# Topics in Graph Colouring

Xinyi Xu

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Department of Mathematics
London School of Economics and Political Science

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## Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, created in collaboration with my supervisor Jan van den Heuvel.

## Abstract

In this thesis, we study two variants of graph (vertex) colourings: multi-colouring and correspondence colouring.

In ordinary graph colouring, each vertex receives a colour. Such a colouring is *proper* if adjacent vertices receive different colours. In $k$*-multi-colouring*, each vertex receives a set of $k$ colours, and such a multi-colouring is proper if adjacent vertices receive disjoint set of colours. A graph is $(n, k)$*-colourable* if there is a proper $k$-multi-colouring of it using $n$ colours.

In the first part of the thesis, we study the following two questions.

1. For given $n$, $k$ and $n'$, $k'$, if a graph is $(n, k)$-colourable, then what is the largest subgraph of it that is $(n', k')$-colourable?

2. For given $n$, $k$, if a graph is $(n, k)$-colourable, then for what $n'$, $k'$ is the whole graph $(n', k')$-colourable?

Question 1 is inspired by a partial colouring conjecture asked by Albertson, Grossman, and Haas [2] in 2000 regarding list colouring. We obtain exact answers for specific values of the parameters, and upper and lower bounds on the largest $(n', k')$-colourable subgraph for general values of $n'$, $k'$.

For Question 2, we first observe how it can be reformulated into a conjecture by Stahl from 1976 regarding Kneser graphs, and prove new results towards Stahl's conjecture.

In the second part of the thesis, we study another variant of colouring, which is known as correspondence colouring.

In *correspondence colouring*, each vertex is associated with a prespecified list of colours, and there is prespecified *correspondence* associated with each edge specifying which pair of colours from the two endvertices *correspond*. (On each edge, a colour on one endvertex *corresponds* to at most one colour on the other endvertex.)

A correspondence colouring is *proper* if each vertex receives a colour from its prespecified list, and that for each edge, the colours on its endvertices do not correspond. A graph is $n$*-correspondence-colourable* if a proper correspondence colouring exist for any prespecified correspondences on any pre-

specified $n$-colour-lists associated to each vertex.

As correspondence colouring is a generalisation of list colouring, it is natural to ask whether Albertson, Grossman, and Haas' conjecture can be generalised to correspondence colouring. Unfortunately, there are graphs on which their conjectured value does not hold, and we will present a series of them.

We then study: for given $n$ and $n'$, how many vertices of a $n$-correspondence-colourable graph can always be properly correspondence-coloured with arbitrary correspondences and arbitrary $n'$-colour-lists on that graph? We generalise some results from the original conjecture in list colouring. Then we discuss some sufficient conditions for a proper correspondence colouring to exist.

The *correspondence chromatic number* of a graph is the smallest $n$ such that the graph is $n$-correspondence colourable. We study how different graph operations affect the correspondence chromatic number of *multigraphs*, in which multiple edges are allowed.

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# Contents

# 1

# Introduction

## 1.1 Graph Theory and Colouring

As abstraction of connections between objects, graphs appear in the study of social or transport networks, data structures, chemical or physical structures and more seemingly unrelated areas. From the Königsberg Seven Bridge problem to the Four Colour Theorem and further, properties of graphs have been widely studied and remain an interesting field of research.

Graph colouring problems are among the most studied topics in structural graph theory. In this thesis, we focus on two generalisations of *vertex colouring*.

Graph colouring problems are concerned with labelling the 'objects' in graphs in a specific way that satisfies certain constraints. Formally, a graph is a collection of *vertices* and *edges*. Each vertex can be viewed as the abstraction of an object, and each edge as the abstraction of connections. Two vertices are *adjacent* if there is an edge between them. The two vertices attached to an edge are the *endvertices* of the edge.

A vertex colouring assigns a colour to each vertex, and such a colouring is *proper* if for every edge in the graph, the colours assigned to its endvertices are different. In this thesis, 'colouring' always refers to vertex colouring.

### 1.1.1 Multi-Colouring and Fractional Colouring

In Chapters 2, 3 and 4, we explore a generalisation of vertex colouring called *multi-colouring.*

Multi-colouring generalises vertex colouring and has been studied extensively; see e.g. [43] for background. In a $k$-multi-colouring, each vertex receives a set of $k$ colours. Such a multi-colouring is *proper* if adjacent vertices receive disjoint $k$-sets of colours. (Two sets are *disjoint* if there is no element that is in both sets.) A graph is $(n, k)$*-colourable* if there is a proper $k$-multi-colouring using at most $n$ colours in total. In the case that $k = 1$, we usually say a graph is $n$-colourable. The $k$-th multi-chromatic number of a graph is the smallest $n$ such that the graph is $(n, k)$-colourable.

*Fractional colouring* is closely associated with multi-colouring. A graph is *fractional-*$\frac{n}{k}$*-colourable* if it is $(tn, tk)$-colourable for some positive integer $t$. The *fractional chromatic number* of a graph is the infimum of $\frac{n}{k}$ such that this graph is fractional-$\frac{n}{k}$-colourable. It is well-known that this infimum is a minimum. (For example, see in [43]. Note this is proved using the equivalent linear programming definition of fractional chromatic number.) I.e. if a graph has fractional chromatic number $\frac{n}{k}$, then it is $(tn, tk)$-colourable for some positive integer $t$.

We study the following questions regarding multi-colouring.

(1.1). For given positive integers $n$, $k$ and $n'$, $k'$, if a graph is $(n, k)$-colourable, then what is the largest induced subgraph of it that is $(n', k')$-colourable?

(1.2). For given positive integers $n$, $k$ and $n'$, $k'$, if a graph is fractional-$\frac{n}{k}$-colourable, then what is the largest induced subgraph of it that is $(n', k')$-colourable?

(1.3). For given positive integers $n$, $k$ and $n'$, $k'$, if a graph is fractional-$\frac{n}{k}$-colourable, then what is the largest induced subgraph of it that is fractional-$\frac{n'}{k'}$-colourable?

2. For given positive integers $n$, $k$, if a graph is $(n, k)$-colourable, then for what $n'$, $k'$ is the whole graph $(n', k')$-colourable?

Questions (1.1), (1.2) and (1.3) are inspired by a partial colouring conjecture formulated by Albertson, Grossman, and Haas [2] regarding list colouring. (List colouring is a variant of vertex colouring, in which each vertex receives a prespecified colour list, and we aim to find a proper list colouring by assigning each vertex a colour from its list. Different vertices can have different list of colours available. We will define it formally later in this chapter.)

In Chapters 3 and 4, we start by showing Albertson et al.'s conjecture does not hold in the setting of fractional colouring. Then we study Questions (1.1)–(1.3) and obtain upper and lower bounds on the largest $(n', k')$-colourable induced subgraph for general values of $n'$, $k'$. In addition to upper and lower bounds, we also obtain exact answers to these questions for specific ranges of $n'$, $k'$ with given $n$, $k$.

In Chapter 3, we focus on the special case that $k' = 1$ of Questions (1.1) and (1.2). That is, we try to answer the question: if a graph is $(n, k)$-colourable (or fractional-$\frac{n}{k}$-colourable), then what is its largest $n'$-colourable induced subgraph?

We are interested to find a general lower bound on the relative order of that subgraph (in terms of a fraction of the order of the original graph). Note that if only the fractional chromatic number is known, a general upper bound for this fraction other than 1 can not be guaranteed: consider a graph that is the disjoint union of an arbitrary large $n'$-colourable graph and a complete graph of $n$ vertices (where $n' < n$ so that both the fractional chromatic number and chromatic number are $n$). Then the relative order of largest $n'$-colourable induced subgraph, as a fraction of the whole graph, can be arbitrarily close to 1.

We now give the formal definitions relevant to our question. Denote $\chi(G)$ and $\chi_f(G)$ as the chromatic number and fractional chromatic number of a graph $G$, respectively. For any positive integer $n'$, the relative order of largest $n'$-colourable induced subgraph in the whole graph is defined as the following

$$\gamma(G,n') := \max\left\{\frac{|V(H)|}{|V(G)|}\;\middle|\; H \leqslant_i G, \chi(H) \leq n'\right\},$$
$$\gamma(r,n') := \inf\left\{\gamma(G,n') \;\middle|\; \chi_f(G) = r\right\}.$$

Here $H \leqslant_i G$ means that $H$ is an induced subgraph of $G$, and $V(G)$ denotes the vertex set of graph $G$.

In Chapter 3, we determine both $\gamma(G,n')$ and $\gamma(r,n')$ as functions of some other graph invariants on $G$ for the case $n' = 0$ or $n' = 1$: for any graph $G$ with at least one vertex, we have $\gamma(G,0) = 0$ and $\gamma(G,1) = \dfrac{\alpha(G)}{|V(G)|}$, where $\alpha(G)$ is the *independence number* of $G$. (The *independence number of a graph* is the order of the its maximum independent set. An *independent* set in a graph is a set of vertices that are mutually nonadjacent.) Similarly, we will prove that $\gamma(r,0) = 0$ and $\gamma(r,1) = \dfrac{1}{r}$ for any rational number $r = 1$ or $r \geq 2$. (There does not exist graph with fractional chromatic number strictly between 1 and 2.) For the more general cases that $n' \geq 2$, we have the following.

*If* $r = \dfrac{n}{k}$ *for some pair of co-primes* $n$ *and* $k$ *and* $2 \leq n' < n - 2k + 2$*, then we have* $\gamma(r,n') = 1 - \left(1 - \dfrac{1}{r}\right)^{n'}$.

For Questions (1.1) and (1.2), we first observe it suffices to study the largest $n'$-colourable induced subgraphs of the Kneser Graph $K(n,k)$. (Which also holds for Question (1.3).) (For $n \geq k \geq 1$, the *Kneser graph* $K(n,k)$ has as vertex set the collection of all $k$-subsets of $[n]$, and there is an edge between two vertices if and only if the two $k$-sets are disjoint. We will always assume $n \geq 2k$, as otherwise the graph is edgeless.) Namely, we will prove that if $G$ is $(n,k)$-colourable, then we have $\gamma(G,n') \geq \gamma(K(n,k),n')$ for all integers $n'$. And hence if we only know $G$ is $(n,k)$-colourable, then $\gamma(K(n,k),n')$ is the best possible upper bound we have. We then explore the upper and lower bounds for $\gamma(K(n,k),n')$ for given $n,k,n'$.

A natural candidate of a 'large' $n'$-colourable induced subgraph of $K(n,k)$ is the subgraph induced by all $k$-subsets of $[n]$ that contain at least one element from $[n']$. This immediately gives

$$\gamma(K(n,k),n') \geq \frac{\binom{n}{k} - \binom{n-n'}{k}}{\binom{n}{k}}.$$

Surprisingly, if $n$ is large compared to $k$, then this is the correct answer of $\gamma(K(n,k),n')$. (We will discuss in detail what 'large' means in Chapter 3.) This type of $n'$-colourable subgraphs are called *trivial* $n'$-colourable subgraphs.

On the other hand, for $n$ closer to $2k$, we can find constructions that are always larger than the above construction, and hence which provide better lower bounds for $\gamma(K(n,k),n')$. However, exact values of $\gamma(K(n,k),n')$ for $n$ closer to $2k$ remain unknown.

We also discuss how we can use different methods to find an upper bound for $\gamma(K(n,k),n')$ and look at their different behaviour. The two main techniques we discuss are

1. structural properties of a Kneser graph, and
2. algebraic method using Cauchy's Interlacing Theorem. (Also known as *Inertia Bound*; see e.g. [24] for background.)

Note that the problem of determining $\gamma(K(n,k),2)$ has been studied by Frankl and Füredi in [18], in which they proved that if $n \geq \frac{1}{2}(3+\sqrt{5})k$, then $\gamma(K(n,k),2)$ is attained by the trivial bipartite subgraph. For $n$ closer to $2k$, they also provided a construction that is better than trivial. Our construction mentioned in the above paragraph is larger than theirs. Another more recent result in [10] leads to answers of $\gamma(K(n,k),n')$ for large enough $n$ compared to $k$ and $n'$. We discuss both, including other related results, in more detail in Chapter 3.

As mentioned, the case $k'=1$ for Questions (1.1) and (1.2) are studied in Chapter 3. We study those questions for general $k'$ in Chapter 4.

Recall that our questions are inspired by a question asked by Albertson et al regarding list colouring. We will prove by examples that the their conjecture does not extends to fractional colouring, nor to multi-colouring, in the sense that there exist graphs $G$ for which strictly less than $\dfrac{n'}{\chi_f(G)}|V(G)|$ vertices can be coloured with $n'$ colours.

Then we study the lower bounds of the fractional version of $\gamma(G, n')$ and $\gamma(r, n')$. Namely, for positive rational number $s$, we define

$$\begin{aligned}\gamma_f(G, s) &:= \max\left\{\frac{|V(H)|}{|V(G)|} \;\middle|\; H \leqslant_i G, \chi_f(H) \leq s\right\},\\ \gamma_f(r, s) &:= \inf\left\{\gamma_f(G, s) \mid \chi_f(G) = r\right\}.\end{aligned}$$

We will prove the following results for any rational number $s \geq 2$. (And we will show that those infimums are not minimums, i.e. they are not attained by any graph.)

*If $2 \leq s < r$, then we have $\gamma_f(r, s) \geq 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$; furthermore, if $\lfloor s \rfloor \leq \frac{r-1}{2}$, then we have $\gamma_f(r, s) = 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$.*

Similar as for $\gamma(G, n')$ and $\gamma(r, n')$, we also study upper and lower bounds of $\gamma_f(G, s)$ and $\gamma_f(r, s)$ for general rationals $r$ and $s$.

Finally, for Question (1.3), we denote by $\pi(G, K(n', k'))$ the order of a largest $(n', k')$-colourable induced subgraph of a graph $G$. Similar as above, we show that if $G$ is $(n, k)$-colourable, then

$$\pi(G, K(n', k')) \geq \pi(K(n, k), K(n', k')).$$

Hence it suffices to study Kneser graphs if the only information we know about $G$ is that it is $(n, k)$-colourable.

We determine exact values of $\pi(K(n, k), K(n', k'))$ for some small values of $n, k, n', k'$, and show that the exact value of $\pi(K(n, k), K(n', k'))$ can be determined by studying the independence number of a special product of graphs. We also study general upper bounds and lower bounds for $\pi(K(n, k), K(n', k'))$.

### Study of Question 2

We study Question 2 in Chapter 2. A first observation is that this question can be reformulated into a question about multi-chromatic number of Kneser graphs, for which Stahl [44] formulated a conjecture in 1976. In particular, we show that in order to answer Question 2 for a given pair $n, k$, it is equivalent to study the $k'$*-th multi-chromatic number* of the Kneser Graph $K(n,k)$ for every $k'$. (The $k'$*-th multi-chromatic number* $\chi_{k'}(G)$ of a graph $G$ is the smallest integer $n'$ such that $G$ is $(n', k')$-colourable.)

Stahl's conjecture [44] states that $\chi_{k'}(K(n,k)) = qn - 2r$, where $k' = qk - r$ for some $q \geq 1$ and $0 \leq r \leq k-1$.

Some simple observations and corollaries (of some not-so-trivial results) can be made regarding this conjecture. For $k = 1$, the Kneser graph $K(n, 1)$ is just the complete graph on $n$ vertices. It is not hard to prove that the $k'$-th multi-chromatic number of a complete graph with $n$ vertices is $k'n$, and hence Stahl's conjecture is true in that case. For $k' = 1$ (hence $q = 1$ and $r = k - 1$), we also find the conjecture to be true, since $\chi_1(G) = \chi(G)$, and we know $\chi(K(n,k)) = n - 2k + 2$ by the well-known Kneser Theorem proved by Lovász [38].

Stahl and a few other authors made quite a number of observations regarding the multi-chromatic number of Kneser graphs. We will reconstruct some of these results in Section 2.2.

Our first contribution on this topic is a simple proof of the following result, which generalises a theorem of Stahl [44] and the main result in Osztényi [42]. Yet our proof is much simpler.

**Theorem 1.1.1.**
*For any $k \geq 2$ and $n > 2k$, if $0 \leq r \leq \dfrac{k}{n-2k}$ and $r \leq k-1$, then we have $\chi_{k'}(K(n,k)) = qn - 2r$ (where $k' = qk - r$ with $q \geq 1$ and $0 \leq r \leq k-1$).*

Our next contribution is to prove that for a fixed $k$, only at most $k^3 - k^2$ values $\chi_{k'}(K(n,k))$ need to be determined in order to conclude whether or not Stahl's conjecture is true for that value of $k$ and for *all $n$ and $k'$*. Hence it is possible to fully resolve this problem for each fixed $k$ in finite time.

**Theorem 1.1.2.**
*Fix $k \geq 1$. Then there exist $n_0(k)$ and $q_0(n,k)$ such that the following holds. If $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$ for all $n \leq n_0(k)$ and for at least one*

*$q \geq q_0(n,k)$, then we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $n \geq 2k$, $q \geq 1$ and $0 \leq r \leq k-1$.*

The functions $n_0(k)$ and $q_0(n,k)$ in Theorem 1.1.2 we obtain are quite complicated; we will give details in Section 2.3. From those values, it is possible to show that we have $n_0(k) < k^3 - k^2 + 2k$ for all $k$, and $q_0(n,k) < \frac{4^k}{\mathrm{e}k}(n-2k)$ for all $k \geq 2$ and $n \geq 2k+1$ (where $\mathrm{e} \approx 2.718$ is Euler's number). (Note for each fixed $k$, we only need to verify at most $k^3 - k^2$ values $\chi_{k'}(K(n,k))$ because Kneser Graphs are edgeless if $n < 2k$.)

As a side note, we can replace $q_0(n,k)$ by $q_0'(k) = \max\{q_0(n,k) \mid 2k \leq n \leq n_0(k)\}$ in Theorem 1.1.2, to remove the dependency of $q_0$ on $n$. We chose to keep $q_0(n,k)$, since for larger values of $n$ we get better bounds for $q_0(n,k)$. For instance, if $n \geq (\log_2 \mathrm{e})\, k^2$, then we can show $q_0(n,k) < n$.

For $k = 4$, our methods show that we only need to find $\chi_{4q-3}(K(n,4))$ for $8 \leq n \leq 10$, $q = 13$, and for $11 \leq n \leq 38$, $q = 12$. The cases $n = 8, 9, 10$ are known to be true by results in [35, 44]. So the first open case is to determine whether or not $\chi_{45}(K(11,4)) = 126$. Note that Stahl already showed $\chi_{45}(K(11,4)) \leq 126$, while our bounds in Chapter 2 give $\chi_{45}(K(11,4)) \geq 124$.

In Section 2.3, we also explain that determining $\chi_{45}(K(11,4))$ can be done by finding the chromatic number of the lexicographic product $K(11,4) \bullet K_{45}$. Unfortunately, $K(11,4) \bullet K_{45}$ is a highly symmetric graph with 14,850 vertices and 12,021,075 edges, and none of the publicly available packages for graph colouring we could find seems to be able to deal with this graph within a reasonable amount of time.

We will discuss the above in full details in Chapter 2.

### 1.1.2 Correspondence Colouring

In the later chapters of this thesis (Chapters 5 and 6), we study another variation of vertex colouring called *correspondence colouring*.

Recall that in a list colouring, each vertex has a prespecified list of colours available, and in a proper list colouring, each vertex is assigned a colour from its prespecified list so that adjacent vertices receive different colours. The special case that all the vertices have an identical prespecified list reduces to ordinary vertex colouring.

Correspondence colouring is a generalisation of list colouring, by introducing more flexible definitions on when a colouring is *proper*. In correspondence colouring, in addition to prespecified colour lists on each vertex, there are prespecified 'correspondences' on each edge that pairs colours from the lists on its two endvertices. (On each edge, one colour from one endvertex is paired with at most another colour from the other endvertex.) A correspondence colouring is proper if each vertex receives a colour from its prespecified list, and 'paired' colours are avoided everywhere. I.e. on each edge, the colours received by its endvertices are not paired in the correspondence on that edge. We will formally define it later in this introduction. Again, a special case of correspondence colouring is that only identical colours are paired in each correspondence. This special case corresponds to list colouring.

Since correspondence colouring generalises list colouring, and list colouring generalises ordinary colouring; it is not hard to note for any graph $G$, we have $\chi(G) \leq \chi_\ell(G) \leq \chi_c(G) \leq \mathrm{degcy}(G)+1$. (Here $\chi(G)$, $\chi_\ell(G)$, $\chi_c(G)$ and $\mathrm{degcy}(G)$ are the chromatic number, list chromatic number and correspondence chromatic number, degeneracy of a graph $G$, respectively. Formal definitions of them are given in Section 1.2.) There has been a large body of literature on correspondence colouring since it was introduced by Dvořák and Postle [9].

It is known that correspondence colouring can behave very differently from ordinary and list colouring. We include some interesting results here, concentrating on results that are related to comparable results on ordinary or list colouring. For examples, it is well-known and not hard to prove that $\chi(C_n) = \chi_\ell(C_n) = 2$ if $n$ is even and $\chi(C_n) = \chi_\ell(C_n) = 3$ if $n$ is odd for a cycle graph $C_n$ of $n$ vertices. But $\chi_c(C_n) = 3$ for all $n \geq 3$. The difference on these three colourings are more obvious on complete bipartite graphs: every complete bipartite graph is 2-colourable, but its list chromatic number can be arbitrarily large; for instance, it is well known that $\chi_\ell(K_{k,l}) = k+1$ if and only if $l \geq k^k$ (see e.g. [15, 46]). In correspondence colouring, we have

$\chi_c(K_{k,l}) = k+1$ for much smaller $l$ compared to $k$; for instance, it is proved in [41] that $\chi_c(K_{k,l}) = k+1$ if $l > (k^k/k!)(\log(k!)+1)$.

In Chapter 5, we explore how certain graph operations will affect the correspondence chromatic number of a multigraph. (In a *multigraph*, we allow multiple edges between any pair of vertices. Since it is possible to have different correspondence on different edges between same pair of vertices, larger chromatic numbers are expected for a multigraph. *Edge multiplicity* of an edge with given endvertices refers to the number of edges connecting those two endvertices.) The change to chromatic numbers or list chromatic numbers by those graph operations are usually trivial to determine or find a good bound of, but their behaviours are much more interesting and very different in correspondence colouring.

In particular, we explore upper and lower bounds on the change of correspondence chromatic numbers after following graph operations
• delete a vertex (and all edges with this vertex as an endvertex);
• delete an edge;
• identify two vertices ('merge' two vertices into a new vertex and remove the two old vertices, so that the new vertex receives all edges with exactly one endvertex coming from the two old vertices);
• take the $m$-th multiple of the original graph (that is, in the resulting graph, the multiplicity of each edge is $m$ times the original multiplicity).

Later in Chapter 5, we explore further the graph operation of taking the $m$-th multiple of the original graph. We try to understand how the correspondence chromatic number increases as the edge multiplicity is multiplied.

Building on results on those $m$-th multiple graphs, we introduce a new graph invariant, which we call *correspondence chromatic limit.* It is defined as the limit of the sequence (in $m$) of a graph's $m$-th multiple's correspondence chromatic number divided by $m$. (I.e. we are interested in $\lim\limits_{m\to\infty} \dfrac{\chi_c\left(G^{(m)}\right)-1}{m}$, where $G^{(m)}$ denotes the $m$-th multiple of $G$.) We prove that this limit is well defined, and is bounded by several other well-studied graph invariants. We also show it is possible to have any fractional part (with some integer part) as this limit, by finding the exact value for a group of special graphs. Several other properties of this limit are also studied.

Finally, we study partial correspondence colouring problems in Chapter 6. That is, we ask a question similar to Question 1:
If a graph is $n$-correspondence-colourable, and $n' < n$ colours are given to each vertex, then how large is the largest induced subgraph of it that can be properly correspondence coloured?

Recall that this question was first asked on list colouring, and correspondence colouring is a generalisation of list colouring. Hence it is natural to ask whether current known results in list colouring can be generalised to this question?

We first show that the conjectured bound for list colouring does not hold for correspondence colouring. That is, there are graphs that are $n$-correspondence-colourable, but strictly less than $\frac{n'}{n}|V(G)|$ vertices can be properly correspondence coloured in some $n'$-correspondence where $n' < n$.

To finish this chapter, we discuss some of the results from list colouring that generalise to this question for correspondence colouring. Then we discuss some sufficient conditions for a proper correspondence colouring to exist.

## 1.2 Preliminaries

We summarise terminology and some more results on multi-colouring and correspondence colouring in this section.

**Graphs**

All graphs in this thesis are finite, undirected and without loops. Additionally, multiple edges are not allowed in most of the chapters, except for Chapter 5. Most of the notation and terminology we use is standard and can be found in any textbook on graph theory. We use $V(G)$ to denote the vertex set and $E(G)$ to denote the edge set of a graph $G$; or simply $V$ and $E$ if there is no ambiguity. The *order* of a graph is the number of its vertices.

A graph $H$ is an *induced subgraph* of a graph $G$ (denoted by $H \leqslant_i G$) if $V(H) \subseteq V(G)$ and $E(H)$ consists of all edges with both endvertices in

$V(H)$, i.e. $V(H) = \{uv : u, v \in V(H) \text{ and } uv \in E(G)\}$. In this case, we also say that $H$ is a subgraph of $G$ induced by $V(H)$. All subgraphs mentioned in this thesis are induced.

The *edge degree* of a vertex $u$ in graph $G$, $\deg_G(u)$, is the total number of edges with $u$ as one of their endvertices. A graph $G$ is $k$*-degenerate* if every subgraph of $G$ has a vertex of edge degree at most $k$. The *degeneracy* of a graph $G$, $\mathrm{degcy}(G)$, is the smallest $k$ such that $G$ is $k$-degenerate. When it is clear what graph $G$ we are considering, we often omit the subscript $G$ and write $\deg(u)$, etc.

An *independent set* in a graph is a set of vertices that is mutually nonadjacent. A *complete graph* of $n$ vertices, denoted by $K_n$ is the $n$-vertex graph with all possible simple edges. A *clique* in a graph is a set of vertices that is mutually adjacent, i.e. a set of vertices that induce a complete subgraph. A *cycle* with $n$ vertices ($n \geq 3$) is the graph whose vertices can be ordered as $v_1, \ldots, v_n$ so that $v_iv_j$ is an edge if and only if $|i - j| = 1$, or $i = 1$ and $j = n$. A *path* from vertex $u$ to $v$ is a list of vertices $u = v_1, \ldots, v_n = v$ so that $v_iv_{i+1}$ is an edge for any $1 \leq i \leq n$. A graph is *connected* if for any two vertices $u, v$ in it, there is a path from $u$ to $v$. A *tree* is a connected graph without any cycle as a subgraph.

In the cases that multiple edges are allowed, we call a graph a *multigraph*. For vertices $u, v$, we denote by $uv$ the collection of all edges with endvertices $u$ and $v$. So if $e \in E(G)$ is an edge, then $e \in uv$ for some (distinct) vertices $u, v \in V(G)$. We usually say just "the edge $uv$" for this collection.

The *multiplicity* of an edge $uv$ in a graph $G$, denoted $\mathfrak{m}_G(uv) = |uv|$, is the number of edges with endvertices $u$ and $v$. We write $\mathfrak{m}_G(uv) = 0$ if there is no edge between $u$ and $v$; we sometimes also write $\mathfrak{m}_G(uu) = 0$ for any $u \in V(G)$ (since loops are not allowed). An edge $uv$ is *simple* if $\mathfrak{m}_G(uv) = 1$; a graph $G$ is *simple* if $\mathfrak{m}_G(uv) \in \{0, 1\}$ for all $u, v \in V(G)$. We omit the subscript $G$ in the cases of no ambiguity.

### Colourings

All colourings in this thesis are vertex colourings, hence a *colouring* of a graph assigns a colour to each vertex. Such a colouring is *proper* if adjacent vertices receive different colours. A graph $G$ is *$n$-colourable* if there is a proper colouring of $G$ using $n$ colours. The *chromatic number* $\chi(G)$ is the smallest $n$ for which $G$ is $n$-colourable.

*Partial colouring* is a variant of vertex colouring. For this we allow the graph to be partially coloured, i.e. some of the vertices may have no colour assigned to them. Such a partial colouring is *proper* if it is a proper colouring in the subgraph induced by the coloured vertices. We usually study what is the largest partial colouring of given graph and given parameters.

We study questions on multi-colouring and correspondence colouring in this thesis. We now give their formal definitions.

As noted in Section 1.1.1, multi-colouring generalises vertex colouring. In a *$k$-multi-colouring*, each vertex receives a set of $k$ colours. And such a multi-colouring is *proper* if adjacent vertices receive disjoint $k$-sets of colours. A graph *is $(n,k)$-colourable* if there is a proper $k$-multi-colouring which uses at most $n$ colours in total. The *$k$-th multi-chromatic number* $\chi_k(G)$ of a graph is the smallest $n$ such that the graph is $(n,k)$-colourable.

*Fractional colouring* is closely associated with multi-colouring. A graph is *fractional-$\frac{n}{k}$-colourable* if it is $(tn,tk)$-colourable for some positive integer $t$. The *fractional chromatic number* $\chi_f(G)$ is the infimum of $\frac{n}{k}$ so that $G$ is fractional-$\frac{n}{k}$-colourable. It is well-known (for example, see [43]) that this infimum is a minimum and hence a rational number. I.e. for any graph $G$, there are positive integers $n,k$ such that $\chi_f(G) = \frac{n}{k}$, and there exists a positive integer $t$ such that $G$ is $(tn,tk)$-colourable.

*List colouring* is another generalisation of vertex colouring. In list colouring, each vertex of a graph has a prespecified list of colours that can be used for that vertex. A *proper* list colouring assigns a colour to each vertex from its list, such that adjacent vertices receive different colours. A graph $G$ is *$n$-choosable* if $G$ can be properly list-coloured with any prespecified $n$-list at each vertex. The *list chromatic number* $\chi_\ell(G)$ (also known as *choice number*) is the smallest $n$ for which $G$ is $n$-choosable.

We study a further variant of graph colouring called *correspondence colouring* or *DP-colouring* (after the authors of the paper in which it first appeared: Dvořák and Postle [9]). Recall that list colouring generalises ordinary colouring, in which all vertices have the same collection of allowed colours. *Correspondence colouring* generalises list colouring further. In addition to each vertex having its own list of allowed colours, each edge comes with a prespecified *correspondence*, which defines what pair of colours on its endvertices cannot be assigned at the same time. In such a correspondence on an edge, each colour from the list of one endvertex is paired with at most one other colour from the list of the other endvertex. The correspondence on each edge can be considered as a matching between the lists of colours of its endvertices; the matching can be partial. Given those lists and correspondences, a *proper correspondence colouring* is an assignment for each vertex of a colour from its own list such that the correspondence on each edge is satisfied.

In the rest of this section, we first define correspondence colouring following the original definition in [9], and discuss some simplification techniques usually applied when studying this type of colourings.

**Definition 1.2.1.**
Given a multigraph $G$, a *correspondence* $\mathcal{C}(G)$ on $G$ consists of two parts:
• for each vertex $u \in V(G)$, there is a list of colours $l(u)$ associated with $u$;
• for each edge $e \in E(G)$ with endvertices $u$ and $v$, there is a *correspondence* $\mathcal{C}(e)$ specifying which pair of colours from the two endvertices *correspond*, such that $\mathcal{C}(e)$ induces a (possibly partial) matching between $\{(u, c) \mid c \in l(u)\}$ and $\{(v, c') \mid c' \in l(v)\}$.

The correspondence $\mathcal{C}(uv)$ on a (multiple) edge $uv$ is the collection of correspondences $\mathcal{C}(e)$ for all edges with endvertices $u$ and $v$.

The correspondence on an edge $e \in uv$ is *full* if the matching $\mathcal{C}(e)$ is perfect. A correspondence on a graph is *full* if the correspondence on every edge is full. If $|l(u)| = n$ for all vertices $u \in V(G)$, then the correspondence $\mathcal{C}(G)$ on $G$ is called an *$n$-correspondence.*

The central idea in correspondence colouring is that the matchings between the list of colours of two vertices indicate combinations of colours that are not allowed. In that sense, ordinary or list colouring are special cases of correspondence colouring in the senses that matchings are between the identical colours in the colour lists.

**Definition 1.2.2.**
Given a multigraph $G$ and a correspondence $\mathcal{C}(G)$ on $G$. A *proper correspondence colouring* on $\mathcal{C}(G)$ is a mapping $\varphi : V(G) \to \bigcup_{u \in V(G)} l(u)$ such that
• for each vertex $u \in V(G)$, we have $\varphi(u) \in l(u)$, and
• for each edge $e$ with endvertices $u, v$, we have $\{(u, \varphi(u)), (v, \varphi(v))\} \notin \mathcal{C}(e)$.

A multigraph $G$ is *$n$-correspondence-colourable* if a proper correspondence colouring exists for any $n$-correspondence on $G$.

The *correspondence chromatic number* $\chi_c(G)$ is the smallest $n$ such that $G$ is $n$-correspondence-colourable.

Before closing this section, we present a useful tool that allows us to assume identical colour list is assigned to each vertex (in the study of correspondence chromatic numbers). This idea first appeared in [9].

**Definition 1.2.3.**
Given a multigraph $G$ and a correspondence $\mathcal{C}(G)$ on $G$. A *renaming function $f$* applied to $\mathcal{C}(G)$ consists of
• a bijective colour-replacement mapping $f_u : l(u) \to l'(u)$ for each $u \in V(G)$ (note $|l'(u)| = |l(u)|$, but $l'(u)$ and $l(u)$ may or may not be identical);
• a replacement of correspondences that sends $\mathcal{C}(G)$ to $f_{\mathcal{C}}(G)$: for each pair of vertices $u, v \in V(G)$, we have $\{(u, d_1), (v, d_2)\} \in f_{\mathcal{C}}(uv)$ if and only if there are colours $c_1 \in l(u), c_2 \in l(v)$ such that $d_1 = f_u(c_1), d_2 = f_v(c_2)$ and $\{(u, c_1), (v, c_2)\} \in \mathcal{C}(uv)$.

Two correspondences $\mathcal{C}(G)$ and $\mathcal{C}'(G)$ on the same graph $G$ are *equivalent* if there exist some renaming function $f$ so that $\mathcal{C}'(G) = f_{\mathcal{C}}(G)$.

If a proper correspondence colouring for some correspondence $\mathcal{C}(G)$ exists, then it exists for any equivalent correspondence $\mathcal{C}'(G)$: assume $\mathcal{C}'(G) = f_{\mathcal{C}}(G)$ and let $p$ be a proper correspondence colouring of $\mathcal{C}$. We will find a proper correspondence colouring $p'$ on $\mathcal{C}'(G)$ using $f$ and $p$: for each vertex $u \in V(G)$, simply set $p'(u) = f_u(p(u))$. Then $p'$ is a proper correspondence colouring of $f_{\mathcal{C}}(G)$.

When studying the correspondence chromatic number of a (multi)graph $G$, we only consider cases where the colour lists assigned to each vertex has the same size. As we can rename colours in the prespecified list of each vertex while keeping the correspondence equivalent, we may assume the colour list assigned to every vertex is identical. We also assume the correspondence on each edge is full: if it isn't, we can add more constraints to make each edge full, and a proper correspondence colouring on the latter correspondence implies a proper correspondence colouring on the original correspondence.

We now have all the general prerequisites for the remainder of this thesis. Further specific definitions will be given when we use them.

# 2

# Multi-Colouring of Graphs: towards Stahl's Conjecture

## 2.1 Introduction

*Multi-colouring* generalises vertex colouring and has been studied extensively; see e.g. [43] for background. Recall that in a *$k$-multi-colouring of a graph*, each vertex receives a set of $k$ colours, and such a colouring is *proper* if adjacent vertices receive disjoint colour sets. A graph $G$ is *$(n, k)$-colourable* if there is a proper $k$-multi-colouring by assigning $k$-subsets of $[n]$ ($=\{1, 2, \ldots, n\}$) to the vertices of $G$. For some integer $k > 0$, the *$k$-th multi-chromatic number* $\chi_k(G)$ is the smallest $n$ such that $G$ is $(n, k)$-colourable.

Note that if $k = 1$, then $k$-multi-colouring is just normal vertex colouring, and $\chi_1(G)$ is just the normal chromatic number $\chi(G)$.

It this chapter, we consider the following question.

**Question 2.1.1.**
*If a graph $G$ is $(n, k)$-colourable (and this is the only information we have), then for what pairs $(n', k')$ is $G$ also $(n', k')$-colourable?*

We note that the corresponding question for more standard $n$-colouring is trivial: if $G$ is $n$-colourable, then it is $n'$-colourable for all $n' \geq n$. Or, more precise: if $\chi(G) = n$, then $G$ is $n'$-colourable if and only if $n' \geq n$. Maybe somewhat surprisingly, the question for multi-colouring appears to be much more challenging, and in fact it is mostly open!

Kneser graphs play a central role in the studies of multi-colouring. Recall that for $n \geq k \geq 1$, the *Kneser graph* $K(n, k)$ has as vertex set the collection of all $k$-subsets of $[n]$ (denoted by $\binom{[n]}{k}$), and there is an edge between two vertices if and only if the two $k$-sets are disjoint. We will usually assume $n \geq 2k$, as otherwise the graph is edgeless.

It is well known and easy to prove (see e.g. [43, Section 3.2]) that a graph $G$ is $(n, k)$-colourable if and only if there is a homomorphism from $G$ to $K(n, k)$. (A *homomorphism* from a graph $G$ to a graph $H$ is a mapping $\varphi : V(G) \to V(H)$ that preserves edges; i.e. if $uv$ is an edge in $G$, then $\varphi(u)\varphi(v)$ is an edge in $H$.)

This means that the following questions are all equivalent to Question 2.1.1.

1. *Given $n, k$, for what $n', k'$ is the Kneser graph $K(n, k)$ also $(n', k')$-colourable?*

2. *Given $n, k$, for what $n', k'$ is there a homomorphism from $K(n, k)$ to $K(n', k')$?*

3. *Given $n, k$, for what $n', k'$ do we have $n' \geq \chi_{k'}(K(n, k))$?*

The last question was studied by Stahl [44], who formulated the following conjecture.

**Conjecture 2.1.2** (Stahl [44])**.**
*For integers $n \geq 2k > 0$, if $k' = qk - r$ where $q \geq 1$ and $0 \leq r \leq k - 1$, then we have $\chi_{k'}(K(n, k)) = qn - 2r$.*

As mentioned in Section 1.1.1, there are some results on special cases of the question, but not much are known in general. For instance, this conjecture is trivially true for $k = 1$; and is true for $k' = 1$ by Lovász's proof [38] of the Kneser conjecture.

In Stahl's original paper, it was proved that the conjectured value is an upper bound. In a follow-up paper [45], Stahl proved the following general lower bound for $\chi_{k'}(K(n,k))$:

$$\chi_{qk-r}(K(n,k)) \geq qn - 2r - (k^2 - 3k + 4). \tag{2.1}$$

The conjecture is also known to be true for some special values of $n$, $k$ and $k'$.

**Theorem 2.1.3.**
(a) *For all $k$ and $k'$, Conjecture 2.1.2 is true for the bipartite Kneser graphs $K(2k,k)$ and for the so-called odd graphs $K(2k+1,k)$ (Stahl [44]).*

(b) *For all $n$ and $k$, Conjecture 2.1.2 is true for any $k'$ that is a multiple of $k$; in other words: $\chi_{qk}(K(n,k)) = qn$ (Stahl [44]).*

(c) *For all $n$ and $k$, Conjecture 2.1.2 is true for all $k' \leq k$; in other words: $\chi_{k-r}(K(n,k)) = n - 2r$ (Stahl [44]).*

(d) *For all $n$ and $k'$, Conjecture 2.1.2 is true for $k = 2$ and $k = 3$ (Stahl [45]).*

Our first results are short proofs of the following theorems. The proofs of our results are given in later sections of this chapter.

**Theorem 2.1.4.**
*For any $k \geq 1$ and $n \geq 2k$, we have $\chi_{k'}(K(n,k)) \geq \dfrac{k'n}{k}$, with equality if and only if $k'$ is a multiple of $k$.*

Note that this theorem extends Theorem 2.1.3 (b). Even then, our proof is considerably simpler and shorter than the proof of Theorem 2.1.3 (b) in [44].

**Theorem 1.1.1.**
*For any $k \geq 1$ and $n > 2k$, if $0 \leq r \leq \dfrac{k}{n-2k}$ and $r \leq k-1$, then for all $q \geq 1$ we have $\chi_{qk-r}(K(n,k)) = qn - 2r$.*

This theorem is a small generalisation of the main result in Osztényi [42]; again with a much simpler and shorter proof.

Another result we present in this chapter is that for a fixed $k$, only at most $k^3 - k^2$ values of $\chi_{k'}(K(n,k))$ need to be determined in order to conclude whether or not Stahl's conjecture is true for that value of $k$ and for *all $n$ and $k'$*.

**Theorem 1.1.2.**
*Fix $k \geq 1$. Then there exist $n_0(k)$ and $q_0(n,k)$ such that the following holds. If $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$ for all $n \leq n_0(k)$ and for at least one $q \geq q_0(n,k)$, then we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $n \geq 2k$, $q \geq 1$ and $0 \leq r \leq k-1$.*

This chapter is structured as following: we recall and present the main ideas, as well as our proof sketches in Section 2.2; formal proofs can be found in Section 2.3. Conclusion and further directions are discussed in Section 2.4.

## 2.2 Main Ideas

In this section, we describe some of the original ideas behind Stahl's conjecture, as developed in [44,45]. To begin with, Stahl [44] proved the following.

**Proposition 2.2.1** (Stahl [44])**.**
*For any integers $n \geq 3$ and $k \geq 2$, there is a homomorphism $\varphi$ from $K(n,k)$ to $K(n-2,k-1)$.*

*Proof.* If $n < 2k$, then any mapping of vertices is a homomorphism, because both $K(n,k)$ and $K(n-2,k-1)$ are edge-less. If $n \geq 2k$, we define $\varphi$ as follows.
*For each $k$-subset $K \in \binom{[n]}{k}$, denote by $\max K$ the maximum element in $K$.*
(a) If $(K \setminus \{\max K\}) \subseteq [n-2]$, then define $\varphi(K) = K \setminus \{\max K\}$;
(b) Otherwise both $n-1$ and $n$ are elements of $K$. Then let $x$ be the largest integer in $[n-2]$ that is not in $K$, and define $\varphi(K) = (\{x\} \cup K) \setminus \{n-1, n\}$.

We will show that $\varphi$ is a homomorphism from $K(n,k)$ to $K(n-2,k-1)$. Denote $K_1, K_2$ as the $k$-sets associated with vertices $u, v$ respectively. If $u, v$ are adjacent in $K(n,k)$, then $K_1, K_2$ are disjoint. If both $K_1$ and $K_2$ satisfy the condition in (a), then it is clear that $\varphi(K_1)$ and $\varphi(K_2)$ are disjoint. Since $K_1, K_2$ are disjoint, at most one of them satisfies condition in (b). Without loss of generality, assume both $n-1$ and $n$ are elements of $K_1$, and $(K_2 \setminus \{\max K_2\}) \subseteq [n-2]$. Let $x$ be the largest integer in $[n-2]$ that is not in $K_1$, then $x$ is either the largest element in $K_2$, or $x$ is not in $K_2$. In either cases, $\varphi(K_1)$ and $\varphi(K_2)$ are disjoint. $\square$

The existence of a homomorphism from $K(n,k)$ to $K(n-2,k-1)$ means that for any graph $G$, if there is a homomorphism from $G$ to $K(n,k)$ (i.e. if $G$ is $(n,k)$-colourable), then there is a homomorphism from $G$ to $K(n-2,k-1)$ (i.e. $G$ is also $(n-2,k-1)$-colourable). And hence for any graph $G$ with at least one edge, we have $\chi_{k-1}(G) \leq \chi_k(G) - 2$. (We require at least one edge in $G$, otherwise $\chi_{k-1}(G) = k-1 = \chi_k(G) - 1$ for all $k$.)

Stahl's conjecture (Conjecture 2.1.2) states that $\chi_{qk-r}(K(n,k)) = qn - 2r$, for any $q \geq 1$ and $0 \leq r \leq k-1$. By Theorem 2.1.3 (b) and the observation in the last paragraph, it is clear that for any fixed $q$, if $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$, then $\chi_{qk-r}(K(n,k)) = qn - 2r$ for any $0 \leq r \leq k-1$ for the same $q$. I.e. in order to prove the conjecture, it suffices to prove it for $r = k-1$. Knowing the Lovász-Kneser Theorem $\chi_1(K(n,k)) = n - 2k + 2$, an immediate corollary is that the conjecture is true for $q = 1$.

On the other hand, we can prove that the conjectured multi-chromatic number is an upper bound using the following result of Geller and Stahl [23].

**Proposition 2.2.2** (Geller and Stahl [23]).
*If a graph $G$ is both $(n_1,k_1)$-colourable and $(n_2,k_2)$-colourable, then $G$ is $(n_1+n_2, k_1+k_2)$-colourable.*

The proof is again constructive.

*Proof.* Let $c_1 : V(G) \to \binom{[n_1]}{k_1}$ and $c_2 : V(G) \to \binom{[n_1+1, n_1+n_2]}{k_2}$ be two multi-colourings of graph $G$. For each vertex $v \in V(G)$, define $c(v) = c_1(v) \cup c_2(v)$. Then $c : V(G) \to \binom{[n_1+n_2]}{k_1+k_2}$ is a proper $(n_1+n_2, k_1+k_2)$-colouring of $G$. $\square$

It follows that we can always construct a $(qk-r)$-multi-colouring of $K(n,k)$ by combining $q-1$ copies of a $(n,k)$-colouring and one copy of a $(n-2r, k-r)$ colouring. This immediately gives

$$\chi_{qk-r}(K(n,k)) \leq qn - 2r. \tag{2.2}$$

Note that other colourings may be possible. For instance, if $k, q, r \geq 2$, we can take $q-2$ copies of a $(n,k)$-colouring, one copy of a $(n-2, k-1)$-colouring and one copy of a $(n-2(r-1), k-(r-1))$-colouring to get the same bound.

### 2.2.1 Main New Ideas

In this section, we sketch some of the ideas behind our results and methods to approach Stahl's conjecture. In fact, many of these ideas have been observed before, but we haven't seen them used in the way we use them.

For any proper $(\chi_{k'}(G), k')$ multi-colouring of a graph $G$, each *colour class* (the set of vertices whose colour set contains a particular colour) is an independent set, hence contains at most $\alpha(G)$ vertices (where $\alpha(G)$ is the *independence number*). Since each vertex appears in at least $k'$ colour classes, we have $\chi_{k'}(G)\alpha(G) \geq k'|V(G)|$ for any $k'$, and hence $\chi_{k'}(G) \geq \left\lceil \dfrac{k'|V(G)|}{\alpha(G)} \right\rceil$.

For Kneser graphs, we have $|V(K(n,k))| = \dbinom{n}{k}$ by definition, while the celebrated Erdős-Ko-Rado Theorem [14] states that $\alpha(K(n,k)) = \dbinom{n-1}{k-1}$ for any $k$ and any $n \geq 2k$. Substituting those values and $k' = qk - r$ in the lower bound for $\chi_{k'}(G)$ above leads to

$$\chi_{k'}(K(n,k)) \geq \left\lceil \frac{k'\binom{n}{k}}{\binom{n-1}{k-1}} \right\rceil = \left\lceil \frac{k'n}{k} \right\rceil = qn - 2r - \left\lfloor \frac{r(n-2k)}{k} \right\rfloor. \tag{2.3}$$

This simple inequality is surprisingly powerful. For instance, it gives a better bound than (2.1) if $n \leq k^2 + 2$. It also more or less directly gives Theorem 1.1.1.

We can obtain further results by using more detailed knowledge about independent sets in Kneser graphs. For instance, in [14] it is also proved that if $n \geq 2k+1$, then the only independent sets of order $\dbinom{n-1}{k-1}$ in the Kneser graph $K(n,k)$ are the so-called *trivial* independent sets: those vertex sets whose vertices correspond to family of $k$-sets in $[n]$ that contain some fixed common element $i \in [n]$. Using that information about the structure of independent sets of order $\alpha(K(n,k))$, in the next section we will show that we can only have equality in (2.3) in very special cases.

**Theorem 2.1.4.**
*For any $k \geq 2$ and $n \geq 2k$, we have $\chi_{k'}(K(n,k)) \geq \dfrac{k'n}{k}$, with equality if and only if $k'$ is a multiple of $k$.*

The statement that $\chi_{pk}(K(n,k)) = pn$ if and only if $p$ is a positive integer is sometimes attributed to Stahl [44] (see, e.g., [42]), but it is not explicitly stated in that paper. It can be found implicitly in the proof of [44, Theorem 9] (using significantly more involved arguments than ours).

As observed in [23], we have that for any graph $G$, $\chi_{k'}(G) = \chi(G \bullet K_{k'})$, where "$\bullet$" denotes the *lexicographic product* of two graphs: $V(G \bullet H) = V(G) \times V(H)$, and $(u_1, v_1)(u_2, v_2) \in E(G \bullet H)$ if and only if either $u_1u_2 \in E(G)$ or $u_1 = u_2$ and $v_1v_2 \in E(H)$. This allows us to translate the problem of finding multi-chromatic numbers to finding chromatic numbers. Since we also have that $|V(G \bullet K_{k'})| = k'|V(G)|$ and $\alpha(G \bullet K_{K'}) = \alpha(G)$, this gives an alternative proof of $\chi_{k'}(G) \geq \left\lceil \dfrac{k'|V(G)|}{\alpha(G)} \right\rceil$.

One of the essential elements in the proof of Theorem 1.1.2 is the result of Hilton and Milner [28] that if $n \geq 2k+1$ and an independent set in the Kneser graph $K(n,k)$ is not trivial, then it has order at most $\dbinom{n-1}{k-1} - \dbinom{n-k-1}{k-1} + 1$. This 'second best' bound is significantly smaller than the Erdős-Ko-Rado bound (details in the next section), which means that for large $n$ and $q$, many of the colours used in a 'good' $(qk-r)$-multi-colouring of $K(n,k)$ must induce trivial independent sets. This observation allows us to prove relations between the multi-chromatic numbers $\chi_{qk-r}(K(n,k))$ for different values of $n$ and $q$, and eventually to prove Theorem 1.1.2.

Finally, we note that Theorem 1.1.2 generalises some known results. Chvátal et al. [6] showed that for fixed $k$, we only need to find $\chi_{k+1}(K(n,k))$ for finitely many $n$ to decide if Stahl's Conjecture holds for $\chi_{k+1}(K(n,k))$ for all $n$. And Stahl [44] proved that for fixed $n, k$ and sufficiently large $k'$, the correctness of the conjecture for $k'$ is equivalent to its correctness for $k' - k$. The proof of that result by Stahl is non-constructive and does not give an explicit bound on the value of $k'$, and hence it can only give a version of Theorem 1.1.2 without a bound on the function $q_0(n,k)$.

## 2.3 Proofs of Our Results

This section contains the full proofs of the results stated in Sections 2.1 and 2.2. In this section, with given $k$, we use $k'$ and $qk-r$ interchangeably ($k' = qk-r$), where $q \geq 1$ and $0 \leq r \leq k-1$.

### 2.3.1 Proof of Theorem 2.1.4

Theorem 2.1.4 is a direct corollary of (2.2) and the following lemma. As explained in Section 2.2, each colour class of an $(n',k')$-colouring of $K(n,k)$ induces an independent set in $K(n,k)$.

**Lemma 2.3.1.**
*For any $k' \geq 1$, $k \geq 1$ and $n \geq 2k$, we have $\chi_{k'}(K(n,k)) \geq \left\lceil \frac{k'n}{k} \right\rceil$. Furthermore, if $\frac{k'n}{k}$ is an integer and $n > 2k$, then $K(n,k)$ is $\left(\frac{k'n}{k}, k'\right)$-colourable if and only if $k'$ is a multiple of $k$.*

*Proof.* We have seen in (2.3) in Section 2.2 that $\chi_{k'}(K(n,k)) \geq \left\lceil \frac{k'\binom{n}{k}}{\binom{n-1}{k-1}} \right\rceil = \left\lceil \frac{k'n}{k} \right\rceil \geq \frac{k'n}{k}$.

If $k' = qk$ for some integer $q$, then we have $\chi_{k'}(K(n,k)) \leq qn = \frac{k'n}{k}$ by (2.2), and hence $\chi_{k'}(K(n,k)) = \frac{k'n}{k} = \left\lceil \frac{k'n}{k} \right\rceil$.

Now suppose $\left\lceil \frac{k'n}{k} \right\rceil = \frac{k'n}{k}$, i.e. $k'n$ is a multiple of $k$, and assume $K(n,k)$ is $\left(\frac{k'n}{k}, k'\right)$-colourable; fix such a multi-colouring. Then the first inequality in (2.3) must be an equality. In particular, every colour class has order $\binom{n-1}{k-1}$, and hence must be trivial. So every colour class is a set of the form $\mathcal{F}_i := \left\{ F \in \binom{[n]}{k} \,\middle|\, i \in F \right\}$, for some $i \in [n]$.

For $i \in [n]$, let $\lambda_i$ be the number of colour classes of the form $\mathcal{F}_i$. For any vertex $F \in \binom{[n]}{k}$ we have that $F$ is in a colour class of the form $\mathcal{F}_i$ if and only if $i \in F$. This means that $\sum_{i \in F} \lambda_i = k'$, for all $F \in \binom{[n]}{k}$. In particular, for any $j_1, j_2 \in [n]$ and any $(k-1)$-set $F'$ in $[n] \setminus \{j_1, j_2\}$, we have

$$\sum_{i \in F' \cup \{j_1\}} \lambda_i = \sum_{i \in F' \cup \{j_2\}} \lambda_i,$$

and hence $\lambda_{j_1} = \lambda_{j_2}$ for any $j_1, j_2 \in [n]$.

On the other hand, we have $\sum_{i \in [n]} \lambda_i = \frac{k'n}{k}$. This means that $\lambda_i = \frac{k'}{k}$ for all $i \in [n]$. In particular we have that $\frac{k'}{k}$ is an integer, completing the proof of the theorem. $\square$

### 2.3.2 Proof of Theorem 1.1.1 and Other Bounds on $\chi_{k'}(K(n,k))$

We first prove Theorem 1.1.1, which states that Stahl's conjecture is true for $0 \le r \le \frac{k}{n-2k}$. Note that this result generalises the following known results.

(a) The conjecture is true if $k'$ is a multiple of $k$, i.e. if $r = 0$ (Stahl [44]).

(b) The conjecture is true if $2k < n < 3k$ and $0 \le r < \frac{k}{n-2k}$ (Osztényi [42]).

**Theorem 1.1.1.**
*For any $k \ge 1$ and $n > 2k$, if $0 \le r \le \frac{k}{n-2k}$ and $r \le k-1$, then for all $q \ge 1$ we have $\chi_{qk-r}(K(n,k)) = qn - 2r$.*

*Proof.* If $0 \le r < \frac{k}{n-2k}$, then $\left\lfloor \frac{r(n-2k)}{k} \right\rfloor = 0$, and hence (2.3) immediately shows that $\chi_{qk-r}(K(n,k)) \ge qn - 2r$. We then have $\chi_{qk-r}(K(n,k)) = qn - 2r$ by (2.2).

If $r = \frac{k}{n-2k}$ is an integer and $0 < r \le k-1$, then $\frac{k'n}{k} = \frac{(qk-r)n}{k} = qn - \frac{n}{n-2k} = qn - 2r - 1$ is an integer. But then $qk - r$ is not a multiple of $k$, since $1 \le r \le k-1$. Therefore by Lemma 2.3.1, we have that $K(n,k)$ is not $(qn-2r-1, qk-r)$-colourable. We can conclude that $\chi_{qk-r}(K(n,k)) = qn - 2r$ by (2.2). $\square$

In [42, Proposition 5], Osztényi also shows a lower bound on multi-chromatic number, which states: *for any* $k, l \geq 2$ *and* $lk < n < 2lk$ $(*)$, *we have*

$$\chi_{qk-r}(K(n,k)) > qn - lr - c, \quad \textit{where } c \geq 1 \textit{ and } c > \frac{lr-1}{\lceil \frac{lk}{n-lk} \rceil - 1}. \tag{2.4}$$

We can restate $(*)$ as: "*for any* $k \geq 2$, $n > 2k$ *and* $\frac{n}{2k} < l < \frac{n}{k}$". We will show that Lemma 2.3.1 is at least as good as (2.4), by rewriting (2.3) as

$$\chi_{qk-r}(K(n,k)) \geq \left\lceil \frac{(qk-r)n}{k} \right\rceil = qn - lr - \left\lfloor \frac{r(n-lk)}{k} \right\rfloor, \tag{2.5}$$

where $k, l \geq 2$ and $n \geq lk$.

For $n \geq lk$, we first show that there is no positive integer $c$ such that $\frac{lr-1}{\lceil \frac{lk}{n-lk} \rceil - 1} < c \leq \left\lfloor \frac{r(n-lk)}{k} \right\rfloor$. Assume such a $c$ exists for some $n, k, r, l$. Then since $c$ is an integer, we have

$$ck \leq r(n-lk). \tag{2.6}$$

Since $lk < n < 2lk$ is equivalent to $lk+1 \leq n \leq 2lk-1$ for integers, we have $\left\lceil \frac{lk}{n-lk} \right\rceil - 1 \geq \left\lceil \frac{lk}{lk-1} \right\rceil - 1 > 0$, which gives $c \left\lceil \frac{lk}{n-lk} \right\rceil - c > lr - 1$. Rearranging leads to $\left\lceil \frac{lk}{n-lk} \right\rceil > \frac{lr+c-1}{c}$, and hence to $\left\lceil \frac{lk}{n-lk} \right\rceil \geq \frac{lr+c}{c} = \frac{lr}{c} + 1$. Since we are working with integers, we can conclude that $\frac{lk}{n-lk} > \frac{lr}{c}$, which gives $ck > r(n-lk)$, contradicting (2.6).

On the other hand, for some values of $n, k, r, l$, there exist integers $c \geq 2$ such that $\left\lfloor \frac{r(n-lk)}{k} \right\rfloor < c \leq \frac{lr-1}{\lceil \frac{lk}{n-lk} \rceil - 1}$. For instance, if $l = 2$, $n = 137$, $k = 56$, and $r = 31$, then $c = 14$ is an example; while if $l = 3$, $n = 145$, $k = 30$, and $r = 17$, then $c = 32$ is an example.

Next we show that sometimes it is possible to improve the lower bound in Lemma 2.3.1 by partitioning $K(n,k)$ into suitable subgraphs, by splitting the ground set $[n]$. Note that for any $m$ that $k \leq m \leq n-k$, we can consider $K(m,k)$ as the subgraph of $K(n,k)$ induced by $\left\{ F \in \binom{[n]}{k} \,\middle|\, F \subseteq [m] \right\}$, and $K(n-m,k)$ as the subgraph induced by $\left\{ F \in \binom{[n]}{k} \,\middle|\, F \subseteq [m+1,n] \right\}$. Moreover, since the vertices of these two subgraphs are disjoint in $K(n,k)$, there is a complete bipartite join between them in $K(n,k)$. These observations immediately give for $k \le m \le n-k$:

$$\chi_{k'}(K(n,k)) \ge \chi_{k'}(K(m,k)) + \chi_{k'}(K(n-m,k)). \tag{2.7}$$

**Theorem 2.3.2.**
*For $k' = qk - r$, with $q \ge 1$ and $1 \le r \le k-1$, let $t$ be the largest integer such that $rt \le k$, i.e. $t = \left\lfloor \frac{k}{r} \right\rfloor$.*

(a) *Write $n = n_0 + \sum_{i=0}^{t} c_i(2k+i)$ for some integers $n_0$, $0 \le n_0 < 2k$, and $c_i \ge 0$, $i = 0, \ldots, t$. Then we have $\chi_{k'}(K(n,k)) \ge q(n-n_0) - 2r\sum_{i=0}^{t} c_i + k' \left\lfloor \frac{n_0}{k} \right\rfloor$.*

(b) *Alternatively, write $n = n_0 + \sum_{i=0}^{t} c_i(2k+i)$ for some integers $n_0$, $2k \le n_0 < 4k$, and $c_i \ge 0$, $i = 0, \ldots, t$. Then we have $\chi_{k'}(K(n,k)) \ge q(n-n_0) - 2r\sum_{i=0}^{t} c_i + \left\lceil \frac{n_0 k'}{k} \right\rceil$.*

*Proof.* Using (2.7), for both cases we have:

$$\begin{aligned}
\chi_{k'}(K(n,k)) &\ge \chi_{k'}(K(n_0,k)) + \chi_{k'}(K(n-n_0,k)) \\
&\ge \chi_{k'}(K(n_0,k)) + \sum_{i=0}^{t} c_i \chi_{k'}(K(2k+i,k)) \\
&= \chi_{k'}(K(n_0,k)) + \sum_{i=0}^{t} c_i(q(2k+i) - 2r) \\
&= \chi_{k'}(K(n_0,k)) + q(n-n_0) - 2r\sum_{i=0}^{t} c_i.
\end{aligned}$$

Then for case (a), we immediately have $\chi_{k'}(K(n_0,k)) = 0$ if $0 \le n_0 < k$, and $\chi_{k'}(K(n_0,k)) = k'$ if $k \le n_0 < 2k$. So we have $\chi_{k'}(K(n_0,k)) = k' \left\lfloor \frac{n_0}{k} \right\rfloor$. For case (b), we have $\chi_{k'}(K(n_0,k)) \ge \left\lceil \frac{n_0 k'}{k} \right\rceil$ by Lemma 2.3.1. □

Note that Lemma 2.3.1 is a special case of Theorem 2.3.2 (b) by taking $n = c_0 \cdot 2k + n_0$ for some $2k \le n_0 < 4k$.

Although Theorem 2.3.2 is based on Lemma 2.3.1, it actually can give better bounds in many cases. For example, if $n = c_1(2k+1)$ for some $c_1$, then Theorem 2.3.2 gives $\chi_{k'}(K(n,k)) \geq qn - 2rc_1$, whilst Lemma 2.3.1 only gives $\chi_{k'}(K(n,k)) \geq qn - 2rc_1 - \left\lfloor \frac{rc_1}{k} \right\rfloor$.

### 2.3.3 Proof of Theorem 1.1.2

Theorem 1.1.2 is a corollary to the following two results.

**Theorem 2.3.3.**
*For any $k \geq 1$, there exists $n_0(k) < k^3 - k^2 + 2k$, such that if $n \geq n_0(k)$, then we have $\chi_{qk-r}(K(n,k)) \geq q + \chi_{qk-r}(K(n-1,k))$ for any $q \geq 1$ and $0 \leq r \leq k-1$.*

**Theorem 2.3.4.**
*For any $k \geq 1$ and $n \geq 2k$, there exist $q_0(n,k)$, such that if $q \geq q_0(n,k)$, then we have $\chi_{qk-r}(K(n,k)) \geq n + \chi_{(q-1)k-r}(K(n,k))$ for any $0 \leq r \leq k-1$.*

We first prove Theorem 1.1.2, as a corollary of Theorems 2.3.3 and 2.3.4.

**Theorem 1.1.2.**
*Fix $k \geq 1$. Then there exist $n_0(k)$ and $q_0(n,k)$ such that the following holds. If $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$ for all $2k \leq n \leq n_0(k)$ and for at least one $q \geq q_0(n,k)$, then we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $n \geq 2k$, $q \geq 1$ and $0 \leq r \leq k-1$.*

*Proof.* Fix $k \geq 1$. First let $n_0(k)$ be the integer as in Theorem 2.3.3. I.e. if $n \geq n_0(k)$, then for any $0 \leq r \leq k-1$, we have

$$\chi_{qk-r}(K(n,k)) \geq q + \chi_{qk-r}(K(n-1,k)). \tag{2.8}$$

For each $n \geq 2k$, let $q_0(n,k)$ be the integer as in Theorem 2.3.4. I.e. if $q \geq q_0(n,k)$, then for any $0 \leq r \leq k-1$, we have

$$\chi_{qk-r}(K(n,k)) \geq n + \chi_{(q-1)k-r}(K(n,k)). \tag{2.9}$$

At the same time, note that for any $n \geq 2k$ and $q' \geq 2$, we have

$$\chi_{q'k-r}(K(n,k)) \leq n + \chi_{(q'-1)k-r}(K(n,k)), \tag{2.10}$$

since combining a copy of a $(\chi_{(q'-1)k-r}(K(n,k)), (q'-1)k-r)$-colouring and a copy of a $(n,k)$-colouring of $K(n,k)$ produces a proper $(q'k-r)$-multi-colouring.

Assume the conditions in Theorem 1.1.2 hold. I.e. for each $2k \le n \le n_0(k)$, we have $\chi_{qk-(k-1)}(K(n,k)) = qn - 2(k-1)$ for some $q \ge q_0(n,k)$.

For any fixed $n$, $2k \le n \le n_0(k)$, find its corresponding $q^* \ge q_0(n,k)$ such that $\chi_{q^*k-(k-1)}(K(n,k)) = q^*n - 2(k-1)$. Then we immediately have $\chi_{q^*k-r}(K(n,k)) = q^*n - 2r$ for any $0 \le r \le k-1$ for that $q^*$, since for any non-empty graph $G$ we have $\chi_{k'+1}(G) \ge \chi_{k'}(G)$, and $\chi_{q^*k}(K(n,k)) = q^*n$. Hence by (2.2), (2.9), and (2.10), we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $q \ge 1$ and $0 \le r \le k-1$.

Therefore by (2.2), (2.8), and the correctness of the conjecture for $n = n_0(k)$, we have $\chi_{qk-r}(K(n,k)) = qn - 2r$ for all $n \ge n_0(k)$, $q \ge 1$ and $0 \le r \le k-1$. Thus for all $n \ge 2k$, $q \ge 1$ and $0 \le r \le k-1$ we can conclude $\chi_{qk-r}(K(n,k)) = qn - 2r$. □

Before proving the two theorems from the start of this section, we recall that for any $n \ge 2k$, the independence number of the Kneser graph $K(n,k)$ is $\alpha(K(n,k)) = \binom{n-1}{k-1}$. Furthermore, if $n \ge 2k+1$, then the only maximum independent sets in $K(n,k)$ are the *trivial* independent sets [11]. Denote by $\alpha^*(K(n,k))$ the order of the maximum independent set of $K(n,k)$ that is not *trivial*. Then we know that $\alpha^*(K(n,k)) = \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ if $n \ge 2k+1$, by results in [28].

In the following proof of Theorem 2.3.4, we also name $q_0(n,k)$. The formula is complicated, but we will discuss its upper bound in Section 2.4.

*Proof of Theorem 2.3.4.* We have seen that Stahl's conjecture holds for bipartite Kneser graphs (i.e. if $n = 2k$), hence Theorem 2.3.4 is true if $n = 2k$. Also note that if $k = 1$, then $K(n,k)$ is the complete graph $K_n$, and in that case the theorem is trivially true. So from now on we assume that $n \ge 2k+1$ and $k \ge 2$.

Fix any $k \ge 2$, $n \ge 2k+1$ and $0 \le r \le k-1$. Consider any $q \ge q_0(n,k)$, where $q_0(n,k)$ is a function we will specify later. Fix a proper $(x, qk-r)$-colouring $C : K(n,k) \to \binom{[x]}{qk-r}$ for some $x \le qn - 2r$. We will find a proper $(x-n, (q-1)k-r)$-colouring of $K(n,k)$.

Denote $y$ as the number of *nontrivial* colour classes in $C$ (that is, those colour classes that cannot be written as a subset of $\left\{ F \in \binom{[n]}{k} \;\middle|\; i \in F \right\}$ for any $i \in [n]$). Hence, there are $x - y$ *trivial* colour classes, and by counting the appearance of each vertex in all colours, we have

$$\begin{aligned}(qk-r)\binom{n}{k} &\le (x-y)\alpha(K(n,k)) + y\alpha^*(K(n,k)) \\ &\le (qn-2r-y)\binom{n-1}{k-1} + y\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right).\end{aligned}$$

Since $(qk-r)\binom{n}{k} = \left(qn - \frac{rn}{k}\right)\binom{n-1}{k-1}$, this gives

$$y \le \frac{\left(\frac{rn}{k} - 2r\right)\binom{n-1}{k-1}}{\binom{n-k-1}{k-1} - 1} = \frac{r(n-2k)\binom{n-1}{k-1}}{k\left(\binom{n-k-1}{k-1} - 1\right)}. \tag{2.11}$$

We claim that if $q \ge q_0(n,k)$, where

$$\begin{aligned}&q_0(n,k) \\ &= \frac{(k-1)(n-2k+1)}{n-k} + \frac{(k-1)(n-2k)(n-1)\left(\binom{n-2}{k-1} - \binom{n-k-1}{k-1}\right)}{k(n-k)\left(\binom{n-k-1}{k-1} - 1\right)} + 1,\end{aligned}$$

then for all $i \in [n]$, there is a *trivial* colour class in $C$ that is *centred* at $i$, i.e. a colour class in $C$ that is a subset of $\left\{ F \in \binom{[n]}{k} \;\middle|\; i \in F \right\}$.

Assume that there is no colour class centred at $i^*$ for some $i^* \in [n]$, i.e. none of the colour classes in $C$ is a subset of $\mathcal{F} = \left\{ F \in \binom{[n]}{k} \;\middle|\; i^* \in F \right\}$. Then each trivial colour class contains at most $\binom{n-2}{k-2}$ vertices in $\mathcal{F}$, and each nontrivial colour class contains at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1}$ vertices in $\mathcal{F}$.

Counting the appearance of all vertices in $\mathcal{F}$, we have

$$(qk-r)|\mathcal{F}| = (qk-r)\binom{n-1}{k-1}$$

$$\leq (x-y)\binom{n-2}{k-2} + y\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1}\right),$$

which leads to

$$\begin{aligned}\frac{(qk-r)(n-1)}{k-1}\binom{n-2}{k-2} &\leq x\binom{n-2}{k-2} + y\sum_{i=3}^{k+1}\binom{n-i}{k-2} \\ &\leq (qn-2r)\binom{n-2}{k-2} + y\sum_{i=3}^{k+1}\binom{n-i}{k-2},\end{aligned}$$

which in turn can be rewritten as

$$y\left(\binom{n-2}{k-1} - \binom{n-k-1}{k-1}\right) \geq \frac{q(n-k) - r(n-2k+1)}{k-1}\binom{n-2}{k-2}. \quad (2.12)$$

Combining (2.11) and (2.12), we have

$$\frac{r(n-2k)\binom{n-1}{k-1}\left(\binom{n-2}{k-1} - \binom{n-k-1}{k-1}\right)}{k\left(\binom{n-k-1}{k-1} - 1\right)} \geq \frac{q(n-k) - r(n-2k+1)}{k-1}\binom{n-2}{k-2},$$

which rearranges to

$$q \leq \frac{r(n-2k+1)}{n-k} + \frac{r(n-2k)(n-1)\left(\binom{n-2}{k-1} - \binom{n-k-1}{k-1}\right)}{k(n-k)\left(\binom{n-k-1}{k-1} - 1\right)}.$$

So if we choose $q_0(n,k)$ as larger than the right-hand side of this last inequality, then we obtain a contradiction.

Since for each $i \in [n]$, non-existence of colour class centred at $i$ leads to a contradiction, we now assume that for all $i \in [n]$, there is a trivial colour class in $C$ that is centred at $i$. By removing one such trivial colour class centred at $i$ for each $i \in [n]$, we remove $n$ colour classes in total and at most $k$ colours for each vertex. Thus we find a proper $(x-n, (q-1)k-r)$-colouring of $K(n,k)$, and consequently have proved $\chi_{(q-1)k-r}(K(n,k)) \leq \chi_{qk-r}(K(n,k)) - n$. $\square$

We need the following technical lemma for the proof of Theorem 2.3.3.

**Lemma 2.3.5.**
*For any $k \geq 2$, there exist $n_0(k) \leq k^3 - k^2 + \frac{\sqrt{41}-3}{2}k - 1$ such that if $n \geq n_0(k)$, then $\frac{\alpha^*(K(n,k))}{\alpha(K(n,k))} < \frac{n}{(n-2k+2)k}$.*

*Proof.* First note that if $k = 2$, then $\dfrac{\alpha^*(K(n,k))}{\alpha(K(n,k))} = \dfrac{3}{n-1}$, which is smaller than $\dfrac{n}{2(n-2)}$ if $n > 4$. I.e. $n_0(2) = 5$ suffices.

Now assume $k \geq 3$. Note that

$$\frac{\alpha^*(K(n,k))}{\alpha(K(n,k))} = \frac{\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1}{\binom{n-1}{k-1}} = 1 - \prod_{i=1}^{k-1} \frac{n-k-i}{n-i} + \frac{1}{\binom{n-1}{k-1}},$$

and

$$\frac{n}{(n-2k+2)k} = \frac{1}{k}\left(1 + \frac{2(k-1)}{n-2k+2}\right).$$

For all $k \geq 3$ and $n \geq 2k$ we have

$$\begin{aligned}
\frac{1}{\binom{n-1}{k-1}} = \frac{k!}{\prod_{i=1}^{k-1}(n-i)} &= \frac{k-1}{k(n-2k+2)} \cdot \frac{k^2(k-2)!(n-2k+2)}{\prod_{i=1}^{k-1}(n-i)} \\
&= \frac{k-1}{k(n-2k+2)} \cdot \frac{2k}{n-1} \cdot \frac{k}{n-2} \cdot \frac{n-2(k-1)}{n-(k-1)} \cdot \prod_{i=3}^{k-2} \frac{k-i+1}{n-i} \\
&< \frac{k-1}{k(n-2k+2)}.
\end{aligned}$$

Also $\displaystyle\prod_{i=1}^{k-1} \frac{n-k-i}{n-i} = \prod_{i=1}^{k-1}\left(1 - \frac{k}{n-i}\right) > \left(1 - \frac{k}{n-k+1}\right)^{k-1}$. Hence it suffices to find $n_0(k) \geq 2k$ such that $1 - \left(1 - \dfrac{k}{n-k+1}\right)^{k-1} \leq \dfrac{1}{k}$ for any $n \geq n_0(k)$.

Define $f(n,k) := \left(1 - \dfrac{k}{n-k+1}\right)^{k-1} - \dfrac{k-1}{k}$. Since for any fixed $k$, $f(n,k)$ increases as $n$ increases, it suffices to find some $n_0(k) \geq 2k$ such that $f(n_0(k), k) \geq 0$.

Let $n_0(k) = k^3 - k^2 + (c+1)k - 1$, where $c = \dfrac{\sqrt{41}-5}{2} \approx 1.702$ and define $g(k) := f(n_0(k), k) = \left(1 - \dfrac{1}{k^2-k+c}\right)^{k-1} - \dfrac{k-1}{k}$. Then for any $k \geq 3$, $g(k) \geq 0$ if and only if $h(k) := (k-1)\ln\left(1 - \dfrac{1}{k^2-k+c}\right) - \ln\dfrac{k-1}{k} \geq 0$.

Note that $\lim_{k\to\infty} g(k) = 0$, hence it suffices to prove $\dfrac{\mathrm{d}h}{\mathrm{d}k} \leq 0$, which is indeed the case with the chosen $c$. □

Now we are ready to prove Theorem 2.3.3.

*Proof of Theorem 2.3.3.* If $k = 1$, then $K(n,k)$ is the complete graph $K_n$ and $\chi_{k'}(K(n,1)) = nk'$ for any $k'$. Hence Theorem 2.3.3 is true for $k = 1$.

Now assume $k \geq 2$. Take $n \geq n_0(k)$, with $n_0(k)$ as in Lemma 2.3.5, and assume there is a proper $(x, qk-r)$-colouring of $K(n,k)$ for some $x \leq qn-2r$. We will prove there is a proper $(x-q, qk-r)$-colouring of $K(n-1,k)$, which shows $\chi_{qk-r}(K(n-1,k)) \leq \chi_{qk-r}(K(n,k)) - q$.

We first claim there are at least $(q-1)n+1$ trivial colour classes in the $(x, qk-r)$-colouring of $K(n,k)$. If this is not the case, then there are at most $(q-1)n$ colour classes that appear on more than $\alpha^*(K(n,k))$ vertices. Hence counting the total number of appearance of each vertex in all colour classes, we have

$$\begin{aligned} (qk-r)\binom{n}{k} &\leq (q-1)n\alpha(K(n,k)) + (x-(q-1)n)\alpha^*(K(n,k)) \\ &\leq (q-1)n\alpha(K(n,k)) + (qn-2r-(q-1)n)\alpha^*(K(n,k)), \end{aligned} \tag{2.13}$$

since each vertex appear in exactly $qk-r$ colour classes.

Rearranging (2.13) leads to $\dfrac{\alpha^*(K(n,k))}{\alpha(K(n,k))} \geq \dfrac{n(k-r)}{k(n-2r)} = \dfrac{n}{2k}\left(1 - \dfrac{n-2k}{n-2r}\right) \geq \dfrac{n}{(n-2k+2)k}$, which contradicts Lemma 2.3.5.

Hence there are at least $(q-1)n+1$ trivial colour classes in the $(x, qk-r)$-colouring, where each trivial colour class is a subset of $\left\{F \in \binom{[n]}{k} \;\middle|\; i \in F\right\}$ for some $i \in [n]$. Therefore there is some $i^* \in [n]$ such that at least $q$ trivial colour classes are subsets of $\left\{F \in \binom{[n]}{k} \;\middle|\; i^* \in F\right\}$. Without loss of generality assume $i^* = n$. Removing those $q$ trivial colour classes, we obtain a $(x-q, qk-r)$-colouring of of $K(n-1,k)$, as required. □

## 2.4 Concluding Remarks

Our results are mainly of the following two types.

We present a simple proof of and extend known cases, using the order of maximum independent set in Kneser graphs. As discussed in Section 2.3, this simple bound is surprisingly strong.

We also show that for each $k$, at most $k^3 - k^2$ values of $\chi_{k'}(n,k)$ need to be examined to either prove the conjecture for $k$ or find a counterexample. As mentioned in Section 2.1, for fixed $k$, larger $n$ yield smaller $q_0(n,k)$. For instance:

(a) if $n \geq ck - 1$ for some constant $c > 2$, then we have
$$q_0(n,k) < \left(\left(\frac{c-1}{c-2}\right)^{k-1} - 1\right)(n-2k) + 1;$$
(b) if $n \geq k^2 + k - 1$, then $q_0(n,k) < (\mathrm{e}-1)(n-2k) + \dfrac{1}{k+1}$;

(c) if $n \geq k^3 + k - 1$, then $q_0(n,k) < \dfrac{n}{k} - 1$.

Note that it is likely these upper bounds for $n_0(k)$ and $q_0(n,k)$ can be improved, which may require some more involved calculations.

On the other hand, if the conjecture is false for some $n, q, k, r$, then we can find more cases that the conjecture is false by Theorems 2.3.3 and 2.3.4.

**Corollary 2.4.1.**
*If for some $n, q, k, r$, we have $\chi_{qk-r}(K(n,k)) \leq qn - 2r - 1$, then for any $r'$ such that $r \leq r' \leq k-1$ and $q' \geq q + r' - r$, we have $\chi_{q'k-r'}(K(n,k)) \leq q'n - 2r' - 1$.*

Csorba and Osztényi [7] proved that the topological lower bounds used by Lovász to prove the Kneser Conjecture cannot be used (immediately) to prove Stahl's conjecture. They proved that the topological lower bound only implies $\chi_{k'}(K(n,k)) \geq k' \left\lfloor \frac{n}{k} \right\rfloor$ for $k' \geq \binom{n}{k}$. With our Theorem 1.1.2, for any fixed $k$ and $n$, we only need to determine one $\chi_{qk-r}(K(n,k))$ with $q > q_0(n,k)$ to either prove or disprove the conjecture, and $q_0(n,k)$ is reasonably small if $n$ is large comparing to $k$. (E.g. we have just seen that if $n \geq k^3 + k - 1$, then $q_0(n,k) < \frac{n}{k} - 1$.) Hence it might still be possible to prove the conjecture for large enough $n$ for each $k$ using the same topological bounds.

On the other hand, although $q_0(n,k)$ decreases as $n$ increases, it never decreases to less then 2 for general $k$. Hence one can not hope to prove a 'large enough $n$' result using only Theorem 2.3.3.

We studied the question that for what pairs of $n'$ and $k'$, is there a homomorphism from $K(n,k)$ to $K(n',k')$? One may ask a similar question: for what pairs of $n'$ and $k'$, is there a homomorphism from $K(n,k)$ to $K(n',k')$ and a homomorphism from $K(n',k')$ to $K(n,k)$ at the same time?

This question is not hard to answer. For simplicity, we write $G \to H$ if there is a graph homomorphism from $G$ to $H$.

**Theorem 2.4.2.**
*For any integers $k$ and $n \geq 2k$, we have $K(n,k) \to K(n',k')$ and $K(n',k') \to K(n,k)$ at the same time if and only if*
(1) $n = n'$ *and* $k = k'$, *or*
(2) $\dfrac{n}{k} = \dfrac{n'}{k'} = 2$.

*Proof.* If $n = n'$ and $k = k'$, then clearly $K(n,k) \to K(n',k')$ and $K(n',k') \to K(n,k)$. Also if $\dfrac{n}{k} = \dfrac{n'}{k'} = 2$, then both $K(n,k)$ and $K(n',k')$ are bipartite, and we can easily map any bipartite graph to a single edge, by mapping all vertices in one part to one endvertex of that edge, and all vertices in the other part to the other endvertex.

Now assume that we have both $K(n,k) \to K(n',k')$ and $K(n',k') \to K(n,k)$, but (2) does not hold. This means that $K(n,k)$ is $(n',k')$-colourable, and hence $\dfrac{n'}{k'} \geq \chi_f(K(n,k)) = \dfrac{n}{k}$. Similarly, we find $\dfrac{n}{k} \geq \chi_f(K(n',k')) = \dfrac{n'}{k'}$. So we get that $\dfrac{n'}{k'} = \dfrac{n}{k} > 2$ (since we assumed (2) does not hold). We can write this as $n' = \dfrac{nk'}{k} > 2k'$ and $n = \dfrac{n'k}{k'} > 2k$. Then the second statement of Lemma 2.3.1 gives that $k'$ is a multiple of $k$ and $k$ is a multiple of $k'$. That must mean $k = k'$, and hence also $n = n'$, so (1) holds. $\square$

# 3

# $n$-Colourable Induced Subgraphs

## 3.1 Introduction

We study partial colouring problems in this and the next chapters. Those questions are inspired by a partial colouring conjecture in list colouring. Graphs in this chapter are undirected and without multiple edges nor loops.

Recall that in list colouring, each vertex of a graph is allocated with a list of colours that can be used for that vertex. A *proper* list colouring assigns a colour to each vertex from its list, such that adjacent vertices receive different colours. A graph $G$ is *$n$-choosable*, if for each assignment of lists of length $n$ to each vertex, a proper list colouring exists. And the *list chromatic number* $\chi_\ell(G)$ is the smallest $n$ for which $G$ is $n$-choosable.

In 2000, Albertson, Grossman, and Haas [2] asked the following question: given a graph $G$ with list chromatic number $n$, if each vertex has been assigned a list with $n'$ colours, $1 \leq n' \leq n$, can we always properly colour at least $\frac{n'}{n}|V(G)|$ vertices of $G$?

This problem is trivial for ordinary colouring. Recall that a graph $G$ is *$n$-colourable* if there is a proper colouring of (the vertices of) $G$ using $n$ colours. If a graph $G$ is $n$-colourable, then we can always partition the graph into $n$

independent sets by using the $n$ colour sets. If we only have $n' \leq n$ colours available, we can always colour at least $\frac{n'}{n}|V(G)|$ vertices by choosing the largest $n'$ independent sets. In general, it is not possible to obtain a better lower bound than $\frac{n'}{n}|V(G)|$, as can be seen by, for example, considering complete graphs.

For the original question on list colouring, Chappell [5] proved that if $G$ is $n$-choosable and $n' \leq n$, then at least $\frac{6}{7} \cdot \frac{n'}{n}|V(G)|$ vertices can always be properly coloured if every vertex has a list of $n'$ colours. The conjecture of Albertson, Grossmann, and Haas has been studied extensively, see e.g. [25, 31, 32], and is still open.

In this and the next chapters, we study the partial colouring problem associated with other variants of graph colouring *multi-colouring* and *fractional colouring.*

Recall that a graph $G$ is $(n, k)$*-colourable* if there is a proper multi-colouring by assigning $k$-subsets of $[n]$ to the vertices of $G$. And the $k$*-th multi-chromatic number* $\chi_k(G)$ is the smallest $n$ such that $G$ is $(n, k)$-colourable. Also recall that a graph $G$ is *fractional-*$\frac{n}{k}$*-colourable* if $G$ is $(\ell n, \ell k)$-colourable for some integer $\ell \geq 1$. And the *fractional chromatic number* $\chi_f(G)$ is the infimum of $\frac{n}{k}$ so that $G$ is fractional-$\frac{n}{k}$-colourable.

It is natural to ask whether the question of Albertson et al. also holds in the setting of multiple colouring or fractional colouring. For instance, for any fractional-$r$-colourable graph $G$ and $s < r$, does it always have a fractional-$s$-colourable subgraph of at least $\frac{s}{r}|V(G)|$ vertices? We show that this bound do not always hold. In Section 4.2.1 we will show that that if $s < r$, then there exists fractional-$r$-colourable graphs $G$ for which strictly less than $\frac{s}{r}|V(G)|$ vertices can be properly fractional-$s$-coloured. Similar properties are also observed on multiple colouring. Thus it is natural to ask the following questions.

**Question 3.1.1.**
(a) *Given a rational number $r > 0$ and integer $n \geq 1$, for what real number $a$ can we guarantee that every fractional-$r$-colourable graph $G$ has an $n$-colourable induced subgraph with at least $a|V(G)|$ vertices?*

(b) *Given rational numbers $r, s > 0$, for what real number $b$ can we guarantee that every fractional-$r$-colourable graph $G$ has a fractional-$s$-colourable induced subgraph with at least $b|V(G)|$ vertices?*

(c) *Given positive integers $n, k, n', k'$, for what real number $c$ can we guarantee that every $(n,k)$-colourable graph $G$ has an $(n',k')$-colourable induced subgraph with at least $c|V(G)|$ vertices?*

We focus on Question 3.1.1 (a) in this chapter. Questions 3.1.1 (b) and (c) will be covered in the next chapter. We present the main results and some proofs in Section 3.2; all other proofs can be found in Section 3.3.

## 3.2 Results

Let $G$ be a graph, $n'$ a positive integer and $r$ a positive rational number. Recall the definition of $\gamma(G,n')$ and $\gamma(r,n')$ from the first chapter:

$$\gamma(G,n') = \max\left\{ \frac{|V(H)|}{|V(G)|} \;\middle|\; H \leqslant_i G,\ H \text{ is } n'\text{-colourable} \right\},$$
$$\gamma(r,n') = \inf\{\gamma(G,n') \mid \chi_f(G) = r\}.$$

(Here $H \leqslant_i G$ means that $H$ is an induced subgraph of $G$.)

Note that a graph $H$ is 1-colourable if and only if $H$ is edgeless. This means that we have for any graph $G$: $\gamma(G,1) = \frac{\alpha(G)}{|V(G)|}$, where $\alpha(G)$ denotes the independence number of $G$ (order of the maximum independent set of $G$). It is well known that $\frac{\alpha(G)}{|V(G)|} \geq \frac{1}{\chi(G)}$, but in fact we have the sharper inequality $\frac{\alpha(G)}{|V(G)|} \geq \frac{1}{\chi_f(G)}$. This inequality follows from the following argument. The fractional chromatic number $\chi_f(G)$ is the infimum of $\frac{n}{k}$ for which $G$ is $(n,k)$-colourable. If $G$ is $(n,k)$-colourable, then simple counting over all vertices and colour classes yields $k|V(G)| \leq n\alpha(G)$, i.e. $\frac{\alpha(G)}{|V(G)|} \geq \frac{1}{n/k}$ for this pair $n,k$. This gives that $\frac{\alpha(G)}{|V(G)|} \geq \frac{1}{\chi_f(G)}$.

This lower bound is also the best possible if only $\chi_f(G)$ is known. For instance, if $G$ is *vertex transitive*, then we have $\frac{\alpha(G)}{|V(G)|} = \frac{1}{\chi_f(G)}$ (see e.g. [43]).

(A graph $G$ is *vertex transitive* if for any two vertices $u, v \in V(H)$, there is an automorphism of $H$ that maps $u$ to $v$.) This means that for rational numbers $r = 1$ or $r \geq 2$, we have $\gamma(r, 1) = \frac{1}{r}$. (There is no graph with fractional chromatic number strictly between 1 and 2.)

Before discussing lower bounds, we note that $n' = 2$ is a special case, since $\gamma(r, 2) = \gamma_f(r, 2)$. This is an easy conclusion by a well-known and easy-to-prove fact: for a graph $H$, $\chi_f(H) = 2$ if and only if $\chi(H) = 2$. I.e. $H$ is $(2k, k)$-colourable for some positive integer $k$ if and only if $H$ is bipartite.

Later in this chapter, we will prove the following result.

**Theorem 3.2.1.**
(a) *If* $r = \frac{n}{k}$ *for some co-prime* $n$ *and* $k$ *and* $2 \leq n' < n - 2k + 2$*, then we have* $\gamma(r, n') = 1 - \left(1 - \frac{1}{r}\right)^{n'}$.
(b) *The bound in part* (a) *is not attained by any graph, i.e. for any fractional-*$r$*-colourable graph* $G$ *we have the strict inequality* $\gamma(G, n') > \gamma(r, n')$.

We start with showing the lower bound for $\gamma(G, n')$ with given $\chi_f(G) = r$ is always attained by a Kneser graph. Recall for $n \geq k \geq 1$, the *Kneser graph* $K(n, k)$ has the collection of all $k$-subsets of $[n]$ (denoted by $\binom{[n]}{k}$) as vertex set, and there is an edge between two vertices if and only if these two $k$-sets are disjoint. We will always assume $n \geq 2k$, as otherwise the graph is edgeless. It is well known and easy to prove (see e.g. [43]) that a graph $G$ is $(n, k)$-colourable if and only if there is a homomorphism from $G$ to $K(n, k)$.

To decide $\gamma(r, n')$ with given $r$, it suffices to only look at Kneser graphs with fractional chromatic number $r$, by the following theorem.

**Theorem 3.2.2.**
*If* $G$ *is* $(n, k)$*-colourable, then* $\gamma(G, n') \geq \gamma(K(n, k), n')$.

In fact, a more general statement can be proved for vertex transitive graphs using similar averaging argument. Then Theorem 3.2.2 is an easy corollary of the well-known fact that Kneser graphs are vertex transitive. We will only prove the following more general theorem.

**Theorem 3.2.3.**
*If there is a graph homomorphism from a graph $G$ to a vertex transitive graph $H$, then for any $n'$ we have $\gamma(G, n') \geq \gamma(H, n')$.*

Note that Theorem 3.2.3 reproduces Albertson and Collins' no-homomorphism Lemma in [1]. We will also generalise it in Section 4.2.

### 3.2.1 $n'$-Colourable Induced Subgraphs of $K(n, k)$

In this section, we study the maximum $n'$-colourable induced subgraphs of $K(n, k)$ in detail. For easy of reading, we will use the equivalent term $n'$-partite instead of $n'$-colourable. In particular, we give upper bounds on the order of such subgraph by structural and algebraic methods. We also give several lower bounds by constructing specific types of subgraphs.

The celebrated Erdős-Ko-Rado Theorem [14] states that, for any $n \geq 2k$, the maximum independent set of the Kneser graph $K(n, k)$ is of order $\binom{n-1}{k-1}$. Moreover, if $n \geq 2k + 1$, then the only such maximum independent set (up to permutations of the ground set $[n]$) is the maximal *trivial independent set* $\left\{F \in \binom{[n]}{k} \;\middle|\; 1 \in F\right\}$.

An $n'$-partite induced subgraph of $K(n, k)$ is *trivial* if there are at most $n'$ elements of $[n]$ so that each vertex in this subgraph contains some of those $n'$ elements. Natural candidates of a 'large' $n'$-partite induced subgraph in $K(n, k)$ will be the maximal *trivial families*: $\left\{F \in \binom{[n]}{k} \;\middle|\; F \cap [n'] \neq \emptyset\right\}$. Interestingly, those maximal trivial families are indeed the largest $n'$-partite induced subgraphs if $n$ is large enough (as a function of $k$ and $n'$). On the other hand, larger $n'$-partite induced subgraphs can always be constructed if $n$ is 'close' to $2k$, as we will discuss later in this section.

For simplicity of formulas, we denote $t(n, k, n') := \dfrac{\binom{n}{k} - \binom{n-n'}{k}}{\binom{n}{k}}$ as the relative size of a maximal trivial $n'$-partite induced subgraph of $K(n, k)$ compared to the whole graph. Then for any $n' \leq n - 2k + 1$ (because

$\chi(K(n,k)) = n - 2k + 2)$, we have

$$\gamma(K(n,k),n') \geq \frac{\binom{n}{k} - \binom{n-n'}{k}}{\binom{n}{k}} =: t(n,k,n').$$

Independent sets in Kneser graphs that are not trivial have also been studied. For a non-trivial independent set $H \leqslant_i K(n,k)$, assume 1 is the element that appears most among all the vertices. Since $H$ is not trivial, we can assume $[2, k+1]$ is a vertex in $H$. Hence a natural candidate of large non-trivial independent set is $\left\{F \in \binom{[n]}{k} \middle| 1 \in F, F \cap [2,k+1] \neq \emptyset\right\} \cup \{[2,k+1]\}$. This set family has $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ vertices. Hilton and Milner [28] showed that if $n > 2k$ and $H'$ is an independent set in $K(n,k)$ that is not trivial, then $|H'| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. They also showed that the maximum such independent sets is either the set we discussed, or another certain structure (in some edge cases). We will not discuss more details here.

In Section 3.3.1, we will prove the following theorem using the Erdős-Ko-Rado Theorem and the Hilton-Milner bound on non-trivial maximum independent set.

**Theorem 3.2.4.**
*For any $n' \geq 1$, $k \geq 2$ and $n \geq n_0(n',k) := \max\{k^2 + n', n'k\}$, we have $\gamma(K(n,k),n') = t(n,k,n')$.*

The main idea behind the proof of this result is that if $n$ is quadratic in $k$, then $t(n,k,n')$ is larger than the upper bound for an $n'$-partite induced subgraph of $K(n,k)$ that is not trivial.

Note that more careful calculations can give better bounds on $n$ than those in Theorem 3.2.4. For instance, it is not hard to prove:

- *if $n \geq \frac{1}{2}k^2 + k - 1$ and $k \geq 4$, then $\gamma(K(n,k),2) = t(n,k,2)$ (and hence $\gamma_f(K(n,k),2) = \gamma(K(n,k),2) = t(n,k,2)$); and*
- *if $n \geq \max\left\{\frac{1}{2}k^2 + k + 2, 3k\right\}$, then $\gamma(K(n,k),3) = t(n,k,3)$.*

But on the other hand, only using the Hilton-Milner Theorem cannot lead to a better than quadratic lower bound of $n_0(2,k)$. We will prove in Section 3.3.1 that for any constants $a$ and $\epsilon > 0$, the condition $n > ak^{2-\epsilon}$ is not sufficient to prove that the non-trivial bipartite upper bound is smaller than the trivial one, if only the Hilton-Milner Theorem is used.

So a natural next question is: can we get a better-than-quadratic $n_0(2,k)$? The answer is yes, but we need to use more information about the structure of Kneser graphs than just their independent sets.

For instance we have the following.

**Theorem 3.2.5** (Frankl and Füredi [18]).
*If $n \geq \frac{1}{2}(3+\sqrt{5})k$, then $\gamma(K(n,k),2) = t(n,k,2)$.*

This result means that if a graph $G$ is $(n,k)$-colourable, for some $n,k$ with $n \geq \frac{1}{2}(3+\sqrt{5})k$, and this is the only information we know, then $\gamma(G,2) \geq t(n,k,2)$ is the best possible lower bound. This also means $\gamma(r,2) = \inf\left\{t(n,k,2) \,\middle|\, \frac{n}{k} = r\right\}$ for rational $r \geq \frac{1}{2}(3+\sqrt{5})$.

Note that in the same paper, Frankl and Füredi also claimed that if $n = 2k + c\sqrt{k}$, then $\gamma(K(n,k),n') \leq (1+c^{-4})t(n,k,2)$ as Theorem 1; but there are errors in their proof, which cannot be easily fixed. Their Theorem 1 was proved using Lemma 4 from the same paper, in which they claimed that if $n < 2k + \frac{k}{2}$, then the order of $\mathscr{A} := \left\{F \in \binom{[n]}{k} \,\middle|\, |F \cap [1,2i]| \geq i \text{ for some } i \geq \frac{n}{4}\right\}$ is bounded above by some small multiple of $\binom{n}{k}$. This upper bound and definition of $\mathscr{A}$ are both critical in their later proof of Theorem 1 (in the sense that their counting technique in Theorem 1 requires the definition of $\mathscr{A}$ to be exactly the same as they claimed). But in the proof of Lemma 4, they instead counted $\left\{F \in \binom{[n]}{k} \,\middle|\, \min\{i : |F \cap [1,2i]| \geq i\} \geq \frac{n}{4}\right\}$, which missed all sets $F$ such that $\min\{i : |F \cap [1,2i]| \geq i\} < \frac{n}{4}$ but still satisfies $|F \cap [1,2i]| \geq i$ for some $i \geq \frac{n}{4}$.

For those $(n,k)$, $n \geq 2k$, that are not covered by Theorem 3.2.5, we can prove the upper bound in the theorem below. We use a 'shifting method' to 're-structure' any given induced bipartite subgraph of a Kneser graph, without reducing its order. This method allows us to assume some structure of a maximum induced bipartite subgraph, and hence to analyse its order. Unfortunately, this method does not apply to $n'$-partite subgraphs with

$n' \geq 3$ in the most obvious way. We will discuss this method further in Section 3.3.2.

**Theorem 3.2.6.**
*If $n = ck$ where $c < \frac{1}{2}(3+\sqrt{5})$, then there exists a $f(c,k) \leq \frac{4}{3}$ such that $\gamma(K(n,k),2) \leq f(c,k)t(n,k,2)$.*

We present and discuss the explicit formula of $f(c,k)$ in Section 3.3.2. Here we just note that for fixed $c < \frac{1}{2}(3+\sqrt{5})$, we have

$$\lim_{k\to\infty} f(c,k) = \min\left\{\frac{c^2}{c^3-3c^2+4c-1}, \frac{-c^3+4c^2-2c}{2c-1}\right\}.$$

Also note that $\dfrac{c^2}{c^3-3c^2+4c-1} = \dfrac{-c^3+4c^2-2c}{2c-1} = 1$ if $c = \frac{1}{2}(3+\sqrt{5})$, which agrees with Frankl and Füredi's result if $n = \frac{1}{2}(3+\sqrt{5})k$.

One may next ask the question that whether a smaller bound on $n_0(n',k)$ is possible. I.e. does there exist $n_0(n',k)$ smaller than $\frac{1}{2}(3+\sqrt{5})k$ such that if $n \geq n_0(n',k)$, then the largest $n'$-partite induced subgraph of $K(n,k)$ is trivial? This seems to be beyond the scope of classic structural methods. Recently, Ellis and Lifshitz [10] use an influence-based method to show that the largest $n'$-partite induced subgraph of $K(n,k)$ is trivial if $n \geq 2k + c(n')k^{2/3}$, where $c(n')$ is a large constant depending only on $n'$. The fact that $c(n')$ only depends on $n'$ allows us to conclude the following.

**Theorem 3.2.7.**
*If $\chi_f(G) = \dfrac{n}{k}$, where $n$ and $k$ are co-prime, then for $2 \leq n' \leq n-2k+1$, $\gamma(G,n') > 1 - \left(1-\dfrac{k}{n}\right)^{n'}$ is the best possible lower bound.*

*In particular, $\gamma\left(\dfrac{n}{k}, n'\right) = 1 - \left(1-\dfrac{k}{n}\right)^{n'}$ for $n,k,n'$ as above.*

We have seen how structural bounds on the maximum $n'$-partite induced subgraph of $K(n,k)$ behave. We can also study the maximum $n'$-partite induced subgraph problem using algebraic methods. We describe our ideas and results in what follows.

In order to be able to use algebraic methods, we first turn the maximum $n'$-partite induced subgraph problem into a maximum independent set problem. For two graphs $G$ and $H$, the *Cartesian product* $G \,\square\, H$ has vertex set $\{(u, v) \mid u \in G, v \in H\}$, and $(u_1, v_1)(u_2, v_2)$ is an edge if and only if $u_1 = u_2$ and $v_1v_2 \in E(H)$, or $v_1 = v_2$ and $u_1u_2 \in E(G)$. For an integer $n'$, there exists an obvious one-to-one correspondence between an $n'$-colourable subgraph of $G$ and an independent set in $G \,\square\, K_{n'}$.

There exist a well-known algebraic upper bound of the maximum size independent set of a graph $G$: The independence number is at most the minimum of the number of non-negative eigenvalues and the number of non-positive eigenvalues of any weighted adjacency matrix of $G$. The *adjacency matrix* $A(G)$ is the $|V(G)| \times |V(G)|$ matrix with rows and columns labelled by vertices, where $(A(G))_{ij} = 1$ if $ij$ is an edge and $(A(G))_{ij} = 0$ otherwise. A *weighted adjacency matrix* of $G$ is a matrix obtained by replacing any of the entries 1 (i.e. any component $ij$ corresponding to an edge) by a real number. See e.g. [24] for information on this approach. This method is known as the 'inertia method' or the 'inertia bound' in the literature.

Eigenvalues of adjacency matrices of Kneser graphs are well-studied. This allows us to find the eigenvalues of certain weighted adjacency matrices of the Cartesian product $K(n, k) \,\square\, K_{n'})$, and hence to find upper bounds on their independence numbers. Note a general formula for general weighted adjacency matrices is more or less impossible to find.

The weighted adjacency matrix of $K(n, k) \,\square\, K_{n'}$ we will consider is the matrix $A(K(n, k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})$, where $\beta$ is any real constant. Here $\otimes$ denotes the tensor product of matrices. Hence $A(K(n, k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})$ is just the weighted adjacency matrix of $K(n, k) \,\square\, K_{n'}$, by assigning weight 1 to edges generated by edges from $K(n, k)$, and weight $\beta$ to edges generated by edges from $K_{n'}$.

For $n' = 2$, by finding the value of $\beta$ in $A(K(n, k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2)$ that gives the best inertia bound, we get the following upper bound for the order of induced bipartite subgraphs of $K(n, k)$. We will later discuss that in a certain sense this bound is a 'pretty good' upper bound for $n$ 'close' to $2k$. The structural bound in Theorem 3.2.6 is better for $n$ a bit further away from $2k$.

**Theorem 3.2.8.**
*If $H \leqslant_i K(n,k)$ is bipartite, then* $|V(H)| \leq \binom{n}{k} - \binom{n-1}{i+1} + \binom{n-1}{i}$, *where* $i = \min\left\{k-1, \left\lfloor \frac{n-1-\sqrt{n+1}}{2} \right\rfloor\right\}$.

More upper bounds for $n' = 2$, when $n$ is close to $2k$ are studied and compared in Section 3.4.1, including a linear programming formulation that strengthens our structural upper bound. We also looked at other existing algebraic methods to bound the independence number, such as Hoffman's bound [29] and the method developed by Wilson [47]. We did not manage to use these methods, though, to give better upper bounds then those mentioned in this section.

At this point we have seen various upper bounds for the order of an induced $n'$-partite subgraph of $K(n,k)$. The next question is whether we have better lower bounds than the trivial one. That is, whether we have a construction of induced $n'$-partite subgraphs that is larger than the trivial.

The answer is yes. If $n - 2k = o(\sqrt{k})$, then we will present a construction showing $\gamma(K(n,k), n') > t(n,k,n')$ for large enough $k$. I.e. if we have a function $f(k)$ such that $f(k) = o(\sqrt{k})$, then for large enough $k$ and $n = 2k + f(k)$ we can always find $n'$-partite induced subgraphs of $K(n,k)$ that are larger than the trivial $n'$-partite induced subgraph induced by $\left\{F \in \binom{[n]}{k} \;\middle|\; F \cap [n'] \neq \emptyset\right\}$. In addition, for $n' = 2$ we can require $f(k) \geq 1$ if $k \geq 3$. In particular this means that we can give larger than trivial bipartite subgraphs of $K(n,k)$ if $n = 2k+1$, for all $k \geq 3$.

We now describe our construction. If $n = 2k$, then the whole Kneser graph is bipartite. If $n > 2k$, we consider the subgraph of $K(n,k)$ induced by the following vertex sets, based on the parity of $k$.

**Example 3.2.9.**
*If $k = 2t+1$ is an odd positive integer and $n \geq 2k+1$, then we take*

$$\begin{aligned}\mathcal{F}_{odd}(n,k) := &\left\{F \in \binom{[n]}{k} \;\middle|\; |F \cap [k]| \geq \tfrac{1}{2}(k+1)\right\}\\ &\cup \left\{F \in \binom{[n]}{k} \;\middle|\; |F \cap [n-k+1,n]| \geq \tfrac{1}{2}(k+1)\right\},\end{aligned}$$

*which has order* $|\mathcal{F}_{odd}(n,k)| = \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{k+j-1}{t}\binom{n-k-j}{t}$.

*If* $k = 2t$ *is an even positive integer and* $n \geq 2k+1$, *then we take*

$$\begin{aligned}\mathcal{F}_{even}(n,k) := &\left\{F \in \binom{[n]}{k} \,\middle|\, |F \cap [k-1]| \geq \tfrac{1}{2}k\right\}\\ &\cup \left\{F \in \binom{[n]}{k} \,\middle|\, |F \cap [n-k,n]| \geq \tfrac{1}{2}k+1\right\},\end{aligned}$$

*which has order* $|\mathcal{F}_{even}(n,k)| = \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{k+j}{t}\binom{n-k-1-j}{t-1}$.

We will prove the following theorem in Section 3.3.4, which shows that our examples above are larger than trivial bipartite subgraphs in the specific range.

**Theorem 3.2.10.**
*For any* $k \geq 3$, *there exists* $T \geq 1$ *such that if* $2k+1 \leq n \leq 2k+T$, *then the* Example 3.2.9 are larger than the trivial bipartite induced subgraphs of $K(n,k)$. Moreover, this $T$ can be arbitrary large if $k$ is large enough.

It is worth noting that even for $n'$-partite induced subgraphs with $n' > 2$, the same examples are still better than trivial for some values of $n$ and $k$.

**Theorem 3.2.11.**
*If* $n - 2k = o(\sqrt{k})$, *then for sufficiently large* $k$, *the subgraphs induced by the vertex sets in* Example 3.2.9 are larger than the trivial $n'$-partite induced subgraphs in $K(n,k)$.

Note that Frankl and Füredi [18] also give a better than trivial construction of bipartite induced subgraphs, but our construction in Example 3.2.9 is always marginally better than theirs. More precisely, our subgraphs are always larger, but if we divide the sizes of the subgraphs by the size of trivial $n'$-partite subgraphs, then asymptotically they behave the same. We will discuss this in details in Section 3.3.4.

It is clear that the above constructions give a lower bound for $\gamma(K(n,k),n')$. Interestingly, both the lower bound and the upper bounds we studied have the property that if $n = 2k + o(\sqrt{k})$, then they approach to $\frac{4}{3}t(n,k,2)$ as $k$ goes to infinity. But in fact, this already is the case for most of the earlier known lower and upper bounds. When we consider the case that $n = 2k + \Omega(\sqrt{k})$, we see some more variation among the different bounds. This will be discussed in Section 3.4.1.

### 3.2.2 Inertia Bounds for $\gamma(K(n,k),n')$

Theorem 3.2.8 provides an algebraic upper bound for $\gamma(K(n,k),2)$ that is also known as the inertia bound. This method generalises to $\gamma(K(n,k),n')$ for any integer $n' \geq 2$, as we will discuss in this subsection. However, as we will see, this general upper bound is not very powerful for larger $n'$, if we use the same edge weights as we did to obtain Theorem 3.2.8.

It is considerably more involved to analyse the cases with more different edge weights. We went a bit further for the case $n' = 3$, i.e. considering the graph $K(n,k) \,\square\, K_3$, assigning weight 1 to edges corresponding to edges from $K(n,k)$ and three different weights to the three edges corresponding to the copies of $K_3$. But we can show that does not lead to a better upper bound than just using one edge weight. Computational experiments indicate that this may also hold for larger $n'$ (i.e. that using multiple edge weights on the $K_{n'}$ part does not lead to a better upper bound). For space reasons, we will not discuss this further.

**Theorem 3.2.12.**
*For any $n' \geq 3$ and $n \geq 2k + n' - 1$ (so $K(n,k)$ is not $n'$-colourable), an $n'$-colourable subgraph in $K(n,k)$ has order at most*

$$\min\left\{\binom{n}{k}, \binom{n}{k} - \binom{n-1}{i+1} + (n'-1)\binom{n-1}{i}\right\},$$

*where* $i = \min\left\{k-1, \max\left\{\left\lfloor\frac{n'k-n}{n'-2}\right\rfloor,\right.\right.$
$$\left.\left.\left\lfloor\frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'}\right\rfloor\right\}\right\}.$$

*Furthermore, this is the best possible inertia bound in the following cases:*
(a) *if we only apply one type of edge weight in $K_{n'}$, i.e. using the eigenvalues*

*of* $A(K(n,k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})$*;*
(b) *if* $n' = 3$ *and we use three different edge weights on edges corresponding to the copies of* $K_3$ *in* $K(n,k) \,\square\, K_3$*.*

We prove Theorem 3.2.12 in Section 3.3.3. In the remainder of this subsection we will discuss the limitations of this bound.

Since we assume $n \geq 2k + n' - 1$ (otherwise the whole graph $K(n,k)$ is $n'$-colourbale), it is clear that we can colour at most $\binom{n}{k} - (n - 2k + 2 - n')$ vertices properly with $n'$ colours. (Otherwise we will have a proper colouring of the whole graph with fewer than $n-2k+2$ colours, contradicting the Kneser Theorem proved by Lovász.) Hence we consider the bound in Theorem 3.2.12 as *useful* if and only if the bound is strictly less than $\binom{n}{k}$.

We will show that if $n \leq \dfrac{n'^2k}{2n'-2}$, then Theorem 3.2.12 does not provide a useful bound. Since $\dfrac{n'^2}{2n'-2} = \dfrac{n'+1}{2} + \dfrac{1}{2n'-2} \geq \dfrac{9}{4}$ if $n' \geq 3$, this leaves only a limited range of $n$ with a useful inertia upper bound for each $k$. On the other hand, recall that Ellis and Lifshitz [10] proved that the largest $n'$-partite induced subgraph of $K(n,k)$ is trivial if $n \geq 2k + c(n')k^{2/3}$, where $c(n')$ is a large constant only depend on $n'$. Hence the inertia bound in Theorem 3.2.12 is only useful for those $n$ with $\dfrac{n'^2k}{2n'-2} < n < 2k + c(n')k^{2/3}$. For fixed $n'$, such $n$ exist for finitely many $k$ only, depending on the value of $c(n')$.

We now give the promised proof that if $n \leq \dfrac{n'^2k}{2n'-2}$, then Theorem 3.2.12 does not provide a useful bound.
As in the theorem, set
$$i^* = \min\left\{k-1, \max\left\{\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor, \left\lfloor \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2n + 4n'^2}}{4n'} \right\rfloor \right\}\right\}.$$
It is clear that our bound in the theorem is useful if and only if
$$-\binom{n-1}{i^*+1} + (n'-1)\binom{n-1}{i^*} < 0,$$
which simplifies to $n > n'(i^*+1)$. That is, the bound is useful if and only

- if $n > n'k$, or
- $n > n'\left(\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor + 1\right)$ and

$$n > n'\left(\left\lfloor \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'} \right\rfloor + 1\right).$$

Note that

$$\begin{aligned}
&n'\left(\left\lfloor \frac{(n'+2)n - 2n' - \sqrt{((s-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'} \right\rfloor + 1\right) \\
&\leq \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4} + n' \\
&\leq \frac{(n'+2)n + 2n' - \sqrt{((n'-2)n + (2n'+4))^2}}{4} \qquad \text{(since } n' \geq 3\text{)} \\
&= n - 1 < n.
\end{aligned}$$

Therefore, the bound in Theorem 3.2.12 is not useful if and only if $n \leq n'\left(\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor + 1\right)$ and $n \leq n'k$. We can simplify this by noting that if $n \leq n' \cdot \frac{n'k-n}{n'-2}$, then $n \leq n'\left(\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor + 1\right)$. On the other hand, by rearranging $n \leq n' \cdot \frac{n'k-n}{n'-2}$, we have $n \leq \frac{n'^2 k}{2n'-2} = \left(\frac{n'+1}{2} + \frac{1}{2n'-2}\right) k \leq n'k$, for all $n' \geq 3$. Thus we conclude that if $n \leq n'\left(\frac{n'k-n}{n'-2}\right)$, then Theorem 3.2.12 does not give a useful bound.

## 3.3 Proofs for the Results in this Chapter

In this section we prove the results presented in the previous section. We first prove Theorem 3.2.3.

**Theorem 3.2.3.**
*If there is a graph homomorphism from a graph $G$ to a vertex transitive graph $H$, then for any $n'$, we have $\gamma(G, n') \geq \gamma(H, n')$.*

*Proof.* Consider any vertex transitive graph $H$, and let $G$ be any graph that admits a homomorphism $\varphi : V(G) \to V(H)$ to $H$.

We will prove $\dfrac{\pi(G, K(n,k))}{|V(G)|} \geq \dfrac{\pi(H, K(n,k))}{|V(H)|}$ for any $n, k$. Recall that $\pi(G, K(n,k))$ denotes the order of a largest induced subgraph of $G$ that is $(n,k)$-colourable. This immediately implies the theorem, since $n'$-colourable is essentially $(n', 1)$-colourable.

For fixed $n, k$, denote $X_1, \ldots, X_t$ as all the maximum $(n,k)$-colourable induced subgraphs of $H$. For a vertex $v \in V(H)$, denote by $m(v)$ the number of $X_i$'s that contain $v$. For any two vertices $v_1, v_2 \in V(H)$, there is an automorphism of $H$ that maps $v_1$ to $v_2$, since $H$ is vertex transitive. In that automorphism, every $X_i$ that contains $v_1$ is mapped to a $X_j$ that contains $v_2$, and different $X_i$'s are mapped to different $X_j$'s, since the mapping is bijective. Hence $m(v_2) \geq m(v_1)$. Similarly we have $m(v_1) \geq m(v_2)$. This means that there is an $m$ such that $m(v) = m$ for all vertices $v \in V(H)$.

Then we have

$$\begin{aligned} t \cdot \pi(H, K(n,k)) = \sum_{i=1}^{t} |X_i| &= \sum_{i=1}^{t} \sum_{v \in V(H)} \mathbb{1}_{v \in X_i} \\ &= \sum_{v \in V(H)} \sum_{i=1}^{t} \mathbb{1}_{v \in X_i} = |V(H)| m, \end{aligned}$$

where $\mathbb{1}$ is the indicator function that returns 1 if the subscript property is true and 0 otherwise.

Since the preimage of any $(n,k)$-colourable graph is $(n,k)$-colourable, we also have

$$\begin{aligned} t \cdot \pi(G, K(n,k)) &\geq \sum_{i=1}^{t} |\varphi^{-1}(X_i)| \\ &= \sum_{i=1}^{t} \sum_{u \in V(G)} \mathbb{1}_{\varphi(u) \in X_i} = \sum_{i=1}^{t} \sum_{v \in V(H)} \sum_{u \in \varphi^{-1}(v)} \mathbb{1}_{\varphi(u) \in X_i} \\ &= \sum_{i=1}^{t} \sum_{v \in V(H)} \mathbb{1}_{v \in X_i} |\varphi^{-1}(v)|, \\ &\qquad \text{(since } v \in X_i \text{ if and only if } \varphi(u) \in X_i \text{ for any } u \in \varphi^{-1}(v)) \\ &= \sum_{v \in V(H)} \sum_{i=1}^{t} |\varphi^{-1}(v)| \mathbb{1}_{v \in X_i} = \sum_{v \in V(H)} \left( |\varphi^{-1}(v)| \left( \sum_{i=1}^{t} \mathbb{1}_{v \in X_i} \right) \right) \end{aligned}$$

$$= \sum_{v \in V(H)} |\varphi^{-1}(v)| m = |V(G)| m.$$

We can conclude $\dfrac{\pi(G, K(n,k))}{|V(G)|} \geq \dfrac{m}{t} = \dfrac{\pi(H, K(n,k))}{|V(H)|}$. $\square$

Then Theorem 3.2.2 is a simple corollary of the above, by the well-known fact that Kneser graphs are vertex transitive.

### 3.3.1 Proof of Theorem 3.2.4 and Strength of Upper Bounds only using Maximum Independent Sets

We prove Theorem 3.2.4 in this section, in which we analyse $\gamma(K(n,k), n')$ using maximum (non-trivial) independent sets of Kneser graphs.

**Theorem 3.2.4.**
*For any $\geq 1$, $k \geq 2$ and $n \geq n_0(n', k) := \max\{k^2 + n', n'k\}$, we have $\gamma(K(n,k), n') = t(n,k,n')$.*

*Proof.* Note this theorem essentially states that if $n \geq \max\{k^2 + n', n'k\}$, then the order of an $n'$-colourable induced subgraph of $K(n,k)$ is bounded above by the trivial $n'$-colourable induced subgraph.

This theorem is true for $n' = 1$ by the Erdős-Ko-Rado Theorem. We first prove the theorem for $n' = 2$ and apply induction on $n' \geq 2$.

For $n' = 2$, we consider each bipartite induced subgraph $H$ as an union of two independent sets $\mathscr{H}_1$ and $\mathscr{H}_2$. If both $\mathscr{H}_1$ and $\mathscr{H}_2$ are of the trivial kind, then clearly $|V(H)| \leq \binom{n}{k} - \binom{n-2}{k}$. If $\mathscr{H}_1$ is of the trivial kind, then by moving all vertices in $\mathscr{H}_2$ that agree with the trivial type of $\mathscr{H}_1$ into $\mathscr{H}_1$ (i.e. if all vertices in $\mathscr{H}_1$ contain element $x \in [n]$, then we move all vertices in $\mathscr{H}_2$ that contain $x$ into $\mathscr{H}_1$), we see the rest of $|\mathscr{H}_2|$ is at most the largest independent set in $K(n-1,k)$ and hence $|V(H)| \leq \binom{n}{k} - \binom{n-2}{k}$.

If neither $\mathscr{H}_1$ nor $\mathscr{H}_2$ is of the trivial kind, then by the Hilton-Milner Theorem, we have $|V(H)| \leq 2\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right)$. Direct calculation shows if $n \geq \max\{k^2+2, 2k\}$, then $2\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right) \leq$

$\binom{n}{k} - \binom{n-2}{k}$ and we are done.

Now assume this Theorem is true for all $n'$ that $2 \leq n' \leq N$. We will prove its correctness for $n' = N+1$.

Consider any $(N+1)$-colourable graph $H \leqslant_i K(n,k)$, where $n \geq \max\{k^2 + N+1, (N+1)k\}$. As $H$ is $(N+1)$-colourable, we can partition $H$ into $N+1$ independent sets $\mathscr{H}_1, \ldots, \mathscr{H}_{N+1}$. Denote $m$ (that $0 \leq m \leq N+1$) as the number of independent sets that is trivial (i.e. there is an element shared by all vertices in that independent set).

If $m = 0$, then we have $|V(H)| \leq (N+1)\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right)$ by the Hilton-Milner bound. Direct calculation shows if $n \geq \max\{k^2 + N+1, (N+1)k\}$, then $(N+1)\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right) \leq \binom{n}{k} - \binom{n-(N+1)}{k}$.

If $m \geq 1$, then without loss of generality, we can assume that $\mathscr{H}_i$ is trivial for $1 \leq i \leq m$; and for each trivial $\mathscr{H}_{i^*}$, the element shared by all vertices in $\mathscr{H}_{i^*}$ is $i^*$. Then similarly, we move all vertices $F \in H$ that contains $i$ for some $1 \leq i \leq m$ into $\mathscr{H}_i$, if that vertex is not already in one of $\mathscr{H}_1, \ldots, \mathscr{H}_m$. (If there are multiple $i \in F$ that $1 \leq i \leq m$, then it doesn't matter which $\mathscr{H}_i$ we move $F$ into.)

Let $H'$ be the graph induced by vertices in $H$ that do not contain any of $1, \ldots, m$. I.e. $V(H') = \{F \in V(H) | F \cap [m] = \emptyset\}$. Then we have $|V(H)| \leq \binom{n}{k} - \binom{n-m}{k} + |V(H')|$. Note $H'$ itself is an $(N+1-m)$-colourable induced subgraph of $K(n-m,k)$. Since $n \geq \max\{k^2+N+1, (N+1)k\}$, we have $n - m \geq \max\{k^2 + N + 1 - m, (N+1)k - m\} \geq \max\{k^2 + N + 1 - m, (N+1-m)k\}$. Hence by our inductive hypothesis, we have $|H'| \leq \binom{n-m}{k} - \binom{n-m-(N+1-m)}{k}$. Therefore $|V(H)| \leq \binom{n}{k} - \binom{n-m}{k} + \binom{n-m}{k} - \binom{n-(N+1)}{k} = \binom{n}{k} - \binom{n-(N+1)}{k}$.
This completes the inductive step. □

We mentioned in Section 3.2.1 that directly comparing independent set orders using Erdős-Ko-Rado Theorem and Hilton-Milner Theorem does not lead to better-than-quadratic lower bound of $n_0(2,k)$. This is proved by Lemma 3.3.1 and Theorem 3.3.2.

**Lemma 3.3.1.**
*If $H$ is a bipartite induced subgraph of $K(n,k)$ ($n > 2k$) that at least one part of $H$ is trivial, then $|V(H)| \le t(n,k,2)$. In general, for any bipartite $|H|$, we have $|V(H)| \le \max\left\{t(n,k,2), 2\binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-3} + 3\right\}$.*

**Theorem 3.3.2.**
*For any $a, \epsilon > 0$, there exists $k_0 = k_0(a,\epsilon)$ such that for any $k > k_0$, there are integers $n > ak^{2-\epsilon}$ that $\binom{n}{k} - \binom{n-2}{k} < 2\binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-3} + 3$.*

The first part of Lemma 3.3.1 is already proved in above. For completeness we still include its proof. The rest is proved using a more fine-grained version of the Hilton-Milner Theorem. Other than the bound we used in above, Hilton and Milner [28] also proved that if $n > 2k$ and $H$ is an independent set in $K(n,k)$ so that any $|V(H)|-t+1$ vertices ($t \ge 0$ and $k \ge \min\{3, t+1\}$) in $H$ do not share a common element, then $|H| \le \binom{n-1}{k-1} - \binom{n-k}{k-1} + n - k$ if $2 < k < t+2$, and $|H| \le \binom{n-1}{k-1} - \binom{n-k}{k-1} + \binom{n-k-t}{k-t-1} + t$ if $k \le 2$ or $k \ge n' + 2$.
(Note general non-trivial independent sets satisfy the above condition with $t = 1$, hence general non-trivial independent sets have order at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.) Similar results for $n'$-colourable induced subgraphs can also be studied, but we don't state them explicitly here.

*Short proof of Lemma 3.3.1.* Fix $n, k$ and let $H$ be a bipartite induced subgraph of $K(n,k)$. A bipartite subgraph is simply a union of two independent sets, denote $H = \mathcal{H}_1 \cup \mathcal{H}_2$ where both $\mathcal{H}_1$, $\mathcal{H}_2$ are independent sets in $K(n,k)$. By the Erdős-Ko-Rado Theorem, we have $|H| \le 2\binom{n-1}{k-1}$.
But simply taking two maximum independent set (the trivial kind) only gives

a bipartite subgraph of order $\binom{n-1}{k-1} + \binom{n-2}{k-1}$, since there are vertices belong to both independent sets. Also if $\mathscr{H}_1$ is of the trivial kind, then without loss of generality we assume all vertices in $\mathscr{H}_1$ contains element 1. Then by moving every vertex in $\mathscr{H}_2$ containing 1 to $\mathscr{H}_1$, the rest of $\mathscr{H}_2$ can at most be the maximum independent set in $K(n-1,k)$ and hence of order at most $\binom{n-2}{k-1}$.

Hence if $|V(H)| > \binom{n-1}{k-1} + \binom{n-2}{k-1}$, then it must be the case that both independent sets $\mathscr{H}_1$ and $\mathscr{H}_2$ are non-trivial. If all but one vertices in $\mathscr{H}_1$ share one common element and same for $\mathscr{H}_2$, then both $\mathscr{H}_1$ and $\mathscr{H}_2$ are the maximum non-trivial independent sets, and

$$|V(H)| \leq 2\left(\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1\right) - \text{`overlap'}.$$

With Hilton and Milner's results, we know those maximum non-trivial independent sets have a specific structure

$$\left\{F \in \binom{[n]}{k} \middle| 1 \in F, F \cap [2,k+1] \neq \emptyset\right\} \cup \{[2,k+1]\}$$

(or $\left\{F \in \binom{[n]}{3} \middle| F \cap [3] \geq 2\right\}$ if $k=3$). Hence we can determine that 'overlap' (i.e. vertices in both) will be at least $\binom{n-2}{k-2} - 2(k-1)$ if $n = 2k+1$, and $\binom{n-2}{k-2} - 2\binom{n-k-2}{k-2} + \binom{n-2k-2}{k-2}$ if $n \geq 2k+2$. (Here $k > 3$ is essential; It is possible to construct two maximum non-trivial independent set without overlap if $k=2$ or $k=3$.) This resulting order is always smaller than $\binom{n-1}{k-1} + \binom{n-2}{k-1}$ if $k > 3$.

If all but one vertices in $\mathscr{H}_1$ share one common element, and all but two vertices in $\mathscr{H}_2$ share one common element, then we have $|V(H)| \leq 2\binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-3} + 3$. (Here the 'overlap' still exists, but becomes harder to analysis, since there will be more cases on possible structure of $\mathscr{H}_2$. We avoid it here since this will not change our later results of $n_0(k)$ by order in $k$.) $\square$

*Proof of Theorem 3.3.2.* Denote LHS $= \binom{n}{k} - \binom{n-2}{k} = \binom{n-1}{k-1} +$

$\binom{n-2}{k-1}$ and RHS $= 2\binom{n-1}{k-1} - \binom{n-k}{k-1} - \binom{n-k-1}{k-1} + \binom{n-k-2}{k-3} + 3.$

Fix any $a, \epsilon > 0$ and let $n = ak^{2-\epsilon} + 2$. We will prove if $k$ is larger than some $k_0(a, \epsilon)$ (defined later), then we have LHS $-$ RHS $< 0$. We can assume $0 < \epsilon < 1$, since if the statement is true for some $\epsilon^*$, then it is true for any $\epsilon > \epsilon^*$.

Note LHS $-$ RHS $= (A(k) - 1)\binom{n-2}{k-2} - 3$, where

$$\begin{aligned}
A(k) &= \frac{\binom{n-k}{k-1} + \binom{n-k-1}{k-1} - \binom{n-k-2}{k-3}}{\binom{n-2}{k-2}} \\
&= \frac{(2n-k-3)(n-2k+1)}{(k-1)(n-2k+1)(n-k)(n-k-1)} \\
&\quad \cdot \frac{(n-k)!}{(k-2)!(n-2k)!} \cdot \frac{(n-k)!(k-2)!}{(n-2)!} \\
&= \frac{2n-k-3}{k-1} \cdot \frac{(n-k-2)!(n-k)!}{(n-2)!(n-2k)!} \\
&= \frac{2n-k-3}{k-1} \cdot \prod_{i=2}^{k-1} \left( \frac{n-k-i}{n-i} \right) \\
&\le \frac{2n-k-3}{k-1} \left( 1 - \frac{k}{n-2} \right)^{k-2} \\
&= \frac{2ak^{2-\epsilon} - k + 1}{k-1} \left( 1 - \frac{k}{ak^{2-\epsilon}} \right)^{k-2} \\
&= 2ak^{1-\epsilon} \left( 1 - \frac{1}{ak^{1-\epsilon}} \right)^{(k-2)^{1-\epsilon}(k-2)^{\epsilon}}
\end{aligned}$$

We will show $\lim_{k \to \infty} A(k) = 0$. Hence there exists positive integer $k_0 \in \mathbb{N}$ that if $k \ge k_0$, then $A(k) < \frac{1}{2}$ and therefore LHS $-$ RHS $< 0$.

Since $0 < \epsilon < 1$, it is easy to prove that $\lim_{k \to \infty} \left(1 - \frac{1}{ak^{1-\epsilon}}\right)^{(k-2)^{1-\epsilon}} = \mathrm{e}^{-1/a}$. Hence there is $k_1 \in \mathbb{N}$ that if $k > k_1$, then

$$-\frac{1}{a} - \frac{1}{2a} < \ln \left( 1 - \frac{1}{ak^{1-\epsilon}} \right)^{(k-2)^{1-\epsilon}} < -\frac{1}{a} + \frac{1}{2a}.$$

That is, if $k > k_1$, then $0 < A(k) < 2a \exp\left( (1-\epsilon)\ln k - \frac{1}{2a}(k-2)^{\epsilon} \right)$. Here the right-most formula goes to 0 as $k$ goes to infinity. Thus $A(k)$ goes to 0 as $k$ goes to infinity. $\square$

### 3.3.2 Proof of Theorems 3.2.6 and 3.2.7

Next, we prove Theorem 3.2.6. We will use more structural properties of Kneser graphs. For this part, we treat the vertex set of each induced subgraph in $K(n,k)$ as a set family (the family of sets that represented by each vertex in the induced subgraph).

We need a bit more background to present our proof. A set family $\mathcal{F}$ is *intersecting* if for any $F_1, F_2 \in \mathcal{F}$, we have $F_1 \cap F_2 \neq \emptyset$. It is clear that a subgraph of $K(n,k)$ is independent if and only if the corresponding set family is intersecting. Also, each bipartite subgraph of $K(n,k)$ is induced by the union of two intersecting families, and union of any two intersecting families in $\binom{[n]}{k}$ induces a bipartite subgraph of $K(n,k)$. A set family $\mathcal{F}'$ is *$x$-intersecting* for some integer $x$ if for any $F_1, F_2 \in \mathcal{F}$, we have $|F_1 \cap F_2| \geq x$.

For two $k$-subsets $F_1, F_2$ of $[n]$, $F_1$ is *lexicographically smaller* than $F_2$ (denote by $F_1 \prec F_2$) if for each $i = 1, \ldots, k$, the $i$-th smallest element in $F_1$ is at most the $i$-th smallest element in $F_2$. I.e. if we order all the elements in $F_1$ from small to large, and order all the elements in $F_2$ from small to large, then for each $i = 1, \ldots, k$, the $i$-th element in $F_1$ is always at most the $i$-th element in $F_2$. Note it is possible that two $k$-sets are not comparable.

A set family $\mathcal{F} \subseteq \binom{[n]}{k}$ is *(left-)shifted* if for any $F \in \mathcal{F}$, and any $F' \in \binom{[n]}{k}$ that $F' \prec F$, we have $F' \in \mathcal{F}$.

Similarly, a set family $\mathcal{F} \subseteq \binom{[n]}{k}$ is *right-shifted* if for any $F \in \mathcal{F}$, and any $F' \in \binom{[n]}{k}$ that $F' \succ F$, we have $F' \in \mathcal{F}$. If the shifting direction is not specified, *left* is always assumed.

A bipartite subgraph of $K(n,k)$ is *maximal* if adding any vertex not in the subgraph will make the graph non-bipartite.

The following lemma allows us to assume some structural properties on our bipartite subgraph. Lemma 3.3.3 uses an idea similar to [18].

**Lemma 3.3.3.**
*If $\mathcal{F} \subseteq \binom{[n]}{k}$ induces a maximal bipartite subgraph in $K(n,k)$, then there exist left-shifted intersecting set family $\mathcal{F}_1$ and right-shifted intersecting set*

*family* $\mathcal{F}_2$ *in* $\binom{[n]}{k}$ *such that* $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$ *and* $|\mathcal{F}_1 \cup \mathcal{F}_2| = |\mathcal{F}|$.

*Proof of Lemma 3.3.3.* Consider any set family $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$ that induces a bipartite subgraph in $K(n,k)$, where both $\mathcal{F}_1$ and $\mathcal{F}_2$ are intersecting. We will rearrange and replace sets in them, so that the resulting $\mathcal{F}_1$ is left-shifted and the resulting $\mathcal{F}_2$ is right-shifted, while both of them are still intersecting.

First note the definition of 'shifted' is equivalent to the following:

- $\mathcal{H} \subseteq \binom{[n]}{k}$ is left-shifted if and only if for all $F \in \mathcal{H}$ and $j \in F$, for any $i < j$ that $i \notin F$, we have $(F \setminus \{j\}) \cup \{i\} \in \mathcal{H}$;
- $\mathcal{H} \subseteq \binom{[n]}{k}$ is right-shifted if and only if for all $F \in \mathcal{H}$ and $j \in F$, for any $i > j$ that $i \notin F$, we have $(F \setminus \{j\}) \cup \{i\} \in \mathcal{H}$.

We will apply a method called 'shifting operator' (defined in next paragraph) to move and update the sets. This method was used in [18], and [22] surveyed its single-sided version.

For any integers $i, j \in [n]$ that $i < j$, the left shifting operator

$$\mathcal{L}_{ij}(F) = \begin{cases} (F \setminus \{j\}) \cup \{i\} & \text{if } j \in F \text{ and } i \notin F \\ F & \textit{otherwise}; \end{cases}$$

and right shifting operator

$$\mathcal{R}_{ij}(F) = \begin{cases} (F \setminus \{i\}) \cup \{j\} & \text{if } i \in F \text{ and } j \notin F \\ F & \textit{otherwise}. \end{cases}$$

Finally, the shifting operator $\mathcal{S}_{ij}(\mathcal{F}_1, \mathcal{F}_2) = (\mathcal{F}_1', \mathcal{F}_2')$ that for any $F \in \mathcal{F}_1$:

- if $\mathcal{L}_{ij}(F) \in \mathcal{F}_2$, then put put $\mathcal{L}_{ij}(F)$ in $\mathcal{F}_1'$ and put $F$ in $\mathcal{F}_2'$;
- otherwise, just put $\mathcal{L}_{ij}(F)$ in $\mathcal{F}_1'$.

And for any $F \in \mathcal{F}_2$:

- if $\mathcal{R}_{ij}(F) \in \mathcal{F}_1$, then put put $\mathcal{R}_{ij}(F)$ in $\mathcal{F}_2'$ and put $F$ in $\mathcal{F}_1'$;

- otherwise, just put $\mathcal{R}_{ij}(F)$ in $\mathcal{F}_2'$.

It is not hard to verify that for any $i, j$, the shifting operator $\mathcal{S}_{ij}(\mathcal{F}_1, \mathcal{F}_2)$ keeps both $\mathcal{F}_1$ and $\mathcal{F}_2$ intersecting, and keeps $|\mathcal{F}_1| + |\mathcal{F}_2|$ unchanged.

We apply the operator on $\mathcal{F}_1, \mathcal{F}_2$ as long as they are not both shifted (to the correct direction). This process terminates, as each effective application (that changes or moves some sets) of the shifting operator makes some sets in $\mathcal{F}_1$ lexicographically smaller, or / and some sets in $\mathcal{F}_2$ lexicographically larger. There are only finitely many sets in $\mathcal{F}_1, \mathcal{F}_2$, and finitely many possible move to each set (so it become lexicographically larger or smaller, depending on which side it belongs to). □

Before we continue, note this method does not directly generalise to tripartite subgraphs. (Hence we don't have a tripartite version of this Theorem.)

We need another famous result regarding the 'shadow' of an intersecting family to prove Theorem 3.2.6. The following intersecting shadow theorem is proved by Kruskal [36] and Katona [33] independently. The theorem holds for more general cases, but we only include what is relevant to us here.

A set family $\mathcal{F}$ is *$t$-intersecting* if for any $F_1, F_2 \in \mathcal{F}$, we have $|F_1 \cap F_2| \geq t$.

**Theorem 3.3.4** (Kruskal-Katona [33, 36])**.**
*For a $t$-intersecting set family $\mathcal{F} \subseteq \binom{[n]}{k}$, denote $\Delta(\mathcal{F})$ as its 'shadow', i.e. $\Delta(\mathcal{F}) := \{H \subseteq F | F \in \mathcal{F}, |H| = k-1\}$. Then we have $\dfrac{|\Delta(\mathcal{F})|}{|\mathcal{F}|} \geq \dfrac{k}{k-t+1}$.*

Now we are ready to present our proof to Theorem 3.2.6.

**Theorem 3.2.6.**
*If $n = ck$ where $c < \dfrac{3+\sqrt{5}}{2}$, then $\gamma(K(n,k),2) \leq f(c,k)t(n,k,2)$ for some $f(c,k) \leq \dfrac{4}{3}$.*

*Proof of Theorem 3.2.6.* Let $\mathcal{F}$ be the vertex set of a bipartite subgraph in $K(n,k)$. By Lemma 3.3.3, we can assume $\mathcal{F} = \mathcal{F}_1 \sqcup \mathcal{F}_2$, where $\mathcal{F}_1$ is left-shifted and $\mathcal{F}_2$ is right-shifted. We can also assume $n \leq \dfrac{\sqrt{5}+3}{2}k$, since otherwise $\gamma(K(n,k),2) = t(n,k,2)$, proved by Frankl and Füredi [18].

We further partition $\mathcal{F}_1 = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$, and $\mathcal{F}_2 = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3$ by the following rules:

- $\mathcal{A}_1 := \{F \in \mathcal{F}_1 | 1 \in F\}$,
- $\mathcal{A}_2 := \{F \in \mathcal{F}_1 | 1 \notin F, n \notin F\}$,
- $\mathcal{A}_3 := \{F \in \mathcal{F}_1 | 1 \notin F, n \in F\} = \mathcal{F}_1 \setminus (\mathcal{A}_1 \cup \mathcal{A}_2)$;
- $\mathcal{B}_1 := \{F \in \mathcal{F}_2 | n \in F\}$,
- $\mathcal{B}_2 := \{F \in \mathcal{F}_2 | n \notin F, 1 \notin F\}$,
- $\mathcal{B}_3 := \{F \in \mathcal{F}_2 | n \notin F, 1 \in F\} = \mathcal{F}_2 \setminus (\mathcal{B}_1 \cup \mathcal{B}_2)$.

Consider an arbitrary $F \in \mathcal{A}_3$, and let $F'(j) := (F \setminus \{n\}) \cup \{j\}$, for some $j \in [2, n] \setminus F$, where $[2, n] = \{2, \ldots, n\}$. Since $\mathcal{F}_1$ is left-shifted, we must have $F'(j) \in \mathcal{A}_2$ for any $j \in [2, n] \setminus F$. Hence $(n-k-1)|\mathcal{A}_3| \leq k|\mathcal{A}_2|$. Similarly we have $(n-k-1)|\mathcal{B}_3| \leq k|\mathcal{B}_2|$, and therefore $|\mathcal{F}| \leq |\mathcal{A}_1| + |\mathcal{B}_1| + \left(1 + \frac{k}{n-k-1}\right)(|\mathcal{A}_2| + |\mathcal{B}_2|) = |\mathcal{A}_1| + |\mathcal{B}_1| + \frac{n-1}{n-k-1}(|\mathcal{A}_2| + |\mathcal{B}_2|)$.

**Claim.** Set family $\mathcal{A}_2$ is 2-intersecting.

*Proof.* Since $\mathcal{A}_2$ is a subset of $\mathcal{F}_1$, so it is intersecting. Assume there are two sets $F_1, F_2 \in \mathcal{A}_2$ that $|F_1 \cap F_2| = 1$. Denote $x$ as the only element in both $F_1$ and $F_2$. Since all sets in $\mathcal{A}_2$ do not contain 1, we have $x > 1$.

As $\mathcal{F}_1$ is left-shifted, and $F_1' := (F_1 \setminus \{x\}) \cup \{1\}$ is lexicographically smaller than $F_1$, so $F_1' \in \mathcal{F}_1$. But then $F_1' \cap F_2 = \emptyset$ contradicts to the assumption that $\mathcal{F}_1$ is intersecting. ∎

Note by the same reasons, set family $\mathcal{B}_2$ is also 2-intersecting.

Consider $\mathcal{A}_2' := \{[2, n-1] \setminus F | F \in \mathcal{A}_2\}$. Note $\mathcal{A}_2' \subseteq \binom{[2, n-1]}{n-k-2}$ and for any $F_1', F_2' \in \mathcal{A}_2'$, we have $F_1' = [2, n-1] \setminus F_1$ and $F_2' = [2, n-1] \setminus F_2$ for some $F_1$ and $F_2$. Hence $|F_1' \cap F_2'| = |[2, n-1] \setminus (F_1 \cup F_2)| \geq n-2-(2k-2) = n-2k$. I.e. $\mathcal{A}_2'$ is $(n-2k)$-intersecting.

Then by the intersecting shadow theorem (Theorem 3.3.4), we have

$$|\Delta(\mathcal{A}_2')| \geq \frac{n-k-2}{(n-k-2)-(n-2k)+1}|\mathcal{A}_2'| = \frac{n-k-2}{k-1}|\mathcal{A}_2|.$$

Note for any $F' \in \Delta(\mathcal{A}'_2)$, $F' \cup \{1\} \notin \mathcal{A}_1$: since by the definition of $\mathcal{A}'_2$, we have $F' \subseteq [2,n] \setminus F$ for some $F \in \mathcal{A}_2 \subseteq \mathcal{F}_1$; and hence $F \cup \{1\}$ is disjoint with $F$.

Denote $\mathcal{B}'_2$ similarly. ($\mathcal{B}'_2 := \{[2,n-1] \setminus F | F \in \mathcal{B}_2\}$.)

Hence $|\mathcal{A}_1| + |\Delta(\mathcal{A}'_2)| + |\mathcal{B}_1| + |\Delta(\mathcal{B}'_2)| \le \binom{n-1}{k-1} + \binom{n-2}{k-1}$. Therefore

$$|\mathcal{A}_1| + |\mathcal{B}_1| + \frac{n-k-2}{k-1}\left(|\mathcal{A}_2| + |\mathcal{B}_2|\right) \le \binom{n-1}{k-1} + \binom{n-2}{k-1}. \tag{3.1}$$

That is, $|\mathcal{A}_2|+|\mathcal{B}_2| \le \frac{k-1}{n-k-2}\left(\binom{n-1}{k-1} + \binom{n-2}{k-1} - |\mathcal{A}_1| - |\mathcal{B}_1|\right)$. And hence

$$\begin{aligned} |\mathcal{F}| \le{}& \frac{(n-1)(k-1)}{(n-k-1)(n-k-2)}\left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right) \\ &- \left(\frac{(n-1)(k-1)}{(n-k-1)(n-k-2)} - 1\right)(|\mathcal{A}_1| + |\mathcal{B}_1|). \end{aligned} \tag{3.2}$$

Note if $n \le \frac{3+\sqrt{5}}{2}k$, then $\frac{(n-1)(k-1)}{(n-k-1)(n-k-2)} > 1$.

On the other hand since $\mathcal{F}_1$ is left-shifted, we have $|\mathcal{A}_2 \cup \mathcal{A}_3| \le \frac{n-k}{k}|\mathcal{A}_1|$. (Which is true because for any set $F$ in $\mathcal{A}_2$ or $\mathcal{A}_3$, and for any $i \in F$, we have $(F \setminus \{i\}) \cup \{1\} \in \mathcal{A}_1$.) Hence we have

$$|\mathcal{F}| \le \frac{n}{k}(|\mathcal{A}_1| + |\mathcal{B}_2|). \tag{3.3}$$

Combine (3.2) and (3.3), we have

$$\begin{aligned} |\mathcal{F}| &\le \frac{n}{k} \cdot \frac{\frac{(n-1)(k-1)}{(n-k-1)(n-k-2)}\left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right)}{\frac{n}{k} + \frac{(n-1)(k-1)}{(n-k-1)(n-k-2)} - 1} \\ &= \frac{n(n-1)(k-1)\left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right)}{n^3 - (3k+3)n^2 + (4k^2+5k+2)n - (k^3+4k^2+k)}. \end{aligned} \tag{3.4}$$

Equivalently, we can rearrange (3.1) into the following,

$$|\mathcal{A}_1| + |\mathcal{B}_1| \le \binom{n-1}{k-1} + \binom{n-2}{k-1} - \frac{n-k-2}{k-1}\left(|\mathcal{A}_2| + |\mathcal{B}_2|\right). \tag{3.5}$$

And combine (3.2) and (3.5), we have

$$|\mathscr{F}| \leq \frac{(n-1)(-n^2+(4k-1)n-2k^2)}{k(k-1)(2n-k-1)} \cdot \left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right). \tag{3.6}$$

If we denote $c = \dfrac{n}{k}$ (note $2 \leq c \leq \dfrac{3+\sqrt{5}}{2}$), then we have the following since both (3.4) and (3.6) need to be satisfied:

$$|\mathscr{F}| \leq f(c,k) \cdot \left(\binom{n-1}{k-1} + \binom{n-2}{k-1}\right),$$

where

$$f(c,k) = \min\left\{\frac{c(ck-1)(k-1)}{c^3k^2 - 3c^2k(k+1) + c(4k^2+5k+2) - (k^2+4k+1)}, \frac{(ck-1)(-c^2k + c(4k-1) - 2k)}{(k-1)((2c-1)k-1)}\right\}.$$

Thus

$$\gamma(K(n,k),2) \leq f(c,k) \cdot \frac{\binom{n-1}{k-1} + \binom{n-2}{k-1}}{\binom{n}{k}} = f(c,k)t(n,k,2).$$

Note

$$\lim_{k\to\infty} f(c,k) = \min\left\{\frac{c^2}{c^3-3c^2+4c-1}, \frac{-c^3+4c^2-2c}{2c-1}\right\}.$$

$\square$

Then we prove Theorem 3.2.7, which provide the exact values of $\gamma(r,n')$ with certain ranges of $r$ and $n'$. This is a corollary following the theorem of Ellis and Lifshitz [10]: if $\mathscr{F}$ is an $n'$-partite subgraph of $K(n,k)$ where $n \geq 2k + c(n')k^{2/3}$ (for some $c(n')$ only depending on $n'$), then $|\mathscr{F}| \leq \binom{n}{k} - \binom{n-n'}{k}$.

**Theorem 3.2.7.**
*If $\chi_f(G) = \dfrac{n}{k}$ (with co-primes $n$, $k$) and this is the only information we have about $G$, then for $2 \leq n' \leq n-2k+1$, $\gamma(G,n') > 1 - \left(1-\dfrac{k}{n}\right)^{n'}$ is the best possible lower bound.*
*In particular, $\gamma\left(\dfrac{n}{k}, n'\right) = 1 - \left(1-\dfrac{k}{n}\right)^{n'}$ for $n'$ in the above range.*

*Proof.* Consider any graph $G$ that $\chi_f(G) = \frac{n}{k}$ with co-primes $n, k$. We will prove $\gamma(G, n') > 1 - \left(1 - \frac{k}{n}\right)^{n'}$, and furthermore, if $2 \leq n' \leq n - 2k + 1$, then this is the best possible lower bound in general. (I.e. the best possible lower bound that holds for all graphs with fractional chromatic number $\frac{n}{k}$.)

Since $\chi_f(G) = \frac{n}{k}$, we know $G$ is $(tn, tk)$-colourable for some integer $t$. If $n' > tn$, then clearly $\gamma(K(tn, tk), n') = 1$, so we can assume $n' \leq tn$ in the calculation. We have

$$\begin{aligned}
\gamma(G, n') \geq \gamma(K(tn, tk), n') &\geq \frac{\binom{tn}{tk} - \binom{tn-n'}{tk}}{\binom{tn}{tk}} \\
&= 1 - \prod_{i=0}^{n'-1} \frac{tn - tk - i}{tn - i} \\
&= 1 - \prod_{i=0}^{n'-1} \left(1 - \frac{k}{n - \frac{i}{t}}\right) \\
&> 1 - \left(1 - \frac{k}{n}\right)^{n'}.
\end{aligned}$$

From now on we consider the case that $n' \leq n - 2k + 1$. We will define a series of graphs with fractional chromatic number $\frac{n}{k}$ and showing the above lower bound is the best possible.

Recall that $c(n')$ (as in Ellis and Lifshitz's result) is a constant only depending on $n'$. Let $T_0 = \frac{c(n')^3 k^2}{(n-2k)^3}$, then for any $t > T_0$, we have $tn > 2tk + c(n')(tk)^{2/3}$. That is, for any $t > T_0$, we have $\gamma(K(tn, tk), n') = t(tn, tk, n') = 1 - \prod_{i=0}^{n'-1} \left(1 - \frac{k}{n - \frac{i}{t}}\right) > 1 - \left(1 - \frac{k}{n}\right)^{n'}$.

Note that the above lower bound is also a limit,

$$\text{i.e. } \lim_{t \to \infty} \left(1 - \prod_{i=0}^{n'-1} \left(1 - \frac{k}{n - \frac{i}{t}}\right)\right) = 1 - \left(1 - \frac{k}{n}\right)^{n'}.$$

Therefore we have $\inf \{\gamma(K(tn, tk), n') | t > T_0\} = 1 - \left(1 - \frac{k}{n}\right)^{n'}$.

Since the lower bound holds for all graphs with fractional chromatic number $\frac{n}{k}$, and by the fact that

$$\gamma\left(\frac{n}{k}, n'\right) = \inf\left\{\gamma(G, n')\middle|\chi_f(G) = \frac{n}{k}\right\} \leq \inf\left\{\gamma(K(tn, tk), n')\middle|t > T_0\right\},$$

we conclude that $\gamma\left(\frac{n}{k}, n'\right) = 1 - \left(1 - \frac{k}{n}\right)^{n'}$. □

### 3.3.3 Proofs of Theorems 3.2.8 and 3.2.12

In this section, we prove some algebraic upper bound of bipartite induced subgraphs of $K(n,k)$.

**Theorem 3.2.8.**
*If $H \leqslant_i K(n,k)$ is bipartite, then $|V(H)| \leq \binom{n}{k} - \binom{n-1}{i+1} + \binom{n-1}{i}$, where $i = \min\left\{k-1, \left\lfloor \frac{n-1-\sqrt{n+1}}{2} \right\rfloor\right\}$.*

*Proof of Theorem 3.2.8.* Denote $A(G)$ as the adjacency matrix of a graph $G$. It is well known that the eigenvalues of $A(K(n,k))$ are

$$(-1)^i\binom{n-k-i}{k-i} \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1},$$

in which for simplicity we use $\binom{n}{-1} = 0$ as a convention.
Next we claim the eigenvalues of $A(K(n,k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2)$ are just

$$(-1)^i\binom{n-k-i}{k-i} + \beta \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1},$$

and

$$(-1)^i\binom{n-k-i}{k-i} - \beta \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1}.$$

Note the eigenvalues of $A(K_2)$ are 1 and $-1$, with eigenvectors $\overrightarrow{v_1} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\overrightarrow{v_2} = \begin{bmatrix}1\\-1\end{bmatrix}$. For any eigenvector $\overrightarrow{w_i}$ of $A(K(n,k))$ that $A(k(n,k))\overrightarrow{w_i} = (-1)^i\binom{n-k-i}{k-i}$, it is not hard to verify that

$$(A(K(n,k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2))(\overrightarrow{w_i} \otimes \overrightarrow{v_1})$$

$$= \left( (-1)^i \binom{n-k-i}{k-i} + \beta \right) (\overrightarrow{w_i} \otimes \overrightarrow{v_1}),$$

and

$$\begin{aligned}
&(A(K(n,k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2))(\overrightarrow{w_i} \otimes \overrightarrow{v_2}) \\
&= \left( (-1)^i \binom{n-k-i}{k-i} - \beta \right) (\overrightarrow{w_i} \otimes \overrightarrow{v_2}).
\end{aligned}$$

Denote $E^+(\beta)$ as the number of non-negative eigenvalues in $A(K(n,k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2)$, and $E^-(\beta)$ as the number of non-positive eigenvalues in $A(K(n,k)) \otimes I_2 + \beta \cdot I_{\binom{n}{k}} \otimes A(K_2)$.

Since each independent set is an all-zero principle submatrix of the original weighted adjacency matrix, we have $|V(H)| \leq \alpha\left(K(n,k) \,\square\, K_2\right) \leq \min\{E^+(\beta), E^-(\beta)\}$ for any $\beta \in \mathbb{R}$ by the Cauchy Interlacing Theorem. (For more background of this, see e.g. [24].) We will find the $\beta$ that minimise $\min\{E^+(\beta), E^-(\beta)\}$ in the following.

Since we want to minimise $\min\{E^+(\beta), E^-(\beta)\}$, we only need to consider the cases that $\beta$ is not exactly equal to some eigenvalues of $A(K(n,k))$. And it suffices to only consider non-negative $\beta$ since the negative cases of $\beta$ is symmetric to the positive cases.

If $\binom{n-k-(i+1)}{k-(i+1)} < \beta < \binom{n-k-i}{k-i}$ for some $0 \leq i \leq k-1$, then

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{ \binom{n}{k} - \binom{n-1}{i+1} + \binom{n-1}{i}, \binom{n}{k} + \binom{n-1}{i+1} - \binom{n-1}{i} \right\} \\
&\quad + \binom{n}{k} - \binom{n-1}{i+1} + \binom{n-1}{i}.
\end{aligned}$$

Note whether $\min\{E^+(\beta), E^-(\beta)\}$ is $E^+(\beta)$ or $E^-(\beta)$ depends on the parity of $i$, but the minimum follows the same formula as above.

If $0 < \beta < \binom{n-k-k}{k-k} = 1$, then

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{ 2\binom{n-1}{k-1}, 2\binom{n-1}{k} \right\}
\end{aligned}$$

$$= 2\binom{n-1}{k-1} = \binom{n}{k} - \binom{n-1}{k} + \binom{n-1}{k-1},$$

which is the same as the case $i = k-1$.

And if $\beta > \binom{n-k}{k}$, then

$$\min\{E^+(\beta), E^-(\beta)\} = \binom{n}{k},$$

which is always larger then the earlier two cases.

That is, to decide the smallest $\min\{E^+(\beta), E^-(\beta)\}$, it suffices to find the $0 \le i \le k-1$ that maximise $\binom{n-1}{i+1} - \binom{n-1}{i}$.

A direct calculation shows for $1 \le i \le k-1$, we have $\binom{n-1}{i+1} - \binom{n-1}{i} > \binom{n-1}{i} - \binom{n-1}{i-1}$ if and only if $i < \frac{n-1-\sqrt{n+1}}{2}$ or $i > \frac{n-1+\sqrt{n+1}}{2}$. But since $1 \le i \le k-1$, we conclude that $\binom{n-1}{i+1} - \binom{n-1}{i}$ is increasing (that 'choosing $i$' is larger than 'choosing $i-1$') whenever $i < \frac{n-1-\sqrt{n+1}}{2}$. That is, $i = \min\left\{k-1, \left\lfloor \frac{n-1-\sqrt{n+1}}{2} \right\rfloor\right\}$ maximises $\binom{n-1}{i+1} - \binom{n-1}{i}$ for $1 \le i \le k-1$. $\square$

The proof of Theorem 3.2.12 is very similar to Theorem 3.2.8.

**Theorem 3.2.12.**
*For any $n' \ge 3$ and $n \ge 2k + n' - 1$ (so $K(n,k)$ is not $n'$-colourable), we have*

$$\pi(K(n,k), K_{n'}) \le \min\left\{\binom{n}{k}, \binom{n}{k} - \binom{n-1}{i^*+1} + (n'-1)\binom{n-1}{i^*}\right\},$$

*where* $i^* = \min\left\{k-1, \max\left\{\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor, \left\lfloor \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'} \right\rfloor\right\}\right\}.$

*Furthermore, for any $n' \ge 3$, this is the best possible inertia bound if we only apply one type of edge weight in $K_{n'}$; for $n' = 3$, this is the best possible inertia bound even if we allow three different edge weights on $K_3$.*

We first prove for any $n' \geq 3$ and $n \geq 2k + n' - 1$, we have

$$\pi(K(n,k), K_{n'}) \leq \min\left\{ \binom{n}{k}, \binom{n}{k} - \binom{n-1}{i^*+1} + (n'-1)\binom{n-1}{i^*} \right\},$$

where $i^*$ is given as in Theorem 3.2.12. And also this is the best possible inertia bound if we only apply one type of edge weight in $K_{n'}$. This part of the proof is similar to Theorem 3.2.8, but we consider $A(K(n,k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})$ here.

*Proof of Theorem 3.2.12, part 1.* With essentially the same proof as in Theorem 3.2.8, we have the following as the eigenvalues of $A(K(n,k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})$:

$$(-1)^i \binom{n-k-i}{k-i} + (n'-1)\beta \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1},$$

and

$$(-1)^i \binom{n-k-i}{k-i} - \beta \text{ with multiplicity } (n'-1)\left(\binom{n}{i} - \binom{n}{i-1}\right).$$

Denote $E^+(\beta)$ as the number of non-negative eigenvalues in $A(K(n,k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})\}$, and $E^-(\beta)$ as the number of non-positive eigenvalues in $A(K(n,k)) \otimes I_{n'} + \beta \cdot I_{\binom{n}{k}} \otimes A(K_{n'})\}$.

As in Theorem 3.2.8, we have $\pi(K(n,k), K_{n'}) \leq \min\{E^+(\beta), E^-(\beta)\}$ for any choice of $\beta \in \mathbb{R}$. We will find the $\beta$ that minimise $\min\{E^+(\beta), E^-(\beta)\}$ in the rest of this proof.

To minimise $\min\{E^+(\beta), E^-(\beta)\}$, we only need to consider the cases that neither $\beta$ nor $-(n'-1)\beta$ is exactly equal to some eigenvalues of $A(K(n,k))$. Note here the cases of choosing positive $\beta$ and negative $\beta$ are no longer symmetric.

There are three cases of the value of $\beta$ that we need to consider.

(1) $\beta$ is strictly between 0 and $(-1)^k \binom{n-2k}{0}$. (I.e. $\beta$ is 'close to 0'.)

(2) $\beta$ is strictly between $(-1)^{i+2} \binom{n-k-(i+2)}{k-(i+2)}$ and $(-1)^i \binom{n-k-i}{k-i}$ for some $0 \leq i \leq k-1$. (Depending on the parity of $i$, $\beta$ can be either negative

or positive. For simplicity of notation, we take $\binom{n-k-(k+1)}{-1} = 0$ as a convention.)

(3) $\beta > \binom{n-k}{k}$ or $\beta < -\binom{n-k}{k}$.

In case (1), $\beta$ is strictly between 0 and $(-1)^k$, hence $|(n'-1)\beta| < n'-1 < n-2k+1 = \binom{n-2k+1}{1}$ by assumption that $n \geq n-2k+n'-1$. That is, none of the eigenvalues of $K(n,k)$ has 'switched sign'. Then

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{n'\binom{n-1}{k-1}, n'\binom{n-1}{k}\right\} \\
&\quad = n'\binom{n-1}{k-1}.
\end{aligned}$$

In case (2), $\beta$ is strictly between $(-1)^{i+2}\binom{n-k-(i+2)}{k-(i+2)}$ and $(-1)^i\binom{n-k-i}{k-i}$ for some $0 \leq i \leq k-1$. By our assumption that neither $\beta$ nor $-(n'-1)\beta$ is equal to some eigenvalue of $A(K(n,k))$, we know one of the following cases holds.

Case (2-1): we have $-(n'-1)\beta$ strictly between $(-1)^{j+2}\binom{n-k-(j+2)}{k-(j+2)}$ and $(-1)^j\binom{n-k-j}{k-j}$ for some $0 \leq j \leq k-2$.

Case (2-2): $-(n'-1)\beta > \binom{n-k}{k}$.

Case (2-3): $-(n'-1)\beta < -\binom{n-k-1}{k-1}$.

Case (2-2) is possible only if $i$ is odd, and hence

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{(n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{i}, (n'-1)\binom{n-1}{i} + \binom{n}{k}\right\}.
\end{aligned}$$

Case (2-3) is possible only if $i$ is even, and hence

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{(n'-1)\binom{n-1}{i} + \binom{n}{k}, (n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{i}\right\}.
\end{aligned}$$

But in both cases, $(n'-1)\binom{n-1}{i} + \binom{n}{k} > \binom{n}{k}$ and $(n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{i} > \binom{n}{k}$ for all $n' \geq 3$ and for all $0 \leq i \leq k-1$. That is, neither (2-2) nor (2-3) will be the best upper bound.

In case (2-1), we have

$$\begin{aligned}
&\min\{E^+(\beta), E^-(\beta)\} \\
&\quad = \min\left\{(n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{i} + \binom{n-1}{j},\right. \\
&\qquad\qquad \left.(n'-1)\binom{n-1}{i} + \binom{n}{k} - \binom{n-1}{j}\right\}.
\end{aligned}$$

Note for any $n' \geq 3$, we have

$$\begin{aligned}
&(n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{i} + \binom{n-1}{j} \\
&\quad > (n'-1)\binom{n}{k} - (n'-1)\binom{n-1}{k-1} > \binom{n}{k}.
\end{aligned}$$

And if $j \leq i$, then $(n'-1)\binom{n-1}{i} + \binom{n}{k} - \binom{n-1}{j} > \binom{n}{k}$.

So in case (2-1), we only need to consider the case that $j > i$ and when

$$\min\{E^+(\beta), E^-(\beta)\} = \binom{n}{k} + (n'-1)\binom{n-1}{i} - \binom{n-1}{j}.$$

(Since the other possibility is always larger than $\binom{n}{k}$, and will never be the best upper bound.)

Note $-(n'-1)\beta$ is of the different sign as $\beta$, so $j$ is of different parity as $i$. Hence $j \geq i+1$ is a necessary condition for $\min\{E^+(\beta), E^-(\beta)\} \leq \binom{n}{k}$.

We have $\beta$ strictly between $(-1)^{i+2}\binom{n-k-(i+2)}{k-(i+2)}$ and $(-1)^i\binom{n-k-i}{k-i}$, and $-(n'-1)\beta$ strictly between $(-1)^{j+2}\binom{n-k-(j+2)}{k-(j+2)}$ and $(-1)^j\binom{n-k-j}{k-j}$. Such $\beta$ exist if and only if

$$\left((n'-1)\binom{n-k-(i+2)}{k-(i+2)}, (n'-1)\binom{n-k-i}{k-i}\right)$$

$$\cap \left(\binom{n-k-(j+2)}{k-(j+2)}, \binom{n-k-j}{k-j}\right) \neq \emptyset.$$

Since $j > i$, we have $\binom{n-k-j}{k-i} < (n'-1)\binom{n-k-i}{k-i}$. Hence such $\beta$ exist if and only if we have

$$\binom{n-k-j}{k-j} > (n'-1)\binom{n-k-(i+2)}{k-(i+2)}. \tag{3.7}$$

It is clear that one necessary condition for (3.7) to hold is $j \leq i+2$. Hence among the cases of interest, we only need to solve for $\binom{n-k-(i+1)}{k-(i+1)} > (n'-1)\binom{n-k-(i+2)}{k-(i+2)}$. This solves to $i > \frac{n'k-n}{n'-2} - 1$. Note $k-1 < \frac{n'k-n}{n'-2} - 1$ for all $n > 2k$.

Therefore, the minimum of $\min\{E^+(\beta), E^-(\beta)\}$ over $\beta$ in case (2) is the minimum of $\binom{n}{k} + (n'-1)\binom{n-1}{i} - \binom{n-1}{i+1}$ over $\frac{n'k-n}{n'-2} - 1 < i \leq k-1$.

In case (3), if $\beta > \binom{n-k}{k}$, then $\min\{E^+(\beta), E^-(\beta)\} = \binom{n}{k}$, which is essentially the whole graph. If $\beta < -\binom{n-k-1}{k-1}$, then there are two cases.

Case (3-1): if $n \leq n'k$, then it is only possible that $-(n'-1)\beta > \binom{n-k}{k}$, in which case we have $\min\{E^+(\beta), E^-(\beta)\} = \binom{n}{k}$.

Case (3-2): if $n > n'k$, then it is also possible that $\beta < -\binom{n-k-1}{k-1}$ and $\binom{n-k-2}{k-2} < -(n'-1)\beta < \binom{n-k}{k}$, in which case we have $\min\{E^+(\beta), E^-(\beta)\} = \binom{n}{k} - \binom{n-1}{0}$. However, for $n$ in this range, choices of $i$ in case (2) leads to better (i.e. smaller) upper bound.

Combining cases (1), (2) and (3), we have

$$\min_{\beta \in \mathbb{R}} \left\{\min\{E^+(\beta), E^-(\beta)\}\right\}$$
$$= \min\left\{\binom{n}{k}, \binom{n}{k} - \max_{\frac{n'k-n}{n'-2}-1<i\leq k-1} \left\{\binom{n-1}{i+1} - (n'-1)\binom{n-1}{i}\right\}\right\}.$$

Direct computation shows that $\binom{n-1}{i+1} - (n'-1)\binom{n-1}{i}$ is larger than $\binom{n-1}{i} - (n'-1)\binom{n-1}{i-1}$ if and only if

$$i < \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'},$$

or

$$i > \frac{(n'+2)n - 2n' + \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'}.$$

Note $\frac{(n'+2)n - 2n' + \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'} > \frac{n-2}{2} > k-1$. We conclude that the maximum of $\binom{n-1}{i+1} - (n'-1)\binom{n-1}{i}$ for $\frac{n'k-n}{n'-2} - 1 < i \leq k-1$ is attained at

$$i^* = \min\left\{k-1, \max\left\{\left\lfloor \frac{n'k-n}{n'-2} \right\rfloor, \left\lfloor \frac{(n'+2)n - 2n' - \sqrt{((n'-2)n)^2 + 4n'^2 n + 4n'^2}}{4n'} \right\rfloor\right\}\right\}.$$

□

We then prove that for $n' = 3$, the given bound is still best possible even if we allow three different edge weights on $K_3$.

*Proof of Theorem 3.2.12, part 2.* By allowing three different edge weights on $K_3$, we are considering the eigenvalues of

$$\mathcal{M}(x,y,z) = A(K(n,k)) \otimes I_3 + \beta \cdot I_{\binom{n}{k}} \otimes \begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}.$$

Denote $\beta_1$, $\beta_2$ and $\beta_3$ as the eigenvalues of $\begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$. It is well-known and easy to prove the sum of eigenvalues equals to the trace (sum of elements in diagonal) of a square matrix, hence we have $\beta_1 + \beta_2 + \beta_3 = 0$ and the eigenvalues of $\mathcal{M}(x,y,z)$ are just

$$(-1)^i \binom{n-k-i}{k-i} + \beta_j \text{ with multiplicity } \binom{n}{i} - \binom{n}{i-1},$$

for each $0 \leq i \leq k$ and each $j = 1, 2, 3$.

We first show that any real value pairs $\beta_1$ and $\beta_2$, together with $-(\beta_1 + \beta_2)$ can be the set of eigenvalues of $\begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$ at the same time, with suitable choices of $x, y, z$.

It suffices to find one set of $x, y, z$ for each fixed pair of $\beta_1$ and $\beta_2$. Without loss of generality, we assume $|\beta_1| \leq |\beta_2|$. Take $x = |\beta_1|$, $y = \sqrt{\dfrac{\beta_1\beta_2 + \beta_2^2}{2}}$ and $z = -\sqrt{\dfrac{\beta_1\beta_2 + \beta_2^2}{2}}$. Note $y$ and $z$ are well defined since $\beta_1\beta_2 + \beta_2^2 = \dfrac{1}{2}(\beta_1 + \beta_2)^2 + \dfrac{1}{2}\beta_2^2 - \dfrac{1}{2}\beta_1^2 \geq 0$. Then it is easy to verify that $\beta_1$, $\beta_2$ and $-(\beta_1 + \beta_2)$ are set of solutions of $\lambda$ in $-\lambda^3 + (x^3 + y^3 + z^3)\lambda + 2xyz = 0$, and hence are set of eigenvalues of $\begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$.

Now denote $E^+(\beta_1, \beta_2)$ as the number of non-negative eigenvalues of

$$m\left(|\beta_1|, \sqrt{\frac{\beta_1\beta_2 + \beta_2^2}{2}}, -\sqrt{\frac{\beta_1\beta_2 + \beta_2^2}{2}}\right),$$

and $E^-(\beta_1, \beta_2)$ as the number of non-positive eigenvalues of

$$m\left(|\beta_1|, \sqrt{\frac{\beta_1\beta_2 + \beta_2^2}{2}}, -\sqrt{\frac{\beta_1\beta_2 + \beta_2^2}{2}}\right).$$

(We assumed that $|\beta_1| \leq |\beta_2|$.)

We can also assume two of $\beta_1$, $\beta_2$ are $-(\beta_1 + \beta_2)$ are of the same sign and the other one is of opposite sign. (Otherwise they must all equal to 0.) Without loss of generality, $\beta_1$ and $\beta_2$ are of the same sign.

Then with the same method as in first part of the proof, we have

$$\begin{aligned} &\min\{E^+(\beta_1, \beta_2), E^-(\beta_1, \beta_2)\} \\ = \min &\left\{ \binom{n}{k}, \binom{n}{k} - \binom{n-1}{i_3} + \binom{n-1}{i_1} + \binom{n-1}{i_2}, \right. \\ &\left. 2\binom{n}{k} - \binom{n-1}{i_1} - \binom{n-1}{i_2} + \binom{n-1}{i_3} \right\} \end{aligned}$$

$$= \min\left\{\binom{n}{k}, \binom{n}{k} - \binom{n-1}{i_3} + \binom{n-1}{i_1} + \binom{n-1}{i_2}\right\}. \tag{3.8}$$

Here $i_1$, $i_2$ and $i_3$ are non-negative integers at most $k-1$ such that

$$\binom{n-k-(i_1+2)}{k-(i_1+2)} < |\beta_1| < \binom{n-k-i_1}{k-i_1}, \tag{3.9}$$

$$\binom{n-k-(i_2+2)}{k-(i_2+2)} < |\beta_2| < \binom{n-k-i_2}{k-i_2}, \tag{3.10}$$

and

$$\binom{n-k-(i_3+2)}{k-(i_3+2)} < |\beta_1+\beta_2| < \binom{n-k-i_3}{k-i_3}. \tag{3.11}$$

Note $|\beta_1+\beta_2| = |\beta_1| + |\beta_2|$ since $\beta_1$ and $\beta_2$ are of the same sign.

If $i_1 = i_2$, then (3.8) is the same as first part of the theorem, i.e. the same as only allow one edge weight on $K_3$.

Hence we assume $i_1 > i_2$ without loss of generality. Since $i_1$ and $i_2$ are of the same parity, we have $i_2 \le i_1 - 2$.

We claim it is not possible that $i_3 \ge i_1 + 1$. If instead we have $i_3 \ge i_1 + 1$, then

$$\begin{aligned}\binom{n-k-i_3}{k-i_3} &\le \binom{n-k-(i_1+1)}{k-(i_1+1)}\\ &\le \binom{n-k-(i_2+3)}{k-(i_2+3)} < \binom{n-k-(i_2+2)}{k-(i_2+2)}.\end{aligned}$$

Which is not possible. And therefore it is not possible for (3.9), (3.10) and (3.11) to hold at the same time.

However, if $i_3 \le i_1 - 1$, then $\binom{n}{k} - \binom{n-1}{i_3} + \binom{n-1}{i_1} + \binom{n-1}{i_2} > \binom{n}{k}$ and hence $\min\{E^+(\beta_1,\beta_2), E^-(\beta_1,\beta_2)\} = \binom{n}{k}$.

Thus we conclude that we will not get a better inertia bound by allowing three different edge weights on $K_3$. $\square$

### 3.3.4 Proof of Theorem 3.2.10

Finally, we prove our Example 3.2.9 is larger than the trivial $n'$-partite induced subgraphs as we claimed.

Recall that for $n \geq 2k+1$ and odd $k$, we have

$$\mathcal{F}_{\text{odd}}(n,k) := \left\{ F \in \binom{[n]}{k} \middle| |F \cap [k]| \geq \left\lceil \frac{k}{2} \right\rceil \right\} \cup \left\{ F \in \binom{[n]}{k} \middle| |F \cap [n-k+1,n]| \geq \left\lceil \frac{k}{2} \right\rceil \right\}.$$

And for $n \geq 2k+1$ and even $k$, we have

$$\mathcal{F}_{\text{even}}(n,k) := \left\{ F \in \binom{[n]}{k} \middle| |F \cap [k-1]| \geq \frac{k}{2} \right\} \cup \left\{ F \in \binom{[n]}{k} \middle| |F \cap [n-k,n]| \geq \frac{k}{2}+1 \right\}.$$

**Theorem 3.2.10.**
*For any $k \geq 3$, there exists $T \geq 1$ that if $2k+1 \leq n \leq 2k+T$, then Example 3.2.9 is larger than the trivial bipartite induced subgraph of $K(n,k)$. Moreover, this $T$ can be arbitrary large for large $k$.*

*Proof of Theorem 3.2.10.* Let $t = \left\lfloor \frac{k}{2} \right\rfloor$. (Hence $k = 2t+1$ if $k$ is odd, and $k = 2t$ if $k$ is even.) We start our proof by showing

$$|\mathcal{F}_{\text{odd}}(n,k)| = \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{k+j-1}{t}\binom{n-k-j}{t}, \tag{3.12}$$

and

$$|\mathcal{F}_{\text{even}}(n,k)| = \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{k+j}{t}\binom{n-k-1-j}{t-1}. \tag{3.13}$$

Firstly, by counting the sets that are not in $\mathcal{F}_{\text{odd}}$ and substract it from $\binom{n}{k}$, we have $|\mathcal{F}_{\text{odd}}| = \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{n-2k}{j} \left( \sum_{i=1}^{j} \binom{k}{t+1-i}\binom{k}{t-j+i} \right)$.(A $k$-subset $F$ of $[n]$ is not in $\mathcal{F}_{\text{odd}}$ if and only if $|F \cap [k]| \leq t$ and $|F \cap [n-k,n]| \leq t$.)

For (3.12), we count the number of sets not in $\mathcal{F}_{\text{odd}}$ in a slightly different way. For each set $F$ that is not in $\mathcal{F}_{\text{odd}}$, consider the $(t+1)$'s smallest element in $F$ and denote it by $j$. Since $F$ contains at most $t$ elements in $[k]$ and at most $t$ elements in $[n-k+1,n]$, we have $j \in [k+1, n-k]$. And for any choice of $F_1 \in \binom{[j-1]}{t}$ and $F_2 \in \binom{[j+1,n]}{t}$, we have $F_1 \cup \{j\} \cup F_2$

a $k$-subset that is not in $\mathscr{F}_{\text{odd}}$. On the other side, any $k$-subset that is not in $\mathscr{F}_{\text{odd}}$ can be represented in this way. By this method of counting we have

$$\begin{aligned}|\mathscr{F}_{\text{odd}}(n,k)| &= \binom{n}{k} - \sum_{j=k+1}^{n-k} \binom{j-1}{t}\binom{n-j}{t} \\ &= \binom{n}{k} - \sum_{j=1}^{n-2k} \binom{k+j-1}{t}\binom{n-k-j}{t}.\end{aligned}$$

The idea for (3.13) is essentially the same, just we consider the $t$'s smallest element as $j$ (instead of the $(t+1)$'s) in any $F$ that is not in $\mathscr{F}_{\text{even}}$.

Now we compare (3.12) and (3.13) with the trivial bipartite order $\binom{n}{k} - \binom{n-2}{k}$. We will first show that if $n = 2k+1$ and $k \geq 3$, then

$$|\mathscr{F}_{\text{odd}}(n,k)| > \binom{n}{k} - \binom{n-2}{k}, \tag{3.14}$$

and

$$|\mathscr{F}_{\text{even}}(n,k)| > \binom{n}{k} - \binom{n-2}{k}. \tag{3.15}$$

For (3.14), it suffice to show that $\binom{k}{t}\binom{k}{t} < \binom{n-2}{k} = \binom{2k-1}{k}$ for each $k \geq 3$.

Denote $f_{\text{odd}}(t) = \binom{2t+1}{t}\binom{2t+1}{t} \Big/ \binom{4t+1}{2t+1}$. We will prove by induction that $f_{\text{odd}}(t) < 1$ for all $t \geq 1$. (Note $k \geq 3$ if and only if $t \geq 1$.)

It is easy to check that $f_{\text{odd}}(1) = 9/10 < 1$. Assume $f_{\text{odd}}(t) < 1$ for some $t \geq 1$. Then

$$\begin{aligned}f_{\text{odd}}(t+1) &= \binom{2t+3}{t+1}\binom{2t+3}{t+1} \Big/ \binom{4t+5}{2t+3} \\ &= \frac{(2t+2)(2t+3)^3}{(4t+5)(4t+3)(t+2)^2} \cdot \frac{\binom{2t+1}{t}\binom{2t+1}{t}}{\binom{4t+1}{2t+1}} \\ &= \frac{16t^4+88t^3+180t^2+162t+54}{16t^4+96t^3+207t^2+188t+60} f(t),\end{aligned}$$

which is less than 1 if $f_{\text{odd}}(t) < 1$. Hence we can conclude $f_{\text{odd}}(t) < 1$ for any $t \geq 1$.

For (3.15), similarly denote $f_{\text{even}}(t) = \binom{2t+1}{t}\binom{2t-1}{t-1}\Big/\binom{4t-1}{2t}$. (Note $k = 2t$ if $k$ is even, and here we will show $f_{\text{even}}(t) < 1$ for any $t \geq 2$.)

It is easy to verify that $f(2) = 30/35 < 1$. Assume $f_{\text{even}}(t) < 1$ for some $t \geq 2$. Then

$$\begin{aligned} f_{\text{even}}(t+1) &= \binom{2t+3}{t+1}\binom{2t+1}{t}\Big/\binom{4t+3}{2t+2} \\ &= \frac{2(2t+1)^2(2t+3)}{(t+2)(4t+1)(4t+3)} \cdot \frac{\binom{2t+1}{t}\binom{2t-1}{t-1}}{\binom{4t-1}{2t}} \\ &= \frac{16t^3+40t^2+28t+6}{16t^3+48t^2+35t+6} f(t), \end{aligned}$$

which is less than 1 if $f_{\text{even}}(t) < 1$. Hence we can conclude $f_{\text{even}}(t) < 1$ for any $t \geq 2$.

As stated in the Theorem, we will have an integer $T \geq 1$ for each $k \geq 3$ such that if $2k+1 \leq n \leq 2k+T$, then Example 3.2.9 is larger than the trivial bipartite. That it, for each $n \in [2k+1, 2k+T]$, we have

$$\sum_{j=1}^{n-2k} \binom{k+j-1}{t}\binom{n-k-j}{t} < \binom{n-2}{k}$$

if $k$ is odd, and

$$\sum_{j=1}^{n-2k} \binom{k+j}{t}\binom{n-k-1-j}{t-1} < \binom{n-2}{k}$$

if $k$ is even. Now we show this $T \geq 1$ can be arbitrary large if $k$ is large. Denote $t = \left\lfloor \frac{k}{2} \right\rfloor$. Since the case $n - 2k = 1$ is proved above, we can assume $n \geq 2k+2$ here. Hence it suffice to prove

$$(n-2k)\binom{n-k}{t}\binom{n-k}{t} < \binom{2k}{k}.$$

We then use Stirling's approximation that $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$. We have

$$\frac{\binom{2k}{k}}{(n-2k)\binom{n-k}{t}\binom{n-k}{t}}$$

$$
\begin{aligned}
&= \left(\frac{(n-k-t)\cdots(k-t+1)}{(n-k)\cdots(k+1)}\right)^2 \cdot \frac{1}{n-2k} \cdot \frac{\binom{2k}{k}}{\binom{k}{t}\binom{k}{t}} \\
&\geq \frac{1}{(n-2k)\cdot 2^{2n-4k}} \cdot \frac{(2k)!t!(k-t)!t!(k-t)!}{k!k!k!k!} \\
&\geq \frac{1}{T\cdot 2^{2T}} \cdot \frac{(2k)!t!(k-t)!t!(k-t)!}{k!k!k!k!} \\
&\sim \begin{cases} \frac{1}{2^{2T}T} \cdot \frac{\sqrt{4\pi k}t(t+1)}{k^2} \cdot \frac{2^{2k}t^{2t}(t+1)^{2t+2}}{k^{2k}}, & \text{if } k \text{ odd} \\ \frac{1}{2^{2T}T} \cdot \frac{\sqrt{\pi k}}{2}, & \text{if } k \text{ even} \end{cases} \\
&= \Omega(\sqrt{k}).
\end{aligned}
$$

That is, for any fixed $T$, there is $k_0(T)$ such that if $k > k_0(T)$, then we have $\binom{2k}{k} < (n-2k)\binom{n-k}{t}\binom{n-k}{t}$ for any $n \leq 2k+T$. $\square$

**Theorem 3.2.11.**
*If $n-2k = o(\sqrt{k})$, then for sufficiently large $k$, Example 3.2.9 is larger than the trivial $n'$-partite induced subgraph in $K(n,k)$.*

*Proof of Theorem 3.2.11.* We can prove using similar technique as Theorem 3.2.10. But a slightly more careful computation is needed. For simplicity of notation assume $n = 2k+x$, with $x = o(\sqrt{k})$.

We will show for any fixed $n'$ and large enough $k$, we have $\binom{n}{k} - |\mathcal{F}_{\text{odd}}| < \binom{n-n'}{k}$ and $\binom{n}{k} - |\mathcal{F}_{\text{even}}| < \binom{n-n'}{k}$. (Note $\binom{n-n'}{k}$ is the number of vertices not in a trivial $n'$-partite family.)

Let $n' \geq 2$ be a fixed integer. Firstly we consider the case that $k = 2t+1$ where $t \geq 1$. Then for each $1 \leq j \leq x$, we have

$$
\begin{aligned}
&\binom{k+j-1}{t}\binom{n-k-j}{t} \Big/ \binom{n-n'}{k} \\
&\quad= \frac{(k+j-1)!(n-k-j)!k!(n-n'-k)!}{t!(k-t+j-1)!t!(n-k-t-j)(n-n')!} \\
&\quad= \frac{(k+j-1)!(k+x-j)!k!(k+x-n')!}{t!(t+j)!t!(t+1+x-j)(2k+x-n')!} \\
&\quad\sim \sqrt{\frac{(k+j-1)(k+x-j)k(k+x-n')}{2\pi t(t+j)t(t+1+x-j)(2k+x-n')}}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \frac{(k+j-1)^{k+j-1}(k+x-j)^{k+x-j}k^k(k+x-n')^{k+x-n'}}{t^t(t+j)^{t+j}t^t(t+1+x-j)^{t+1+x-j}(2k+x-n')^{2k+x-n'}} \\
\sim &\sqrt{\frac{2}{\pi t}} \cdot 2^{n'-1} \cdot \frac{(2t+j)^{2t+j}(2t+1+x-j)^{2t+1+x-j}}{(2t)^t(2t+2j)^{t+j}(2t)^t(2t+2+2x-2j)^{t+1+x-j}} \\
&\qquad \cdot \frac{(2k)^k(2k+2x-2n')^{k+x-n'}}{(2k+x-n')^{2k+x-n'}} \\
&= 2^{n'-1}\sqrt{\frac{2}{\pi t}} \cdot \left(1+\frac{j}{2t}\right)^t \left(1-\frac{j}{2t+2j}\right)^{t+j} \\
&\qquad \cdot \left(1+\frac{x-j+1}{2t}\right)^t \left(1-\frac{x-j+1}{2t+2+2x-2j}\right)^{t+1+x-j} \\
&\qquad \cdot \left(1-\frac{x-n'}{2k+x-n'}\right)^k \left(1+\frac{x-n'}{2k+x-n'}\right)^{k+x-n'} \\
&= \frac{2^{n'}}{\sqrt{2\pi t}} \cdot \left(1+\frac{j^2}{(2t)(2t+2j)}\right)^t \left(1-\frac{j}{2t+2j}\right)^j \\
&\qquad \cdot \left(1+\frac{(x-j+1)^2}{(2t)(2t+2+2x-2j)}\right)^t \left(1-\frac{x-j+1}{2t+2+2x-2j}\right)^{x-j+1} \\
&\qquad \cdot \left(1-\frac{(x-n')^2}{(2k+x-n')^2}\right)^k \left(1+\frac{x-n'}{2k+x-n'}\right)^{x-n'} \\
&\sim \frac{2^{n'}}{\sqrt{\pi k}}(\text{since } x = o(\sqrt{k})).
\end{aligned}
$$

Hence

$$\sum_{j=1}^{x}\left(\binom{k+j-1}{t}\binom{n-k-j}{t}\Big/\binom{n-n'}{k}\right) \sim \frac{x\cdot 2^{n'}}{\sqrt{\pi k}}.$$

Now assume $k = 2t$ is even and $t \geq 2$. Then for each $1 \leq j \leq n-2k$, with essentially the same calculation as above, we have

$$\binom{k+j}{t}\binom{n-k-1-j}{t-1}\Big/\binom{n-n'}{k} \sim \frac{2^{n'}}{\sqrt{\pi k}},$$

and hence

$$\sum_{j=1}^{x}\left(\binom{k+j}{t}\binom{n-k-1-j}{t-1}\Big/\binom{n-n'}{k}\right) \sim \frac{x\cdot 2^{n'}}{\sqrt{\pi k}}.$$

Since $x = o(\sqrt{k})$, it is clear that for any fixed $n'$, we have $\lim_{k\to\infty} \frac{x\cdot 2^{n'}}{\sqrt{\pi k}} = 0$. Hence for any fixed $n'$, there exist $k_0(n')$ such that if $k > k_0(n')$, then

$$\sum_{j=1}^{x}\left(\binom{k+j-1}{t}\binom{n-k-j}{t}\Big/\binom{n-n'}{k}\right) < 1,$$

and
$$\sum_{j=1}^{x}\left(\binom{k+j}{t}\binom{n-k-1-j}{t-1}\binom{n-n'}{k}\right)<1.$$
That is, $|\mathcal{F}_{\text{odd}}|>\binom{n}{k}-\binom{n-n'}{k}$ and $|\mathcal{F}_{\text{odd}}|>\binom{n}{k}-\binom{n-n'}{k}$ with those given $n$, $k$, $n'$. □

We finish this section by a short proof showing our Example 3.2.9 is better than the construction in [18]. Recall the construction in [18] is
$$\mathcal{H}(n,k)=\left\{F\in\binom{[n]}{k}\middle|\left|F\cap\left[\frac{n}{2}\right]\right|>\frac{n}{4}\text{ or }\left|F\cap\left[\frac{n}{2}+1,n\right]\right|>\frac{n}{4}\right\},$$
is only for even values of $n$. And note
$$|\mathcal{H}(n,k)|=\binom{n}{k}-\sum_{k-n/4\le j\le n/4}\binom{n/2}{j}\binom{n/2}{k-j}.$$

Denote $\mathcal{F}(n,k,u)=\left\{F\in\binom{[n]}{k}\middle|F\cap[u]>\frac{u}{2}\right\}$. We will prove the following stronger result.

**Lemma 3.3.5.**
*For any $k\ge 1$, $n\ge 2k$ and $u_1,u_2\ge 1$,*
(a) *if $|u_1-u_2|=1$ and $u_1$ is even, then $|\mathcal{F}(n,k,u_1)|\le|\mathcal{F}(n,k,u_2)|$;*
(b) *if both $u_1$ and $u_2$ are odd, and $u_1\ge u_2$, then $|\mathcal{F}(n,k,u_1)|\le|\mathcal{F}(n,k,u_2)|$.*

For any odd $k\ge 1$ and $n\ge 2k$, there does not exist set $F\in\binom{[n]}{k}$ such that $|F\cap[k]|\ge\left\lceil\frac{k}{2}\right\rceil$ and $|F\cap[n-k+1,n]|\ge\left\lceil\frac{k}{2}\right\rceil$. Hence we have $|\mathcal{F}_{\text{odd}}(n,k)|=2|\mathcal{F}\left(n,k,k\right)|$. Similarly, for any even $k\ge 1$ and $n\ge 2k$, we have $|\mathcal{F}_{\text{even}}(n,k)|=|\mathcal{F}\left(n,k,k-1\right)|+|\mathcal{F}\left(n,k,k+1\right)|$. And for any $k\ge 1$ and even $n\ge 2k$, we have $|\mathcal{H}(n,k)|=2\left|\mathcal{F}\left(n,k,\frac{n}{2}\right)\right|$. Thus Lemma 3.3.5 immediately implies Example 3.2.9 is larger than $\mathcal{H}(n,k)$.

*Proof of Lemma 3.3.5.* We first compare $\mathcal{F}(n,k,u+1)$ and $\mathcal{F}(n,k,u)$ for any $u\ge 1$.
If $F\in(\mathcal{F}(n,k,u+1)\setminus\mathcal{F}(n,k,u))$, then $F\cap[u+1]>\frac{u+1}{2}$ and $F\cap[u]\le\frac{u}{2}$. This is true only if $u+1\in F$ and $|F\cap[u+1]|=\frac{u+2}{2}$ (i.e. $u$ is even).

Hence,

$$|\mathcal{F}(n,k,u+1)\setminus\mathcal{F}(n,k,u)| = \begin{cases} \binom{u}{u/2}\binom{n-u-1}{k-u/2-1} & \text{, if } u \text{ is even;} \\ 0 & \text{, otherwise.} \end{cases}$$

If $F \in (\mathcal{F}(n,k,u)\setminus\mathcal{F}(n,k,u+1))$, then $F\cap[u] > \frac{u}{2}$ and $F\cap[u+1] \leq \frac{u+1}{2}$. This happens only if $|F\cap[u]| = \frac{u+1}{2}$ (i.e. $u$ is odd) and $u+1 \notin F$. Hence

$$|\mathcal{F}(n,k,u)\setminus\mathcal{F}(n,k,u+1)| = \begin{cases} \binom{u}{(u+1)/2}\binom{n-u-1}{k-(u+1)/2} & \text{, if } u \text{ is odd;} \\ 0 & \text{, otherwise.} \end{cases}$$

Therefore $|\mathcal{F}(n,k,u)| < |\mathcal{F}(n,k,u+1)|$ if $u$ is even, and $|\mathcal{F}(n,k,u)| > |\mathcal{F}(n,k,u+1)|$ if $u$ is odd.

With a similar counting approach, for any odd integer $u \geq 1$, we have

$$\begin{aligned} |\mathcal{F}(n,k,u)\setminus\mathcal{F}(n,k,u+2)| &= \binom{u}{(u+1)/2}\binom{n-u-2}{k-(u+1)/2} \\ |\mathcal{F}(n,k,u+2)\setminus\mathcal{F}(n,k,u)| &= \binom{u}{(u-1)/2}\binom{n-u-2}{k-(u+3)/2}. \end{aligned}$$

Hence our claim (b) is also proved since $\binom{u}{(u-1)/2} = \binom{u}{(u+1)/2}$ and $\binom{n-u-2}{k-(u+3)/2} < \binom{n-u-2}{k-(u+1)/2}$. $\square$

## 3.4 Discussion and Open Questions

We start this section by further comparison on various upper and lower bounds that we obtained.

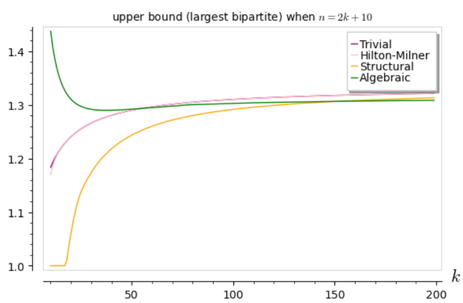
### 3.4.1 More Upper and Lower Bounds of $\gamma(K(n,k),2)$

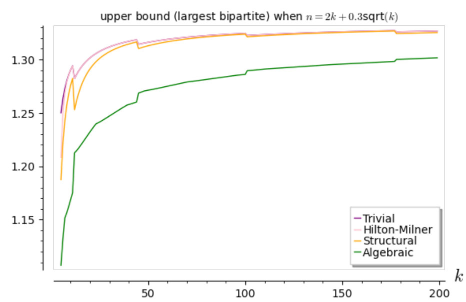
In this short section, we discuss a few more techniques that we studied for the upper bound of $\gamma(K(n,k),2)$, and compare them for $n$ in different ranges as functions of $k$. We also compare then with the lower bounds we presented in Section 3.2.1.

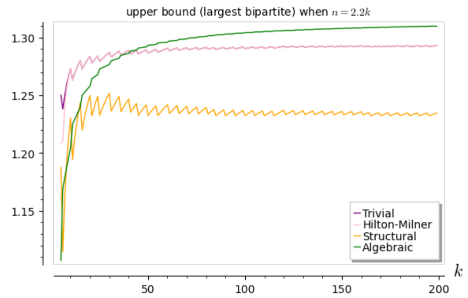
In the following plots, we present the comparison of four upper bounds on $\gamma(K(n,k),2)$, with $k$ as x-axis and $n$ as a function of $k$, shown on each plot's title. To get a better understanding of how those bounds compare to the trivial bipartite graphs, the values shown are already divided by $t(n,k,2)$.

Consider any fixed graph $K(n,k)$. Since each bipartite graph is an union of two independent sets, its order is always bounded above by two times the maximum independent set. In the following plot, 'Trivial' denote the value of $2\binom{n-1}{k-1}\Big/\left(\binom{n}{k}-\binom{n-2}{k}\right)$. Lemma 3.3.1 gives an slightly better upper bound for $\gamma(K(n,k),2)$ by considering the fact that each maximum independent set in $K(n,k)$ has a special structure, and the union of two maximum independent sets are significantly smaller than $2\binom{n-1}{k-1}$. Hence we can use Hilton and Milner's result on 'second largest' independent set in $K(n,k)$ to achieve a better bound. In the following plot, 'Hilton-Milner' denote the value of $\max\left\{t(n,k,2), 2\binom{n-1}{k-1}-\binom{n-k}{k-1}-\binom{n-k-1}{k-1}+\binom{n-k-2}{k-3}+3\right\}$ divided by $\binom{n}{k}-\binom{n-2}{k}$. Note for each choice of $n$, the plot of 'Trivial' and 'Hilton-Milner' almost equals, apart from very small $k$.

Theorem 3.2.6 presents a better structural bounds for $\gamma(K(n,k),2)$, and is shown as 'Structural' in the plots. Theorem 3.2.8 provides another upper bound using algebraic methods, and is shown as 'Algebraic' in the following plots. As we can see, this upper bound is better (i.e. smaller) if $n = 2k+10$ and larger $k$, or $n = 2k + 0.3\sqrt{k}$. We had other numerical experiments showing this is a better upper bound if $n - 2k$ is in $o(\sqrt{k})$. For space reasons, we will not discuss this further.






Recall that in Theorem 3.2.6, we bound the maximum order of a bipartite subgraph $H \leqslant_i K(n,k)$ by first partition $H$ into smaller parts, each satisfying certain conditions, then we find the maximum possible total order $|H|$, under structural constraints on each part. We can find more constraints if we refine the partition of $H$.

Fixed $n, k$ and a bipartite $H \leqslant_i K(n,k)$. As in Lemma 3.3.3, we can assume the vertex set of $H$ is a disjoint union of a left-shifted intersecting family $\mathcal{F}_1$ and a right-shifted intersecting family $\mathcal{F}_2$. In addition to what we've done in the proof of Theorem 3.2.6, we further partition $\mathcal{F}_1 = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3 \sqcup \mathcal{A}_4$, and $\mathcal{F}_2 = \mathcal{B}_1 \sqcup \mathcal{B}_2 \sqcup \mathcal{B}_3 \sqcup \mathcal{B}_4$. For simplicity of formulas, for a set family $\mathcal{X} \subseteq \binom{[n]}{k}$, we use $\mathcal{X}(i) := \{F \setminus \{i\} | i \in F, F \in \mathcal{X}\}$ and $\mathcal{X}(\bar{i}) := \{F | i \in F, F \in \mathcal{X}\}$. Note we may have multiple arguments in the function bracket. For instance, $\mathcal{X}(i, \bar{j}) := (\mathcal{X}(i))(\bar{j})$.

The partition is defined by the following rules:

- $\mathcal{A}_1 := \mathcal{F}_1(1, n)$, $\mathcal{A}_2 := \mathcal{F}_1(1, \bar{n})$, $\mathcal{A}_3 := \mathcal{F}_1(\bar{1}, n)$, $\mathcal{A}_4 := \mathcal{F}_1(\bar{1}, \bar{n})$;
- $\mathcal{B}_1 := \mathcal{F}_2(n, 1)$, $\mathcal{B}_2 := \mathcal{F}_2(n, \bar{1})$, $\mathcal{B}_3 := \mathcal{F}_1(\bar{n}, 1)$, $\mathcal{B}_4 := \mathcal{F}_1(\bar{n}, \bar{1})$.

Following similar ideas as in the proof of Theorem 3.2.6, we have the following constraints.

- For $\mathcal{A}_i$, $1 \leq i \leq 4$, we have $|\mathcal{A}_1| \leq \frac{k-1}{n-k}|\mathcal{A}_2|$, $|\mathcal{A}_3| \leq \frac{n-k}{k-1}|\mathcal{A}_1|$, $|\mathcal{A}_3| \leq |\mathcal{A}_2|$, $|\mathcal{A}_3| \leq \frac{k}{n-k-1}|\mathcal{A}_4|$, $|\mathcal{A}_4| \leq \frac{n-k-1}{k}|\mathcal{A}_2|$, $|\mathcal{A}_3| + |\mathcal{A}_4| \leq \frac{n-k}{k}(|\mathcal{A}_1|+|\mathcal{A}_2|)$, and $|\mathcal{A}_1|+|\mathcal{A}_2|+\frac{n-k-1}{k-1}(|\mathcal{A}_3| + |\mathcal{A}_4|) \leq \binom{n-1}{k-1}$;

- for $\mathcal{B}_i$, $1 \le i \le 4$, we have $|\mathcal{B}_1| \le \frac{k-1}{n-k}|\mathcal{B}_2|$, $|\mathcal{B}_3| \le \frac{n-k}{k-1}|\mathcal{B}_1|$, $|\mathcal{B}_3| \le |\mathcal{B}_2|$, $|\mathcal{B}_3| \le \frac{k}{n-k-1}|\mathcal{B}_4|$, $|\mathcal{B}_4| \le \frac{n-k-1}{k}|\mathcal{B}_2|$, $|\mathcal{B}_3| + |\mathcal{B}_4| \le \frac{n-k}{k}(|\mathcal{B}_1|+|\mathcal{B}_2|)$, and $|\mathcal{B}_1|+|\mathcal{B}_2|+\frac{n-k-1}{k-1}(|\mathcal{B}_3| + |\mathcal{B}_4|) \le \binom{n-1}{k-1}$;

- moreover, we have $|\mathcal{A}_1|+|\mathcal{A}_2|+|\mathcal{B}_1|+|\mathcal{B}_2|+\frac{n-k-2}{k-1}(|\mathcal{A}_4| + |\mathcal{B}_4|) \le \binom{n}{k} - \binom{n-2}{k}$, $|\mathcal{A}_1| + |\mathcal{A}_2| + |\mathcal{B}_1| + |\mathcal{B}_2| + \frac{n-k-1}{k-1}(|\mathcal{A}_3| + |\mathcal{B}_3|) \le \binom{n}{k} - \binom{n-2}{k}$.

Some additional constrains involving above parts relies on the following results regarding *cross-intersecting* families. The concept of *cross-intersecting* set families was first introduced by Hilton and Milner [28] in their proof to the maximum non-trivial intersecting families. Two set families $\mathcal{X}$ and $\mathcal{Y}$ are cross-intersecting if for any sets $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, we have $X \cap Y \neq \emptyset$. The order of cross-intersecting families has been extensively studied. We only mention results relevant to our studies here.

**Theorem 3.4.1** (Frankl and Tokushige [21])**.**
*If* $\mathcal{X} \subseteq \binom{[m]}{k_1}$ *and* $\mathcal{Y} \subseteq \binom{[m]}{k_2}$ *are cross-intersecting,* $m \ge k_1+k_2$, $k_1 \le k_2$, *then* $|\mathcal{X}| + |\mathcal{Y}| \le \binom{m}{k_2} - \binom{m-k_1}{k_2} + 1$.

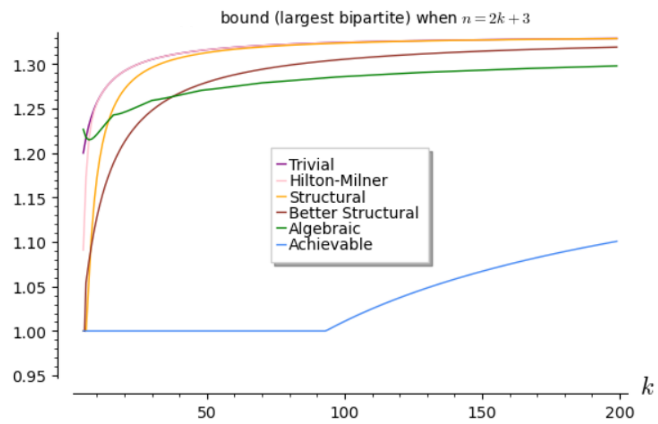
**Theorem 3.4.2** (Kupavskii [37])**.**
*If* $\mathcal{X} \subseteq \binom{[m]}{k_1}$ *and* $\mathcal{Y} \subseteq \binom{[m]}{k_2}$ *are cross-intersecting, then* $|\mathcal{X}| \Big/ \binom{m}{k_1} + |\mathcal{Y}| \Big/ \binom{m}{k_2} \le 1$.

With the above results, we have the following additional constraints.

- For $\mathcal{A}_i$, $1 \le i \le 4$, we have $|\mathcal{A}_1|+|\mathcal{A}_4| \le \binom{n-2}{k} - \binom{n-k}{k} + 1$, $|\mathcal{A}_2| + |\mathcal{A}_3| \le \binom{n-2}{k-1} + \binom{n-k}{k-1} + 1$ and $|\mathcal{A}_1| \Big/ \binom{n-2}{k-2} + |\mathcal{A}_4| \Big/ \binom{n-2}{k} \le 1$;

- for $\mathcal{B}_i$, $1 \le i \le 4$, we have $|\mathcal{B}_1| + |\mathcal{B}_4| \le \binom{n-2}{k} - \binom{n-k}{k} + 1$, $|\mathcal{B}_2| + |\mathcal{B}_3| \le \binom{n-2}{k-1} + \binom{n-k}{k-1} + 1$ and $|\mathcal{B}_1| \Big/ \binom{n-2}{k-2} + |\mathcal{B}_4| \Big/ \binom{n-2}{k} \le 1$.

An analytical maximum $|H|$ to the above set of formulas involving $\mathscr{A}_i$'s and $\mathscr{B}_i$'s will be complicated. Instead, we plot how the upper bound of $|H|$ achieved by above constraints as following (using existing linear programming solvers). Here two additional plots are introduced, where 'Better Structural' is obtained by maximising $\sum_{i=1}^{4} |\mathscr{A}_i| + \sum_{i=1}^{4} |\mathscr{B}_i|$ subject to above formulas. And 'Achievable' denote the maximum over trivial bipartite subgraphs and the subgraph constructed in Examples 3.2.9.




Although it may not seem to be clear from the plot, recall that both the lower bound and the upper bounds we studied have the property that if $n = 2k + o(\sqrt{k})$, then they approach to $\frac{4}{3}t(n, k, 2)$ as $k$ goes to infinity.

When we consider the case that $n = 2k+\Omega(\sqrt{k})$, we see some more variation among the different bounds. For instance, if $n = 2k + o(k)$, then 'Trivial', 'Hilton-Milner', and 'Structural' still approach to $\frac{4}{3}t(n, k, 2)$ as $k$ goes to infinity, but the behaviour of 'Algebraic' becomes hard to track. If $n = 2k+ck$, then those mentioned upper bounds will approach to different constant multiple of $t(n, k, 2)$, where each multiple is a function of $c$. Note if $n = 2k + \Omega(\sqrt{k})$, then our construction in Example 3.2.9 is not better than trivial bipartite subgraph, hence 'Achievable' is equal to $t(n, k, 2)$. For space reasons, we will not discuss this further.

In Example 3.2.9, we present a special construction of bipartite subgraph of $K(n, k)$, which is larger then the construction in [18]. It is worth asking: is there any bipartite subgraph of $K(n, k)$ that is larger than what we have?

A family of natural candidates, which is also a generalisation of Example 3.2.9 is

$$\begin{aligned}\mathcal{F}(n,k,t_1,t_2) := &\left\{F \in \binom{[n]}{k} \;\middle|\; |F \cap [t_1]| \geq \tfrac{1}{2}(t_1+1)\right\} \\ &\cup \left\{F \in \binom{[n]}{k} \;\middle|\; |F \cap [n-t_2+1,n]| \geq \tfrac{1}{2}(t_2+1)\right\}.\end{aligned}$$

We have the following observations, each follows a similar approach as in Lemma 3.3.5, but with slightly more involved ideas.

**Lemma 3.4.3.**
*For fixed positive integers $n,k$ such that $n \geq 2k+1$, there is a pair of $t_1^*, t_2^*$ maximising $|\mathcal{F}(n,k,t_1,t_2)|$ such that both $t_1^*$, $t_2^*$ are odd, and $t_1^* + t_2^* \leq 2k$.*

**Lemma 3.4.4.**
*For fixed positive integers $k \geq 3$ and $n = 2k+1$, we have*
(a) $|\mathcal{F}(n,k,t_1,t_2)| \geq |\mathcal{F}(n,k,t_1-2,t_2)|$ for odd integers $t_1,t_2$ that $3 \leq t_1 \leq k$, $1 \leq t_2 \leq k$;
(b) $|\mathcal{F}(n,k,k+x,k-x)| \leq |\mathcal{F}(n,k,k+x-2,k-x+2)|$ for each $1 \leq x \leq k-3$ with the different parity as $k$ (i.e. such that $k-x$ is odd).

Lemmas 3.4.3 and 3.4.4 implies the following.

**Theorem 3.4.5.**
*For fixed positive integer $k$ and $n = 2k+1$, if $k$ is odd, then $t_1 = t_2 = k$ maximises $|\mathcal{F}(n,k,t_1,t_2)|$; if $k$ is even, then $t_1 = k-1$, $t_2 = k+1$ maximises $|\mathcal{F}(n,k,t_1,t_2)|$.*

But they are not sufficient for us to conclude which pair of $t_1,t_2$ maximises $|\mathcal{F}(n,k,t_1,t_2)|$ for general pairs of $n,k$. It is tempting to ask whether our $\mathcal{F}_{\text{odd}}$ and $\mathcal{F}_{\text{even}}$ as in Example 3.2.9 always maximise $|\mathcal{F}(n,k,t_1,t_2)|$? Unfortunately, direct computation using computer programs suggest not, even for $n$ close enough to $2k$.

Instead, direct computation suggest for each small fixed $c$ and $n = 2k+c$, if $k$ is relatively large, then the best choice of $t_1,t_2$ in $|\mathcal{F}(n,k,t_1,t_2)|$ is the choice as in $\mathcal{F}_{\text{odd}}$ and $\mathcal{F}_{\text{even}}$. But for smaller $k$ (with each $c$), the best choice is either $t_1 = t_2 = 1$ (which is the trivial bipartite subgraph), or some $t_1,t_2$

close to but not equal to $k$. For instance fix $c = 4$, if $k = 63$, $n = 130$, then $|\mathcal{F}(n,k,t_1,t_2)|$ is maximised at $t_1 = t_2 = 1$; if instead $k = 64$, $n = 132$, then $|\mathcal{F}(n,k,t_1,t_2)|$ is maximised at $t_1 = 63, t_2 = 61$; similarly if $k = 65$, $n = 134$, then $|\mathcal{F}(n,k,t_1,t_2)|$ is maximised at $t_1 = t_2 = 63$; finally for larger $k$, in this case $k = 113, n = 230$, we have $|\mathcal{F}(n,k,t_1,t_2)|$ maximised at $t_1 = t_2 = 113$, agreeing with the construction of $\mathcal{F}_{\text{odd}}$.

In the other direction, our computation also seem to suggest for each fixed $k$ and $n = 2k+c$, then as $c$ increases, the best choice of $t_1, t_2$ always start as the choice in $\mathcal{F}_{\text{odd}}$ and $\mathcal{F}_{\text{even}}$, and becomes something smaller but close enough to $k$, and then immediately drop to $t_1 = t_2 = 1$. For example, if $k = 161$ and $n = 2k + c$, then for $1 \le c \le 4$, we have $|\mathcal{F}(n,k,t_1,t_2)|$ maximised at $t_1 = t_2 = 161$. But if $c = 5$, then the maximum is attained at $t_1 = 159, t_2 = 161$; if $c = 6$, then the maximum is attained at $t_1 = 155, t_2 = 157$; and if $c \ge 7$, then the maximum is attained at $t_1 = t_2 = 1$. For space reasons, we will not discuss this further, and leave the rest to interested readers.

*Proof of Lemma 3.4.3.* We will show that if any of $t_1$, $t_2$ is even, then there is another pair of $t_1'$, $t_2'$ such that $|\mathcal{F}(n,k,t_1',t_2')| \ge |\mathcal{F}(n,k,t_1,t_2)|$.

Without loss of generality, we assume $t_1$ is even. For each $F \in \mathcal{F}(n,k,t_1,t_2)$, we have $F \in \mathcal{F}(n,k,t_1-1,t_2)$, and hence $|\mathcal{F}(n,k,t_1-1,t_2)| \ge |\mathcal{F}(n,k,t_1,t_2)|$.

Now we suppose $t_1, t_2$ are both odd and $t_1 + t_2 > 2k$, then we have $t_1 + t_2 \ge 2k + 2$ by the parity. Then at least one of $t_1, t_2$ is at least 3. Without loss of generality we assume $t_1 \ge 3$ since $2k + 2 \ge 4$. Direct comparison shows $|\mathcal{F}(n,k,t_1-2,t_2)| \ge |\mathcal{F}(n,k,t_1,t_2)|$. Repeat this process and we will get a pair of $t_1', t_2'$ such that $|\mathcal{F}(n,k,t_1',t_2')| \ge |\mathcal{F}(n,k,t_1,t_2)|$, and $t_1' + t_2' \le 2k$.

That is, there is a pair of $t_1^*$, $t_2^*$ maximising $|\mathcal{F}(n,k,t_1,t_2)|$ such that both $t_1^*$, $t_2^*$ are odd and $t_1^* + t_2^* \le 2k$. $\square$

*Proof of Lemma 3.4.4.* For part (a), fix positive integers $k \ge 3$, $3 \le t_1 \le k$, $1 \le t_2 \le k$, and let $n = 2k + 1$. By direct comparison of set families, we have

$$|\mathcal{F}(2k+1,k,t_1,t_2) \setminus \mathcal{F}(2k+1,k,t_1-2,t_2)|$$

$$= \binom{t_1 - 2}{(t_1-3)/2} \sum_{i=0}^{(t_2-1)/2} \binom{t_2}{i} \binom{2k+1-t_1-t_2}{k-(t_1-3)/2-2-i},$$

and

$$|\mathcal{F}(2k+1,k,t_1-2,t_2) \setminus \mathcal{F}(2k+1,k,t_1,t_2)|$$
$$= \binom{t_1 - 2}{(t_1-1)/2} \sum_{i=0}^{(t_2-1)/2} \binom{t_2}{i} \binom{2k+1-t_1-t_2}{k-(t_1-1)/2-i}.$$

Hence $|\mathcal{F}(2k+1,k,t_1,t_2)| \geq |\mathcal{F}(2k+1,k,t_1-2,t_2)|$ if and only if for each $i$ that $0 \leq i \leq \dfrac{t_2-1}{2}$, we have

$$\binom{2k+1-t_1-t_2}{k-(t_1-1)/2-i-1} \geq \binom{2k+1-t_1-t_2}{k-(t_1-1)/2-i}.$$

Which is true if and only if $\dfrac{(t_2-1)}{2} \geq i$, which is always true.

For part (b), fix $k \geq 3$, $x \geq 1$ and let $n = 2k+1$. By direct comparison of set families, we have

$$|\mathcal{F}(2k+1,k,k+x,k-x) \setminus \mathcal{F}(2k+1,k,k+x-2,k-x+2)|$$
$$= \binom{k+x-2}{(k+x-3)/2}\binom{k-x+1}{(k-x-1)/2} + \binom{k+x-2}{(k+x-3)/2}\binom{k-x}{(k-x+1)/2},$$

and

$$|\mathcal{F}(2k+1,k,k+x-2,k-x+2) \setminus \mathcal{F}(2k+1,k,k+x,k-x)|$$
$$= \binom{k+x-2}{(k+x-1)/2}\binom{k-x}{(k-x-1)/2} + \binom{k+x-1}{(k+x-3)/2}\binom{k-x}{(k-x-1)/2}.$$

Hence $|\mathcal{F}(2k+1,k,k+x,k-x)| \leq |\mathcal{F}(2k+1,k,k+x-2,k-x+2)|$ if and only if

$$\binom{k+x-2}{(k+x-3)/2}\binom{k-x+1}{(k-x-1)/2} + \binom{k+x-2}{(k+x-3)/2}\binom{k-x}{(k-x+1)/2}$$
$$\leq \binom{k+x-2}{(k+x-1)/2}\binom{k-x}{(k-x-1)/2} + \binom{k+x-1}{(k+x-3)/2}\binom{k-x}{(k-x-1)/2},$$

which is true if and only if $x \geq 1$. $\square$

### 3.4.2 Open Questions

We discussed when inertia bounds on the largest $n'$-partite subgraph of $K(n,k)$ is useful, and its limitations in this chapter. Since the inertia bounds is built on Cauchy Interlacing Theorem, we can assign any weights in the weighted adjacency matrix in any position with an edge. (Non-edge positions remains zero.) Naturally, one may ask whether a more flexible weighting technique leads to better inertia bounds than Theorem 3.2.8 and 3.2.12.

We explored the bipartite case by allowing different weights in edges from different 'layers' of the graph product $K(n,k) \square K_2$. That is, we studied the eigenvalues of $A(K(n,k)) \otimes \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} + I_{\binom{n}{k}} \otimes \begin{bmatrix} 0 & y \\ y & 0 \end{bmatrix}$. Unfortunately, we receive the same best bound as in Theorem 3.2.8. It seems to suggest that in order to receive better inertia bound, one need to introduce more flexibility on those weights. We leave rest of the exploration to interested readers.

We have better induced bipartite subgraphs of $K(n,k)$ comparing to Frankl and Füredi's result in [18] if $n < 2k + \frac{\sqrt{\pi}}{4}\sqrt{k}$. Whilst our upper bound is approximately $1.4t(n,k,2)$, our construction achieves $1.3t(n,k,2)$ when $k$ is large. Hence it is also interesting to ask:

**Question 3.4.6.**
*If $G$ is $(n,k)$-colourable and $n < 2k+\frac{\sqrt{\pi}}{4}\sqrt{k}$, then what is the largest induced bipartite subgraph in $G$?*

Naturally, the next question following above regards tripartite induced subgraphs. Is there better general constructions (than Examples 3.2.9 and the *trivial* tripartite) of a 'large' tripartite induced subgraph of $K(n,k)$? For smaller values of $n$ and $k$, it is possible to compute the exact maximum tripartite subgraphs of $K(n,k)$ using computer programs. For instance, the largest tripartite subgraph of $K(8,3)$ has 47 vertices, and one such tripartite subgraph attaining 47 is by taking a maximum independent set in $K(8,3)$ (which has $\binom{7}{2} = 21$ vertices), and the largest bipartite subgraph in the remaining $K(7,3)$ (which has 26 vertices, as we will see in Section 4.2.1). Note this example is larger than the maximal trivial tripartite subgraph of

$K(8,3)$, which has $\binom{8}{3} - \binom{5}{3} = 46$ vertices.

In general, upper bounds on the largest tripartite subgraph of $K(n,k)$ comes from the maximum of following:

• three non-*trivial* independent sets in $K(n,k)$;

• a largest independent set in $K(n,k)$ and a largest bipartite subgraph in the rest $K(n-1,k)$;

• the *trivial* tripartite subgraph of $K(n,k)$.

Direct computation on the above ideas, using the Erdős-Ko-Rado and Hilton-Milner bounds does not leads better results than what are already discussed.

Also note that Frankl and Füredi's shifting method cannot be directly applied on $n'$-partite subgraph of $K(n,k)$ if $n' \geq 3$. (In the sense that 'shifted' $n'$-colourable subgraph may no longer be $n'$-colourable, for any $n' \geq 2$.) Hence it is also of interest to study if an analogous method or other structural may lead to better upper bounds.

# 4

# Partial Multi-Colouring and Fractional Colouring

## 4.1 Introduction

In this chapter, we study more general partial colouring problems associated with *multi-colouring* and *fractional colouring*. Graphs in this chapter are undirected and with neither multiple edges nor loops.

Recall that in a $k$*-multi-colouring* of a graph, each vertex receives a set of $k$ different colours, and such a colouring is *proper* if adjacent vertices receive disjoint colour sets. A graph $G$ is $(n,k)$*-colourable* if there is a proper $k$-multi-colouring of $G$ using $n$ colours in total. And the $k$*-th multi-chromatic number* $\chi_k(G)$ is the smallest $n$ such that $G$ is $(n,k)$-colourable.

Also recall that a graph $G$ is *fractional-*$\frac{n}{k}$*-colourable* if $G$ is $(tn,tk)$-colourable for some $t \geq 1$. The *fractional chromatic number* $\chi_f(G)$ is the infimum of $\frac{n}{k}$ so that $G$ is fractional-$\frac{n}{k}$-colourable. It is well-known (for example, see in [43]) that this infimum is a minimum, i.e. if $\chi_f(G) = \frac{n}{k}$, then $G$ is $(tn,tk)$-colourable for some $t \geq 1$.

In this chapter, we study the following two sub-questions.

**Question 3.1.1.**

(b) *Given rational numbers $r, s > 0$, for what real number $b$ can we guaran-*

*tee that every fractional-r-colourable graph $G$ has a fractional-s-colourable induced subgraph with at least $b|V(G)|$ vertices?*

(c) *Given positive integers $n, k, n', k'$, for what real number $c$ can we guarantee that every $(n, k)$-colourable graph $G$ has an $(n', k')$-colourable induced subgraph with at least $c|V(G)|$ vertices?*

We present main results as well as proofs in Section 4.2. Discussion on results, open problems and further interests are in Section 4.3.

## 4.2 Main Results and Their Proofs

We present our main results regarding Question 3.1.1 (b) and (c) in this section. As most of their proofs are reasonably short, we also include the proofs in this section.

Let $G$ be a graph, $n'$ a positive integer and $s$ a positive rational number. Recall the definition of $\gamma(G, n')$ that we studied in the last chapter, as well as $\gamma_f(G, s)$ that we introduced in the first chapter:

$$\gamma(G, n') = \max\left\{\frac{|V(H)|}{|V(G)|}\ \middle|\ H \leqslant_i G,\ H \text{ is } n'\text{-colourable}\right\},$$
$$\gamma_f(G, s) = \max\left\{\frac{|V(H)|}{|V(G)|}\middle|H \leqslant_i G,\ \chi_f(H) \leq s\right\}.$$

(Again, $H \leqslant_i G$ means $H$ is an induced subgraph of $G$.)

We discussed $\gamma(G, n')$ with $n' < 2$ in the last chapter. Note that if $s$ is strictly smaller than 2, then $\gamma_f(G, s)$ equals to $\gamma(G, \lfloor s \rfloor)$. In particular, for any $s < 1$, if $H \leqslant_i G$ has fractional chromatic number at most $s$, then we know $H$ has no vertex. Which leads to $\gamma_f(G, s) = \gamma(G, 0) = 0$ if $s < 1$. For $1 \leq s < 2$, if $H \leqslant_i G$ has fractional chromatic number at most $s$, then $H$ is edgeless. Hence $\gamma_f(G, s) = \gamma(G, 1) = \dfrac{\alpha(G)}{|V(G)|} \geq \dfrac{1}{\chi_f(G)}$. (Recall that $\alpha(G)$ denotes the *independence number* of $G$.) Here $\dfrac{\alpha(G)}{|V(G)|} \geq \dfrac{1}{\chi_f(G)}$ holds with essentially the same reason as last chapter: the fractional chromatic number $\chi_f(G)$ is the infimum of $\dfrac{n}{k}$ that $G$ is $(n, k)$-colourable. If a graph $G$ is $(n, k)$-colourable, then simple counting gives $k|V(G)| \leq n\alpha(G)$, i.e. $\dfrac{\alpha(G)}{|V(G)|} \geq \dfrac{1}{n/k}$

for any pair of $n, k$ that $G$ is $(n, k)$-colourable. This lower bound is also the best possible with given fractional chromatic number by the cases that $G$ with the given fractional chromatic number is *vertex transitive*.

For general $s \geq 2$, we are interested in deciding the best possible lower bounds of $\gamma_f(G, s)$ with given fractional chromatic number. Recall that

$$\gamma_f(r, s) = \inf \{\gamma_f(G, s) | \chi_f(G) = r\}.$$

Later in this chapter, we will prove the following results. Note those infimums ($\gamma_f(r, s)$ where $r > s \geq 2$) are not minimum, i.e. not attained by any graph.

**Theorem 4.2.1.**
*For any rational numbers $r > s \geq 2$, we have $\gamma_f(r, s) \geq 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$; furthermore, if $\lfloor s \rfloor \leq \frac{r-1}{2}$, then we have $\gamma_f(r, s) = 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$.*

We start by showing the lower bound for $\gamma_f(G, s)$ with given fractional chromatic number of $G$ are always observed by some extensively studied graphs called *Kneser graphs*, as similar to $\gamma(G, s)$.

Recall that for $n \geq k \geq 1$, the *Kneser graph* $K(n, k)$ has the collection of all $k$-subsets of $[n]$ (denoted by $\binom{[n]}{k}$) as vertex set, and there is an edge between two vertices if and only if these two $k$-sets are disjoint. We will always assume $n \geq 2k$, as otherwise the graph is edgeless. It is well-known (see e.g. [43]) that a graph $G$ is $(n, k)$-colourable if and only if there is a homomorphism from $G$ to $K(n, k)$: in a proper $(n, k)$-colouring of $G$, the set of $k$ colours on each vertex $x$ of $G$ gives which vertex of $K(n, k)$ is $x$ mapped to; and a homomorphism from $G$ to $K(n, k)$ naturally gives which $k$ colours should be assigned to each vertex in a proper $k$-multi-colouring.

To determine $\gamma_f(r, s)$ with given $r$, it suffices to only consider Kneser graphs with fractional chromatic number $r$ by Theorem 3.2.2 and its extension to $\gamma_f(G, s)$.

**Theorem 3.2.2 (extended).**
*If $G$ is $(n, k)$-colourable, then $\gamma(G, s) \geq \gamma(K(n, k), s)$ and $\gamma_f(G, s) \geq \gamma_f(K(n, k), s)$.*

Extended Theorem 3.2.2 is again an easy corollary of the following, and the well-known fact that Kneser graphs are vertex transitive.

**Theorem 3.2.3 (extended).**
*If there is a graph homomorphism from a graph $G$ to a vertex transitive graph $H$, then for any $s$, we have $\gamma(G,s) \geq \gamma(H,s)$ and $\gamma_f(G,s) \geq \gamma_f(H,s)$.*

The same proof of Theorem 3.2.3 applies to the above extension: in the proof, we proved that if there is a homomorphism from a graph $G$ to a vertex transitive graph $H$, then for any $n \geq 2k$, the largest $(n,k)$-colourable subgraph of $G$ has at least the same portion of any $(n,k)$-colourable subgraph of $H$. We can then conclude the extended Theorem 3.2.3 by definition of $\gamma(G,s)$ and $\gamma_f(G,s)$.

### 4.2.1 Counterexamples of the AGH Conjecture in Fractional Colouring

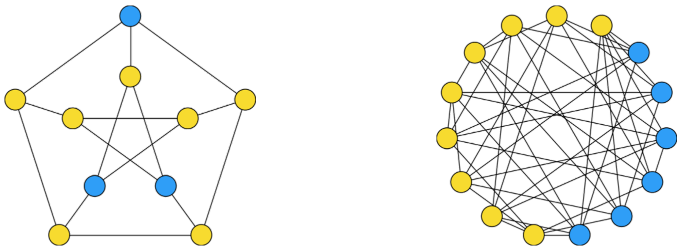
Recall that our Question 3.1.1 is inspired by a conjecture of by Albertson et al. [2] regarding partial list colouring. In this section, we show by examples that the generalisation of that conjecture in fractional colouring is not true.

They asked: for a given graph $G$ with list chromatic number $n$, if each vertex has $n'$ colours where $1 \leq n' \leq n$, can we always properly colour at least $\frac{n'}{n}|V(G)|$ vertices of $G$?

We present the following examples that if a graph has fractional chromatic number $r$, and $n' \leq r$ colours are given, then we cannot always properly colour $\frac{n'}{r}|V(G)|$ vertices of $G$. Those examples are also highlighted by light yellow in the images following.

• The bipartite subgraph induced by $\left\{F \in \binom{[5]}{2} \middle| F \cap \{1,2\} \neq \emptyset\right\}$ in $K(5,2)$ is maximum. I.e. $\gamma(K(5,2),2) = \frac{7}{10}$. Which also means if $G$ is $(5,2)$-colourable, then at least $\frac{7}{10}|V(G)|$ vertices of $G$ can be properly coloured with 2 colours. But we cannot guarantee any ratio larger than $\frac{7}{10}$. (In particular, if $G$ has fractional chromatic number $\frac{5}{2}$, then we cannot always properly colour $\frac{2}{5/2}|V(G)|$ vertices of $G$.)

- The bipartite subgraph induced by $\left\{F \in \binom{[6]}{2} \middle| F \cap \{1,2\} \neq \emptyset\right\}$ in $K(6,2)$ is maximum. I.e. $\gamma(K(6,2),2) = \frac{3}{5}$. Which also means if $G$ is $(6,2)$-colourable, then at least $\frac{3}{5}|V(G)|$ vertices of $G$ can be properly coloured with 2 colours. But we cannot guarantee more.



*One of the maximum bipartite subgraphs in $K(5,2)$ and in $K(6,2)$. $K(5,2)$ is also famously known as the Petersen Graph.*

No more than $\frac{7}{10}$ of $K(5,2)$, nor more than $\frac{3}{5}$ of $K(6,2)$ can be properly coloured by 2 colours. Note both ratios are smaller than $\frac{2}{\chi_f(G)}|V(G)|$.

It is not hard to prove the following. We assume $n \geq 5$ since if $n = 4$ and $G$ is $(4,2)$-colourable, then this whole graph is bipartite, i.e. 2-colourable. (Note this implies a lower bound on $\gamma\left(\frac{n}{2}, 2\right)$, but not the exact value. Since fractional-$\frac{n}{2}$-colourable graphs are not necessarily $(n,2)$-colourable.)

**Lemma 4.2.2.**
*For any given $n \geq 5$, if a graph $G$ is $(n,2)$-colourable, then $\gamma_f(G,2) = \gamma(G,2) \geq \frac{4n-6}{n(n-1)}$. This is the best possible lower bound if we only know $G$ is $(n,2)$-colourable.*

*Proof.* First note the bipartite subgraph induced by

$$\left\{F \in \binom{[n]}{2} \middle| F \cap \{1,2\} \neq \emptyset\right\}$$

has $2n-3$ vertices, hence $\gamma_f(G,2) = \gamma(G,2) \geq \frac{2n-3}{n(n-1)/2}$ by Theorem 3.2.2.

We then prove $\gamma_f(K(n,2),2) = \gamma(K(n,2),2) = \dfrac{4n-6}{n(n-1)}$. It is not hard to claim the largest independent set in $K(n,2)$ is of order $n-1$ and if $n \geq 5$, then it must be the case that all vertices share one same element. If there are three vertices in an independent set that do not share any common element, then it is not possible to add a fourth vertex while keeping them independent (i.e. keeping sets represented by those vertices pairwise intersecting).

That is, since each bipartite subgraph is the union of two independent sets, if there is a bipartite subgraph of $K(n,k)$ with at least $2n-2$ vertices, then both independent sets must be exactly of order $n-1$. But then their union is of order $2n-3$. □

The largest bipartite subgraphs in $K(n,2)$ are the so-called *trivial* bipartite subgraphs. But the above observations does not generalise to $K(7,3)$ in the most obvious way: the maximal bipartite subgraphs of $K(7,3)$ following above construction is $H_1 = \left\{F \in \binom{[7]}{3} \middle| F \cap \{1,2\} \neq \emptyset\right\}$ with 25 vertices, but $H_2 = \left\{F \in \binom{[7]}{3} \middle| |F \cap \{1,2,3\}| \geq 2 \text{ or } |F \cap \{4,5,6\}| \geq 2\right\}$ with 26 vertices is also bipartite. Furthermore, it is not hard to prove that $\gamma(K(7,3),2) = \dfrac{26}{35}$. (We can prove it by careful case analysis, or simply apply the Hilton-Milner bound, as 26 is exactly 2 times the maximum non-*trivial* maximum independent set in $K(7,3)$.)

That is, if graph $G$ is $(7,3)$-colourable, then at least $\dfrac{26}{35}|V(G)|$ vertices of $G$ can be properly coloured with 2 colours. And this is the best possible lower bound if we only know $G$ is $(7,3)$-colourable. This is another example showing the generalisation of AGH Conjecture in fractional colouring does not hold.

### 4.2.2 $s$-Fractional-colourable induced subgraph of $K(n,k)$

In this section, we study $\gamma_f(G,s)$ and $\gamma_f(r,s)$ more carefully.

Recall that if $\chi_f(G) = \dfrac{n}{k}$ (that $n,k$ are co-primes), then $G$ is $(tn,tk)$-colourable for some integer $t$. And if an induced subgraph $H \leqslant_i K(n,k)$ is fractional-$s$-colourable, then $K_{\lfloor s\rfloor+1}$ is not a subgraph of $H$, i.e. $V(H)$

contains no more than $\lfloor s \rfloor$ pairwise disjoint sets. On the other hand, any $\lfloor s \rfloor$-colourable subgraph of $K(n, k)$ is clearly fractional-$s$-colourable.

Following above observations, we have that the problem of largest fractional-$s$-colourable induced subgraph problem is closely related to the Erdős Matching Conjecture [12].

**Conjecture 4.2.3** (Erdős [12])**.**
*For any $k \geq 2$, $s \geq 0$ and $n \geq (s+1)k-1$, if there is no $s+1$ pairwise disjoint $k$-sets in $\mathcal{F} \subset \binom{[n]}{k}$, then we have $|\mathcal{F}| \leq \binom{n}{k} - \binom{n-s}{k}$.*

It is easy to see the conjectured upper bound is attained by the *trivial $s$-partite subgraph* of $K(n, k)$: $\left| \left\{ F \in \binom{[n]}{k} \middle| F \cap [s] \neq \emptyset \right\} \right| = \binom{n}{k} - \binom{n-s}{k}$. The Erdős Matching Conjecture (Conjecture 4.2.3) has been studied extensively. Erdős proved the correctness of Conjecture 4.2.3 for large enough $n > n_0 = n_0(s, k)$ in his original paper. There have been massive number of results on decreasing this $n_0(s, k)$, see e.g. in [4, 19, 30]. The most recent progress that works for all $k$ is in [16] that the Erdős Matching Conjecture is true if $n \geq (2s+1)k - s$ (for integer $s$). It is also proved to be true if $k = 2$ in [13] or $k = 3$ in [17, 20, 39].

Those known cases of the Matching Conjecture allows us to determine the exact $\gamma_f(r, s)$ for $r, s$ in certain range of values.

**Theorem 4.2.1 (full).**
*For any positive rational number $r$, if a graph $G$ has fractional chromatic number $r$, then $\gamma_f(G, s) > 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$. Furthermore, if $r \geq 2\lfloor s \rfloor + 1$, then $\gamma_f(G, s) > 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$ is the best possible lower bound with given $r$.*

*That is, for any positive rationals $r, s$, we have $\gamma_f(r, s) \geq 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$; furthermore, if $r > s \geq 2$ and $r \geq 2\lfloor s \rfloor + 1$, then $\gamma_f(r, s) = 1 - \left(1 - \frac{1}{r}\right)^{\lfloor s \rfloor}$.*

*Proof of Theorem 4.2.1 (full).* For given rational number $r > 0$, consider any graph $G$ that $\chi_f(G) = r = \frac{n}{k}$, with co-primes $n$, $k$. Hence $G$ is $(tn, tk)$-colourable for some positive integer $t$.

With the same proof as in Theorem 3.2.7, we have

$$\gamma_f(G,s) \geq \gamma_f(K(tn,tk),\lfloor s \rfloor) > 1-\left(1-\frac{1}{r}\right)^{\lfloor s \rfloor}.$$

Now consider any $s$-fractional-colourable induced subgraph $H$ of $K(tn,tk)$. Clearly $H$ does not contain the $(\lfloor s \rfloor+1)$-clique as a subgraph. (As otherwise $\chi_f(H) \geq \chi_f\left(K_{\lfloor s \rfloor+1}\right) = \lfloor s \rfloor + 1$.) That is, $V(H) \subseteq \binom{[tn]}{tk}$ is a set family without $\lfloor s \rfloor + 1$ pairwise disjoint $tk$-sets.

Therefore if $r \geq 2\lfloor s \rfloor+1$, then $tn \geq (2\lfloor s \rfloor+1)tk > (2\lfloor s \rfloor+1)tk - \lfloor s \rfloor$, and by those known cases of the Matching Conjecture, we conclude that $|V(H)| \leq \binom{tn}{tk} - \binom{tn-\lfloor s \rfloor}{tk}$. Following the same calculation as in Theorem 3.2.7, we have $\gamma_f(K(tn,tk),s) > 1-\left(1-\frac{1}{r}\right)^{\lfloor s \rfloor}$ is the best possible general lower bound that holds for all possible $t$. Thus $\gamma_f(r,s) \geq 1-\left(1-\frac{1}{r}\right)^{\lfloor s \rfloor}$. $\square$

Before closing this section, we note the known cases that the Matching Conjecture is true if $k=2$ or $k=3$ allows us to conclude the following. We omit the proofs since it is essentially the same as Theorem 4.2.1 but with concrete numbers.

**Corollary 4.2.4.**
*For any integer $n$ and real number $s$,*
*if $n \geq 2(\lfloor s \rfloor + 1)$, then $\gamma_f(K(n,2),s) = t(n,2,\lfloor s \rfloor)$;*
*if $n \geq 3(\lfloor s \rfloor + 1)$, then $\gamma_f(K(n,3),s) = t(n,3,\lfloor s \rfloor)$.*

### 4.2.3 $(n',k')$-Colourable induced subgraph of $K(n,k)$

In this section, we study the maximum induced subgraph of $K(n,k)$ that is $(n',k')$-colourable (have a homomorphism to $K(n',k')$) with given $n'$ and $k'$. Note $\frac{n'}{k'} > \frac{n}{k}$ does not guarantee a $(n,k)$-colourable graph is $(n',k')$-colourable: for example $\chi(K(21,7)) = 9$, hence not $(6,1)$-colourable whilst $6 > \frac{21}{7}$.

Recall that $\pi(G,K(n',k')) := \max\left|\left\{|V(H)| : H \leqslant_i G, H \to K(n',k')\right\}\right|$.
(Note $H_1 \to H_2$ stands for a homomorphism from graph $H_1$ to graph $H_2$.)

In Theorem 3.2.3, we proved that if there is a graph homomorphism from a graph $G$ to a vertex transitive graph $H$, then for any $n' \geq 2k'$, we have $\dfrac{\pi(G, K(n', k'))}{|V(G)|} \geq \dfrac{\pi(H, K(n', k'))}{|V(H)|}$. In particular, if we know $G$ is $(n, k)$-colourable for some $n, k$ (but nothing else is known about $G$), then we only need to study $\pi(K(n, k), K(n', k'))$ to decide a 'guaranteed portion' of $G$ that is $(n', k')$-colourable: since $\dfrac{\pi(K(n, k), K(n', k'))}{|V(K(n, k))|}$ is a lower bound for $\dfrac{\pi(G, K(n', k'))}{|V(G)|}$ for any $(n, k)$-colourable graph $G$, and this is attained if for example, $G$ is $K(n, k)$.

Not surprisingly, progress on the Erdős Matching Conjecture answers $\pi(K(n, k), K(n', k'))$ for some ranges of parameters, and hence answers Question 3.1.1 (c) for those ranges of parameters.

**Theorem 4.2.5.**

*For any $n \geq 2k$ and $n' \geq 2k'$, we have*

$$\pi(K(n, k), K(n', k')) \geq \max_{1 \leq b \leq k} \left\{ \sum_{i=0}^{\min\{a-b, k-b\}} \binom{a}{b+i} \binom{n-a}{k-b-i} \right\},$$

*where $a = 2b + \left\lfloor \dfrac{n' - 2k'}{\lceil k'/b \rceil} \right\rfloor$.*

*Furthermore, if $n \geq \left(2 \left\lfloor \dfrac{n'}{k'} \right\rfloor + 1\right) k - \left\lfloor \dfrac{n'}{k'} \right\rfloor$, then*

$$\pi(K(n, k), K(n', k')) = \binom{n}{k} - \binom{n - \lfloor n'/k' \rfloor}{k}.$$

**Corollary 4.2.6.**

*For any $n \geq 2k$ and $n' \geq 2k'$, if a graph $G$ is $(n, k)$-colourable, then*

$$\pi(G, K(n', k')) \geq \max_{1 \leq b \leq k} \left\{ \sum_{i=0}^{\min\{a-b, k-b\}} \binom{a}{b+i} \binom{n-a}{k-b-i} \right\} \cdot \frac{|V(G)|}{\binom{n}{k}},$$

*where $a = 2b + \left\lfloor \dfrac{n' - 2k'}{\lceil k'/b \rceil} \right\rfloor$.*

*Furthermore, if $n \geq \left(2 \left\lfloor \dfrac{n'}{k'} \right\rfloor + 1\right) k - \left\lfloor \dfrac{n'}{k'} \right\rfloor$, then $\left(1 - \dfrac{\binom{n - \lfloor n'/k' \rfloor}{k}}{\binom{n}{k}}\right) |V(G)|$ is the best possible lower bound of $\pi(G, K(n', k'))$.*

*Proof of Theorem 4.2.5.* Proof to the first part of the theorem is by construction.

For any $b \le k$ and $a \le n$, denote $\mathcal{F}(n,k,a,b) = \left\{ F \in \binom{[n]}{k} \middle| |F \cap [a]| \ge b \right\}$. It is not hard to see that

$$|\mathcal{F}(n,k,a,b)| = \sum_{i=0}^{\min\{a-b,k-b\}} \binom{a}{b+i}\binom{n-a}{k-b-i}.$$

Note

$$|\mathcal{F}(n,k,\lfloor n'/k' \rfloor,1)| = \binom{n}{k} - \binom{n-\lfloor n'/k' \rfloor}{k}$$

and

$$|\mathcal{F}(n,k,n',k')| = \sum_{i=0}^{\min\{n'-k',k-k'\}} \binom{n'}{k'+i}\binom{n-n'}{k-k'-i},$$

where the second formula is only valid if $k' \le k$ and $n' \le n$.

It suffices to show that $\mathcal{F}(n,k,a,b)$ is $(n',k')$-colourable for any $b \le k$ and $a = 2b + \left\lfloor \frac{n'-2k'}{\lceil k'/b \rceil} \right\rfloor$. This follows from the fact that Kneser graph $K(a,b)$ is $(xa-2r, xb-r)$-colourable for any $x \ge 1$ and $0 \le r \le b-1$ (by Equation (2.2) in Section 2.2).

Firstly, $\mathcal{F}(n,k,a,b)$ is $(a,b)$-colourable since for any $F \in \mathcal{F}(n,k,a,b)$, we simply assign any $b$ colours from $F \cap [a]$. Hence if $F_1 \cap F_2 = \emptyset$ (i.e. there is an edge between $F_1$ and $F_2$), then the colours assigned to $F_1$ and $F_2$ are disjoint. I.e. this is a proper $(a,b)$-colouring on $\mathcal{F}(n,k,a,b)$.

Let $x = \left\lceil \frac{k'}{b} \right\rceil$ and $r = \left\lceil \frac{k'}{b} \right\rceil b - k'$,
then $K(a,b)$ is $\left( \left\lceil \frac{k'}{b} \right\rceil a - 2\left\lceil \frac{k'}{b} \right\rceil b + 2k', k' \right)$-colourable. Then since

$$\left\lceil \frac{k'}{b} \right\rceil \left( 2b + \left\lfloor \frac{n'-2k'}{\lceil k'/b \rceil} \right\rfloor \right) - 2\left\lceil \frac{k'}{b} \right\rceil b + 2k' = \left\lceil \frac{k'}{b} \right\rceil \cdot \left\lfloor \frac{n'}{\lceil k'/b \rceil} \right\rfloor \le n',$$

we have that $K(a,b)$ is $(n',k')$-colourable, and hence $\mathcal{F}(n,k,a,b)$ is $(n',k')$-colourable.

The second part of the theorem follows from current progress on the Matching Conjecture that the conjecture if true for integer $s$ if $n \ge (2s+1)k - s$.

Similar to Theorem 4.2.1, if a subgraph $H \leqslant_i K(n,k)$ is $(n',k')$-colourable, then $V(H) \subseteq \binom{[n]}{k}$ does not contain $\left\lfloor \frac{n'}{k'} \right\rfloor + 1$ pairwise disjoint sets. Hence if $n \geq \left(2\left\lfloor \frac{n'}{k'} \right\rfloor + 1\right)k - \left\lfloor \frac{n'}{k'} \right\rfloor$, then $|V(H)| \leq \binom{n}{k} - \binom{n - \lfloor n'/k' \rfloor}{k}$. □

For those parameters not included in Theorem 4.2.5, first note that the Erdős Matching Conjecture is also proved to be true for $k = 2$, $k = 3$ and $n \geq sk + 1$ in [17, 20, 39]. These lead to the following.

**Theorem 4.2.7.**
$\pi(K(6,2), K(5,2)) = 10$, *and for any* $n \geq 7$, *we have* $\pi(K(n,2), K(5,2)) = \binom{n}{2} - \binom{n-2}{2}$.
$\pi(K(7,3), K(5,2)) = 35$, *and for any* $n \geq 10$, *we have* $\pi(K(n,3), K(5,2)) = \binom{n}{3} - \binom{n-2}{3}$.

*Proof of Theorem 4.2.7.* Since $K(5,2)$ is an induced subgraph of $K(6,2)$, we have

$$\pi(K(6,2), K(5,2)) \geq |V(K(5,2))| = 10.$$

By Pigeonhole's Principle, any 11 sets in $\binom{[6]}{2}$ contains three pairwise disjoint sets, hence forming a $K_3$ which is not $(5,2)$-colourable. Therefore $\pi(K(6,2), K(5,2)) = 10$.

The whole graph $K(7,3)$ is $(5,2)$-colourable, hence $\pi(K(7,3), K(5,2)) = 35$.

Note any $(5,2)$-colourable subgraph of $K(n,k)$ does not have $K_3$ as a subgraph, hence there is no three pairwise disjoint sets in the vertex set. Hence by the proven cases of Matching Conjecture, we have $\pi(K(n,2), K(5,2)) = \binom{n}{2} - \binom{n-2}{2}$ for $n \geq 7$ and $\pi(K(n,3), K(5,2)) = \binom{n}{3} - \binom{n-2}{3}$ for $n \geq 10$. □

We can also prove $\pi(K(7,3), K(6,3)) = 25$ and $\pi(K(10,4), K(8,4)) = 140$ using the well-known fact that $H$ is $(2k', k')$-colourable for some $k'$ if and only if $H$ is bipartite, and careful case analysis. We omit the proofs for these two small results, as they can be easily verified, but do not lead to any further cases.

Some other parameters can be checked with the assistance of computer program. Recall that in Section 3.2.1, we formulate the partial colouring problem into an independent set problem. Similarly, we can also formulate the problem of 'maximum induced subgraph of $K(n,k)$ that has a homomorphism to $K(n',k')$' into the problem of 'maximum independent set in some graph product of $K(n,k)$ and $K(n',k')$'.

In the next proposition, we use the following notations:

• $G \,\Box\, H$ is the *Cartesian product* between graphs $G$ and $H$;

• $G \times H$ is the *tensor product* between graphs $G$ and $H$: we have $\{(u,v), u \in G, v \in H\}$ as the vertex set, and $(u_1,v_1)(u_2,v_2)$ is an edge in $G \times H$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$;

• $G + H$ combines edge sets of two graphs with the same vertex set, we can do so whenever there is a natural one to one correspondence between the vertex set of graphs $G$ and $H$.

Also recall that $\alpha(G)$ is the *independence number* of $G$ (maximum order of an independent set of $G$), and $\overline{G}$ denotes the *complement graph* of $G$ (that is, $\overline{G}$ has the same vertex set as $G$, but $uv$ ($u \neq v$) is an edge in $\overline{G}$ if and only if $uv$ is not an edge in $G$).

**Proposition 4.2.8.**
*For any positive integers $n \geq 2k$ and $n' \geq 2k'$, we have*
$$\pi(K(n,k), K(n',k')) = \alpha\left(K(n,k) \,\Box\, K_{\binom{n'}{k'}} + K(n,k) \times \overline{K(n',k')}\right).$$

*Proof of Proposition 4.2.8.* First note in our graph product, we represent the vertex set of $K(n,k)$ and $\overline{K(n',k')}$ as usual, and we also use $\binom{[n']}{k'}$ as the vertex set of $K_{\binom{n'}{k'}}$. (There is no ambiguity since complete graph has all the possible edges.) Hence the summation of graph $K(n,k) \,\Box\, K_{\binom{n'}{k'}}$ and graph $K(n,k) \times \overline{K(n',k')}$ is well defined.

It is not hard to find an one-to-one correspondence of a $(n',k')$-colourable induced subgraph of $K(n,k)$ and an independent set in the graph product $K(n,k) \,\Box\, K_{\binom{n'}{k'}} + K(n,k) \times \overline{K(n',k')}$.

Assume $H \leqslant_i K(n,k)$ is $(n',k')$-colourable and $f : V(H) \to \binom{[n']}{k'}$ is a proper $(n',k')$-colouring of $H$. Then it is not hard to verify that $\{(v, f(v)) | v \in$

$V(H)\}$ is an independent set in $K(n,k) \,\square\, K_{\binom{n'}{k'}} + K(n,k) \times \overline{K(n',k')}$.

Similarly, if $H$ is an independent set in $K(n,k) \,\square\, K_{\binom{n'}{k'}} + K(n,k) \times \overline{K(n',k')}$, then $H' = \{v_1 | (v_1, v_2) \in V(H)$ for some $v_2\}$ is a $(n', k')$-colourable induced subgraph of $K(n,k)$. (Note it is not possible that both $(v_1, v_2)$ and $(v_3, v_4)$ are in $V(H)$, but $v_1 = v_3$ while $v_2 \neq v_4$, nor $v_2 = v_4$ while $v_1 \neq v_3$.) And we have a proper $(n', k')$-colouring of $H'$ by taking $f(v_1) = v_2$ for each $(v_1, v_2) \in V(H)$. $\square$

Proposition 4.2.8 allows us to use existing independence number computer programs to compute results for smaller pairs of $(n,k)$ and $(n',k')$. Since the product of graphs we consider are highly symmetric, for each specific pair of parameters, we can break the symmetry (to allow faster computation) by removing vertices that are always included in some of the largest independent set. For instance, since the resulting graph product is also vertex transitive, we can always assume any fixed vertex is included in some maximum independent set. That is, we can delete any fixed vertex and its neighbours, then find the largest independent set in the resulting graph and add the first fixed vertex back.

We verified that $\pi(K(8,3), K(5,2)) = 40$ and $\pi(K(9,4), K(2,1)) = 96$ using computer programs. (It is noted in [24] that $96 \leq \pi(K(9,4), K(2,1)) \leq 98$.) Additionally, Leonard Soicher helped us verified that $\pi(K(9,3), K(5,2)) = 50$ using his group theoretic computer program, which was built for highly symmetric graphs.

In the rest of this section, we provide some examples of maximum partially colourable subgraphs.

We are aware of one $(5,2)$-colourable subgraph (up to isomorphism) of $K(8,3)$ that attains $\pi(K(8,3), K(5,2)) = 40$: $\left\{F \in \binom{[8]}{3} \middle| |F \cap [5]| \geq 2\right\}$.

We are aware of many subgraphs that attain $\pi(K(9,4), K(2,1)) = 96$. But (those we are aware) all have $\left\{F \in \binom{[9]}{4} \middle| |F \cap [3]| \geq 2\right\}$ as one side (an independent set in the bipartite graph), and $\binom{[4,9]}{4} \cup \{H \cup \{x\} | H \in \mathscr{H}, x \in [3]\}$ for the other side. Here $\mathscr{H}$ can be any of the maximum independent sets of

$\binom{[4,9]}{3}$. (There are many such $\mathscr{H}$ because $|[4,9]| = 6 = 2 \times 3$.)
For larger graphs, we have lower bounds by construction and several upper bounds, but exact computation is still hard. For instance, we know $362 \leq \pi(K(11,5), K(2,1)) \leq 372$, but the product graph after deleting some vertices are still too large for a computer program.

One may ask whether similar algebraic bounds as in Sections 3.2.1 and 3.2.2 can be useful here. Unfortunately, simply weighting Hoffman's bound [29], inertia method (discussed in Section 3.2.1) or alternated Wilson's method [47] does not give us better upper bounds than considering the maximum $(n' - 2k' + 2)$-colourable subgraph of $K(n,k)$. (This is an upper bound since if $H$ is $(n', k')$-colourable, then $H$ is $(n' - 2k' + 2)$-colourable.) It is possible to study general inertia bounds by studying the number of positive and negative eigenvalues of a weighted version of $A(K(n,k) \ \square\ K_{\binom{n'}{k'}} + K(n,k) \times \overline{K(n',k')}) = A(K(n,k)) \otimes I_{\binom{n'}{k'}} + I_{\binom{n}{k}} \otimes A(K_{\binom{n'}{k'}}) + A(K(n,k)) \otimes A(\overline{K(n',k')})$, but the analysis become over complicated very soon. For instance, it is not hard to find all the eigenvalues of $\alpha A(K(n,k)) \otimes I_{\binom{n'}{k'}} + \beta I_{\binom{n}{k}} \otimes A(K_{\binom{n'}{k'}}) + \gamma A(K(n,k)) \otimes A(\overline{K(n',k')})$ as functions of $\alpha, \beta, \gamma$, but the analysis on finding best $\alpha, \beta, \gamma$ is very complicated. We hence don't include more details for space reasons.

## 4.3 Discussions

In this chapter, we studied upper and lower bounds of $\gamma_f(r,s)$ and $\pi(K(n,k), K(n',k'))$. We also derived some exact values with given ranges of parameters. But the following more general question is still unsolved.

**Question 4.3.1.**
*For given integer $k \geq 1$ and rational number $s \geq 2$, if $n \geq 2k$ is close to $sk$ (for example, if $n < 2sk$) and $\chi_f(G) = \dfrac{n}{k}$, then what is the best possible lower bound of $\gamma_f(G,s)$?*

Some parameters of $\pi(K(n,k), K(n',k'))$ were studied, and we showed some subgraphs that attain $\pi(K(8,3), K(5,2))$ or $\pi(K(9,4), K(2,1))$. It is of interest to understand whether there are other constructions attaining these $\pi(K(n,k), K(n',k'))$. Before closing this chapter, we also note the problem we studied in Chapter 2 is a specification of the questions we studied here.

# 5

# Effects of Graph Operations on Correspondence Chromatic Numbers

## 5.1 Introduction

In this and the next chapter, we study another variant of graph colouring called *correspondence colouring* or *DP-colouring* (after the authors of the paper in which it first appeared: Dvořák and Postle [9]). All graphs we consider will be undirected and finite; multiple edges are allowed, but not loops.

We use the notation $xy$ for the collection of all edges with endvertices $x$ and $y$. So if $e \in E(G)$ is an edge, then $e \in xy$ for some (distinct) vertices $x, y \in V(G)$. We usually say just "the edge $xy$" for this collection. The *multiplicity* of an edge $xy$ in a graph $G$, denoted $\mathfrak{m}_G(xy) = |xy|$, is the number of edges with endvertices $x$ and $y$. We write $\mathfrak{m}_G(xy) = 0$ if there is no edge between $x$ and $y$; we sometimes also write $\mathfrak{m}_G(xx) = 0$ for any $x \in V(G)$ (since loops are not allowed). An edge $xy$ is *simple* if $\mathfrak{m}_G(xy) = 1$; a graph $G$ is *simple* if $\mathfrak{m}_G(xy) \in \{0, 1\}$ for all $x, y \in V(G)$.

Recall that correspondence colouring generalises ordinary colouring. In correspondence colouring, each vertex is associated with a prespecified list of available colours, and each edge is associated with a prespecified correspondence, specifying which pair of colours from the two endvertices correspond. (On each edge, a colour on one endvertex corresponds to at most one colour on the other endvertex.) A correspondence colouring is *proper* if each vertex receives a colour from its prespecified list, and that for each edge, the colours on its endvertices do not correspond.

Also recall the formal definition of correspondence colouring.

**Definition 1.2.1.**
Given a multigraph $G$, a *correspondence* $\mathcal{C}(G)$ on $G$ consists of two parts:
• for each vertex $x \in V(G)$, there is a list of colours $l(x)$ associated with $x$;
• for each edge $e \in E(G)$ with endvertices $x$ and $y$, there is a *correspondence* $\mathcal{C}(e)$ specifying which pair of colours from the two endvertices *correspond*, such that $\mathcal{C}(e)$ induces a (possibly partial) matching between $\{(x,c) \mid c \in l(x)\}$ and $\{(y,c') \mid c' \in l(y)\}$.

The correspondence $\mathcal{C}(xy)$ on a (multiple) edge $xy$ is the collection of correspondences $\mathcal{C}(e)$ for all edges with endvertices $x$ and $y$.

Recall that the correspondence on an edge $e \in xy$ is *full* if the induced matching $\mathcal{C}(e)$ is perfect. A correspondence assignment on a graph is *full* if the correspondence on every edge is full.

If $|l(x)| = n$ for all vertices $x \in V(G)$, then $\mathcal{C}(G)$ is a $n$*-correspondence*. A graph $G$ is $n$*-correspondence-colourable* if a proper correspondence colouring on $G$ exists for any $n$-correspondence on it. The correspondence chromatic number $\chi_c(G)$ of a (multi)graph is the smallest such $n$.

As discussed in Section 1.2, since study of correspondence chromatic numbers assumed equal-order colour lists, and the fact that we can always 're-name' colours on each vertex while keeping the original correspondence on each edge, we may assume the list of colours associated to each vertex are identical.

We can also assume the correspondence on each edge is full. Since if it is not, adding more correspondences only makes finding a proper correspondence colouring 'harder': if a proper correspondence colouring exist on the later correspondence, then the same colouring is proper on the earlier correspondence. On the other hand, if the earlier correspondence is an $n$-correspondence, then the later correspondence stays $n$-correspondence, and *$n$-correspondence-colourable* required a proper correspondence colouring exists on all $n$-correspondence of that graph.

We present our main results in Section 5.2 and proofs in Section 5.3. Discussions, open problems and area of further interests are in Section 5.4.

## 5.2 Change of Correspondence Chromatic Numbers

In this section, we present our results on the effects of certain graph operations on the correspondence chromatic numbers. As mentioned in Section 5.1, graphs refer to multigraphs that may have multiple edges but not loops. Unless otherwise stated, edges refer to (multiple) edges with multiplicity at least 1. We first investigate how vertex and edge deletion may change the correspondence chromatic number of a graph.

For a graph $G$ and a vertex $x \in V(G)$, let $G - x$ be the resulting graph after removing $x$ and all edges incident to $x$; and let $G - xy$ be the resulting graph of removing all edges with endvertices $x, y$ (vertex set unchanged).

Denote $M_G(x) := \max\{\mathfrak{m}_G(xv) : v \in V(G)\}$ as the maximum multiplicity of edges with one of the endvertices $x$. We have the following bounds for edge or vertex deletion on any graph $G$. We will see in Section 5.3.1 for proofs as well as examples showing our bounds are best possible for general graphs.

**Theorem 5.2.1.**
*For any graph $G$ and any vertex $x$ in $G$, we have $\chi_c(G - x) \leq \chi_c(G) \leq \chi_c(G - x) + M_G(x)$.*

**Theorem 5.2.2.**
*For any graph $G$ and any vertices $x, y$ in $G$, we have $\chi_c(G - xy) \leq \chi_c(G) \leq \chi_c(G - xy) + \min\{M_G(x), M_G(y)\}$.*

We can extend Theorem 5.2.2 to cover the operations of 'change multiplicity of an edge'.

**Theorem 5.2.3.** *Let any graph $G$ and edge $xy$ with $\mathfrak{m}(xy) = m$ be given. If we decrease $\mathfrak{m}(xy)$ to $m' < m$ and denote the new graph as $G'$, then* $\chi_c(G') \leq \chi_c(G) \leq \chi_c(G') + \max\{m - m', \min\{M_{G-xy}(x), M_{G-xy}(y)\}\}$.
*If we increase $\mathfrak{m}(xy)$ to $m'' > m$ and denote the new graph as $G''$, then* $\chi_c(G) \leq \chi_c(G'') \leq \chi_c(G) + \max\{m - m', \min\{M_{G-xy}(x), M_{G-xy}(y)\}\}$.

Our above results provides an upper bound on the change of correspondence chromatic number if we increase the multiplicity of an edge. In ordinary or list colouring, adding one edge increases the relevant chromatic number by at most 1. One may ask, is correspondence so different that our bounds make sense? The answer is yes. And the following example shows that for a general graph, adding one edge can increase $\chi_c$ by arbitrarily much. Here $\nabla_{a,b,c}$ denotes the graph with three vertices, and edge multiplicities of $a, b, c$ respectively.

**Theorem 5.2.4.**
*For any positive integer $l \geq 1$ and $m \geq (l+1)^2$, we have*

$$\chi_c\left(\nabla_{lm,lm,m}\right) - \chi_c\left(\nabla_{lm,lm,m-1}\right) = l + 1.$$

One can also ask, does adding an edge between non-adjacent vertices leads to a similar or different result? We leave it as a conjecture.

**Conjecture 5.2.5.**
*Given any integer $l \geq 1$, there exist graph $G$ such that adding one edge between two non-adjacent vertices in $G$ increase its correspondence chromatic number by more than $l$.*

The next question is what will happen to the correspondence chromatic number of we identify two vertices of a graph? Denote $G/xy$ as the resulting graph after identifying vertices $x, y$: remove $x, y$ and create a new vertex $z$ adopts all edges incident to $x$ or $y$ (in $G$). Note $x, y$ can be adjacent or non-adjacent in $G$. Denoting $N_G(v)$ as the collection of all vertices adjacent to $v$ in $G$, we have $N_{G/xy}(z) = N_G(x) \cup N_G(y)$, and $\forall v \in V(G/xy) \setminus \{z\}$,

$\mathfrak{m}(zv) = \mathfrak{m}_G(xv) + \mathfrak{m}_G(yv)$. The following shows the relationship between $\chi_c(G)$ and $\chi_c(G/xy)$.

**Theorem 5.2.6.**
*For any graph $G$ and adjacent vertices $x, y$, we have*
$\chi_c(G) \leq \max\left\{\chi_c(G/xy), \mathfrak{m}(xy) + \left\lceil \frac{1}{2}\chi_c(G/xy)\right\rceil\right\}$.

In the case that the two vertices identified were connected by a simple edge, we have:

**Corollary 5.2.7.**
*For any graph $G$ and any simple edge $xy \in E(G)$, if $G/xy$ has at least one edge, then $\chi_c(G) \leq \chi_c(G/xy)$.*

Note that $\chi_c(G/xy) \geq 2$ is essential in Corollary 5.2.7 (in other words, we need at least one edge in $G/xy$): if $\chi_c(G/xy) = 1$, consider graph $G$ of two vertices and one simple edge $xy$, then $\chi_c(G) = 2$ but $\chi_c(G/xy) = 1$, i.e. Corollary 5.2.7 no longer holds.

Following Theorem 5.2.6, we ask the following question: can we bound $\chi_c(G)$ and $\chi_c(G/xy)$ from the other side? Is there a constant $k$ so that $\chi_c(G/xy) \leq \chi_c(G) + k$? The answer is yes but $k$ depends on the choice of $x, y$.

Denote $M_G(xy) = \max\{\mathfrak{m}(xv) + \mathfrak{m}(yv) : v \in V(G), v \neq x, y\}$.
(Note $M_{G/xy}(z) = M_G(xy)$.) We have the following:

**Theorem 5.2.8.**
*For any graph $G$ and vertices $x, y$ in $G$, we have $\chi_c(G/xy) \leq \chi_c(G - xy) + M_G(xy)$.*

Another interesting operation on a multigraph is about multiplying the whole edge set. We define the *$m$-multiple copy* $G^{(m)}$ of graph $G$ as the graph with the same vertex set as $G$, but multiplicity of each edge is multiplied by $m$, i.e. $\mathfrak{m}_{G^{(m)}}(xy) = m \cdot \mathfrak{m}_G(xy)$ for each edge $xy$ in $G$. By convention $G^{(0)}$ is the empty graph with vertex set $V(G)$ and no edges. The following observation provides some idea on how this operation changes correspondence chromatic number, more detailed results follow. (Note that in the process of preparing this thesis, we found that Ciletti [40] also studied a relevant topic and proved Theorem 5.2.9. Our research has been done independently from theirs.)

**Theorem 5.2.9.**
*For any graph $G$ and $m \geq 1$, we have* $m\left(\chi_c(G)-1\right) \leq \chi_c\left(G^{(m)}\right)-1 \leq \operatorname{degcy}\left(G^{(m)}\right)$.

Can Theorem 5.2.9 be generalised to any two multiple copies of a graph? We leave the general question as a conjecture, but will show its correctness for some classes of graphs.

**Conjecture 5.2.10.**
*For any $k, l \geq 1$, we have* $\chi_c\left(G^{(k+l)}\right)-1 \geq \left(\chi_c\left(G^{(k)}\right)-1\right)+\left(\chi_c\left(G^{(l)}\right)-1\right)$.

We understand that $\chi_c\left(G^{(m)}\right)$ is bounded between $m\left(\chi_c(G)-1\right)+1$ and $\operatorname{degcy}\left(G^{(m)}\right)+1$ by Theorem 5.2.9. The following results show the asymptotic behaviour of $\chi_c\left(G^{(m)}\right)$.

**Theorem 5.2.11.**
*For any graph $G$, the sequence* $\left\{\mathfrak{a}_m : \mathfrak{a}_m = \dfrac{\chi_c\left(G^{(m)}\right)-1}{m}, m \in \mathbb{N}\right\}$ *converges as $m$ goes to infinity.*

In the proof that the sequence in Theorem 5.2.11 converges to its supremum, we will note that the sequence is not monotone in general. With Theorem 5.2.11, we introduce and study the following graph invariant that measures the correspondence chromatic number of graph $G^{(m)}$, $m \in \mathbb{N}$ asymptotically.

**Definition 5.2.12.**
*The* correspondence chromatic limit *of a base graph $G$ (that may or may not be simple) is* $\mathfrak{a}_\infty(G) := \lim\limits_{m \to \infty} \dfrac{\chi_c\left(G^{(m)}\right)-1}{m}$.

Note the ordinary or list colouring version of Definition 5.2.12 is trivial: their chromatic number do not change by changing edge multiplicities (as long as not change from or to 0), so the sequences always converge to 0.

By Theorem 5.2.11 and Theorem 5.2.9, we naturally conclude the following:

**Corollary 5.2.13.**
*For any graph $G$, we have $\chi_c(G) - 1 \le \mathfrak{a}_\infty(G) \le \operatorname{degcy}(G)$.*

So we ask: Can the inequality in Corollary 5.2.13 be strict? What values can the correspondence chromatic limit attain?

For the first question, we will see examples that $\mathfrak{a}_\infty(G)$ lies strictly between $\chi_c(G) - 1$ and $\operatorname{degcy}(G)$ for some graphs in Section 5.3.4. For the second question, we determine the exact correspondence chromatic number of 'isosceles triangles': graphs with three vertices, and edge multiplicities $m, m, n$ (namely, $\nabla_{m,m,n}$). And hence prove the correspondence chromatic limit can attain any given fractional part with some determined integer part, as a corollary of following theorems.

**Theorem 5.2.14.**
*For any integers $m, n > 0$, we have*

$$\chi_c(\nabla_{m,m,n}) = \max\left\{m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil n + 1\right\}.$$

**Theorem 5.2.15.**
*For any integers $m, n > 0$, we have $\mathfrak{a}_\infty(\nabla_{m,m,n}) = \max\left\{m + \frac{m}{\lceil m/n \rceil}, \left\lceil \frac{m}{n} \right\rceil n\right\}$.*

**Corollary 5.2.16.**
*For any integers $q > p \ge 0$, there exists positive integers $N, m, n$ so that $\mathfrak{a}_\infty(\nabla_{m,m,n}) = N + \frac{p}{q}$.*

We include the proof to Corollary 5.2.16 here as it is short.

*Proof.* Given $q > p \ge 0$, let $m = q^2 + p$ and $n = q + 1$.
Hence $\left\lceil \frac{m}{n} \right\rceil = \left\lceil (q-1) + \frac{p+1}{q+1} \right\rceil = q$. Then $m + \frac{m}{\lceil m/n \rceil} = q^2 + p + q + \frac{p}{q}$ and $\left\lceil \frac{m}{n} \right\rceil n = q(q+1) \le q^2 + p + q + \frac{p}{q}$.
Therefore $\mathfrak{a}_\infty(\nabla_{m,m,n}) = q^2 + p + q + \frac{p}{q}$. □

Note Corollary 5.2.16 suggests: with any integer denominator $q$ (or any fractional part $\frac{p}{q}$), there are graphs $G$ that $\mathfrak{a}_\infty(G)$ is of denominator $q$ (or the fractional part $\frac{p}{q}$), and $\mathfrak{a}_\infty(G) = \mathfrak{a}_k(G)$ for some $k$. On the other hand, there are also graphs that $\mathfrak{a}_\infty(G) < \mathfrak{a}_k(G)$ for any $k$: a simple example will be $\mathfrak{a}_\infty(C_n) = 2$. It is natural to ask: can the correspondence chromatic limit $\mathfrak{a}_\infty(G)$ attain any positive rational number?

For positive rationals smaller than 2, we know it is not the case: consider any positive rational number $x < 2$, because $\mathfrak{a}_\infty(G) \geq \mathfrak{a}_1(G) = \chi_c(G) - 1$, we have $\chi_c(G) = 1$ or $\chi_c(G) = 2$. In the first case, $G$ is an independent set, then $\mathfrak{a}_\infty(G) = 0$. In the second case, $G$ is cycle-less, so it is a simple tree, and it is not hard to prove $\mathfrak{a}_\infty(G) = 2$. But what about larger rational numbers? Or even irrationals? In general, we think there are always some rational numbers that is not the correspondence chromatic limit of any graph. We leave this as a conjecture.

**Conjecture 5.2.17.**
*For any $k \geq 1$, there are some rational number $x \in (k, k+1)$ such that there does not exist graph $G$ with $\mathfrak{a}_\infty(G) = x$.*

We also conclude the following from Theorem 5.2.14 and Theorem 5.2.15. Note Corollary 5.2.19 proves Conjecture 5.2.10 for all 'isosceles triangle' graphs.

**Corollary 5.2.18.**
*For any $m, n \geq 0$, we have $\chi_c(\nabla_{m,m,n}) - 1 \leq \mathfrak{a}_\infty(\nabla_{m,m,n}) \leq \chi_c(\nabla_{m,m,n})$.*

**Corollary 5.2.19.**
*For any $m, n, k, l \geq 0$, we have*

$$\chi_c\left(\nabla^{(k+l)}_{m,m,n}\right) - 1 \geq \left(\chi_c\left(\nabla^{(k)}_{m,m,n}\right) - 1\right) + \left(\chi_c\left(\nabla^{(l)}_{m,m,n}\right) - 1\right).$$

## 5.3 Proofs

In this section, we prove results presented in Section 5.2 and provide examples showing some of our bounds are best possible for general graphs. We will use the following definition.

**Definition 5.3.1.**
*Let $G$ be a graph and $\mathcal{C}(G)$ be any correspondence associated with $G$. Let*

*$x, y$ be any two vertices of $G$. The* correspondence graph *on $xy$, denoted by $\mathcal{C}(xy)$, is a simple graph with the vertex-colour pairs on $x, y$ as vertices, and the correspondence on $xy$ as edges:*
- *$V(\mathcal{C}(xy)) = \{(x,c) : c \in l(x)\} \cup \{(y,c) : c \in l(y)\}$, and*
- *$E(\mathcal{C}(xy)) = \{\{(x,c_x),(y,c_y)\} : \{(x,c_x),(y,c_y)\} \in \mathcal{C}(G)\}$.*

If there is no edge with endvertices $x$ and $y$, then $\mathcal{C}(G)$ is just an empty graph. Also note the correspondence graph on any edge is bipartite and simple, with maximum vertex degree at most $\mathfrak{m}_G(xy)$.

### 5.3.1 Proofs of Theorems 5.2.1 and 5.2.2

We first prove Theorem 5.2.1. Note vertex deletion is a special case of repeated edge deletions, in which we delete all edges incident to one vertex; the remaining isolated vertex does not affect the chromatic number of resulting graph, as long as there are other vertices in the graph.

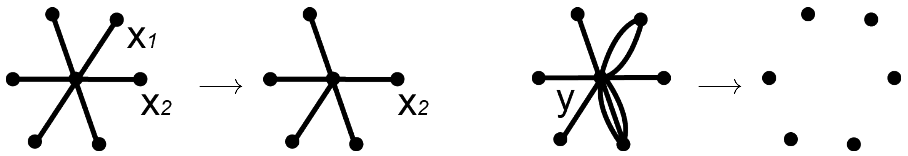
Recall that $M_G(x) := \max\{\mathfrak{m}_G(xv) : v \in V(G)\}$.

**Theorem 5.2.1.**
*For any graph $G$ and vertex $x$ in $G$, we have $\chi_c(G-x) \leq \chi_c(G) \leq \chi_c(G-x) + M_G(x)$.*

*Proof.* Fix an arbitrary graph $G$ with correspondence chromatic number $\chi_c(G) = n$. To prove $\chi_c(G-x) \leq \chi_c(G)$, fix any $n$-correspondence $\mathcal{C}(G-x)$ on $G-x$. We fix a correspondence $\mathcal{C}(G)$ by copying the correspondence on each edge in $E(G-x)$ to each edge in $E(G)$, and leaving empty correspondence on edges with endvertex $x$. Then a proper correspondence colouring on $\mathcal{C}(G)$ exist since $\chi_c(G) = n$, and we can copy the colouring back to $G-x$. And hence $\chi_c(G-x) \leq n$.

To prove the second inequality, we fix any $x \in V(G)$ and assume $\chi_c(G-x) = n'$. Now for any $(n'+M_G(x))$-correspondence on $G$, copy the correspondence on each edge of $G$ to each edge of $G-x$, whenever an edge exists. Fix a colour $c^* \in l(x)$ and let $l'(x) = l(x) \setminus \{c^*\}$. For each original neighbour $v \in V(G-x)$ (that $vx \in E(G)$), remove colours corresponded to $c^*$ and let $l'(v) = l(v) \setminus \{c : \{(x,c^*),(v,c)\} \in \mathcal{C}(G)\}$; for any other $u \in V(G-x)$, let $l'(u) = l(u)$. Now the correspondence induced by $l'(v)$ (for all $v \in V(G)$) is a correspondence on $G - x$ with at least $n'$ colours in each vertices' colour lists, so a proper correspondence colouring exists. We copy the proper correspondence colouring from $V(G - x)$ back to $V(G)$ and colour vertex $x$ by $c^*$. Hence $\chi_c(G) \leq n' + M_G(x)$. $\square$

Our bounds on both sides are best possible for general graphs by the following examples. Following left graphs show that $\chi_c(G - x_1) = \chi_c(G)$ (both graphs have correspondence chromatic number 2) and right side graphs show $\chi_c(G) = \chi_c(G - y) + M_G(y)$ (here $\chi_c$ changes from 4 to 1).




For Theorem 5.2.2, the first inequality that $\chi_c(G - xy) \leq \chi_c(G)$ is easy to verify and follows a similar idea as in Theorem 5.2.1. We will prove the second inequality that $\chi_c(G) \leq \chi_c(G - xy) + \min\{M_G(x), M_G(y)\}$.

**Theorem 5.2.2.**
*For any graph $G$ and vertices $x, y$ in $G$, we have $\chi_c(G - xy) \leq \chi_c(G) \leq \chi_c(G - xy) + \min\{M_G(x), M_G(y)\}$.*

*Proof.* Fix any graph $G$ with at least one edge, as otherwise there is nothing to prove. Fix arbitrary $xy \in E(G)$ that $\mathfrak{m}(xy) \geq 1$. Without loss of generality, we assume $M_G(x) \leq M_G(y)$. Let $n = \chi_c(G - xy) + M_G(x)$.

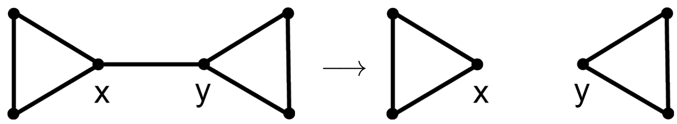
Let $\mathcal{C}(G)$ be an arbitrary $n$-correspondence on $G$. Copy the correspondence on each edge of $G$ to each of $G - xy$. Fix an arbitrary colour $c^* \in l(x)$. Let

- $l'(x) = l(x) \setminus \{c^*\}$,
- $\forall u \in N_G(x)$, $l'(u) = l(u) \setminus \{c : \{(u, c), (x, c^*)\} \in \mathcal{C}(G)\}$, and
- $\forall u \notin N_G(x)$, $l'(u) = l(u)$.

Denote the remaining correspondence (induced by $l'(v)$ for all $v \in V(G)$) on $G - xy$ with colour lists $l'$ as $\mathcal{C}'(G - xy)$. Note that $\forall v \in V(G)$, $|l'(v)| \geq n - M_G(x) \geq \chi_c(G - xy)$, so there is a proper correspondence colouring on $\mathcal{C}'(G - xy)$. Denote it by $p : V(G - xy) \to \bigcup l'(v)$.

Since $\{(x, c^*), (y, p(y))\} \notin \mathcal{C}(G)$, there is a proper correspondence colouring on $\mathcal{C}(G)$ by assigning colour $c^*$ to $x$ and colour $p(v)$ to any other vertex $v \neq x$. $\square$

Theorem 5.2.2 is best possible for general graphs. For the first inequality, we have the following graphs that $\chi_c(G) = \chi_c(G - xy) = 3$ as an evidence.




To show that $\chi_c(G) = \chi_c(G - xy) + \min\{M_G(x), M_G(y)\}$ is best possible for some graphs, we consider $G = C_k^{(m)}$ with some $k \geq 3$ and $m \geq 0$, so $\chi_c(G) = 2m+1$ (proof follows from Lemma 5.3.6 in later of this section). For any adjacent $x, y \in V(G)$, we have $M_G(x) = M_G(y) = m$ and $\chi_c(G - xy) = m + 1$.

### 5.3.2 Proofs of Theorems 5.2.6 and 5.2.8

We start with definitions specific to this section. Consider an arbitrary correspondence $\mathcal{C}(G)$ on a graph $G$. For two vertices $x, y$, vertex-colour pairs $(x, c)$ and $(y, c')$ are *anti-matched* if $c \in l(x)$, $c' \in l(y)$ and they do not correspond in $\mathcal{C}(G)$. An *anti-matching* (on $\mathcal{C}(xy)$) of size $K$ is a pairing $P$ of anti-matched vertex-colour pairs between $\{(x, c) : c \in l(x)\}$ and $\{(y, c) : c \in l(y)\}$, so that $P \cap \mathcal{C}(xy) = \emptyset$ and $|P| = K$.

If vertices $x, y$ are not adjacent in $G$, then an anti-matching of size $|l(x)|$ exists. Hence any $\chi_c(G/xy)$-correspondence $\mathcal{C}(G)$ on $G$ has a proper correspondence colouring: we can copy correspondence $\mathcal{C}(G)$ to $G/xy$ by identifying anti-matched vertex-colour pairs on $x, y$, then a proper correspondence colouring on $\mathcal{C}(G/xy)$ exist and can be copied back to $G$. That is, $\chi_c(G) \leq \chi_c(G/xy)$ if $\mathfrak{m}(xy) = 0$. In the rest of this section, we study how identifying $x, y$ (where $\mathfrak{m}(xy) \geq 1$) affects the correspondence chromatic numbers of a graph by studying existence and size of anti-matching.

We first derive Theorem 5.2.6 from Lemma 5.3.2.

**Theorem 5.2.6.**
*For any graph $G$ and adjacent vertices $x, y$, we have*

$$\chi_c(G) \leq \max\left\{\chi_c(G/xy), \mathfrak{m}(xy) + \left\lceil \tfrac{1}{2}\chi_c(G/xy)\right\rceil\right\}.$$

**Lemma 5.3.2.** *Let $H$ be a simple bipartite graph with vertex partition $X \sqcup Y$ that $|X| = |Y|$. If $\deg_H(v) \geq k$ for any vertex $v \in X \sqcup Y$, then there is a matching of size $\min\{2k, |X|\}$ in $H$.*

*Proof of Theorem 5.2.6.* We first consider the case that $\mathfrak{m}(xy) \leq \frac{1}{2}\chi_c(G/xy)$. Fix an arbitrary $(\chi_c(G/xy))$-correspondence $\mathcal{C}(G)$ on $G$. Because $\mathfrak{m}(xy) \leq \frac{1}{2}\chi_c(G/xy)$, the correspondence graph $\mathcal{C}(xy)$ is a simple bipartite graph with maximum degree at most $\mathfrak{m}(xy)$, and $2(\chi_c(G/xy) - \mathfrak{m}(xy)) \geq \chi_c(G/xy)$. By Lemma 5.3.2, we can find an anti-matching of size $\chi_c(G/xy)$ on $\mathcal{C}(xy)$. Without loss of generality we assume the anti-matching is between identical colours. (We can always achieve this by renaming colours on $\mathcal{C}(G)$ while keeping the original correspondence.)

Denote the vertex generated by identifying $x, y$ as $z$. Define correspondence $\mathcal{C}(G/xy)$ by adopting $\mathcal{C}(G)$, while $z$ replacing all appearance of $x, y$. Hence a proper correspondence colouring on $\mathcal{C}(G/xy)$ exists. Assume $z$ is coloured by $c^*$, we have a proper correspondence colouring on $\mathcal{C}(G)$ by copying all colour assignments in $\mathcal{C}(G/xy)$ and colour both $x, y$ by $c^*$.

If instead $\mathfrak{m}(xy) > \frac{1}{2}\chi_c(G/xy)$, then firstly by the integrality, we have $\mathfrak{m}(xy) \geq \lceil \frac{1}{2}\chi_c(G/xy)\rceil$. Let $\mathcal{C}'(G)$ be an arbitrary $\left(\mathfrak{m}(xy) + \lceil \frac{1}{2}\chi_c(G/xy)\rceil\right)$-correspondence on $G$. By Lemma 5.3.2, there is an anti-matching of size $2\lceil \frac{1}{2}\chi_c(G/xy)\rceil \geq \chi_c(G/xy)$ on $\mathcal{C}(xy)$. Assume the anti-matching is between some pairs of identical colours, and remove colours in $l(x)$, $l(y)$ that are not in this anti-matching. Denote the resulting correspondence by $\mathcal{C}''(G)$. With essentially the same argument as above, a proper correspondence colouring on $\mathcal{C}''(G)$ exists, and hence a proper correspondence colouring on $\mathcal{C}'(G)$ exists. Therefore $\chi_c(G) \leq \mathfrak{m}(xy) + \lceil \frac{1}{2}\chi_c(G/xy)\rceil$.

Note $\mathfrak{m}(xy) \leq \frac{1}{2}\chi_c(G/xy)$ if and only if $\chi_c(G/xy) \geq \mathfrak{m}(xy) + \lceil \frac{1}{2}\chi_c(G/xy)\rceil$. We hence conclude

$$\chi_c(G) \leq \max\left\{\chi_c(G/xy), \mathfrak{m}(xy) + \left\lceil \tfrac{1}{2}\chi_c(G/xy)\right\rceil\right\}.$$

□

Lemma 5.3.2 can be derived from the König's Theorem (the maximum size of a matching equals the minimum size of a vertex cover in any bipartite simple graphs), see in e.g. Diestel's book [8]. We include a direct proof to Lemma 5.3.2 for completeness.

*Proof of Lemma 5.3.2.* There is nothing to prove if $k = 0$, so we can assume $k \geq 1$.

Let $H = (X \sqcup Y, E)$ be a simple bipartite graph with $|X| = |Y|$ and minimum vertex degree $k$. Assume $M = \{x_1y_1, x_2y_2, \ldots, x_Ky_K\}$ is one of the maximum matching in $H$. Note that $K \geq 1$ since $\deg_H(v) \geq k \geq 1$ for any $v \in V(H)$.

Assume $K < \min\{2k, |X|\}$, then there exist $x^* \in X \setminus \{x_1, \ldots, x_K\}$ and $y^* \in Y \setminus \{y_1, \ldots, y_K\}$. As $M$ is a maximum matching, so $N_H(x^*) \cap (Y \setminus \{y_1, \ldots, y_K\}) = \emptyset$ and $N_H(y^*) \cap (X \setminus \{x_1, \ldots, x_K\}) = \emptyset$. I.e. all neighbours of $x^*$ and $y^*$ are in $\{x_1, \ldots, x_K, y_1, \ldots, y_K\}$. But since $deg_H(x^*) \geq k$ and $deg_H(y^*) \geq k$, and the fact that $K < 2k$, there exist $i^* \in \{i \in [K] : x^*y_i \in E(H)\} \cap \{i \in [K] : y^*x_i \in E(H)\}$, then we can build a matching of size $K + 1$ by an augmenting path. □

The bound achieved in Lemma 5.3.2 is the best possible for general graphs. It is clear if $|X| \leq 2k$. And if $|X| > 2k$, consider the graph that all vertices in $X$ only adjacent to the first $k$ vertices in $Y$, and all vertices in $Y$ only adjacent to the first $k$ vertices in $X$, then the maximum order of a matching in this graph is $2k$.

There are many graphs with $\chi_c(G) = \chi_c(G - xy)$. For an example, consider a cycle of at least 3 vertices, by identifying two adjacent vertices, its correspondence chromatic number does not change. We also have examples that $\chi_c(G) = \mathfrak{m}_G(xy) + \lceil \frac{1}{2}\chi_c(G/xy) \rceil$: consider $G$ as a graph of two vertices and one edge, then $\chi_c(G) = \mathfrak{m}(xy) + 1$ and $\chi_c(G/xy) = 1$.

We have a strengthened version of Theorem 5.2.6.

**Theorem 5.3.3.**
*For any graph $G$, we have*

$$\chi_c(G) \leq \max\left\{\chi_c\left(G/xy\right), \min\left\{\mathfrak{m}(xy) + \left\lceil \tfrac{1}{2}\chi_c(G/xy)\right\rceil, \right.\right.$$
$$\left.\left.\max\left\{\mathfrak{m}(xy), \chi_c(G/xy)\right\} + \max\{M_G(xy), 1\}\right\}\right\}.$$

*Proof.* Recall $M_G(xy) = \max\{\mathfrak{m}(xv) + \mathfrak{m}(yv) : v \in V(G), v \neq x, y\}$. With Theorem 5.2.6, we only need to show our theorem holds in the case $\max\{\mathfrak{m}(xy), \chi_c(G/xy)\} + \max\{M_G(xy), 1\} < \mathfrak{m}(xy) + \lceil \frac{1}{2}\chi_c(G/xy)\rceil$.

Fix arbitrary vertices $x \neq y$ in graph $G$, and let $K = \max\{\mathfrak{m}(xy), \chi_c(G/xy)\}+ \max\{M_G(xy), 1\}$. Consider an arbitrary $K$-correspondence $\mathcal{C}(G)$ on $G$ and fix any colour $c^* \in l(x)$. Find colour $c^{**} \in l(y)$ so that $\{(x, c^*), (y, c^{**})\} \notin \mathcal{C}(xy)$ (note such $c^{**}$ exists because $|l(y)| = K \geq \mathfrak{m}(xy) + 1$). We can assume $c^{**} = c^*$: if not, we just apply a renaming function $f$ to $\mathcal{C}(G)$ setting $f_y(c^{**}) = c^*$ and all other $f_v$ as identity.

So now $\{(x, c^*), (y, c^*)\} \notin \mathcal{C}(xy)$. Denote the new vertex generated by identifying $x, y$ as $z$. Copy $\mathcal{C}(G)$ to $G/xy$ and apply the following alterations:
- $l'(z) = l(z) \setminus \{c^*\}$,
- $\forall v \in N_{G/xy}(z)$, $l'(v) = l(v) \setminus \{c : \{(z, c^*), (v, c)\} \in \mathcal{C}(zv)\}$, and
- $\forall u \in V(G) \setminus (N_{G/xy}(z) \cup \{z\})$, $l'(u) = l(u)$.

Denote $C'(G/xy)$ as the correspondence on $G/xy$ induced by $l'$. Note each colour list is of size at least $\chi_c(G/xy)$, so a proper correspondence colouring exists, say $p : V(G/xy) \to \bigcup\limits_{v \in G/xy} l'(v)$. Then we have a proper correspondence colouring $p^*$ on $\mathcal{C}(G)$: let $p^*(x) = p^*(y) = c^*$, and $p^*(v) = p(v)$ for any $v \neq x, y$. $\square$

For Theorem 5.2.8, we first present and prove the following result, which uses a similar method.

**Lemma 5.3.4.**
*For any simple graph $G$ and $x, y \in V(G)$, we have $\chi_c(G/xy) \leq \chi_c(G - xy) + 2$. If $x, y$ has at most one common neighbour, then $\chi_c(G/xy) \leq \chi_c(G - xy) + 1$.*

*Proof.* Fix vertices $x \neq y$ in simple graph $G$ and denote the new vertex generated by identifying $x, y$ as $z$. The case that $x, y$ do not have a common neighbour (i.e. $N_G(x) \cap N_G(y) = \emptyset$) is easier and we will discuss it later.

Assume there exists $w \in N_G(x) \cap N_G(y)$. Let $K = \chi_c(G - xy) + 1$ if $|N_G(x) \cap N_G(y)| = 1$, or $K = \chi_c(G - xy) + 2$ if $|N_G(x) \cap N_G(y)| > 1$. Let $\mathcal{C}(G/xy)$ be an arbitrary $K$-correspondence on $G/xy$. Copy $\mathcal{C}(G/xy)$ to $G - xy$ and leave correspondence on $xy$ empty (denote by $\mathcal{C}(G)$).

Fix an arbitrary colour $c^* \in l(x)$, find colours $c_i \in l(w)$ and $c_j \in l(y)$ so that $\{(x, c^*), (w, c_i)\} \in \mathcal{C}(xw)$ and $\{(y, c_j), (w, c_i)\} \in \mathcal{C}(yw)$. Note $c_i, c_j$ exist by the assumption of full correspondence on each edge other than $xy$. Let

- $l'(x) = l(x) \setminus \{c^*\}$, $l'(y) = l(y) \setminus \{c_j\}$, $l'(w) = l(w) \setminus \{c_i\}$,
- for each $v \in (N_G(x) \cup N_G(y)) \setminus \{x, y, w\}$,
$l'(v) = l(v) \setminus \{c : \{(x, c^*), (v, c)\} \in \mathcal{C}(xv) \text{ or } \{(y, c_j), (v, c)\} \in \mathcal{C}(yv)\}$, and
- for any other vertex $u$, $l'(u) = l(u)$.

Update the correspondence on each edge accordingly, and denote the resulting correspondence (induced by $l'$) as $\mathcal{C}'(G)$.

A proper correspondence colouring exists on $\mathcal{C}'(G)$ as each colour list is of order at least $\chi_c(G - xy)$ (and the fact that $\mathcal{C}(xy)$ is empty). Denote the proper correspondence colouring as $p : V(G) \to \bigcup_{v \in G} l'(v)$. Then a proper correspondence colouring $p^*$ on $\mathcal{C}(G/xy)$ exist by letting $p^*(z) = c^*$, and $\forall v \neq z, p^*(v) = p(v)$.

If $x, y$ do not have a common neighbour in $G$, then essentially the same steps skipping all $w$ related parts applies with $K = \chi_c(G - xy) + 1$. $\square$

The proof of Theorem 5.2.8 is essentially the same as the $|N_G(x) \cap N_G(y)| > 1$ case of above, except we take $K = \chi_c(G - xy) + M_G(xy)$ instead. We skip the proof here.

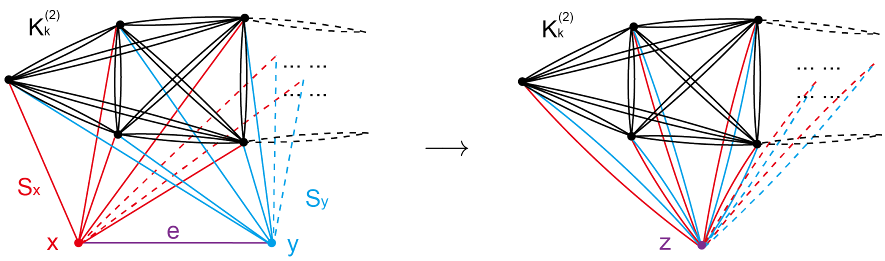
The following example shows that our bounds in Theorem 5.2.8 are the best possible. We'll need Theorem 5.3.5 in the steps of verification.

**Theorem 5.3.5.**
*For $k \geq 2$, $m \geq 1$, we have $\chi_c\left(K_k^{(m)}\right) = m(k-1) + 1$ and $\chi_c\left(C_k^{(m)}\right) = 2m + 1$.*

Theorem 5.3.5 can be proved directly using Theorem 5.2.9, or as a conclusion of Lemma 5.3.6 proved by Bernshteyn, Kostochka and Pron [3]. We omit their proof here. (Recall $G^{(m)}$ is the resulting graph by multiplying edge multiplicity of each edge in $G$ by $m$.)

**Lemma 5.3.6** (Bernshteyn, Kostochka and Pron [3])**.**
*Let $G$ be a connected multigraph. There is a correspondence on $G$ that does not have a proper correspondence colouring and each vertex has a colour list of order at least of its degree, if and only if each block of $G$ is $K_n^{(m)}$ or $C_n^{(m)}$.*

We construct a graph $H$ with $\chi_c(H/xy) = \chi_c(H-xy) + M_H(xy)$. Let $m, k, p \geq 1$ be given that $p \leq \dfrac{m}{2}$. Let $H$ be a graph with vertex set $V(H) = V\left(K_k^{(m)}\right) \cup \{x, y\}$.




*Multigraphs $H$ and $H/xy$ with $m = 2$*

Denote $S_x = \left\{xv : v \in V\left(K_k^{(m)}\right)\right\}$ and $S_y = \left\{yv : v \in V\left(K_k^{(m)}\right)\right\}$ as the edge sets. Then the edge set of $H$ is

$$E(H) = E\left(K_k^{(m)}\right) \cup S_x^{\left(\left\lfloor \frac{m}{2} \right\rfloor\right)} \cup S_y^{\left(\left\lceil \frac{m}{2} \right\rceil\right)} \cup \{e\}^{(p)}.$$

Note $M_H(xy) = m$. We will use Theorem 5.3.5 to show

$$\chi_c(H/xy) = \chi_c(H-xy) + M_H(xy).$$

*Proof.* For any $m, k \geq 1$, we have $\chi_c(H-xy) \geq \chi_c\left(K_k^{(m)}\right) = m(k-1)+1 =$ degcy$(H-xy)+1$ (as long as $p \leq \dfrac{m}{2}$), so $\chi_c(H-xy) = m(k-1)+1$. Since $\chi_c(H/xy) = \chi_c\left(K_k^{(m+1)}\right) = mk+1$, we conclude $\chi_c(H/xy) = \chi_c(H-xy) + m = \chi_c(H-xy) + M_H(xy)$. □

### 5.3.3 Proofs of Theorems 5.2.9 and 5.2.11

We first prove Theorem 5.2.9.

**Theorem 5.2.9.**
*For any graph $G$ and $m \geq 1$, we have $\chi_c\left(G^{(m)}\right) - 1 \geq m\left(\chi_c(G) - 1\right)$.*

*Proof.* Consider an arbitrary graph $G$. If $\chi_c(G) = 1$ then $G$ is an independent set, and there is nothing to prove. So we assume $\chi_c(G) \geq 2$. We will construct a $(m(\chi_c(G) - 1))$-correspondence on $G^{(m)}$ that does not have a proper correspondence colouring.

Denote $K = \chi_c(G)$. Let $\mathcal{C}(G)$ be an $(K-1)$-correspondence on $G$ that does not have a proper correspondence colouring. We now define an $(m(K-1))$-correspondence $\mathcal{C}^{(m)}$ on $G^{(m)}$: assign colour list $\{c_{i,j} : i \in [K-1], j \in [m]\}$ to each vertex $v$, and correspondence $\{(u, c_{i_u,j_u}), (v, c_{i_v,j_v})\} \in \mathcal{C}^{(m)}$ if and only if $\{(u, c_{i_u}), (v, c_{i_v})\} \in \mathcal{C}(G)$.

Then $G^{(m)}$ that does not have a proper correspondence colouring under $\mathcal{C}^{(m)}$, as any proper correspondence colouring on $\mathcal{C}^{(m)}$ deduces a proper correspondence colouring on $\mathcal{C}(G)$, which is impossible. □

We conclude the following corollaries from Theorem 5.2.9.

**Corollary 5.3.7.**
*If $\chi_c(G) = \operatorname{degcy}(G) + 1$, then $\chi_c(G^{(m)}) = m \cdot \operatorname{degcy}(G) + 1$.*

*Proof.* Let $G$ be a graph with $\chi_c(G) = \operatorname{degcy}(G) + 1$. Then Theorem 5.2.9 implies

$$\chi_c\left(G^{(m)}\right) \geq m(\operatorname{degcy}(G) - 1) + 1 = m \cdot \operatorname{degcy}(G) + 1 = \operatorname{degcy}\left(G^{(m)}\right) + 1,$$

and hence $\chi_c\left(G^{(m)}\right) = m \cdot \operatorname{degcy}(G) + 1$. □

**Corollary 5.3.8.**
*For any graph $G$ and $m \geq 1$, we have $\chi_c(G) - 1 \leq \dfrac{\chi_c\left(G^{(m)}\right) - 1}{m} \leq \operatorname{degcy}(G)$.*

The first inequality directly follows from Theorem 5.2.9 and the second inequality follows from $\chi_c\left(G^{(m)}\right) \leq m \cdot \operatorname{degcy}(G) + 1$.

We now prove in Theorem 5.2.11 that the correspondence chromatic limit is well defined. Note this sequence converges to its supremum, but is not monotone in general.

**Theorem 5.2.11.**
*For any graph $G$, the sequence $\{\mathfrak{a}_m : \mathfrak{a}_m = \dfrac{\chi_c\left(G^{(m)}\right) - 1}{m}, m \in \mathbb{N}\}$ converges as $m$ goes to infinity.*

*Proof.* Consider an arbitrary graph $G$. Recall $\mathfrak{a}_m := \dfrac{\chi_c\left(G^{(m)}\right) - 1}{m}$, and hence $\mathfrak{a}_1 = \chi_c(G) - 1$.

We first show the sequence $\{\mathfrak{a}_m\}_{m\in\mathbb{N}}$ is bounded both above and below. For any positive integer $m$, we have

$$\chi_c(G) - 1 = \mathfrak{a}_1 = \frac{m\left(\chi_c(G) - 1\right)}{m} \leq \frac{\chi_c\left(G^{(m)}\right) - 1}{m} = \mathfrak{a}_m,$$

and

$$\mathfrak{a}_m = \frac{\chi_c\left(G^{(m)}\right) - 1}{m} \leq \frac{(m \cdot \operatorname{degcy}(G) + 1) - 1}{m} = \operatorname{degcy}(G).$$

Hence by the least upper bound property of bounded sequence of real numbers, there exists $R \in \mathbb{R}$ such that $R = \sup\{\mathfrak{a}_m : m \in \mathbb{N}\}$.

For any given $s, t \in \mathbb{N}$ (without loss of generality assume $s \geq t$), there exist integers $q = q(s,t)$ and $r = r(s,t)$ $(0 \leq r(s,t) \leq t-1)$ so that $s = qt + r$. (Precisely, $q(s,t) = \left\lfloor \dfrac{s}{t} \right\rfloor$ and $r(s,t) = s - \left\lfloor \dfrac{s}{t} \right\rfloor t$ but we will not use it here.) Then

$$\begin{aligned}
\forall s, t \in \mathbb{N},\ s \geq t,\ s \cdot \mathfrak{a}_s =& \chi_c\left(G^{(qt+r)}\right) - 1 \\
\geq& \chi_c\left(G^{(qt)}\right) - 1 \\
\geq& q\left(\chi_c\left(G^{(t)}\right) - 1\right) \\
=& qt \cdot \mathfrak{a}_t, \\
\text{i.e. } \mathfrak{a}_s - \mathfrak{a}_t \geq \frac{qt}{s} \cdot \mathfrak{a}_t - \mathfrak{a}_t =& -\frac{r}{s} \cdot \mathfrak{a}_t \geq -\frac{r}{s} \cdot \operatorname{degcy}(G).
\end{aligned}$$

By the definition of supremum, for any given $\epsilon > 0$, there exists at least one $N_0 \in \mathbb{N}$ such that $R - \frac{\epsilon}{2} < \mathfrak{a}_{N_0} \le R$. Let $N_1 = \max\{N_0, \lceil \mathrm{degcy}(G) \cdot 2N_0/\epsilon \rceil\}$. For any $m \ge N_1 \ge N_0$, denote $r' = r(m, N_0) := m - \left\lfloor \frac{m}{N_0} \right\rfloor \cdot N_0$. Note $0 \le r(m, N_0) \le N_0 - 1$ and

$$\frac{r'}{m} \le \frac{r'}{N_1} < \frac{N_0}{N_1} \le \frac{N_0}{\mathrm{degcy}(G) \cdot 2N_0/\epsilon} = \frac{\epsilon}{2 \cdot \mathrm{degcy}(G)}.$$

Therefore we have

$$R \ge \mathfrak{a}_m \ge \mathfrak{a}_{N_0} - \frac{r'}{m} \cdot \mathrm{degcy}(G) > \mathfrak{a}_{N_0} - \frac{\epsilon}{2} > R - \epsilon.$$

I.e. $\forall \epsilon > 0$, $\exists N_1 \in \mathbb{N}$ such that $m \ge N_1 \implies |\mathfrak{a}_m - R| < \epsilon$. Thus $\mathfrak{a}_m$ converges to $R = \sup\{\mathfrak{a}_m : m \in \mathbb{N}\}$ as $m$ goes to infinity. □

### 5.3.4 Proofs of Theorems 5.2.4, 5.2.14 and 5.2.15

Recall that for integers $a, b, c \ge 0$, the graph $\nabla_{a,b,c}$ has three vertices and three (possibly empty) edges of multiplicity $a, b, c$ respectively. In this subsection, we always denote the vertices of $\nabla_{a,b,c}$ as $A, B, C$, and edge multiplicities satisfy $\mathfrak{m}_{AB} = a$, $\mathfrak{m}_{AC} = b$ and $\mathfrak{m}_{BC} = c$.

In the cases that $a = b = c$, Theorem 5.3.5 shows that $\chi_c(\nabla_{a,a,a}) = \chi_c\left(C_3^{(a)}\right) = 2a + 1$. And it is not hard to prove for any $a \ge 0$, we have $\chi_c(\nabla_{a,a,0}) = a + 1$. In this section (Theorem 5.2.14), we will determine the exact $\chi_c(\nabla_{a,b,c})$ for the cases that $a = b$ or $b = c$.

We first prove Theorem 5.2.4 using Theorem 5.2.14.

**Theorem 5.2.14.**
*For any $m, n > 0$, we have*

$$\chi_c(\nabla_{m,m,n}) = \max\left\{m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil n + 1\right\}.$$

**Theorem 5.2.4.**
*For any positive integers $l$ and $m \ge (l+1)^2$, we have*

$$\chi_c(\nabla_{lm,lm,m}) - \chi_c(\nabla_{lm,lm,m-1}) = l + 1.$$

*Proof of Theorem 5.2.4.* By Theorem 5.2.14, we calculate

$$\chi_c(\nabla_{lm,lm,m}) = \max\{lm + m + 1, lm + 1\} = lm + m + 1.$$

We have $\left\lceil \frac{lm}{m-1} \right\rceil = \left\lceil l + \frac{l}{m-1} \right\rceil = l + 1$ since $0 < \frac{l}{m-1} \leq \frac{l}{l^2+2l} \leq \frac{1}{3}$. And hence

$$\left\lfloor \frac{lm}{\lceil lm/(m-1) \rceil} \right\rfloor = \left\lfloor \frac{lm}{l+1} \right\rfloor = \left\lfloor m - \frac{m}{l+1} \right\rfloor \leq m - (l+1).$$

Then

$$lm + \left\lfloor \frac{lm}{\lceil ml/(m-1) \rceil} \right\rfloor + 1 \leq lm + m - (l+1) + 1 = lm + m - l,$$

and

$$\left\lceil \frac{lm}{m-1} \right\rceil (m-1) + 1 = (l+1)(m-1) + 1 = lm + m - l.$$

Thus $\chi_c(\nabla_{lm,lm,m-1}) = lm + m - l = \chi_c(\nabla_{lm,lm,m}) - (l+1)$. □

We then prove Corollary 5.2.19 as a corollary of Theorem 5.2.14.

**Corollary 5.2.19.**
*For any $m, n, k, l \geq 0$, we have*

$$\chi_c\left(\nabla^{(k+l)}_{m,m,n}\right) - 1 \geq \left(\chi_c\left(\nabla^{(k)}_{m,m,n}\right) - 1\right) + \left(\chi_c\left(\nabla^{(l)}_{m,m,n}\right) - 1\right).$$

*Proof.* If any of $m, n, k, l$ is 0, then the inequality automatically hold. For the more general case, fix arbitrary $m, n, k, l \geq 1$ and denote $G = \nabla_{m,m,n}$ for simplicity.

If $n \geq m$, then $\chi_c\left(G^{(k+l)}\right) - 1 = \max\{2m(k+l) + 1, n(k+l) + 1\} = \left(\chi_c\left(G^{(k)}\right) - 1\right) + \left(\chi_c\left(G^{(l)}\right) - 1\right)$.

If $m > n$, then by Theorem 5.2.14, we have

$$\chi_c\left(G^{(k)}\right) - 1 = \max\left\{mk + \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor, \left\lceil \frac{m}{n} \right\rceil nk\right\},$$

and

$$\chi_c(G^{(l)}) - 1 = \max\left\{ml + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor, \left\lceil \frac{m}{n} \right\rceil nl\right\}.$$

Note $\left\lfloor \frac{m(k+l)}{\lceil m/n \rceil} \right\rfloor \geq \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor$ always holds. We hence have the following three cases.

Case 1: $\chi_c\left(G^{(k)}\right) - 1 = mk + \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor \geq \left\lceil \frac{m}{n} \right\rceil nk$ and $\chi_c\left(G^{(l)}\right) - 1 = ml + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor \geq \left\lceil \frac{m}{n} \right\rceil nl$.

Then $m(k+l) + \left\lfloor \frac{m(k+l)}{\lceil m/n \rceil} \right\rfloor \geq mk + \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor + ml + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor \geq \left\lceil \frac{m}{n} \right\rceil n(k+l)$. So $\chi_c\left(G^{(k+l)}\right) = m(k+l) + \left\lfloor \frac{m(k+l)}{\lceil m/n \rceil} \right\rfloor \geq \left(\chi_c\left(G^{(k)}\right) - 1\right) + \left(\chi_c\left(G^{(l)}\right) - 1\right)$.

Case 2: $\chi_c\left(G^{(k)}\right) - 1 = \left\lceil \frac{m}{n} \right\rceil nk \geq mk + \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor$ and $\chi_c\left(G^{(l)}\right) - 1 = \left\lceil \frac{m}{n} \right\rceil nl \geq ml + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor$.

If $\chi_c\left(G^{(k+l)}\right) = \left\lceil \frac{m}{n} \right\rceil n(k+l)$, then $\chi_c(G^{(k+l)}) - 1 = \left(\chi_c\left(G^{(k)}\right) - 1\right) + \left(\chi_c\left(G^{(l)}\right) - 1\right)$.

If $\chi_c\left(G^{(k+l)}\right) = m(k+l) + \left\lfloor \frac{m(k+l)}{\lceil m/n \rceil} \right\rfloor$, then $\chi_c(G^{(k+l)}) - 1 \geq \left\lceil \frac{m}{n} \right\rceil n(k+l) = \left(\chi_c\left(G^{(k)}\right) - 1\right) + \left(\chi_c\left(G^{(l)}\right) - 1\right)$.

Case 3: $\chi_c\left(G^{(k)}\right) - 1 = mk + \left\lfloor \frac{mk}{\lceil m/n \rceil} \right\rfloor > \left\lceil \frac{m}{n} \right\rceil nk$ and $\chi_c\left(G^{(l)}\right) - 1 = \left\lceil \frac{m}{n} \right\rceil nl > ml + \left\lfloor \frac{ml}{\lceil m/n \rceil} \right\rfloor$.

Note the above cannot happen if $n > m$ or $n$ divides $m$, so we have $m = nq+r$ for some integers $q$ and $1 \leq r \leq n-1$. Then $\left\lfloor \frac{mk}{q+1} \right\rfloor > (q+1)nk - mk = (n-r)k$ implies $\frac{mk}{q+1} > (n-r)k$, and hence $n-r < \frac{m}{q+1}$. At the same time $(q+1)nl > ml + \left\lfloor \frac{ml}{q+1} \right\rfloor$, so $(n-r)l > \left\lfloor \frac{ml}{q+1} \right\rfloor$. In other words, positive integers $k, l$ satisfy $\frac{ml}{q+1} > (n-r)l > \left\lfloor \frac{ml}{q+1} \right\rfloor$, which is not possible because both second and third terms are integers, and $\frac{ml}{q+1} < \left\lfloor \frac{ml}{q+1} \right\rfloor + 1$.

Thus for all $m, n, k, l \geq 0$, we have

$$\chi_c\left(\nabla^{(k+l)}_{m,m,n}\right) - 1 \geq \left(\chi_c\left(\nabla^{(k)}_{m,m,n}\right) - 1\right) + \left(\chi_c\left(\nabla^{(l)}_{m,m,n}\right) - 1\right).$$

$\square$

Also by Theorem 5.2.14, we can calculate the correspondence chromatic limit $\mathfrak{a}_\infty(G)$ of 'isosceles triangles' directly. For examples:

- $\chi_c\left(\nabla_{3m,3m,2m}\right) = \lfloor 4.5m \rfloor + 1$ for any $m \geq 1$, hence $\mathfrak{a}_\infty(\nabla_{3,3,2}) = 4.5$;
- $\chi_c\left(\nabla_{5m,5m,3m}\right) = \lfloor 7.5m \rfloor + 1$ for any $m \geq 1$, hence $\mathfrak{a}_\infty(\nabla_{5,5,3}) = 7.5$.

We then prove Theorem 5.2.15 again using Theorem 5.2.14.

**Theorem 5.2.15.**
*For any* $m, n > 0$, $\mathfrak{a}_\infty(\nabla_{m,m,n}) = \max\left\{m + \dfrac{m}{\lceil m/n \rceil}, \left\lceil \dfrac{m}{n} \right\rceil n\right\}$.

*Proof.* Recall that $\mathfrak{a}_\infty(G) := \lim\limits_{m\to\infty} \dfrac{\chi_c\left(G^{(m)}\right) - 1}{m}$. We will use Theorem 5.2.14 to show

$$\mathfrak{a}_{\lceil m/n \rceil}(\nabla_{m,m,n}) = \max\left\{m + \frac{m}{\lceil m/n \rceil}, \left\lceil \frac{m}{n} \right\rceil n\right\}.$$

And then we prove

$$\mathfrak{a}_\infty(\nabla_{m,m,n}) = \mathfrak{a}_{\lceil m/n \rceil}(\nabla_{m,m,n}).$$

For simplicity, we denote $k_t^* := \chi_c\left(\nabla_{mt,mt,nt}\right)$ for each $t \geq 1$.

Firstly, by definition and Theorem 5.2.14, we have

$$\begin{aligned}
\mathfrak{a}_{\lceil m/n \rceil}(\nabla_{m,m,n}) =& \frac{k^*_{\lceil m/n \rceil} - 1}{\lceil m/n \rceil} \\
=& \frac{\max\left\{m\lceil m/n \rceil + \left\lfloor \dfrac{m\lceil m/n \rceil}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \dfrac{m}{n} \right\rceil n\lceil m/n \rceil + 1\right\} - 1}{\lceil m/n \rceil} \\
=& \frac{\max\left\{m\lceil m/n \rceil + m, \left\lceil \dfrac{m}{n} \right\rceil n\lceil m/n \rceil\right\}}{\lceil m/n \rceil} \\
=& \max\left\{m + \frac{m}{\lceil m/n \rceil}, \left\lceil \frac{m}{n} \right\rceil n\right\}.
\end{aligned}$$

As we proved in Theorem 5.2.11, the correspondence chromatic limit $\mathfrak{a}_\infty(G)$ evaluates to the supremum of sequence $\{\mathfrak{a}_t(G) : t \in \mathbb{N}\}$. Since $\mathfrak{a}_{\lceil m/n \rceil}$ is clearly a member of the sequence, we will show $\mathfrak{a}_{\lceil m/n \rceil}$ is the maximum of the sequence. On the contrast, we assume there is some positive integer $T$ such that $\mathfrak{a}_T > \mathfrak{a}_{\lceil m/n \rceil}$. Therefore

$$\frac{k_T^* - 1}{T} > \frac{k^*_{\lceil m/n \rceil} - 1}{\lceil m/n \rceil} = \max\left\{m + \frac{m}{\lceil m/n \rceil}, \left\lceil \frac{m}{n} \right\rceil n\right\},$$

$$k_T^* > \max\left\{mT + \frac{mT}{\lceil m/n \rceil} + 1, \left\lceil \frac{m}{n} \right\rceil nT + 1\right\}.$$

The above contradicts to the fact that

$$k_T^* = \max\left\{mT + \left\lfloor \frac{mT}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil nT + 1\right\},$$

so there is no such $T$.

Thus $\mathfrak{a}_\infty(\nabla_{m,m,n}) = \mathfrak{a}_{\lceil m/n \rceil}(\nabla_{m,m,n}) = \max\left\{m + \frac{m}{\lceil m/n \rceil}, \left\lceil \frac{m}{n} \right\rceil n\right\}$. □

We now start to prove Theorem 5.2.14. We will determine $\chi_c(\nabla_{m,m,n})$ with $n \geq m \geq 0$ and $m \geq n \geq 0$ respectively, and show that the formulas in both cases agree with the Theorem. We first prove the easier case $n \geq m \geq 0$.

**Lemma 5.3.9.**
*If $n \geq m \geq 0$, then $\chi_c(\nabla_{n,m,m}) = \max\{n+1, 2m+1\}$.*

*Proof.* Firstly, $\chi_c(\nabla_{n,m,m}) \geq n+1$ since we have an edge of multiplicity $n$. If $m \leq \frac{n}{2}$, then $\operatorname{degcy}(\nabla_{n,m,m}) \leq n$ and hence $\chi_c(\nabla_{n,m,m}) = n+1$. (Note in this case, we have $n+1 \geq 2m+1$.)
If instead $m > \frac{n}{2}$, then $\chi_c(\nabla_{n,m,m}) \geq \chi_c\left(C_3^{(m)}\right) = 2m+1$, but at the same time $\operatorname{degcy}(\nabla_{n,m,m}) = 2m$, so $\chi_c(\nabla_{n,m,m}) = 2m+1$. (Note in this case, we have $2m+1 \geq n+1$.) □

For the other case $m \geq n$:

**Lemma 5.3.10.**
*If $m \geq n \geq 0$, then $\chi_c(\nabla_{m,m,n}) = \min\left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}$.*

Denote $S = \left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}$. We first show that $\min S$ exists, and for and two positive integers $k_1 \leq k_2$, if $k_1 \in S$, then $k_2 \in S$. As

$$\left\lfloor \frac{m+n+1}{m+n+1-m} \right\rfloor n \leq \frac{mn+n^2+n}{n+1} = m+n-\frac{m}{n+1} < m+n+1,$$

we have $m+n+1 \in S$, and in particular $S \neq \emptyset$. Clearly $S$ is bounded below and consist of integers, therefore $\min S$ exists.

Consider any two positive integers $k_1 \leq k_2$ such that $k_1 \in S$. Denote $t = k_2 - k_1 \geq 0$, then we have

$$\begin{aligned}\left\lfloor \frac{k_2}{k_2 - m} \right\rfloor n &= \left\lfloor \frac{k_1 + t}{k_1 + t - m} \right\rfloor n \\ &\leq \left\lfloor \frac{k_1 + \frac{k_1}{k_1 - m} t}{k_1 - m + t} \right\rfloor n = \left\lfloor \frac{k_1}{k_1 - m} \right\rfloor n < k_1 \leq k_2,\end{aligned}$$

i.e. for any positive integer $k_2 \geq \min S$, we have $k_2 > \left\lfloor \frac{k_2}{k_2 - m} \right\rfloor$ (and clearly $k_2 \geq \min S > m$). Hence $k_2 \in S$.

Then we prove the easier direction of Lemma 5.3.10 as the following Lemma.

**Lemma 5.3.11.**
*For any $m \geq n > 0$, if $k \leq \left\lfloor \frac{k}{k-m} \right\rfloor n$, then $\nabla_{m,m,n}$ is not $k$-correspondence colourable.*

*Proof.* Fix any given $m, n, k \geq 1$ such that $m \geq n > 0$ and $k \leq \left\lfloor \frac{k}{k-m} \right\rfloor n$. It suffices to construct an $k$-correspondence $\mathcal{C}(\nabla_{m,m,n})$ that does not have a proper correspondence colouring. Note $\chi_c(\nabla_{m,m,n}) \leq (m + n + 1)$ and $\left\lfloor \frac{m+n+1}{m+n+1-m} \right\rfloor n < m+n+1$, so $k < m+n+1$ by our earlier observations. Denote colour lists on the three vertices of $\nabla_{m,m,n}$ as $l(A) = \{a_i : 1 \leq i \leq k\}$, $l(B) = \{b_i : 1 \leq i \leq k\}$ and $l(C) = \{c_i : 1 \leq i \leq k\}$. Recall $\mathfrak{m}_{AB} = m$, $\mathfrak{m}_{AC} = m$ and $\mathfrak{m}_{BC} = n$.
Because $\left\lfloor \frac{k}{k-m} \right\rfloor (k - m) \leq \frac{k}{k-m}(k - m) = k \leq \left\lfloor \frac{k}{k-m} \right\rfloor n$, we can express $k$ as a sum of integers between $k - m$ and $n$, i.e. $k = \sum_{s=1}^{\lfloor k/(k-m) \rfloor} h_s$, $k - m \leq h_s \leq n$ for each $1 \leq s \leq \left\lfloor \frac{k}{k-m} \right\rfloor$. For referring purpose later, we let $h_0 = 0$ and write $k = \sum_{s=0}^{\lfloor k/(k-m) \rfloor} h_s$.

Construct the correspondence $\mathcal{C}(BC)$ on edge $BC$ be as following:
• $\{(B, b_i), (C, c_j)\} \in \mathcal{C}(BC)$ (colour $b_i$ and colour $c_j$ corresponds) if and only if there exist a positive integer $s^* \leq \left\lfloor \frac{k}{k-m} \right\rfloor$ such that

$$1 + \sum_{s=0}^{s^*-1} h_s \leq i, j \leq \sum_{s=0}^{s^*} h_s.$$

That is, for each $s^* \geq 1$, there is a complete bipartite subgraph $K_{h_{s^*},h_{s^*}}$ in the correspondence graph $\mathcal{C}(BC)$, induced by all colours $b_i, c_j$ where $1 + \sum_{s=0}^{s^*-1} h_s \leq i, j \leq \sum_{s=0}^{s^*} h_s$. The correspondence on $BC$ is well-defined, since $k - m \leq h_s \leq n$ for each $s$, $1 \leq s \leq \left\lfloor \frac{k}{k-m} \right\rfloor$, and each $c \in l(C)$ corresponds to at most $n = \mathfrak{m}_{BC}$ colours in $l(B)$, vice versa.

Now for $\mathcal{C}(AB)$ and $\mathcal{C}(AC)$:

- $\{(A, a_i), (B, b_j)\} \in \mathcal{C}(AB)$ if and only if $\{(C, c_i), (B, b_j)\} \notin \mathcal{C}(BC)$,
- $\{(C, c_i), (A, a_j)\} \in \mathcal{C}(CA)$ if and only if $\{(C, c_i), (B, b_j)\} \notin \mathcal{C}(BC)$.

These correspondences are also well-defined, because the degree of each vertex-colour pair in $\mathcal{C}(BC)$ is at least $k-m$, and hence at most $k-(k-m) = m$ in $\mathcal{C}(AB)$ or $\mathcal{C}(AC)$.

There is no proper correspondence colouring on $\mathcal{C}(G)$, because for each colour $a_i \in l(A)$, all the colours in $l(B), l(C)$ that do not correspond to $a_i$ are on the same $K_{h_s,h_s}$ subgraph in $\mathcal{C}(BC)$. Hence for any colour $a_i \in l(A)$, it is not possible to find $b_j \in l(B), c_k \in l(C)$ such that none of them correspond. $\square$

Then we prove the harder direction of Lemma 5.3.10 that if $m \geq n$, then $\min\left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}$ colours are sufficient for $\nabla_{m,m,n}$ (to have a proper correspondence colouring) with any prespecified correspondence. We need the following four lemmas.

**Lemma 5.3.12.**
*If $k > \left\lfloor \frac{k}{k-m} \right\rfloor n$ and $k > m \geq n$, then $k > m + \frac{2}{3}n$.*

*Proof.* Fix any positive integers $k > m \geq n$ such that $k > \left\lfloor \frac{k}{k-m} \right\rfloor n$. If $k > m+n$, then the statement is immediately true. So we assume $k = m+\alpha n$ for some rational number $\alpha \in (0, 1)$.

Substitute $k = m + \alpha n$ into the formula, then we have
$m + \alpha n > \left\lfloor \frac{m + \alpha n}{m + \alpha n - m} \right\rfloor n = \left\lfloor \frac{m + \alpha n}{\alpha n} \right\rfloor n$, which implies:
(1) $m + \alpha n > \left\lfloor \frac{m + \alpha n}{\alpha n} \right\rfloor n \geq 2n$ since $m + \alpha n \geq n + \alpha n > 2\alpha n$,

(2) $m + \alpha n > \left\lfloor \dfrac{m+\alpha n}{\alpha n} \right\rfloor n \geq \left( \dfrac{m+\alpha n}{\alpha n} - \dfrac{\alpha n - 1}{\alpha n} \right) n = \dfrac{m+1}{\alpha}$, note the second inequality holds because $\alpha n = k - m$ is an integer.

Here (1) shows $m > (2-\alpha)n$ and (2) shows $m < \dfrac{\alpha^2 n - 1}{1-\alpha}$. So if such a positive integer $m$ exists, we have $(2-\alpha)n < \dfrac{\alpha^2 n - 1}{1-\alpha}$ for some $n$, i.e. $(2-3\alpha)n < -1$. Which is only possible if $2 - 3\alpha < 0$, i.e. $\alpha > \dfrac{2}{3}$.

Thus for any positive integers $m$ and $n$, there is no positive integer $k$ satisfying both $k \leq m + \dfrac{2}{3}n$ and $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$. In other words, if $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$ and $k > m \geq n$, then $k > m + \dfrac{2}{3}n$. ☐

**Lemma 5.3.13.**

*If a $k$-correspondence $\mathcal{C}(\nabla_{m,m,n})$ does not have a proper correspondence colouring, then for each colour $b^* \in l(B)$, there are at least $k-m$ colours $a_i \in l(A)$ that do not correspond to $(B, b^*)$; and for each such $a_i$, the subgraph of $\mathcal{C}(BC)$ induced by colours do not correspond to $(A, a_i)$ (i.e. $\{(B,b) : \{(B,b),(A,a_i)\} \notin \mathcal{C}\} \sqcup \{(C,c) : \{(C,c),(A,a_i)\} \notin \mathcal{C}\}$), is a complete bipartite subgraph containing $(B, b^*)$, and of at least $k-m$ vertices in each part.*

*Proof.* Fix any $b^* \in l(B)$, denote $l^*(A)$ as the colours in $l(A)$ not corresponding to $b^*$. Then $|l^*(A)| \geq k - m$, as $\mathfrak{m}_{AB} = m$. For each $a_i \in l^*(A)$, because colouring $A$ by $a_i$ does not lead to a proper correspondence colouring of $G$, if we consider the colours do not correspond to $a_i$ and denote them as $l_i(B)$, $l_i(C)$ respectively, then none of the combinations between $l_i(B)$ and $l_i(C)$ is valid. In other words, $(\{B\} \times l_i(B)) \cup (\{C\} \times l_i(C))$ induces a complete bipartite subgraph in $\mathcal{C}(BC)$. It is clear that this subgraph contains $(B, b^*)$, and at least $k-m$ vertex-colour pairs in each part, i.e. $|l_i(B)| \geq k-m$ and $|l_i(C)| \geq k-m$. ☐

**Lemma 5.3.14.**

*If a $k$-correspondence $\mathcal{C}(\nabla_{m,m,n})$ does not have a proper correspondence colouring and $k > m + \dfrac{n}{2}$, then there exist a naming of $l(B) = \{b_i : 1 \leq i \leq k\}$ (i.e. a way of renaming the colours in $l(B)$ while keeping the original* *correspondence) and a naming of* $l(C) = \{c_i : 1 \le i \le k\}$ *such that for each* $1 \le s \le \left\lfloor \frac{k}{k-m} \right\rfloor$*, we have*

$$(s-1)(k-m)+1 \le i, j \le s(k-m) \implies \{(B, b_i), (C, c_j)\} \in \mathcal{C}(BC).$$

*Proof.* Assume $k > m + \frac{n}{2}$ and $\mathcal{C}(\nabla_{m,m,n})$ is a $k$-correspondence that does not have a proper correspondence colouring. Since every $\nabla_{m,m,n}$ is $(m+n+1)$-correspondence colourable, we can assume $k \le m+n$.

Because colouring $A$ by $a_1$ (any arbitrarily fixed colour) does not lead to a proper correspondence colouring on $\mathcal{C}(\nabla_{m,m,n})$, the colours in $l(B)$ and $l(C)$ that do not correspond to $a_1$ induce a complete bipartite subgraph of $\mathcal{C}(BC)$ of at least $k-m$ vertex-colour pairs in each part. Denote an arbitrarily chosen $K_{k-m,k-m}$ subgraph as $H_1$. Name the colours in $V(H_1) \cap (\{B\} \times l(B))$ as $b_1, \ldots, b_{k-m}$ and the colours in $V(H_1) \cap (\{C\} \times l(C))$ as $c_1, \ldots, c_{k-m}$ respectively.

Now we have our base case ($s = 1$). Assume we have already named $s(k-m)$ colours in $l(B)$ and $l(C)$, say $b_1, \ldots, b_{s(k-m)}$ and $c_1, \ldots, c_{s(k-m)}$, for some $s \le \left\lfloor \frac{k}{k-m} \right\rfloor - 1 = \left\lfloor \frac{m}{k-m} \right\rfloor$. Denote the collection of unnamed colours as $l_R(B)$ and $l_R(C)$ respectively.

Considering colours in $l(A)$, we claim there exist some $a^* \in l(A)$ that does not correspond to at least $n-(k-m)+1$ colours in $l_R(B)$. Note $|l(A)| = k$ and $|l_R(B)| = k - s(k-m)$, so for each $a \in l(A)$, the average number of edges (on the correspondence graph) between it and $l_R(B)$ is at most $\frac{1}{k}(m(k-s(k-m)))$; and hence there exists $a^* \in l(A)$ that does not correspond with at least $(k-s(k-m)) - \left\lfloor \frac{1}{k}(m(k-s(k-m))) \right\rfloor$ colours in $l_R(B)$. Note

$$\begin{aligned}
& k - s(k-m) - \left\lfloor \frac{1}{k}(m(k-s(k-m))) \right\rfloor - (n-(k-m)) \\
= & \left\lceil \frac{1}{k}\left((k-s(k-m))(k-m)\right) \right\rceil - (n-(k-m)) \\
\ge & \left\lceil \frac{1}{k}\left((k - \left\lfloor \frac{m}{k-m} \right\rfloor (k-m))(k-m)\right) \right\rceil - (n-(k-m)) \\
\ge & \left\lceil \frac{1}{k}\left((k - \left(\frac{m}{k-m}\right) (k-m))(k-m)\right) \right\rceil - (n-(k-m))
\end{aligned}$$

$$= \left\lceil \frac{1}{k}\left(k^2 - 2km + m^2 - kn + k^2 - km\right) \right\rceil$$
$$= \left\lceil \frac{2}{k}\left(\left(k - \frac{3}{4}m - \frac{1}{4}n\right)^2 - \frac{9m^2}{16} - \frac{n^2}{16} + \frac{m^2}{2}\right) \right\rceil$$
since $m^2 + n^2 \leq (m+n)^2$,
$$\geq \left\lceil \frac{2}{k}\left(\left(k - \frac{3}{4}m - \frac{1}{4}n\right)^2 - \frac{(m+n)^2}{16}\right) \right\rceil$$
$$= \left\lceil \frac{2}{k}\left(\left(k - m - \frac{n}{2}\right)\left(k - \frac{m}{2}\right)\right) \right\rceil,\ \text{taking ceiling of positive products,}$$
$$\geq 1.$$

So $a^* \in l(A)$ does not correspond to at least $(n - (k - m)) + 1$ colours in $l_R(B)$. Name those colours in $l_R(B)$ that do not correspond to $a^*$ as $b_{s(k-m)+1}, \ldots, b_{s(k-m)+t}$, if $t = |l_R(B)|$ that $(n - (k - m)) + 1 \leq t \leq k - m$; if instead we have $|l_R(B)| > k - m$, then just take $t = k - m$ and name arbitrary $k - m$ of them.

Because colouring $A$ by $a^*$ also does not lead to a proper correspondence colouring on $\chi_c(\nabla_{m,m,n})$ by choosing any $b \in l(B)$ and $c \in l(C)$, the colours in $l(B)$, $l(C)$ that do not correspond to $a^*$ (including $b_{s(k-m)+1}$, $\ldots$, $b_{s(k-m)+t}$) induce a complete bipartite subgraph in $\mathcal{C}(BC)$, which has at least $k - m$ vertices on both sides. Denote $H_{s+1}$ as a $K_{k-m,k-m}$ in this complete bipartite subgraph that contains $b_{s(k-m)+1}, \ldots, b_{s(k-m)+t}$.

Because $\mathfrak{m}_{BC} = n$ and $(k - m) + t > n$, there does not exist $c \in l(C) \setminus l_R(C)$ that is also in $H_{s+1}$. (Otherwise there will be some vertex-colour pairs in $l(C) \setminus l_R(C)$ that correspond to more than $n$ colours in $l(B)$.) Name all $k - m$ colours in $H_{s+1} \cap l(C)$ as $c_{s(k-m)+1}, \ldots, c_{(s+1)(k-m)}$. Also, for exactly the same reason there does not exist $b \in l(B) \setminus l_R(B)$ that is also in $H_{s+1}$. Name the unnamed ones as $b_{s(k-m)+t+1}, \ldots, b_{(s+1)(k-m)}$.

Repeat the above procedure for each $s \leq \left\lfloor \frac{k}{k-m} \right\rfloor - 1$, then we find an ordering of the first $\left\lfloor \frac{k}{k-m} \right\rfloor (k - m)$ colours in $l(B)$ and $l(C)$ respectively. We pick an arbitrary order for the unnamed colours to complete the required ordering. □

**Lemma 5.3.15.**
*Let $k > m + \frac{2}{3}n$, and the naming of $l(B)$ and $l(C)$ be as in Lemma 5.3.14. If $(s_1 - 1)(k-m)+1 \le i \le s_1(k-m)$, $(s_2-1)(k-m)+1 \le j \le s_2(k-m)$ for some $s_1 \neq s_2$, then there does not exist a $K_{k-m,k-m}$ subgraph of $\mathcal{C}(BC)$ containing both $b_i$ and $c_j$, nor both $b_i$ and $b_j$, nor both $c_i$ and $c_j$.*

*Proof.* Assume the colour names in $l(B)$ and $l(C)$ are in accordance to Lemma 5.3.14. Denote $H_s$ as the $K_{k-m,k-m}$ subgraph of $\mathcal{C}(BC)$ induced by colours indexed $(s-1)(k-m)+1, \ldots, s(k-m)$. Fix arbitrary pair of indices $i$, $j$ such that $(s_1-1)(k-m)+1 \le i \le s_1(k-m)$ and $(s_2-1)(k-m)+1 \le j \le s_2(k-m)$, for some $s_1 \neq s_2$. So $(B, b_i), (C, c_i) \in V(H_{s_1})$ and $(B, b_j), (C, c_j) \in V(H_{s_2})$. We discuss the following subcases.

(1) Note $b_i$ corresponds to at most $n-(k-m)$ vertex-colour pairs outside $H_{s_1}$, and $c_j$ corresponds to at most $n-(k-m)$ vertex-colour pairs outside $H_{s_2}$. If there exists $H^* = K_{k-m,k-m} \subseteq \mathcal{C}(BC)$ that $(B, b_i) \in V(H^*)$ and $(C, c_j) \in V(H^*)$, then $H^*$ contains at least $(k-m)-(n-(k-m))$ colours in the $l(C)$ side of $H_{s_1}$, and contains at least $(k-m)-(n-(k-m))$ colours in the $l(B)$ side of $H_{s_2}$. Consider $c^* \in l(C)$ such that $(C, c^*) \in V(H^*) \cap V(H_{s_2})$. Then it corresponds to at least $(k-m)+((k-m)-(n-(k-m))) = 3(k-m)-n$ colours in $l(B)$. But $k > m + \frac{2}{3}n$ implies $3(k-m)-n > n$, so the existence of $c^*$ contradicts to $\mathfrak{m}_{BC} = n$. I.e. there does not exist a $K_{k-m,k-m}$ subgraph in $\mathcal{C}(BC)$ containing both $b_i$ and $c_j$.

(2) If there is some $H^{**} = K_{k-m,k-m} \subseteq \mathcal{C}(BC)$ where both $(B, b_i) \in V(H^{**})$ and $(B, b_j) \in V(H^{**})$, then with similar arguments as in (1), $H^{**}$ contains at least $(k-m)-(n-(k-m))$ colours in the $l(C)$ side of $H_{s_1}$. But now $b_j$ corresponds to at least $3(k-m)-n > n$ colours in $l(C)$, which contradicts $\mathfrak{m}_{BC} = n$. I.e. there does not exist a $K_{k-m,k-m}$ subgraph in $\mathcal{C}(BC)$ containing both $b_i$ and $b_j$.

(3) With essentially the same arguments as in (2), there does not exist a $K_{k-m,k-m}$ subgraph in $\mathcal{C}(BC)$ containing both $c_i$ and $c_j$. $\square$

Now we can prove Lemma 5.3.10. Recall that

**Lemma 5.3.10.**
*If* $m \geq n \geq 0$*, then* $\chi_c(\nabla_{m,m,n}) = \min\left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}$.

*Proof.* Fix any $m \geq n \geq 0$ and denote $G = \nabla_{m,m,n}$. Let $k^* = \min\left\{k \in \mathbb{N} : k > m,\ k > \left\lfloor \frac{k}{k-m} n \right\rfloor\right\}$.

Assume on the contrast there exists an $k^*$-correspondence $\mathcal{C}(G)$ that does not have a proper correspondence colouring. Note Lemma 5.3.12 suggests $k^* > m + \frac{2}{3}n$, so the conditions in Lemmas 5.3.13, 5.3.14 and 5.3.15 are satisfied. Also let the colour names of $l(B)$ and $l(C)$ be in accordance to Lemma 5.3.14.

For each $s$ such that $1 \leq s \leq \left\lfloor \frac{k^*}{k^*-m} \right\rfloor$, let $l_s(A) \subseteq l(A)$ denote the sublist of colours in $l(A)$ that does not correspond to colour $b_{(s-1)(k^*-m)+1}$, i.e. $l_s(A) = \{a \in l(A) : \{(A,a),(B,b_{(s-1)(k^*-m)+1})\} \notin \mathcal{C}(AB)\}$. Note $|l_s(A)| \geq k^* - m$, and Lemma 5.3.15 suggests that if $i \neq j$, then $l_i(A) \cap l_j(A) = \emptyset$.

Denote the colours in $l(C)$ that does not correspond to some of $l_s(A)$ as $l_s(C)$. For any $s$, since $\mathfrak{m}_{BC} = n$, we have $b_{(s-1)(k^*-m)+1}$ corresponds to at most $n$ colours in $l(C)$, and hence each $l_s(C)$ contains at most $n$ distinct colours, otherwise a proper correspondence colouring on $\mathcal{C}(G)$ exists. I.e. $|l_s(C)| = |\{c \in l(C) : \{(C,c),(A,a)\} \notin \mathcal{C}(AC) \text{ for some } a \in l_s(A)\}| \leq n$.

As $k^* > \left\lfloor \frac{k^*}{k^*-m} \right\rfloor n$, there exists at least one colour $c^* \in l(C)$ that is not in any of the $l_s(C)$. So $c^*$ corresponds to all colours in $\bigcup_s l_s(A)$. But since all $l_s(A)$ are disjoint and of cardinality at least $k^*-m$, we have $\left|\bigcup_s l_s(A)\right| \geq \left\lfloor \frac{k^*}{k^*-m} \right\rfloor (k^*-m) \geq \left(\frac{k^*}{k^*-m} - \frac{k^*-m-1}{k^*-m}\right)(k^*-m) = m+1 > m$, which contradicts to $\mathfrak{m}_{AC} = m$ that $c^*$ corresponds to at most $m$ colours in $l(A)$. So such an uncolourable $k^*$-correspondence on $G$ does not exist, and hence $\chi_c(G) \leq k^*$.

Together with Lemma 5.3.11, we conclude that

$$\chi_c(G) = \min\left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}.$$

□

Now we have all the tools required for Theorem 5.2.14. It is not hard to see Theorem 5.2.14 agrees with Lemma 5.3.9 for $m, n \neq 0$: if $0 < m \leq n$, then $\lceil m/n \rceil = 1$, and hence $m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1 = 2m + 1$ and $\left\lceil \frac{m}{n} \right\rceil n + 1 = n + 1$. We need the following two lemmas to show it agrees with Lemma 5.3.10.

**Lemma 5.3.16.**
*For $m \geq n > 0$, if $k = \max\left\{m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil n + 1\right\}$, then $k > m$ and $k > \left\lfloor \frac{k}{k-m} \right\rfloor n$.*

*Proof.* Fix any $m \geq n > 0$, and let $k = \max\left\{m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor, \left\lceil \frac{m}{n} \right\rceil n\right\} + 1$. It is clear that $k > m$, so we will prove that $k - \left\lfloor \frac{k}{k-m} \right\rfloor n$ is positive.

If $n$ divides $m$, then $m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1 = m + n + 1$ and $\left\lceil \frac{m}{n} \right\rceil n + 1 = m + 1$. So $k = m + n + 1$ and $k - \left\lfloor \frac{k}{k-m} \right\rfloor n = m + n + 1 - n - \left\lfloor \frac{m}{n+1} \right\rfloor n \geq 1$.

If $n$ does not divide $m$, we write $m = nq + r$ for some integer $q$ and $1 \leq r \leq n - 1$. So $\left\lceil \frac{m}{n} \right\rceil = q + 1$.

Case 1: $m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor \geq \left\lceil \frac{m}{n} \right\rceil n$, i.e. $m + \left\lfloor \frac{m}{q+1} \right\rfloor \geq (q+1)n = m + n - r$. Which means $\left\lfloor \frac{m}{q+1} \right\rfloor \geq n - r$, i.e. $m \geq (q+1)(n-r)$, or $n \leq (q+2)r$.

In this case $k = m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1$. Denote $r' = n - \left\lfloor \frac{m}{q+1} \right\rfloor$. We know $1 \leq r' \leq r$ because $n > \left\lfloor \frac{m}{q+1} \right\rfloor \geq n - r$. So $(n - r')(q+1) \leq m < (n - r' + 1)(q+1)$, and we have

$$
\begin{aligned}
k - \left\lfloor \frac{k}{k-m} \right\rfloor n &= m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1 - n - \left\lfloor \frac{m}{\left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1} \right\rfloor n \\
&= m + (n - r') + 1 - n - \left\lfloor \frac{m}{n - r' + 1} \right\rfloor n \\
&\geq m - r' + 1 \\
&\geq 1.
\end{aligned}
$$

Here the second last inequality holds since $m < (n-r'+1)(q+1)$, which leads to $\left\lfloor \dfrac{m}{n-r'+1} \right\rfloor \leq q$.

Case 2: $m + \left\lfloor \dfrac{m}{\lceil m/n \rceil} \right\rfloor < \left\lceil \dfrac{m}{n} \right\rceil n$, i.e. $m + \left\lfloor \dfrac{m}{q+1} \right\rfloor < (q+1)n = m+n-r$.

Which means $\left\lfloor \dfrac{m}{q+1} \right\rfloor < n-r$, i.e. $m < (q+1)(n-r)$, or $n > (q+2)r$.

In this case $k = \left\lceil \dfrac{m}{n} \right\rceil n + 1$. And then

$$\begin{aligned} k - \left\lfloor \frac{k}{k-m} \right\rfloor n = & \left\lceil \frac{m}{n} \right\rceil n + 1 - n - \left\lfloor \frac{m}{\lceil m/n \rceil n + 1 - m} \right\rfloor n \\ &= (q+1)n + 1 - n - \left\lfloor \frac{m}{(q+1)n - (nq+r) + 1} \right\rfloor n \\ &= qn + 1 - \left\lfloor \frac{m}{n-r+1} \right\rfloor n \\ &\geq qn + 1 - qn \\ &= 1, \end{aligned}$$

Here the inequality holds because $\dfrac{m}{n-r} < q+1$, which leads to $\dfrac{m}{n-r+1} < q+1$.

That is, in all cases of $k = \max\left\{ m + \left\lfloor \dfrac{m}{\lceil m/n \rceil} \right\rfloor, \left\lceil \dfrac{m}{n} \right\rceil n \right\} + 1$, we have $k > m$ and $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$. $\square$

Then we prove the other direction.

**Lemma 5.3.17.**
*For any integers $k > m \geq n > 0$, if $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$, then $k > m + \dfrac{m}{\lceil m/n \rceil}$ and $k > \left\lceil \dfrac{m}{n} \right\rceil n$.*

*Proof.* Fix any integers $m \geq n > 0$, and integer $k$ such that $k > m$ and $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$.

Consider functions $f_1 : \mathbb{R}_{>m} \setminus \{m\} \to \mathbb{Z}$, $f_1(x) = \left\lfloor \dfrac{m}{x-m} \right\rfloor$ and $f_2 : \mathbb{R}_{>0} \to \mathbb{R}$, $f_2(x) = \dfrac{x}{n} - 1$. (Here $\mathbb{R}_{>\alpha}$ denotes all the real numbers larger than $\alpha$.) Note $f_1$ is the floor function of a reciprocal function and $f_2$ is linear. Also note $k > m$ and $k > \left\lfloor \dfrac{k}{k-m} \right\rfloor n$ translates to $k \geq m+1$ and $f_2(k) > f_1(k)$.

For each non-negative integer $y$ in the image of $f_1(x)$, $x > m$, we have

$$\begin{aligned} f_1(x) = y \Leftrightarrow y &\le \frac{m}{x-m} < y+1, \\ &\Leftrightarrow \frac{(y+2)m}{y+1} < x \le \frac{(y+1)m}{y}, \\ &\Leftrightarrow \frac{(y+2)m}{(y+1)n} - 1 < f_2(x) \le \frac{(y+1)m}{yn} - 1. \end{aligned}$$

Hence for some non-negative integer $y$, There exists $x$ (not necessarily an integer) satisfying both $f_1(x) = y$ and $f_2(x) > f_1(x)$ only if $\frac{(y+1)m}{yn} - 1 > y$. Since for all $x > m$, we have $f_2$ monotonically increasing and $f_1$ is non-increasing, hence the largest such $y$ (that exist $x$ satisfying both $f_1(x) = y$ and $f_2(x) > f_1(x)$) gives a strict lower bound on possible $x$ values (that $f_2(x) > f_1(x)$). Although we cannot determine the 'smallest' such $x$, because $x$ does not have to be integer and the inequality is strict, we can determine the largest such $y$ since it is an integer. Denote $y^*$ as the largest $y$ such that there exist $x$ with both $f_1(x) = y^*$ and $f_2(x) > f_1(x)$.

Solving $\frac{(y+1)m}{yn} - 1 > y$, we have $y < \frac{m}{n}$, so $y^* = \left\lceil \frac{m}{n} \right\rceil - 1$. That is, the preimage $f_1^{-1}(y)$ where $x$ first satisfies $f_2(x) > f_1(x)$ is $f_1^{-1}(y^*) = f_1^{-1}\left(\left\lceil \frac{m}{n} \right\rceil - 1\right)$. In other words, because $f_1$ is non-increasing for $x > m$, we have $f_1(x) \le y^*$ implies $x \ge \frac{(y^*+2)m}{y^*+1} = \frac{(\lceil m/n \rceil + 1)m}{\lceil m/n \rceil} = m + \frac{m}{\lceil m/n \rceil}$. And $f_2(x) > y^*$ implies $\frac{x}{n} - 1 \ge y^* = \left\lceil \frac{m}{n} \right\rceil - 1$, i.e. $x > \left\lceil \frac{m}{n} \right\rceil n$.

For each $k$ satisfying both $k > m$ and $f_2(k) > f_1(k)$, there are $x$ satisfying $x > m$ and $f_1(x) > f_2(x)$ with $\lceil x \rceil = k$. Hence $k > m$ and $k > \left\lfloor \frac{k}{k-m} \right\rfloor n$ implies $k > m + \frac{m}{\lceil m/n \rceil}$ and $k > \left\lceil \frac{m}{n} \right\rceil n$. □

Finally, we can conclude for the $m \ge n$, the formula in Lemma 5.3.10 also agrees with the formula in Theorem 5.2.14.

**Theorem 5.2.14.**
*For any integers $m, n > 0$, we have*

$$\chi_c(\nabla_{m,m,n}) = \max\left\{ m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil n + 1 \right\}.$$

*Proof.* Lemma 5.3.17 and Lemma 5.3.16 suggests

$$\begin{aligned}\chi_c(\nabla_{m,m,n}) &= \min\left\{k \in \mathbb{N} : k > m, k > \left\lfloor \frac{k}{k-m} \right\rfloor n\right\}\\ &= \min\left\{k \in \mathbb{N} : k > m + \frac{m}{\lceil m/n \rceil}, k > \left\lceil \frac{m}{n} \right\rceil n\right\}.\end{aligned}$$

Since $\left\lceil \frac{m}{n} \right\rceil n$ is an integer, and integer $k > m + \frac{m}{\lceil m/n \rceil}$ if and only if $k \geq \left\lfloor m + \frac{m}{\lceil m/n \rceil} \right\rfloor + 1$, we have

$$\chi_c(\nabla_{m,m,n}) = \max\left\{m + \left\lfloor \frac{m}{\lceil m/n \rceil} \right\rfloor + 1, \left\lceil \frac{m}{n} \right\rceil n + 1\right\}.$$

□

## 5.4 Discussions

In this chapter, we studied the effects of certain graph operations on the correspondence chromatic numbers, including vertex / edge deletion, identification and multiplying the whole edge set of a graph. We also find the exact correspondence chromatic number of a class of graphs. We discuss some possible ways of improvement and more open problems in this section.

Recall that we proved the following theorem in Section 5.3.2.

**Theorem 5.3.3.**
*For any graph $G$ and vertices $x, y \in V(G)$, we have*

$$\chi_c(G) \leq \max\left\{\chi_c(G/xy), \min\left\{\mathfrak{m}(xy) + \left\lceil \frac{1}{2}\chi_c(G/xy) \right\rceil, \max\{\mathfrak{m}(xy), \chi_c(G/xy)\} + \max\{M_G(xy), 1\}\right\}\right\}.$$

If we analyse Theorem 5.3.3 a bit further, it means:

- if $\chi_c(G/xy) \geq 2\mathfrak{m}(xy)$, then $\chi_c(G) \leq \chi_c(G/xy)$;
- if $\mathfrak{m}(xy) < \chi_c(G/xy) < 2\mathfrak{m}(xy)$, then $\chi_c(G) \leq \min\left\{\mathfrak{m}(xy) + \left\lceil \frac{1}{2}\chi_c(G/xy) \right\rceil, \chi_c(G/xy) + \max\{M_G(xy), 1\}\right\}$;
- if $\chi_c(G/xy) \leq \mathfrak{m}(xy)$ and $\chi_c(G/xy) \geq 2M_G(xy)$, then

$\chi_c(G) \leq \mathfrak{m}(xy) + \max\{M_G(xy), 1\}$;

- if $\chi_c(G/xy) \leq \mathfrak{m}(xy)$ and $\chi_c(G/xy) < 2M_G(xy)$, then $\chi_c(G) \leq \mathfrak{m}(xy) + \left\lceil \frac{1}{2}\chi_c(G/xy) \right\rceil$.

It is of interests to know whether our bounds in Theorem 5.3.3 can be further improved. We discussed in Section 5.3.2 for the cases that

$$\max\left\{\chi_c\left(G/xy\right), \mathfrak{m}_{xy}\right\} + \max\{M_{xy}(G), 1\} < \mathfrak{m}_{xy} + \left\lceil \frac{1}{2}\chi_c(G/xy) \right\rceil,$$

and the first term will provide a better bound for $\chi_c(G)$ than the second. But it is still not understood whether there are graphs $G$ satisfying $\chi_c(G) = \max\left\{\chi_c\left(G/xy\right), \mathfrak{m}_{xy}\right\} + \max\{M_{xy}(G), 1\}$.

In Section 5.3.4, we determined the exact correspondence chromatic numbers of $\nabla_{a,b,c}$ with $a = b$ or $b = c$. It is also of interest to understand $\chi_c(\nabla_{a,b,c})$ for general positive integers $a, b, c$. We present the following observations here and leave the rest as an open problem.

**Corollary 5.4.1.**
*For any given $a \geq b \geq c > 0$, we have*

$$\max\left\{a+1, 2c+1, b + \left\lfloor \frac{b}{\lceil b/c \rceil} \right\rfloor + 1, \left\lceil \frac{b}{c} \right\rceil c + 1\right\} \leq \chi_c(\nabla_{a,b,c})$$
$$\leq \min\left\{\max\left\{a + \left\lfloor \frac{a}{\lceil a/c \rceil} \right\rfloor + 1, \left\lceil \frac{a}{c} \right\rceil c + 1\right\}, \max\{b+c+1, a+1\}\right\}.$$

Corollary 5.4.1 is a direct corollary of Theorem 5.2.14 and the fact that

$$\max\{\chi_c(\nabla_{a,c,c}), \chi_c(\nabla_{b,b,c})\} \leq \chi_c(\nabla_{a,b,c}) \leq \min\{\chi_c(\nabla_{a,a,c}), \mathrm{degcy}(\nabla_{a,b,c}){+}1\}.$$

**Corollary 5.4.2.**
*Fix any positive integers $a > b > c$.*
*(1) If $a \geq b + c$, then $\chi_c(\nabla_{a,b,c}) = a + 1$.*
*(2) If $c \mid b$ and $a \leq b + c$, then $\chi_c(\nabla_{a,b,c}) = b + c + 1$.*
*(3) If $b + c = a + 1$, then $\chi_c(\nabla_{a,b,c}) = b + c + 1 = a + 2$.*
*(4) If $a \leq \frac{4}{3}c$, then $\chi_c(\nabla_{a,b,c}) = 2c + 1$.*

Corollary 5.4.2 (1), (2) and (4) are derived from Corollary 5.4.1. Corollary 5.4.2 (3) is a bit different but also simple. We give the proofs as following.

*Proof.* (1) Note $\chi_c(\nabla_{a,b,c}) \geq a+1$ considering the edge with multiplicity $a$. And as $a \geq b+c$, $\text{degcy}(\nabla_{a,b,c}) = a$, so $\chi_c(\nabla_{a,b,c}) = a+1$.

(2) Because $c \mid b$, we have $\chi_c(\nabla_{a,b,c}) \geq \chi_c(\nabla_{b,b,c}) = b+c+1$. And since $a \leq b+c$, $\text{degcy}(\nabla_{a,b,c}) = b+c$. So $\nabla_{a,b,c} = b+c+1$.

(3) Because $b+c = a+1$, we have $\text{degcy}(\nabla_{a,b,c}) = b+c$. We construct a $(b+c)$-correspondence without a proper correspondence colouring:

- $\{(A, a_i), (B, b_j)\} \in \mathcal{C}(\nabla_{a,b,c})$ if $j \equiv i+k \mod b+c$, for $k = 1, \ldots, a$;
- $\{(A, a_i), (C, c_j)\} \in \mathcal{C}(\nabla_{a,b,c})$ if $j \equiv i+k \mod b+c$, for $k = 0, \ldots, b-1$;
- $\{(B, b_i), (C, c_j)\} \in \mathcal{C}(\nabla_{a,b,c})$ if $j \equiv i+k \mod b+c$, for $k = b, \ldots, b+c-1$.

So $\chi_c(\nabla_{a,b,c}) = b+c+1 = a+2$.

(4) As $a \leq \frac{4}{3}c < 2c$, $\chi_c(\nabla_{a,b,c}) \geq \chi_c(\nabla_{a,c,c}) = \max\{a+1, 2c+1\} = 2c+1$. At the same time

$$\chi_c(\nabla_{a,b,c}) \leq \chi_c(\nabla_{a,a,c}) = \max\left\{a + \left\lfloor \frac{a}{\lceil a/c \rceil} \right\rfloor + 1, \left\lceil \frac{a}{c} \right\rceil c + 1\right\} = 2c+1.$$

So $\chi_c(\nabla_{a,b,c}) = 2c+1$. $\square$

**Question 5.4.3.** *What are the exact correspondence chromatic numbers of $\nabla_{a,b,c}$, for $a, b, c$ that have not been considered by Corollary 5.4.2?*

# 6

# Partial Correspondence Colouring

## 6.1 Introduction

Recall that a graph $G$ is $n$-correspondence-colourable if a proper correspondence colouring exists for any $n$-correspondence on $G$; and the correspondence chromatic number $\chi_c(G)$ is the smallest $n$ so that $G$ is $n$-correspondence-colourable.

Also recall in correspondence colouring, we can always 'rename' colours in each colour list while keeping its original correspondence on each edge. Hence we may assume all colour lists assigned to each vertex are identical, given that they are of the same order. We can also assume all correspondences are full, because otherwise we can always add more correspond to make it full, and a proper correspondence colouring on the new correspondence easily implies a proper correspondence colouring on the original correspondence.

In this chapter, we study *partial* correspondence colouring problems. All graphs in this chapter are undirected and without multiple edges nor loops. Formally, we study the following question.

**Question 6.1.1.**
*Given $n$ and a graph $G$ that is $n$-correspondence colourable. For any positive integer $n' < n$ and arbitrary $n'$-correspondence on $G$, how many vertices of $G$ can we guarantee to be properly correspondence coloured?*

Similar to a few earlier chapters, this question is inspired by a partial colouring conjecture asked by Albertson et al. [2] regarding list colouring: given a graph $G$ that is $n$-choosable, if $n'$ colours are given to each vertex, $1 \leq n' \leq n$, can we always properly list colour at least $\frac{n'}{n}|V(G)|$ vertices of $G$?

Recall that this problem is trivial for ordinary colouring: if a graph $G$ is $n$-colourable, then we can always partition the graph into $n$ independent sets using the $n$ colour classes. If we only have $n' \leq n$ colours available, we can always colour at least $\frac{n'}{n}|V(G)|$ vertices by choosing the largest $n'$ independent sets. It is not possible to obtain a better lower bound than $\frac{n'}{n}|V(G)|$ for general $n$-colourable graphs, as can be seen by, for example, considering complete graphs.

Also recall that for the original question on list colouring, Chappell [5] proved that if $G$ is $n$-choosable and $n' \leq n$, then at least $\frac{6}{7} \cdot \frac{n'}{n}|V(G)|$ vertices can always be properly list coloured if every vertex has a list of $n'$ colours. The conjecture of Albertson et al. has been studied extensively, see e.g. [25, 31, 32], and is still open.

In Section 6.3, we will explain by a series of examples showing the conjecture of Albertson et al. does not hold if we extend it to correspondence colouring. That is, there exist graphs $G$ with correspondence chromatic number $n$, and an $n'$-correspondence with another integer $n' < n$ on $G$ such that less than $\frac{n'}{n}|V(G)|$ vertices can be properly correspondence coloured.

We then study Question 6.1.1. (Note that in the process of preparing this thesis, we found that the authors in [34] also studied partial correspondence colouring.) Our research has been done independently from theirs.

Unfortunately, the technique Chappell used to prove the $\frac{6}{7} \cdot \frac{n'}{n}|V(G)|$ lower bound for list colouring cannot be directly applied to prove a similar bound for correspondence colouring. We discuss some of the partial colouring (on ordinary or list colouring) results that extend to correspondence colouring in Section 6.2. Then we discuss some sufficient conditions for a proper correspondence colouring to exist in Sections 6.4 and 6.5.

## 6.2 Partial Correspondence Colouring

In this section, we generalise some results on partial list colouring [5, 25, 31] to correspondence colouring. Recall that with the renaming function discussed in Section 1.2, we can rename the colours associated to each vertex while keeping the same number of colours and correspondence. We use the following notations in this chapter.

**Definition 6.2.1.**
*For a graph $G$ and a correspondence $\mathcal{C}(G)$ on it, the* partial correspondence colourable number $\lambda_c(\mathcal{C}(G))$ *of $G$ with respect to $\mathcal{C}$ is the maximum number of vertices on $G$ that can be properly correspondence coloured respecting $\mathcal{C}(G)$. For any given positive integer $n'$ and a graph $G$, the* $n'$-th certainly colourable number $\lambda_c(G, n')$ *is the smallest $\lambda_c(\mathcal{C}(G))$ over all possible $n'$-correspondence $\mathcal{C}(G)$.*

**Lemma 6.2.2.**
*Let $G$ be a graph with correspondence chromatic number $\chi_c(G) = n$. For any given integer $n' \leq n$, we have $\lambda_c(G, n') \geq \dfrac{|V(G)|}{\lceil n/n' \rceil}$.*

*Proof.* Given graph $G$ with correspondence chromatic number $\chi_c(G) = n$, assign an arbitrary $n'$-correspondence on $G$ where $n' \leq n$ and denoted by $\mathcal{C}(G)$. Let $p := \lceil n/n' \rceil$.

Denote $l(v) = \{c_1, c_2, ..., c_{n'}\}$ as the colour list associated with each vertex $v$. Now we define correspondence $\mathcal{C}'(G)$:

- For each vertex $x$ in $V(G)$, we extend its colour list to $l'(x) = L' = \bigcup_{i=1}^{p} l^i(x)$ where $l^1(x) = l(x) = \{c_j : 1 \leq j \leq n'\}$ and $l^i(x) = \{c_j^i : 1 \leq j \leq n'\}$ for $2 \leq i \leq p$;

- The correspondence on each edge keeps original: for adjacent vertices $x$ and $y$, $\{(x, c_k^i), (y, c_l^j)\} \in \mathcal{C}'(G)$ if and only if $i = j$ and $\{(x, c_k), (y, c_l)\} \in \mathcal{C}(G)$.

By the definition of $p$, we have $n'p \geq n$, so there is a proper correspondence colouring $f : V(G) \to L'$ under $\mathcal{C}'(G)$. Using this proper correspondence colouring, $V(G)$ can be partitioned into $p$ colour classes $C^1, C^2, ..., C^p$: a vertex $x$ is in $C^i$ if and only if $f(x) \in l^i(x)$. Now one of the $p$ classes contains at least the average number of vertices over all classes (i.e. $\frac{|V(G)|}{p}$). Without loss of generality, we assume $C^1$ contains at least $\frac{|V(G)|}{p}$ vertices. As the correspondence $\mathcal{C}'(G)$ is copied from $\mathcal{C}(G)$, we can properly correspondence colour each vertex $x \in C^1$ by colour $f(x)$, which will be a proper partial colouring in $\mathcal{C}(G)$. Thus we have $\lambda_c(G, n') \geq \frac{|V(G)|}{p} = \frac{|V(G)|}{\lceil n/n' \rceil}$. □

**Corollary 6.2.3.**
*Let $G$ be a graph with correspondence chromatic number $\chi_c(G)$. For each positive integer $n'$, if $n' < \chi_c(G)$, then we have $\lambda_c(G, n') > \frac{n'|V(G)|}{2\chi_c(G)}$, or if $n' \geq \chi_c(G)$, then we have $\lambda_c(G, n') = |V(G)|$.*

*Proof.* The $n' \geq \chi_c(G)$ part is trivial. If $n' < \chi_c(G)$, then we have $\lceil \chi_c(G)/n' \rceil < \frac{\chi_c(G)}{n'} + 1 \leq \frac{2\chi_c(G)}{n'}$, and hence by Lemma 6.2.2, we have $\lambda_c(G, n') \geq \frac{|V(G)|}{\lceil \chi_c(G)/n' \rceil} > \frac{n'|V(G)|}{2\chi_c(G)}$. □

**Lemma 6.2.4.**
*Given any graph $G$ with correspondence chromatic number $\chi_c(G)$. For any integers $r, s$ such that $1 \leq r, s \leq \chi_c(G)$, we have $\lambda_c(G, r) + \lambda_c(G, s) \geq \lambda_c(G, r + s)$.*

*Proof.* By our definition of certainly colourable numbers, it is sufficient to construct a $(r + s)$-correspondence $\mathcal{C}_{r+s}(G)$ such that $\lambda_c(G, r) + \lambda_c(G, s) \geq \lambda_c(\mathcal{C}_{r+s}(G))$, and hence $\lambda_c(G, r) + \lambda_c(G, s) \geq \lambda_c(G, r + s)$.

Let $\mathcal{C}_r(G)$ and $\mathcal{C}_s(G)$ be arbitrary $r$-correspondence and $s$-correspondence on $G$, with colour lists $l_r = \{c_1, \ldots, c_r\}$ and $l_s = \{c_{r+1}, \ldots, c_{r+s}\}$ respectively.

We construct the $(r+s)$-correspondence $\mathcal{C}_{r+s}(G)$ with colour lists $l_{r+s}(v)$ assigned to each vertex:

- $l_{r+s}(v) = l_r(v) \cup l_s(v) = \{c_1, ..., c_r, c_{r+1}, ..., c_{r+s}\}$,
- $\{(u, c_i), (v, c_j)\} \in \mathcal{C}_{r+s}(G)$ iff $\{(u, c_i), (v, c_j)\} \in \mathcal{C}_r(G) \cup \mathcal{C}_s(G)$.

Note that for any $i, j$ such that $1 \leq i \leq r$, $1 \leq j \leq s$, there is no correspondence between $(u, c_i)$ and $(v, c_{r+j})$.

Let $f$ be the partial correspondence colouring of $G$ so that exactly $\lambda_c(\mathcal{C}_{r+s}(G))$ vertices are coloured. We may partition the coloured vertices to subsets $R = \{v \in V(G) : f(v) \in l_r\}$ and $S = \{v \in V(G) : f(v) \in l_s\}$. Note that $|R| \leq \lambda_c(\mathcal{C}_r(G))$ and $|S| \leq \lambda_c(\mathcal{C}_s(G))$. Hence

$$\begin{aligned}\lambda_c(G, r+s) &\leq \lambda_c(\mathcal{C}_{r+s}(G)) = |R| + |S| \\ &\leq \lambda_c(\mathcal{C}_r(G)) + \lambda_c(\mathcal{C}_s(G)) = \lambda_c(G, r) + \lambda_c(G, s).\end{aligned}$$

□

**Corollary 6.2.5.**
*Given graph $G$ and positive integer $n'$ such that $1 \leq n' \leq \chi_c(G)$. Then at least one of inequalities $\lambda_c(G, n') \geq \dfrac{n'|V(G)|}{\chi_c(G)}$ or $\lambda_c(G, \chi_c(G) - n') \geq \dfrac{(\chi_c(G) - n')|V(G)|}{\chi_c(G)}$ is true.*

*Proof.* Let graph $G$ be given and denote $n = \chi_c(G)$. On the contrary, assume there is some $n'$, $1 \leq n' \leq n$ such that $\lambda_c(G, n') < \dfrac{n'|V(G)|}{n}$ and $\lambda_c(G, n - n') < \dfrac{(n - n')|V(G)|}{n}$. Then according to the Lemma 6.2.4, we have

$$\lambda_c(G, n) \leq \lambda_c(G, n') + \lambda_c(G, n - n') < \frac{n'|V(G)|}{n} + \frac{(n - n')|V(G)|}{n} = |V(G)|.$$

Which gives $\lambda_c(G, n) < |V(G)|$, contradicting $n = \chi_c(G)$. □

We need the following definition for the rest of the results.

**Definition 6.2.6.**
*Let $G$ be a graph and $H$ its subgraph, $\mathcal{C}(G)$ is a correspondence on $G$ with*

*colour list $l(v)$ to each vertex $v \in V(G)$. The* restriction *of $\mathcal{C}(G)$ to $H$, denote by $\mathcal{C}(G)|_H$, is the correspondence on $H$ with the same colour list $l(v)$ as $\mathcal{C}(G)$ and correspondence inherited from $\mathcal{C}(G)$: for each edge $uv \in E(H)$, we have $\{(u, c_u), (v, d_v)\} \in \mathcal{C}(G)|_H$ if and only if $\{(u, c_u), (v, d_v)\} \in \mathcal{C}(G)$.*

**Corollary 6.2.7.**
*Given a graph $G$ and $H$ an induced subgraph of $G$. For any positive integer $n'$, we have $\lambda_c(G, n') \geq \lambda_c(H, n')$.*

*Proof.* Given graph $G$ and let $H \leqslant_i G$ be its induced subgraph. Given any positive integer $n'$. Let $\mathcal{C}(G)$ be the correspondence on $G$ that $\lambda_c(\mathcal{C}(G)) = \lambda_c(G, n')$. Let correspondence $\mathcal{C}(H)$ be $\mathcal{C}(G)|_H$ on $H$, then $\lambda_c(H, n') \leq \lambda_c(\mathcal{C}(H)) \leq \lambda_c(\mathcal{C}(G)) = \lambda_c(G, n')$. □

**Lemma 6.2.8.**
*Given graph $G$ and any integer $n'$ such that $1 \leq n' \leq \chi_c(G)$, we have:*
(1) if there is an induced subgraph $H$ of $G$ such that $\chi_c(H) = n'$, then $\lambda_c(G, n') \geq |H|$;
(2) if $H'$ is an induced subgraph of $G$ such that $|V(H')| = \lambda_c(G, n')$, then $\chi_c(H') \geq n'$.

*Proof.* (1) Assume by contrary that $\lambda_c(G, n') < |H|$. Let $\mathcal{C}_{n'}(G)$ be the $n'$-correspondence on $G$ so that $\lambda_c(G, n') = \lambda_c(\mathcal{C}_{n'}(G))$. Thus we can colour fewer than $|H|$ vertices in $G$ under $\mathcal{C}_{n'}(G)$. But $\chi_c(H) = n'$ and hence all of $V(H)$ can be coloured under $\mathcal{C}_{n'}|_H$, contradicts to $\lambda_c(\mathcal{C}_{n'}(G)) < |H|$. We conclude that $\lambda_c(G, n') \geq |H|$.

(2) Suppose on the contrary that $\chi_c(H') = s < n'$, then a proper correspondence colouring exists for any $s$-correspondence $\mathcal{C}_s(H')$ on $H'$. Consider an $n'$-correspondence $\mathcal{C}_{n'}(G)$ on $G$ that $\lambda_c(G, n') = \lambda_c(\mathcal{C}_{n'}(G))$. Fix a vertex $x \in V(G \setminus H')$ and colour $c \in l_{n'}(x)$. (Such $x$ exists as otherwise $H' = G$, $|H'| = \lambda_c(G, n')$, and hence $\chi_c(H') = \chi_c(G) = n'$.) Let $\mathcal{C}'(H')$ be the correspondence inherit from $\mathcal{C}(H')$ by removing any colours corresponded to colour $c$, i.e. for each vertex $v \in V(H')$, we have $l'(v) = l(v) \setminus \{d : \{(v, d), (x, c)\} \in \mathcal{C}_{n'}(G)\}$. Then by the fact that $\chi_c(H') = s \leq n' - 1$ and

for each vertex $v \in V(H')$, we have $|l'(v)| \geq n' - 1$, we see $H'$ is $\mathcal{C}'(H')$-colourable. Hence we can colour $|H'|+1$ vertices of $G$ under $\mathcal{C}_{n'}(G)$ by assign colour $c$ to vertex $x$, which contradict to our assumption of $\lambda_c(\mathcal{C}_{n'}(G)) = |H'|$. We conclude that $\chi_c(H') \geq n'$. (Note that it is 'at least' because we may not be able to colour exactly the subgraph $H'$ under chosen correspondence.) $\square$

Recall that a graph is *k-degenerate* if every subgraph of it has a vertex of degree at most $k$. We denote the degeneracy of graph $G$ as $\text{degcy}(G)$. Also recall that for any graph $G$, we have $\chi(G) \leq \chi_l(G) \leq \chi_c(G) \leq \text{degcy}(G)+1$.

A *chord* in a cycle of a graph is an edge between two vertices that are not adjacent in the cycle. A graph is *chordless* if it has no chords.

**Lemma 6.2.9.**
*Let $G$ be a chordless graph with correspondence chromatic number $n$, then for any integer $n'$ where $0 < n' < n$, we have $\lambda_c(G, n') \geq \dfrac{n'|V(G)|}{n}$.*

We need the following lemma to prove Lemma 6.2.9.

**Lemma 6.2.10** (Janssen et al. [32])**.**
*Let $G$ be a minimally 2-connected (2-connected and chordless) graph. Then for any vertex $x \in V(G)$, we can find vertices $u, v, w \in V(G)$ such that $v, w \in N_G(u)$, $\deg_G(v) = \deg_G(w) = 2$ and $x \notin \{v, w\}$.*

*Proof of Lemma 6.2.9.* We have $n \leq 3$ by the 2-degeneracy property of chordless graphs. The lemma for $n \leq 2$ is trivial: we only need to discuss $n' = 1$, which holds by choosing the largest single-coloured vertex subset under each correspondence.

Now we consider $n = 3$. The case that $n' = 1$ is trivial as the largest single-coloured vertex subset of $G$ has size at least $\lceil |V(G)|/3 \rceil$. If $n' = 2$, we will use induction on number of vertices to prove $\lambda_c(G, 2) \geq \dfrac{2|V(G)|}{3}$.

Assume the hypothesis holds for graphs with at most $|V(G)| - 1$ vertices, i.e. $\lambda_c(G, 2) \geq \dfrac{2|V(G)|}{3}$ for $|V(G)| \leq |V(G)| - 1$.

If there is a vertex $x \in V(G)$ with degree at most 1, denote $G'$ as graph $G$ after removing $x$. By our assumption, we have $\lambda_c(G', 2) \geq \dfrac{2(|V(G)| - 1)}{3}$.

Look at any correspondence of $G'$, we can always add $x$ back with at least one colour available as $x$ has at most 1 neighbour in $G$. Hence we have $\lambda_c(G,2) \geq \dfrac{2(|V(G)|-1)}{3} + 1 > \dfrac{2|V(G)|}{3}$.

If every vertex in $G$ has at least 2 neighbours, we consider the block graph of $G$ and a leaf block $B$ in $G$. (The block graph of a undirected graph is the intersection graph of its bi-connected components, note that such graph must be a block graph as its bi-connected components for each articulation vertex must be a clique.) The leaf $B$ is 2-connected as $G$ does not have isolated vertices and the chordless property of $G$ assures that $B$ is minimally 2-connected. As $B$ is a leaf block, we can find cut vertex $x$ so that removing $x$ separate $B$ from the main graph. By Lemma 6.2.10 we can find vertices $u, v, w \in V(B)$ so that $v, w \in N_B(u)$, $x \notin \{v, w\}$, and $\deg_B(v) = \deg_B(w) = 2$, then neither $v$ nor $w$ is a cut vertex, and hence their degree in $G$ stays, i.e. $\deg_G(v) = \deg_G(w) = 2$. Now let $S = \{u, v, w\}$ and let $G''$ denote graph $G$ after removing $S$. By induction hypothesis $\lambda_c(G'', 2) \geq \dfrac{2(|V(G)|-3)}{3}$. When adding $u, v$ and $w$ back to $G$, if we don't colour $u$, both $v$ and $w$ can be properly correspondence coloured as they only have at most one coloured neighbour. Thus $\lambda_c(G,2) \geq \dfrac{2(|V(G)|-3)}{3} + 2 = \dfrac{2|V(G)|}{3}$. $\square$

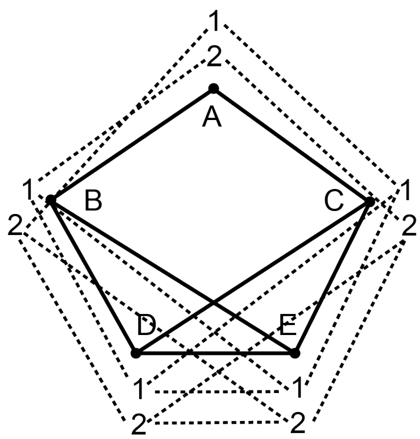
Let $\mathcal{G}$ be a hereditary graph family that all graphs in $\mathcal{G}$ satisfy $\chi(G) = \chi_c(G)$. Then for any integer $n'$ that $0 < n' \leq \chi_c(G)$, we have $\lambda_c(G, n') \geq \dfrac{n'|V(G)|}{\chi_c(G)}$: we can always find an induced subgraph $H$ satisfying $\chi(H) = n'$ and $|V(H)| \geq \dfrac{n'|V(G)|}{\chi(G)}$ by introducing a $\chi(G)$-partition of $V(G)$ with independent sets and combining the largest $n'$ such independent partitions. By the hereditary property $\chi_c(H) = \chi(H) \leq n'$ and hence $\lambda_c(G, n') \geq \dfrac{n'|V(G)|}{\chi_c(G)}$.

Note that family of chordal graphs (graphs where all cycles of length four and above has a chord) and family of odd cycles are two examples of hereditary graph families with $\chi_c(G) = \chi(G)$.

## 6.3 Counterexamples of the AGH Conjecture in Correspondence Colouring

As a chordless graph is 2-degenerate, it is natural to ask whether all 2-degenerate graphs satisfies the extended conjecture. The answer is no by the following counterexample.

The graph as shown is 2-degenerate, with correspondence chromatic number 3, but only $3 < \frac{2}{3} \cdot 5$ vertices can be coloured with the given 2-correspondence as indicated in the picture: each vertex is indexed by a letter and each edge by a solid line; colours assigned to each vertex are denoted by integers, and two integers are linked by a dotted line if they correspond in the 2-correspondence.




*Counterexample to the extended AGH conjecture*

To prove that at most 3 vertices can be properly correspondence coloured, assume 4 vertices can be correspondence coloured properly on the contrary. In this case at least one of the bottom vertices ($D$ or $E$) must be coloured.

If only one of $D$ and $E$ is coloured, all of $A$, $B$ and $C$ must be coloured. Without loss of generality assume $D$ is coloured by 1 and then $B$ and $C$ must both be coloured by 2, but then we cannot correspondence colour $A$ properly.

If both $D$ and $E$ are coloured, then they must be of different colours and hence none of $B$ and $C$ can be properly correspondence coloured.

We can construct a series of counter examples $G_n$ of similar structure by following steps:

(1) Fix $n \geq 2$, denote the output graph by $G_n$ and start with $K_n$,
i.e. $V_1 = \{v_i : 1 \leq i \leq n\}$, $E_1 = \{v_i v_j : 1 \leq i, j \leq n, i \neq j\}$;

(2) add $n$ vertices to the graph and connect each of them to each $v_i$,
i.e. $V_2 = V_1 \cup \{w_i : 1 \leq i \leq n\}$, $E_2 = E_1 \cup \{v_i w_j : 1 \leq i, j \leq n\}$;

(3) add vertex $w_0$ and connect it to every other $w_i$ added in step 2,
i.e. $V_3 = V_2 \cup \{w_0\}$, $G_3 = E_2 \cup \{w_0 w_i : 1 \leq i \leq n\}$;

(4) output $G_n = (V_3, E_3)$.

And we define the $n$-correspondence $\mathcal{C}(G_n)$ on $G_n$:

(1) Let all correspondence be straight except for edges $w_0 w_i$, $1 \leq i \leq n$,
i.e. $\mathcal{C}_1 = \{\{(x, i)(y, j)\} : xy \in E_2, 1 \leq i, j \leq n\}$;

(2) add the correspondence on $w_0 w_i$ so every colour in $l(w_0)$
corresponds with $n$ different colours along each of $\{w_0 w_i : 1 \leq i \leq n\}$,
i.e. $\mathcal{C}_2 = \mathcal{C}_1 \cup \{\{(w_0, j), (w_i, i + j - 1 \mod n)\} : 1 \leq i \leq n, 1 \leq j \leq n\}$;

(3) output $\mathcal{C}(G_n) = \mathcal{C}_2$.

**Theorem 6.3.1.**
*Each of the graphs $G_n$ ($n \leq 2$) is $n$-degenerate, with correspondence chromatic number $n + 1$, but at most $2n - 1 < \dfrac{n}{n+1}(2n + 1)$ vertices can be properly correspondence coloured with the given $n$-correspondence.*

*Proof.* On the contrary, suppose we can properly correspondence colour at least $2n$ vertices in $G_n$ under $\mathcal{C}(G_n)$. So at least $n - 1$ vertices from $\{v_1 : 1 \leq i \leq n\}$ are coloured.

If all vertices in $\{v_1 : 1 \leq i \leq n\}$ are coloured, they must be of different colours, so no vertex from $w_1$ to $w_n$ can be coloured, then at most $n+1 < 2n$ vertices in $G_n$ are coloured.

If $n - 1$ vertices of $\{v_1 : 1 \leq i \leq n\}$ are coloured, then they must be of different colours and we denote the only unchosen colour by $k$; we must colour all of $\{w_i : 0 \leq i \leq n\}$, but all of $\{w_i : 1 \leq i \leq n\}$ can only be coloured by the $k$, and hence $w_0$ can not be coloured under $\mathcal{C}(G)$, as each of the colours in $l(w_0)$ correspond with colour $k$ of some $w_i$, $1 \leq i \leq n$. Then at most $2n - 1 < 2n$ vertices are coloured.

We can conclude that at most $2n-1$ vertices of $G_n$ can be properly correspondence coloured under $\mathcal{C}(G)$, and hence $\lambda_c(G_n, n) \leq \lambda_c(\mathcal{C}(G_n)) = 2n-1$. $\square$

## 6.4 Condition on the size of Correspondence

We present a lemma that guarantees the existence of a proper correspondence colouring by restricting the correspondence between adjacent vertices.

**Lemma 6.4.1.**
*Let $G$ be a graph and $k$ a positive integer. Denote $l(v)$ as the lists of colours associated to each vertex $v \in V(G)$ and $\mathcal{C}(G)$ as the correspondence on $G$. If $|l(v)| \geq 2k$ for each vertex $v \in V(G)$ and each vertex-colour pair $(u, c)$ corresponds to at most $k$ other vertex-colour pairs, i.e. for any $(u, c)$ such that $u \in V(G)$ and $c \in l(v)$, we have $|\{\{(u, c), (v, c')\} : \{(u, c), (v, c')\} \in \mathcal{C}\}| \leq k$. Then there exist a proper correspondence colouring of $G$ with respect to $\mathcal{C}(G)$.*

In order to complete the proof, we need the following result by Haxell [27].

**Lemma 6.4.2** (Haxell [27])**.**
*Let $k$ be a positive integer. Let $H$ be a graph of maximum degree at most $k$, and let $V(H) = \bigcup_{i=1}^{n} V_i$ be a partition of the vertex set $V(H)$. If $|V_i| \geq 2k$ for each $i$, then there is an independent set $\{v_i : 1 \leq i \leq n, v_i \in V_i\}$ of $H$.*

Haxell derived the above result in [27] from another theorem regarding 'dominating subsets', whose proof was noted essentially the same as another complicated Theorem. A more generalised version of Haxell's result was proved as Theorem 4.1 in [26], where the existence of specific independent set was generalised to existence of induced subgraph with component sizes at most $r$, while the condition on size of each partition was generalised to 'at least $k + \lfloor k/r \rfloor$'. Proof of the later theorem can be found in [26] in full.

Now we prove Lemma 6.4.1.

*Proof.* Consider a given graph $G$ and positive integer $k$, colour lists $l(v)$

assigned to each vertex with $|l(v)| = 2k$, and correspondence $\mathcal{C}(G) \subseteq \{\{(u, c_1), (v, c_2)\} : \{u, v\} \in E(G), c_1 \in l(u), c_2 \in l(v)\}$. We define the auxiliary graph $A = (V(A), E(A))$ where

- $V(A) = \{(v, c) : v \in V(G), c \in l(v)\}$,
- $\{(u, c_1), (v, c_2)\} \in E(A)$ if and only if $\{(u, c_1), (v, c_2)\} \in \mathcal{C}(G)$.

By our construction, finding a proper correspondence colouring of $G$ is equivalent to finding an independent set of size $|V(G)|$ in $A$, with one vertex from each set of $2k$ vertices $(v, c)$ associated to the same $v$.

Now by our assumption, the maximum degree of $A$ is at most $k$, and for each vertex $u \in V(G)$, $V(u) := \{(u, c) : c \in l(u)\}$ is of cardinality at least $2k$ and $\bigcup_{u \in V(G)} V(u)$ is a pairwise disjoint partition of $V(A)$.

Hence by Haxell's result, we can find colour $c_u \in l(u)$ for each $u \in V(G)$ such that $I = \{(u, c_u) : u \in V(G)\}$ is an independent set in the auxiliary graph $A$ we built. Here we have exactly one vertex $(u, c_u) \in I$ comes from one subset $V(u)$ of the partition. Therefore we have a proper correspondence colouring of $G$ with respect to $\mathcal{C}(G)$ by assigning $c_u$ to each $u \in V(G)$ following the pairs in $I$. □

**Definition 6.4.3.**
*Let $G$ be a graph and $\mathcal{C}(G)$ the correspondence assigned to $G$. We say $\mathcal{C}(G)$ is* consistent *if there is no series of vertex-colour pairs $(u_1, c_1), (u_2, c_2), \ldots, (u_k, c_k)$ such that $u_1 = u_k$, and for each $i$, $1 \leq i \leq k - 1$, we have $\{(u_i, c_i), (u_{i+1}, c_{i+1})\} \in \mathcal{C}(G)$, but $c_1 \neq c_k$. $\mathcal{C}(G)$ is* inconsistent *if such series exists.*

If a correspondence is consistent, then after certain steps of colour renaming, we can make it equivalent to a list colouring while keeping correspondence unchanged: for each series of vertex-colour pairs $(u_1, c_1), (u_2, c_2), \ldots, (u_k, c_k)$ such that $\{(u_i, c_i), (u_{i+1}, c_{i+1})\} \in \mathcal{C}(G)$ for $1 \leq i \leq k-1$, we rename $c_1, \ldots, c_k$ all to a new colour $c^*$; this will not cause different colours from list of the same vertex be renamed to the same new colour, since the correspondence is consistent. Hence a correspondence colouring problem can be regarded as list colouring problem if the correspondence is consistent. This essentially means inconsistent correspondence are more interesting to study. We also note that, a correspondence colouring problem is equivalent to a ordinary colouring problem if it is consistent, each colour list have the same number of colours, and correspondence on all the edges are full. (Since the correspondence on all edges are full, this means the list colouring problem we have after renaming have the same list on any edge's endvertices, i.e. all colour lists are the same.)

## 6.5 Discussion on $\lambda_c(G, 2)$ and Related Properties

Let $G$ be a graph with $\chi_c(G) = n$. Fix $n' \leq n$, recall that $\lambda_c(G, n')$ is the guaranteed number of properly correspondence colourable vertices in $G$ under any $n'$-correspondence $\mathcal{C}(G)$.

It is clear that $\lambda_c(G, 1) = \alpha(G)$ where $\alpha(G)$ is the Independence number of $G$: trivially $\lambda_c(G, 1) \geq \alpha(G)$, and the we cannot have $\lambda_c(G, 1) > \alpha(G)$ by considering the all-straight correspondence on $G$. By the inequality $\lambda_c(G, n') + \lambda_c(G, n - n') \geq \lambda_c(G, n) = |V(G)|$, we also have $|V(G)| - \alpha(G) \leq \lambda_c(G, n - 1) \leq |V(G)| - 1$.

In Chapter 4, we defined $\pi(G, K_{n'})$ as the number of vertices in a largest induced subgraph of $G$ that has a homomorphism to $K_{n'}$, i.e. that is $n'$-colourable. Here we define it similarly for correspondence colouring.

**Definition 6.5.1.**
*Given graph $G$ and $n' \leq \chi_c(G)$, the size of the maximum $n'$-correspondence-colourable induced subgraph of $G$ is $\pi_c(G, K_{n'})$. Since we will only study $n'$-correspondence-colourable for integer $n'$, we also denote it as $\pi_c(G, n')$ for simplicity of notations.*

From the definition, we have $\pi_c(G, n') \leq |V(G)| - (n - n')$. We also directly have $\lambda_c(G, n') \geq \pi_c(G, n')$ and that the equality holds for $n' = 1$. It is natural to ask: how does $\lambda_c(G, 2)$ compare to $\pi_c(G, 2)$? We first discuss the same question in ordinary or list colouring.

We define/recall both $\lambda$ and $\pi$ in ordinary colouring or list colouring:

$$\begin{aligned}
\lambda(G,n') &:= \min_{\text{arbitrary } n' \text{ colours}} \big\{ \max \big\{ |H| \big| H \leqslant_i G,\, H \text{ properly colourable} \\
&\qquad\qquad\qquad\qquad \text{by given } n' \text{ colours} \big\}\big\} \\
\pi(G,n') &:= \max \big\{ |H| \big| H \leqslant_i G, \chi(H) \le n' \big\} \\
\lambda_l(G,n') &:= \min_{\text{arbitrary } n'\text{-lists on each vertex}} \big\{ \max \big\{ |H| \big| H \leqslant_i G,\, H \text{ properly} \\
&\qquad\qquad\qquad\qquad \text{list colourable with given } n'\text{-lists} \big\}\big\} \\
\pi_l(G,n') &:= \max \big\{ |H| \big| H \leqslant G, \chi_l(H) \le n' \big\}.
\end{aligned}$$

First note $\pi(G,2)$ in ordinary colouring gives the size of maximum induced bipartite subgraph, while in the correspondence colouring setting it evaluates the size of maximum induced sub-forest, as any cycle has correspondence chromatic number 3.

In terms of comparing $\lambda$ and $\pi$, it is clear that for any positive integer $n'$, we have $\lambda(G,n') = \pi(G,n')$. And for the case that $n' = 1$, we know all $\lambda, \pi, \lambda_l, \pi_l, \lambda_c$ and $\pi_c$ of graph $G$ equals to the independence number of $G$.

Interestingly, from $n' = 2$, we have examples such that $\lambda_l(G,2) > \pi_l(G,2)$. Consider a complete bipartite graph $G = K_{k,k}$ with $k \ge 5$ vertices on each side. We will show that $\pi_l(G,2) = k+1$ and $\lambda_l(G,2) \ge k+2$. Denote $V(G) = A \sqcup B$, where $A, B$ are the two sides of the bipartite graph, with $k$ vertices each. For $\pi_l(G,2)$, it is not hard to show that $K_{2,4}$ and $K_{3,3}$ are not 2-list-colourable, and since $k \ge 5$, the largest 2-list-colourable induced subgraph of $G$ has is either $A \cup \{v\}$ for some $v \in B$, or $B \cup \{u\}$ for some $u \in A$. For $\lambda_l(G,2)$, consider arbitrary 2-lists on each vertex, and denote the 2-list on each vertex $v \in V(G)$ as $l(v)$. If any two vertices $u_1, v_1 \in A$ have the a common colour $c_1$ in their list, then we can colour $u_1, v_1$ by $c_1$ and the whole vertex set $B$ by any colour not the same as $c_1$; the same applies if any two vertices $u_2, v_2 \in B$ have a common colour $c_2$ in their list. Instead, if none of the above cases is true, then all vertices in $A$ have disjoint colour lists, and all vertices in $B$ have disjoint colour lists. Hence by the pigeonhole principle, there exist vertices $u_3, v_3 \in A$ and colours $c_3$ from $l(u_3)$, colour $c_3'$ from list $l(v_3)$ such that $l(u) \ne \{c_3, c_3'\}$ for any $u \in B$. That is, we can always colour $B \cup \{u_3, v_3\}$ properly in this case. I.e. $\lambda_l(G,2) \ge k+2$.

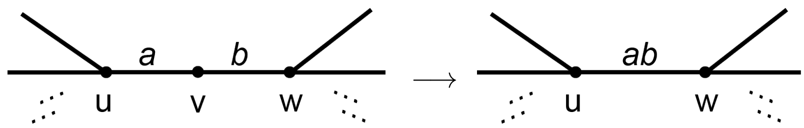
For the same $G$, we also have $\pi_c(G, 2) = k+1$. However, the same idea does not immediately give $\lambda_c(G, 2)$. It is still interesting to understand whether the same example gives $\lambda_c(G, 2) > \pi_c(G, 2)$, and what the relationship between $\lambda_c$ and $\pi_c$ is.

**Lemma 6.5.2.**
*Let $G$ be a graph and denote the graph after contracting both neighbours of a degree-2 vertex as $G^-$, then we have $\lambda_c(G, 2) = \lambda_c(G^-, 2) + 1$ and $\pi_c(G, 2) = \pi_c(G^-, 2) + 1$.*

*Proof.* Let $v \in V(G)$ be a vertex of degree 2 and denote $N_G(v) = uw$. Denote $G^-$ as the resulting graph after contracting $uv$ and $\{w, v\}$.

The lemma is equivalent to the following statements
(1) $\pi_c(G, 2) = k \Rightarrow \pi_c(G^-, 2) \geq k - 1$,
(2) $\pi_c(G^-, 2) = k \Rightarrow \pi_c(G, 2) \geq k + 1$,
(3) $\lambda_c(G, 2) = k \Rightarrow \lambda_c(G^-, 2) \geq k - 1$, and
(4) $\lambda_c(G^-, 2) = k \Rightarrow \lambda(G, 2) \geq k + 1$.

We consider cases of $uw \notin E(G)$ and $uw \in E(G)$ separately.




*Contract edges $a$ and $b$: case $uw \notin E(G)$*

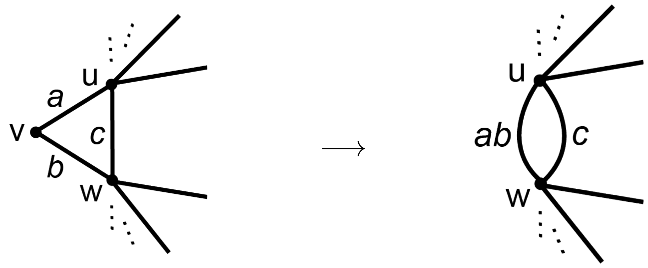
We first prove (1) and (2) under the case $uw \notin E(G)$, which is shown as above (we only demonstrate the interesting part of graph $G$ and $G^-$). Here edges $a = uv \in E(G)$, $b = vw \in E(G)$ and $ab = uw \in E(G^-)$.

For statement (1), we have the following cases (we say a vertex is *chosen* if the vertex is in the maximum 2-correspondence-colourable induced subgraph of $G$ or $G^-$):

- If all of $u$, $v$ and $w$ are chosen in $G$, i.e. $u -_a v -_b w$ is included in a forest subgraph of size $k$ in $G$, then both $u$ and $w$ can be included in a forest induced subgraph of size at least $k - 1$ in $G^-$;

- If $v$ and exactly one of $u$, $w$ are chosen in $G$, then the chosen $u$ or $w$ is included in a forest induced subgraph of size at least $k-1$ in $G^{-}$;
- If $u$ and $w$ are chosen in $G$, but not $v$, then at least one of $u, w$ can be included in a forest induced subgraph of size at least $k-1$ in $G^{-}$;
- If only $v$, $u$, $w$ is chosen in $G$, then take the rest part of the forest induced subgraph chosen in $G$ gives a forest induced subgraph of size $k-1$ in $G^{-}$;
- None of $u$, $v$, $w$ is chosen in $G$. This case cannot hold: if neither of $u$ and $w$ is chosen in $G$, then $v$ is guaranteed to be chosen for the maximality.

Now we are left with the case that there is an edge $c = uw \in E(G)$ before contraction:




*Contract edges $a$ and $b$: case $uw \in E(G)$*

For this case, first note that at most two of $u, v, w$ can be chosen in the chosen forest induced subgraph of $G$.

- If exactly one of $u, w$ is chosen in the sub-forest of $G$, then $v$ must also be chosen as otherwise the induced sub-forest is not maximal; then we can simply keep $u$ or $w$ in the forest induced subgraph of $G^{-}$;
- If both $u, w$ are chosen, we keep exactly one of $u$ or $w$ in the forest induced subgraph of $G^{-}$;
- If none of $u, w$ is chosen in the sub-forest of $G$, for the maximality of forest induced subgraph, $v$ must be included, hence $G^{-}$ can use the same forest induced subgraph as $G$ by just losing $v$.

Now (1) is proved.

Proof of (2) is similar, we first consider the case $uw \notin E(G)$:

- If both $u$ and $w$ are chosen in $G^-$: so $u -_{ab} w$ is part of a forest induced subgraph of size $k$ in $G^-$, then $u -_a v -_b w$ is part of a forest induced subgraph of size at least $k+1$ in $G$;
- If only $u$ is chosen in $G^-$, then $u -_a v$ is part of a forest induced subgraph of size at least $k+1$ in $G$;
- If only $w$ is chosen in $G^-$, then $v -_b w$ is part of a forest induced subgraph of size at least $k+1$ in $G$;
- If neither $u$ nor $w$ is chosen in $G^-$, then $v$ can be added to the forest induced subgraph to make its size at least $k+1$ in $G$.

For the case $c = uw \in E(G)$ before contraction, note that at most one of $u, w$ can be chosen in the forest induced subgraph of $G^-$. And no matter one or none of $u, w$ is chosen in $G^-$, we can always add $v$ to the chosen forest in $G^-$ to build a forest induced subgraph of size at least $k+1$ in $G$. Here (2) is proved.

Now we look at statements (3) and (4) regarding $\lambda(G, 2)$.

Here we say a vertex is chosen if it is coloured in the partial correspondence colouring that provides $\lambda(G, 2)$. For a given 2-correspondence with colour list $\{1, 2\}$, we say an edge is *straight* if identical colours correspond along the edge, an edge is *crossed* if there are different colours correspond along this edge in the 2-correspondence.

For statement (3):

Assume on the contrary that $\lambda(G, 2) = k$ but $\lambda(G^-, 2) \leq k-2$. Let $\mathcal{C}(G^-)$ be the correspondence on $G^-$ that only $k-2$ vertices can be properly correspondence coloured. We copy the correspondence on each edge of $G^-$ to $G$ (i.e. for $\{x, y\} \in E(G) \setminus \{a, b\}$, $\{(x, c_1), (y, c_2)\} \in \mathcal{C}(G)$ if and only if $\{(x, c_1), (y, c_2)\} \in \mathcal{C}(G^-)$), and we copy the correspondence on $ab$ to $a$.

Note that by our assumption and definition of $\lambda(G, 2)$, we should be able to properly correspondence colour at least $k$ vertices of $G$ under any 2-

correspondence, hence the behaviour of correspondence on $a$ and $b$ should not lower the number of colourable vertices.

In both cases (regarding whether $uw$ is in $E(G)$), all of $V(G) - \{v, w\}$ and $V(G^-) - \{w\}$ should behave the same on its chosen-unchosen status. So the only possibility that $\lambda(G, 2) = k$ and $\lambda(G^-, 2) \leq k - 2$ is $v, w$ are both chosen in $G$ while $w$ is not chosen in $G^-$.

In both cases, let edge $b$ be crossed, that is both $v, w$ in $G$ need to be coloured by the same colour, without loss of generality assume both are coloured by '1'. Then $w$ can also be coloured by '1' in $G^-$, contradiction.

For statement (4):

Here $\lambda(G^-, 2) = k$ and we need $\lambda(G, 2) \geq k$. Let an arbitrarily correspondence on $G$ be given and denoted by $\mathcal{C}(G)$. We copy all correspondence on $E(G) \setminus \{a, b\}$ to $E(G^-) \setminus \{ab\}$ (i.e. for $\{x, y\} \in E(G^-) \setminus \{ab\}$, $\{(x, c_1), (y, c_2)\} \in \mathcal{C}(G^-)$ if and only if $\{(x, c_1), (y, c_2)\} \in \mathcal{C}(G)$); and for edge $ab$, if $a$ and $b$ are both straight, or both crossed, then let $ab$ be straight in $\mathcal{C}(G^-)$; if one of $a$, $b$ is straight and the other crossed, then let $ab$ be crossed in $\mathcal{C}(G^-)$.

By assumption, at least $k$ vertices of $G^-$ can be properly correspondence coloured. In both cases (regarding whether $uw$ is in $E(G)$), if at most one of $u$, $w$ is chosen in $G^-$, we also choose it in $G$ and add $v$ (vertex $v$ can be chosed as it has at most one chosen neighbour).

In the case that $uw \notin E(G)$, if both $u$ and $w$ are chosen in $G^-$:

- If $ab$ is crossed, then both $u, w$ should be coloured by the same colour, then we copy all chosen vertices in $G^-$ to $G$; $v$ is also colourable as $a$ and $b$ are both straight or both crossed, and $u$, $w$ are of the same colour (i.e. without loss of generality assume $u$ and $w$ are coloured by 1, then $v$ can be coloured by 1 if $a$, $b$ both crossed, 2 if $a$, $b$ both straight);

- If $ab$ is straight, then $u$, $w$ are coloured by different colours, without loss of generality assume $u$, $w$ are coloured by 1, 2 respectively; we copy all coloured vertices in $G^-$ to $G$ and see that $v$ could also be coloured (if $a$ straight, $b$ crossed then $v$ can be coloured by 2; if $a$ crossed, $b$ straight then $v$ can be coloured by 2).

In the case that $uw \in E(G)$, if both $u$ and $w$ are coloured in $G^{-}$, the only possibility is that $c$ and $ab$ are both straight or both crossed. Then same derivations as previous case hold. Here (4) is cleared.

Now we've seen the proof to all 4 statements, and we can conclude that $\lambda(G, 2) = \lambda(G^{-}, 2) + 1$ and $\pi_c(G, 2) = \pi_c(G^{-}, 2) + 1$. $\square$

Theorem 6.5.3 is a simple conclusion of Lemma 6.5.2.

**Theorem 6.5.3.**
*If $G$ is a graph with minimum number of vertices that $\lambda_c(G, 2) > \pi_c(G, 2)$, then $G$ does not have vertex of degree 2.*

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