion W^* as defined earlier, it is known that

$$\begin{aligned} f(x) &= \mathbb{E}_x^* \left[\int_0^\infty g(W^*) e^{-\alpha t} e^{-\int_0^t k(W^*) ds} dt \right] \\ &= \mathbb{E}_x \left[\int_0^\infty g(W_t) e^{-\alpha t} e^{-\int_0^\infty k(W_s) ds} e^{-\sum_{i=1}^n \gamma_i L_t^{(a_i)}} dt \right], \end{aligned} \tag{4.1}$$

where $g : \mathbb{R} \to \mathbb{R}$, $k : \mathbb{R} \to [0, \infty)$ are piecewise continuous functions and $\alpha \in \mathbb{R}_+$ is a constant. With this and some appropriate functions g and k, we can use this to compute the joint distribution of the Brownian local times at levels a_i. For more details regarding elastic Brownian motion, see Karatzas and Shreve (1991) and Grebenkov (2006).

4.2.2 Brownian Motion with Drift

Consider the process $W^{(\mu)} = \{W_t^{(\mu)},\ t \geq 0\}$, a Brownian motion with drift defined as

$$W_t^{(\mu)} = \mu t + \sigma W_t$$

with $\mu \in \mathbb{R}$ the drift, $\sigma \in \mathbb{R}_+$ the dispersion coefficient and $W = \{W_t,\ t \geq 0\}$ the standard Brownian motion defined in (1.0.1). We let $X = \{X_t,\ t \geq 0\}$ to be the process $W^{(\mu)}$ with a reflecting boundary $b > 0$. We want to derive the expression for $1 \leq i \leq n$,

$$\mathbb{E}\left[e^{-\sum_{i=1}^{n} \gamma_i L_{\Gamma_0(l)}(a_i)}\right], \tag{4.2}$$

for positive coefficients γ_i, n distinct levels a_i such that $0 < a_1 < \ldots < a_n < b$ and the inverse local time of X defined as

$$\Gamma_0(l) = \inf\{t \geq 0\ :\ L_t(0) = l\}, \tag{4.3}$$

where $L_t(0)$ is the local time process of X at 0. The quantity in (4.2) is of interest as it finds applications in the field of counterparty credit risk which is further discussed in Subsection 4.2.3.

Construction

In the effort to derive (4.2), we first compute for $\epsilon > 0$,

$$f(\epsilon) = \mathbb{E}_\epsilon\left[e^{-\sum_{i=1}^{n} \gamma_i L_{T_0}(a_i)}\right], \tag{4.4}$$

where we have $0 < a_1 < \ldots < a_n < b$ and the first hitting time of the level 0 of X defined as

$$T_0 := \inf\{t \geq 0\ :\ X_t = 0\}. \tag{4.5}$$

Using the expression (4.1) from the elastic Brownian motion and following the construction in Karatzas and Shreve (1991), we see that the expectation (4.4) can be derived by first finding a function on $[0, b]$ such that it is bounded, continuous and satisfies the conditions for

$1 \leq i \leq n$.

$$f'(a_i+) - f'(a_i-) = \gamma_i f(a_i) \tag{4.6}$$

$$f(0) = 1 \tag{4.7}$$

$$f'(b) = 0, \tag{4.8}$$

where the first condition (4.6) is for the elasticity condition for the Local time process to make sense at levels a_i and the third condition (4.8) corresponds to the reflection boundary at level b. We see that the function f has to be of the form

$$f(x) = \frac{\sigma^2}{\mu} C_i e^{-\frac{2\mu}{\sigma^2}x} + D_i,$$

for C_i and D_i some deterministic constants, in each of the interval $(a_i, a_{i+1}]$ for $0 \leq i \leq n$ where $a_{n+1} = b$.

We can then compute the expression (4.4) by deriving

$$f(\epsilon) = \mathbb{E}_\epsilon \left[e^{-\sum_{i=1}^n \gamma_i L_{T_0}(a_i)} \right] = \frac{\sigma^2}{\mu} C_0 e^{-\frac{2\mu}{\sigma^2}\epsilon} + D_0,$$

for some constants C_0 and D_0 to be determined using (4.6), (4.7) and (4.8). We solve the above system for the case when $n = 2$ in order to find the distribution of the Brownian local times at levels a_1 and a_2 for $0 < a_1 < a_2 < b$,

$$\mathbb{E}_\epsilon \left[e^{-\gamma_1 L_{T_0}(a_1)} e^{-\gamma_2 L_{T_0}(a_2)} \right].$$

We follow the aforementioned steps with $n = 2$.

Lemma 4.2.1. *For positive parameters γ_1, γ_2, two levels a_1, a_2 such that $0 < a_1 < a_2 < b$ and stopping time T_0 defined in (4.5), the joint Laplace transform of the Brownian local times* *at levels a_1 and a_2, evaluated at T_0 can be derived as*

$$\mathbb{E}_\epsilon\left[e^{-\gamma_1 L_{T_0}(a_1)}e^{-\gamma_2 L_{T_0}(a_2)}\right]$$
$$=\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}\epsilon}-1\right)\frac{\frac{2\gamma_2 e^{-\frac{2\mu}{\sigma^2}a_1}}{2e^{-\frac{2\mu}{\sigma^2}a_2}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}+\gamma_1}{2e^{-\frac{2\mu}{\sigma^2}a_1}-\gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)-\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}{2e^{-\frac{2\mu}{\sigma^2}a_2}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}+1.$$

Proof. When $n=2$, we have that condition (4.6) gives us for $1\le i\le 2$

$$\begin{aligned}&-2C_ie^{-\frac{2\mu}{\sigma^2}(a_i+\epsilon)}+2C_{i-1}e^{-\frac{2\mu}{\sigma^2}(a_i-\epsilon)}=\gamma_i\frac{\sigma^2}{\mu}C_{i-1}e^{-\frac{2\mu}{\sigma^2}a_i}+\gamma_iD_{i-1}\\ &\Rightarrow\begin{cases}-2C_1e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)}+2C_0e^{-\frac{2\mu}{\sigma^2}(a_1-\epsilon)} &=\gamma_1\frac{\sigma^2}{\mu}C_0e^{-\frac{2\mu}{\sigma^2}a_1}+\gamma_1D_0\\ -2C_2e^{-\frac{2\mu}{\sigma^2}(a_2+\epsilon)}+2C_1e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)} &=\gamma_2\frac{\sigma^2}{\mu}C_1e^{-\frac{2\mu}{\sigma^2}a_2}+\gamma_2D_1.\end{cases}\end{aligned}\tag{4.9}$$

The condition in (4.7) gives

$$\frac{\sigma^2}{\mu}C_0+D_0=1,\tag{4.10}$$

and condition (4.8) gives

$$-2C_2e^{-\frac{2\mu}{\sigma^2}b}=0.$$

With the continuity of f, we have that for $1\le i\le 2$,

$$\begin{aligned}&\frac{\sigma^2}{\mu}C_{i-1}e^{-\frac{2\mu}{\sigma^2}a_i}+D_{i-1}=\frac{\sigma^2}{\mu}C_ie^{-\frac{2\mu}{\sigma^2}a_i}+D_i\\ &\Rightarrow\begin{cases}\frac{\sigma^2}{\mu}C_0\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)-\frac{\sigma^2}{\mu}e^{-\frac{2\mu}{\sigma^2}a_1}C_1+1 &=D_1\\ \frac{\sigma^2}{\mu}C_1e^{-\frac{2\mu}{\sigma^2}a_2}+D_1 &=D_2.\end{cases}\end{aligned}\tag{4.11}$$

Substituting $C_2=0$ into (4.9), we have that,

$$\begin{aligned}-2C_2e^{-\frac{2\mu}{\sigma^2}(a_2+\epsilon)}+2C_1e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)}&=\gamma_2\frac{\sigma^2}{\mu}C_1e^{-\frac{2\mu}{\sigma^2}a_2}+\gamma_2D_1\\ 2C_1e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)}&=\gamma_2\frac{\sigma^2}{\mu}C_1e^{-\frac{2\mu}{\sigma^2}a_2}+\gamma_2D_1\\ C_1&=\gamma_2\frac{\frac{\sigma^2}{\mu}C_0\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)+1}{2e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}.\end{aligned}\tag{4.12}$$

From (4.9) and (4.10), we have that

$$-2C_1 e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)} + 2C_0 e^{-\frac{2\mu}{\sigma^2}(a_1-\epsilon)} = \gamma_1 \frac{\sigma^2}{\mu} C_0 e^{-\frac{2\mu}{\sigma^2}a_1} + \gamma_1 D_0$$

$$C_0\left[2e^{-\frac{2\mu}{\sigma^2}(a_1-\epsilon)} - \gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right] = \frac{2e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)}\gamma_2\frac{\sigma^2}{\mu}C_0\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right) + 2e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2} - e^{-\frac{2\mu}{\sigma^2}a_1}\right)} + \gamma_1.$$

Finally, we can see that

$$C_0 = \frac{\frac{2e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)} + \gamma_1}{2e^{-\frac{2\mu}{\sigma^2}(a_1-\epsilon)} - \gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right) - \frac{2e^{-\frac{2\mu}{\sigma^2}(a_1+\epsilon)}\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}{2e^{-\frac{2\mu}{\sigma^2}(a_2-\epsilon)} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}.$$

Taking the limit when ϵ approaches 0 gives us C_0 of the form

$$C_0 = \frac{\frac{2\gamma_2 e^{-\frac{2\mu}{\sigma^2}a_1}}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)} + \gamma_1}{2e^{-\frac{2\mu}{\sigma^2}a_1} - \gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right) - \frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}.$$

Using the expressions derived, we can derive that the joint Laplace transform of the Brownian local times at levels a_1 and a_2 up till the first time the process the level 0 as

$$\begin{aligned}
f(\epsilon) &= \mathbb{E}_\epsilon\left[e^{-\gamma_1 L_{T_0}(a_1)}e^{-\gamma_2 L_{T_0}(a_2)}\right] \\
&= \frac{\sigma^2}{\mu}C_0 e^{-\frac{2\mu}{\sigma^2}\epsilon} + D_0 \\
&= \frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}\epsilon}-1\right)\frac{\frac{2\gamma_2 e^{-\frac{2\mu}{\sigma^2}a_1}}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)} + \gamma_1}{2e^{-\frac{2\mu}{\sigma^2}a_1} - \gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right) - \frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}} + 1.
\end{aligned}$$

$\square$

Theorem 4.2.2. *For positive parameters a_1, a_2, positive levels a_1, a_2 where $a_1 < a_2 < \infty$ and the right inverse of the local time of X at 0 as defined in (4.3), the joint Laplace transform* *of the Brownian local times at levels a_1 and a_2 evaluated at $\Gamma_0(l)$ can be computed as*

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\gamma_1 L_{\Gamma_0(l)}(a_1)} e^{-\gamma_2 L_{\Gamma_0(l)(a_2)}}\right)\\
&= \exp\left(\frac{\frac{-4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}}{\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)} - \gamma_1 - \frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2} - e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}\right) \cdot \exp\left(\frac{-2l}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right).
\end{aligned}
$$

Proof. The proof can be obtained by considering the independent excursions as we have with a Brownian motion with drift starting from $\epsilon > 0$ with a reflecting boundary at $b > 0$ until the first time the process reaches 0. This tells us that we can derive the joint Laplace transform of the Brownian local times at levels a_1 and a_2 evaluated at the right inverse of the Brownian local time at 0 from the joint Laplace transform of the same quantities evaluated at the first time the process reaches level 0. We observe the process until we accumulate an amount of l on the Brownian local clock at the level 0. We need to cross this level for l times, this is the same as having l many excursions that starts from $\epsilon > 0$ that is being reflected at level b and stopped at the first time the process gets to 0.

$$
\mathbb{E}\left(e^{-\gamma_1 L_{\Gamma_0(l)}(a_1)} e^{-\gamma_2 L_{\Gamma_0(l)(a_2)}}\right) = \lim_{\epsilon \to 0}\left(\mathbb{E}_\epsilon\left[e^{-\gamma_1 L_{T_0}(a_1)} e^{-\gamma_2 L_{T_0}(a_2)}\right]^{\frac{l}{\epsilon}}\right).
$$

Substituting our results from Lemma (4.2.1), we have that

$$
\begin{aligned}
&\mathbb{E}\left(e^{-\gamma_1 L_{\Gamma_0(l)}(a_1)} e^{-\gamma_2 L_{\Gamma_0(l)(a_2)}}\right)\\
&= e^{-2lC_0}\\
&= \exp\left(-2l\frac{\frac{2\gamma_2 e^{-\frac{2\mu}{\sigma^2}a_1}}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2} - e^{-\frac{2\mu}{\sigma^2}a_1}\right)} + \gamma_1}{2e^{-\frac{2\mu}{\sigma^2}a_1} - \gamma_1\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right) - \frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2} - e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}\right)\\
&= \exp\left(\frac{\frac{-4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}}{\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)} - \gamma_1 - \frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}a_2} - \gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2} - e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}\right) \cdot \exp\left(\frac{-2l}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right).
\end{aligned}
$$

$\square$

Theorem 4.2.3. *The joint distribution of the Brownian local times at positive levels a_1 and a_2 evaluated at $\Gamma_0(l)$ as defined in (4.3) can be obtained as*

$$
\begin{aligned}
&f_{L_{\Gamma_0(l)}(a_1),L_{\Gamma_0(l)}(a_2)}(z_1,z_2)\\
&= \exp\left(\frac{-2l}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right)\sqrt{\frac{\frac{4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}}{z_1}}\sqrt{\frac{\frac{4z_1e^{-\frac{2\mu}{\sigma^2}a_1}e^{-\frac{2\mu}{\sigma^2}a_2}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)\right]^2}}{z_2}}\\
&\quad\cdot\exp\left(-\frac{2z_1e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}\right)\exp\left(\frac{2z_1e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right)\exp\left(\frac{2e^{-\frac{2\mu}{\sigma^2}a_2}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}z_2\right)\\
&\quad\cdot J_1\left(2i\sqrt{\frac{4z_1e^{-\frac{2\mu}{\sigma^2}a_1}e^{-\frac{2\mu}{\sigma^2}a_2}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)\right]^2}z_2}\right)J_1\left(2i\sqrt{\frac{4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}z_1}\right),
\end{aligned}
$$

where J is the Bessel function of the first kind.

Proof. The joint density can be obtained by inverting

$$\mathbb{E}\left(e^{-\gamma_1 L_{\Gamma_0(l)}(a_1)}e^{-\gamma_2 L_{\Gamma_0(l)(a_2)}}\right)$$

derived in Theorem (4.2.2) with respect to γ_1 and γ_2. Inverting wrt γ_1 gives us

$$\begin{aligned}
&\mathcal{L}_{\gamma_1}^{-1}\left(\mathbb{E}\left(e^{-\gamma_1 L_{\Gamma_0(l)}(a_1)}e^{-\gamma_2 L_{\Gamma_0(l)(a_2)}}\right)\right)(z_1)\\
&=\exp\left(\frac{-2l}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right)\\
&\quad\cdot\mathcal{L}_{\gamma_1}^{-1}\left(\exp\left(\frac{\frac{-4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}}{\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}-\gamma_1-\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}a_2}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}}\right)\right)(z_1)\\
&=\exp\left(\frac{-2l}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}\right)(-i)\sqrt{\frac{\frac{4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}}{z_1}}J_1\left(2i\sqrt{\frac{4le^{-\frac{2\mu}{\sigma^2}a_1}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)\right]^2}z_1}\right)\\
&\quad\cdot\exp\left(z_1\left(\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_1}-1\right)}-\frac{2e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}a_2}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}\right)\right).
\end{aligned}$$

Taking only the term with γ_2 and invert the Laplace transform with respect to γ_2, we have

$$\begin{aligned}
&\mathcal{L}_{\gamma_2}^{-1}\left(\exp\left(-\frac{2z_1e^{-\frac{2\mu}{\sigma^2}a_1}\gamma_2}{2e^{-\frac{2\mu}{\sigma^2}a_2}-\gamma_2\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}\right)\right)(z_2)\\
&=\exp\left(-\frac{2z_1e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}\right)\cdot\mathcal{L}_{\gamma_2}^{-1}\left(\exp\left(-\frac{\frac{4z_1e^{-\frac{2\mu}{\sigma^2}a_1}e^{-\frac{2\mu}{\sigma^2}a_2}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)\right]^2}}{\frac{2e^{-\frac{2\mu}{\sigma^2}a_2}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}-\gamma_2}\right)\right)(z_2)\\
&=\exp\left(-\frac{2z_1e^{-\frac{2\mu}{\sigma^2}a_1}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}\right)\cdot(-i)\sqrt{\frac{\frac{4z_1e^{-\frac{2\mu}{\sigma^2}a_1}e^{-\frac{2\mu}{\sigma^2}a_2}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)\right]^2}}{z_2}}\\
&\quad\cdot\exp\left(\frac{2e^{-\frac{2\mu}{\sigma^2}a_2}}{\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)}z_2\right)J_1\left(2i\sqrt{\frac{4z_1e^{-\frac{2\mu}{\sigma^2}a_1}e^{-\frac{2\mu}{\sigma^2}a_2}}{\left[\frac{\sigma^2}{\mu}\left(e^{-\frac{2\mu}{\sigma^2}a_2}-e^{-\frac{2\mu}{\sigma^2}a_1}\right)\right]^2}z_2}\right).
\end{aligned}$$

Combining these expressions, we can finally obtain the required density. $\square$

4.2.3 Application and Concluding Remarks

In response to the Financial Crisis in 2007-2009 which exhibited the event where more than 60% of losses in default came from the exposure to the credit quality of the counterparty instead of the actual default, the Basel Committee on Banking Supervision (BCBS) strengthened its management of counterparty credit risk by including an additional counterparty credit valuation adjustment (CVA) capital charge on top of the traditional default capital charge. For this chapter, we focus on the Over The Counter (OTC) derivatives, which can be traded in the forms of options, futures, forwards, swaps and many more. It is well known that these derivatives have different types of risks associated to them, and in particular, we look at counterparty credit risk of derivatives, which depends on market condition and counterparty behaviour. The definition of counterparty credit risk detailed by the BCBS in Basel (2004) reads:

Definition 4.2.4. *Counterparty credit risk is the risk that the counterparty to a transaction could default before the final settlement of the transaction's cash flows.*

Basel (2004) provided two approaches for the computation of banks capital requirements: A Standardized Approach and an Internal Model Approach. The details on Standardized Counterparty Credit Risk Approach can be found in Committee et al. (2014) which gives:

$$\text{Counterparty Credit Capital Charge} = \text{EAD} \cdot \text{RW} \cdot 8\%,$$

where RW is the risk weight and 8% is to reflect the obligation in Pillar 1 of Basel and EAD is the exposure at default computed as:

$$\text{EAD} = 1.4 \cdot (\text{RC+PFE}),$$

for RC the replacement cost and PFE the potential future exposure.

The Internal Model Approach is detailed in Basel III (2011) and Committee et al. (2016) to present banks with the freedom to model internally the EAD for the OTC derivatives. Let $T \in \mathbb{R}_+$ be the maturity date, time grid $0 < t_1 < t_2 < \ldots < t_N = T$ for $N \in \mathbb{N}_+$. The

Expected Positive Exposure (EPE) is defined as

$$EPE := \frac{1}{T} \sum_{n=1}^{N} EE_n \cdot \Delta_n,$$

where EE_n is the expected exposure of the derivative and $\Delta_n = t_n - t_{n-1}$. For an equally spaced time grid, the EPE becomes

$$EPE = \frac{1}{N} \sum_{n=1}^{N} EE_n.$$

We follow Bonollo et al. (2017) who utilize the Brownian local time to evaluate the counterparty credit for the Accumulator derivative. The accumulator derivative is a instrument with the payoff P_i on observation day t_i, $i \in \mathbb{N}_+$ given as

$$P_i = \begin{cases} 0, & \text{if } \max_{0 \le s \le t_i} S_s \ge b, \\ Q(S_{t_i} - K), & \text{if } \max_{0 \le s \le t_i} S_s < b,\ S_s \ge K, \\ gQ(S_{t_i} - K), & \text{if } \max_{0 \le s \le t_i} S_s < b,\ S_s < K, \end{cases}$$

where b is the knock-out barrier level, K is the strike price, Q is the purchase quantity and g the gearing ratio. See Lam et al. (2009) and Bonollo et al. (2017) for more details. We extend their model and consider that the payoff of the derivative is computed at:

$$\Gamma_0(l) = \inf\{t \ge 0 \ :\ L_t(0) = l\}.$$

For ease of computation, we set $Q = 1, g = 2$ and assume that the barrier b is reflective. Then the payoff of the derivative is

$$\int_0^\infty L_{\Gamma_0(l)}(x)[(x-K)_+ - 2(K-x)_+]dx,$$

which tells us the fair value on day t_i, $i \in \mathbb{N}_+$ can be obtained as

$$V_{t_i} = \mathbb{E}\left(e^{-r(\Gamma_0(l) - t_i)} \int_0^\infty L_{\Gamma_0(l)}(x)[(x-K)_+ - 2(K-x)_+]dx \right).$$

Setting $r = 0$ and using Fubini's Theorem, the fair value is

$$V = \int_0^\infty \mathbb{E}\left[L_{\Gamma_0(l)}(x)\right][(x-K)_+ - 2(K-x)_+]dx,$$

and the expectation can be obtained using the density derived in Theorem (4.2.3). The expected exposure of the derivative can then be obtained as

$$EE = \mathbb{E}\left[V\right].$$

We are aware that it might not be realistic to set $r = 0$. This procedure can be easily extended to the case when $r \neq 0$. We can derive for positive parameters $\alpha, \gamma_1, \gamma_2$, two levels $a_1, a_2 \in \mathbb{R}$ such that $0 < a_1 < a_2 < b$ for $b \in \mathbb{R}$ and stopping time T_0 as defined in (4.5)

$$\mathbb{E}_\epsilon\left[e^{-\alpha T_0} e^{-\gamma_1 L_{T_0}^{(a_1)}} e^{-\gamma_2 L_{T_0}^{(a_2)}}\right].$$

This is left as future research.

4.3 Compound Poisson Process with Drift

In this section, we look at the claim surplus process and the distribution of its number of downcrossing for some deterministic level a. A risk reserve process, $U = \{U_t,\ t \geq 0\}$ defined as

$$U_t = u + \delta t - \sum_{i=1}^{N_t} \xi_i,$$

where u is the initial reserve, δ is the premium rate, $\{N_t,\ t \geq 0\}$ is a Poisson process with intensity $\lambda \in \mathbb{R}_+$ and $\{\xi_i,\ i \geq 1\}$ is a sequence of independent and identically distributed random variables which are independent of N and $\xi_i \sim F$ for some law F with no atoms at zero. This process is most commonly used to describe the risk process of an insurance company in the time interval $[0, \infty)$ which receives premiums continuously at a constant rate δ from the policyholders and pays an amount of ξ_i when a claim happens. We take the constant δ to be positive. The main quantity of interest of the study of the risk process revolves around computing the probability of ruin which is the event when the reserve U drops below zero, indicating that the sum of the claims that the company has to pay out is more than the initial reserve and premium that the company is receiving. The probability of ultimate ruin, denoted by $\psi(u)$ is defined as

$$\psi(u) = \mathbb{P}\left[\inf_{t\geq 0} U_t < 0 \mid U_0 = u\right].$$

For computing purposes, it is more convenient the aggregate loss process $X = \{X_t,\ t \geq 0\}$ defined as

$$X_t = u - U_t = \sum_{i=1}^{N_t} \xi_i - \delta t. \tag{4.13}$$

A Lévy process is a process $Z = \{Z_t,\ t \geq 0\}$ defined on probability space $(\Omega, \mathcal{F}, P)$ with paths $\mathbb{P}-$almost surely right continuous with left limits, $\mathbb{P}[Z_0 = 0] = 1$ and has independent and stationary increments. The distribution of a Lévy process Z can be uniquely determined by its characteristic exponent $\Psi(\theta)$ for $\theta \in \mathbb{R}$

$$\mathbb{E}\left[e^{i\theta Z_t}\right] = e^{-t\Psi(\theta)}.$$

The Lévy Khintchine formula for Lévy processes guarantees the existence of a Lévy triplet (a, σ, Π) where $a \in \mathbb{R}, \sigma \geq 0$ and Π a measure concentrated on $\mathbb{R}\backslash\{0\}$ with $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ such that for $\theta \in \mathbb{R}$

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 - \int_{\mathbb{R}}(e^{i\theta z} - 1 - i\theta y \mathbb{1}_{\{|z|<1\}})\Pi(dz).$$

With this formula, we can see that the loss process X can also be seen as a Lévy process with Lévy triplet (a, σ, Π) with $a = \mu, \sigma = 0$ and $\Pi(dy) = \lambda F(dy)$. We will refer to Kyprianou (2014) for the computations in this section.

4.3.1 Local Time at One Level

We are interested in computing the Laplace transform of the number of downcrossing of a level $a \in \mathbb{R}_+$ until the first time the local time at 0 exceeds an amount l.

$$\mathbb{E}_{e_\theta}\left[e^{-\beta N_{\Gamma_0(l)}}\right],$$

where e_θ is an exponential random variable with parameter $\theta > 0$, $N_t = \#\{X_t = a\}$ and $\Gamma_0(l)$ is such that

$$\Gamma_0(l) = \inf\{t \geq 0 \ : \ N_t^{(0)} = l\}. \tag{4.14}$$

We will compute this for the case when the starting point $X_0 = x$ is below a, at a and above a. Let us define the following stopping times:

$$T_0 := \inf\{t \geq 0 \ : \ X_t = 0\} = \inf\{t \geq 0 \ : \ X_t \leq 0\} =: T_0^-, \tag{4.15}$$

$$\tau_a^+ := \inf\{t \geq 0 \ : \ X_t > a\}. \tag{4.16}$$

Theorem 4.3.1. *For $\beta \in \mathbb{R}_+$ and T_0 as defined in (4.15), we can compute*

$$\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right] = \begin{cases} \frac{W^{(0)}(a-x)}{W^{(0)}(a)} + e^{-\beta}\left(1 - \frac{W^{(0)}(a-x)}{W^{(0)}(a)}\right)\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & 0 < x < a, \\ \frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & x = a, \\ e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & x > a. \end{cases}$$

The function $W^{(q)} : \mathbb{R}_+ \to \mathbb{R}$ is the q-scaled function for $q \geq 0$ such that it is the unique function satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q},$$

for all $\beta > \sup\{y : \psi(y) = q\}$ where ψ is the Laplace exponent of X.

Proof. For the case when $0 < x < a$, using T_0 and τ_a^+ as defined in (4.15) and (4.16), we have

$$\begin{aligned} \mathbb{E}_x\left[e^{-\beta N_{T_0}}\right] &= \mathbb{E}_x\left[\mathbb{1}_{\{T_0 < \tau_a^+\}} e^{-\beta N_{T_0}}\right] + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+ < T_0\}} e^{-\beta N_{T_0}}\right] \\ &= \mathbb{P}_x\left[T_0 < \tau_a^+\right] + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+ < T_0\}} e^{-\beta N_{T_0}}\right]. \end{aligned}$$

Let us define for $Y = \{Y_t = -X_t, \ t \geq 0\}$

$$\tilde{\tau}_{x-a}^- = \inf\{t \geq 0 : Y_t < x - a\}, \tag{4.17}$$

$$\tilde{\tau}_x^+ = \inf\{t \geq 0 : Y_t > x\}. \tag{4.18}$$

Conditioning with respect to the filtration at time τ_a^+, the expression in $\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right]$ becomes

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right] &= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+ < T_0\}}\mathbb{E}_{X_{\tau_a^+}}\left[e^{-\beta N_{T_0}}\right]\right] \\
&= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + \mathbb{E}_x\left[\mathbb{1}_{\{\tau_a^+ < T_0\}} \cdot e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\right] \\
&= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\mathbb{P}_x\left[\tau_a^+ < T_0\right] \\
&= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\mathbb{P}_0\left[\tilde{\tau}_{x-a}^- < \tilde{\tau}_x^+\right] \\
&= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\left(1 - \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right]\right).
\end{aligned}$$

From Kyprianou (2014), we have that for a spectrally negative Lévy process $X_t = \mu t - S_t$ with any $x \leq a$ and $q \geq 0$

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbb{1}_{\{\tau_0^- > \tau_a^+\}}\right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{4.19}$$

where the function W is the scale function given for compound Poisson process with rate $\lambda > 0$ and exponential jump distribution with parameter $\delta > 0$

$$W(x) = \frac{1}{\delta}\left(1 + \frac{\lambda}{\mu\delta - \lambda}\left[1 - e^{-(\mu - \frac{\lambda}{\delta})x}\right]\right).$$

Putting everything together, we have that for $0 < x < a$

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right] &= \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\left(1 - \mathbb{P}_0\left[\tilde{\tau}_x^+ < \tilde{\tau}_{x-a}^-\right]\right) \\
&= \tilde{\mathbb{P}}_{a-x}\left[\tilde{\tau}_a^+ < \tilde{\tau}_0^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\left(1 - \tilde{\mathbb{P}}_{a-x}\left[\tilde{\tau}_a^+ < \tilde{\tau}_0^-\right]\right) \\
&= \frac{W^{(0)}(a-x)}{W^{(0)}(a)} + e^{-\beta}\left(1 - \frac{W^{(0)}(a-x)}{W^{(0)}(a)}\right)\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] \\
&= \frac{W^{(0)}(a-x)}{W^{(0)}(a)} + e^{-\beta}\left(1 - \frac{W^{(0)}(a-x)}{W^{(0)}(a)}\right)\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1 - e^{-\beta}\left(1 - \frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}.
\end{aligned}$$

The expression $\tilde{\mathbb{P}}_{a-x}\left[\tilde{\tau}_a^+ < \tilde{\tau}_0^-\right]$ can be obtained from (4.19) and $\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]$ is calculated in the next part.

For the case when $x = a$, we can compute using similar reasoning that

$$\begin{aligned}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] &= \mathbb{E}_a\left[\mathbb{1}_{\{T_0^- < \tau_a^+\}} e^{-\beta N_{T_0}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}}\right] \\ &= e^{-\beta}\mathbb{E}_a\left[\mathbb{1}_{\{T_0^- < \tau_a^+\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}}\right] \\ &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tau_{-a}^- < \tau_0^+\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}}\right],\end{aligned}$$

with

$$\begin{aligned}\tilde{\tau}_a^+ &:= \inf\{t \geq 0 \;:\; Y_t \geq a\}, \\ \tilde{\tau}_0^- &:= \inf\{t \geq 0 \;:\; Y_t < 0\}.\end{aligned}$$

The expression becomes

$$\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] = e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}}\right].$$

Conditioning with respect to the filtration at time τ_a^+, we can compute

$$\begin{aligned}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}}\right] \\ &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + \mathbb{E}_a\left[\mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta N_{T_0}} | \mathcal{F}_{\tau_a^+}\right]\right] \\ &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} \mathbb{E}_{X_{\tau_a^+}}\left[e^{-\beta N_{T_0}}\right]\right] \\ &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + \mathbb{E}_a\left[\mathbb{1}_{\{\tau_a^+ < T_0^-\}} e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\right] \\ &= e^{-\beta}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_a^+ < \tilde{\tau}_0^-\}}\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_0^- < \tilde{\tau}_a^+\}}\right] \\ &= e^{-\beta}\mathbb{P}_0\left[\tilde{\tau}_a^+ < \tilde{\tau}_0^-\right] + e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\mathbb{P}_0\left[\tilde{\tau}_0^- < \tilde{\tau}_a^+\right].\end{aligned}$$

Some rearranging gives

$$\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] = \frac{e^{-\beta}\mathbb{P}_0\left[\tilde{\tau}_a^+ < \tilde{\tau}_0^-\right]}{1 - e^{-\beta}\mathbb{P}_0\left[\tilde{\tau}_0^- < \tilde{\tau}_a^+\right]} \stackrel{(4.19)}{=} \frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1 - e^{-\beta}\left(1 - \frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}. \tag{4.20}$$

For the case when $x > a$, conditioning on the filtration with respect to time T_a, we can compute

$$\begin{aligned}
\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right] &= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\beta N_{T_0}}|\mathcal{F}_{T_a}\right]\right] \\
&= \mathbb{E}_x\left[e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right]\right] \\
&= e^{-\beta}\mathbb{E}_a\left[e^{-\beta N_{T_0}}\right] \\
&= e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}.
\end{aligned}$$

where the last equality is obtained from (4.20). □

Remark 4.3.2. *For $0 < x < a$, N_{T_0} can be viewed as the product of a Bernoulli random variable with success probability p_1 and an independent Geometric random variable with support on the set $\{1, 2, \ldots\}$ and success probability p_2 where p_1 and p_2 are given as*

$$\begin{aligned}
p_1 &= 1 - \frac{W^{(0)}(a-x))}{W^{(0)}(a)}, \\
p_2 &= \frac{W^{(0)}(0)}{W^{(0)}(a)}.
\end{aligned}$$

Proof. We can compute

$$\mathbb{P}_x\left[N_{T_0} = n\right] = \begin{cases} \frac{W^{(0)}(a-x)}{W^{(0)}(a)}; & n = 0, \\ \frac{W^{(0)}(0)}{W^{(0)}(a)}\left(1 - \frac{W^{(0)}(a-x)}{W^{(0)}(a)}\right)\left(1 - \frac{W^{(0)}(0)}{W^{(0)}(a)}\right)^{n-1}; & n \geq 1, \end{cases}$$

where $\lambda > 0$ is the rate of the compound Poisson process and $\alpha > 0$ is the parameter of the exponential jump distribution. From there, we can easily identify the distribution of N_{T_0}. □

Theorem 4.3.3. *For $\beta \in \mathbb{C}_+$, e_θ an independent exponential variable with parameter $\theta > 0$ and $\Gamma_0(l)$ as defined in (4.14), we have*

$$\mathbb{E}_{e_\theta}\left[e^{-\beta N_{\Gamma_0(l)}}\right] = \begin{cases} \left[g(\theta, \lambda, \delta, \mu, a, \beta)\right]^l; & 0 < x < a, \\ \left[\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}\right]^l; & x = a, \\ \left[e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}\right]^l; & x > a, \end{cases}$$

where the function g is given as

$$g(\theta,\lambda,\delta,\mu,a,\beta)$$
$$= \frac{1-e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}}{W^{(0)}(a)}\left[\frac{1+\frac{\lambda}{\mu\delta-\lambda}}{\delta}-\frac{\theta\lambda e^{-(\mu-\frac{\lambda}{\delta})a}}{\delta(\mu\delta-\lambda)(\theta+\mu-\frac{\lambda}{\delta})}\right]+\frac{e^{-2\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)},$$

for $\theta+\mu-\frac{\lambda}{\delta}>0$.

Proof. From Theorem (4.3.1), we computed

$$\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right]=\begin{cases}\frac{W^{(0)}(a-x)}{W^{(0)}(a)}\left(1-e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}\right) & \\ \quad +e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & 0<x<a,\\ \frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & x=a,\\ e^{-\beta}\frac{e^{-\beta}\frac{W^{(0)}(0)}{W^{(0)}(a)}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}; & x>a.\end{cases}$$

This tells us the number of times we have crossed the level $a>0$ in this particular excursion starting from $X_0=x$ until the first time the processes goes below 0. As we can see from the expression in Theorem (4.3.1), the Laplace transform of N_{T_0} depends on the position of the starting point. Therefore, assuming that the starting point $X_0=x$ follows an exponential distribution with parameter $\theta>0$, we have

$$\mathbb{E}_{e\theta}\left[e^{-\beta\Gamma_0(l)}\right]=\left(\mathbb{E}_{e\theta}\left[e^{-\beta N_{T_0}}\right]\right)^l=\left(\int_0^\infty\mathbb{E}_x\left[e^{-\beta N_{T_0}}\right]\theta e^{-\theta x}dx\right)^l.$$

since the excursions are independent. Using the results in Theorem (4.3.1), for $0<x<a$, we have

$$W(a-x)\theta e^{-\theta x}=\frac{1}{\delta}\left(1+\frac{\lambda}{\mu\delta-\lambda}\left[1-e^{-(\mu-\frac{\lambda}{\delta})(a-x)}\right]\right)\theta e^{-\theta x}$$
$$=\frac{1+\frac{\lambda}{\mu\delta-\lambda}}{\delta}\theta e^{-\theta x}-\frac{\theta\lambda e^{-(\mu-\frac{\lambda}{\delta})a}}{\delta(\mu\delta-\lambda)}e^{-(\theta+\mu-\frac{\lambda}{\delta})x}.$$

Therefore, for $0 < x < a$, we have

$$\begin{aligned}
&\mathbb{E}_{e_\theta}\left[e^{-\beta\Gamma_0(l)}\right] \\
&= \left(\int_0^\infty \mathbb{E}_x\left[e^{-\beta N_{T_0}}\right]\theta e^{-\theta x}dx\right)^l \\
&= \left(\frac{1-e^{-\beta}\frac{e^{-\beta\frac{W^{(0)}(0)}{W^{(0)}(a)}}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}}{W^{(0)}(a)}\left[\frac{1+\frac{\lambda}{\mu\delta-\lambda}}{\delta}-\frac{\theta\lambda e^{-(\mu-\frac{\lambda}{\delta})a}}{\delta(\mu\delta-\lambda)(\theta+\mu-\frac{\lambda}{\delta})}\right]+\frac{e^{-2\beta\frac{W^{(0)}(0)}{W^{(0)}(a)}}}{1-e^{-\beta}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a)}\right)}\right)^l.
\end{aligned}$$

$\square$

4.3.2 Local Times at Two Levels

We are interested in computing the Laplace transform of the numbers of downcrossing of two level $a_1, a_2 \in \mathbb{R}_+$ such that $0 < a_1 < a_2 < \infty$ until the first time the local time at 0 exceeds an amount $l > 0$:

$$\mathbb{E}_{e_\theta}\left[e^{-\beta_1 N^{(a_1)}_{\Gamma_0(l)}}e^{-\beta_2 N^{(a_2)}_{\Gamma_0(l)}}\right],$$

where e_θ is an exponential random variable with parameter $\theta > 0$, $N_t^{(a_1)} = \#\{X_t = a_1\}$, $N_t^{(a_2)} = \#\{X_t = a_2\}$ and $\Gamma_0(l)$ is as defined in (4.14). Let us define as in the case for one level

$$\begin{aligned}
T_0 &=:= \inf\{t \geq 0 \ : \ X_t = 0\} = \inf\{t \geq 0 \ : \ X_t \leq 0\} =: T_0^-, \\
\tau_a^+ &:= \inf\{t \geq 0\ ;\ X_t > a\}.
\end{aligned}$$

We will compute this for the case when $x > a_1$, $x = a_1$ and $0 < x < a_1$ with the exponential jump distribution rate, $\delta = 1$ for simplicity.

Theorem 4.3.4. *For $\beta_1, \beta_2 \in \mathbb{C}_+$, $0 < a_1 < a_2 < \infty$ and $\delta = 1$, we can derive*

$$\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right]$$
$$= \begin{cases} \frac{W^{(0)}(a-x)}{W^{(0)}(a)} \\ \qquad +\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right] e^{-\beta_1} \int_{-\infty}^{0} \mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right] f_{Z_\tau}(z)dz; & 0 < x < a_1, \\ \frac{e^{-\beta_1}\frac{W^{(0)}(0)}{W^{(0)}(a_1)}}{1-e^{-\beta_1}\left(\int_{-\infty}^{-(a_2-a_1)} + \int_{-(a_2-a_1)}^{0}\right)\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right] f_{Y_\tau}(z)dz}; & x = a_1, \\ e^{-\beta_1}\mathbb{E}_{x-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right]; & x > a_1, \end{cases}$$

where

$$\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]$$
$$= \begin{cases} e^{-\beta_2}\frac{e^{-\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}; & z \in (-\infty, -(a_2-a_1)), \\ \frac{W^{(0)}(a_2-a_1+z)}{W^{(0)}(a_2-a_1)} \\ \qquad +e^{-\beta_2}\left(1-\frac{W^{(0)}(a_2-a_1+z)}{W^{(0)}(a_2-a_1)}\right)\frac{e^{-\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}; & z \in (-(a_2-a_1), 0), \end{cases}$$

and the density f_{Y_τ} is given as

$$\mathbb{P}_x\left[Y_\tau \in dz\right]$$
$$= \mu e^{\mu z} dz \frac{\lambda}{\mu-\lambda}\left(\left[\frac{W(0)}{W(a_1)} - 1\right]\left(1-e^{-\mu a_1}\right) - \left[\frac{W(0)}{W(a_1)}e^{-(\mu-\lambda)a_1} - 1\right]\left(1-e^{-\lambda a_1}\right)\right),$$

and the function $W(x)$ is such that

$$W(x) = \frac{1}{\delta}\left[1 + \frac{\lambda}{\delta\mu-\lambda}\left(1-e^{-\left(\mu-\frac{\lambda}{\delta}\right)x}\right)\right] \overset{(\delta=1)}{=} 1 + \frac{\lambda}{\mu-\lambda}\left(1-e^{-(\mu-\lambda)x}\right).$$

Proof. We start with the case when $x > a_1$. Conditioning with respect to the filtration at time $\tau_{a_1}^-$ and using the strong Markov property, we can derive

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}} | \mathcal{F}_{\tau_{a_1}^-}\right]\right] \\
&= e^{-\beta_1}\mathbb{E}_x\left[e^{-\beta_2 N^{(a_2)}_{\tau_{a_1}^{(-)}}} \mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\right] \\
&= e^{-\beta_1}\mathbb{E}_{x-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right],
\end{aligned}
$$

where $\mathbb{E}_{x-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]$ can be obtained from Theorem (4.3.1). In order to compute the joint Laplace transform, we have to compute when $x = a_1$,

$$
\begin{aligned}
&\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right] \\
&= \mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}} \mathbb{1}_{\{T_0<\tau_{a_1}^+\}}\right] + \mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}} \mathbb{1}_{\{\tau_{a_1}^+<T_0\}}\right] \\
&= e^{-\beta_1}\mathbb{E}_{a_1}\left[\mathbb{1}_{\{T_0<\tau_{a_1}^+\}}\right] + \mathbb{E}_{a_1}\left[\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}} \mathbb{1}_{\{\tau_{a_1}^+<T_0\}} | \mathcal{F}_{\tau_{a_1}^+}\right]\right] \\
&= e^{-\beta_1}\mathbb{E}_0\left[\tilde{\tau}_{a_1}^+ < \tilde{\tau}_0^-\right] + \mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\mathbb{E}_{a_1}\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}} e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right],
\end{aligned}
$$

by conditioning with respect to the filtration at time $\tau_{a_1}^+$ and strong Markov property. We have from the calculations for the local time at one level that

$$
\mathbb{E}_0\left[\tilde{\tau}_{a_1}^+ < \tilde{\tau}_0^-\right] = \frac{W^{(0)}(0)}{W^{(0)}(a_1)}.
$$

The joint Laplace transform then becomes

$$
\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}} e^{-\beta_2 N_{T_0}^{(a_2)}}\right] = \frac{e^{-\beta_1}\frac{W^{(0)}(0)}{W^{(0)}(a_1)}}{1 - \mathbb{E}_{a_1}\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}} e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]}. \tag{4.21}
$$

What is left to do is to compute the expectation in the denominator.

$$
\begin{aligned}
&\mathbb{E}_{a_1}\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\mathbb{E}_0\left[\mathbb{1}_{\{\tilde{\tau}_0^-<\tilde{\tau}_{a_1}^+\}}\mathbb{E}_{-Y_{\tilde{\tau}_0^-}}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\int_{-\infty}^{0}\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]f_{Y_\tau}(z)dz\\
&=e^{-\beta_1}\left(\int_{-\infty}^{-(a_2-a_1)}+\int_{-(a_2-a_1)}^{0}\right)\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]f_{Y_\tau}(z)dz.
\end{aligned}
$$

The density of $Y_{\tilde{\tau}_0^-}$ can be computed by referring to Kyprianou (2014)

$$
\begin{aligned}
&\mathbb{P}_x\left[Y_\tau\in dz\right]\\
&=\int_0^a\mathbb{P}_x\left[Y_\tau\in dz, Y_{\tau_-}\in dy\right]\\
&=\int_0^a\lambda\mu e^{\mu(z-y)}dz\left(\frac{W(x)W(a-y)-W(a)W(x-y)}{W(a)}\right)dy\\
&=\frac{\lambda\mu}{\delta}e^{\mu z}\frac{W(x)}{W(a)}dz\int_0^a e^{-\mu y}\left(1+\frac{\lambda}{\delta\mu-\lambda}-\frac{\lambda}{\delta\mu-\lambda}e^{-(\mu-\frac{\lambda}{\delta})(a-y)}\right)dy\\
&\quad-\frac{\lambda\mu}{\delta}e^{\mu z}\int_0^a e^{-\mu y}dz\left(1+\frac{\lambda}{\delta\mu-\lambda}-\frac{\lambda}{\delta\mu-\lambda}e^{-(\mu-\frac{\lambda}{\delta})(x-y)}\right)dy\\
&=\lambda\mu e^{\mu z}dz\left(1+\frac{\lambda}{\mu-\lambda}\right)\left[\frac{W(0)}{W(a_1)}-1\right]\frac{1-e^{-\mu a_1}}{\mu}\\
&\quad-\lambda\mu e^{\mu z}dz\frac{\lambda}{\mu-\lambda}\left[\frac{W(0)}{W(a_1)}e^{-(\mu-\lambda)a_1}-1\right]\frac{1-e^{-\lambda a_1}}{\lambda}\\
&=\mu e^{\mu z}dz\frac{\lambda}{\mu-\lambda}\left(\left[\frac{W(0)}{W(a_1)}-1\right]\left(1-e^{-\mu a_1}\right)-\left[\frac{W(0)}{W(a_1)}e^{-(\mu-\lambda)a_1}-1\right]\left(1-e^{-\lambda a_1}\right)\right),
\end{aligned}
$$

for $\delta=1$, $x=0$ and $a=a_1$. We can compute the expectation using the results with one level by having $-z$ instead of x as the starting point, and a_2-a_1 instead of a as the crossing level.

$$\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]$$

$$=\begin{cases} \dfrac{e^{-2\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}; & z\in(-\infty,-(a_2-a_1)),\\ \dfrac{W^{(0)}(a_2-a_1+z)}{W^{(0)}(a_2-a_1)}\left[1-\dfrac{e^{-2\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}\right] & \\ \qquad +\dfrac{e^{-2\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}; & z\in(-(a_2-a_1),0). \end{cases}$$

Putting everything together gives us

$$\begin{aligned}
&\mathbb{E}_{a_1}\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\left(\int_{-\infty}^{-(a_2-a_1)}+\int_{-(a_2-a_1)}^{0}\right)\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]f_{Y_\tau}(z)dz\\
&=\frac{e^{-\beta_1}e^{-2\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}\frac{\lambda}{\mu-\lambda}\\
&\qquad\cdot\left(\left[\frac{W(0)}{W(a_1)}-1\right]\left(1-e^{-\mu a_1}\right)-\left[\frac{W(0)}{W(a_1)}e^{-(\mu-\lambda)a_1}-1\right]\left(1-e^{-\lambda a_1}\right)\right)\\
&\quad+\frac{1-\dfrac{e^{-2\beta_2}\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}}{1-e^{-\beta_2}\left(1-\frac{W^{(0)}(0)}{W^{(0)}(a_2-a_1)}\right)}}{W^{(0)}(a_2-a_1)}\left(\frac{1-e^{-\mu(a_2-a_1)}}{\mu-\lambda}-\frac{e^{-(\mu-\lambda)(a_2-a_1)}-e^{-\mu(a_2-a_1)}}{\mu-\lambda}\right)\frac{\mu\lambda e^{-\beta_1}}{\mu-\lambda}\\
&\qquad\cdot\left(\left[\frac{W(0)}{W(a_1)}-1\right]\left(1-e^{-\mu a_1}\right)-\left[\frac{W(0)}{W(a_1)}e^{-(\mu-\lambda)a_1}-1\right]\left(1-e^{-\lambda a_1}\right)\right).
\end{aligned}$$

Substituting this into (4.21) gives us the expression of the joint Laplace transform when $x=a_1$.

Finally, let us look at the case when $0 < x < a_1$.

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\\
&=\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\mathbb{1}_{\{T_0<\tau_{a_1}^+\}}\right]+\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}\right]\\
&=\mathbb{P}_x\left[T_0<\tau_{a_1}^+\right]+\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}\right]\\
&=\mathbb{P}_x\left[T_0<\tau_{a_1}^+\right]+\mathbb{E}_x\left[\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}|\mathcal{F}_{\tau_{a_1}^+}\right]\right]\\
&=\frac{W^{(0)}(a-x)}{W^{(0)}(a)}+\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\mathbb{E}_x\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right],
\end{aligned}
$$

by conditioning on the filtration at time $\tau_{a_1}^+$ and the last equality comes from the case when $x > a_1$. The expression for $\mathbb{E}_{a_1}\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\right]$ is the case that we have computed for $x = a_1$ so let us focus on the last expectation. Defining $Z_t = Y_t + a_1 - x$ gives

$$
\begin{aligned}
&\mathbb{E}_x\left[\mathbb{1}_{\{\tau_{a_1}^+<T_0\}}e^{-\beta_1}\mathbb{E}_{X_{\tau_{a_1}^+}-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\mathbb{E}_0\left[\mathbb{1}_{\{\tau_{a_1-x}^+<\tau_{-x}^-\}}\mathbb{E}_{X_{\tau_{a_1-x}^+}+x-a_1}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\mathbb{E}_{a_1-x}\left[\mathbb{1}_{\{\tilde{\tau}_0^-<\tilde{\tau}_{a_1}^+\}}\mathbb{E}_{-Z_{\tilde{\tau}_0^-}}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]\right]\\
&=e^{-\beta_1}\int_{-\infty}^{0}\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]f_{Z_\tau}(z)dz\\
&=e^{-\beta_1}\left(\int_{-\infty}^{-(a_2-a_1)}+\int_{-(a_2-a_1)}^{0}\right)\mathbb{E}_{-z}\left[e^{-\beta_2 N_{T_0}^{(a_2-a_1)}}\right]f_{Z_\tau}(z)dz.
\end{aligned}
$$

We can then proceed using the same steps as described for the case when $x = a_1$. □

Remark 4.3.5. *We can compute the joint Laplace transform of the same quantity at time $\Gamma_0(l)$ as defined in (4.14) with an exponentially distributed starting point with parameter $\theta > 0$ by setting*

$$
\begin{aligned}
\mathbb{E}_{e_\theta}\left[e^{-\beta_1 N_{\Gamma_0 l}^{(a_1)}}e^{-\beta_2 N_{\Gamma_0(l)}^{(a_2)}}\right]&=\left(\mathbb{E}_{e_\theta}\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\right)^l\\
&=\left(\int_0^\infty\mathbb{E}_x\left[e^{-\beta_1 N_{T_0}^{(a_1)}}e^{-\beta_2 N_{T_0}^{(a_2)}}\right]\theta e^{-\theta x}dx\right)^l.
\end{aligned}
\tag{4.22}
$$

The reasoning behind this is that for the joint Laplace transform computed in Theorem (4.3.4), we have an excursion starting from $X_0 = x$ and stopping the first time the process goes below *the level 0. Since each of these excursions are independent of each other, by assuming the starting point, we can use (4.22) to derive the joint Laplace transform evaluated at the time when we have passed the level* 0 *for* l *many times by having* l *many excursions of the same kind. The exponential assumption for the starting point is important as we replicate the excursion for* l *many times, which mean that the starting point for each of these excursion may vary.*

Chapter 5

Hitting and Exit Times for Other Diffusions

5.1 Introduction

5.1.1 Introduction

Let $X = \{X_t,\ t \geq 0\}$ be a one-dimensional Brownian motion starting in x and is defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}_x)$ taking values in $(\mathbb{R}, \mathcal{B})$. Given a bounded open space $D \subseteq \mathbb{R}$, define the first exit time from D as

$$\tau_D := \inf\{t \geq 0 \ : \ X_t \notin D\}.$$

For $x \in \mathbb{R}$, consider the function f defined as

$$f(x) = \mathbb{E}_x\left[e^{-\beta X_{\tau_D}}\right], \tag{5.1}$$

where $\beta \in \mathbb{C}_+$ is the killing rate. Following Peskir and Shiryaev (2006), we see that the function f is the solution to the Dirichlet problem:

$$\begin{aligned} \mathcal{A}f &= \beta f; \qquad \text{in } D, \\ f|_{\partial D} &= 1, \end{aligned}$$

where ∂_D is the boundary of D and $\mathcal{A}$ is the infinitesimal operator defined as

$$\mathcal{A}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}_x\left[f(X_t)\right] - f(x)}{t},$$

for $x \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. It is well known that, from Dynkin (1965) that for X continuous, the infinitesimal generator $\mathcal{A}$ has the differential form:

$$\mathcal{A}f(x) = \beta f(x) + \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2}.$$

This differential form of the infinitesimal generator forms a bridge that connects the probability and analysis as our task of finding the expectation in (5.1) resolves around solving the differential equation presented in the Dirichlet problem.

There are many tools for obtaining solutions to differential equations, for example if the differential equation is separable, then the differential equation can be easily solved by separating the dependent and independent variables. A closer inspection reveals that the underlying method allowing the separation technique to be possible is the presence of a Lie group symmetry. The theory of Lie group symmetry, developed by Marius Sophus Lie is a topic that has been extensively studied by many, see for example Lie (1970), Bluman and Kumei (1989), Stephani (1989), Olver (1993), Hydon and Hydon (2000), Starrett (2007) to name a few.

The development of Lie group symmetry has a significant impact in, but not limited to the fields of pure and applied Mathematics, Physics and Engineering. The Lie group symmetry finds applications in vast areas of studies, such as algebraic topology, differential geometry, invariant theory, bifurcation theory, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics and others. The steps to apply Lie group symmetry will be detailed in the next chapter. We also discuss the diffusion that links to the Nicholson's integral. Finally, we construct a conditioned Brownian motion and find the Laplace transform of its first exit time as well as its density function.
Before moving on to the application of Lie group symmetry, we provide some important definitions surrounding this topic.

5.1.2 Preliminaries/Definitions

Definition 5.1.1. ***One-Parameter Lie Group of Transformation.***

On the Euclidean plane, let $\boldsymbol{x} = (x, y)$ *and* $\hat{\boldsymbol{x}} = (\hat{x}, \hat{y})$ *be some points on the plane. For* $\epsilon \in \mathbb{C}$*, the transformation*

$$\Gamma_\epsilon : \boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x}, \epsilon),$$

such that

$$\hat{x} = f(x, y, \epsilon), \qquad \hat{y} = g(x, y, \epsilon),$$

is the one-parameter group of transformation with parameter ϵ*, if the following hold:*

I. Γ_0 *gives the identity transformation, i.e.*

$$f(x, y, 0) = x, \qquad g(x, y, 0) = y.$$

II. Γ *is closed under composition. This means that for* $\epsilon_1, \epsilon_2 \in \mathbb{C}$*,*

$$\Gamma_{\epsilon_2}\Gamma_{\epsilon_1} = \Gamma_{\epsilon_2+\epsilon_1},$$

i.e.

$$x^* = f(\hat{x}, \hat{y}, \delta) = f(x, y, \epsilon_1 + \epsilon_2), \qquad y^* = f(\hat{x}, \hat{y}, \delta) = f(x, y, \epsilon_1 + \epsilon_2)$$

.

III. Γ_ϵ^{-1} *gives the inverse transformation:*

$$\Gamma_\epsilon^{-1} = \Gamma_{-\epsilon}.$$

IV. *Each* $\hat{\boldsymbol{X}}$ *can be represented as a Taylor series in* ϵ*.*

Definition 5.1.2. ***Orbits of Solutions.***

Under the transformation Γ *as defined in Definition 5.1.1, for a suitable choice of* ϵ*, the orbit through a point* (x, y) *is the set of all points that* (x, y) *can be mapped to. This means that* *the points on the orbit through (x, y) can be represented as*

$$(\hat{x}, \hat{y}) = (\hat{x}(x, y, \epsilon), \hat{y}(x, y, \epsilon)).$$

Definition 5.1.3. ***Tangent Vector and Tangent Vector Field****.*

For a one-parameter group of transformation Γ as defined in Definition 5.1.1 for $\epsilon \in \mathbb{C}$ such that

$$\hat{x} = f(x, y, \epsilon), \qquad \hat{y} = g(x, y, \epsilon).$$

According to Property (IV) of Definition 5.1.1, expanding $(\hat{x}, \hat{y})$ about the identity gives

$$\hat{x} = x + \epsilon\xi(x, y) + \mathcal{O}(\epsilon^2),$$
$$\hat{y} = y + \epsilon\eta(x, y) + \mathcal{O}(\epsilon^2),$$

such that

$$\xi(x, y) = \left.\frac{d\hat{x}}{d\epsilon}\right|_{\epsilon=0}, \qquad \eta(x, y) = \left.\frac{d\hat{y}}{d\epsilon}\right|_{\epsilon=0}.$$

(ξ, η) is called the tangent vector at (x, y) to the curve described by the transformed points $(\hat{x}, \hat{y})$, and it is the tangent vector field of the one-parameter Lie group of transformation Γ.

Definition 5.1.4. ***Infinitesimal Generator.***

The infinitesimal generator of the Lie group transformation is given by

$$\mathcal{A} = \xi(x, y)\frac{\partial}{\partial_x} + \eta(x, y)\frac{\partial}{\partial_y}. \tag{5.2}$$

Definition 5.1.5. ***Symmetry Condition****.*

The symmetry condition for a first order ordinary differential equation (ODE) is the condition required to make sure that any transformation maps the set of solution curve of a differential equation to another solution curve that also satisfies the original equation. Consider a first order ODE of the form

$$\frac{dy}{dx} = \omega(x, y),$$

where ω is an arbitrary function of x and y. Then the symmetry condition for ODE is

$$\frac{d\hat{y}}{d\hat{x}} = \omega(\hat{x}, \hat{y}), \qquad \textit{when} \qquad \frac{dy}{dx} = \omega(x, y).$$

Using the total derivative operator D_x defined by

$$D_x = \frac{dx}{dx}\frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{dy'}{dx}\frac{\partial}{\partial y'} + \ldots = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'},$$

we have that with $y' = \omega(x, y)$ that

$$\frac{d\hat{y}}{d\hat{x}} = \frac{D_x\hat{y}}{D_x\hat{x}} = \frac{\hat{y}_x + \omega(x, y)\hat{y}_y}{\hat{x}_x + \omega(x, y)\hat{x}_y} = \omega(\hat{x}, \hat{y}). \tag{5.3}$$

Definition 5.1.6. ***Linearized Symmetry Condition.***

The Lie symmetries of

$$\frac{dy}{dx} = \omega(x, y),$$

can be represented by

$$\begin{aligned} \hat{x} &= x + \epsilon\xi(x, y) + \mathcal{O}(\epsilon^2), \\ \hat{y} &= y + \epsilon\eta(x, y) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{5.4}$$

and we note that we can obtain linearized form of $\hat{x}$ and $\hat{y}$ when ignoring terms of order ϵ^2 and higher. In order to obtained a linearized form of the symmetry condition in Definition 5.1.5, we substitute (5.4) into (5.3) to obtain

$$\frac{\omega(x, y) + \epsilon[\eta_x + \omega(x, y)\eta_y] + \mathcal{O}(\epsilon^2)}{1 + \epsilon[\xi_x + \omega(x, y)\xi_y] + \mathcal{O}(\epsilon^2)} = \omega(x + \epsilon\xi + \mathcal{O}(\epsilon^2), y + \epsilon\eta + \mathcal{O}(\epsilon^2)).$$

Ignoring terms of order ϵ^2 and higher and equating the terms or order $\mathcal{O}(\epsilon)$ gives us the linearized symmetry condition for first order ODE:

$$\eta_x + (\eta_y - \xi_x)\omega(x, y) - \xi_y\omega^2(x, y) = \xi\omega_x(x, y) + \eta\omega_y(x, y).$$

Definition 5.1.7. ***Canonical Coordinates.***

Any coordinates $(r(x,y), s(x,y))$ *satisfying*

$$\begin{aligned} \xi(x,y)r_x + \eta(x,y)r_y &= 0, \\ \xi(x,y)s_x + \eta(x,y)s_y &= 1, \end{aligned} \tag{5.5}$$

and

$$r_x s_y - r_y s_x \neq 0,$$

is called the canonical coordinates. The canonical coordinates can be obtained from (5.5) by using the method of characteristics such that

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = ds.$$

The ODE of the form $\frac{dy}{dx} = \omega(x,y)$ *can then be transformed to its canonical coordinates:*

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}. \tag{5.6}$$

5.2 Construction

Lemma 5.2.1. *Consider a general first order differential equation of the form*

$$g'(x) = g^2(x) + p(x) =: \omega(x,g), \tag{5.7}$$

for some function $p : \mathbb{R} \to \mathbb{R}$ *such that*

$$p(x) = -b^2(x) - b'(x) - 2\beta. \tag{5.8}$$

Then, its reduced form in terms of canonical coordinates $(r,s) = (r(x,g), s(x,g))$ *can be obtained as*

$$\frac{ds}{dr} = \frac{1}{r^2 + \psi},$$

where we define ψ *as*

$$\psi = \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + p(x)\alpha^2(x).$$

Proof. For the ODE (5.7), we take symmetries of the form as defined in (5.4)

$$\begin{aligned}\hat{x} &= x + \epsilon\xi(x, g) + \mathcal{O}(\epsilon^2),\\ \hat{g} &= g + \epsilon\eta(x, g) + \mathcal{O}(\epsilon^2).\end{aligned} \tag{5.9}$$

With $\omega := \omega(x, g)$, this gives us the linearized symmetry condition as defined in Definition (5.1.6):

$$\eta_x + (\eta_g - \xi_x)\omega - \xi_g\omega^2 = \xi\omega_x + \eta\omega_g, \tag{5.10}$$

where ω_x and ω_g can be computed from (5.7) by treating $g(x)$ as g:

$$\begin{aligned}\omega_x &= p'(x) = -2b(x)b'(x) - b''(x),\\ \omega_g &= 2g(x).\end{aligned} \tag{5.11}$$

Substituting (5.11) in (5.10), we have

$$\eta_x + (\eta_g - \xi_x)[g^2(x) + p(x)] - \xi_g[g^2(x) + p(x)]^2 = \xi p'(x) + \eta(2g(x)).^1 \tag{5.12}$$

In order to solve the above system for ξ and η, we consider the symmetry tangent vector of the form:

$$\begin{aligned}\xi(x, g) &= \alpha(x),\\ \eta(x, g) &= \gamma(x) + \kappa(x)g(x),\end{aligned} \tag{5.13}$$

The left hand side of (5.12) then becomes

$$[\kappa(x) - \alpha'(x)]g^2(x) + \kappa'(x)g(x) + \gamma'(x) + [\kappa(x) - \alpha'(x)]p(x), \tag{5.14}$$

and the right hand side of (5.12)

$$2\kappa(x)g^2(x) + 2\gamma(x)g + \alpha(x)p'(x),$$

[1]Note: Equation (5.12) has 2 dependent variables ξ and η, therefore we have infinitely many solutions.

giving us the following:

$$\begin{cases} \kappa(x) & = -\alpha'(x), \\ \kappa'(x) & = 2\gamma(x), \\ \eta(x,g) & = -\frac{1}{2}\alpha''(x) - \alpha'(x)g(x), \\ \alpha'''(x) + 4\alpha'(x)p(x) + 2\alpha(x)p'(x) & = 0. \end{cases} \tag{5.15}$$

We can now find the canonical coordinates as defined in Definition 5.1.7. Let $(r, s) = (r(x, g), s(x, g))$ be the canonical coordinates such that

$$(\hat{r}, \hat{s}) = (r(\hat{x}, \hat{g}), s(\hat{x}, \hat{g})) = (r, s + \epsilon).$$

Then, under this new coordinates, the tangent vector at this point is such that

$$\left.\frac{d\hat{r}}{d\epsilon}\right|_{\epsilon=0} = 0, \qquad \left.\frac{d\hat{s}}{d\epsilon}\right|_{\epsilon=0} = 1. \tag{5.16}$$

This means that we need the conditions $\mathcal{A}r = 0$ and $\mathcal{A}s = 1$ to be satisfied, where $\mathcal{A}$ is the infinitesimal generator as defined in Definition 5.1.4. This gives us the condition as defined in (5.5):

$$\begin{aligned} \xi(x,g)r_x + \eta(x,g)r_g &= 0, \\ \xi(x,g)s_x + \eta(x,g)s_g &= 1. \end{aligned} \tag{5.17}$$

Substituting (5.13) and (5.15) into (5.17), we get

$$s = \int \frac{1}{\alpha(x)} dx, \qquad r = \frac{1}{2}\alpha'(x) + \alpha(x)g(x). \tag{5.18}$$

With these, the original ODE (5.7) can be reduced in terms of the canonical coordinates as given in (5.6):

$$
\begin{aligned}
\frac{ds}{dr} &= \frac{s_x + \omega(x,g)s_g}{r_x + \omega(x,g)r_g} \\
&= \frac{1}{\frac{1}{2}\alpha''(x)\alpha(x) + \alpha'(x)\alpha(x)g(x) + g^2(x)\alpha^2(x) + p(x)\alpha^2(x)} \\
&= \frac{1}{g^2(x)\alpha^2(x) + \alpha'(x)\alpha(x)g(x) + \frac{1}{4}[\alpha'(x)]^2 - \frac{1}{4}[\alpha'(x)]^2 + \frac{1}{2}\alpha''(x)\alpha(x) + p(x)\alpha^2(x)} \\
&= \frac{1}{r^2 + \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + p(x)\alpha^2(x)} \\
&=: \frac{1}{r^2 + \psi}.
\end{aligned}
\tag{5.19}
$$

□

With the simplified result from Lemma 5.2.1, we are ready to solve the original problem (5.7) for different cases.

Remark 5.2.2. *The original problem as specified in (5.7) corresponds to a non-linear Riccati equation, which can be converted to a second order linear ordinary differential equation. This correspondence means that the Riccati equation can be solved by obtaining quadrature, when a particular solution is known. This section aims to provide another method to obtain the solution via the Lie group theory.*

Theorem 5.2.3. *For an ODE of the form in (5.7) given as*

$$
g'(x) = g^2(x) + p(x) =: \omega(x,g),
$$

the solution can be obtained as

$$
g(x) = \begin{cases}
\frac{1}{\alpha(x)}\left(\frac{1}{\int \frac{1}{\alpha(x)}dx - c_1} - \frac{1}{2}\alpha'(x)\right); & \psi = 0, \\
\frac{\sqrt{\psi}\tan\left(\left[\int \frac{1}{\alpha(x)}dx - c_2\right]\sqrt{\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}; & \psi > 0, \\
\frac{-\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx - c_3\right]\sqrt{-\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}; & \psi < 0,
\end{cases}
$$

where $\alpha(x)$ is the solution to

$$
\alpha'''(x) + 4\alpha'(x)p(x) + 2\alpha(x)p'(x) = 0, \tag{5.20}
$$

$p(x)$ satisfies

$$p(x) = -b^2(x) - b'(x) - 2\beta,$$

and $\tanh(x)$ *is the hyperbolic tangent function.*

Proof. From Lemma 5.2.1, we see that the solving the differential equation (5.7) is equivalent to solving

$$\frac{ds}{dr} = \frac{1}{r^2 + \psi},$$

where $\psi = \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + p(x)\alpha^2(x)$. We will do this for 3 cases.

Case 1: $\psi = 0$. The differential equation in (5.19) becomes

$$\frac{ds}{dr} = \frac{1}{r^2}.$$

This can be solved directly to obtain

$$s = \frac{1}{r} + c_1,$$

where c_1 is an integrating constant. Using

$$s = \int \frac{1}{\alpha(x)} dx; \qquad r = \frac{1}{2}\alpha'(x) + \alpha(x)g(x),$$

as derived in (5.18), we see that

$$\int \frac{1}{\alpha(x)} dx = \frac{1}{\frac{1}{2}\alpha'(x) + \alpha(x)g(x)} + c_1.$$

Then the solution to the original differential equation can be derived as

$$g(x) = \frac{1}{\alpha(x)} \left(\frac{1}{\int \frac{1}{\alpha(x)} dx - c_1} - \frac{1}{2}\alpha'(x) \right). \tag{5.21}$$

Case 2: $\psi > 0$. The equation (5.19) can be written as

$$\frac{ds}{dr} = \frac{1}{r^2 + [\sqrt{\psi}]^2}.$$

So, we have

$$s = \frac{\tan^{-1}\left(\frac{r}{\sqrt{\psi}}\right)}{\sqrt{\psi}} + c_2,$$

where c_2 is an integrating constant. Using (5.18), we can derive

$$\left(\int \frac{1}{\alpha(x)}dx - c_2\right)\sqrt{\psi} = \tan^{-1}\left(\frac{\frac{1}{2}\alpha'(x) + \alpha(x)g(x)}{\sqrt{\psi}}\right),$$

which gives us

$$g(x) = \frac{\sqrt{\psi}\tan\left(\left[\int \frac{1}{\alpha(x)}dx - c_2\right]\sqrt{\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}. \tag{5.22}$$

Case 3: $\psi < 0$. With the differential equation in (5.19), we can solve it to obtain

$$s = -\frac{\tanh^{-1}\left(\frac{r}{\sqrt{-\psi}}\right)}{\sqrt{-\psi}} + c_3$$
$$(s - c_3)(-\sqrt{-\psi}) = \tanh^{-1}\left(\frac{r}{\sqrt{-\psi}}\right),$$

where c_3 is an integrating constant. Using (5.18), we have that

$$\tanh\left(\left[\int \frac{1}{\alpha(x)}dx - c_3\right]\left(-\sqrt{-\psi}\right)\right) = \frac{\frac{1}{2}\alpha'(x) + \alpha(x)g(x)}{\sqrt{-\psi}},$$

which gives us the solution to the original differential equation

$$\begin{aligned} g(x) &= \frac{\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx - c_3\right]\left(-\sqrt{-\psi}\right)\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)} \\ &= \frac{-\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx - c_3\right]\sqrt{-\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}. \end{aligned} \tag{5.23}$$

□

We have seen the use of Lie algebra in solving differential equations above. We will now explore this method for diffusion theory to help us identify martingales.

Theorem 5.2.4. *For $X = \{X_t;\ t \geq 0\}$ a standard Brownian motion, we can derive a* *martingale of the form*

$$e^{-\beta t}f(x) := e^{-\beta t}e^{-h(X_t)},$$

where $h(x)$ is given as

$$h(x) = \begin{cases} \int \frac{1}{\alpha(x)}\left(\frac{1}{\int \frac{1}{\alpha(x)}dx - c_1} - \frac{1}{2}\alpha'(x)\right)dx + \int b(x)dx; & \psi = 0, \\ \int \frac{\sqrt{\psi}\tan\left(\left[\int \frac{1}{\alpha(x)}dx - c_2\right]\sqrt{\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}dx + \int b(x)dx; & \psi > 0, \\ \int \frac{-\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx - c_3\right]\sqrt{-\psi}\right) - \frac{1}{2}\alpha'(x)}{\alpha(x)}dx + \int b(x)dx; & \psi < 0, \end{cases}$$

where $\alpha(x)$ is the solution to (5.20).

Proof. We use the martingale approach discussed in Remark 2.2.1 to derive the required martingale. For a diffusion $X = \{X_t;\ t \geq 0\}$ with stochastic differential equation given as

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t; \qquad X_0 = x, \tag{5.24}$$

where $W = \{W_t,\ t \geq 0\}$ is a standard Brownian motion defined in Definition 1.0.1, $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}_+$ are functions of class $\mathcal{C}^1$ and $\mathcal{C}^2$ and representing the drift and dispersion coefficients respectively so that

$$b'(x) - \frac{1}{2}\sigma(x)\sigma''(x) - \frac{b(x)\sigma'(x)}{\sigma(x)},$$

is bounded and $\frac{1}{\sigma}$ is non-integrable at $\pm\infty$. Then the stochastic differential equation (5.24) admits a unique and strong solution.

Let the state space of X be either non-negative real half-line or the real line. The infinitesimal generator of the process (t, X_t) acting on a function $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ belonging to its domain is given by

$$\mathcal{A}f(x,t) = \frac{\partial f(x,t)}{\partial t} + b(x)\frac{\partial f(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}, \tag{5.25}$$

where b is a function such that $b : \mathbb{R} \to \mathbb{R}$. Let us assume now that the function $f(x,t)$ takes the form of

$$f(x,t) = e^{-\beta t}f(x), \tag{5.26}$$

where $f:\mathbb{R}\to\mathbb{R}$. As in Remark 2.2.1, substituting (5.26) into (5.25) and setting it to 0, it is easy to obtain

$$-\beta f(x)+b(x)f'(x)+\frac{1}{2}f''(x)=0. \tag{5.27}$$

A transformation of the form $f(x)=e^{-h(x)}$ gives us

$$-\beta-b(x)h'(x)+\frac{1}{2}(h'(x))^2-\frac{1}{2}h''(x)=0.$$

We observe that this is a non-linear second order differential equation, which can then be easily transformed into the form of the differential equation in (5.7) given as

$$g'(x)=g^2(x)-b^2(x)-b'(x)-2\beta.$$

From Theorem 5.2.3, we see that the solutions to this differential equation are given as (5.21), (5.22) and (5.23). From this, we can easily revert the transformations applied to retrieve $h(x)$.

$$h(x)=\begin{cases}\int \frac{1}{\alpha(x)}\left(\frac{1}{\int\frac{1}{\alpha(x)}dx-c_1}-\frac{1}{2}\alpha'(x)\right)dx+\int b(x)dx; & \psi=0,\\ \int\frac{\sqrt{\psi}\tan\left(\left[\int\frac{1}{\alpha(x)}dx-c_2\right]\sqrt{\psi}\right)-\frac{1}{2}\alpha'(x)}{\alpha(x)}dx+\int b(x)dx; & \psi>0,\\ \int\frac{-\sqrt{-\psi}\tanh\left(\left[\int\frac{1}{\alpha(x)}dx-c_3\right]\sqrt{-\psi}\right)-\frac{1}{2}\alpha'(x)}{\alpha(x)}dx+\int b(x)dx; & \psi<0,\end{cases}$$

with $\alpha(x)$ satisfying equation (5.20). □

Finally, we see that we need the expression for $\alpha(x)$ in Lemma 5.2.1, Theorems 5.2.3 and 5.2.4. We have derived in (5.15) that $\alpha(x)$ has to satisfy the third order differential equation (5.20)

$$\alpha'''(x)+4\alpha'(x)p(x)+2\alpha(x)p'(x)=0.$$

Therefore, in order to obtain a solution for $\alpha(x)$, we have to solve this differential equation. Following van Hoeij (2007), let K be a differential field of characteristic 0 which is a field equipped with a derivation operator ∂ such that

$$L=a_n\partial^n+a_{n-1}\partial^{n-1}+\ldots+a_1\partial+a_0.$$

When $a_n \neq 0$ and $L(y) = 0$, we say that L is a homogeneous linear differential equation of order n with coefficients in K. Let $K[\partial]$ be ring formed by a set of differential operators over K and $C_K =: \{x \in K \ : \ \partial(x) = 0\}$ be the field of constants. For $L \in K\partial$, if we have a third order linear differential equation of the form:

$$L(y) = y'''(x) + a_2 y''(x) + a_1 y'(x) + a_0 y(x), \tag{5.28}$$

then this can be reduced to a second order linear differential equation taking the form

$$L_2(y) = y''(x) + b_1 y'(x) + b_0 y(x), \tag{5.29}$$

where

$$b_1 = \frac{a_2}{3}; \qquad b_0 = -\frac{b_1' - a_1 + 2b_1^2}{4},$$

provided that L is the symmetric square of L_2. For more details on derivation, see Singer (1985) and van Hoeij (2007). Comparing (5.20) to (5.28), the coefficients a_2, a_1 and a_0 are such that

$$a_2(x) = 0; \qquad a_1(x) = 4p(x); \qquad a_0(x) = 2p'(x), \tag{5.30}$$

and this tells us that if $L(\alpha)$ defined as

$$L(\alpha) := \alpha'''(x) + 4\alpha'(x)p(x) + 2\alpha(x)p'(x) = 0,$$

is a symmetric square of a second order operator L_2, then we can reduce $\alpha(x)$ to the following second order differential equation:

$$L_2(y) := y''(x) + [-b^2(x) - b'(x) - 2\beta]y(x) = 0. \tag{5.31}$$

where $b(x)$ represents the drift component in (5.25). With this, we can see that if $\{y_1, y_2\}$ is the basis of the solution space for $L_2(y) = 0$, then $\{y_1^2, y_2^2, y_1 y_2\}$ is the basis of the solution space for $L(y) = 0$. This gives a way to simplify our third order differential equation for $\alpha(x)$ and what we need to do is to check that L is the symmetric square of L_2. Following the proof of Singer (1985), we see that L is the symmetric square of L_2 if and only if

$$4b_0b_1 + 2b_0' = a_0, \tag{5.32}$$

is satisfied, where a_0, b_0 and b_1 are coefficients in (5.28) and (5.29). In our case for the coefficients as defined in (5.30), we can see that we have $b_1 = 0$ and $b_0 = p(x)$ and therefore condition (5.32) is satisfied.

5.3 Nicholson's Integral

Let us consider a one-dimensional diffusion process $X = \{X_t,\ t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with infinitesimal generator on $C^2(\mathbb{R}_+)$ given as:

$$\mathcal{A}_X = \frac{b}{x}\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2},$$

where b is an arbitrary constant. For $b = \frac{1-2\alpha}{2}$ with $\alpha \in (0,1)$, this coincides with the infinitesimal generator of a Bessel process of dimension $2(1-\alpha)$. See Borodin and Salminen (2002) and Jeanblanc et al. (2009) for a more detailed investigation of the Bessel process. The infinitesimal generator of the process (t, X_t) acting on a bounded function $f : \mathbb{R}_+^2 \to \mathbb{R}$ is given by:

$$\begin{aligned} \mathcal{A}f(x,t) &= \frac{\partial f(x,t)}{\partial t} + \mathcal{A}_X f(x,t) \\ &= \frac{\partial f(x,t)}{\partial t} - \frac{b}{x}\frac{\partial f(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}. \end{aligned} \tag{5.33}$$

Comparing (5.33) to (5.25), we see the the drift term $b(x)$ becomes

$$b(x) = -\frac{b}{x},$$

where b is an arbitrary constant. As derived in (5.31), we know we have to consider the following second order ordinary differential equation to obtain the values for $\alpha(x)$:

$$\begin{aligned} y''(x) + [-b^2(x) - b'(x) - 2\beta]y(x) = 0 \\ y''(x) + \left(-\frac{b^2}{x^2} - \frac{b}{x^2} - 2\beta\right)y(x) = 0. \end{aligned} \tag{5.34}$$

Rearranging equation (5.34) gives us

$$x^2y''(x) + \left(2\left(-\frac{1}{2}\right) + 1\right) xy'(x) + \big(-2\beta x^2 - b^2 - b\big)y(x) = 0.$$

We can easily see that this is of the form of a transformed Bessel differential equation proposed by Bowman (2012) given as :

$$x^2y''(x) + (2p+1)xy'(x) + (q^2x^{2r} + s^2)y(x) = 0,$$

with $p = -\frac{1}{2}$, $q^2 = -2\beta, r = 1$ and $s^2 = -(b(b+1))$. The basis of the solution space to this differential equation is $\{y_1(x), y_2(x)\}$ where $y_1(x)$ and $y_2(x)$ are such that

$$y_1(x) = \sqrt{x}J_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right),$$
$$y_2(x) = \sqrt{x}Y_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right),$$

where $J_n(x)$ and $Y_n(x)$ are the Bessel functions of the first and second kinds. As derived earlier, the third order differential for $\alpha(x)$ in (5.20) has $\{y_1^2(x), y_2^2(x), y_1(x)y_2(x)\}$ as the basis of the solution space where $y_1^2(x), y_2^2(x)$ and $y_1(x)y_2(x)$ given as :

$$\begin{cases} x\, J^2_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right), \\ x\, Y^2_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right), \\ x\, J_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right) \cdot Y_{\sqrt{(b+\frac{1}{2})^2}}\left(-i\sqrt{2\beta}x\right). \end{cases} \tag{5.35}$$

We will now explore the connection of this diffusion X with the Nicholson's Integral. The famous Nicholson's integral is given as

$$J_n^2(z) + Y_n^2(z) = \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh(t))\cosh(2vt)dt$$

where $\sinh(x)$ and $\cosh(x)$ are the hyperbolic sine and cosine functions respectively and $K_0(x)$ is the modified Bessel function of the second kind. The Nicholson's integral has been proven by many, such as Watson (1922) using Hardy's theory of generalized integrals and integration over contours in the complex plane, and Dixon and Ferrar (1930) using a transformation.

In an attempt to prove the Nicholson's integral, Wilkins Jr (1948) considered the following differential equation:

$$z^2y'''(z) + 3zy''(z) + (1 - 4n^2 + 4z^2)y'(z) + 4zy(z) = 0. \tag{5.36}$$

This is equation (3) of the paper. By setting

$$\begin{cases} y(z) & = \frac{1}{x}\alpha(x), \\ z & = -i\sqrt{2\beta}x, \\ n & = b + \frac{1}{2}. \end{cases} \tag{5.37}$$

We see that equation (5.36) of the paper can be transformed into our equation for $\alpha(x)$ in (5.20). Using chain rule, we can derive the expressions for $y'(x), y''(x)$ and $y'''(x)$ in terms of $\alpha(x)$. For $y'(x)$, we have

$$\begin{aligned} y'(z) &= \frac{d}{dz}y(z) \\ &= \frac{d}{dz}\frac{1}{x}\alpha(x) \\ &= \left(-\frac{1}{x^2}\alpha(x) + \frac{1}{x}\alpha'(x)\right) \cdot \frac{i}{\sqrt{2\beta}}. \end{aligned} \tag{5.38}$$

Similarly, for $y''(z)$, we obtain

$$y''(z) = \frac{-1}{2\beta}\left(\frac{2}{x^3}\alpha(x) - \frac{2}{x^2}\alpha'(x) + \frac{1}{x}\alpha''(x)\right). \tag{5.39}$$

Finally for $y'''(z)$, we have

$$y'''(z) = \frac{-1}{2\beta} \cdot \frac{i}{\sqrt{2\beta}}\left(\frac{1}{x}\alpha'''(x) - \frac{3}{x^2}\alpha''(x) + \frac{6}{x^3}\alpha'(x) - \frac{6}{x^4}\alpha(x)\right). \tag{5.40}$$

Substituting (5.37), (5.38), (5.39) and (5.40) into (5.36) and rearranging the terms, we obtain

$$\alpha'''(x) + 4\biggl(-\frac{b^2}{x^2} - \frac{b}{x^2} - 2\beta\biggr)\alpha'(x) + 2\biggl(\frac{2b^2}{x^3} + \frac{2b}{x^3}\biggr)\alpha(x) = 0. \tag{5.41}$$

Comparing this to our original equation for $\alpha(x)$ in (5.20):

$$\alpha'''(x) + 4p(x)\alpha'(x) + 2p'(x)\alpha(x) = 0,$$

we see that $p(x)$ is such that

$$\begin{aligned} p(x) &= -b^2(x) - b'(x) - 2\beta \\ &= -\frac{b^2}{x^2} - \frac{b}{x^2} - 2\beta. \end{aligned}$$

This tells us that $b(x) = -\dfrac{b}{x}$, which is the same as the drift term of our diffusion X. This means that we can use what we derived earlier for the diffusion to solve the Nicholson's integral using the method proposed by Wilkins Jr (1948). Then, if we set $z = -i\sqrt{2\beta}x$, we have $x = \frac{i}{\sqrt{2\beta}}z$, then

$$\begin{aligned} y(z) &= \frac{1}{x}\alpha(x) \\ &= \frac{1}{\frac{i}{\sqrt{2\beta}}z}\alpha\left(\frac{i}{\sqrt{2\beta}}z\right) \\ &= -\frac{i\sqrt{2\beta}}{z}\alpha\left(\frac{i}{\sqrt{2\beta}}z\right). \end{aligned} \tag{5.42}$$

Using values of $\alpha(x)$ derived in (5.35), we have that

$$y(z) = \begin{cases} J^2_{\sqrt{(b+\frac{1}{2})^2}}(z), \\ Y^2_{\sqrt{(b+\frac{1}{2})^2}}(z), \\ J_{\sqrt{(b+\frac{1}{2})^2}}(z) \cdot Y_{\sqrt{(b+\frac{1}{2})^2}}(z), \end{cases} \tag{5.43}$$

as given in the paper. In addition, following the argument in the paper, we have that for $A = \frac{\pi^2}{8}$, $B = A$ and $C = 0$, we can obtain

$$\begin{aligned} y(z) &= AJ^2_{\sqrt{(b+\frac{1}{2})^2}}(z) + BY^2_{\sqrt{(b+\frac{1}{2})^2}}(z) + CJ_{\sqrt{(b+\frac{1}{2})^2}}(z) \cdot Y_{\sqrt{(b+\frac{1}{2})^2}}(z) \\ &= \frac{\pi^2}{8}\left(J_{\sqrt{(b+\frac{1}{2})^2}}(z) + Y_{\sqrt{(b+\frac{1}{2})^2}}(z)\right) + 0, \end{aligned} \tag{5.44}$$

which completes the proof of the Nicholson integral.

5.4 Conditioned Brownian Motion with Drift

Let us consider a one-dimensional Brownian motion $X = \{X_t,\ t \geq 0\}$ with drift parameter $\mu \in \mathbb{R}_+$, scale parameter $\sigma = 1$. Since this is a time-homogeneous Markov process, then according to Karatzas and Shreve (1991), X has an infinitesimal generator $\mathcal{A}$ that satisfies

$$\mathcal{A} = \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2}.$$

With the assumption that h^* is a C^2 function on the domain $\mathbb{R}_+$, we can perform Doob's h-transform as detailed in Theorem (3.2.1), where the function h is obtained as

$$h^*(x) = 1 + be^{-2\mu x}.$$

As detailed in the Theorem (3.2.1), the conditioned process $\{X_t,\ t \geq 0\}$ is a diffusion on $\mathbb{R}_+$ and has drift term of the form

$$\mu + \frac{1}{h^*(x)}\frac{\partial h^*}{\partial x} = \mu\frac{1 - be^{-2\mu x}}{1 + be^{-2\mu x}},$$

and the infinitesimal generator $\mathcal{A}^*$ of the conditioned process is given as

$$\mathcal{A}^* f(x) = \mu \cdot \frac{1 - be^{-2\mu x}}{1 + be^{-2\mu x}}\frac{\partial f(x)}{\partial x} + \frac{1}{2}\frac{\partial^2 f(x)}{\partial x^2},$$

for a function $f \in \mathcal{D}(\mathcal{A}^*)$ where $\mathcal{D}(\mathcal{A})$ is

$$\mathcal{D}(\mathcal{A}^*) = \{f \in C^2(\mathbb{R}_+)\}.$$

It is well established from Ikeda and Watanabe (1989) that the operator $(\mathcal{A}^*, \mathcal{D}(\mathcal{A}^*))$ generates a unique family of (strongly Markovian) measures $\{\mathbb{P}^*_x; x \in \mathbb{R}_+\}$ on the space $(C_+, \mathcal{B}(C_+)$ with $C_+ = C([0,\infty), \mathbb{R}_+)$ such that

$$f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)ds, \tag{5.45}$$

is a martingale under $\mathbb{P}^*_x$ for every $f \in \mathcal{D}(\mathcal{A}^*)$. Note that $C_+ = C([0,\infty), \mathbb{R}_+)$ denotes the family of continuous functions from the time set $[0,\infty)$ into the state space $_+$, and the symbol

$\mathcal{B}(C_+)$ denotes the Borel σ-algebra on C_+, i.e. the smallest σ-algebra containing all Borel cylinder subsets of C_+. Note also that (X_t, t) is again Markovian with infinitesimal generator $\mathcal{A}^{**}$ such that

$$\mathcal{A}^{**}f(x,t) = \frac{\partial f(x,t)}{\partial t} + \mathcal{A}^{*}f(x,t),$$

for a function $h(x,t) \in \mathcal{D}(\mathcal{A}^{**})$, i.e. $f(x,\cdot)$ has a continuous first derivative for each x and $f(\cdot,t)$ is in the domain of $\mathcal{A}^{*}$ for each t. We then have for our conditioned process an infinitesimal generator given as

$$\mathcal{A}^{**}f(x,t) = \frac{\partial f(x,t)}{\partial t} + \mu \cdot \frac{1-be^{-2\mu x}}{1+be^{-2\mu x}}\frac{\partial f(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2 f(x,t)}{\partial x^2}, \tag{5.46}$$

for $f(x,t) \in \mathcal{D}(\mathcal{A}^{**})$. Following our construction, a quick comparison between (5.25) and (5.46) tells us that the drift terms $b(x)$ becomes

$$b(x) = \mu\frac{1-be^{-2\mu x}}{1+be^{-2\mu x}}, \tag{5.47}$$

and in order to obtain a form of martingale, we can consider

$$-\beta f(x) + \mu\frac{1-be^{-2\mu x}}{1+be^{-2\mu x}}f'(x) + \frac{1}{2}f''(x) = 0. \tag{5.48}$$

From (5.31), we know we have to solve the following differential equation to obtain the solutions for $\alpha(x)$:

$$\begin{aligned} y''(x) + [-b^2(x) - b'(x) - 2\beta]y(x) &= 0 \\ y''(x) + \left[-\mu^2\frac{(1-be^{-2\mu x})^2}{(1+be^{-2\mu x})^2} - \mu\frac{4\mu be^{-2\mu x}}{(1+be^{-2\mu x})^2} - 2\beta\right]y(x) &= 0 \\ y''(x) + \left[\frac{-\mu}{(1+be^{-2\mu x})^2}\left(\mu - 2\mu be^{-2\mu x} + \mu b^2e^{-4\mu x} + 4\mu be^{-2\mu x}\right) - 2\beta\right]y(x) &= 0 \\ y''(x) + \left[\frac{-\mu}{(1+be^{-2\mu x})^2}\cdot\mu\cdot\left(1+be^{-2\mu x}\right)^2 - 2\beta\right]y(x) &= 0 \\ y''(x) + (-\mu^2 - 2\beta)y(x) &= 0. \end{aligned} \tag{5.49}$$

We can easily solve equation (5.49) by realizing that its solution is proportional to $e^{\gamma x}$. So, letting $y(x) = e^{\gamma x}$ and substituting into equation (5.49), we have

$$\begin{aligned} \frac{d^2}{dx^2}\left(e^{\gamma x}\right) &= (\mu^2 + 2\beta)e^{\gamma x} \\ \gamma^2 e^{\gamma x} &= (\mu^2 + 2\beta)e^{\gamma x}, \end{aligned} \tag{5.50}$$

we obtain that $\gamma = \sqrt{\mu^2 + 2\beta}$ or $\gamma = -\sqrt{\mu^2 + 2\beta}$ and hence,

$$y(x) = \begin{cases} e^{\sqrt{\mu^2+2\beta}x}, \\ \\ e^{-\sqrt{\mu^2+2\beta}x}. \end{cases} \tag{5.51}$$

Then, from our arguments earlier, equation (5.20) has solutions y_1^2, y_2^2 and $y_1 y_2$.

$$\alpha(x) = \begin{cases} e^{2\sqrt{\mu^2+2\beta}x}, \\ e^{-2\sqrt{\mu^2+2\beta}x}, \\ e^{\sqrt{\mu^2+2\beta}x} \cdot e^{-\sqrt{\mu^2+2\beta}x} = 1. \end{cases} \tag{5.52}$$

From (5.19), with $b(x)$ as derived in (5.47) and $\alpha(x)$ as obtained in (5.52),

$$\begin{aligned} \psi &= \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + p(x)\alpha^2(x) \\ &= \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + \left[-b^2(x) - b'(x) - 2\beta\right]\alpha^2(x) \\ &= \frac{1}{2}\alpha''(x)\alpha(x) - \frac{1}{4}[\alpha'(x)]^2 + \left(-\mu^2 - 2\beta\right)\alpha^2(x) \\ &= -a^2 C_2^2, \end{aligned} \tag{5.53}$$

where C_2 is an arbitrary constant.

5.4.1 First Exit Time

Consider the conditioned process $X = \{X_t,\ t \geq 0\}$ defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, starting at x for $x \in \mathbb{R}_+$ under the associated probability $\mathbb{P}_x$ with $\mathbb{E}_x$ as corresponding the expectation operator. Let T_D be the first exit time of X from D defined as

$$T_D = \inf\{t \geq 0\ :\ X_t \notin D\}, \tag{5.54}$$

where D is an open, bounded domain. We are interested in finding the Laplace transform of the first exit time T_D given as

$$f(x) := \mathbb{E}_x\left[e^{-\beta T_D}\right], \tag{5.55}$$

for $x \in \bar{D}$. Then following Peskir and Shiryaev (2006) or Karatzas and Shreve (1991), we have that for D and T_D as defined earlier, f is the solution of the Dirichlet problem:

$$\begin{aligned} \mathcal{A}f - \beta f &= 0; \qquad \text{in } D, \\ f &= 1; \qquad \text{on } \partial D. \end{aligned} \tag{5.56}$$

Theorem 5.4.1. *For $\beta \in \mathbb{C}_+$, $m_1, m_2 \in \mathbb{R}_+$ such that $0 < m_1 < x < m_2$ and the domain $D = (m_1, m_2)$, the Laplace transform of T_D can be obtained as*

$$\begin{aligned} &\mathbb{E}_x\left[e^{-\beta T_D}\right] \\ &= e^{-\mu(m_1-x)}\frac{b+e^{2\mu m_1}}{b+e^{2\mu x}}\frac{\sinh(\sqrt{\mu^2+2\beta}(m_2-x)))}{\sinh(\sqrt{\mu^2+2\beta}(m_2-m_1))} \\ &\qquad + e^{-\mu(m_2-x)}\frac{b+e^{2\mu m_2}}{b+e^{2\mu x}}\frac{\sinh(\sqrt{\mu^2+2\beta}(x-m_1))}{\sinh(\sqrt{\mu^2+2\beta}(m_2-m_1))}, \end{aligned}$$

where $\sinh(x)$ *is the hyperbolic sine function.*

Proof. For $D = (m_1, m_2)$ and the stopping times T_{m_1} and T_{m_2} defined as

$$\begin{aligned} T_{m_1} &= \inf\{t \geq 0 \ : \ X_t = m_1\}, \\ T_{m_2} &= \inf\{t \geq 0 \ : \ X_t = m_2\}, \end{aligned}$$

and since the first exit time of the process $\{X_t,\ t \geq 0\}$ from the interval $D = (m_1, m_2)$ can be computed as

$$T_D = T_{m_1} \wedge T_{m_2} := \min(T_{m_1}, T_{m_2}),$$

then we see that the Laplace transform of the first exit time can be split into 2 cases

$$\mathbb{E}_x\left[e^{-\beta T_D}\right] = \mathbb{E}_x\left[e^{-\beta T_{m_1}}\mathbb{1}_{\{T_{m_1}<T_{m_2}\}}\right] + \mathbb{E}_x\left[e^{-\beta T_{m_2}}\mathbb{1}_{\{T_{m_2}<T_{m_1}\}}\right].$$

For the first expectation on the right hand side, using the same reasoning as in Karatzas and

Shreve (1991), we have that

$$\mathbb{E}_x\left[e^{-\beta T_{m_1}}\mathbb{1}_{\{T_{m_1}<T_{m_2}\}}\right]=f(x),$$

with the boundary conditions

$$\begin{cases} f(m_1) & =1, \\ f(m_2) & =0, \end{cases} \tag{5.57}$$

whereas the second term can be computed by

$$\mathbb{E}_x\left[e^{-\beta T_{m_2}}\mathbb{1}_{\{T_{m_2}<T_{m_1}\}}\right]=f(x),$$

with the boundary conditions

$$\begin{cases} f(m_2) & =1, \\ f(m_1) & =0, \end{cases} \tag{5.58}$$

where $f(x)$ is the solution to (5.48). In order to derive $f(x)$, we follow the same steps as in Theorem 5.2.4. From equation (5.53) and Theorem (5.2.3), we see that $C_2 \neq 0$ corresponds to the case when $\psi < 0$ which gives us

$$g(x)=\frac{-\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx-c_3\right]\sqrt{-\psi}\right)-\frac{1}{2}\alpha'(x)}{\alpha(x)},$$

and $h(x)$ of the form

$$h(x)=\int \frac{-\sqrt{-\psi}\tanh\left(\left[\int \frac{1}{\alpha(x)}dx-c_3\right]\sqrt{-\psi}\right)-\frac{1}{2}\alpha'(x)}{\alpha(x)}dx+\int b(x)dx,$$

and we can derive $f(x)$ using the expressions for ψ in (5.53), $\alpha(x)$ in (5.52) and $b(x)$ in (5.47)

$$\begin{aligned} f(x) &= e^{-h(x)} \\ &= C_1\frac{e^{(\mu-\sqrt{\mu^2+2\beta})x}}{b+e^{2\mu x}}+C_2\frac{e^{(\mu+\sqrt{\mu^2+2\beta})x}}{\sqrt{\mu^2+2\beta}(b+e^{2\mu x})}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants to be determined using the boundary conditions (5.57) and (5.58). For the first expectation $\mathbb{E}_x\left[e^{-\beta T_{m_1}}\mathbb{1}_{\{T_{m_1}<T_{m_2}\}}\right]=f(x)$ with the following

boundary conditions, we can determine

$$\begin{cases} f(m_1) &= 1, \\ f(m_2) &= 0, \end{cases}$$

$$\begin{cases} 1 &= C_1 \frac{e^{(\mu-\sqrt{\mu^2+2\beta})m_1}}{b+e^{2\mu m_1}} + C_2 \frac{e^{(\mu+\sqrt{\mu^2+2\beta})m_1}}{\sqrt{\mu^2+2\beta}(b+e^{2\mu m_1})}, \\ 0 &= C_1 \frac{e^{(\mu-\sqrt{\mu^2+2\beta})m_2}}{b+e^{2\mu m_2}} + C_2 \frac{e^{(\mu+\sqrt{\mu^2+2\beta})m_2}}{\sqrt{\mu^2+2\beta}(b+e^{2\mu m_2})}, \end{cases}$$

$$\Rightarrow \begin{cases} C_1 &= e^{-\mu m_1} e^{\sqrt{\mu^2+2\beta}m_1} \frac{b+e^{2\mu m_1}}{1-e^{-2\sqrt{\mu^2+2\beta}(m_2-m_1)}}, \\ C_2 &= -\sqrt{\mu^2+2\beta} C_1 e^{-2\sqrt{\mu^2+2\beta}m_2}. \end{cases}$$

Then, for $0 < m_1 < x < m_2$, we can derive

$$\begin{aligned} &\mathbb{E}_x \left[e^{-\beta T_{m_1}} \mathbb{1}_{\{T_{m_1} < T_{m_2}\}} \right] \\ &= f(x) \\ &= e^{-\mu(m_1-x)} \frac{b+e^{2\mu m_1}}{b+e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2+2\beta}(m_2-x)))}{\sinh(\sqrt{\mu^2+2\beta}(m_2-m_1))}. \end{aligned} \tag{5.59}$$

For the second expectation $\mathbb{E}_x \left[e^{-\beta T_{m_2}} \mathbb{1}_{\{T_{m_2} < T_{m_1}\}} \right] = f(x)$ with the appropriate boundary conditions, we have

$$\begin{cases} f(m_2) &= 1, \\ f(m_1) &= 0, \end{cases}$$

$$\begin{cases} 1 &= C_1 \frac{e^{(\mu-\sqrt{\mu^2+2\beta})m_2}}{b+e^{2\mu m_2}} + C_2 \frac{e^{(\mu+\sqrt{\mu^2+2\beta})m_2}}{\sqrt{\mu^2+2\beta}(b+e^{2\mu m_2})} \\ 0 &= C_1 \frac{e^{(\mu-\sqrt{\mu^2+2\beta})m_1}}{b+e^{2\mu m_1}} + C_2 \frac{e^{(\mu+\sqrt{\mu^2+2\beta})m_1}}{\sqrt{\mu^2+2\beta}(b+e^{2\mu m_1})} \end{cases},$$

$$\Rightarrow \begin{cases} C_1 &= e^{-\mu m_2} e^{\sqrt{\mu^2+2\beta}m_2} \frac{b+e^{2\mu m_2}}{e^{-2\sqrt{\mu^2+2\beta}(m_1-m_2)}} \\ C_2 &= -\sqrt{\mu^2+2\beta} C_1 e^{-2\sqrt{\mu^2+2\beta}m_1} \end{cases}.$$

Then for $0 < m_1 < x < m_2$, we can derive

$$\begin{aligned} & \mathbb{E}_x \left[e^{-\beta T_{m_2}} \mathbb{1}_{\{T_{m_2} < T_{m_1}\}} \right] \\ & = f(x) \\ & = e^{-\mu(m_2 - x)} \frac{b + e^{2\mu m_2}}{b + e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2 + 2\beta}(x - m_1))}{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - m_1))}. \end{aligned} \tag{5.60}$$

The expression for the Laplace transform of the first exit time can then be obtained from (5.59) and (5.60)

$$\begin{aligned} & \mathbb{E}_x \left[e^{-\beta T_D} \right] \\ & = \mathbb{E}_x \left[e^{-\beta T_{m_1}} \mathbb{1}_{\{T_{m_1} < T_{m_2}\}} \right] + \mathbb{E}_x \left[e^{-\beta T_{m_2}} \mathbb{1}_{\{T_{m_2} < T_{m_1}\}} \right] \\ & = e^{-\mu(m_1 - x)} \frac{b + e^{2\mu m_1}}{b + e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - x)))}{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - m_1))} \\ & \quad + e^{-\mu(m_2 - x)} \frac{b + e^{2\mu m_2}}{b + e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2 + 2\beta}(x - m_1))}{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - m_1))}. \end{aligned} \tag{5.61}$$

□

With the expression of the Laplace transform of the first exit time, we can easily invert the Laplace transform to obtain its density.

Theorem 5.4.2. *For $t \geq 0$, $\mu \in \mathbb{R}_+$ and $0 < m_1 < x < m_2$, the density function of the first exit time can be derived as*

$$\begin{aligned} & \mathbb{P}_x \left[T_D \in dt \right] \\ & = e^{\mu(m_1 - x)} \frac{1 + b e^{-2\mu m_1}}{1 + b e^{-2\mu x}} e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{x - m_1 + 2k(m_2 - m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(x - m_1 + 2k(m_2 - m_1))^2}{2t}} dt \\ & \quad + e^{\mu(m_2 - x)} \frac{1 + b e^{-2\mu m_2}}{1 + b e^{-2\mu x}} e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{m_2 - x + 2k(m_2 - m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(m_2 - x + 2k(m_2 - m_1))^2}{2t}} dt. \end{aligned}$$

Proof. From (5.61), we have

$$\begin{aligned} \mathbb{E}_x \left[e^{-\beta T_D} \right] = & \; e^{-\mu(m_1 - x)} \frac{b + e^{2\mu m_1}}{b + e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - x)))}{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - m_1))} \\ & + e^{-\mu(m_2 - x)} \frac{b + e^{2\mu m_2}}{b + e^{2\mu x}} \frac{\sinh(\sqrt{\mu^2 + 2\beta}(x - m_1))}{\sinh(\sqrt{\mu^2 + 2\beta}(m_2 - m_1))}. \end{aligned}$$

We want to invert this with respect to β. Using Borodin and Salminen (2002), we know that

$$\mathcal{L}_\beta^{-1}\left(\frac{\sinh(a\sqrt{2\beta})}{\sinh(b\sqrt{2\beta})}\right)(t) = \sum_{k=-\infty}^{\infty} \frac{b-a+2kb}{\sqrt{2\pi t^3}} e^{-\frac{(b-a+2kb)^2}{2t}}; \qquad a < b.$$

Then, we can easily derive that for $0 < m_1 < x < m_2$,

$$\mathcal{L}_\beta^{-1}\left(\frac{\sinh(\sqrt{\mu^2+2\beta}(m_2-x)))}{\sinh(\sqrt{\mu^2+2\beta}(m_2-m_1))}\right)(t) = e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{x-m_1+2k(m_2-m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(x-m_1+2k(m_2-m_1))^2}{2t}},$$

$$\mathcal{L}_\beta^{-1}\left(\frac{\sinh(\sqrt{\mu^2+2\beta}(x-m_1))}{\sinh(\sqrt{\mu^2+2\beta}(m_2-m_1))}\right)(t) = e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{m_2-x+2k(m_2-m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(m_2-x+2k(m_2-m_1))^2}{2t}}.$$

Therefore, we can derive that

$$\begin{aligned}
&\mathbb{P}_x\left[T_D \in dt\right] \\
&= e^{\mu(m_1-x)}\frac{1+be^{-2\mu m_1}}{1+be^{-2\mu x}} e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{x-m_1+2k(m_2-m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(x-m_1+2k(m_2-m_1))^2}{2t}} dt \\
&\quad + e^{\mu(m_2-x)}\frac{1+be^{-2\mu m_2}}{1+be^{-2\mu x}} e^{-\mu^2 t} \sum_{k=-\infty}^{\infty} \frac{m_2-x+2k(m_2-m_1)}{\sqrt{2\pi t^3}} e^{-\frac{(m_2-x+2k(m_2-m_1))^2}{2t}} dt.
\end{aligned}$$

□

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