**The London School of Economics and Political Science**

# Macro-Finance and the Open Economy

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# Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).



I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

I declare that my thesis consists of 25547 words.

## Statement of co-authored work

I confirm that Chapter 2 was jointly co-authored with Chang He and Chapter 4 was jointly co-authored with Edoardo Leonardi. I contributed 50% of both work.

# Abstract

This thesis studies macro-fiance and the open economy. Chapter 1 analyzes whether rising asset prices make savers better off. I study the effect of fundamental drivers of rising asset prices (a fall in time discount rate, an increase in productivity, financial innovation, or a bubble driven by financial frictions) on top *welfare* inequality between super rich entrepreneurs and savers through *leverage*. Given the rising asset prices, falling risk-free rates, and rising top wealth inequality observed in the U.S., my theoretical model suggests that the falling time discount rate of the super-rich is the main driver of the trend, and therefore savers are worse off.

Chapter 2 investigates sovereign bond safety. Using a continuous-time two-country Lucas tree model with equity constraint, this chapter argues that the country-size effect and the equity-rebalancing effect are the key determinants of sovereign bond safety. Model predictions also reconcile with the empirical facts of flight-to-safety and the covered interest parity (CIP) in both normal and crisis times.

Chapter 3 focuses on equity rebalancing. Given the peak of country size of G-10 currency group relative to the US in 2008, the theoretical model accounts for 1) exchange rate risk hedging, 2) equity valuation and diversification 3) US net foreign asset position (NFA) changes, and 4) global wealth transfers, that are consistent with empirical facts, both before and after the Great Financial Crisis.

Chapter 4 investigates international diversification. This chapter finds that small countries that are already largely exposed to international risk via trade and investment channel would find it optimal to find refuge in domestic equity and safe assets. Openness strengthens domestic equity and currency while illiquid international equity market weakens them.

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# Chapter 1

# Asset Prices, Welfare Inequality, and Leverage

## 1.1 Introduction

Increasing top wealth inequality, rising asset prices and falling interest rate over the past few decades have raised important questions about their causes and effects. It has also led to policy debates on the impact of rising asset prices on welfare inequality. This paper answers the question: Do rising asset prices make savers better off?

In answering this question, the object of the inequality measure matters. Traditionally, focus has been on wealth inequality. However, the welfare effect of rising wealth inequality is rather obscure.[1] In response, a small but growing literature has emerged to study welfare inequality. But they are either in a partial equilibrium framework or do not distinguish different drivers of rising asset prices.[2] This paper takes one step further by studying the effect of *fundamental drivers* of rising asset prices on welfare inequality. To the best of my knowledge, this the first paper to show that different fundamental drivers of rising asset prices have *different* implications on *wealth inequality* and *welfare* in a general equilibrium framework. Taking advantage of the theoretical framework, this paper makes three contributions. In the first place, it identifies the main driver of the *joint* trend of rising asset prices, declining

[1] Saez, Yagan, and Zucman 2021 and John H Cochrane 2020 hold two opposite views on whether rising asset prices benefit the rich or just "on paper".

[2] Fagereng, Gomez, et al. 2022 study the effect of rising asset prices on welfare inequality mainly in a partial equilibrium framework. Greenwald et al. 2021 study the effect of falling interest rates on wealth inequality and welfare inequality, but do not focus on studying various fundamental drivers of the falling interest rates.

interest rate and increasing top wealth inequality. In the second place, it provides tractable *welfare* analysis in a fully-fledged general equilibrium environment. Last but not least, it highlights the importance of *leverage* in the mechanism, which has been less studied in the literature.

This paper considers an environment in which there are two types of agents – entrepreneurs and savers – and two assets – productive capital and risk-free bond.[3] Only entrepreneurs can invest in productive capital and own private business. And I refer to entrepreneurs as the super-rich.[4] In equilibrium, entrepreneurs borrow from savers in risk-free bond to finance their risky assets, that is, they use *leverage*.[5] Therefore, there are two endogenous asset prices, the price of productive capital and the price of risk-free bond, i.e. the risk-free rate. In this environment, I analyze the impact of fundamental drivers of asset price changes on wealth inequality, and ultimately on savers' welfare.

The source of changes in asset prices matter as well. I consider an economy where financial frictions limit risk-sharing and asset bubbles may occur. In this economy, there are four types of fundamental drivers of asset price changes: time discount rate ("patience")[6], productivity[7], financial innovation (or regulation)[8], and a bubble driven by financial frictions. I show that different fundamental drivers of asset price changes have different implications on wealth inequality and savers' welfare by affecting entrepreneurs' leverage differently. Financial frictions in my model play three important roles. First, they have been proved important for asset price spikes and collapses over time. Second, financial frictions can impact inequality by reducing allocation efficiency. Third, they also create bubbles when the market value of an asset exceeds its fundamental value (the net present value of all its future cash flows). Allowing for bubbles is important because recent studies have shown that the dotcom bubble and the Great Recession resulted in spikes in wealth inequality fluctuations (Gomez 2019). An, Lou, and Shi 2022 provide empirical evidence that the burst of stock market bubbles

[3]In the full model in section 1.4, there are three asset classes: private equity, public equity, and risk-free bond. Productive capital is the underlying asset of both private equity and public equity.

[4]In steady state, entrepreneurs are ensured to be richer than savers. I will discuss model details in later sections.

[5]Entrepreneurs save in high return risky assets but borrow in risk-free bond.

[6]The time discount rate is composed of not only the actual patience of the agent, but also other elements like death rate as in perpetual youth model. I discuss the supporting empirical evidence of "patience" change later in this section.

[7]That is, asset pay-offs in an endowment economy. The net present value of an asset increases when its productivity or future pay-offs increase.

[8]Financial innovation in this paper refers to less severe financial frictions and/or a lower volatility of idiosyncratic risk, while financial regulation refers to the opposite of financial innovation.

has large redistribution effects. The gap in this literature is a lack of a general equilibrium framework that takes into account the effect of asset bubbles on inequality.

To begin with, I consider a deterministic two-period model with an exogenous interest rate and an endogenous asset price (the price of productive capital).[9] The asset price changes via two channels in this simple model: productivity channel and interest rate channel. I show that a rising asset price always increases wealth inequality. However, depending on which channel is at work, it can have different consequences for the welfare of savers. In one instance, higher asset price due to higher productivity directly increases wealth inequality because the super-rich hold all the assets whose value increase. Savers' welfare is not affected since they do not hold the asset. This is the productivity channel. In the other instance, higher asset price due to lower interest rate increases wealth inequality and benefits the super-rich who borrow at the expense of the savers. The magnitude of this welfare effect depends on agents' borrowing and lending positions. This is the interest rate channel. In a partial equilibrium analysis, I show that even though savers do not own assets, the effect of changing asset price spills over to them through the abovementioned interest rate channel.[10]

In order to understand how asset prices and leverage interact, and how they jointly affect wealth inequality and the welfare of savers, it is essential that they are both endogenously determined. The next step therefore is to endogenize the interest rate in the two-period model, which allows me to study the welfare effect of fundamental drivers of rising asset prices. I show that falling time discount rate (rising "patience") of super rich entrepreneurs increases asset prices, decreases leverage, increases wealth inequality, and makes savers worse off. When super rich entrepreneurs become more "patient", their saving demand increases, and they borrow less. In general equilibrium, the interest rate decreases and wealth inequality rises. As a result, the falling time discount rate of super rich entrepreneurs decreases the welfare of savers.[11]

Given the intuition from the two-period model, I subsequently enrich the analysis by considering three more important elements: uncertainty, bubbles, and endogenous feedback

[9] There are no bubbles in the deterministic two-period model.

[10] Using Norwegian data, Fagereng, Gomez, et al. 2022 show that the rich borrows against their private business and debt is an important asset class that accounts for welfare gains and losses.

[11] Since this experiment constitutes a change in the entrepreneurs' preferences, we cannot say whether they are better off or not.

from wealth inequality. The motivation is three-fold. Uncertainty creates a risk premium that decreases asset prices, and generates precautionary saving motives that reduce leverage. By contrast, bubbles raise the value of an asset but reduce precautionary saving motives. Because these forces work in conflicting directions, they each have a different prediction for wealth inequality. The relative strength of the competing effects are determined by the fundamental drivers of rising asset prices in the richer model.

As the final step, I develop an infinite-horizon model with financial frictions, idiosyncratic risks and endogenous bubbles. I characterize the long-run level of wealth inequality and welfare both in an economy both with and without bubbles. I also discuss the fluctuations of wealth inequality and welfare in response to the four fundamental drivers of asset price changes: time discount rate, productivity, financial innovation, and a bubble driven by financial frictions.

The first fundamental driver, falling time discount rate of entrepreneurs, increases asset prices, decreases leverage, increases wealth inequality. Savers are worse off. This is in an echo of the two-period model's results.

The second fundamental driver, an increase in productivity, increases asset prices as well as leverage, but has no impact on wealth inequality. However, savers benefit from higher productivity due to positive income effect. This result shows that even though wealth inequality does not change, rising asset prices can still have an effect on welfare.

Unlike conventional wisdom that the third or the fourth fundamental driver, financial innovation or a bubble, has the tendency to increase wealth inequality, I find instead that they reduce wealth inequality and increase the welfare of savers. This result can be explained intuitively in two ways. From a portfolio choice perspective, a bubble increases the market value of an asset[12] and financial innovation increases asset prices. Keeping leverage fixed, a bubble or financial innovation would directly increase wealth inequality since super rich entrepreneurs hold assets that are rising in value. This is the classic intuition in a partial equilibrium analysis. However the story is incomplete in general equilibrium, because leverage changes as well. In fact, super rich entrepreneurs borrow more from savers when there is a bubble or financial innovation because their precautionary saving motives decrease. This is the *leverage channel*. In equilibrium, the negative effect of increase in leverage dominates the

[12]To be precise, I use "market value of an asset" rather than "asset price" when there is a bubble. Recall that the market value of an asset is the sum of the fundamental value of an asset (net present value of all future cash flows) and the value of bubble.

positive effect of increase in asset price on entrepreneurs' wealth.[13] From a risk perspective, by taking more risks and receiving a higher return, the super rich entrepreneurs accumulate more wealth relative to the savers. Financial innovation or a bubble reduces the total idiosyncratic risks in the economy which are entirely borne by the super-rich.[14] Consequently, wealth inequality decreases and savers are better off.

Through the lens of my theory, the observed rising asset prices, falling risk-free rates and rising top wealth inequality in the past few decades in the U.S. suggest that the falling time discount rate of super rich entrepreneurs is the main driver of the trend, and savers are worse off.

To better understand this result, I discuss the theoretical intuition, the relevant empirical interpretations, and related results in the literature. The theoretical intuition is straightforward and robust: In the most simple asset pricing equation, asset price is the discounted value of future cash flows. When super rich entrepreneurs become more "patient", they discount future cash flows less. As a result, asset price, that is, the net present value of the asset, increases. Since super rich entrepreneurs are borrowers in terms of risk-free bond, as they become more "patient", the borrowing demand for risk-free bond falls and therefore the risk-free rate falls. And as super rich entrepreneurs become more "patient", they accumulate more wealth in the long run and thus wealth inequality increases.

Falling time discount rate of entrepreneurs can be interpreted as a slowdown in firm dynamism (lower firm entry/exit rate) or a change in demographic characteristics (growing dispersion of life expectancy by wealth groups, see Isaacs and Choudhury 2017 for supportive empirical evidence).[15] My theoretical framework implies that the lower firm exit rate as observed in the secular stagnation and the dispersion of demographic changes across wealth groups to be the most promising underlying changes in the economy that drive the joint trend of rising asset prices, declining interest rate, and increasing top wealth inequality.

My result is complementary to the literature where the time discount rate of the whole

[13]This theoretical result that both financial innovation and bubbles expand leverage is also consistent with empirical evidence on bubbles, credit cycles, and financial crisis (see Schularick and Taylor 2012, Jordà, Schularick, and Taylor 2015, M. K. Brunnermeier and Oehmke 2013 among others).

[14]I will elaborate on why bubbles reduce idiosyncratic risks later in the paper.

[15]Behavioral elements can also be an underlying reason for the change in time discount rate. However, it is hard to find empirical evidence for a long-run trend of changes in behavioural elements.

economy is considered to be the driver of increasing inequality.[16] The result is also consistent with A. R. Mian, Straub, and Sufi 2020, who empirically document the saving increase of the rich and dissaving of the savers. Moreover, my model further shows that the underlying driver of the saving glut of the rich is associated with a decline in their time discount rate.

Empirically, there are time periods in which asset prices and wealth inequality do not co-move. My theory suggests that this is due to productivity changes. The theory also suggests that in periods of negative comovement between asset prices and wealth inequality, there are changes in financial innovation or bubbles at work. In both these cases, rising asset prices make savers better off.

### 1.1.1 Literature review

This paper contributes to the growing literature on understanding the impact of rising asset prices on inequality, the literature on asset bubbles driven by financial frictions, and the macro-finance literature highlighting the importance of credit market and leverage.

Inequality has been extensively studied in a large and evolving literature. This paper contributes to the recent studies on how asset prices impact wealth inequality, but with a focus on top inequality. Kuhn, Schularick, and Steins 2020 show that asset prices are significant factors in wealth inequality in the US. Fagereng, Holm, et al. 2019 show that capital gains play an important role in saving behavior. Albuquerque 2022 shows that portfolio changes matter for wealth inequality as well. Cioffi 2021 and Xavier 2021 study wealth inequality by incorporating heterogeneity in risk exposure and asset returns in partial equilibrium models. Gomez et al. 2016 studies the role of aggregate risk in shaping wealth inequality and asset prices. Gomez and Gouin-Bonenfant 2020 studies the long-run effect of low interest rate on wealth inequality. This paper is also related to the small but growing literature studying welfare inequality. Fagereng, Gomez, et al. 2022 and Greenwald et al. 2021 study wealth inequality and welfare inequality mainly in a partial equilibrium setting. Complementary to their result, I consider a general equilibrium framework with financial frictions, idiosyncratic risks, and unequal capital income and characterize the leverage channel with endogenous interest rate. I show that different fundamental drivers of rising asset prices affect wealth

[16] Liu, A. Mian, and Sufi 2022 argues that a reduction in time discount factor of the whole economy can help explain the rising profit share and declining productivity growth following the decline in the interest rate. Greenwald et al. 2021 argues that declining interest rates due to a combination of declining time discount rate, growth rate, and growth uncertainty can explain a large fraction of the increasing wealth inequality.

inequality and welfare differently through the leverage channel.

This paper also contributes to the literature on rational bubbles that have positive value due to financial frictions. I characterize a new type of bubble and study the effect of bubble on *inequality*. In my model, bubbles expands leverage by lowering precautionary saving motives in an economy with idiosyncratic risks. A long literature studies rational bubbles in line with Samuelson 1958 and Tirole 1985, such as A. Martin and Ventura 2012 on growth and Farhi and Tirole 2012 on liquidity among many others, see A. Martin and Ventura 2018 for a comprehensive survey. Recent works like Reis 2021 and M. K. Brunnermeier, Merkel, and Sannikov 2022 show that bubbles can explain the high level of government debt that cannot be sustained by fiscal surplus. Miao and Wang 2018 studies stock price bubbles that relax the borrowing limit which is directly given by credit constraint. While in my model, there is no borrowing constraint and bubbles relax the limit on public stock market which is *indirectly* given by an equity constraint.

Finally, this paper contributes to the macro-finance literature highlighting financial frictions and credit market by studying the leverage effect on wealth inequality and welfare in a *long-run* horizon of three decades rather than at business cycle frequency. The equity constraint in the model is in the same spirit as M. K. Brunnermeier and Sannikov 2014, where entrepreneurs have to keep some fraction of the firms as private equity because of agency problems. The idiosyncratic risks associated with private capital is related to Di Tella and Hall 2020. I depart from them and have a richer financial market structure by differentiating private equities and public equities.[17]

This paper is organized as follows. Section 1.2 provides motivating facts for key modelling elements, section 1.3 discusses the two-period model, section 1.4 sets up the full model and discusses fundamental equilibrium without bubbles, section 1.5 discusses bubble equilibrium, section 1.6 analyzes how changes in asset price affect inequality and welfare, and finally section 1.7 concludes.

[17] The financial market structure is also related to M. K. Brunnermeier, Merkel, and Sannikov 2022. I differ from them by having two types of agents, entrepreneurs and savers, which is important for wealth inequality and the existence of the type of bubble in my model.

## 1.2 Motivating facts

Motivated by the well-established fact in the literature that wealth inequality is rising and exploding at the top end in recent decades, as well as a long strand of research showing that asset valuations have also been rising in recent decades, I study how financial assets affect top inequality. This section provides some motivations and rationales for key modelling elements.

**A focus on capital** A large literature studies the driving forces of rising wealth inequality at the top. While differences in labor income and saving rates are considered important factors driving the exploding trend in many studies, De Nardi and Fella 2017 show that labor income differences only can not explain the wealth concentration at the top and Fagereng, Holm, et al. 2019 show that saving rates only differ by wealth groups when capital gains are included. Rising asset prices and capital gains in recent decades have become the focus of a growing literature to understand the wealth concentration at the top. Benhabib, Bisin, and Zhu 2011 show theoretically that capital income risk, rather than labor income, drives the properties of right tail of wealth distribution. Thus, my model features an economy where capital income uncertainty is one of the key elements determining inequality. I show in appendix that the main result and mechanism still go through in an extension with labor.

**Financial market structure and frictions** It has been shown that there are systematic differences in portfolio compositions and rates of return along the wealth distribution: The super rich entrepreneurs group is characterized by a heavy portfolio share in high-return assets, especially private business equity, while the savers group holds mainly public equity, such as stock market index fund, and safe assets such as deposit (Fagereng, Holm, et al. 2019,Kuhn, Schularick, and Steins 2020, Martınez-Toledano 2020,Xavier 2021, and Albuquerque 2022).[18] To capture such portfolio heterogeneity, I include three classes of assets in the model: private equity, public equity, and risk-free bond. I also assume restricted participation in equity market: savers cannot hold private equity, but can hold public equity *inactively*. Since the access to private equity market is the access to high-return assets, the restricted private equity market participation gives rise to return heterogeneity of different groups. The heterogeneous portfolios and returns arise in the model are consistent with the empirical facts discussed

[18] I focus on financial asset in this paper and do not consider housing explicitly.

earlier. While I interpret the *inactive* participation of savers in public stock market as their holdings of public equity through pension, which account for a non-negligible proportion in the data for some countries, U.S. for example.

**Idiosyncratic risk** The important role of idiosyncratic risk in explaining the top wealth concentration has been studied both theoretically and empirically (Campbell, Ramadorai, and Ranish 2019, Gomez 2019,Benhabib, Bisin, and Luo 2019, and Atkeson and Irie 2020). Kartashova 2014 has documented that private equities on average earn a premium over public equities due to idiosyncratic risks and such return difference varies with economic fundamentals. Di Tella and Hall 2020 show that idiosyncratic risks affect the return of capital and create inefficient recessions. I incorporate idiosyncratic risks associated with private equities as an important ingredient in the model for asset prices and inequality.

**Type dependence and size dependence** As shown by Gabaix, Lasry, et al. 2016 and Fagereng, Guiso, et al. 2020, "type dependence" (persistence heterogeneity in returns) and "size dependence" (positive correlation between return and wealth) are important to generate both the high level and the fast rise of top wealth inequality in the past few decades. In the model, agents are born either to be an entrepreneur or a saver throughout their life. This is the "type dependence" needed to generate wealth inequality. And entrepreneurs hold a portfolio with higher return than savers. This is the "size dependence" needed to make entrepreneurs indeed richer than savers in equilibrium.

## 1.3 Two-period Model

In this section, I develop a two-period model with two agents, an entrepreneur and a saver. This two-period model is a simplified version of the full model in section 1.4. I start with partial equilibrium analysis with exogenous interest rate and endogenous asset price. I show that rising asset prices always increase wealth inequality and hurt the savers. I then proceed to general equilibrium analysis with endogenous interest rate. I show the main result that declining "impatience" of the entrepreneurs raises asset prices, increases wealth inequality, and hurts the savers. At the end of this section, I discuss the limitation of the two-period model.

### 1.3.1 Model Set-up

In the two-period model, time $t \in \{0, 1\}$. There are two agents, an entrepreneur and a saver, representing for the top 1 percent wealth group and the group in between top 1 and 10 percent of the wealth distribution.

**Preferences** Entrepreneur and saver differ in "patience". Saver's discount factor is denoted as $\frac{1}{1+\rho}$ and entrepreneur's discount factor is denoted as $\frac{1}{1+\rho^e}$, where $\rho$ and $\rho^e$ are discount rates of saver and entrepreneur respectively.[19] Entrepreneur is more "impatient" than saver, that is $\rho^e = \rho + \delta^e$ where $\delta^e$ captures the relative "impatience" of entrepreneur. One can intuitively think that entrepreneur and saver share a common discount rate $\rho$, but entrepreneur may die or become bankrupt in period $t = 1$ at a Poisson rate of $\delta^e$.[20] For consistency with later sections, I assume logarithmic utility for both entrepreneur and saver.[21]

**Technology** There is productive capital in fixed supply $K$. Capital $K$ produces $aK$ units of consumption good at time $t = 1$, where $a$ is the productivity of capital and is an exogenous parameter. Entrepreneur and saver are endowed with $W_0^e$ and $W_0^s$ respectively at time $t = 0$.

**Financial assets** There are productive capital $K$ in fixed supply and risk-free bond $B$ in zero net supply in the economy. One can also think of capital as an asset like stock since it delivers cash flows over time and $a$ is the dividend paid out per unit of the asset. Denote $q$ as the price of per unit capital $K$ and $1 + r^f$ as the (gross) return of the risk-free bond $B$. I refer to $q$ as capital price or asset price interchangeably and $r^f$ as the (net) interest rate.

**Financial market structure** Importantly, saver is restricted from stock market participation and entrepreneur holds all the capital in the economy.[22] At time $t = 0$, entrepreneur and saver cannot trade the productive capital $K$ due to market segmentation, but can trade risk-free bond $B$ freely.

[19] I denote discount factors in this way to keep consistency with the full model in section 1.4

[20] This intuition can be proved when time is continuous and infinite. The assumption that entrepreneurs are less patient is standard in the macro-finance literature to prevent the entrepreneurs from eventually taking over all the wealth in the economy and becoming fully self-financed. See Kiyotaki and Moore 1997 and Bernanke, Gertler, and Gilchrist 1999 for example.

[21] The main result and key mechanism does not depend on utility function forms.

[22] I leave the distinction between private equity and public equity for later sections.

**Optimization problems** Now I can write optimization problems of entrepreneur and saver. The entrepreneur's problem is as follows:

$$
\begin{aligned}
V^e = \max_{C_0^e, C_1^e, K_1, B^e} \quad & U(C_0^e) + \frac{1}{1+\rho^e} U(C_1^e) \\
\text{s.t.} \quad & C_0^e + qK_1 + B^e = qK + W_0^e \\
& C_1^e = aK_1 + (1+r^f)B^e
\end{aligned}
\tag{1.1}
$$

where $C_t^e$ is entrepreneur's consumption at time $t$, $K_t$ is entrepreneur's capital holding at time $t$, and $B^e$ is entrepreneur's bond holding at time 0. Entrepreneur chooses how much capital to hold at $t = 1$, how much to invest in bond at time $t = 0$, and the consumption plan over time. Since $t = 1$ is the final period, agents consume everything that they own at $t = 1$.

Similarly, saver's optimization problem is as follos:

$$
\begin{aligned}
V^s = \max_{C_0^s, C_1^s, B^s} \quad & U(C_0^s) + \frac{1}{1+\rho} U(C_1^s) \\
\text{s.t.} \quad & C_0^s + B^s = W_0^s \\
& C_1^s = (1+r^f)B^s
\end{aligned}
\tag{1.2}
$$

where $C_t^s$ is saver's consumption at time $t$, $B^s$ is saver's bond holding at time 0. Without access to capital, saver simply chooses how much to invest in bond to smooth consumption over time.

**Lemma 1** (Asset price)**.** *The price of capital $q$ is given by the following asset pricing equation*

$$
q = \frac{a}{1+r^f} \tag{1.3}
$$

*Proof.* From first-order conditions of entrepreneur's problem. See appendix. ■

From the asset pricing equation (1.3), one can see that changes of asset price come from *two channels*. The first channel is *interest rate*, that is, changes of interest rate $r^f$. The second channel is *productivity*, that is, changes of productivity $a$.

**Wealth inequality** Interested in how asset price changes affect inequality, I introduce wealth inequality. Denote $\eta_t$ as entrepreneur's wealth share at time $t$. Entrepreneur's wealth share $\eta_0$ at $t = 0$ is exogenous. While entrepreneur's wealth share $\eta_1$ at $t = 1$ is endogenous and as follows

$$\eta_1 = \frac{qK_1 + B^e}{qK_1} = \frac{aK_1 + B^e(1+r^f)}{aK_1 + (B^e + B^s)(1+r^f)} \tag{1.4}$$

In the two-agent model, I refer to entrepreneur's wealth share as *wealth inequality.* A rise in entrepreneur's wealth share is an increase in wealth inequality.

### 1.3.2 Partial equilibrium

**Proposition 1** (Partial equilibrium)**.** *In the partial equilibrium analysis where the interest rate $r^f$ is exogenous and asset price $q$ is endogenous, entrepreneur's wealth share tomorrow $\eta_1$ responds to changes of asset price $q$ as follows:*

$$\frac{d\eta_1}{dq} > 0 \tag{1.5}$$

*And saver's welfare $V^s$ respond to changes of asset price $q$ as follows*[23]*:*

$$\begin{aligned} \frac{\partial V^s}{\partial a} &= 0 \\ \frac{\partial V^s}{\partial r^f} &= \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} > 0 \end{aligned} \tag{1.6}$$

*where $B^s$ is saver's bond holding, $K_1$ is entrepreneur's asset holding at $t = 1$, $r^f$ is the (net) interest rate, and $a$ is productivity.*

*Proof.* See appendix. ■

The interest rate ($r^f$) channel and productivity ($a$) channel of asset price changes emerge for welfare changes as well.

From (1.6), one can see that saver's welfare is affected by the change of interest rate $r^f$, not by the change of productivity $a$. Since saver can not hold any capital but can trade risk-free bond with entrepreneur freely. The saver's welfare is only affected through the *interest rate channel.*

I summarize the partial equilibrium results as follows: The rising asset price $q = \frac{a}{1+r^f}$ due to a lower interest rate $r^f$ increases wealth inequality and hurts the saver who now saves at a lower interest rate.

[23] Here I am using the market clearing condition for capital and the exogenous interest rate is such that entrepreneur is a borrower.

### 1.3.3 General equilibrium

Partial equilibrium analysis shows that even though the saver does not own the asset, the effect of rising asset prices spills over to them through leverage and interest rates. I proceed to link asset price changes and leverage in general equilibrium by imposing market clearing conditions for risk-free bonds to have endogenous interest rate.

**Market clearing** For capital with fixed supply $K$ and risk-free bond with net zero supply:

$$K_1 = K \tag{1.7}$$

$$B^s + B^e = 0 \tag{1.8}$$

**Proposition 2** (General equilibrium)**.** *In general equilibrium, I solve for asset price $q$, wealth inequality tomorrow $\eta_1$, and welfare of the saver $V^s$, given exogenous parameters of the model $\{a, \rho^e, \rho, K, W_0^e, W_0^s\}$ where $a$ is productivity, $\rho^e$ and $\rho$ are discount rates of entrepreneur and saver, $W_0^e$ and $W_0^s$ are the endowment of entrepreneur and saver today, and $K$ is the total supply of capital. I study the comparative statics with respect to entrepreneur's relative "impatience" (a decrease in $\delta^e = \rho^e - \rho$),*

1. *Asset price $q$ and interest rate $r^f$:*

$$\frac{\partial q}{\partial \delta^e} < 0 \tag{1.9}$$

$$\frac{\partial r^f}{\partial \delta^e} > 0 \tag{1.10}$$

2. *Wealth inequality (entrepreneur's wealth share) tomorrow $\eta_1$:*

$$\frac{\partial \eta_1}{\partial \delta^e} < 0 \tag{1.11}$$

3. *Saver's welfare $V^s$ is as follows:*

$$\frac{\partial V^s}{\partial \delta^e} > 0 \tag{1.12}$$

*Proof.* See appendix. ■

A decline in entrepreneur's relative "impatience" (a decrease in $\delta^e = \rho^e - \rho$) increases entrepreneur's saving demand, decreases the interest rate $r^f$, and increases asset price $q$. Wealth inequality tomorrow increases because the saver now faces a lower interest rate $r^f$.

A decline in entrepreneur's relative "impatience" (a decrease in $\delta^e$) hurts the saver due to lower interest rate. Using envelop theorem, one can write the welfare changes as follows,

$$\frac{\partial V^s}{\partial \delta^e} = \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} \underbrace{\frac{\partial r^f}{\partial \delta^e}}_{>0} > 0$$

where $B^s > 0$ is the saving of saver and $B^e < 0$ is the borrowing of entrepreneur.

I summarize the general equilibrium results as follows: A decline in entrepreneur's relative "impatience" (a decrease in $\delta^e$) raises asset price, increases wealth inequality tomorrow, and hurts the saver.

### 1.3.4 What's missing

While the analysis is sharp and intuitive in the simple two-period model, some important elements that affect asset prices, inequality and welfare are still missing. I discuss three key elements in this section: uncertainty, bubbles, and endogenous feedback from wealth inequality.

**Uncertainty** A volatile economic environment makes it difficult to ignore uncertainty. Uncertainty creates precautionary saving motive and requires a risk premium. Previous studies have shown that precautionary saving motive is important for determining consumption plans and asset prices are heavily influenced by risk premium.

Uncertainty can result in conflicting direct effect and indirect effect on asset prices, wealth inequality and welfare. A lower level of uncertainty leads to a lower risk premium, which directly increases asset prices. A lower level of uncertainty also leads to a lower level of precautionary saving motive, which increases the interest rate and decreases entrepreneur's borrowing. The direct effect of less uncertainty tends to increase wealth inequality and the welfare of the rich, whereas the indirect effect tends to decrease them.

**Bubble** A bubble emerges when the market value of an asset exceeds its fundamental value, which is the net present value of all future cash flows. In the two-period model, I cannot identify how bubbles affect asset prices and inequality differently from the fundamental value of an asset.

Bubbles also result in conflicting direct effect and indirect on asset prices, wealth inequality and welfare. A rise in the value of bubble directly increases the market value of an asset. While it also increases interest rate because of inter-temporal substitution effect. The increased interest rate indirectly decreases asset prices. The direct effect of a bubble tends to increase wealth inequality and the welfare of the rich, while the indirect effect tends to decrease them.

**Endogenous feedback from wealth inequality** An important mechanism that is missing in the two-period model is the endogenous feedback from wealth inequality $\eta$ to the economy. When there are more than two periods, wealth inequality becomes an endogenous state variable that asset prices and people's consumption-saving plans depend on. The endogenous feedback from wealth inequality in a multi-period model will turn out to be important in section 1.4.

In the following section, I develop an infinite-horizon model that takes into account uncertainty and endogenous bubbles and characterize the endogenous feedback from wealth inequality to the economy.

## 1.4 Full model

In this section, I develop a continuous-time infinite-horizon version of the two-period model in section 1.3 with a few key deviations:[24]

1. There are two groups of agents, entrepreneurs $i \in [0, 1]$ and savers $j \in [0, 1]$.
2. There are public equities and private equities on financial market.
3. There is uncertainty in the economy.
4. Asset bubbles are endogenously formed.

[24] Time is set to be continuous for tractability.

### 1.4.1 Full Model Set-up

**Preferences** There are two groups of agents, entrepreneurs and savers. Both groups of agents have logarithmic utility for tractability.[25] Entrepreneurs (discount rate $\rho^e = \rho + \delta^e$) are more "impatient" than savers (discount rate $\rho$).

**Technology** Entrepreneurs and savers live in an endowment economy with productive capital (or a tree).[26] Per unit capital produces $a$ units of output and $a$ is productivity. Only entrepreneurs can manage private capital and private capital is exposed to idiosyncratic risks. When managed by any individual entrepreneur $i$, private capital $k_t^i$ evolves according to the following Ito process

$$\frac{dk_t^i}{k_t^i} = gdt + \tilde{\sigma} d\tilde{Z}_{i,t} \tag{1.13}$$

where $g$ is the expected growth rate of capital, and $\tilde{\sigma}$ is the volatility of idiosyncratic risk $d\tilde{Z}_{i,t}$. Idiosyncratic shock $d\tilde{Z}_{i,t}$ is specific to each entrepreneur $i$. The idiosyncratic shocks $d\tilde{Z}_{i,t}$, $\forall i$ are independent and they cancel out in the aggregate, i.e. $\int_0^1 d\tilde{Z}_{i,t} = 0$. One can think of each entrepreneur as running a private firm using capital to produce. Entrepreneurs can buy and sell capital on the market. The price of capital per unit is denoted as $q_t$, which is an endogenous process. Postulate the process for capital price $q_t$ which I will solve for as follows,

$$\frac{dq_t}{q_t} = \mu_t^q dt$$

The capital price $q_t$ does not carry any idiosyncratic risk because it is determined in the aggregate.

**Financial market** The financial market consists of private equities, public equities, and risk-free bonds.

Entrepreneurs can issue outside equities to a public stock market but there is a maximum $1 - \underline{\chi}$ fraction of capital that can be promised as outside equity due to agency frictions arising from incentive problems. That is, entrepreneurs must hold at least $\underline{\chi}$ fraction of the value of their private firm as private equity. This constraint is also known as the "skin in the game".[27]

[25] I show in appendix that the main result hold with general CRRA utility function.

[26] See appendix for an extension with labor and investment.

[27] This type of equity constraint is wildly used in the macro-fiance literature, micro-founded in the corporate finance literature, and receives supportive empirical evidence.

The value of outside equity issued by entrepreneur $i$ is denoted as $V_t^{oe,i} = (1 - \chi_{it}) q_t k_t^i$, where $1 - \chi_{it}$ is the fraction of capital issued as outside equity by entrepreneur $i$.

Idiosyncratic risks cancel out after the public stock market pools outside equities together. And the diversified outside equities form a stock market index fund free from idiosyncratic risks, S&P 500 index fund for example. I refer to public equity and the stock market index fund interchangeably. Savers are restricted from holding private capital but they can hold the stock market index fund in an *inactive* way.[28] One can interpret this assumption as the savers hold the stock market index fund through pension. The value of the stock market index fund (public equity) is denoted as $V_t^{mf}$ and will be determined in equilibrium.

The fraction of public equity held by savers is denoted as $1 - \kappa$, where $\kappa$ is a parameter of the model.[29] Using the market clearing condition for public equity, entrepreneurs hold $\kappa$ fraction of the public equity in equilibrium.

Both entrepreneurs and savers can trade risk-free bond freely. Risk-free bond is in zero net supply and is denoted as $B_t$.

Figure 1.1 shows the balance sheets of entrepreneurs and savers respectively and figure 1.2 shows the financial market structure.

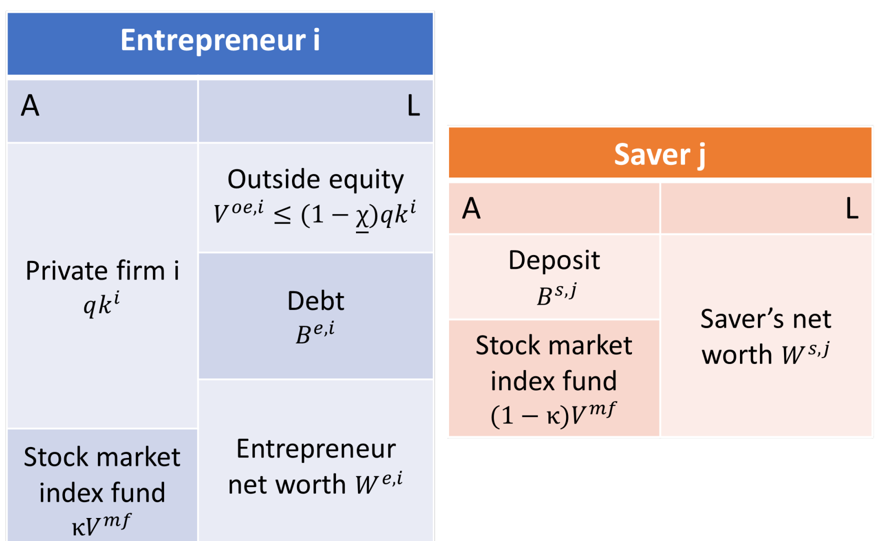


Figure 1.1: Balance sheets of entrepreneur and saver

[28] Allowing savers to optimally choose their portfolio share in public equity will create indeterminacy between leverage and public equity holding. However, all other results (asset prices, wealth inequality, bubbles) remains the same.

[29] I restrict the value of $\kappa$ such that $\kappa > \max\{\frac{\chi}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\sqrt{\delta^e}} - 1), \underline{\chi}(1 - \frac{\tilde{\sigma}}{\sqrt{\delta^e}})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho+\sqrt{\delta^e}}\frac{\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho}\}, \frac{\chi}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\delta^e} - 1)\}$. I discuss this restriction in appendix.

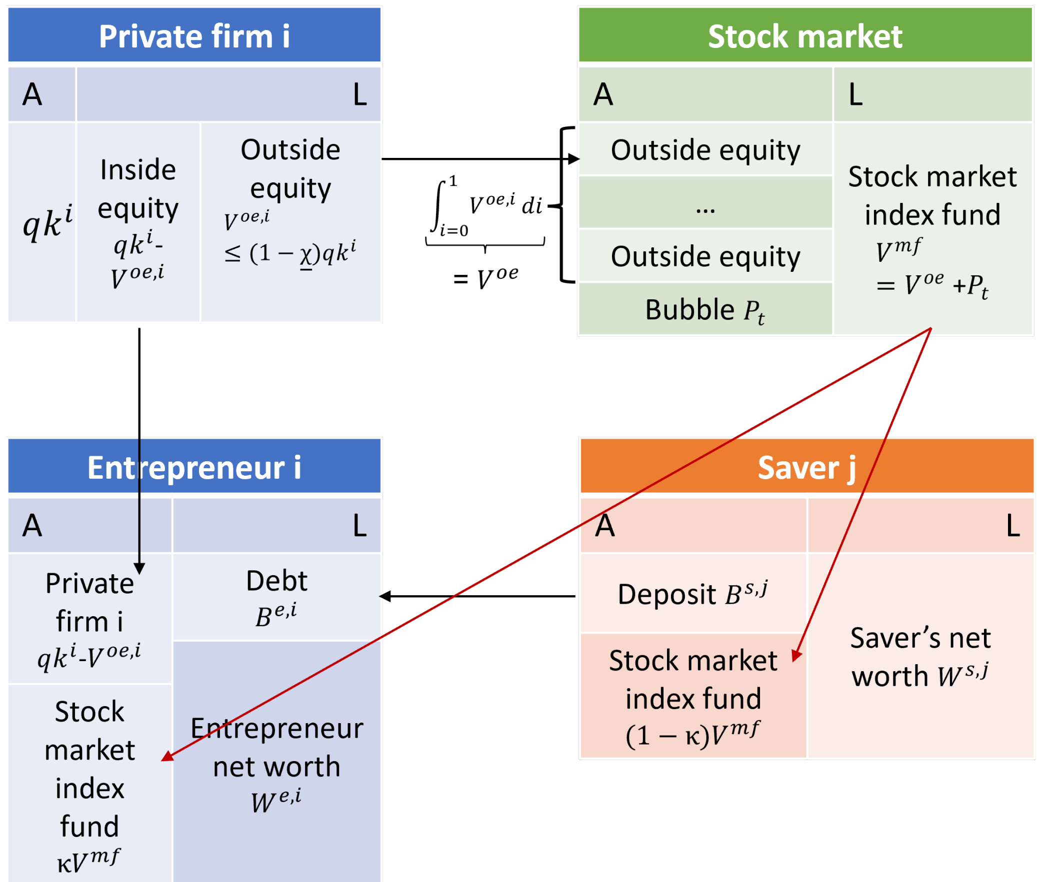


Figure 1.2: Financial market structure

**Asset returns** I introduce notations for asset returns which are *endogenous* processes. The return of capital when held by entrepreneur $i$, i.e. private equity, is denoted as follows

$$\begin{aligned} dr_t^{k,i} &= \underbrace{\frac{a}{q_t}}_{\text{dividend yield}} dt + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gain}} \\ &= (\frac{a}{q_t} + g + \mu_t^q)dt + \tilde{\sigma} d\tilde{Z}_t^i \end{aligned} \tag{1.14}$$

The return of outside equity issued by entrepreneur $i$ is

$$dr_t^{oe,i} = \mathbb{E}[dr_t^{oe,i}] + \tilde{\sigma} d\tilde{Z}_t^i$$

where the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is determined in equilibrium. Outside equity has the same risk characteristic as inside equity but may have a different expected return due to the equity constraint. Without the equity constraint, the expected return of outside equity should equal the return of inside equity, $\mathbb{E}[dr_t^{oe,i}] = \mathbb{E}[dr_t^{k,i}]$. However, when the equity constraint binds, the expected return of outside equity is lower than the return of inside equity $\mathbb{E}[dr_t^{oe,i}] < \mathbb{E}[dr_t^{k,i}]$.

Finally, the return of public equity (the stock market index fund) is denoted as $dr_t^{mf}$, and the return of risk-free bond is denoted as $\frac{dB_t}{B_t} = r_t^f dt$. The return of public equity and risk-free rate are the same for each individual.

**Wealth and portfolio shares** I introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the wealth of entrepreneur $i$ as $W_t^{e,i}$ and the wealth of saver $j$ as $W_t^{h,j}$. The portfolio share of capital of entrepreneur $i$ is denoted as $\theta_t^{k,i} = \frac{q_t k_t^i}{W_t^{e,i}}$. Outside equity's portfolio share of an entrepreneur's wealth is denoted as $\theta_t^{oe,i} = \frac{-(1-\chi_{it})q_t k_t^i}{W_t^{e,i}}$.[30] The portfolio share of public equity (the stock market index fund) held by entrepreneur $i$ is denoted as $\theta_t^{mf,i}$. And the portfolio share of public equity (the stock market index fund) held by saver $j$ is denoted as $\alpha_t^{mf,i} = \frac{(1-\kappa)V_t^{mf}}{W_t^{s,j}}$. Entrepreneurs can optimally choose their portfolio share in public equity $\theta_t^{mf,i}$, while savers take their portfolio share in public equity $\alpha_t^{mf,i} = \frac{(1-\kappa)V_t^{mf}}{W_t^{s,j}}$ as given.

**Wealth inequality** Define entrepreneurs' wealth share as

$$\eta_t = \frac{W_t^e}{W_t^e + W_t^s} \tag{1.15}$$

where $W_t^e = \int_{i=0}^1 W_t^{e,i}\, di$ and $W_t^s = \int_{j=0}^1 W_t^{s,j}\, dj$ are the aggregate wealth of entrepreneurs and savers at time $t$ respectively. I refer to $\eta_t$ as *wealth inequality.* If entrepreneurs' wealth share increases, wealth inequality increases.

**Optimization problems** The optimization problem for entrepreneur $i$ is as follows:

$$\begin{aligned}
\max_{\{c_t^{e,i},\theta_t^{k,i},\theta_t^{oe,i},\theta_t^{mf}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[\int_0^\infty e^{-\rho^e t} \log c_t^{e,i} dt\right] \\
\text{s.t.} \quad & \frac{dW_t^{e,i}}{W_t^{e,i}} = r_t^f dt + \theta_t^{k,i}(dr_t^{k,i} - r_t^f dt) + \theta_t^{oe,i}(dr_t^{oe,i} - r_t^f dt) + \theta_t^{mf,i}(dr_t^{mf} - r_t^f dt) - \frac{c_t^{e,i}}{W_t^{e,i}} d \\
& -\theta_t^{oe,i} \le (1-\underline{\chi})\theta_t^{k,i}
\end{aligned} \tag{1.16}$$

[30] I use a minus sign because outside equities are issued by entrepreneurs.

An entrepreneur optimally choose the consumption plan and portfolio shares of private (inside) equity, outside equity, public equity, and risk-free bond, taking the returns of the assets as given.

The optimization problem for saver $j$ is as follows[31]:

$$\begin{aligned} \max_{\{c_t^{s,j}\}_{t=0}^{\infty}} \quad & \int_0^{\infty} e^{-\rho t} \log c_t^{s,j} \, dt \\ \text{s.t.} \quad & \frac{dW_t^{s,j}}{W_t^{s,j}} = r_t^f \, dt + \alpha_t^{mf,j} (dr_t^{mf} - r_t^f \, dt) - \frac{c_t^{s,j}}{W_t^{s,j}} dt \end{aligned} \tag{1.17}$$

A saver optimally choose the consumption and saving plan, taking the risk-free rate as given. I leave the HJB equations for optimization problems and first-order conditions in appendix.

**Market clearing condition** The market for consumption clears as follows

$$C_t^e + C_t^s = aK_t \tag{1.18}$$

Equation (1.18) is the market clearing condition for consumption good. The left-hand side of (1.18) is the total demand of consumption good in the economy, where $C_t^e = \int_i c_t^{e,i} \, di$ and $C_t^s = \int_j c_t^{s,j} \, dj$ are the aggregate consumption of entrepreneurs and savers respectively. The right-hand side of (1.18) is the total supply of consumption good in the economy, where $K_t = \int_i k_t^i \, di$ is the aggregate capital and total production at time $t$ in the economy is $aK_t$.

**Definition of bubble** I define bubble as follows

$$P_t = V_t^{mf} - \int_i V_t^{oe,i} \, di \tag{1.19}$$

Recall that $V_t^{mf}$ is the total value of public equity which is held by entrepreneurs, or market value. The second term $\int_i V_t^{oe,i} \, di$ the value of all outside equities issued by entrepreneurs, that is, the fundamental value of public equity. On the left-hand side, $P_t$ is the wedge between the market value and fundamental value of public equity which can arise endogenously. I refer to $P_t$ as the value of *bubble*, which will be determined in equilibrium.

For the value of bubble, I will also work with $p_t \equiv \frac{P_t}{(1-\chi_t)K_t}$ for easier mathematical

[31]Note that savers do not carry any idiosyncratic risks because the stock market index fund diversifies idiosyncratic risks.

expressions. Postulate a process for $p_t$ which I will solve for,

$$\frac{dp_t}{p_t} = \mu_t^p dt$$

Note that the process of $p_t$ does not contain idiosyncratic risks, because the value of bubble is determined in the aggregate.

For the rest of the paper, I use superscripts $\{f, b\}$ to distinguish variables in fundamental equilibrium (defined in the following section 1.4.2) and bubble equilibrium (defined in section 1.5.2) when necessary. And I use an overline to denote variables in steady states.

### 1.4.2 Fundamental Equilibrium

In this section, I focus on and solve for fundamental equilibrium which is defined later. I define and solve for bubble equilibrium in section 1.5.

**Definition 1** (Fundamental equilibrium). *A fundamental equilibrium is a process of capital price $q_t$, a process of outside equity return $dr_t^{oe}$, a process of risk-free rate $r_t^f$, a process of public equity return $dr_t^{mf}$, and a process of entrepreneurs' wealth share $\eta_t^f$, given exogenous parameters of the model $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$, such that*

1. *entrepreneurs solve optimization problem* (A.155)

2. *savers solve optimization problem* (1.17)

3. *consumption good's market clears* (1.18)

4. *the value of bubble is zero, $P_t = 0$*

**Key equations for fundamental equilibrium** I provide some key equations for solving the fundamental equilibrium with intuition and leave the technical details in appendix.

From first-order conditions with respect to consumption, entrepreneurs and savers optimally consume a constant fraction of their wealth with logarithmic utility. The constants are their discount rates $\rho$ and $r$:

$$\frac{c_t^{e,i}}{W_t^{e,i}} = \underbrace{\rho + \delta^e}_{\rho^e}, \quad \frac{c_t^{s,j}}{W_t^{s,j}} = \rho \tag{1.20}$$

The expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is pivotal for equilibrium because the maximum issuance of outside equity is limited by the equity constraint. In fundamental equilibrium, the equity constraint binds, it follows that

$$\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}] = dr_t^{mf} \tag{1.21}$$

The first inequality $\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}]$ shows that the expected return of private equity (inside equity) is higher than the expected return of outside equity. This is because of the binding equity constraint. The second equality $\mathbb{E}[dr_t^{oe,i}] = dr_t^{mf}$ shows that the expected return of outside equity is equal to the return of public equity. This comes directly from equation (1.19) when the value of bubble is zero ($P_t = 0$). As one will see in next section, equation (1.21) only holds true in fundamental equilibrium and changes in bubble equilibrium. Entrepreneurs' wealth share, i.e. wealth inequality, which is the important endogenous state variable of the model, evolves as follows,

$$\frac{d\eta_t^f}{\eta_t^f} = (1 - \eta_t^f)\left(-\delta^e + \left(\frac{\chi\tilde{\sigma}}{\eta_t^f}\right)^2\right) dt \tag{1.22}$$

Equation (1.22) shows the decisive forces for wealth inequality: the patience gap between entrepreneurs and savers $\delta^e$ and the effect of uncertainty $\left(\frac{\chi\tilde{\sigma}}{\eta_t^f}\right)^2$. Entrepreneurs are more impatient than savers, they consume a larger fraction of their wealth than savers, $\delta^e > 0$. The patience gap decreases entrepreneurs' wealth share. While entrepreneurs earn a higher return that compensates for the idiosyncratic risks associated with their private capital, $\left(\frac{\chi\tilde{\sigma}}{\eta_t^f}\right)^2 > 0$. The risk premium increases entrepreneurs' wealth share, thus wealth inequality. Another interpretation of $\left(\frac{\chi\tilde{\sigma}}{\eta_t^f}\right)^2$ is entrepreneurs' precautionary saving motive.[32] With a higher level of precautionary saving motive, entrepreneurs borrow less, which increases their wealth share and wealth inequality.

I solve for the steady state[33] of fundamental equilibrium and summarize the results in the following proposition.

**Proposition 3** (Fundamental equilibrium steady state)**.** *In steady state of fundamental equi-*

[32]In this model, risk premium of capital and precautionary saving motive happen to be the same, because the only source of uncertainty in the economy is the idiosyncratic risk associated with private capital.

[33]Or a balanced growth path with respect to aggregate capital which is the only source of growth in the economy. Steady-state is regarding to the lower-case variables that are scaled to be per unit of capital.

*librium, capital price, wealth inequality, and the value of bubble are as follows*

$$\overline{q}^f = \frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{1.23}$$

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} \tag{1.24}$$

$$\overline{p}^f = 0 \tag{1.25}$$

*where* $\rho$ *and* $r$ *are discount rates of entrepreneurs and savers respectively,* $a$ *is the productivity of capital,* $\underline{\chi}$ *is the minimum fraction of the private firm that must be kept by entrepreneurs as inside equity, and* $\tilde{\sigma}$ *is the volatility of idiosyncratic risks associated with private capital.*

*For a non-degenerate wealth distribution in steady state where entrepreneurs are wealthier than savers, the following parameter restriction is required:*

$$\frac{1}{2} < \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} < 1 \tag{1.26}$$

*Risk premium of capital and risk-free rate in steady state are as follows*

$$\frac{\mathbb{E}[d\overline{r}^{k,f,i} - \overline{r}^f dt]}{dt} = \underline{\chi}\tilde{\sigma}\sqrt{\delta^e} \tag{1.27}$$

$$\overline{r}^f = \rho + g \tag{1.28}$$

*where* $g$ *is the expected growth rate of capital.*

*Proof.* See appendix. ∎

DSfundamental.png

Figure 1.3: Capital price in fundamental equilibrium steady state

Figure 1.3 showed how capital price is determined by capital structure (inside equity and outside equity) in fundamental equilibrium steady state. The $x$-axis, $\chi$, is the fraction of the private firm that is kept as inside equity. The $y$-axis, $q$, is the price of capital. In equilibrium, entrepreneurs would like to issue maximum amount of outside equity to offload idiosyncratic risks. The equity constraint always binds, $\chi = \underline{\chi}$. Capital price $q^f$ is decreasing in $\chi$ because the more inside equity kept, the more idiosyncratic risk, the higher risk premium of capital, and thus lower price.

### 1.4.3 Connection with the two-period model

In this section, I connect the full model with the two-period model in section 1.3.

**Proposition 4** (Connection with the two-period model)**.** *Assume there is no uncertainty, $\tilde{\sigma} = 0$, and the expected growth rate of capital $g = 0$, the partial equilibrium and general equilibrium results in the two-period model in section 1.3 in the full model are recovered as follows,*

1. *In partial equilibrium with an exogenous constant risk-free rate $r^f$ and endogenous asset price $q^f$:*

   (a) *Asset price $q^f$ and wealth inequality $\eta_t^f$ are*

$$q^f = \frac{a}{r^f} \tag{1.29}$$

$$\eta_t^f = 1 - \frac{B_t}{q^f K_t} \tag{1.30}$$

   (b) *Wealth inequality $\eta_t^f$ changes with respect to asset price $q^f$ as follows*

$$\frac{d\eta_t^f}{dq^f} > 0 \tag{1.31}$$

   (c) *Savers' welfare $V^{s,f}$ change with respect to asset price $q^f$ as follows*

$$\frac{dV^{s,f}}{dq} = \int_0^\infty -e^{-\rho t} U'(C_t^s) \rho K_t q^f \frac{d\eta_t^f}{dq} \, dt < 0 \tag{1.32}$$

2. *Solving for general equilibrium with endogenous risk-free rate $r_t^f$ and asset price $q_t^f$:*

   (a) *capital price $q_t^f$,*

$$q_t^f = \frac{a}{\delta^e \eta_t^f + \rho} \tag{1.33}$$

$$\frac{\partial q_t^f}{\partial \delta^e} < 0 \tag{1.34}$$

   (b) *risk-free rate $r_t^f$,*

$$r_t^f = \frac{(\rho + \delta^e)\delta^e \eta_t^f}{\delta^e \eta_t^f + \rho} + \rho \tag{1.35}$$

*(c) wealth inequality* $\eta_t^f$*,*

$$\eta_t^f = \frac{1}{e^{\delta^e t + \log \frac{1-\eta_0}{\eta_0}} + 1} \tag{1.36}$$

$$\frac{\partial \eta_t^f}{\partial \delta^e} < 0 \tag{1.37}$$

*(d) and welfare of savers* $V^{s,f}$*,*

$$V^{s,f} = \int_0^\infty e^{-\rho t} \log(\rho(1-\eta_t^f) q_t^f K_t)\, dt \tag{1.38}$$

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial \delta^e} > 0 \tag{1.39}$$

*where* $a$ *is the productivity of capital,* $\rho + \delta^e$ *and* $\rho$ *are discount rates of entrepreneurs and savers respectively,* $\delta^e$ *is the relative "impatience" of entrepreneurs, and* $\eta_0$ *is the initial wealth inequality.*

*Proof.* See appendix. ■

**Consistency with the two-period model** Partial equilibrium results of the infinite-horizon model are consistent with the two-period model (see proposition 1): rising asset prices always increases wealth inequality and hurt the savers.

General equilibrium results of the infinite-horizon model are also consistent with the two-period model: Declining relative "impatience" of entrepreneurs (a decrease in $\delta^e$) raises asset price $q_t$, increases wealth inequality $\eta_t$, and decreases welfare of savers $V^{s,f}$.

## 1.5 Bubble equilibrium

I start this section by providing intuitions for bubbles. Then I define and solve for bubble equilibrium, and compare bubble equilibrium with fundamental equilibrium. I finish this section by discussing the effect of bubble on inequality.

### 1.5.1 Intuition for bubbles

Before solving for bubble equilibrium, I provide some intuition why there can be a bubble on public equity by identifying critical reasons in my model, as well as making an analogy to the existing literature.

There are three critical reasons in my model for bubbles to exist. First, idiosyncratic risks are diversified away in the aggregate on public stock market. Entrepreneurs are willing to offload their idiosyncratic risks by issuing outside equities of their private firms to the public stock market. Second, there is an equity constraint that limits outside equity issuance. Entrepreneurs still carry some idiosyncratic risks that create precautionary saving motive for them. Third, there is a trade-off for impatient entrepreneurs between borrowing from savers to consume and leveraging up idiosyncratic risks on their wealth which increases their precautionary saving motive. Bubbles on public equity reduce entrepreneurs' precautionary saving motive because the value of bubble does not carry idiosyncratic risks. Impatient entrepreneurs can thus borrow more from savers compared to when there is no bubble.

I make an analogy and show the subtle difference between bubbles in my model and bubbles that have positive value because they directly relax some constraint (see Kocherlakota 2009 and Miao and Wang 2018 for example). Recall that in my model there is an equity constraint that limits how much outside equities entrepreneurs can issue. However, this constraint also *indirectly* limits the supply of public equity. The value of bubble increases the supply of public equity, which breaks the indirect limitation of (outside) equity constraint on public equity.

$$\underbrace{V_t^{mf}}_{\text{value of public equity}} \leq \underbrace{(1-\underline{\chi})q_t K_t}_{\text{limit on outside equity}} + \underbrace{P_t}_{\text{value of bubble}} \tag{1.40}$$

As shown in equation (1.40), bubbles relax the indirect limit on public equity supply but not the direct limit on outside equity issuance.

Bubbles in this paper are stable as in the literature on rational bubbles driven by financial frictions. There is no *endogenous* switches from fundamental equilibrium to bubble equilibrium. However, as in the literature, one can think of there is an *exogenous* probability $\pi$ for fundamental equilibrium to realize and a probability $1-\pi$ for bubble equilibrium to realize. In the baseline model, I do not talk about stochastic bubbles but simply compare fundamental equilibrium and bubble equilibrium.

### 1.5.2 Bubble equilibrium

**Definition 2** (bubble equilibrium)**.** *A bubble equilibrium is a capital price process $q_t$, a process of outside equity return $dr_t^{oe}$, a process of risk-free rate $r_t^f$, a process of public equity return*

*$dr_t^{mf}$, and a process of entrepreneurs' wealth share $\eta_t^b$, given exogenous parameters of the model $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, a, g\}$, such that*

1. *entrepreneurs solve optimization problem* (A.155)
2. *savers solve optimization problem* (1.17)
3. *consumption good's market clears* (1.18)
4. *the value of bubble is positive, $P_t > 0$*
5. *the equilibrium is trembling-hand perfect.*[34]

**Key equations for bubble equilibrium** savers and entrepreneurs have the same optimal consumption plans as constant fractions of their wealth in bubble equilibrium as in fundamental equilibrium (1.20).

However, the pivotal equation for the expected return of outside equity in bubble equilibrium becomes

$$\mathbb{E}[dr_t^{k,i}] = \mathbb{E}[dr_t^{oe,i}] > dr_t^{mf} \tag{1.41}$$

In equation (1.41), the first equality $\mathbb{E}[dr_t^{k,i}] = \mathbb{E}[dr_t^{oe,i}]$ shows that the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$ is equal to the expected return of private equity $\mathbb{E}[dr_t^{k,i}]$. Entrepreneurs are indifferent between inside equities and outside equities, and the equity constraint is not binding.[35] The second equality $\mathbb{E}[dr_t^{oe,i}] > dr_t^{mf}$ shows that the expected return of outside equity is higher than the return of public equity. This is because outside equities earn a risk-premium for idiosyncratic risks. Figure 1.4 showed the value of bubble and capital price is determined by capital structure in bubble equilibrium steady state.

Recall that in fundamental equilibrium, equation (1.21) shows that the expected return of outside equity is lower than the expected return of private equity due to the binding equity constraint and equal to the return of public equity, $\mathbb{E}[dr_t^{k,i}] > \mathbb{E}[dr_t^{oe,i}] = dr_t^{mf}$. The comparison between equation (1.21) and (1.41) shows the origin of the bubble: the wedge between the return of public equity $dr_t^{mf}$ and the expected return of outside equity $\mathbb{E}[dr_t^{oe,i}]$. It is exactly this wedge in return that creates a wedge in the market value and the fundamental

[34] I use the trembling-hand perfect equilibrium as an equilibrium refinement to resolve indeterminacy. See the following section and appendix for details.

[35] The issuance of outside equity is thus indeterminate without equilibrium refinement.

value of public equity, $P_t > 0$, as shown in equation (1.19). Figure 1.5 showed the supply and demand for public equity in bubble equilibrium steady state.

The evolution of wealth inequality, i.e. entrepreneurs' wealth share, in bubble equilibrium is as follows,

$$\frac{d\eta_t^b}{\eta_t^b} = (1-\eta_t^b)\Bigg( -\delta^e + \left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2 \Bigg) dt \tag{1.42}$$

Similar intuition as in fundamental equilibrium applies to the wealth share evolution in bubble equilibrium: Impatience of entrepreneurs decrease their wealth share, $\delta^e > 0$. While the risk premium of idiosyncratic risks associated with private capital increases entrepreneurs' wealth share, $\left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2 > 0$. Note that given the same level of wealth inequality $\eta_t$, the risk premium or the precautionary saving motive is *lower* in bubble equilibrium compared to fundamental equilibrium. To achieve the same level of risk premium or the precautionary saving motive as in fundamental equilibrium, a *lower* level of wealth inequality is needed in bubble equilibrium.

I solve for the steady state of bubble equilibrium and summarize the results in the following proposition.

**Proposition 5** (Bubble equilibrium steady state)**.** *In steady state, capital price, wealth inequality, and the value of bubble are as follows*

$$\overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{1.43}$$

$$\overline{\eta}^b = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi})\frac{\tilde{\sigma}\delta^e}{\rho}} \tag{1.44}$$

$$\overline{p}^b = \frac{a}{\rho} - \overline{q}^b \tag{1.45}$$

*where $\rho + \delta^e$ and $\rho$ are discount rates of entrepreneurs and savers respectively, $a$ is the productivity of capital, $\underline{\chi}$ is the minimum fraction of the private firm that must be kept by entrepreneurs as inside equity, and $\tilde{\sigma}$ is the volatility of idiosyncratic risks associated with private capital.*

*For a non-degenerate steady state wealth distribution, the following parameter restriction is required:*

$$[\underline{\chi} - (1-\underline{\chi})\frac{\delta^e}{\rho}]\tilde{\sigma} < \sqrt{\delta^e} \tag{1.46}$$

*Risk premium of capital and risk-free rate in steady-state are as follows*

$$\frac{\mathbb{E}[d\overline{r}^{k,b,i} - \overline{r}^f dt]}{dt} = \tilde{\sigma}\sqrt{\delta^e} \tag{1.47}$$

$$\overline{r}^f = \rho + g \tag{1.48}$$

*where $g$ is the expected growth rate of capital.*

*Proof.* See appendix. ■

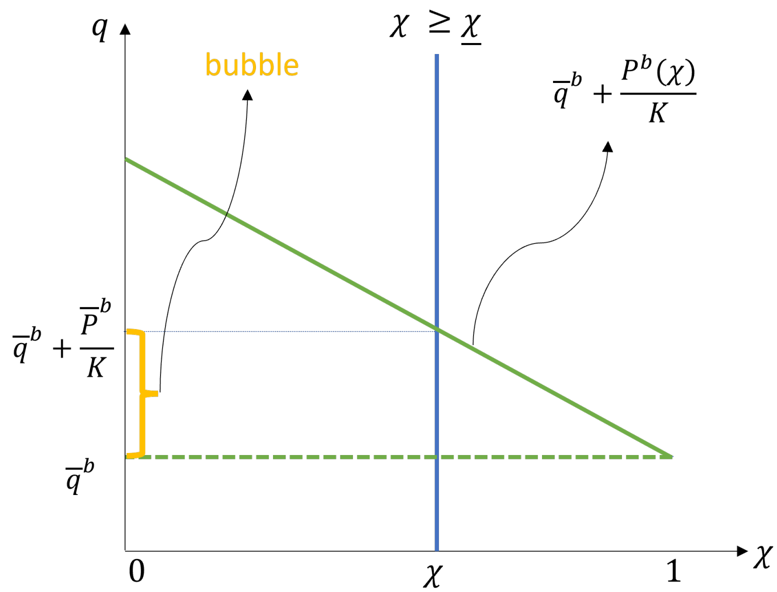


Figure 1.4: Capital price in bubble equilibrium steady state

Figure 1.4 shows how the value of bubble is determined by capital structure in bubble equilibrium steady state. And figure 1.5 shows the comparison between fundamental equilibrium and bubble equilibrium. The $x$-axis, $\chi$, is the fraction of the private firm that is kept as inside equity. The $y$-axis, $q$, is price. Note that in bubble equilibrium, price of private capital $\overline{q}^b$ does not depend on capital structure. Because entrepreneurs are indifferent between inside equity and outside equity in bubble equilibrium. The value of bubble $\frac{P^b}{K}$ is linearly decreasing in $\chi$.

### 1.5.3 Comparison of fundamental and bubble equilibrium

I compare steady states of bubble equilibrium and fundamental equilibrium in the following proposition, and discuss the mechanism with intuition.

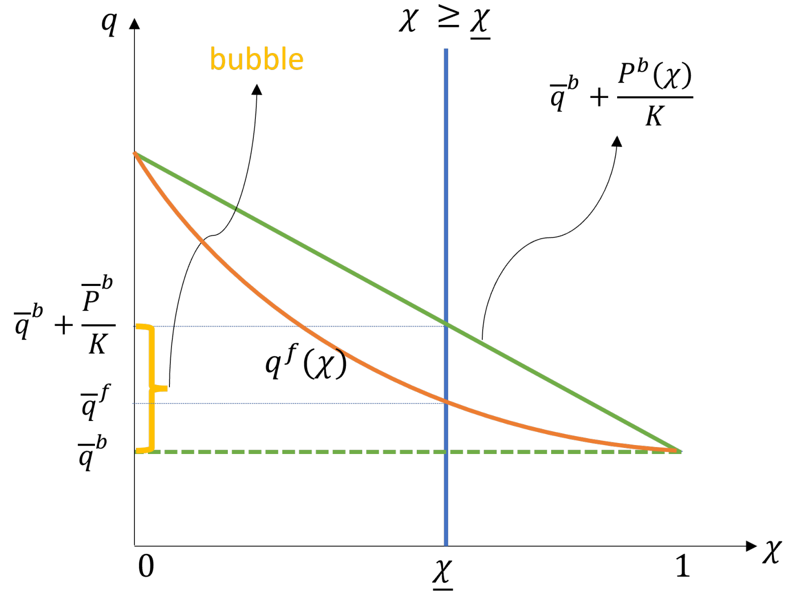


Figure 1.5: Comparison of capital price in bubble and fundamental equilibrium

**Proposition 6** (Comparison of fundamental and bubble equilibrium)**.** *Comparing steady states of fundamental and bubble equilibrium as follows,*

1. *Capital price:*

$$\overline{q}^b < \overline{q}^f \tag{1.49}$$

2. *Total wealth in the economy:*

$$\overline{q}^b K_t + \overline{P}_t > \overline{q}^f K_t \tag{1.50}$$

3. *Risk-free bond issued by entrepreneurs:*

$$\overline{B}_t^b > \overline{B}_t^f \tag{1.51}$$

4. *Wealth inequality:*

$$\overline{\eta}^b < \overline{\eta}^f \tag{1.52}$$

5. *Risk premium on capital:*

$$\frac{\mathbb{E}[d\overline{r}^{k,b,i} - \overline{r}^f dt]}{dt} > \frac{\mathbb{E}[d\overline{r}^{k,f,i} - \overline{r}^f dt]}{dt} \tag{1.53}$$

Equation (1.49) shows that capital price is lower in bubble equilibrium than in fundamental equilibrium. In fundamental equilibrium, the expected return of outside equity equals the return of public equity (1.21). In bubble equilibrium, the expected return of outside equity equals the return of inside equity, which is higher than the return of public equity (1.41). Higher expected return of outside equity lowers capital price in bubble equilibrium. As a corollary, the total amount of idiosyncratic risks associated with private capital is reduced in bubble equilibrium, $\overline{q}^b k_t^i \tilde{\sigma} < \overline{q}^f k_t^i \tilde{\sigma}$.

Equation (1.50) shows that total wealth is higher in bubble equilibrium than in fundamental equilibrium, since bubbles reduce the total amount of idiosyncratic risks that cannot be diversified which is the key friction in the economy.

Equation (1.51) shows entrepreneurs borrow more from savers in bubble equilibrium than in fundamental equilibrium. Entrepreneurs' precautionary saving motive decreases in bubble equilibrium as the value of bubble does not carry idiosyncratic risks.

Equation (1.52) shows that wealth inequality is *lower* in bubble equilibrium than in fundamental equilibrium. This result seems counter-intuitive at the first sight. I provide two intuitive explanations. If one looks at the balance sheet of entrepreneurs (see figure 1.1), their total asset value is indeed higher in bubble equilibrium. This is the *price channel*. Nevertheless, on the liability side, entrepreneurs' borrowing also increases in bubble equilibrium. A bubble increases the total value of entrepreneurs' assets and decreases their precautionary saving motive. Entrepreneurs borrow more from savers in bubble equilibrium than in fundamental equilibrium. This is the *leverage channel*. In equilibrium, leverage channel dominates. As a result, wealth inequality is lower in bubble equilibrium. Another explanation is to look at entrepreneurs' wealth accumulation process (1.42). Entrepreneurs accumulate their wealth by taking idiosyncratic risks and earning a higher return. As bubbles reduce the total amount of idiosyncratic risks in the economy, entrepreneurs' wealth share decreases. Thus wealth inequality decreases.

Equation (1.53) shows that risk premium on capital is higher in bubble equilibrium than in fundamental equilibrium. Bubble equilibrium is achieved when entrepreneurs are willing to hold a bubble. In order to clear the equity market, the expected return of capital is higher and the risk premium on capital is higher in bubble equilibrium.

### 1.5.4 Safety and liquidity effect of bubble on wealth inequality

After solving for both fundamental equilibrium and bubble equilibrium, I am able to delve deeper into how bubbles affect inequality. In this section, I study some extreme cases and provide more intuition.

A bubble increases the *safety* of the economy by reducing idiosyncratic risks and increases the *liquidity* of the economy by allowing entrepreneurs to borrow more. As a result, wealth inequality decreases. I examine some extreme cases in the following propositions to identify the safety effect and the liquidity effect of bubble on wealth inequality.

**Proposition 7** (Safety effect)**.** *I examine the steady states of two extreme cases with no uncertainty,*

1. *In an economy with no equity constraint ($\underline{\chi} = 0$): price of capital $q$, wealth inequality $\eta$, and value of bubble $P$ in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = \frac{a}{\rho} \qquad \overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho}$$

$$\overline{\eta}^f = 0 \qquad \overline{\eta}^b = 0$$

$$\overline{P}_t^f = 0 \qquad \overline{P}_t^b = (\frac{a}{\rho} - \overline{q}^b)K_t$$

*and*

$$\lim_{\underline{\chi}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1 + (1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = \frac{1}{1 + \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}}$$

2. *In an economy with no idiosyncratic risk ($\tilde{\sigma} = 0$): price of capital $q$, wealth inequality $\eta$, and value of bubble $P$ in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = \overline{q}^b = \frac{a}{\rho}$$

$$\overline{\eta}^f = \overline{\eta}^b = 0$$

$$\overline{P}_t^f = \overline{P}_t^b = 0$$

*and*

$$\lim_{\tilde{\sigma}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1 + (1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = 1$$

*where $\rho$ and $r$ are discount rates of entrepreneurs and savers respectively, $\underline{\chi}$ is the minimum fraction of the private capital that must be kept by entrepreneurs, $\tilde{\sigma}$ is the volatility of idiosyncratic risks, $a$ is the productivity of capital, and $K_t$ is the aggregate capital in the economy at time $t$.*

In both cases, there are no idiosyncratic risks in equilibrium. The fundamental equilibrium in both cases are the same. Entrepreneurs can borrow against all their wealth, so liquidity is perfect in both cases. The value of bubble $P$ differs. While the value of bubble is zero when idiosyncratic risk is zero, $\tilde{\sigma} = 0$, the bubble can still sustain a positive value in the case of $\underline{\chi} = 0$. The wealth share of entrepreneurs become 0 in steady state in both cases. The level of wealth inequality, which is first-order, does not seem to be affected by safety. However, in both cases, $\eta$ converges to zero, but at different rates,

$$\lim_{\underline{\chi}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = \frac{1}{1+\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} \tag{1.54}$$

$$\lim_{\tilde{\sigma}\to 0}\left(\frac{\overline{\eta}^b}{\overline{\eta}^f}\right) = \frac{1}{1+(1-\underline{\chi})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}} = 1 \tag{1.55}$$

This comparison captures the subtle *second-order safety effect* of bubble on wealth inequality.

**Proposition 8** (Liquidity effect)**.** *I examine the steady states of two extreme cases with maximum level of uncertainty and frictions,*

1. *In an economy with no public stock market ($\underline{\chi} = 1$): price of capital $q$, wealth inequality $\eta$, and value of bubble $P$ in fundamental equilibrium and bubble equilibrium are as follows,*

$$\overline{q}^f = \overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e}+\rho}$$

$$\overline{\eta}^f = \overline{\eta}^b = \frac{\tilde{\sigma}}{\sqrt{\delta^e}}$$

$$\overline{P}_t^f = \overline{P}_t^b = 0$$

2. *In an economy with infinite volatility of idiosyncratic risks ($\tilde{\sigma} = +\infty$ ): price of capital $q$, wealth inequality $\eta$, and value of bubble $P$ in fundamental equilibrium and bubble*

*equilibrium are as follows,*

$$\overline{q}^f = 0 \qquad \overline{q}^b = 0$$

$$\overline{\eta}^f = 1 \qquad \overline{\eta}^b = \frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e}$$

$$\overline{P}_t^f = 0 \qquad \overline{P}_t^b = (1-\underline{\chi})\frac{a}{\rho}K_t$$

*where* $\rho+\delta^e$ *and* $\rho$ *are discount rates of entrepreneurs and savers respectively,* $\delta^e$ *is the relative "impatience" of entrepreneurs,* $\underline{\chi}$ *is the minimum fraction of the private capital that must be kept by entrepreneurs,* $\tilde{\sigma}$ *is the volatility of idiosyncratic risks,* $a$ *is the productivity of capital, and* $K_t$ *is the aggregate capital in the economy at time* $t$*.*

In the case of $\underline{\chi} = 1$, there is no public stock market access and idiosyncratic risks can not be diversified. As idiosyncratic risk goes up, $\tilde{\sigma} > \sqrt{\delta^e}$, the liquidity in the economy stops and all the wealth are held by entrepreneurs, $\overline{\eta}^f = \overline{\eta}^b = 1$. The value of bubble is zero, $\overline{P}_t^b = 0$. In the case of $\underline{\chi} < 1$, the public stock market operates to the extent of $1 - \underline{\chi} > 0$, and the bubble has a positive value, $P_t^b > 0$. In fundamental economy, as idiosyncratic risk goes up, $\underline{\chi}\tilde{\sigma} > \sqrt{\delta^e}$, all the wealth are held by entrepreneurs, $\overline{\eta}^f = 1$, and liquidity in the economy stops. However, there can still be liquidity in the bubble economy even as idiosyncratic risk goes to infinity, $\tilde{\sigma} \to +\infty$,

$$\lim_{\tilde{\sigma}\to+\infty} \underbrace{\frac{\tilde{\sigma}}{\sqrt{\delta^e}}}_{\text{wealth inequality } \overline{\eta}^b \text{ in the case of } \underline{\chi}=1} = 1 \tag{1.56}$$

$$\lim_{\tilde{\sigma}\to+\infty} \underbrace{\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi})\frac{\tilde{\sigma}(\delta^e)}{\rho}}}_{\text{wealth inequality } \overline{\eta}^b \text{ in the case of } \underline{\chi}<1} = \frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e} \tag{1.57}$$

and

$$\frac{\underline{\chi}\rho}{(1-\underline{\chi})\delta^e} < 1 \quad \text{if} \quad \underline{\chi}\rho < (1-\underline{\chi})\delta^e \tag{1.58}$$

This comparison captures the *first-order liquidity effect* of bubble on wealth inequality.

In general cases, safety effect and liquidity effect work together. By looking at extreme cases in this section, I try to provide sharper intuition for these effects. Also note that $\underline{\chi} = 1$ case corresponds to the two-period model in section 1.3. In this case, the value of the endogenous bubble is zero. The public stock market that pools outside equities is the key to form endogenous bubbles that have positive value.

## 1.6 Inequality and welfare

In this section, I analyze the effect of rising asset prices due to different shocks on inequality measured in wealth, consumption and welfare both in fundamental equilibrium and bubble equilibrium.

Recall that I defined $\delta^e = \rho^e - \rho$ which captures the relative "impatience" of entrepreneurs to savers. One can interpret $\delta^e$ as the entry/exit or the motality/fertility rate of the entrepreneurs and $\rho$ as a common discount rate shared by all agents in the economy.

I will work with a set of parameters $\{\delta^e, \rho, \tilde{\sigma}, \underline{\chi}, g, a\}$, where besides $\delta^e$ and $\rho$, $\underline{\chi}$ is the minimum fraction of the private capital that must be kept by entrepreneurs, $\tilde{\sigma}$ is the volatility of idiosyncratic risks, $a$ is the productivity of capital, and $g$ is the growth rate of aggregate capital in the economy. The shocks are categorized into three types: the relative "impatience" of entrepreneurs ($\delta^e$), financial innovation and regulation ($\underline{\chi}\tilde{\sigma}$), productivity ($a$), as well as a bubble ($p_t > 0$). I will focus on steady-state analysis.

### 1.6.1 Level of inequality

In this section, I compare the steady-state level of wealth inequality, consumption inequality, and welfare in fundamental equilibrium and bubble equilibrium.

**Wealth inequality** Recall that wealth inequality is defined as entrepreneurs' wealth share relative to total wealth in the economy:

$$\eta_t = \frac{W_t^e}{W_t^e + W_t^s}$$

Since entrepreneurs and savers consume a constant fraction of their wealth with logarithmic utility, and the constants are their discount rates $\rho + \delta^e$ and $\rho$.[36] Interestingly, if we define consumption inequality as the aggregate consumption of entrepreneurs relative to the total consumption of the economy, it does not always move in the same direction as wealth

[36] Using logarithmic utility allows me to characterize equilibrium in closed-form. However, the leverage channel remains with more general utility functions.

inequality. Because consumption inequality is affected by discount rates as well as wealth inequality. Keeping the discount rates fixed, consumption inequality increases when wealth inequality increases. When there are shocks to discount rates, consumption inequality is not only directly affected by changes in discount rates themselves, but also indirectly affected by changes in wealth inequality. I expands on consumption inequality in the appendix.

**Welfare** The value function of savers in fundamental economy as an example for illustration is as follows

$$V^{s,f} = \int_0^\infty e^{-\rho t} \left( \underbrace{\log(1-\eta_t^f)}_{\text{wealth share}} + \underbrace{\log \rho}_{\text{consumption rate}} + \underbrace{\log \left( q_t^f K_t \right)}_{\text{total wealth}} \right) dt \tag{1.59}$$

As shown in equation (1.59), savers' welfare is affected by their wealth share, consumption rate, and total wealth in the economy. Savers do not carry any risk, so their welfare is not affected by precautionary saving motive.

The value functions for savers starting from steady state in fundamental economy and bubble economy are denoted as $\overline{V}^{s,f}$ and $\overline{V}^{s,b}$ respectively.

**Proposition 9** (Level of inequality)**.** *Comparing the level of wealth inequality, consumption inequality and welfare,*

1. *Wealth inequality:*
$$\overline{\eta}^b < \overline{\eta}^f \tag{1.60}$$

2. *Leverage:*
$$\overline{B}_t^b > \overline{B}_t^f \tag{1.61}$$

3. *Savers' welfare:*
$$\overline{V}^{s,b} > \overline{V}^{s,f} \tag{1.62}$$

*Proof.* See appendix. ■

Wealth inequality *decreases*, and welfare of the savers *increases* in bubble equilibrium compared to fundamental equilibrium in steady state.

The decrease in wealth inequality is discussed in proposition 6. Consumption inequality also decreases as there is no discount rate shocks.

Leverage increases in bubble equilibrium as the positive value of bubble decreases the precautionary saving motive of entrepreneurs, as shown in equation (1.42) and discussed in proposition 6.

Savers are better off in steady state of bubble equilibrium compared to fundamental equilibrium. The increase in savers' welfare follows from the decreased consumption inequality in bubble equilibrium.

### 1.6.2 Rising asset prices on inequality and welfare

I consider small changes of parameters in steady state and ignore the transition to new steady state which happens fast. The following proposition states how rising asset prices due to different shocks affect wealth inequality and welfare in a fundamental economy and is presented in Figure 1.6.

**Proposition 10.** *I identify the effect of rising asset prices due to different shocks on inequality in fundamental equilibrium steady state as follows,*

1. *When the relative "impatience" of entrepreneurs ($\delta^e$) decreases, asset price $q^f$ rises, wealth inequality $\eta^f$ increases. Savers' welfare $V^{s,f}$ always decreases.*

$$d\delta^e < 0 \implies \begin{cases} dq^f > 0 \\ dB^f < 0 \quad d\eta^f > 0 \\ dV^{s,f} < 0 \end{cases}$$

2. *When there is financial innovation ($\underline{\chi}\tilde{\sigma}$ decreases), asset price $q^f$ rises, wealth inequality $\eta^f$ decreases. Savers' welfare $V^{s,f}$ always increases.*

$$d(\underline{\chi}\tilde{\sigma}) < 0 \implies \begin{cases} dq^f > 0 \\ dB^f > 0 \quad d\eta^f < 0 \\ dV^{s,f} > 0 \end{cases}$$

3. *When productivity increases (a increases), asset price $q^f$ rises, wealth inequality $\eta^f$ do*

*not change. Savers' welfare* $V^{s,f}$ *increases.*

$$da > 0 \implies \begin{cases} dq^f > 0 \\ dB^f > 0 \quad d\eta^f = 0 \\ dV^{s,f} > 0 \end{cases}$$

*Proof.* See appendix. ■

| Asset price (public equity) ↑ | Leverage | Wealth inequality | welfare of savers |
|---|---|---|---|
| Entrepreneurs' relative "impatience" ↓ | ↓ | ↑ | ☹ |
| Financial innovation | ↑ | ↓ | ☺ |
| Productivity ↑ | ↑ | — | ☺ |

Figure 1.6: Rising asset price on inequality in fundamental equilibrium

As figure 1.6 shows, we can theoretically identify the exact shock to the asset price $q^f$ by looking at the co-movement of the asset price $q^f$ and wealth inequality $\overline{\eta}^f$, and conclude how the rising asset price $q^f$ affect savers' welfare.

**Relative "impatience" of entrepreneurs** When entrepreneurs become less impatient relative to savers ($\delta^e$ decreases), they consume a smaller fraction of their wealth everyday which directly decreases their consumption share, however, they also borrow less from savers (leverage $B^f$ decreases) which increases their wealth share and indirectly increases their consumption share. As entrepreneurs are wealthier than savers $\overline{\eta}^f > \frac{1}{2}$, the indirect effect dominates the direct effect, the consumption inequality increases. As a result, the welfare of savers decreases because their consumption falls.

Through the lens of my theory, the main trend of rising asset prices and rising wealth inequality in the past a few decades suggests that the relative "impatience" of the super rich entrepreneurs to savers declines. In this case, savers are worse off.

**Financial innovation** Financial innovation (a decrease in $\underline{\chi}\tilde{\sigma}$) increases the asset value and reduces the precautionary saving motive of entrepreneurs. Entrepreneurs borrow more from savers than before (leverage $B^f$ increases). As a result, wealth inequality and consumption inequality decrease. The welfare of savers increases as their consumption increases.

This is a general equilibrium result. On one hand, financial innovation raises asset prices and increases the asset value of the super rich entrepreneurs. This is the *price channel*. On the other hand, it reduces the precautionary saving motive of the entrepreneurs who are borrowers, and they borrow more from the savers. This is the *leverage channel*. In equilibrium, leverage channel dominates. Wealth inequality and consumption inequality decreases. Savers are better off.

**Productivity** An increase in productivity $a$ increases the asset price $q^f$ and leverage $B^f$, but not wealth inequality nor consumption inequality in fundamental economy[37]. However, savers benefit from higher productivity due to a positive income effect. This result shows that even though wealth inequality and consumption inequality do not change, rising asset prices can still have welfare effect.

### 1.6.3 Rising asset prices on inequality in bubble economy

The following proposition states how rising asset prices due to different shocks affect inequality in bubble economy and is presented in Figure 1.7.

**Proposition 11.** *I identify the effect of rising asset prices due to different shocks on inequality in bubble equilibrium steady state as follows,*

1. *When the relative "impatience" of entrepreneurs ($\delta^e$) decreases, the price of private equity $q^b$ rises, the price of public equity $(q^b + p^b)$ does not change, wealth inequality $\eta^b$ increases. Savers' welfare $V^{s,b}$ decreases.*

$$d\delta^e < 0 \implies \begin{cases} dq^b > 0 & d(q^b + p^b) = 0 \\ dB^f < 0 & d\eta^b > 0 \\ dV^{s,b} < 0 & \end{cases}$$

[37]This result holds true for CRRA utility functions

2. *When there is financial innovation ($\underline{\chi}\tilde{\sigma}$ decreases), the price of private equity $q^b$ rises, the price of public equity $(q^b+p^b)$ does not change, wealth inequality $\eta^b$ decreases. Savers' welfare $V^{s,b}$ increases.*

$$d(\underline{\chi}\tilde{\sigma}) < 0 \implies \begin{cases} dq^b > 0 & d(q^b + p^b) = 0 \\ dB^f > 0 & d\eta^b < 0 \\ dV^{s,b} > 0 & \end{cases}$$

3. *When productivity increases ($a$ increases), the price of private equity $q^b$ rises, the price of public equity $(q^b + p^b)$ rises, wealth inequality $\eta^b$ do not change. Savers' welfare $V^{s,b}$ increases.*

$$da > 0 \implies \begin{cases} dq^b > 0 & d(q^b + p^b) > 0 \\ dB^f > 0 & d\eta^b = 0 \\ dV^{s,b} > 0 & \end{cases}$$

*Proof.* See appendix. ■

| private equity price↑ | public equity price | Leverage | Wealth inequality | welfare of savers |
|---|---|---|---|---|
| Entrepreneurs' relative "impatience" ↓ | — | ↓ | ↑ | ☹ |
| Financial innovation | — | ↑ | ↓ | ☺ |
| Productivity ↑ | ↑ | ↑ | — | ☺ |

Figure 1.7: Rising asset price on inequality in bubble equilibrium

In bubble economy, as figure 1.7 shows, we can theoretically identify the exact shock to asset prices $q^b$ and $(q^b + p^b)$ by looking at the co-movement of the price of private equity $q^b$, the price of public equity $(q^b + p^b)$, and wealth inequality $\overline{\eta}^b$, and conclude how the rising asset prices affect savers' welfare.

**Public equity** In bubble economy, the price of private equity $q^b$ and public equity $(q^b+p^b)$ differ and they do not necessarily move together. The price of public equity (per unit of capital) in steady state is $(\overline{q}^b + \overline{p}^b) = \frac{a}{\rho}$. Only changes to productivity $a$ and the common impatience of the economy $\rho$ affect the price of public equity. In order to identify the shock in bubble economy, we need to look at both the changes of private equity price $q^b$ and public equity price $(q^b + p^b)$.

### 1.6.4 The effect of bubble on comparative statics of inequality

In this section, I compare the comparative statics of inequality in fundamental economy and bubble economy.

**Proposition 12** (Financial shocks)**.** *I consider comparative statics with respect to the volatility of idiosyncratic risk and the tightness of the equity constraint* $\{\tilde{\sigma}, \underline{\chi}\}$*:*

1. *Wealth inequality in response to financial shocks:*

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} > \frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0$$

$$\frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} > 0, \quad \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} > 0$$

*and*

$$\frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} < \frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} \iff \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}(1-\underline{\chi})^2 + 2(1-\underline{\chi}) - 1 > 0$$

2. *Savers' welfare in response to financial shocks:*

$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} < \frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0$$

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < 0, \quad \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0$$

*and*

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} \iff \frac{\tilde{\sigma}}{\delta}\underline{\chi}^2 - 2\underline{\chi} + 1 > 0$$

*Proof.* See appendix. ■

In bubble equilibrium, the total value of asset that do not carry idiosyncratic risks is higher and the idiosyncratic risks on entrepreneurs' balance sheet is lower than in fundamental equilibrium. Hence, the effect of financial innovation on wealth inequality is mitigated by a higher value of safe asset in bubble economy in the first place.

In steady state of a bubble economy, the risk-premium earned by entrepreneurs decreases as the idiosyncratic risk $\tilde{\sigma}$ decreases, the value of bubble relative to the price of private equity also decreases, which further mitigates the decreasing effect of lower idiosyncratic risk on wealth inequality.

For a decreases of the tightness of the equity constraint $\underline{\chi}$, risk-premium decreases, while the value of bubble relative to the price of private equity increases, which amplifies the decreasing effect of lower idiosyncratic risk on wealth inequality.

**Proposition 13** (Discount rate shock)**.** *I consider comparative statics with respect to the relative "impatience" of entrepreneurs* $\{\delta^e\}$*,*

1. *Wealth inequality in response to discount rate shocks:*

$$\frac{\partial \overline{\eta}^f}{\partial \delta^e} < \frac{\partial \overline{\eta}^b}{\partial \delta^e} < 0$$

2. *Welfare in response to discount rate shocks:*

$$\frac{\partial \overline{V}^{s,f}}{\partial \delta^e} < \frac{\partial \overline{V}^{s,b}}{\partial \delta^e} \iff \frac{-1}{1-\overline{\eta}^{C,f}} \frac{\partial \overline{\eta}^{C,f}}{\partial \delta^e} < \frac{-1}{1-\overline{\eta}^{C,b}} \frac{\partial \overline{\eta}^{C,b}}{\partial \delta^e}$$

*where* $\overline{\eta}^C = \frac{(\rho+\delta^e)\overline{\eta}}{\delta^e\overline{\eta}+\rho}$ *is the consumption inequality.*

*Proof.* See appendix. ■

A decrease in the relative "impatience" of entrepreneurs ($\delta^e$) in steady state increases wealth inequality more in fundamental equilibrium than in bubble equilibrium, because the value of bubble "buffers" for savers. A lower $\delta^e$ directly decreases consumption inequality, while the increasing wealth inequality indirectly increases consumption inequality. Both in fundamental economy and bubble economy, a decrease of $\delta^e$ decreases consumption inequality as well as the welfare of savers. Whereas the relative magnitude of decreasing $\delta^e$ on savers' welfare in the two types of economies depends on parameters of the model.

## 1.7 Conclusion

This paper studies the effect or fundamental drivers of rising asset prices on top inequality through the leverage channel. Through the lens of my theory, the observed rising asset prices and rising wealth inequality at the top end in the past a few decades suggest that the declining relative impatience of the super rich entrepreneurs is the main driver of the trend. Savers are worse off.

Taking advantage a stylized model for clear mechanism and sharp intuition, this paper also leaves opportunities for future research on extensions of the model, optimal stabilization policies, as well as empirical and quantitative exercises.

# Chapter 2

# A Theory of Sovereign Bond Safety: Country Size and Equity Rebalancing Channel

## 2.1 Introduction

Sovereign bonds valued as safe assets by global investors pay lower expected returns that can not be compensated by exchange rates movements. Such persistent difference in sovereign bond return, reflecting variations in sovereign bond safety, is also known as the failure of the *uncovered interest rate parity* (UIP). The failure of UIP to hold in data has been a long-standing puzzle in International Finance since the pioneering work of Fama (1984). Moreover, recent literature has documented that the UIP premium reverses sign and that the reversal seems to be systematically correlated with the period of crisis (Corsetti and Marin 2020).

What determines sovereign bonds safety, reflected by their relative returns (UIP premium), in both crisis and normal times? There's yet a unifying theory that jointly explains the sovereign bonds safety both in normal and crisis times, especially through the dynamics in equity markets. Both the international bonds market and equity markets experienced dramatic fall during the period of crisis. However, the literature has either focused on the bond markets to study the exchange rate dynamics while leaving the equity markets unattended (Gabaix and Maggiori 2015 Itskhoki and Mukhin 2021), or on the portfolio rebalancing dynamics in the equity markets solely (Hau and Rey 2008; Camanho, Hau and Rey 2021).

This paper leverage the insights on portfolio rebalancing from the equity markets to generate rich dynamics in the foreign exchange and the sovereign bonds market. We provide a theory that the relative size of the country (measured by GDP) as well as the equity rebalancing channel jointly determine sovereign bond safety. Using a two-country Lucas tree model with equity constraints, we characterize the model mechanism in closed-form and reconcile the observed UIP patterns both in normal and in crisis times. We propose that the interaction between *country-size effect* and the *equity-rebalancing effect* due to equity constraint is the key driver of UIP patterns.

In normal times, the two countries can perfectly share consumption risks through freely adjusting their equity and bond holdings. The *country-size effect* makes the larger country's bond (U.S) a global safe asset in normal times as the larger country constitutes most of total world consumption risks through the international trade and financial market. Therefore, investors are willing to pay a safety premium for the larger country bond (U.S) by receiving lower expected returns. This rationalizes the observed UIP premium during normal times.

In the period of crisis, both countries are constrained in their equity holdings and have to take on more home risks in consumption than they would ideally prefer. However, the two model mechanisms – namely the country-size and equity-rebalancing effect – work differently for the smaller and the larger country in crisis. For investors in the smaller country (G-10), the equity-rebalancing effect *competes* with the country-size effect. If the country-size effect dominates, home bond becomes safer for home investors in crisis and UIP reversal occurs; if the equity-rebalancing effect dominates, the larger country's bond remains safer for home investors and the flight-to-safety occurs. For investors in the larger country (U.S.), the equity-rebalancing effect *collaborates* with the country-size effect and the safety of the larger country's bond is strengthened for home investors in crisis.

Both the country-size spillover mechanism and the equity-rebalancing effect during the period are well-founded by empirical evidence. On country-size effect, we found that the UIP premium has a strong and negative correlation with the relative size of a country within the G-10 currency group, consistent with the empirical evidence of advanced countries from Hassan 2013. By comparison, we didn't find such correlation for currencies in the EME group. On equity-rebalancing effect, we refer to evidence that equity home bias dropped for both developed and emerging countries during the great financial crisis, while the US investors increased their equity home bias (Wynter 2019).

The equity constraint is the key and only departure of our model from a standard two-country Lucas tree model. Each country has to hold at least a fraction of their domestic equity - they can not issue as much domestic equity share as they would like to. The equity constraints deliver observationally equivalent equity home which is well-documented in the literature (French and Poterba 1991; Hau and Helene Rey 2008; Coeurdacier and Helene Rey 2013). In this paper, we argue that shocks that tightens the equity constraint facing home country drives the system in to crisis. That is, the maximum holding of foreign equity by home country decreases during crisis. The equity constraints in our model fall into the balance sheet constraints class that is supported by empirical evidence: 2018 uses banking regulation to test the balance sheet constraints and shows that the balance sheet constraints have impact on asset prices. 2019 provides direct evidence that the risk of balance sheet constraints becoming tighter is priced.

Our model predictions also reconcile with the empirical facts on deviations from the *covered interest rate parity* (CIP) and convenient yields. The failure of CIP implies a breakdown of the no-arbitrage condition, contrasts the friction-less market assumption, and points to models with financial frictions. There is market segmentation during crisis time in our model with equity constraints. Such market segmentation limits arbitrage across markets and law of one price is violated in crisis time, which generates the deviations from CIP.

### 2.1.1 Literature review

This paper builds on and contributes to the study of currency risk premia, safe asset determination, and portfolio rebalancing.

The currency risk premia is related to the discussion of safe asset, UIP puzzles, and exchange rate risk hedging. Gopinath and Stein 2020 shows that a currency that hedges exchange rate risk endogenously has lower return and becomes the dominant currency due to the complementarity in trade invoicing and banking. In their model, exchange rate is modeled as exogenous. Gabaix and Maggiori 2015 studies exchange rate determination in an imperfect financial market with global financiers facing credit constraints and explains the UIP violation, taking the households' Dollar bond deman as given. Itskhoki and Mukhin 2021 introduces segmented international financial market with noise traders into a standard international real business cycle model and accounts for the UIP violation puzzle, taking the noise traders' Dollar bond demand as exogenous. In my model, demand for both bonds and equities are endogenously determined. Farhi and Gabaix 2016 proposes rare disasters as an determinant for exchange rates. In their model, countries' different exposures to disaster, modeled as exogenous parameters, determine their currency risk premia. Less work have looked at the UIP reversal in crisis. Corsetti and Marin 2020 documents the robust fact that the sign of the UIP puzzle changes in crisis and uses rare disaster to explain the UIP reversal in crisis taking the UIP violation in normal times as given.

Existing literature has proposed various fundamental determinants of bond safety: co-ordination of investors(Z. He, Krishnamurthy, and Milbradt 2019, M. Brunnermeier et al., 2011,2016,2017), financial depth(Maggiori 2017), heterogeneous risk aversion coefficients(Gourinchas and Helene Rey 2007), rare disaster and heterogeneous disaster resilience (Farhi and Gabaix 2016, Corsetti and Marin 2020), and country size effect solely (Hassan 2013, I. Martin 2011). While each of the existing theory can explain only one of the empirical facts mentioned above, our paper jointly explain UIP violation in normal times, UIP reversal, flight to safety, CIP deviations and convenience yields in crisis.

The portfolio rebalancing literature has either focused on the bond markets to study the exchange rate dynamics while leaving the equity markets unattended (Gabaix and Maggiori 2015; Itskhoki and Mukhin 2021), or on the portfolio rebalancing dynamics in the equity markets solely (Hau and Hélène Rey 2004; Camanho, Hau, and Hélene Rey 2022). This paper bridges this gap.

Our model builds on classic continuous-time asset pricing framework. Starting from fiction-less models: 2007 solves a two-tree model with perfect substitutable goods because of which there is no space for exchange rate. Pavlova and Rigobon 2007 solves a two-tree model with log-linear preference, which is a knife-edge case of CES consumption where the country size spillover effect does not show up because the fixed expenditure on two goods when consumption is of Cobb-Douglas form. I. Martin 2011 solves the price levels in a two-trees model with general CRRA utility, CES consumption, and shocks following any Levy process whereas I instead focus on optimal portfolio trade-off and solve for intertemporal risk pricing and Euler equations. Continuing to models with financial frictions: Pavlova and Rigobon 2008 builds a center-periphery three-country model with exogenous country size parameters and general portfolio constraints to study contagion and exchange rate movements in crisis. Garleanu and Pedersen 2011 shows that deviations from law of one price emerges in a heterogeneous risk-averse agents model with linear margin constraints.

## 2.2 Motivating Empirical Facts

### 2.2.1 UIP Deviation and Reversal

To fix ideas, let us define UIP premium as the excess return of home currency asset against the U.S. Dollar (foreign currency). The UIP premium in logs is therefore:

$$\mathbb{E}_t[\lambda_{t+h}] \equiv (i_t - i_t^{\text{US}}) \;-\; (\mathbb{E}_t \; s_{t+h} - s_t) \tag{2.1}$$

where $i_t$ and $i_t^{\text{US}}$ are local and U.S. annualized one-year government bond yields; $h$ is the 12-month horizon. Exchange rate $s$ is in units of local currency per USD; an increase in $s$ would imply local currency depreciation against the USD. When $\mathbb{E}_t[\lambda_{t+h}] = 0$, UIP condition holds and there's no excess return from the currency carry trade. If $\mathbb{E}_t \; \lambda_{t+h} > 0$, there's positive excess returns for the currency trade that home currencies and shorts USD; vice versa for $\mathbb{E}_t[\lambda_{t+h}] < 0$.

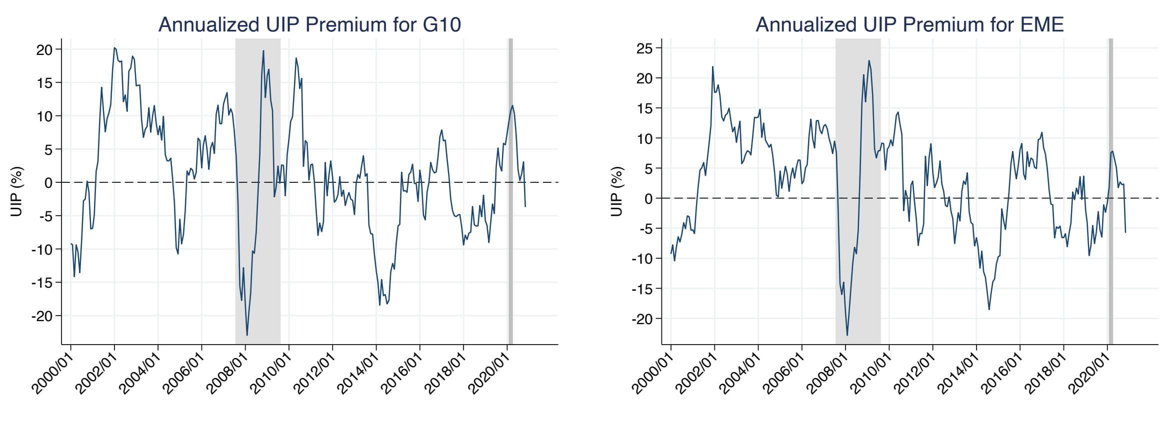


Table 2.1: Annualized UIP Premium for G10 and EME Group

**Note**: This panel presents the annualized average UIP premium of local currency against the USD for G10 currency group (left) and emerging market economies. UIP premiums are in log points and calculated for one-year government bond yields. The grey area corresponds to months of the Great Financial Crisis.

Our work is motivated by the empirical facts on UIP reversals in the period of crisis, as shown in Table 1. The pattern of UIP reversals during the the period of crisis are robust for both advanced and emerging market economies[1]. Average realized local-currency premium – mostly positive in normal times – falls dramatically in the start of the great financial crisis of 2008/09 before reverting back to positive. We also plot UIP premmium at the currency level and found that most currencies share the same feature of UIP reversal during the crisis, as reported in Table A.1 and A.2 in the Appendix.

### 2.2.2 Country Size and UIP Premium

Consistent with Hassan 2013, we found negative and statistically significant correlation between UIP premium and country size measured by GDP for the advanced economies (our G10 currency group). However, the relation between UIP and country size breaks down for the group of emerging market economies, suggesting that the country-size efffect might not be the main determinant of UIP pattern in normal or crisis times for emerging countries. Scatter plots in Table A.5 attest to this claim. OLS regression with year and country fixed effects confirms the statistically significant relation, as reported in Table A.3 and A.4 in the Appendix B.

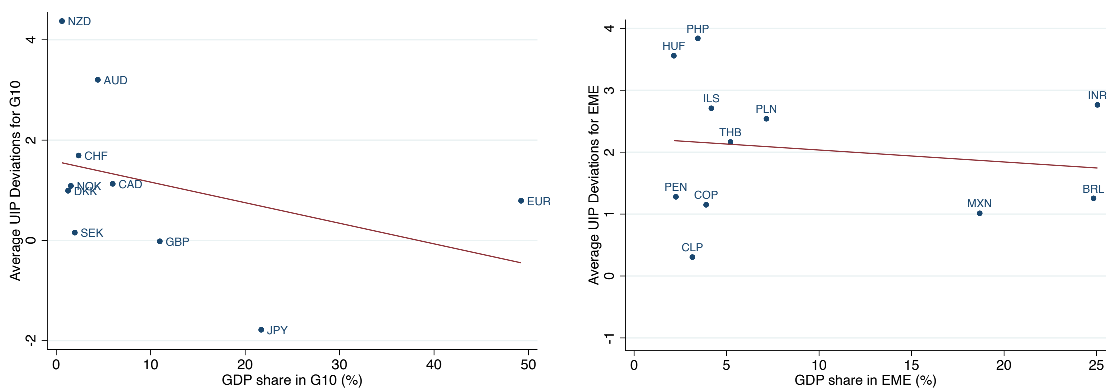


Table 2.2: Average UIP Premium and GDP Share

**Note**: This panel of figures present the scatter plot of average UIP premium against average GDP share for each currency in their respective G10 (left) or EME group (right). UIP deviations are in log points and are annualized and averaged across time for the sample period of 2000-2021. Each dot represent a currency.

We also split the sample before and after the great financial crisis for robustness checks. While the correlation between UIP premium and country size for EMEs remains weak, we found that the relation is robust for G10 group within subsamples. In addition, the slope coefficient becomes more than three times steeper after the great financial crisis. When we

[1]There are 17 emerging market currencies in our sample. They are BRL, CLP, COP, HUF, IDR, ILS, INR, KRW, MXN, MYR, PEN, PHP, PLN, RUB, THB, TRY and ZAR.

plot the average size of the G10 currency (or EME) relative to the size of US, we found that the size of G10 currency size peaks at 2008, as shown in Table A.5 in the appendix.

### 2.2.3 Equity Rebalancing During Crisis

We provide evidence that the equity home bias, defined as the residual share of foreign portfolio investments over the optimal share (approximated using market capitalization), changes over the great financial crisis. Specifically, equity home bias *drops* for most both advanced and emerging market economies alike during the crisis while the *increases* for the United States, as reported in Table 3 below.

Table 2.3: Change in Equity Home Bias During 2008

| G10 | Home Bias Change (%) | EMEs | Home Bias Change (%) |
|---|---|---|---|
| USD | 4.76 | BRL | -0.30 |
| AUD | 2.70 | CLP | -15.53 |
| CAD | -1.92 | COP | 1.50 |
| CHF | 4.90 | HUF | -10.14 |
| DKK | -2.89 | IDR | -0.07 |
| EUR | -3.05 | IDN | -2.23 |
| JPY | 3.26 | MXN | 0.24 |
| NZD | -9.33 | MYR | -1.79 |
| SEK | -9.78 | PHP | 0.25 |
| | | RUB | -0.44 |
| | | THB | -0.05 |
| | | TRY | -0.03 |
| | | ZAR | -2.79 |

**Note**: This table reports the change in equity home bias for selected currencies during 2008. The numbers for equity home bias in the table are from Wynter 2019 and calculated based on the Coordinated Portfolio Investment Survey (CPIS) published by IMF. Home bias is defined as:

$$\text{Home Bias}_i = 1 - \left(\frac{\text{Foreign Portfolio Weight}_i}{\text{Share of Mkt. Cap in the World}_i}\right)$$

The fact that equity home bias drops on average for both the G10 and EME group is the most direct evidence that supports that equity rebalancing effects during the crisis. Unlike the country-size effect that only works the advanced economies, the equity-rebalancing effects are present for both currency groups. During the period of crisis, investors in both G10 and emerging countries increase their holdings of foreign (USD) assets while decrease home-currency assets.[2]

[2]Note that the total change in portfolio shares of foreign equity comes from both active rebalancing and valuation effect.

## 2.3 Model

### 2.3.1 Model Set-up

Time is continuous and infinite horizon, $t \in [0, +\infty)$. There are two countries in the world, home country (denoted by $H$) and foreign country (denoted by $F$). For ease of illustration, I will call home country the UK and foreign country the US.

**Technology** Each country is endowed with a tree producing domestic good. The two trees evolve as follows,

$$\frac{dY_{H,t}}{Y_{H,t}} = \mu_{H,t}\, dt + \sigma_{H,t}\, dZ_{H,t}$$
$$\frac{dY_{F,t}}{Y_{F,t}} = \mu_{F,t}\, dt + \sigma_{F,t}\, dZ_{F,t}$$

where $\{\mu_{H,t}, \mu_{F,t}, \sigma_{H,t}, \sigma_{F,t}\}$ are exogenous parameters (or processes). For simplicity, we assume throughout the paper that $\mu_{H,t} = \mu_{F,t} = \mu$ and $\sigma_{H,t} = \sigma_{F,t} = \sigma$.

**Preferences** In order to highlight my mechanism, I assume homogeneous preference, logarithmic utility, and no consumption home bias for the representative agents of the two countries. The final consumption is a CES aggregate of the two goods produced by the two countries. The expected utility of the representative agent in country $i$, takes the form

$$\mathbb{E} \int_0^\infty e^{-\rho t} \log C_{H,t}\, dt$$

where

$$C_{H,t} = \left[ \alpha^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}}$$

$\alpha$ is the share parameter [3]. $\eta$ is the elasticity of substitution between the two goods, assumed to be greater than 1 and smaller than infinity.

[3] Unlike Pavlova and Rigobon 2008 where $\alpha$ represents the country size, here in my model $\alpha$ is not a key parameter of interest.

**Numeraire** Define 1 unit of the CES basket of total output $\overline{Y}_t$ as numeraire throughout the paper,

$$\overline{Y}_t \equiv \left[\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

Denote the process of total output $\overline{Y}_t$ as

$$\frac{d\overline{Y}_t}{\overline{Y}_t} = \overline{\mu}_t dt + \overline{\sigma}_t dZ_t$$

where $dZ_t = [dZ_{H,t} \;\; dZ_{F,t}]^T$.

**International trade market and exchange rate** The international trade market (of home and foreign goods) is frictionless. Denote $p_t^H$ as the price of home good and $p_t^F$ as the price of foreign good. The real exchange rate is given by the relative price of good 1 and good 2[4],

$$e_t \equiv \frac{p_t^H}{p_t^F} \tag{2.2}$$

and $e_t$ is also the terms of trade in this model. And denote the endogenous process of real exchange rate, $e_t$, as

$$\frac{de_t}{e_t} = \mu_t^e dt + (\sigma_t^e)^T dZ_t$$

**Equity** Each country can issue domestic equity shares in unit supply. The equities are risky claims to domestic trees. Denote $S_t^H$ and $S_t^F$ as the total value of domestic equity and foreign equity respectively. Define $\chi_t^{H,H}$ as the the share of home stock market (apple tree) held by home investor, $\chi_t^{H,F}$ as the share of foreign stock market (orange tree) held by home investor. And similarly define $\chi_t^{F,H}$ as the share of home stock market (apple tree) held by foreign investor and $\chi_t^{F,F}$ as the share of foreign stock market (orange tree) held by foreign investor.

**Equity constraint** Importantly, equity constraint for home country:

$$0 \leq \chi_t^{H,F} \leq \overline{\chi}^F \tag{2.3}$$

[4]An increase of $e_t$ corresponds to an appreciation of home currency relative to foreign currency.

This equation is saying that home investor can not hold more than $\overline{\chi}^F$ share of foreign equity, nor short-sell foreign equity[5].

And similarly we have equity constraint for foreign country:

$$0 \leq \chi_t^{F,H} \leq \overline{\chi}^H \tag{2.4}$$

That is, foreign investor can not hold more than $\overline{\chi}^H$ share of home equity, nor short-sell home equity.

**Sovereign Bond** Each country can issue (sovereign) bond in zero net supply. The bonds, denoted as $B_t^H$ and $B_t^F$, are instantaneously *risk-free in domestic goods* but *not* risk-free in terms of numeraire[6]. The price of home (foreign) bond $B_t^H$ ($B_t^F$) in terms of numeraire is the same as the price of home (foreign) good $p_t^H$ ($p_t^F$). Denote $B_t^{H,H}$ as the home bond held by home investors[7] and $B_t^{F,H}$ as the home bond held by foreign investors. And denote $B_t^{H,F}$ as the foreign bond held by home investors and $B_t^{F,F}$ the foreign bond held by foreign investors.

**Asset returns** I introduce notations for asset returns which are *endogenous* processes. Recall that $B_t^H$ is instantaneously risk-less bond in home good and denote the return process of home bond (in terms of numeraire) as:

$$dr_t^{B^H} = \frac{d(p_t^H B_t^H)}{p_{H,t} B_t^H} = (\mu_{p^H,t} + r_t^H)\, dt + \sigma_{p^H,t}\, dZ_t$$

where $\mu_{p^H,t}$ and $\sigma_{p^H,t}$ are given by the endogenous process

$$\frac{dp_t^H}{p_t^H} = \mu_{p^H,t} t\, dt + \sigma_{p^H,t}\, dZ_t$$

Similarly denote the return process of foreign bond (in terms of numeraire) as:

$$dr_t^{B^F} = \frac{d(p_t^F B_t^F)}{p_{F,t} B_t^F} = (\mu_{p^F,t} + r_t^F)\, dt + \sigma_{p^F,t}\, dZ_t$$

Recall that $S_t^H$ is the total value of home equity and define $q_t^H$ as the per unit price of home

[5]Here one can replace the lower bound 0 to a negative number, say $\underline{\chi}^F$. The key thing is that $\chi_t^{H,F}$ (the share of foreign equity held by home investor) is lower bounded.

[6]Their returns are subject to exchange rate risks through price changes

[7]$B_t^{H,H} > 0$ means lending and $B_t^{H,H} < 0$ means borrowing

equity in terms of numeraire $\overline{Y}_t$, that is, $S_t^H = q_t^H \overline{Y}_t$. And postulate the endogenous process of $q_t^H$ as follows

$$\frac{dq_t^H}{q_t^H} = \mu_{q^H,t}dt + \sigma_{q^H,t}dZ \tag{2.5}$$

The return of home equity in terms of numeraire is given by

$$dr_t^{S^H} = \underbrace{\frac{p_t^H Y_{H,t}}{q_t^H \overline{Y}_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^H \overline{Y}_t)}{q_t^H \overline{Y}_t}}_{\text{capital gain}}$$

and similarly

$$dr_t^{S^F} = \underbrace{\frac{p_t^F Y_{F,t}}{q_t^F \overline{Y}_t}dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^F \overline{Y}_t)}{q_t^F \overline{Y}_t}}_{\text{capital gain}}$$

**Forward market** Since there is no friction on the bond markets, there naturally exists a FX forward market for home bond and foreign bond. Home investor can enter a FX forward contract (long in home currency and short in foreign currency) with zero cost today which will deliver an instantaneous return $dr_t^{B^H} - dr_t^{B^F}$.[8].

**Wealth and portfolio shares** I introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the aggregate wealth of home country as $W_t^H$ and the aggregate wealth of foreign country as $W_t^F$.

Denote $\theta_t^{H,S^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H}$ as the portfolio share of home equity for home country, $\theta_t^{H,S^F} = \frac{\chi_t^{H,F} S_t^F}{W_t^H}$ as the portfolio share of foreign equity for home country. And similarly, denote $\theta_t^{F,S^H} = \frac{\chi_t^{F,H} S_t^H}{W_t^F}$ as the portfolio share of home equity for foreign country and $\theta_t^{F,S^F} = \frac{\chi_t^{F,F} S_t^F}{W_t^F}$ as the portfolio share of foreign equity for foreign country.

We can similarly define portfolio shares of bonds in home and foreign country. Denote $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$ as the portfolio share of home bond for home country, $\theta_t^{H,B^F} = \frac{p_t^H B_t^{H,F}}{W_t^H}$ as the portfolio share of foreign bond for home country, And similarly, denote $\theta_t^{F,B^H} = \frac{p_t^H B_t^{F,H}}{W_t^F}$ as the portfolio share of home bond for foreign country and $\theta_t^{F,B^F} = \frac{p_t^F B_t^{F,F}}{W_t^F}$ as the portfolio share of foreign bond for foreign country.

[8]In equilibrium, investors will exactly do so as discussed in appendix

**Country Size** Define relative size of home country as follows.

$$s_t = \frac{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = \alpha^{\frac{1}{\eta}} \left( \frac{Y_{H,t}}{\overline{Y}_t} \right)^{\frac{\eta-1}{\eta}} \tag{2.6}$$

**Optimization problems** The optimization problem for home country is as follows:

$$\begin{aligned} \max_{\{C_{HH,t}, C_{HF,t}, \chi_t^{H,H}, \chi_t^{H,f}, \theta_t^{H,B^H}, \theta_t^{H,B^F}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[ \int_0^\infty e^{-\rho t} \log\left( \left[ \alpha^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \right) dt \right] \\ \text{s.t.} \quad \frac{dW_t^H}{W_t^H} &= \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ &\quad - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \\ 1 &= \frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} + \theta_t^{H,B^H} + \theta_t^{H,B^F} \\ 0 &\le \chi_t^{H,F} \le \overline{\chi}^F \end{aligned} \tag{2.7}$$

The optimization problem for foreign country is similar and discussed in appendix.

**Market clearing conditions** Home equity market clears,

$$\chi_t^{H,H} + \chi_t^{F,H} = 1 \tag{2.8}$$

Foreign equity market clears,

$$\chi_t^{H,F} + \chi_t^{F,F} = 1 \tag{2.9}$$

Home bond market clears,

$$B_t^{H,H} + B_t^{F,H} = 0 \tag{2.10}$$

And foreign bond market clears,

$$B_t^{H,F} + B_t^{F,F} = 0 \tag{2.11}$$

Total consumption of home (foreign) good equals total production of home (foreign) good,

$$C_{HH,t} + C_{FH,t} = Y_{H,t} \tag{2.12}$$

$$C_{HF,t} + C_{FF,t} = Y_{F,t} \tag{2.13}$$

## 2.4 Complete market model

Before solving the model with equity constraints, it is useful to solve for the complete market case which works as a clear illustration of the country size spillover effect.

Postulate two stochastic discount factor processes for the two countries, $\xi_{H,t} = e^{-\rho t}\frac{1}{C_{H,t}}$ and $\xi_{F,t} = e^{-\rho t}\frac{1}{C_{F,t}}$, as

$$\frac{d\xi_{H,t}}{\xi_{H,t}} = -r^f_{H,t}\,dt - m^T_{H,t}\,dZ_t$$

$$\frac{d\xi_{F,t}}{\xi_{F,t}} = -r^f_{F,t}\,dt - m^T_{F,t}\,dZ_t$$

respectively. $m_{H,t}$ is the vector of risk prices in home country and also the consumption risk[9] in the logarithmic utility case.

In the complete market case, there exists a unique stochastic discount factor $\xi_t$ such that $\frac{d\xi_{H,t}}{\xi_{H,t}} = \frac{d\xi_{F,t}}{\xi_{F,t}} = \frac{d\xi_t}{\xi_t}$.

### 2.4.1 Sovereign bond safety and country size spillover effect

**Proposition 14** (Sovereign bond safety)**.** *Expected return difference between home bond and foreign bond is given by*

$$\frac{\mathbb{E}_t\left[dr_t^{B^H} - dr_t^{B^F}\right]}{dt} = m^T_{H,t}\sigma^e_t = m^T_{F,t}\sigma^e_t \tag{2.14}$$

*where $dr_t^{B^H}$ is the return process for home bond, $dr_t^{B^F}$ is the return process for foreign bond, $\sigma^e_t$ is the exchange rate risk.*

*If home country's consumption risk is positively correlated with exchange rate risk (domestic consumption is low when domestic currency depreciates), then home bond earns a positive risk premium.*

*If home country's consumption risk is negatively correlated with its exchange rate (domestic consumption is high when domestic currency depreciates), then home bond earns a negative safety premium.*

*Proof.* see appendix ■

[9] Consumption risk of a country is defined as the volatility vector of consumption process of that country, $\frac{dC_{H,t}}{C_{H,t}}$.

The intuition is as follows: A bond is considered safe if it has high value when consumption is low, because the bond insures investors against bad times. In my example, US treasury pays lower expected return than UK government bond if GBP depreciates against USD when consumption is low. Because in this case, US treasury is a good hedge for consumption risk and is viewed as safe, while UK government bond does not hedge consumption risk and is viewed as risky. Lustig and Verdelhan 2007 provides empirical evidence for proposition 24.

In my model, uncertainty comes from production fluctuations of the trees. When UK production declines due to a negative shock, the supply of UK good declines, and the relative price of UK good should go up, implying a higher expected return of UK bond. However, this is not the whole story for bond safety. Because the final consumption is an aggregate of both countries' goods, another competing force emerges: the demand for US good increases because of consumption smoothing motive. The final consumption shifts more towards US good than before due to a supply drop of UK good. This positive demand shock for US good will put upward pressure on the expected return of US bond. The next proposition shows that the relative magnitude of the supply force and demand force is determined by the relative country size.

**Proposition 15** (Country size spillover effect)**.** *Solving for* (2.14)*, we have*

$$\frac{\mathbb{E}_t[dr_t^{B^H} - dr_t^{B^F}]}{dt} = \overline{\sigma}_t^T \sigma_t^e = \begin{bmatrix} s_t\sigma_H & (1-s_t)\sigma_F \end{bmatrix} \begin{bmatrix} -\frac{1}{\eta}\sigma_H \\ \frac{1}{\eta}\sigma_F \end{bmatrix} \tag{2.15}$$

$$= \frac{1}{\eta}(-s_t\sigma_H^2 + (1-s_t)\sigma_F^2) \tag{2.16}$$

*and the safety threshold*

$$s^C = \frac{\sigma_F^2}{\sigma_H^2 + \sigma_F^2} \tag{2.17}$$

*If* $s_t < s^C$*, home country is a relatively small country and home bond is riskier than foreign bond.*

*If* $s_t > s^C$*, home country is a relatively large country, country bond is safer than foreign bond.*

*Proof.* see appendix ■

Country size spillover effect states that larger country's bond is safer. US treasury is safer than UK government bond because the size of US economy is a larger than the size of UK economy. Since US contributes a larger share to world consumption, the world consumption risk also consists largely US risk. US treasury becomes a safe asset and pays lower expected return because it is a better hedge for world consumption risk. This is often referred to as the exorbitant privilege of the US Dollar. Going back to the supply force and demand force discussed earlier: the larger the country's share in the world consumption, the larger the magnitude of supply or demand change of its good. When the small country becomes smaller, the large country's production becomes more dominant in world consumption, strengthening the hedging benefit of the large country's bond. On the other hand, when the small country grows larger, world consumption depends less on the large country's production, reducing the hedging benefit of the large country's bond. Country size spillover effect is stronger when there is more asymmetry in $s_t$ and $1 - s_t$. Expected return differences of sovereign bonds are sizable and persistent across countries as discussed in Hassan 2013, which also provides empirical evidence for country size spillover effect.

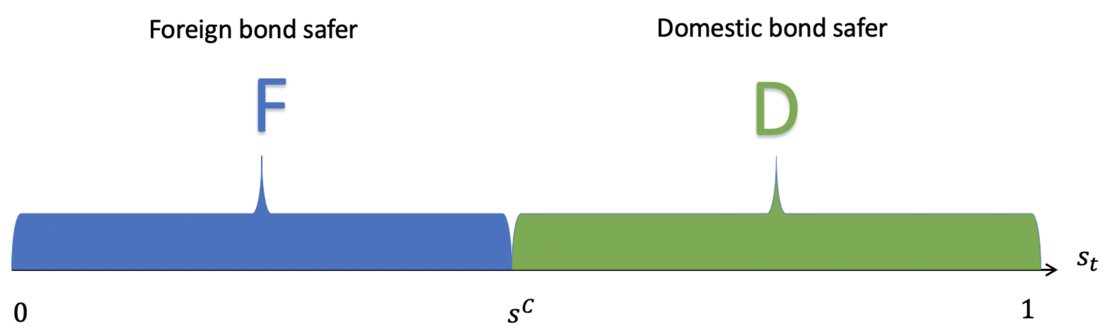


Figure 2.1: safety region

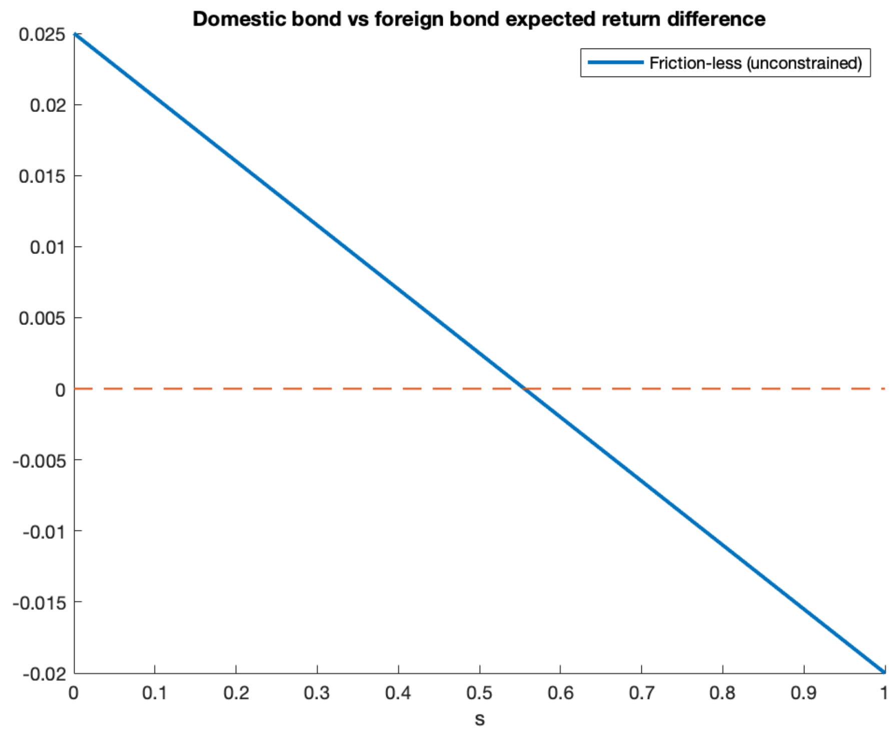


Figure 2.2: Domestic and foreign bond expected return difference

Note: $\sigma_H^2 = 0.04$, $\sigma_F^2 = 0.05$, $\eta = 2$ and $s^C = \frac{5}{9}$

### 2.4.2 Persistence and asymmetry

In the simplest complete market case, on top of country size spillover effect, there are also persistence effect and asymmetry effect on prices of risks through changes in country size $s_t$.

The volatility [10] of country size $s_t$ and the prices of risks during normal times are given by

$$\sigma_{s_t} = \frac{\eta - 1}{\eta}(1 - s_t)\begin{bmatrix} \sigma_H \\ -\sigma_F \end{bmatrix} \tag{2.18}$$

and

$$m_{H,t} = m_{F,t} = \overline{\sigma}_t = \begin{bmatrix} \underbrace{s_t\sigma_H}_{\text{price of home country risk}} \\ \underbrace{(1 - s_t)\sigma_F}_{\text{price of foreign country risk}} \end{bmatrix} \tag{2.19}$$

[10]Throughout this paper, I denote the volatility of the process $\frac{dX_t}{X_t}$ as volatility of $X_t$.

**Persistence** A temporary negative shock to home country's production immediately reduces home country's relative country size, decreases the price of home country risk. In addition, a smaller $s_t$ also affects the magnitude of future shocks on country size $s_t$ as well as prices of risks $\overline{\sigma}_t$. Unlike classic works in macro-finance literatrue (Bernanke, Gertler, and Gilchrist 1999 etc), where the persistence of a temporary shock is due to changes in current and future investment, the persistence here in my benchmark model without investment is purely from changes in country size $s_t$.

**Asymmetry** A negative shock to home country production affects both price of home country risk and price of foreign country risk through country size spillover effect. This shock affect prices of risks asymmetrically through changes in country size $s_t$. As in equation (2.19), the decline of home risk price is mitigated by the smaller size of home country, $s_t$, while increase of foreign country risk price is amplified by the larger size of foreign country, $1 - s_t$. The same shock thus affects prices of the two countries' bonds and equities asymmetrically.

## 2.5 Model with equity constraints

Adding another key ingredient to the model, the equity constraints, I proceed in two steps.

First step, I explore what happens with only one equity constraint for foreign country's holding of home equity, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^H$. There is a maximum limit on home equity share held by foreign investors and no short-selling of home country's equity is allowed. As shown in the following proposition 16, the two countries can still perfectly share consumption risk and have the same prices of risks as in the complete market case.

Second step, I explore the full model with equity constraints for both countries. There exists an endogenous crisis regime in the model which results in asymmetry and instability of the system.

To highlight the mechanism and simplify some algebra for illustration purpose, I assume symmetric parameters for the two trees $\mu_1 = \mu_2 = \mu$ and $\sigma_H = \sigma_F = \sigma$ hereafter. And taking advantage of symmetry, I focus on analysing home country (Home). The symmetric assumption makes sense in the example of UK and US, as the two countries have similar growth rates and volatilities. And we will focus on empirically relevant case where home country (UK) is a small country relative to foreign country (US).

### 2.5.1 Normal regime

**Proposition 16.** *With only one equity constraint for foreign country's holding of home equity, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^H$, the two countries can perfectly share consumption risk and replicate the complete market case result in the sense that proposition 24 and proposition 25 still hold true.*

*Proof.* see appendix ■

An intuitive way to look at proposition 16 is to count the risks and assets. There are two sources of risks, from the two trees. Even with one equity holding constraint, there are still another three assets that can be freely traded which can span all the possible states of the world. So investors in the two countries can still replicate first best risk-sharing through portfolio re-balancing. Similar to the complete market case, there is indeterminacy in the model with respect to portfolio holdings but not asset returns.

**Proposition 17.** *With the only equity constraint for foreign country's holding of home equity, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^H$, a special specification for discount rate $\rho = (\frac{\eta-1}{\eta}\sigma)^2$, initial condition $s_0$, home country's equity shares and bond holdings are given by*

$$
\begin{aligned}
\chi_t^{H,H} &\in [1-\overline{\chi}^H, 1] \\
\chi_t^{H,F} &= \frac{n_0 - \chi_t^{H,H}\rho q_t^H}{1-\rho\chi_t^{H,H} q_t^H} \\
\theta_t^{H,B^H} &= \frac{\rho (q_t^H)'(s_t) s_t (1-s_t)(\chi_t^{H,H} - \chi_t^{H,F})(\eta - 1)}{n_0} \\
\theta_t^{H,B^F} &= -\theta_t^{H,B^H}
\end{aligned}
$$

*where*

$$q_t^H(s) = \frac{1}{2\rho}\Big(1 + \frac{1-s}{s}\ln(1-s) - \frac{s}{1-s}\ln(s)\Big)$$

*is the per unit price of home equity and taking derivatives with respect to $s_t$, we have*

$$(q_t^H)'(s_t) = -\frac{1}{2\rho}\frac{1}{s(1-s)}\Big(1 + \frac{1-s}{s}\ln(1-s) + \frac{s}{1-s}\ln(s)\Big)$$

*And $n_0 = q^H(s_0)$ is the initial wealth share of home country.*

*Proof.* see appendix ■

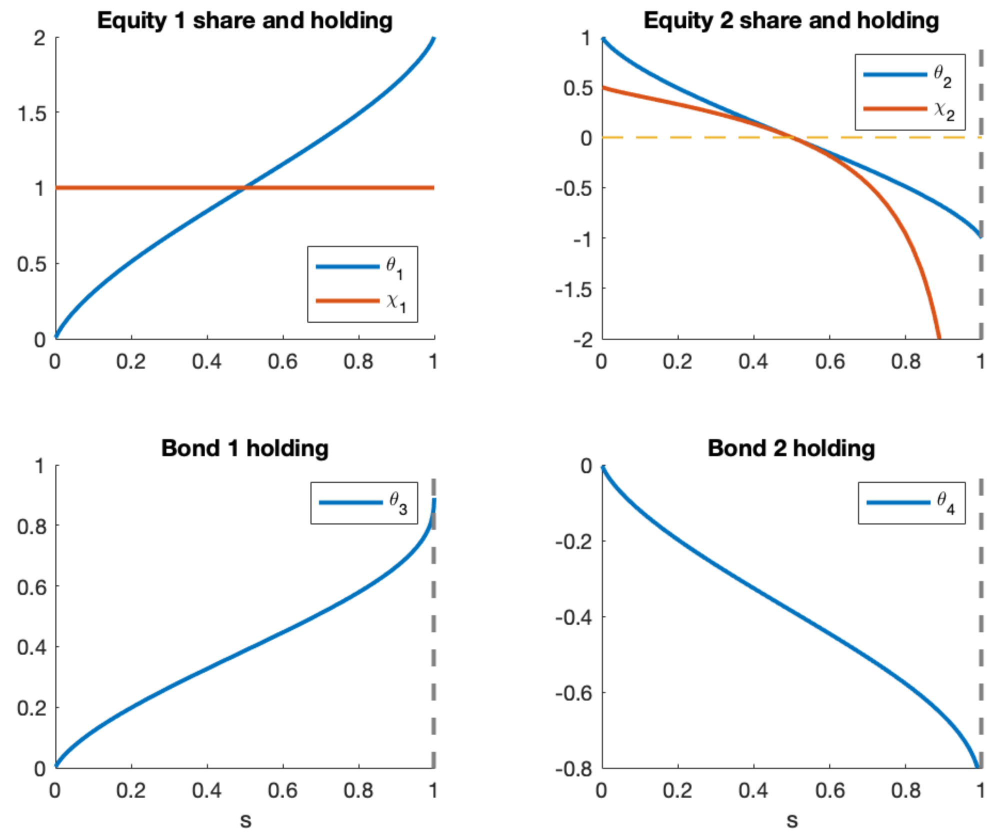


Figure 2.3: home country portfolio

Note: $\chi_t^{H,H} = 1$, $\sigma^2 = 0.04$, $\eta = 2$

From proposition 17 and figure 2.3, we see that the net borrowing in bonds between the two countries is zero. The two countries smooth their consumption by holding equities and use bonds to help achieve perfect risk-sharing. Both countries go short in foreign bond and long in domestic bond. Because domestic bond is a better hedge for domestic risk, the two countries can offload the extra domestic risk from domestic equity holding requirement by lending in domestic bond and borrowing in foreign bond.

Comparing figure 2.2 and the right-bottom panel of figure 2.3, we see that countries borrow more in foreign bond when their country size grows larger and domestic bond becomes safer, fixing home country's holding of domestic equity share. Because when a country grows larger, its domestic equity price increases, leading to a heavier portfolio weight on domestic equity and thus more domestic risk exposure, which requires more hedging. This is consistent with the empirical fact documented by 2020. Until now, the model with only country size spillover effect can explain the UIP violation in normal times and find empirical support from earlier work. However, the model does not have space for crisis yet and is thus silent about what happens in crisis.

### 2.5.2 Crisis regime

Moving on to second step, with equity holding constraints for both countries, an endogenous crisis regime emerges and the system moves into the crisis regime when one country falls too small.

**Proposition 18** (Crisis regime)**.** *The system moves in to crisis regime if* $s_t \in [0, s^U] \cup [1 - s^U, 1]$*, where* $s^U$ *is the crisis boundary and solves*

$$q_1(s^U) = \frac{n_0 - \overline{\chi}^F}{\rho(1 - \overline{\chi}^F)} \tag{2.20}$$

*If* $s_t \in [0, s^U]$*, we have*

$$\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^F$$

*If* $s_t \in [1 - s^U, 1]$*, we have*

$$\chi_t^{H,H} = 1 - \overline{\chi}^H, \quad \chi_t^{H,F} = 0$$

*Proof.* see appendix. ■

The crisis boundary $s^U$ is the left margin where both countries' equity holding constraints bind and $1 - s^U$ is the right margin where the equity holding constraints bind in the opposite direction. When $s_t < s^U$, the two countries can perfectly share exchange rate risk through freely adjusting their equity holdings and trading on the FX market. The gains and losses from FX market will be delivered by capital flows induced by equity trading, until both equity constraints bind. I refer to the constrained region as *crisis regime*. In crisis regime, risk-sharing is limited, and asset returns vary discontinuously from in normal regime due to the constraint on equity rebalancing.

**Safety spectrum**

In crisis regime, equity reblancing is constrained and risk-sharing is limited. This market segmentation drives a wedge between normal time SDF and crisis time SDF, thus a wedge of risk prices between normal times and crisis time. We refer to the effect of the constraints on equity reblancing as the *equity rebalancing effect*[11].

**Proposition 19** (equity rebalancing effect)**.** *In crisis region, there exists a wedge between the normal time SDF and crisis time SDF, due to lack of equity rebalancing. For home country investors, denote this wedge as* $\sigma_{n_t}$*.*

*If* $0 < s_t < s^U$*,*

$$\sigma_{n_t} = \frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\frac{\eta-1}{\eta}(1-s_t)\begin{bmatrix}\sigma_H \\ -\sigma_F\end{bmatrix} \tag{2.21}$$

*If* $1 - s^U < s_t < 1$*,*

$$\sigma_{n_t} = \frac{\eta-1}{\eta}(1-s_t)\begin{bmatrix}\sigma_H \\ -\sigma_F\end{bmatrix} \tag{2.22}$$

*symmetrically for foreign investors.*

*Proof.* see appendix ■

With two equity constraints and the crisis region, the model exhibits a safety spectrum for each country with four regions identified by three key safety thresholds.

**Proposition 20** (Safety spectrum)**.** *With equity holding constraints for both countries,* $0 \leq \chi_t^{F,H} \leq \overline{\chi}^H$ *and* $0 \leq \chi_t^{H,F} \leq \overline{\chi}^F$*, and reasonable parameter restrictions on* $(\overline{\chi}^H, \overline{\chi}^F, \eta)$*, there are three key thresholds, normal time safety threshold* $s^C$*, crisis boundary* $s^U$*, and crisis time safety threshold* $s^A$*,*

$$s^C = \frac{1}{2} \tag{2.23}$$

$$q^H(s^U) = \frac{n_0 - \overline{\chi}^F}{\rho(1-\overline{\chi}^F)} \tag{2.24}$$

$$\frac{2(1-\overline{\chi}^F)s^2 + (2\overline{\chi}^F\eta + (1-\overline{\chi}^F)(\eta-2))s - \eta\overline{\chi}^F}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F} = 0 \tag{2.25}$$

*such that*

$$0 < s^A < s^U < s^C \tag{2.26}$$

[11] To be precise, this is “the lack of equity rebalacing” effect

*When $s^U < s_t < 1 - s^U$, the system stays in normal regime, country size determines bond safety as in proposition 25:*

*If $s^U < s_t < s^C$, home country is a relatively small country, home country's bond is risky.*

*If $s^C < s_t < 1 - s^U$, home country is a relatively large country, home country's bond is safe.*

*When $0 < s_t < s^U$ or $1 - s^U < s_t < 1$, the system moves into crisis regime. Country size spillover effect and the equity rebalancing effect will jointly determin sovereign bond safety.*

*When $0 < s_t < s^U$, the system moves into crisis regime where country size spillover effect competes with equity rebalancing effect:*

*If $s^A < s_t < s^U$, equity rebalancing effect dominates, home country's bond is safe for domestic investors.*

*If $0 < s_t < s^A$, country size spillover effect dominates, home country's bond is risky for domestic investors.*

*When $1 - s^U < s_t < 1$, the system moves into crisis regime where country size spillover effect joins forces with equity rebalancing effect: home country's bond is safe for domestic investors.*

*Proof.* see appendix ■

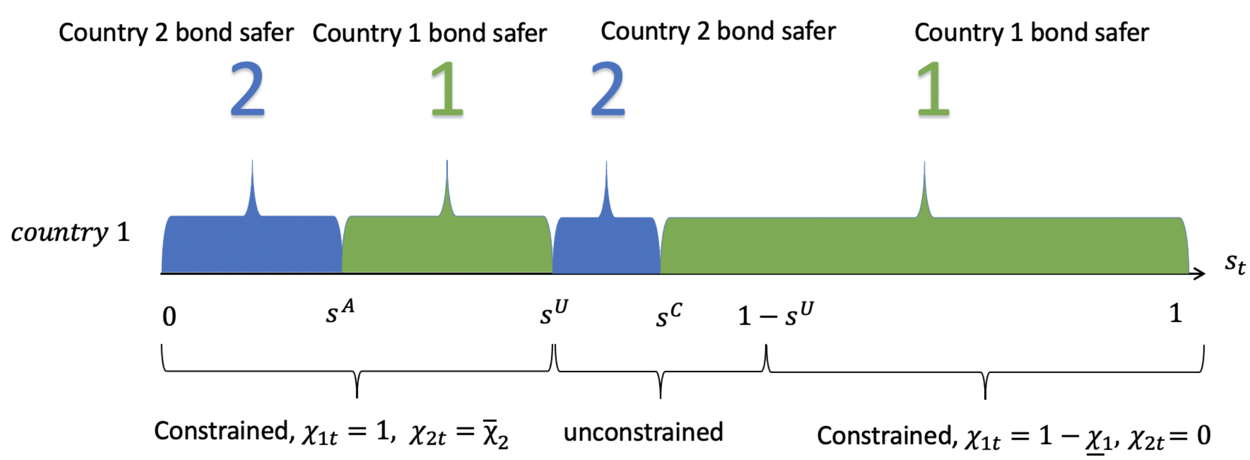


Figure 2.4: safety spectrum for home country investors

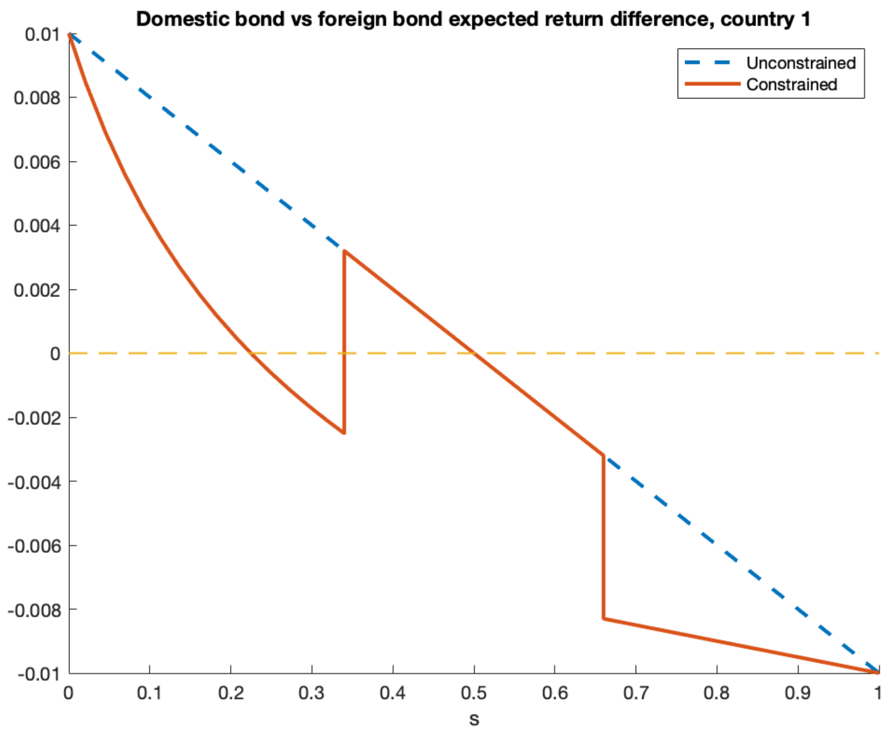


Figure 2.5: Home country: expected return difference of bonds

Note: $\eta = 4$, $1 - \overline{\chi}^H = 1 - \overline{\chi}^F = 0.8$, $\sigma^2 = 0.04$, $s^A = 0.225$, $s^U = 0.34$, $s^C = 0.5$

As shown in figure 2.4, cut by the three thresholds, there are four regions along the safety spectrum. Blue regions represent where foreign bond is safer for domestic investors, and green regions represent where domestic bond is safer. Figure 2.5 shows the expected return difference between domestic bond and foreign bond for home country investors.

In normal regime where $s^U < s_t < 1 - s^U$, there is only the familiar country size spillover effect: the larger country's bond is safer. If home country's size continues falling below $s^U$, the risk-sharing is limited by the equity holding constraints.

In crisis regime, investors in both countries are forced to hold more domestic risk and less foreign risk compared to the perfect risk-sharing scenario in normal regime. Because of limited risk-sharing, domestic bond becomes safer for domestic investors in crisis regime than in normal regime, as it is a better hedge for domestic risk.

equity rebalancing effect improves safety of the domestic bond for domestic investors in crisis regime while country size spillover effect improves safety of the larger country's bond. So in crisis regime, country size spillover effect competes with equity rebalancing effect for the smaller country's investors but collaborates for the larger country's investors.

If the smaller country's size falls in between $s^A$ and $s^U$, equity rebalancing effect dominates country size spillover effect. The smaller country's domestic bond is safe for domestic investors. If the smaller country falls below $s^A$, country size spillover effect dominates equity rebalancing effect. The smaller country's domestic bond is risky for domestic investors.

Whereas for the larger country when $1-s_t > s^C$, its domestic bond is always safe for domestic investors. The safety of the larger country's domestic bond is discontinuously strengthened when $1-s_t > s^U$ due to equity rebalancing effect.

### Domestic amplification

equity rebalancing amplifies the effect of domestic shock. This domestic amplification exists both in the "time series" (compared to normal regime) and in the "cross secrion" (compared to foreign country).

In the crisis regime $[0, s^U]$, the prices of risks for home country investors is given by

$$m_{H,t} = \frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\sigma_{s_t} + \overline{\sigma}_t = \begin{bmatrix} \underbrace{\left[\frac{(1-\overline{\chi}^F)}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\frac{\eta-1}{\eta}(1-s_t)+1\right] s_t\sigma_H}_{\geq s_t\sigma_H,\text{ normal time price of home country risk}} \\ \underbrace{\left[1-\frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t+\overline{\chi}^F}\frac{\eta-1}{\eta}\right](1-s_t)\sigma_F}_{\leq (1-s_t)\sigma_F,\text{ normal time price of foreign country risk}} \end{bmatrix} \tag{2.27}$$

and the prices of risks for foreign country investors is given by

$$m_{F,t} = \sigma_{1-s_t} + \overline{\sigma}_t = \begin{bmatrix} \underbrace{\frac{1}{\eta}s_t\sigma_H}_{\leq s_t\sigma_H,\text{ normal time price of home country risk}} \\ \underbrace{\left[\frac{\eta-1}{\eta}s_t + (1-s_t)\right]\sigma_F}_{\geq (1-s_t)\sigma_F,\text{ normal time price of foreign country risk}} \end{bmatrix} \tag{2.28}$$

For home country investors, compared to in normal times, the effect of a shock on domestic risk price is amplified by the factor

$$\frac{(1-\overline{\chi}^F)}{(1-\overline{\chi}^F)s_t+\overline{\chi}^F}\frac{\eta-1}{\eta}(1-s_t)+1>1$$

and the effect of a shock on foreign risk price is mitigated by the factor

$$1 - \frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\frac{\eta - 1}{\eta} < 1$$

Similarly for foreign country investors, the effect of a shock on domestic risk price is amplified by the factor

$$\frac{\eta - 1}{\eta}\frac{s_t}{1-s_t} + 1 > 1$$

and the effect of a shock on foreign risk price is mitigated by the factor

$$\frac{1}{\eta} < 1$$

In crisis regime, domestic risk price response to a shock is amplified and foreign risk price response to a shock is mitigated compared to in normal times due to lack of equity rebalancing. Domestic amplification improves the hedging benefit of domestic bond in crisis compared to in normal times.

In another dimension, comparing the prices of risks between home country investors and foreign country investors, we have

$$\left[\frac{(1-\overline{\chi}^F)}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\frac{\eta - 1}{\eta}(1-s_t) + 1\right] s_t\sigma_H > s_t\sigma_H > \frac{1}{\eta}s_t\sigma_H$$

where the first term is domestic risk price for home country investors in crisis regime $\left[0, s^U\right]$, the second term is home country risk price in normal regime $\left[s^U, s^C\right]$, and the third term is foreign risk price for foreign country investors in crisis regime $\left[0, s^U\right]$. And similarly

$$\left[1 - \frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F}\frac{\eta - 1}{\eta}\right](1-s_t)\sigma_F < (1-s_t)\sigma_F < \left[\frac{\eta-1}{\eta}s_t + (1-s_t)\right]\sigma_F$$

where the first term is foreign risk price for home country investors in crisis regime $\left[0, s^U\right]$, the second term is foreign country risk price in normal regime $\left[s^U, s^C\right]$, and the third term is domestic risk price for foreign country investors in crisis regime $\left[0, s^U\right]$. In crisis regime, domestic investors hold more domestic risk than foreign investors and thus require a higher risk premium than foreign investors. Domestic amplification drives up domestic risk price and pushes down foreign risk price in crisis for investors in both countries.

### Domestic and global safety

In crisis regime, because of domestic amplification, domestic safety status of bonds may or may not coincide with global safety status of bonds. As shown in the following proposition 21, the smaller country's bond is domestically safe in mild crisis when country sizes are mildly asymmetric and the larger country's bond is globally safe in deep crisis when country sizes are sufficiently asymmetric.

**Proposition 21** (Domestic and global safety)**.** *Assume that home country is the smaller country,* $s_t < s^C$*.*

*The smaller country's bond is domestically safe if* $s_t \in [s^A, s^U]$*.*

*The larger country's bond is globally safe if and only if* $s_t \in [0, s^A] \cup [s^U, s^C]$*.*

*Proof.* see appendix. ■

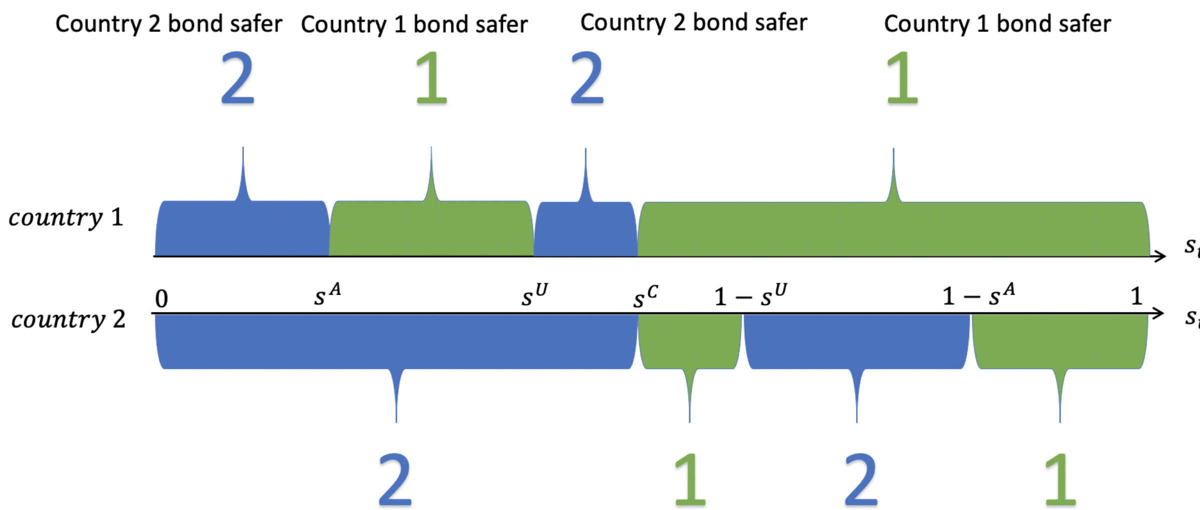


Figure 2.6: global spectrum, symmetric case

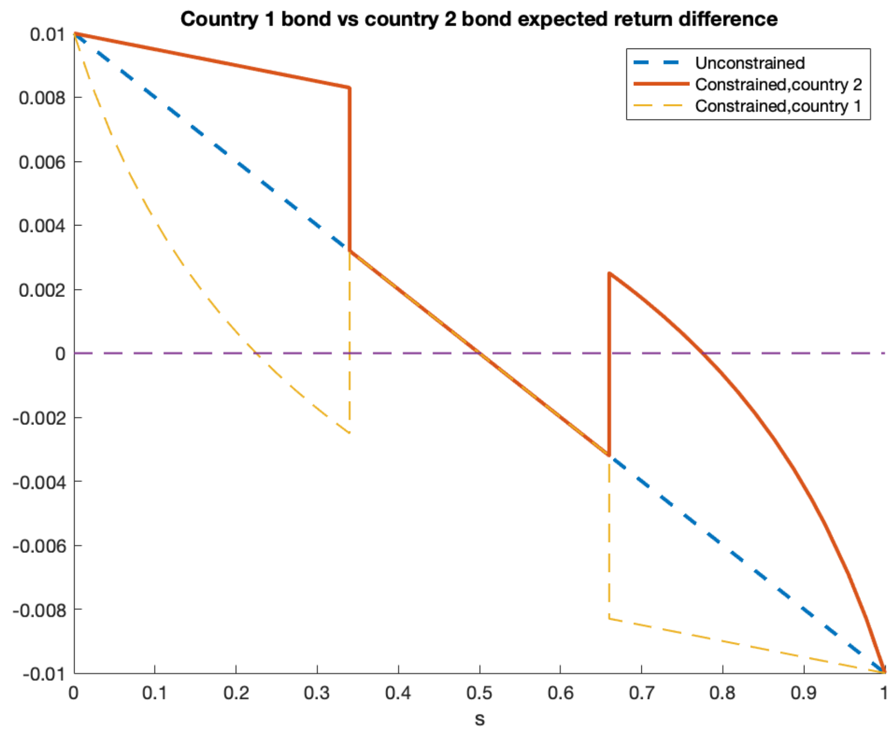


Figure 2.7: Home and foreign country: expected return difference of bonds

Note: $\eta = 4$, $1 - 1 - \overline{\chi}^H = \overline{\chi}^F = 0.2$, $\sigma^2 = 0.04$, $s^A = 0.225$, $s^U = 0.34$, $s^C = 0.5$

In crisis regime, biding equity constraints drives a wedge between SDFs of home and foreign investors. Different pricing kernels result in different returns for the same asset. Investors in the two countries disagree on expected return difference between bonds in crisis, as shown in figure 2.7. Technically, the heterogeneity in SDFs comes from heterogeneity in constraints. The two countries face asymmetric complementary margin requirements on their equity holdings which results in asymmetric Lagrangian multipliers associated with the binding constraints.

For the smaller country, when falling into crisis, country size spillover effect competes with equity rebalancing effect. In mild crisis, equity rebalancing effect dominates country size spillover effect. The smaller country's bond becomes domestically safe. In deep crisis, country size spillover effect dominates equity rebalancing effect. The larger country's bond becomes safe for investors in the smaller country. For investors in the larger country, domestic bond is always safe and domestic bond safety status gets strengthened upon entering crisis regime, see the jumps in figure 2.7. So the larger country's bond is globally safe in normal times and in deep crisis, as shown in figure 2.6.

### Market segmentation

**Proposition 22** (Market segmentation)**.** *In crisis regime, bond holdings of the two countries are given by*

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = 0, \quad \theta_t^{F,B^H} = \theta_t^{F,B^F} = 0$$

*Proof.* see appendix ■

Forced by the constraints, investors in both countries hold more domestic risk than desired and would like to offload domestic risk to foreign investors. However, no such security is available because domestic amplification is resulted from dispersion in consumption prices (in terms of numeraire) of the two countries created by binding equity constraints and applies to any real asset. Real bond returns for investors bear the extra domestic risk coming from consumption price and no bond is held or traded between the two countries in crisis even though there is no friction in bond markets. Liquidity drains between the two countries in every asset market[12] and financial dichotomy emerges in crisis regime.

### Non-linearity and systemic risk

**Non-linearity** The non-linearity in asset returns shows up for investors in the smaller country who hold both domestic equity and foreign equity. The non-linearity factor for home country is given by

$$\frac{(1-\overline{\chi}^F)s_t}{(1-\overline{\chi}^F)s_t + \overline{\chi}^F} \tag{2.29}$$

Taking derivative with respect to $s_t$, we have

$$\frac{\overline{\chi}^F(1-\overline{\chi}^F)}{\left[(1-\overline{\chi}^F)s_t + \overline{\chi}^F\right]^2}$$

where we see the non-linearity effect gets stronger when $s_t$ is smaller, consistent with figure 2.5.

[12] There is still trade happening between the two countries and potential trade in assets within domestic investors.

**Systemic risk** At crisis boundaries $s^U$ and $1-s^U$, there are endogenous jumps between normal regime and crisis regime for both countries, which is the systemic risk in the model. The discontinuous change in expected return difference between sovereign bonds for home country investors at $s^U$ (in absolute value), denoted as $\Delta_{11}$ is given by

$$\Delta_{11} = \frac{(1-\overline{\chi}^F)s^U}{(1-\overline{\chi}^F)s^U + \overline{\chi}^F}\frac{\eta - 1}{\eta^2}(1-s^U)(\sigma_H^2 + \sigma_F^2) \tag{2.30}$$

and similarly denote $\Delta_{21}$ as the absolute change in expected return difference between sovereign bonds for foreign country investors at $s^U$, where

$$\Delta_{21} = \frac{\eta - 1}{\eta^2}s^U(\sigma_H^2 + \sigma_F^2) \tag{2.31}$$

Since $s^U < \frac{1-\overline{\chi}^F}{2(1-\overline{\chi}^F)}$, we have

$$\Delta_{11} > \Delta_{21}$$

that is, the smaller country suffers greater systemic risk instability when it falls into crisis.

## 2.6 UIP reversal, flight-to-safety, CIP deviations and convenience yields

The model with crisis regime can explain several puzzles strongly associated with crisis periods: UIP reversal, flight to safety, CIP deviations and convenience yields.

First, in crisis regime, both countries price risks differently from normal regime. Binding equity constraints distort asset returns and asset safety in crisis, which explains the UIP reversal and flight to safety in crisis. Second, in crisis regime, the two countries disagree on prices of risks with each other, which explains the CIP deviations and convenience yields: in crisis, UK investors and US investors perceive different returns for exactly the same bond, the US Treasury. The gap between the actual return of US Treasury and the perceived return of US Treasury by foreign investors is also known as the CIP deviations or the convenience yield of the US Treasury, which is a signature of the 2008 Great Financial Crisis (GFC).

### 2.6.1 UIP reversal

UIP reversal is the opposite direction of UIP violation in normal times. As documented in Corsetti and Marin 2020, in normal times, US Treasury is a safer asset than UK government bond for UK investors and pays lower expected return as its safety premium. While when crisis hit, UK government bond pays lower expected return and becomes safer than US Treasury for UK investors.

Mapping into the model, in normal times, UK country size $s_t$ is above the crisis boundary $s^U$ but below the normal time safety threshold $s^C$. US Treasury is safer than UK government bond because of country size spillover effect. If UK economy or US economy suffers a rare loss, $s_t$ falls into $[s^A, s^U]$ or rises to above $s^C$, UK government bond reverses to become a domestic safe bond and pays lower expected return for domestic investors compared to the US Treasury. In addition, if $s_t$ rises to $[s^C, 1-s^U]$ or $[1-s^A, 1]$, UK government bond becomes a global safe bond. As shown in figure 2.8, UIP reversal between UK and US government bond happens when UK country size $s_t$ falls into the left blue area $[0, s^A]$ from the green area $[s^A, s^U]$.

The model can also speak to emerging economies. For emerging economies whose initial country size is too small and is below $s^A$, if $s_t$ rises into $[s^A, s^U]$ because the emerging economy grows larger due to its higher growth rate or the larger country, say US, falls into crisis due to rare losses, the emerging economy's domestic bond reverses from being the riskier bond to become the safer bond than US Treasury for domestic investors. As shown in figure 2.8, UIP reversal happens for emerging countries when their domestic country size $s_t$ rises into the green area $[s^A, s^U]$ from the left blue area $[0, s^A]$.

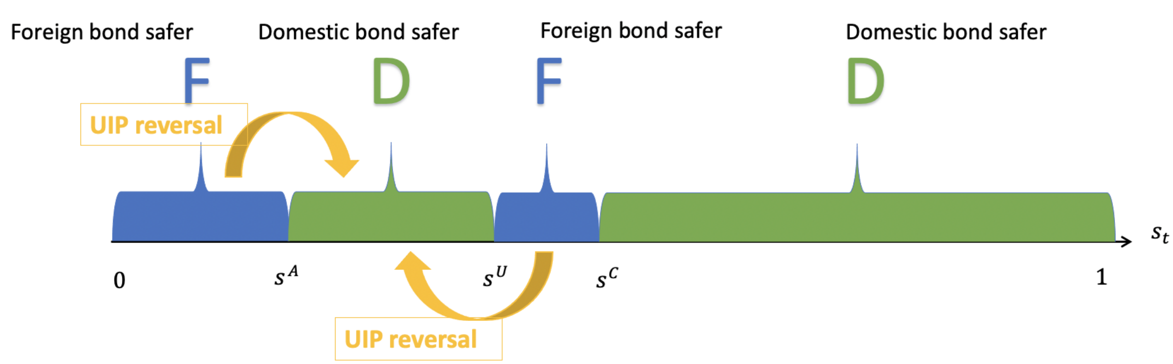


Figure 2.8: UIP reversal

### 2.6.2 Flight to safety

In a flight-to-safety phenomena, investors demand safe assets and push down the return of safe assets in crisis. US Treasury and German bond are good examples for being the global safe asset in crisis and pay lower expected returns than in normal times.

Mapping into the model, if UK country size $s_t$ falls below $s^U$, the safety of US Treasury for US domestic investors is strengthened by equity rebalancing effect and US Treasury yield drops. If UK country size $s_t$ falls below $s^A$, US Treasury is the safe asset for both UK investors and US investors, the global safe asset. The smaller UK country size $s_t$ falls, the safer the US Treasury. Because country size spillover effect, which improves US Treasury safety, gets stronger with falling $s_t$. When UK country size $s_t$ falls into crisis regime $[0, s^U]$, US investors find US Treasury even safer than in normal times and UK investors find US Treasury the safer bond than UK government bond if $s_t$ falls below $s^A$.

In the case of the European debt crisis, German bond is the safe asset and pays historically low return. In the model, when periphery countries falls deep in crisis and their country size $s_t$ drops below $s^A$, German bond is the global safe asset and investors are willing to accept an extra low return as the safety premium.

In a deep crisis, country size spillover effect dominates equity rebalancing effect and the larger country's bond is the global safe asset. From the point of view of investors in the smaller country, a flight-to-safety to the larger country's bond happens when $s_t$ falls into the left blue area as shown in figure 2.9.

For emerging countries whose initial country size $s_t$ is in the green area $[s^A, s^U]$, if US economy suffers from rare losses and $s_t$ rises into blue area $[s^U, s^C]$, US Treasury becomes a global safe asset. As shown in figure 2.9, a flight-to-safety to the larger country's bond also happens when $s_t$ rises into the blue area $[s^U, s^C]$ from the green area $[s^A, s^U]$.

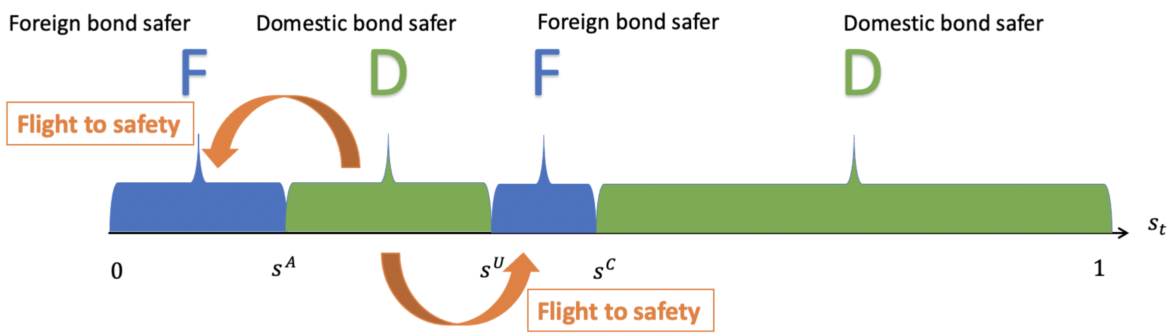


Figure 2.9: flight to safety

### 2.6.3 CIP deviations and convenience yields

CIP deviations is a failure of the Law of One Price: assets with the same underlying dividend flow pay different returns. Government bond convenience yield is the return difference between risk-free rate and government bond yield. The relative convenience yield between sovereign bonds are often related to CIP deviations. 2018 studies the US Treasury premium which is defined as the relative convenience yield between US Treasury and other countries' government bonds by measuring CIP deviations between government bond yields. In my model, the larger country's bond enjoys a positive convenience yield relative to the smaller country's bond in crisis regime.

In crisis region, if home (G-10) investors want to borrow US dollar, they can not directly borrow from US dollar cash market with rate $dr_t^{F,B^F}$ because of market segmentation. However, they can borrow domestic currency at rate $dr^{H,B^H}$ and simultaneously enter a forward contract $-dr^{H,B^H} + dr^{H,B^F}$ to sell domestic currency for US dollar in the future. The implied US dollar rate from FX swap market (or the synthetic dollar rate) is thus $dr^{H,B^F}$.

**Proposition 23.** *The CIP condition violated in crisis regime. The direct US dollar rate from cash market is lower than the synthetic dollar rate implied from FX swap market, that is*

$$\frac{\mathbb{E}_t\left[dr_t^{F,B^F} - dr^{H,B_t^F}\right]}{dt} = r_{F,t}^f - r_{H,t}^f - \frac{1}{1-n_t}\sigma_{n_t}\sigma_{e_t} < 0 \tag{2.32}$$

*Proof.* see appendix ■

## 2.7 Financial development and trade elasticity

With a stable country size $s_t$, bond safety can also change with a shift of the safety spectrum due to changes in financial fiction parameters $1 - \overline{\chi}^H$, $\overline{\chi}^F$, and trade elasticity $\eta$.

### 2.7.1 Financial development

As discussed in section 2.5.2, it is the foreign country's financial development that matters for bond safety when domestic country size shrinks and falls into crisis regime. In crisis regime $[0, s^U]$, a tightening of the larger country's equity holding constraint (i.e., a smaller $\overline{\chi}^F$), has impact on two safety thresholds, the crisis boundary $s^U$ and crisis time safety threshold $s^A$, the non-linear domestic amplification for home country investors, and the systemic risk instability. Tighter equity constraint makes it harder for consumption smoothing and risk-sharing when there is asymmetry in country sizes, which shifts $s^U$ to the right and expands the crisis regime. Chance of entering and the time spent in the crisis regime are increased. Meanwhile, a tighter constraint strengthens equity rebalancing effect and shifts crisis time safety threshold $s^A$ to the left. The safety region of domestic bond in crisis regime is expanded due to increased hedging benefit of domestic bond.

On the other hand, financial development (i.e., a larger $\overline{\chi}^F$) reduces the crisis regime coverage, as well as the safety region of domestic bond in crisis regime until the left threshold $s^A$ exceeds the right threshold $s^U$. With a sufficient loose constraint, equity rebalancing effect is too weak to reverse country size spillover effect and foreign bond is still the safer bond in crisis regime for domestic investors. See equation (3.14), (2.25), (2.21) and figure 2.10.

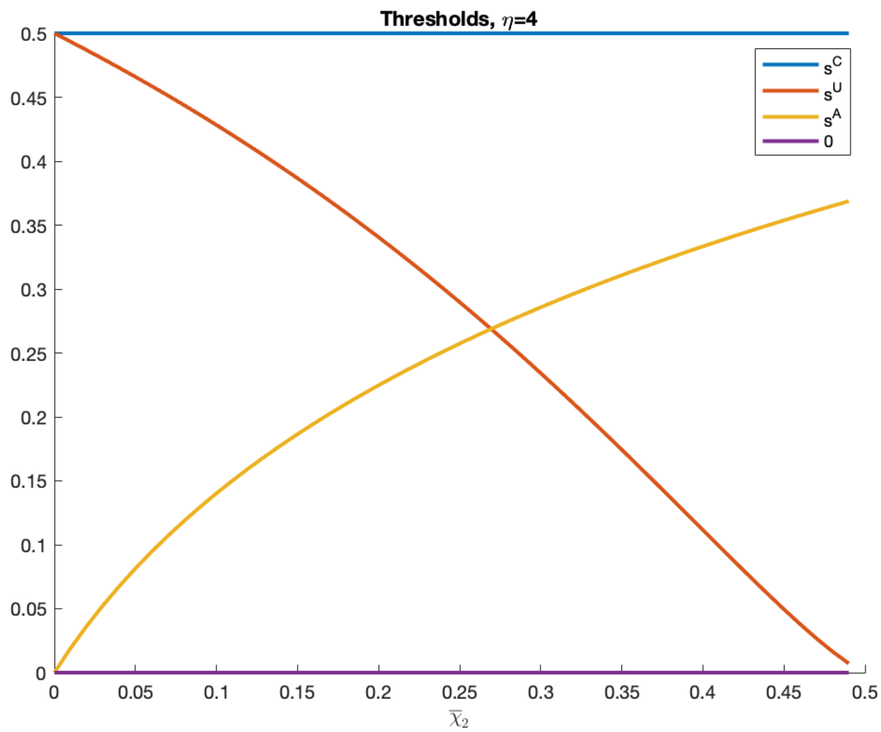


Figure 2.10: Changes of safety thresholds with respect to $\overline{\chi}^F$

Financial development also matters for the non-linear domestic amplification effect and systemic risk instability, see figure 2.11. The larger country's financial development reduces systemic risk instability for both countries when the smaller country falls too small, see equation (2.30) and (2.31). And the domestic non-linear effect weakens with foreign financial development, see equation (2.29).

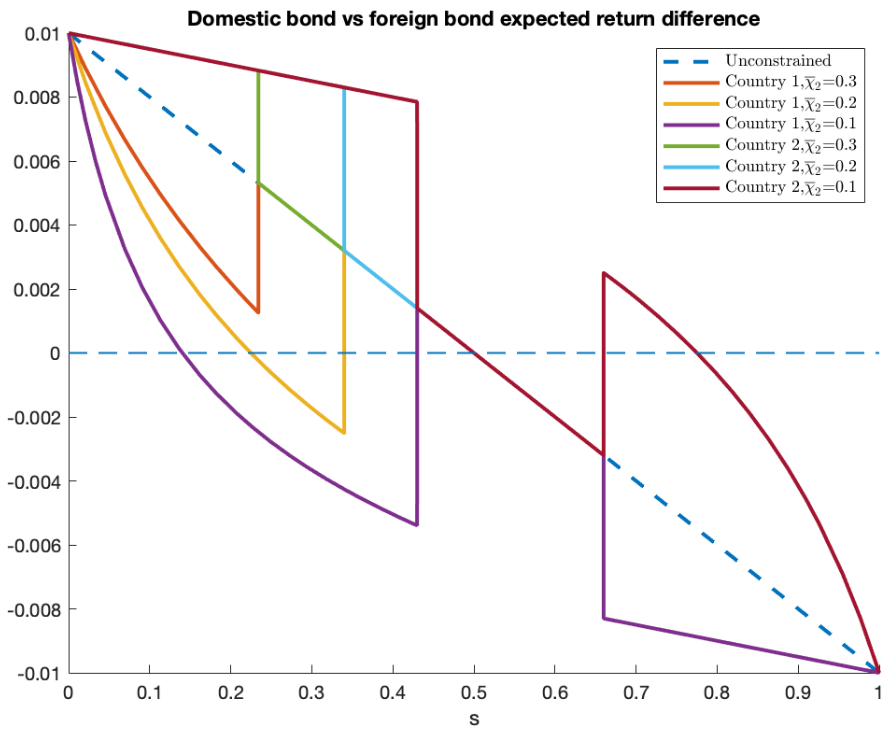


Figure 2.11: Changes with repect to financial development

Note: $\eta = 4$, $\sigma^2 = 0.04$, $1 - \overline{\chi}^H = 0.8$

### 2.7.2 trade elasticity

A larger trade elasticity $\eta$, which means domestic good and foreign good are more substitutable, does not affect the crisis boundary $s^U$ but shifts the crisis time safety threshold $s^A$ to the left, because substitutability between two goods produced by the two countries weakens country size spillover effect and strengthens equity rebalancing effect. See equation (2.16), (2.21), and figure 2.12. Notice that there is discontinuity at $\overline{\chi}^F = 0$ when equation (2.25) degenerates.

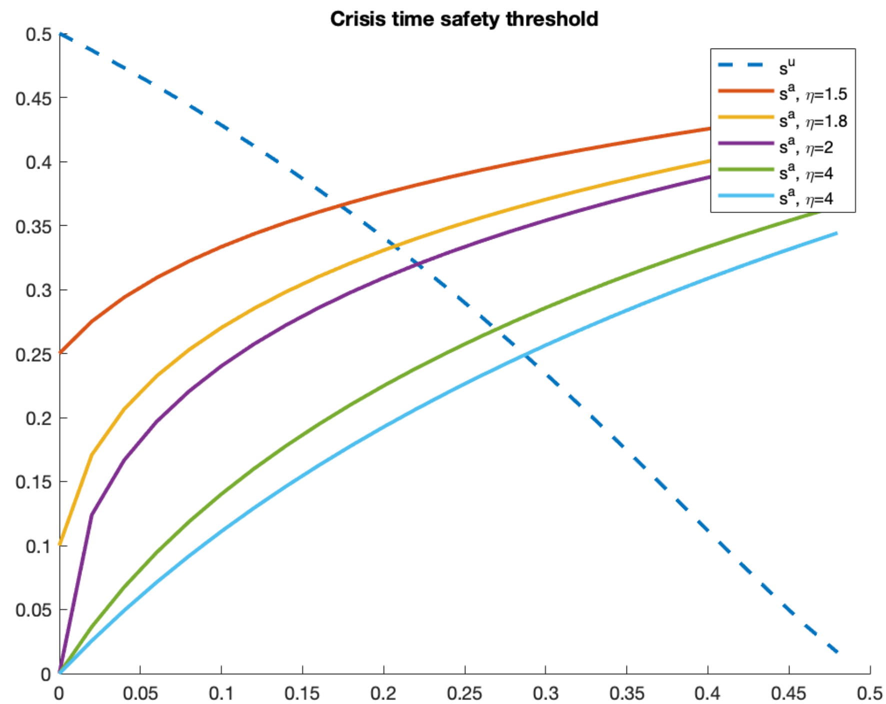


Figure 2.12: Changes of threshold with respect to trade elasticity

## 2.8 Conclusion

This paper provides a theory of sovereign bond safety which is jointly determined by country size and equity rebalancing. Country size spillover effect improves the safety of the larger country's bond, which explains normal time UIP violation. Equity rebalancing, the equity holding constraints in the model, creates endogenous systemic risk instability between normal regime and crisis regime where domestic risk is amplified. The interaction between country size and equity rebalancing in crisis regime explains UIP reversal, flight to safety, sovereign bond CIP deviations and convenience yields at the same time.

# Chapter 3

# A Theory of Equity Rebalancing: Before and After Financial Crisis

## 3.1 Introduction

Before the Great Financial Crisis (GFC), global equity market was under active financial integration and witnessed high level of capital flows. However, the international financial markets experienced dramatic fall in cross-border transactions during the period of crisis (Lane 2013,McQuade and Schmitz 2017). Moreover, new empirical puzzles related to the international equity market post GFC challenge several well-documented and well-understood facts before 2008. I outline these stylized facts ( both before and after the GFC) as follows:

1. The relative GDP of G-10 currency group (compared to the U.S.) peaks at the global financial crisis in 2008. (C. He and Hui 2023)

2. Portfolio-rebalancing strategies aim at mitigating the risk exposure changes due to asset price and exchange rate changes (Hau and Hélène Rey 2004, Camanho, Hau, and Hélene Rey 2022).

3. There is persistent equity home bias (French and Poterba 1991). In 2008, the US equity home bias increased (Wynter 2019).

4. The US NFA position benefits from valuation effect before crisis (Gourinchas and Helene Rey 2007; Gourinchas, Helene Rey, Govillot, et al. 2010) but worsens due to valuation effect during and post crisis(Atkeson, Heathcote, and Perri 2022)

5. US wealth share and consumption share rise relative to the rest of the world during crisis (Dahlquist et al. 2022;Kim 2022).

This paper provides a theoretical framework of equity rebalancing with endogenous *regime change*. Given the peak of country size (measured by GDP) of G-10 currency group relative to the US in 2008 (stylized fact 1), the model generates predictions that are consistent with stylized fact 2 to 5.

Using a continuous-time two-country Lucas tree model augmented with equity constraints, I propose that shocks that tightens equity constraints is the key driver of the regime change into crisis. Although the literature has tackled these new puzzles related to crisis through the lens of cyclicality, it is not clear that these crisis patterns repeat with business cycles. In this paper, I leverage the insights and extends previous work on portfolio rebalancing (Hau and Hélène Rey 2004; Camanho, Hau, and Hélene Rey 2022) to study normal times and crisis time as completely different regimes.

The key modelling element, i.e. the equity constraints, is the only departure of my model from a standard two-country Lucas tree model. Each country has to hold at least a fraction of their domestic equity - they can not issue as much domestic equity share as they would like to. I argue that shocks that tightens the equity constraint facing home country drives the system in to crisis. That is, the maximum holding of foreign equity by home country decreases during crisis. And this shock to equity constraint has several micro-foundations in the literature which I will discuss in section 3.3.3. The equity constraints deliver persistent home bias. Given the country size changes (stylized fact 1), the equity constraints help generate empirically consistent dynamics of equity portfolio changes in 2008 financial crisis. These predictions together make sense of stylized fact 3.

The mechanism of the model works as follows. In normal times, the two countries can perfectly share exchange rate risk through freely adjusting their equity holdings and trading on the FX market.[1] This prediction from the model is consistent with stylized fact 2 (Portfolio-rebalancing strategies aim at mitigating the risk exposure changes due to asset price and exchange rate changes).

Due to the asymmetric holding towards domestic equity relative to foreign equity, home

[1] The model allows for heterogeneous re-balancing behaviours due to an extra degree of freedom which will be explained in later sections.

country will hedge exchange rate risk with foreign country. The gains and losses from FX risk hedging will be delivered by capital flows induced by equity trading - until equity constraints bind. I refer to the constrained region as *crisis regime*.

In crisis regime, both countries are constrained in their equity holdings and equity tradings are limited. Equity market segmentation emerges endogenously. In crisis regime, cross-border capital flows drop significantly and remain in the very low level, consistent with Lane 2013 and McQuade and Schmitz 2017.

Given the country size changes (stylized fact 1), the model predicts that US NFA benefits from positive valuation effect of G10 equity before 2008, but suffers losses due to positive valuation effect of US equity after 2008. This is consistent with stylized fact 4. The intuition is simple: The larger the country (as measured by production), the more dividends delivered by its equity. Since equity price is the net present value of all future dividends, country size is therefore *positively* correlated with higher domestic equity valuation. Since US equity increases in value during crisis, the model also predicts that US global wealth share and consumption share rises, making sense of stylized fact 5.

On a broader level, the model also delivers predictions that can reconcile with the empirical facts on the currency risk premia (C. He and Hui 2023), including the failure of uncovered interest parity (UIP), UIP reversals, and the failure of covered interest parity (CIP). In this paper, I focus on the equity rebalancing side.

### 3.1.1 Literature review

This paper is related to international finance literature studying international capital market including international shock transmission through portfolio shares (Kraay and Ventura 2000 and Kraay and Ventura 2002), valuation effect, exorbitant previlege of the US Dollar (Gourinchas and Helene Rey 2007, Gourinchas, Helene Rey, Govillot, et al. 2010), portfolio reblancing and capital flows (Hau and Hélène Rey 2004 and Camanho, Hau, and Hélene Rey 2022), US NFA position changes and wealth transfers before and after GFC (Atkeson, Heathcote, and Perri 2022). Dahlquist et al. 2022) uses model with consumption home bias, deep habit and time-varying risk appetite to explain US NFA position changes while leaving international equity market dynamics simplified in a complete market setting. Previous works on how incomplete market affect exchange rate and equity prices study partial market segmentation on foreign exchange market (Hau and Hélène Rey 2004), limit to arbitrage ( Gabaix and Maggiori 2015), and portfolio constraint. These work do not focus on different patterns of market dynamics before and after tthe GFC.

The model builds on classic continuous-time asset pricing framework. 2007 solves a two-tree model with perfect substitutable goods because of which there is no space for exchange rate. Pavlova and Rigobon 2007 solves a two-tree model with log-linear preference, which is a knife-edge case of CES consumption where the country size spillover effect does not show up because the fixed expenditure on two goods when consumption is of Cobb-Douglas form. I. Martin 2011 solves the price levels in a two-trees model with general CRRA utility, CES consumption, and shocks following any Levy process whereas I instead focus on optimal portfolio trade-off and solve for intertemporal risk pricing and Euler equations.

The rest of the paper is organized as follows. Section 3.2 sets up the model, Section 3.3 solves the model. Section 3.4 discusses model predictions and reconciles with empirical facts, and section 3.5 concludes.

## 3.2 :Model Set-up

This paper borrows the model from C. He and Hui 2023 to study international equity market dynamics. Readers who are familiar with the model set-up should skip this section.

Time is continuous and infinite horizon, $t \in [0, +\infty)$. There are two countries in the world, country $H$ and country $F$. For ease of illustration, I will call country $H$ the G-10 and country $F$ the US.

**Technology** Each country is endowed with a tree producing domestic good. The two trees evolve as follows,

$$\frac{dY_{H,t}}{Y_{H,t}} = \mu_{H,t}\, dt + \sigma_{H,t}\, dZ_{H,t}$$
$$\frac{dY_{F,t}}{Y_{F,t}} = \mu_{F,t}\, dt + \sigma_{F,t}\, dZ_{F,t}$$

where $\{\mu_{H,t}, \mu_{F,t}, \sigma_{H,t}, \sigma_{F,t}\}$ are exogenous parameters (or processes). For simplicity, we assume throughout the paper that $\mu_{H,t} = \mu_{F,t} = \mu$ and $\sigma_{H,t} = \sigma_{F,t} = \sigma$.

**Preferences** In order to highlight my mechanism, I assume homogeneous preference, logarithmic utility, and no consumption home bias for the representative agents of the two countries. The final consumption is a CES aggregate of the two goods produced by the two countries. The expected utility of the representative agent in country $i$, takes the form

$$\mathbb{E}\int_0^\infty e^{-\rho t}\log C_{H,t}\,dt$$

where

$$C_{H,t}=\left[\alpha^{\frac{1}{\eta}}C_{HH,t}^{\frac{\eta-1}{\eta}}+(1-\alpha)^{\frac{1}{\eta}}C_{HF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$\alpha$ is the share parameter [2]. $\eta$ is the elasticity of substitution between the two goods, assumed to be greater than 1 and smaller than infinity.

**Numeraire** Define 1 unit of the CES basket of total output $\overline{Y}_t$ as numeraire throughout the paper,

$$\overline{Y}_t\equiv\left[\alpha^{\frac{1}{\eta}}Y_{H,t}^{\frac{\eta-1}{\eta}}+(1-\alpha)^{\frac{1}{\eta}}Y_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

Denote the process of total output $\overline{Y}_t$ as

$$\frac{d\overline{Y}_t}{\overline{Y}_t}=\overline{\mu}_t dt+\overline{\sigma}_t dZ_t$$

where $dZ_t=[dZ_{H,t}\;\; dZ_{F,t}]^T$.

**International trade market and exchange rate** The international trade market (of home and foreign goods) is frictionless. Denote $p_t^H$ as the price of home good and $p_t^F$ as the price of foreign good. The real exchange rate is given by the relative price of good 1 and good 2[3],

$$e_t\equiv\frac{p_t^H}{p_t^F}\tag{3.1}$$

and $e_t$ is also the terms of trade in this model. And denote the endogenous process of real exchange rate, $e_t$, as

$$\frac{de_t}{e_t}=\mu_t^e dt+(\sigma_t^e)^T dZ_t$$

[2] Unlike Pavlova and Rigobon (2008b) where $\alpha$ represents the country size, here in my model $\alpha$ is not a key parameter of interest.

[3] An increase of $e_t$ corresponds to an appreciation of home currency relative to foreign currency.

**Equity** Each country can issue domestic equity shares in unit supply. The equities are risky claims to domestic trees. Denote $S_t^H$ and $S_t^F$ as the total value of domestic equity and foreign equity respectively. Define $\chi_t^{H,H}$ as the the share of home stock market (apple tree) held by home investor, $\chi_t^{H,F}$ as the share of foreign stock market (orange tree) held by home investor. And similarly define $\chi_t^{F,H}$ as the share of home stock market (apple tree) held by foreign investor and $\chi_t^{F,F}$ as the share of foreign stock market (orange tree) held by foreign investor.

**Equity constraint** Importantly, equity constraint for home country:

$$0 \leq \chi_t^{H,F} \leq \overline{\chi}^F \tag{3.2}$$

This equation is saying that home investor can not hold more than $\overline{\chi}^F$ share of foreign equity, nor short-sell foreign equity[4].

And similarly we have equity constraint for foreign country:

$$0 \leq \chi_t^{F,H} \leq \overline{\chi}^H \tag{3.3}$$

That is, foreign investor can not hold more than $\overline{\chi}^H$ share of home equity, nor short-sell home equity.

**Sovereign Bond** Each country can issue (sovereign) bond in zero net supply. The bonds, denoted as $B_t^H$ and $B_t^F$, are instantaneously *risk-free in domestic goods* but *not* risk-free in terms of numeraire[5]. The price of home (foreign) bond $B_t^H$ ($B_t^F$) in terms of numeraire is the same as the price of home (foreign) good $p_t^H$ ($p_t^F$). Denote $B_t^{H,H}$ as the home bond held by home investors[6] and $B_t^{F,H}$ as the home bond held by foreign investors. And denote $B_t^{H,F}$ as the foreign bond held by home investors and $B_t^{F,F}$ the foreign bond held by foreign investors.

**Asset returns** I introduce notations for asset returns which are *endogenous* processes. Recall that $B_t^H$ is instantaneously risk-less bond in home good and denote the return process

[4] Here one can replace the lower bound 0 to a negative number, say $\underline{\chi}^F$. The key thing is that $\chi_t^{H,F}$ (the share of foreign equity held by home investor) is lower bounded.

[5] Their returns are subject to exchange rate risks through price changes

[6] $B_t^{H,H} > 0$ means lending and $B_t^{H,H} < 0$ means borrowing

of home bond (in terms of numeraire) as:

$$dr_t^{B^H} = \frac{d(p_t^H B_t^H)}{p_{H,t} B_t^H} = (\mu_{p^H,t} + r_t^H)\, dt + \sigma_{p^H,t}\, dZ_t$$

where $\mu_{p^H,t}$ and $\sigma_{p^H,t}$ are given by the endogenous process

$$\frac{dp_t^H}{p_t^H} = \mu_{p^H,t} t\, dt + \sigma_{p^H,t}\, dZ_t$$

Similarly denote the return process of foreign bond (in terms of numeraire) as:

$$dr_t^{B^F} = \frac{d(p_t^F B_t^F)}{p_{F,t} B_t^F} = (\mu_{p^F,t} + r_t^F)\, dt + \sigma_{p^F,t}\, dZ_t$$

Recall that $S_t^H$ is the total value of home equity and define $q_t^H$ as the per unit price of home equity in terms of numeraire $\overline{Y}_t$, that is, $S_t^H = q_t^H \overline{Y}_t$. And postulate the endogenous process of $q_t^H$ as follows

$$\frac{dq_t^H}{q_t^H} = \mu_{q^H,t} dt + \sigma_{q^H,t} dZ \tag{3.4}$$

The return of home equity in terms of numeraire is given by

$$dr_t^{S^H} = \underbrace{\frac{p_t^H Y_{H,t}}{q_t^H \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^H \overline{Y}_t)}{q_t^H \overline{Y}_t}}_{\text{capital gain}}$$

and similarly

$$dr_t^{S^F} = \underbrace{\frac{p_t^F Y_{F,t}}{q_t^F \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^F \overline{Y}_t)}{q_t^F \overline{Y}_t}}_{\text{capital gain}}$$

**Wealth and portfolio shares** I introduce notations for wealth and portfolio shares, which will be determined in equilibrium. Denote the aggregate wealth of home country as $W_t^H$ and the aggregate wealth of foreign country as $W_t^F$.

Denote $\theta_t^{H,S^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H}$ as the portfolio share of home equity for home country, $\theta_t^{H,S^F} = \frac{\chi_t^{H,F} S_t^F}{W_t^H}$ as the portfolio share of foreign equity for home country. And similarly, denote $\theta_t^{F,S^H} = \frac{\chi_t^{F,H} S_t^H}{W_t^F}$ as the portfolio share of home equity for foreign country and $\theta_t^{F,S^F} = \frac{\chi_t^{F,F} S_t^F}{W_t^F}$ as the portfolio share of foreign equity for foreign country.

We can similarly define portfolio shares of bonds in home and foreign country. Denote

$\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$ as the portfolio share of home bond for home country, $\theta_t^{H,B^F} = \frac{p_t^H B_t^{H,F}}{W_t^H}$ as the portfolio share of foreign bond for home country, And similarly, denote $\theta_t^{F,B^H} = \frac{p_t^H B_t^{F,H}}{W_t^F}$ as the portfolio share of home bond for foreign country and $\theta_t^{F,B^F} = \frac{p_t^F B_t^{F,F}}{W_t^F}$ as the portfolio share of foreign bond for foreign country.

**Country Size** Define relative size of home country as follows.

$$s_t = \frac{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = \alpha^{\frac{1}{\eta}} \left( \frac{Y_{H,t}}{\overline{Y}_t} \right)^{\frac{\eta-1}{\eta}} \tag{3.5}$$

**Optimization problems** The optimization problem for home country is as follows:

$$\begin{aligned}
\max_{\{C_{HH,t}, C_{HF,t}, \chi_t^{H,H}, \chi_t^{H,f}, \theta_t^{H,B^H}, \theta_t^{H,B^F}\}_{t=0}^{\infty}} \quad & \mathbb{E}\left[ \int_0^\infty e^{-\rho t} \log \left( \left[ \alpha^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}} \right]^{\frac{\eta}{\eta-1}} \right) dt \right] \\
\text{s.t.} \quad & \frac{dW_t^H}{W_t^H} = \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\
& \qquad - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \\
& 1 = \frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} + \theta_t^{H,B^H} + \theta_t^{H,B^F} \\
& 0 \le \chi_t^{H,F} \le \overline{\chi}^F
\end{aligned} \tag{3.6}$$

The optimization problem for foreign country is similar and discussed in appendix.

**Market clearing conditions** Home equity market clears,

$$\chi_t^{H,H} + \chi_t^{F,H} = 1 \tag{3.7}$$

Foreign equity market clears,

$$\chi_t^{H,F} + \chi_t^{F,F} = 1 \tag{3.8}$$

Home bond market clears,

$$B_t^{H,H} + B_t^{F,H} = 0 \tag{3.9}$$

And foreign bond market clears,

$$B_t^{H,F} + B_t^{F,F} = 0 \tag{3.10}$$

Total consumption of home (foreign) good equals total production of home (foreign) good,

$$C_{HH,t} + C_{FH,t} = Y_{H,t} \tag{3.11}$$

$$C_{HF,t} + C_{FF,t} = Y_{F,t} \tag{3.12}$$

## 3.3 Solving the model

### 3.3.1 Normal regime

**Proposition 24.** *In normal regime (at least one equity constraint is not binding), home and foreign country can perfectly hedge exchange rate risk through equity re-balancing and trading on the FX market.*

*Proof.* see appendix ■

An intuitive way to look at proposition 24 is to count the risks and assets. There are two sources of risks, from the two trees. Even with one equity holding constraint, there are still another three assets that can be freely traded which can span all the possible states of the world. So investors in the two countries can still replicate first best risk-sharing through portfolio re-balancing. Similar to the complete market case, there is indeterminacy in the model with respect to portfolio holdings but not asset returns.

**Proposition 25** (Country size and equity valuation)**.** *The excess return of home equity relative to foreign equity is positively correlated with home country size.*

$$\text{Cov}(dr_t^{S^H} - dr_t^{S^F}, s_t) > 0 \tag{3.13}$$

*Proof.* See appendix ■

As we find in C. He and Hui 2023, the relative GDP of G-10 currency group to the US has been increasing until 2008. 25 predicts that G-10 currency group equities should experience a boom in valuation compared to US equity. This is consistent with the valuation effects which increased the value of foreign assets held by U.S. residents relative to the value of U.S. assets held by foreigners before the crisis (Gourinchas, Helene Rey, Govillot, et al. 2010, Gourinchas, Helene Rey, Govillot, et al. 2010).

### 3.3.2 Crisis regime

**Proposition 26** (Crisis regime)**.** *The system moves in to crisis regime if* $s_t \in [0, s^U]$*, where* $s^U$ *is the crisis boundary and solves*

$$q_1(s^U) = \frac{n_0 - \overline{\chi}^F}{\rho(1 - \overline{\chi}^F)} \tag{3.14}$$

*and we have that*

$$\frac{ds^U(\overline{\chi}^F)}{d\overline{\chi}^F} \leq 0 \tag{3.15}$$

*That is, a tightening in the equity constraint facing home country* ***expands*** *the crisis region.*

*Proof.* see appendix. ■

We focus on this empirically relevant crisis region $s_t < s^U$ and argue that the system moves into the crisis regime due to a shock that tightens the equity constraint on maximum holding of US equity for G-10 currency group investors.

### 3.3.3 Shocks to equity constraint

In this paper, I argue that a shock that tightens equity constraint facing home country ($\overline{\chi}^F$ decreases) drives the system into the crisis region. I discuss several potential micro-foundations in international and finance literature about the equity constraint.

1. Hedging non-tradable income risk (wages): Baxter and Jermann 1995, Coeurdacier, Kollmann, and P. Martin 2010, Heathcote and Perri 2013

2. Informational frictions: Razin, Sadka, and Yuen 1999 assume that domestic investors can observe the productivity of domestic firms before making their loan decisions, while foreign investors cannot. This results in foreign underinvestment and domestic oversaving.

3. Behavioural biases: Dumas, Kurshev, and Uppal 2009, Dumas, Lewis, and Osambela 2017 differences in beliefs is observationally equivalent to existing models of segmented markets due to asymmetric information

4. Agency frictions (hidden effort or hidden saving): Z. He and Krishnamurthy 2013,Z. He and Krishnamurthy 2018

The origin of a shock to equity constraint can potentially come from all sources discussed above and taking a stand on which one is the underlying source is beyond the scope of this paper.

## 3.4 Model predictions

### 3.4.1 Market segmentation

In crisis regime, both countries' equity markets are constrained and risk-sharing is limited. This market segmentation drives a wedge between normal time SDF and crisis time SDF, thus a wedge between normal time prices of risks and crisis time prices of risks, for both countries. The wedge between normal time SDF and crisis time SDF is what I call financial friction effect.

**Proposition 27** (Market segmentation)**.** *In crisis region $s_t \in [0, s^u]$, we have that*

$$\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^F$$

$$p_t^H B_t^{H,H} = p_t^F B_t^{H,F} = 0$$

*Proof.* see appendix in C. He and Hui 2023. ∎

This prediction is consistent with the sharp contrast of cross-border capital flows during the period of crisis as documented in Lane 2013 and McQuade and Schmitz 2017.

In crisis regime, binding equity constraints drive a wedge between SDFs of home and foreign investors. Different pricing kernels result in different returns for the same asset. There are price wedges on financial assets between home and foreign country due to market segmentation. That is, law of one price is violated for financial assets in crisis region. There exists an *implied* nominal exchange rate defined as follows.

**Proposition 28** (Implied nominal exchange rate)**.** *Define the implied nominal exchange rate $\mathcal{E}_t$ as the ratio of Foreign SDF to Home SDF, $\frac{\xi_t^F}{\xi_t^H}$, and we have*

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = \frac{d\frac{\xi_t^F}{\xi_t^H}}{\frac{\xi_t^F}{\xi_t^H}} \tag{3.16}$$

*In normal regime, we have*

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = 0 \tag{3.17}$$

*In crisis regime, we have*

$$\frac{d\mathcal{E}_t}{\mathcal{E}_t} = (\frac{1}{1-n_t}\mu_{n_t} + \frac{n_t}{(1-n_t)^2}\sigma_{n_t}^2)\, dt + \frac{1}{1-n_t}\sigma_{n_t}\, dZ_t \tag{3.18}$$

*Following tradition in the literature, an increase in $\mathcal{E}_t$ corresponds to a nominal depreciation of the home currency currency. In crisis regime $s_t < s^u$, we have*

$$\mathbb{E}\left[\frac{d\mathcal{E}_t}{\mathcal{E}_t}\right] = (\frac{1}{1-n_t}\mu_{n_t} + \frac{n_t}{(1-n_t)^2}\sigma_{n_t}^2) > 0 \tag{3.19}$$

*That is, in crisis region, home currency is expected to depreciate while foreign currency (US dollar) is expected to appreciate nominally.*

This implied nominal exchange rate makes the segmented market behaves *as if* the market is complete but with inflations (or deflation) in both countries. This is because the composition of home good and foreign good in the numeraire changes over time and carries an aggregate risk $\overline{\sigma}_t$. Recall that the numeraire is defined as 1 unit of the CES basket of total output $\overline{Y}_t \equiv \left[\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$ and we have

$$\frac{d\overline{Y}_t}{\overline{Y}_t} = \overline{\mu}_t dt + \overline{\sigma}_t dZ_t \tag{3.20}$$

In complete market, this aggregate risk is priced the same in home and foreign country. In the segmented market where equity constraints bind, this aggregate risk is priced differently in home and foreign country. That is, home and foreign country will have different price levels with respect to the numeraire over time. Note that this is not in contradiction with frictionless trade market of home and foreign good. Because the two countries always agree on the *relative* price of home good and foreign good ($e_t = \frac{p_t^H}{p_t^F}$) and they simply exchange home good for foreign good to minimize expenditure on the intratemporal trade market.

### 3.4.2 Equity home bias

$S_t^H$ and $S_t^F$ are market capitalization of home and foreign country. Optimal portfolio shares of equities should be given by the share of the equity in the world market portfolio

$$\theta_t^{H,optimal} = \frac{S_t^H}{S_t^H + S_t^F}$$

$$\theta_t^{F,optimal} = \frac{S_t^F}{S_t^H + S_t^F}$$

Following tradition in the literature, define equity home bias as

$$EHB_t^H = 1 - \frac{\chi_t^{H,F} S_t^F}{\theta_t^{F,optimal}} = 1 - \frac{\chi_t^{H,F}}{n_t} \tag{3.21}$$

$$EHB_t^F = 1 - \frac{\chi_t^{F,H} S_t^F}{\theta_t^{H,optimal}} = 1 - \frac{\chi_t^{F,H}}{1 - n_t} \tag{3.22}$$

for home country an foreign country respectively.

**Proposition 29** (Equity home bias)**.** *There is persistent equity home bias,*

$$EHB_t^H > 0; \qquad EHB_t^F > 0$$

*for non-degenerate home country size $s_t$.*

*Proof.* see appendix. ■ Because of equity constraints, both countries will have a asymmetric equity portfolio with higher weight on domestic equity, consistent with the well-documented fact in the literature. These predictions are consistent with the first part of stylized fact 3: there is persistent equity home bias as is well-documented in the literature since French and Poterba 1991. These portfolio weights and equity home bias will change with relative country size in crisis regime.

**Proposition 30.** *Foreign country (US) equity home bias (weakly) increases in crisis region $s \in [0, s^U)$ relative to normal region,*

$$EHB_t^{H,normal} \leq EHB_t^{H,crisis}$$

*Home country (G-10 currency group) experienced a passive increase in their foreign portfolio* *share (that is, how the home bias would have changed for investors in a country if they had not made any trades) in crisis region $s \in [0, s^U)$ relative to normal region.*

*Proof.* see appendix ■

These predictions are jointly consistent with stylized fact 1 and the second part of stylized fact 3.

### 3.4.3 NFA position

**Proposition 31** (NFA position)**.** *In normal regime, NFA position of foreign country (US) is given by*

$$NFA_t^{US,normal} = \chi_t^{FH} S_t^H - \chi_t^{HF} S_t^F = S_t^H - n_0 \tag{3.23}$$

*Foreign (US) NFA increases as home country (G-10 currency group) size $s_t$ increases due to positive valuation effect of home equity*

$$\frac{dS_t^H}{ds_t} > 0 \tag{3.24}$$

*In crisis region $s \in [0, s^U)$, NFA position of foreign country (US) is given by*

$$NFA_t^{US,crisis} = -\overline{\chi}^F \tag{3.25}$$

*Foreign (US) NFA decreases as home country size $s_t$ decreases due to positive valuation effect of foreign (US) equity*

$$\frac{S_t^{FF}}{ds_t} < 0 \tag{3.26}$$

*Proof.* see appendix. ■

These predictions are jointly consistent with stylized fact 1 documented in C. He and Hui 2023 and 4 documented in Atkeson, Heathcote, and Perri 2022.

### 3.4.4 Wealth share

**Proposition 32** (Wealth share)**.** *Foreign country (US) wealth share and consumption share increases in crisis region $s \in [0, s^U)$ relative to normal region,*

$$\frac{W_t^{F,crisis}}{W_t^{H,crisis} + W_t^{F,crisis}} > \frac{W_t^{F,nomral}}{W_t^{H,nomral} + W_t^{F,nomral}} \tag{3.27}$$

$$\frac{C_{F,t}^{crisis}}{C_{H,t}^{crisis} + C_{F,t}^{crisis}} > \frac{C_{F,t}^{normal}}{C_{H,t}^{normal} + C_{F,t}^{normal}} \tag{3.28}$$

*Proof.* see appendix ■

These predictions are consistent with stylized fact 5 documented in Dahlquist et al. 2022 and Kim 2022: US wealth share and consumption share rise relative to the rest of the world in crisis.

## 3.5 Conclusion

This paper provides a theory of equity rebalancing that jointly account for five stylized facts before and after the Great Financial Crisis. In a standard two-country Lucas tree model augmented with equity constraints, I argue that shocks that tightens equity constraints drives the system into the crisis region. The model generates joint dynamics of relative country size (measured by GDP) with 1) exchange rate risk hedging, 2) equity market dynamics (diversification and valuation effect), 3) US net foreign asset position (NFA) changes, and 4) global wealth transfers, that are consistent with documented empirical facts.

# Chapter 4

# International Diversification Puzzle: Trade and Investment Channel

## 4.1 Introduction

This paper studies how exposure to international risk through trade, invoicing and investment would affect a small country's portfolio choice in the financial market. We first show that a sizable exposure of quoted domestic firm to international risk, for example due to a large share of trade invoicing in foreign currency, inward foreign direct investment or international supply chain relationships, could generate strong equity home bias, high demand for safe assets with a safe asset home bias. Hence, once the exposure to international risk is taken into account, equity home bias is not a puzzle as previously considered in the literature but an optimal portfolio choice, jointly with a high demand for safe assets. Intuitively, rational investors will factor this dependence in their portfolio choice, understanding that foreign equities are not as good a hedge as conventionally thought and will not necessarily use such asset as such, but turn to international safe asset and domestic bond for hedging purposes.

We then take one step further to look at the interaction between openness and safe asset demand due to international equity market illiquidity. We show that illiquid international equity markets, e.g. due to a global crisis, initiate a crisis regime for a highly open economy where demand for the international safe asset and the price of international risk surge, whereas the demand for domestic bond and price of domestic risk plummet compared to the normal regime. In time of crisis, openness reverses the safe asset home bias, creates excess demand for international safe asset and greatly weakens domestic currency.

The motivation for the paper starts from the classical observation that the share of domestic equity in foreign portfolios is high across emerging economies and has remained stably so in recent times as evidenced by Figure 4.1, which we will need to update to more recent periods.

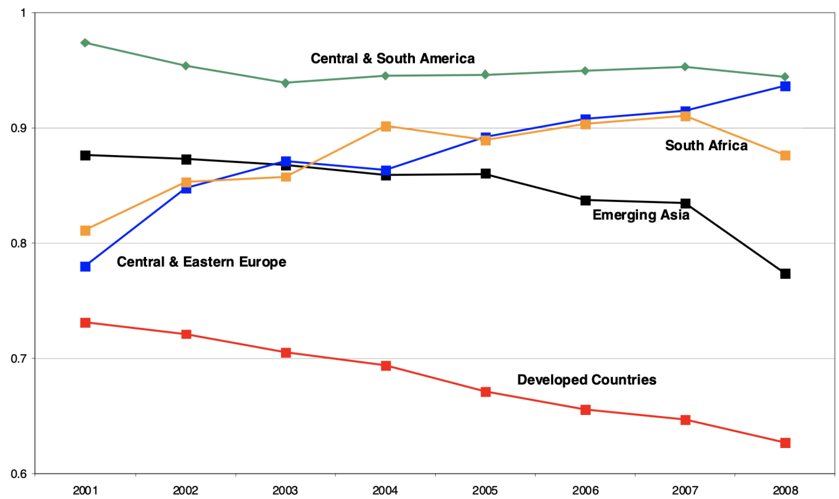


Figure 2: Home Bias in Equities measures across emerging countries (the country measure $EHB_i$ is Market Capitalization-weighted for each region; source: IFS and FIBV. See appendix for the list of countries included)

Figure 4.1: Source: Coeurdacier and Hélène Rey 2013

We couple this empirical fact with the observation that domestic firms in Emerging Markets are largely exposed to international factors of risks such as, for example, inward foreign direct investment and dominant currency invoicing, respectively in Figure 4.2 and 4.3.

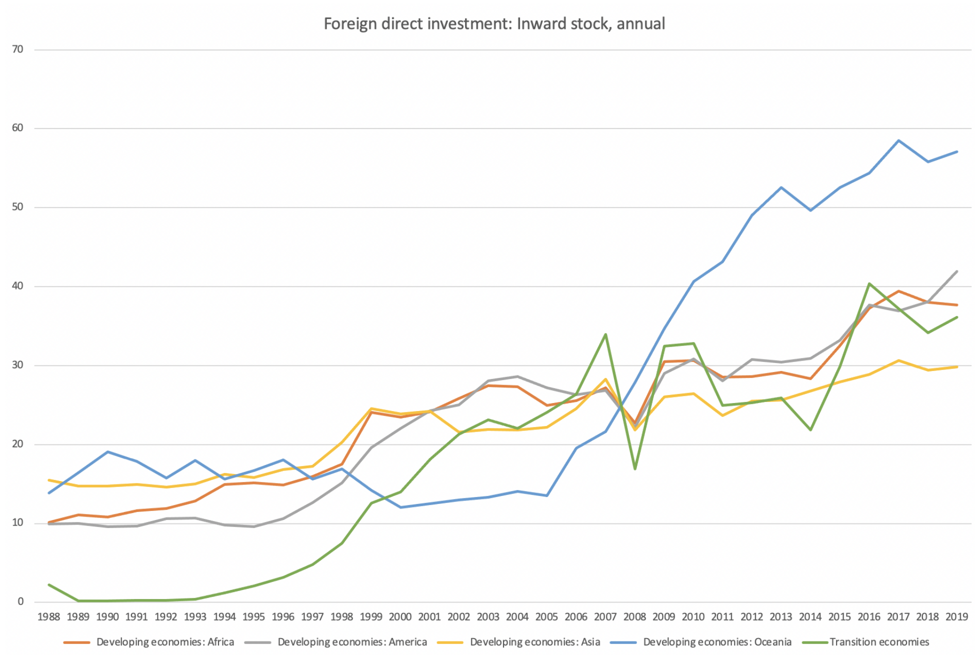


Figure 4.2: Source: UNCAD statistics

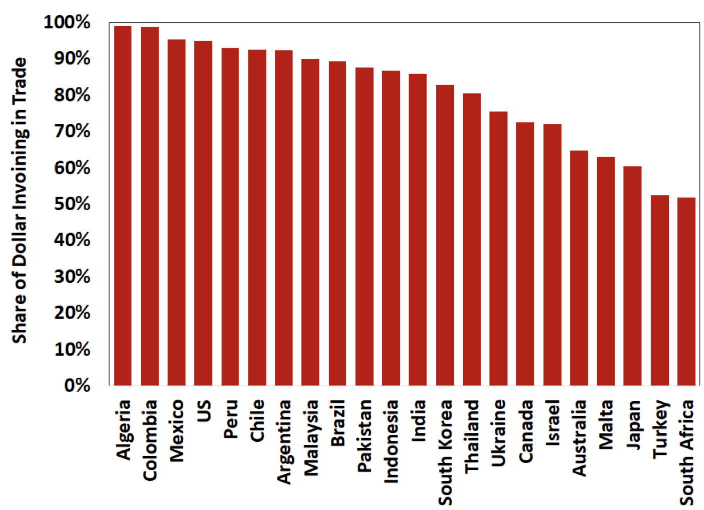


FIGURE 2: Dollar Dominance in Trade Invoicing

Source: National sources, Kamps (2006), Goldberg and Tille (2009), Gopinath (2016), Castellares (2017), Labbe (2018), and Giuliano and Luttini (2019).

Figure 4.3: Source: Shousha 2019

For the time being, however, we are not being specific on the type of international risk that firms may be exposed to. Other examples that we could consider include trade linkages and global value chain. We remain open to alternative sources that could provide interesting complementarities with the channel at hand. One interesting aspect about trade invoicing and credit in foreign currency is that in the classic explanation, such as the one presented in Gopinath and Stein 2018, trade invoicing in the dominant currency represents a source of insurance for the individual firm to hedge against currency risk. Therefore, this could be an interesting channel to explore in that invoicing firms might be exposing domestic investors to the whims of the global financial cycle, a form of externality. In that respect, it could also be interesting to study how the equilibrium level of "invoicing" differs from the "optimal" one and what intervention policy can take to minimize such distance. This would probably be more of a companion project than an extension of the current one.

Given that the current project tries to provide a novel explanation to a well documented empirical observation in the literature, our motivation will have to feature a clear description of the progress we make vis-à-vis the current main theories. For the time being, we would like to make two remarks. First, our explanation can be viewed as complementary to the literature. Second, we recognize that the current explanations are not short of critiques. For example, high cross-border financial transaction costs seem hard to justify given the large volume of flows. Similarly, the role of imperfect information diffusion is arguably weaker given the new means of communications, e.g. the Internet. However, besides these *ex-ante* critiques, we need to find an *ex-post* result that shows, for example, that our theory does a better job at explaining a specific phenomenon than the ongoing debate.

## 4.2 Model

### 4.2.1 Benchmark

In this section, we outline the main features of the model, which is built on a simplified version of M. K. Brunnermeier and Sannikov 2019. For the time being, we have developed the investor side of the economy whilst leaving the firms' one in reduced form. In future iterations of the paper, the aim is that of introducing a more complicated international transmission mechanism for international shocks also for these agents.

We consider a small open economy with three different sources of risk, each associated with a Brownian motion: domestic risk, $dZ_t$, international risk, $dZ_t^*$, and idiosyncratic risk, $d\tilde{Z}_t$, which will wash out in aggregation. In order to partially insure themselves, the investors make decisions on the share of their wealth to devote to each of four financial assets: domestic stock with return $dr_t^{K^i}$, domestic currency/bond in fixed supply with return $dr_t^P$, dollar (numeraire) with exogenous risk free rate $r^*$ and international stock with exogenous return $dr_t^S$. The portfolio weights associated with the four assets are $\theta^K$, $\theta^P$, $\theta^*$ and $\theta^S$ respectively, with $\theta^K + \theta^P + \theta^* + \theta^S = 1$.

Domestic investors all have log utility with discount rate $\rho$, and therefore solve the following maximization problem

$$\max_{\substack{c_t^i, \theta_t^{K,i}, \theta_t^{*,i} \\ \theta_t^{S,i}, \theta_t^{P,i}}} \quad \mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \log(c_t^i) dt \right] \tag{4.1}$$

$$\text{s.t.} \quad \frac{dn_t^i}{n_t} = \theta^K dr_t^k + \theta^P dr_t^P + \theta^* r^* dt + \theta^S dr_t^S - \frac{c_t^i}{n_t^i} dt \tag{4.2}$$

$$+ \text{ Return processes} \tag{4.3}$$

where the return processes are specified below.

First, we assume the existence of two types of domestic firms that differ in the type of capital they use. Specifically, a fraction $\eta$ of them employs exclusively "domestic capital" which is entirely exposed to domestic shocks, and a fraction $1 - \eta$ employs "international capital" exposed to international shocks. In the current setting, the distinction between the two types of firms is very coarse in that each is exposed to only one shock and $\eta$ is exogenous.

Given this, capital held by individual investors can be shown to evolve according to the following Ito process

$$\frac{dk_t^i}{k_t^i} = (\Phi(\iota)) - \delta)dt + \eta\sigma dZ_t + (1 - \eta)\sigma^* dZ_t^* \tag{4.4}$$

where $\Phi(\iota)$ represents a form of capital adjustment cost. The price of capital per unit is $q_t$ and will be a constant in equilibrium, as shown below, hence we will drop the time subscript on this variable. Capital produces $a - \iota$ units of output per unit, and $\iota$ is the optimal investment rate given $q$. Moreover, the return on capital, i.e. on the domestic stock market, for the individual investor will have an idiosyncratic risk component, which approximates a financial friction related to an imperfect equity-investment technology, and therefore will evolve according to

$$dr_t^{K^i} = \underbrace{(\frac{a-\iota}{q} + \Phi(\iota) - \delta)}_{\equiv \mu^K} dt + \eta\sigma dZ_t + (1-\eta)\sigma^* dZ_t^* + \tilde{\sigma} d\tilde{Z}_t^i \tag{4.5}$$

Second, the return on the foreign stock is assumed to follow an exogenous process of the form

$$dr_t^S = \mu^S dt + \sigma^S dZ_t^* \tag{4.6}$$

Third, the return of the domestic currency bond is going to be the same as the return on aggregate domestic wealth

$$dr_t^P = \frac{dN_t}{N_t} = \theta^K dr_t^K + \theta^P dr_t^P + \theta^* r^* dt + \theta^S dr_t^S - \rho dt \tag{4.7}$$

where we have already substituted for the equilibrium condition $\frac{c_t^i}{n_t^i} = \rho$, which we obtain given log utility. Given these elements, we can turn to the solution of the model through the martingale approach.

### 4.2.2 Solution

Using the martingale approach, it can be shown that, given log utility, $\frac{\theta^K}{1-\theta^P}\eta\sigma$ is the price of domestic risk, $\frac{\theta^K(1-\eta)\sigma^*+\theta^S\sigma^S}{1-\theta^P}$ is the price of international risk and $\theta^K\tilde{\sigma}$ is the price of idiosyncratic risk. For easiness of notation, notice that we drop the $i$ superscript on the portfolio weight, given that it will be shown below that such shares are equal across agents. The pricing conditions are as follows.

- Domestic relative to foreign currency/bond:

$$\mu^N - r^* = (\sigma^N)^T \sigma^N = (\frac{\theta^K}{1-\theta^P}\eta\sigma)^2 + (\frac{\theta^K(1-\eta)\sigma^* + \theta^S\sigma^S}{1-\theta^P})^2 \tag{4.8}$$

  where $\sigma^N$ is the diffusion coefficient for aggregate wealth.

- Individual entire portfolio relative to foreign bond:

$$\rho + \mu^N - r^* = (\sigma^N)^T \sigma^N + (\theta^K\tilde{\sigma})^2 \tag{4.9}$$

- International stock relative to foreign bond:

$$\mu^S - r^* = \sigma_2^N \sigma^S = \frac{\theta^K(1-\eta)\sigma^* + \theta^S\sigma^S}{1-\theta^P}\sigma^S \tag{4.10}$$

From (4.8)-(4.10), we can solve for all the portfolio weights. We get portfolio weight on domestic stock $\theta^K$ by subtracting equation (4.9) from (4.8),

$$\theta^K = \frac{\sqrt{\rho}}{\tilde{\sigma}} \tag{4.11}$$

For $\theta^P$ and $\theta^S$, we have from (4.8)

$$1-\theta^P = \frac{(\theta^K\eta\sigma)^2 + (\theta^K(1-\eta)\sigma^* + \theta^S\sigma^S)^2}{\theta^K(\mu^K - r^*) + \theta^S(\mu^S - r^*) - \rho} \tag{4.12}$$

and from (4.10)

$$1-\theta^P = \frac{(\theta^K(1-\eta)\sigma^* + \theta^S\sigma^S)\sigma^S}{\mu^S - r^*} \tag{4.13}$$

and we can solve for $\theta^S$ by equating (4.12) and (4.13), we get

$$\theta^S = \frac{(\mu^S - r^*)[(\theta^K\eta\sigma)^2 + (\theta^K(1-\eta)\sigma^*)^2] - (\theta^K(\mu^K - r^*) - \rho)[\theta^K(1-\eta)\sigma^*\sigma^S]}{(\theta^K(\mu^K - r^*) - \rho)(\sigma^S)^2 - \theta^K(\mu^S - r^*)(1-\eta)\sigma^S\sigma^*} \tag{4.14}$$

### 4.2.3 Equity Home Bias

Before going into general discussion about parameter values, let us look at some special cases to get some intuition. Now let $\sigma = \sigma^* = \sigma^S$ and $\theta^K(\mu^S - r^*) = \theta^K(\mu^K - r^*) - \rho$ (see appendix for details), we have immediately

$$\theta^S = \theta^K(2\eta - 1) \tag{4.15}$$

Remember $\eta$ is the fraction of domestic firms and $(1-\eta)$ the fraction of invoicing firms.

When $\eta = 1$, that is all the firms are domestic firms, we have $\theta^S = \theta^K$ and $EHB = 50\%$. This is the standard theory predicting diversification of stock investment - domestic investors should invest partly domestically and partly internationally to optimally share risks.

However, with foreign direct investment and international trade, the larger the fraction of firms exposed to international risk, the smaller the optimal portfolio weight on international stock. Because these firms have exposed investors to international risk through domestic equity market, international stocks become less safe but more risky for domestic investors.

When $\eta = 0$, that is all the firms are exposed to international risk, we have $\theta^S = -\theta^K$ and $EHB = -50\%$. Domestic investors would optimally short the international stock to hedge international risk, if they could possibly do so. This is our story for the international diversification puzzle - which is not a puzzle when invoicing firms and international/dollar risk are taken into account. Additionally, since it is costly to short international/foreign stock market due to financial market restrictions, in most cases the best domestic investors can do is to hold zero international stock in their portfolio. We study below a version of the benchmark model incorporating this type of portfolio constraint and show that it leads to some further interesting insights.

**Safe Asset Demand**

Solving for all the portfolio weights, we have

$$\theta^K = \frac{\sqrt{\rho}}{\tilde{\sigma}} \tag{4.16}$$

$$\theta^S = \theta^K(2\eta - 1) \tag{4.17}$$

$$\theta^P = 1 - \frac{\eta\sigma^2}{\mu^S - r^*}\theta^K \tag{4.18}$$

$$\theta^* = \frac{\sigma^2 - 2(\mu^S - r^*)}{\mu^S - r^*}\eta\theta^K \tag{4.19}$$

For domestic bond to have a positive value, and consequentially $\theta^P > 0$, we need

$$\eta < \frac{\mu^S - r^*}{\theta^K\sigma^2} \tag{4.20}$$

Larger fraction of international capital (small $\eta$) actually helps stabilize domestic currency and prevents a sudden stop. The intuition is that domestic bonds (or alternatively, currency) is useful in hedging against idiosyncratic risk. Therefore, as idiosyncratic risk increases, $\theta^K$ falls and the threshold for which a domestic bond can be sustained is higher, as in M. K. Brunnermeier and Sannikov 2019. However, for a given level of idiosyncratic risk, higher levels of $\eta$ mean a closer economy, largely exposed to domestic risk. This decreases the willingness to hold domestic bonds, which are basically claims on the domestic wealth. This is a novel insight of this project.

Focusing on equilibrium where domestic bond has a positive value, thus for demand of international safe asset, we have $\theta^* < 0$ if $\eta > \frac{1}{2\theta^K}$ (see appendix). With less international capital (larger $\eta$), the small country would optimally lend to the world in the international safe asset.

### 4.2.4 Portfolio Constraint

We want to take one step further to study the interaction between openness and safe assets demand due to market frictions. Therefore, in this section, we depart from the benchmark model by imposing a portfolio constraint for international stock holding $\theta^S \geq \underline{\theta} \geq 0$. The idea is as follows. In a global crisis, international stocks become extremely illiquid since investors across the world want to sell their equity and turn to safe assets. We capture this idea of liquid safe asset and illiquid international stock in an extreme way, which is to impose a portfolio constraint on the international stock market, that is $\theta^S \geq \underline{\theta}$. If $\underline{\theta} = 0$, we have a no-short-selling constraint. We also allow for $\underline{\theta}$ to be positive, which could arise if the country has some fixed international financial linkage, such as sovereign-wealth funds or stock-market margins enabling investors to participate.

#### Solution with constraint

For simplicity and without loss of generality we assume $\underline{\theta} = 0$. With an exogenous and fixed $\eta$ and logarithmic utility, the solution is simple. The constraint is slack when $\eta > \frac{1}{2}$, meaning more than half of the firms are only exposed to domestic risk and less than half of the firms are using international capital and exposed to international risk. Then we have the solution as in the benchmark model.

The constraint is binding when $\eta < \frac{1}{2}$. The solutions for the portfolio weights in this setting, which we denote with an underline, are

$$\underline{\theta}^K = \theta^K = \frac{\sqrt{\rho}}{\tilde{\sigma}} \tag{4.21}$$

$$\underline{\theta}^S = 0 \tag{4.22}$$

$$\underline{\theta}^P = 1 - \frac{(\sigma\eta)^2 + ((1-\eta)\sigma^*)^2}{\theta^K(\mu^K - r^*) - \rho}(\theta^K)^2 \tag{4.23}$$

$$\underline{\theta}^* = (\frac{(\sigma\eta)^2 + ((1-\eta)\sigma^*)^2}{\theta^K(\mu^K - r^*) - \rho}\theta^K - 1)\theta^K \tag{4.24}$$

**Excess demand of safe asset**

By comparing the current setting with the unconstrained case and using our special parameter assumption, we get

$$\underline{\theta}^* - \theta^* = -\frac{(\eta - \eta^2 - (1-\eta)^2)\sigma^2 - (2\eta - 1)(\mu^K - r^* - \frac{\rho}{\theta^K})}{\mu^K - r^* - \frac{\rho}{\theta^K}} \tag{4.25}$$

Notice $\underline{\theta}^* - \theta^*$ is the excess demand of international safe assets in the constrained economy compared to the unconstrained economy, which we identify with normal times. Given parameter restrictions (see Appendix), we have $\underline{\theta}^* - \theta^* > 0$, that is the excess demand due to financial friction is always positive.

It follows that the excess demand for domestic bond is negative

$$\underline{\theta}^P - \theta^P = -\underline{\theta}^* + \theta^* + \theta^S < 0 \tag{4.26}$$

that is, the demand for domestic bond is depressed because of illiquid international stock market. For domestic bond to have a positive value, it requires

$$\eta^2 + (1-\eta)^2 < \frac{\mu^S - r^*}{\theta^K \sigma^2} \tag{4.27}$$

Compared to the unconstrained case where

$$\eta < \frac{\mu^S - r^*}{\theta^K \sigma^2} \tag{4.28}$$

We see that in the constrained economy, openness greatly weakens domestic currency instead of stabilizing as in the unconstrained economy. In fact, the left hand-side in (4.27) is increasing in $\eta$ (remember, falling $\eta$ means increasing opennes), whereas it is falling in this same parameter for the unconstrained case, as evidenced by (4.28). The intuition behind lies in the impossibility of unloading international risk by selling international equities and therefore reverting to the international safe asset. The latter has two pieces of advantage compared with domestic bonds, i.e. it hedges against international risk and also is not exposed to domestic risk.

**Risk premia**

Constrained case: $\frac{\theta^K}{1-\underline{\theta}^P}\eta\sigma$ is the price of domestic risk, $\frac{\theta^K(1-\eta)\sigma^*}{1-\underline{\theta}^P}$ is the price of international/dollar risk and $\theta^K\tilde{\sigma}$ is the price of idiosyncratic risk.

Unconstrained case: $\frac{\theta^K}{1-\theta^P}\eta\sigma$ is the price of domestic risk, $\frac{\theta^K(1-\eta)\sigma^*+\theta^S\sigma^S}{1-\theta^P}$ is the price of international risk and $\theta^K\tilde{\sigma}$

The price of domestic risk in the constrained economy is lower than in the unconstrained economy, whereas the price of international risk in the constrained economy is higher than unconstrained case. This is consistent with what we have shown in the preceding subsection, as when you cannot off-load international risk the price of such risk will necessarily increase. Moreover, in the constrained case, the demand for domestic bonds drops dramatically, which lowers the price of domestic risk.

**Switching regimes**

In normal times, international equity market is liquid enough for investors to buy and sell. However, liquidity drains when there is a crisis. As a result, we think of the unconstrained economy as normal regime and constrained economy as crisis regime.

If the small country's exposure to international risk $1-\eta$ is large, that is high openness, a sudden global shock can have a big impact on the economy and create domestic turmoils. Switching from the normal regime to crisis one will depress demand for domestic bond, lower the price of domestic risk, increase the demand for international safe assets as well as the price of international risk, as in Figure 4.4. Capital globalization and high openness is good in normal times but bad in crisis, both in terms of welfare and domestic currency stabilization. This could potentially contribute to the discussion of capital control policy.

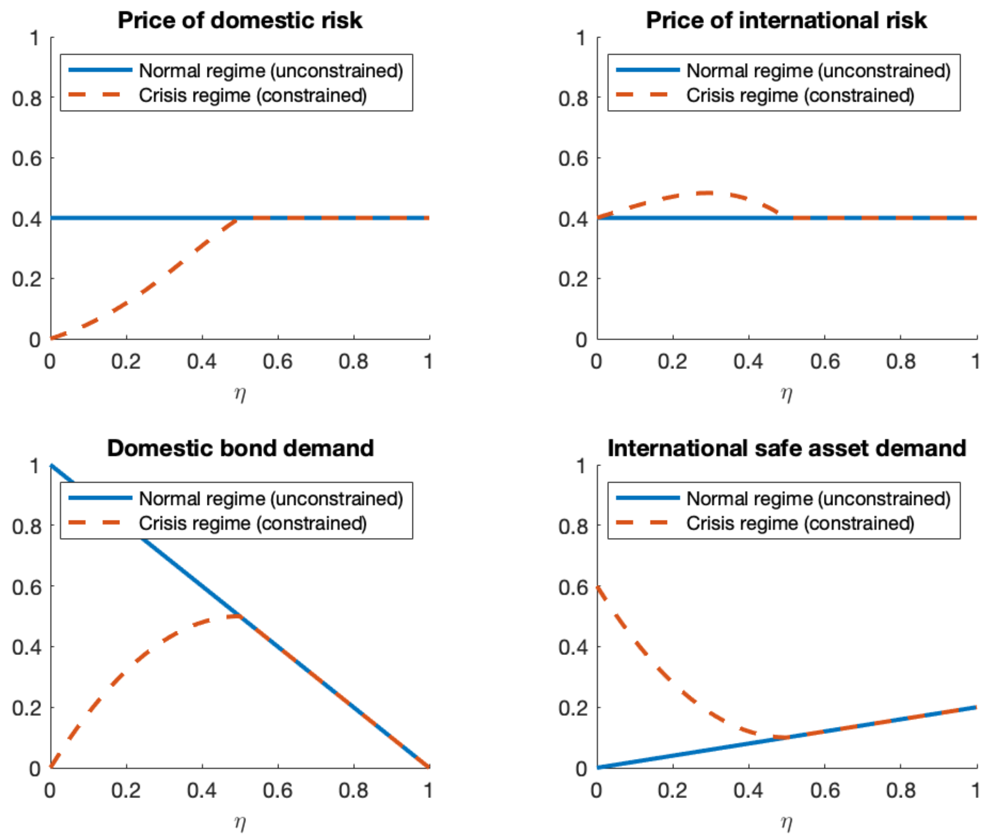


Figure 4.4: Normal vs crisis regime: $\frac{\mu^S - r^*}{\sigma^2} = 0.4$ and $\theta^K = 0.4$

# Appendix A

# Supplement proof and graphs

## A.1 Technical details and extensions

### A.1.1 Two-period model

First-order conditions

$$1 + r^f = \frac{(1+\rho^e)U'(C_0^e)}{U'(C_1^e)} = \frac{(1+\rho)U'(C_0^s)}{U'(C_1^s)} \tag{A.1}$$

$$q = \frac{aU'(C_1^e)}{(1+\rho)U'(C_0^e)} = \frac{a}{1+r^f} \tag{A.2}$$

$$B^s = \frac{1}{2+\rho}W_0^s \tag{A.3}$$

Wealth inequality tomorrow:

$$\eta_1 = \frac{aK + B^e(1+r^f)}{aK + B^e(1+r^f) + B^s(1+r^f)} = \frac{1}{1 + \frac{B^s}{qK+B^e}} \tag{A.4}$$

Using market clearing condition for bonds and capital,

$$1 + r^f = \frac{(1+\rho^e)(2+\rho)aK}{W_0^s(2+\rho^e) + W_0^e(2+\rho)}$$

$$q = \frac{a}{1+r^f}$$

$$\eta_1 = 1 - \frac{W_0^s(1+\rho^e)}{W_0^s(2+\rho^e) + W_0^e(2+\rho)}$$

Consumption of entrepreneurs and savers are

$$C_0^e = W_0^e + \frac{1}{2+\rho} W_0^s \tag{A.5}$$

$$C_1^e = \eta_1 a K \tag{A.6}$$

$$C_0^s = \frac{1+\rho}{2+\rho} W_0^s \tag{A.7}$$

$$C_1^s = (1-\eta_1) a K \tag{A.8}$$

where $\rho^e = \rho + \delta^e$. Comparative statics with respect to $\delta^e$:

$$\frac{\partial \eta_1}{\partial \delta^e} < 0 \tag{A.9}$$

$$\frac{\partial V^s}{\partial \delta^e} = \frac{1}{1+\rho} U'(C_1^s) \underbrace{B^s}_{>0} \underbrace{\frac{\partial r^f}{\partial \delta^e}}_{>0} > 0 \tag{A.10}$$

### A.1.2 Full model

The HJB equation for entrepreneurs' problem (A.155) is

$$\rho^e V^{e,i}(W^{e,i}) = \max_{\{c_t^{e,i}, \theta_t^{k,i}, \theta_t^{oe,i}, \theta_t^{mf}\}_{t=0}^{\infty}} \left\{ \log c^{e,i} + V'(W^{e,i}) W^{e,i} \mu_t^{w,e,i} + \frac{1}{2} V''(W^{e,i}) (W^{e,i} \tilde{\pi}^{w,e,i})^2 + \lambda_t^i \left[ (1-\underline{\chi}) \theta_t^{k,i} + \theta_t^{oe,i} \right] \right\}$$

where

$$\frac{dW^{e,i}}{W^{e,i}} = \mu_t^{w,e,i} dt + \tilde{\pi}^{w,e,i} d\tilde{Z}_t^i \tag{A.11}$$

$$\mu_t^{w,e,i} = \frac{-c^{e,i}}{W^{e,i}} + r_t^f + \theta_t^{k,i} \frac{\mathbb{E}\left[ dr_t^{k,i} - r_t^f dt \right]}{dt} + \theta_t^{oe,i} \frac{\mathbb{E}\left[ dr_t^{oe,i} - r_t^f dt \right]}{dt} + \theta_t^{mf} \frac{\mathbb{E}\left[ dr_t^{mf} - r_t^f dt \right]}{dt} \tag{A.12}$$

$$\tilde{\pi}^{w,e,i} = (\theta_t^{k,i} + \theta_t^{oe,i}) \tilde{\sigma} \tag{A.13}$$

Guess a value function $V^{E,i}(W^{e,i}) = \gamma_t + \rho^e \log W_t^{e,i}$ and take first order conditions, we have

$$c_t^{e,i} = \rho^e W_t^{e,i} \tag{A.14}$$

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - \lambda_t^i(1 - \underline{\chi}) \tag{A.15}$$

$$\frac{\mathbb{E}\left[dr_t^{oe,i} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - \lambda_t^i \tag{A.16}$$

$$\frac{\mathbb{E}\left[dr_t^{mf} - r_t^f dt\right]}{dt} = 0 \tag{A.17}$$

The HJB equation for savers' problem (1.17) is

$$\rho V^{s,j}(W^{s,j}) = \max_{\{c^{s,j}\}_{t=0}^{\infty}} \left\{ \log c^{s,j} + V'(W^{s,j}) W^{s,j} \mu_t^{w,s,j} \right\}$$

where

$$\frac{dW^{s,j}}{W^{s,j}} = \mu_t^{w,s,j} dt \tag{A.18}$$

$$\mu_t^{w,s,j} = \frac{-c^{s,j}}{W^{s,j}} + r_t^f + \alpha_t^{mf} \frac{dr_t^{mf} - r_t^f dt}{dt} \tag{A.19}$$

$$\tag{A.20}$$

Guess a value function $V^{s,j}(W^{s,j}) = \beta_t + \rho \log W_t^{s,j}$ and take first order conditions, we have

$$c_t^{s,j} = \rho W_t^{s,j} \tag{A.21}$$

Note that savers do not carry any idiosyncratic risks because the stock market index fund diversifies idiosyncratic risks.

**Fundamental equilibrium**

Market clearing conditions for fundamental economy:

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = (\rho^e \eta_t^f + \rho(1 - \eta_t^f)) q_t K_t \tag{A.22}$$

$$V_t^{mf} = V_t^{oe} \tag{A.23}$$

Public equity is exactly the pooled outside equities issued by entrepreneurs, so we have the return of public equity equals to the expected return of outside equity.

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right] \tag{A.24}$$

And since the public equity does not carry idiosyncratic risks, the return of public equity also equals to the risk-free rate in equilibrium,

$$\mathbb{E}_t\left[dr_t^{oe,i}\right] = dr_t^{mf} = r_t^f dt \tag{A.25}$$

From entrepreneurs' first order conditions, we have

$$\lambda_t^i = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0$$

so the equity constraint binds, we have

$$\theta_t^{k,i} + \theta_t^{oe,i} = \underline{\chi}\theta_t^{k,i} \tag{A.26}$$

**Steady state of fundamental economy**

In fundamental economy, the total wealth in the economy is $q_t K_t$. The value of bubbble $P_t = 0$. Define entrepreneurs' wealth share as $\eta_t = \frac{W_t^e}{W_t^e + W_t^s} = \frac{W_t^e}{q_t K_t}$.

From first order conditions, we have asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\eta_t^f} \tag{A.27}$$

In equilibrium we also have

$$dr_t^{mf} = \mathbb{E}_t\left[dr_t^{oe,i}\right] = r_t^f dt \tag{A.28}$$

Optimal consumption ratio with logarithmic utilities: $\frac{c_t^e}{W_t^e} = \rho^e$ and $\frac{C_t^s}{W_t^s} = \rho$.

We can now derive the evolution of entrepreneurs wealth share using Ito's lemma,

$$\frac{d\eta_t^f}{\eta_t^f} = \underbrace{(1-\eta_t^f)\left(-\delta^e + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right)}_{\equiv \mu_t^{\eta,f}} dt \tag{A.29}$$

Consumption good's market clearing condition

$$aK_t = C_t^e + C_t^s = \rho^e W_t^e + \rho W_t^s = (\rho^e \eta_t^f + \rho(1-\eta_t^f))q_t K_t \tag{A.30}$$

from which we can solve for capital price $q_t$ as a function of entrepreneurs' wealth share $\eta_t^f$,

$$q_t^f = \frac{a}{\delta^e \eta_t^f + \rho} \tag{A.31}$$

The risk-free rate is given by

$$r_t^f = \rho + \mu_t^{c,s,f} = \rho^e + \mu_t^{c,e,f} - \underbrace{\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2}_{\text{precautionary saving motive}} \tag{A.32}$$

where $\mu_t^{c,s,f}$ and $\mu_t^{c,e,f}$ are the growth rates of savers' consumption and entrepreneurs' consumption respectively. Since in equilibrium we have $C_t^e = \rho^e \eta_t^f q_t K_t$ and $C_t^s = \rho(1-\eta_t^f)q_t K_t$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,f} dt + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i = \big(\mu_t^{\eta,f} + \mu_t^q + g\big)dt + +\tilde{\pi}^{c,e,i} d\tilde{Z}_t^i \tag{A.33}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,f} dt = \big(-\frac{\eta_t^f}{(1-\eta_t^f)}\mu_t^{\eta,f} + \mu_t^q + g\big)dt \tag{A.34}$$

Solving for steady state, that is when $\mu_t^{\eta,f} = 0$ and $\mu_t^q = 0$. Combining (1.22) and (A.30), we have

$$\overline{q}^f = \frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{A.35}$$

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} \tag{A.36}$$

$$\overline{p}^f = 0 \tag{A.37}$$

Here we focus on a non-degenerate steady state wealth distribution and requires that

$$\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} < 1 \tag{A.38}$$

And at the steady state we have

$$\overline{r}^f = \rho + g \tag{A.39}$$

and we also have the risk premium of capital at the steady state

$$\frac{\mathbb{E}[d\overline{r}^{k,i} - \overline{r}^f dt]}{dt} = \frac{(\underline{\chi}\tilde{\sigma})^2}{\overline{\eta}^f} = \underline{\chi}\tilde{\sigma}\sqrt{\delta^e} \tag{A.40}$$

**Connection with two-period model: a special deterministic case**

Initial conditions: Entrepreneurs have initial wealth $W_0^e = \eta_0 aK$ and savers have initial wealth $W_0^s = (1-\eta_0)aK$.

In this case, we have in general equilibrium:

$$\eta_t^f = \frac{1}{e^{\delta^e t + c} + 1} \tag{A.41}$$

$$r_t^f = \frac{(\rho+\delta^e)\delta^e\eta_t^f}{\delta^e\eta_t^f + \rho} + \rho \tag{A.42}$$

where $c = \log\frac{1-\eta_0}{\eta_0}$ and I set $c = 1$ hereafter without loss of generality. we have

$$\frac{\partial\eta_t^f}{\partial\delta^e} = -t(1-\eta_t^f)\eta_t^f < 0 \tag{A.43}$$

and

$$\int_0^t r_s^f ds = -\ln\left(\rho e^{\delta^e t} + \rho + \delta^e\right) + (\rho+\delta^e)t + \ln(2\rho+\delta^e) \tag{A.44}$$

$$\frac{\partial r_t^f}{\partial\delta^e} = \frac{\eta_t^f[(\delta^e)^2\eta_t^f - \rho(\rho+\delta^e)\delta^e(1-\eta_t^f)t + 2\rho\delta^e]}{[\delta^e\eta_t^f + \rho]^2} \tag{A.45}$$

$$\frac{\partial(\int_0^t r_s^f ds)}{\partial\delta^e} = \frac{(\rho+\delta^e)t - 1}{\rho e^{\delta^e t} + \rho + \delta^e} + \frac{1}{2\rho+\delta^e} > 0 \tag{A.46}$$

$$V^s = \frac{\log C_0^s}{\rho} + \int_0^\infty e^{-\rho t} \underbrace{\int_0^t \frac{\rho(\delta^e)\eta_s^{W,f}}{\delta^e \eta_s^{W,f} + \rho} ds}_{R_t^h} \, dt \tag{A.47}$$

$$R_t^h = \delta^e t - \ln\left(\frac{\rho e^{\delta^e t} + \rho + \delta^e}{\rho + \delta^e}\right) + c_1 \tag{A.48}$$

$$\frac{\partial R_t^h}{\partial \delta^e} = \frac{\rho t}{\rho e^{\delta^e t} + \rho + \delta^e} + \frac{\rho e^{\delta^e t}}{\rho + \delta^e(\rho e^{\delta^e t} + \rho + \delta^e)} - \frac{\rho}{(\rho + \delta^e)(2\rho + \delta^e)} \geq 0 \tag{A.49}$$

In general equilibrium with endogenous risk-free rate $r_t^f$, wealth inequality $\eta_t^f$ changes respond to changes of asset pay-off $\{a\}$ and relative "impatience" of entrepreneurs $\{\delta^e\}$ as follows,

$$\frac{\partial \eta_t^f}{\partial a} = 0; \quad \frac{\partial \eta_t^f}{\partial \delta^e} < 0$$

Savers' welfare $V^{s,f}$ responds to changes of asset pay-off $\{a\}$ and relative "impatience" of entrepreneurs $\{\delta^e\}$ as follows,

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial a} > 0$$

and

$$\frac{\partial (V^{s,f}\big|_0^\infty)}{\partial \delta^e} = \int_0^\infty e^{-\rho t} U'(C_t^s) \rho \underbrace{B_t \frac{\partial (\int_0^t r_s^f ds)}{\partial \delta^e}}_{>0} dt > 0 \tag{A.50}$$

### A.1.3 Full model: bubble equilibrium

#### Intuition for bubble

As shown in figure 1.2, the value of the stock market index fund is

$$V_t^{mf} = \underbrace{(1-\chi_t) q_t K_t}_{\text{Value of outside equities}, V_t^{oe}} + \underbrace{P_t}_{\text{Bubble}} \tag{A.51}$$

where $1-\chi_t$ is the fraction of capital issued as outside equity in the aggregate. Since entrepreneurs are identical before idiosyncratic risk is realized, we have $\chi_{it} = \chi_t$. The value of outside equities is denoted as $V_t^{oe} = (1-\chi_t) q_t K_t$ and the value of bubble is denoted as $P_t$, which is also an endogenous process.

Since $\chi_t = \chi_{it} \geq \underline{\chi}$, we have

$$V_t^{oe} \leq (1-\underline{\chi}) q_t K_t$$

If there is no bubble, we simply have outside equity market clears as

$$V_t^{oe} = V_t^{mf}$$

The value of stock market index fund held by entrepreneurs is also constrained as

$$V_t^{oe} = V_t^{mf} \leq (1 - \underline{\chi}) q_t K_t \tag{A.52}$$

A bubble with positive value relaxes this constraint for the stock market index fund because now we have

$$V_t^{mf} = V_t^{oe} + P_t$$

and the constraint (A.52) becomes

$$V_t^{mf} \leq (1 - \underline{\chi}) q_t K_t + P_t \tag{A.53}$$

Bubbles relax the indirect limit on public equity due to the skin-in-the-game constraint.

**Bubble equilibrium**

Market clearing conditions for bubble economy:

$$aK_t = (\rho^e \eta_t^b + \rho(1 - \eta_t^b))(q_t K_t + P_t) \tag{A.54}$$

$$V_t^{mf} = V_t^{oe} + P_t \tag{A.55}$$

Since $V_t^{mf} = V_t^{oe} + P_t$, the public equity is the pooled outside equity plus the bubble, the return of public equity is now

$$dr_t^{mf} = \frac{V_t^{oe}}{V_t^{mf}} \mathbb{E}_t \left[ dr_t^{oe,i} \right] + \frac{P_t}{V_t^{mf}} \frac{dP_t}{P_t} \tag{A.56}$$

The public equity does not carry any idiosyncratic risk, so we have in equilibrium

$$dr_t^{mf} = r_t^f dt \tag{A.57}$$

We need to use other equilibrium conditions to determine the return of outside equity and the amount of outside equity issuance. And in equilibrium, we have the return of outside equity equals to the return of inside equity.

$$\frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} = \frac{\mathbb{E}\left[dr_t^{oe} - r_t^f dt\right]}{dt} = (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \tag{A.58}$$

and it follows that

$$\lambda_t^i = 0 \tag{A.59}$$

which implies $\chi_t$ can be any value in $[\underline{\chi}, 1]$.

**Equilibrium refinement** To determine the amount of outside equity issuance, we perturb the bubble equilibrium by allowing "trembling hands" of agents. Assume that there is $\epsilon > 0$ chance that agents play for the fundamental equilibrium and $1 - \epsilon$ chance for the bubble equilibrium. We have

$$\frac{\mathbb{E}\left[dr_t^{oe} - r_t^f dt\right]}{dt} = (1-\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \tag{A.60}$$

which implies

$$\lambda_t^i = \epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 > 0 \tag{A.61}$$

and the equity constraint binds, $\chi_t = \underline{\chi}$. The return of capital is as follows

$$\begin{aligned} \frac{\mathbb{E}\left[dr_t^{k,i} - r_t^f dt\right]}{dt} &= (\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 - (1-\underline{\chi})\epsilon(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \\ &= (1 - \epsilon + \underline{\chi}\epsilon)(\theta_t^{k,i} + \theta_t^{oe,i})\tilde{\sigma}^2 \end{aligned} \tag{A.62}$$

Taking the limit of $\chi_t$ as $\epsilon \to 0$, we have

$$\lim_{\epsilon \to 0} \chi_t = \underline{\chi} \tag{A.63}$$

As long as there is a positive possibility of the fundamental equilibrium, we can not find a sequence of mixed strategies converging to $\chi_t \neq \underline{\chi}$. And we focus on the trembling-hand perfect equilibrium where $\chi_t = \underline{\chi}$ for the following analysis. This trembling-hand perfect equilibrium creates the highest value of bubble.

**Steady state in bubble equilibrium**

In bubble economy, the total wealth of the economy is $q_tK_t + P_t = q_tK_t + (1-\underline{\chi})p_tK_t$.

From entrepreneur's optimization problem, we have asset pricing equation for capital

$$\frac{\mathbb{E}[dr_t^{k,i} - r_t^f dt]}{dt} = \underbrace{\frac{q_t\underline{\chi}\tilde{\sigma}}{\eta_t^b[\underline{\chi}q_t + (1-\underline{\chi})(p_t+q_t)]}}_{\text{price of risk}} \underbrace{\tilde{\sigma}}_{\text{risk}} \tag{A.64}$$

In equilibrium, we still have

$$dr_t^{mf} = r_t^f dt \tag{A.65}$$

as these assets are all risk-free.

And to determine the value of bubble, we write the return of public equity as

$$\begin{aligned} dr_t^{mf} &= \underbrace{\frac{(1-\underline{\chi})aK_t}{V_t^{mf}}}_{\text{dividend yield}} dt + \underbrace{\frac{dV_t^{mf}}{V_t^{mf}}}_{\text{capital gain}} = \frac{a}{p_t+q_t}dt + \frac{d((p_t+q_t)K_t)}{(p_t+q_t)K_t} \\ &= \left(\frac{a + p_t\mu_t^p + q_t\mu_t^q}{p_t+q_t} + g\right)dt \end{aligned} \tag{A.66}$$

where $1-\underline{\chi}$ is the fraction of capital issued as outside equity, and $V_t^{mf}$ is the value of mutual fund.

And we have consumption good market clearing condition

$$aK_t = (\rho^e\eta_t^b + \rho(1-\eta_t^b))(q_tK_t + P_t) \tag{A.67}$$

Derive the evolution of entrepreneurs' wealth share in the bubbly economy, we have

$$\frac{d\eta_t^b}{\eta_t^b} = \underbrace{(1-\eta_t^b)\left(-\delta^e + \left(\frac{\underline{\chi}q_t\tilde{\sigma}}{\eta_t^b[\underline{\chi}q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2\right)}_{\equiv\mu_t^{\eta,b}} dt \tag{A.68}$$

The risk-free rate is given by

$$r_t^f = \rho + \mu_t^{c,s,b} = \rho^e + \mu_t^{c,e,b} - \underbrace{\left(\frac{\underline{\chi}q_t\tilde{\sigma}}{\eta_t^b[\underline{\chi}q_t + (1-\underline{\chi})(q_t+p_t)]}\right)^2}_{\text{precautionary saving motive}} \tag{A.69}$$

where $\mu_t^{c,s,b}$ and $\mu_t^{c,e,b}$ are the growth rates of savers' consumption and entrepreneurs' consumption respectively. Since in equilibrium we have $C_t^e = \rho^e \eta_t^b (q_t K_t + P_t)$ and $C_t^s = \rho(1 - \eta_t^b)(q_t K_t + P_t)$, we have

$$\frac{dc_t^{e,i}}{c_t^{e,i}} = \mu_t^{c,e,b} dt + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i = \big(\mu_t^{\eta,b} + \frac{(1-\underline{\chi})p_t \mu_t^p + q_t \mu_t^q}{q_t + (1-\underline{\chi})p_t} + g\big)dt + + \tilde{\pi}^{c,e,i} d\tilde{Z}_t^i \tag{A.70}$$

$$\frac{dc_t^{s,j}}{c_t^{s,j}} = \mu_t^{c,s,b} dt = \big(-\frac{\eta_t^b}{(1-\eta_t^b)}\mu_t^{\eta,b} + \frac{(1-\underline{\chi})p_t \mu_t^p + q_t \mu_t^q}{q_t + (1-\underline{\chi})p_t} + g\big)dt \tag{A.71}$$

We solve for prices $q_t$ and $p_t$ as functions of the state variable $\eta_t^b$. Combining (A.64), (A.65), (A.66), and (A.67), and use Ito's lemma, we have

$$\frac{a}{q_t} + \mu_t^q - \frac{a + p_t \mu_t^p + q_t \mu_t^q}{p_t + q_t} = \frac{\underline{\chi} q_t (\tilde{\sigma})^2}{\eta_t^b [\underline{\chi} q_t + (1-\underline{\chi})(p_t + q_t)]} \tag{A.72}$$

$$\mu_t^q = q'(\eta_t^b)\eta_t^b \mu_t^{\eta,b} \tag{A.73}$$

$$\mu_t^p = p'(\eta_t^b)\eta_t^b \mu_t^{\eta,b} \tag{A.74}$$

After some algebra, we have an ordinary differential equation as follows

$$q'(\eta) - p'(\eta) = \left(\frac{\underline{\chi}\tilde{\sigma}^2 q}{\eta(q + (1-\underline{\chi} p))}\frac{p+q}{p} - \frac{a}{q}\right)\frac{1}{\eta \mu^\eta} \tag{A.75}$$

and recall that in bubble equilibrium we have

$$\mu^\eta = (1-\eta_t^b)\left(-\delta^e + \left(\frac{\underline{\chi} q_t \tilde{\sigma}}{\eta_t^b[\underline{\chi} q_t + (1-\underline{\chi})(q_t + p_t)]}\right)^2\right) \tag{A.76}$$

Combining equation (A.75), (A.76) and market clearing condition

$$a = (\rho^e \eta_t^b + \rho(1-\eta_t^b))(q_t + (1-\underline{\chi})p_t) \tag{A.77}$$

one can solve for $q(\eta)$, $p(\eta)$ and find the transition path for $\eta$ in bubble equilibrium.

Solving for steady state, that is, $\mu_t^{\eta,b} = 0$, $\mu_t^q = 0$, and $\mu_t^p = 0$, we have

$$\overline{q}^b = \frac{a}{\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{A.78}$$

$$\overline{\eta}^b = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e} + (1-\underline{\chi})\frac{\tilde{\sigma}\delta^e}{\rho}} \tag{A.79}$$

$$\overline{p} = \frac{a}{\rho} - \overline{q}^b \tag{A.80}$$

And at the steady state we have risk-free rate

$$\overline{r}^f = \rho + g \tag{A.81}$$

we also have the risk premium of capital at the steady state

$$\frac{\mathbb{E}[d\overline{r}^{k,i} - \overline{r}^f dt]}{dt} = \frac{\underline{\chi}\overline{q}\tilde{\sigma}^2}{\overline{\eta}_t^b[\underline{\chi}\overline{q} + (1-\underline{\chi})(\overline{p}+\overline{q})]} = \tilde{\sigma}\sqrt{\delta^e} \tag{A.82}$$

Note that at the steady state, the risk-free rate in the bubble economy is the same as in the fundamental economy. But capital price, wealth inequality and the amount of borrowing and lending are different.

For a non-degenerate wealth distribution, we need

$$[\underline{\chi} - (1-\underline{\chi})\frac{\delta^e}{\rho}]\tilde{\sigma} < \sqrt{\delta^e} \tag{A.83}$$

**Parameter restriction on $\kappa$**

The first restriction

$$\kappa > \frac{\underline{\chi}}{1-\underline{\chi}}(\frac{\tilde{\sigma}}{\sqrt{\delta^e}} - 1) \tag{A.84}$$

is to make sure that entrepreneurs are borrowers. Given that $\tilde{\sigma} < \frac{\sqrt{\delta^e}}{\underline{\chi}}$ (for non-degenerate wealth distribution), the right hand side of equation (A.84) is strictly smaller than 1.

And the second restriction

$$\kappa > \underline{\chi}(1 - \frac{\tilde{\sigma}}{\sqrt{\delta^e}})\frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho + \sqrt{\delta^e}}\frac{\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} + \rho} \tag{A.85}$$

is to ensure that entrepreneurs borrow more in steady state of bubble equilibrium than in fundamental equilibrium. One can see that the right hand side of equation (A.85) is strictly smaller than 1.

### A.1.4 Inequality and welfare

Recall that

$$\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}$$

we can derive wealth share changes with respect to parameters:

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} = \frac{\underline{\chi}}{\sqrt{\delta^e}} \tag{A.86}$$

$$\frac{\partial \overline{\eta}^f}{\partial \delta^e} = -\frac{\underline{\chi}\tilde{\sigma}}{2\delta^e\sqrt{\delta^e}} \tag{A.87}$$

$$\frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} = \frac{\tilde{\sigma}}{\sqrt{\delta^e}} \tag{A.88}$$

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \overline{\eta}^f}{\partial \delta} < 0; \quad \frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} > 0$$

Recall that in the bubble economy, the entrepreneurs' wealth share at steady state is

$$\overline{\eta}^b = \frac{1}{\frac{\sqrt{\delta^e}}{\underline{\chi}\tilde{\sigma}} + \frac{(1-\underline{\chi})}{\underline{\chi}}\frac{\delta^e}{\rho}}$$

and we can derive the comparative statics of wealth inequality with respect to parameters

$$\frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} = \frac{\underline{\chi}\sqrt{\delta^e}}{(\sqrt{\delta^e} + (1-\underline{\chi})\frac{\delta}{\rho}\tilde{\sigma})^2} \tag{A.89}$$

$$\frac{\partial \overline{\eta}^b}{\partial \delta} = -\frac{\underline{\chi}\tilde{\sigma}(\frac{1}{2\sqrt{\delta^e}} + (1-\underline{\chi})\frac{\tilde{\sigma}}{\rho})}{(\sqrt{\delta^e} + (1-\underline{\chi})\frac{\delta^e}{\rho}\tilde{\sigma})^2} \tag{A.90}$$

$$\frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} = \frac{\rho\tilde{\sigma}(\rho\sqrt{\delta^e} + \delta^e\tilde{\sigma})}{(\rho\sqrt{\delta^e} + (1-\underline{\chi})\delta^e\tilde{\sigma})^2} \tag{A.91}$$

$$\frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \overline{\eta}^b}{\partial \delta} < 0; \quad \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} > 0$$

Comparing with fundamental economy, we have

$$\frac{\partial \overline{\eta}^f}{\partial \tilde{\sigma}} > \frac{\partial \overline{\eta}^b}{\partial \tilde{\sigma}} > 0$$

$$\frac{\partial \overline{\eta}^f}{\partial \delta^e} < \frac{\partial \overline{\eta}^b}{\partial \delta^e} < 0$$

$$0 < \frac{\partial \overline{\eta}^b}{\partial \underline{\chi}} < \frac{\partial \overline{\eta}^f}{\partial \underline{\chi}} \quad \text{if} \quad \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho}(1-\underline{\chi})^2 + 2(1-\underline{\chi}) - 1 > 0$$

Consumption inequality is different from wealth inequality as entrepreneurs and savers have different consumption rates.

In fundamental economy, the aggregate consumption of entrepreneurs and savers are as follows,

$$\overline{C}^{e,f} = (\rho+\delta^e)\overline{\eta}^f \overline{q}^f K_t = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)} aK_t \tag{A.92}$$

$$\overline{C}^{s,f} = \rho(1-\overline{\eta}^f)\overline{q}^f K_t = \left(1 - \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)}\right) aK_t \tag{A.93}$$

Define consumption share as

$$\overline{\eta}^{c,f} = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)} \tag{A.94}$$

and we have

$$\frac{\partial \overline{\eta}^{c,f}}{\partial \tilde{\sigma}} = \frac{\underline{\chi}\rho(\rho+\delta^e)}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{A.95}$$

$$\frac{\partial \overline{\eta}^{c,f}}{\partial \delta^e} = \frac{\underline{\chi}\tilde{\sigma}\rho(\delta^e - 2\underline{\chi}\tilde{\sigma}\sqrt{\delta^e} - \rho)}{2\delta^e\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{A.96}$$

$$\frac{\partial \overline{\eta}^{c,f}}{\partial \underline{\chi}} = \frac{\rho(\rho+\delta^e)\tilde{\sigma}}{\sqrt{\delta^e}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{A.97}$$

we require that steady state wealth inequality $\overline{\eta}^f = \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}} > \frac{1}{2}$. We have

$$\frac{\partial \overline{\eta}^{c,f}}{\partial \tilde{\sigma}} > 0; \quad \frac{\partial \overline{\eta}^{c,f}}{\partial \delta} < 0; \quad \frac{\partial \overline{\eta}^{c,f}}{\partial \underline{\chi}} > 0$$

In bubble economy, the consumption of entrepreneurs and savers are as follows,

$$\overline{C}_t^{e,b} = (\rho+\delta^e)\overline{\eta}^b(\overline{q}^b K_t + \overline{P}_t) = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+r)} aK_t \tag{A.98}$$

$$\overline{C}_t^{s,b} = \rho(1-\overline{\eta}^b)(\overline{q}^b K_t + \overline{P}_t) = \left(1 - \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+r)}\right) aK_t \tag{A.99}$$

we have

$$\overline{C}^{e,b} < \overline{C}^{e,f}$$
$$\overline{C}^{s,b} > \overline{C}^{s,f}$$

Define consumption shares as

$$\overline{\eta}^{c,b} = \frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)} \tag{A.100}$$

and comparative statics with respect to parameters

$$\frac{\partial\overline{\eta}^{c,b}}{\partial\tilde{\sigma}} = \frac{\underline{\chi}\rho(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{A.101}$$

$$\frac{\partial\overline{\eta}^{c,b}}{\partial\delta^e} = \frac{\underline{\chi}\tilde{\sigma}\rho(\delta^e-2\tilde{\sigma}\sqrt{\delta^e}-\rho)}{2\delta\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)^2} \tag{A.102}$$

$$\frac{\partial\overline{\eta}^{c,b}}{\partial\underline{\chi}} = \frac{(\rho+\delta^e)\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)} \tag{A.103}$$

$$\frac{\partial\overline{\eta}^{c,b}}{\partial\tilde{\sigma}} > 0; \quad \frac{\partial\overline{\eta}^{c,b}}{\partial\delta^e} < 0; \quad \frac{\partial\overline{\eta}^{c,b}}{\partial\underline{\chi}} > 0$$

Note that $\frac{\partial\overline{\eta}^{c,b}}{\partial\delta^e} < 0$ because entrepreneurs are richer than savers in fundamental equilibrium which requires $\underline{\chi}\tilde{\sigma} > \frac{1}{2}$.

The value function of savers in fundamental economy is as follows

$$V^{s,f} = \int_0^\infty e^{-\rho t}\left(\underbrace{\log(1-\eta_t^f)}_{\text{wealth distribution}} + \underbrace{\log\rho}_{\text{consumption rate}} + \underbrace{\log\left(q_t^f K_t\right)}_{\text{total wealth}}\right)dt \tag{A.104}$$

The value function of savers in bubble economy is as follows

$$V^{s,b} = \int_0^\infty e^{-\rho t}\left(\underbrace{\log(1-\eta_t^b)}_{\text{wealth distribution}} + \underbrace{\log\rho}_{\text{consumption rate}} + \underbrace{\log(q_t^b K_t + P_t)}_{\text{total wealth}}\right)dt \tag{A.105}$$

And the value function of savers starting from fundamental equilibrium steady state:

$$\begin{aligned}\overline{V}^{s,f} &= \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\log(1-\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}) + \log\frac{a\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho} + \log K_0 + \frac{g}{\rho}\right)dt\right] \\ &= \frac{\log(1-\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}) + \log\frac{a\rho}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho} + \log K_0 + \frac{g}{\rho}}{\rho}\end{aligned} \tag{A.106}$$

We can derive savers' welfare changes with respect to parameters:

$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} < 0; \quad \frac{\partial \overline{V}^{s,f}}{\partial \delta^e} > 0; \quad \frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < 0$$

For discount shocks, we have

$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} = -\frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}}{\underline{\chi}} - \tilde{\sigma}} + \frac{1}{\tilde{\sigma} + \frac{\rho}{\underline{\chi}\sqrt{\delta^e}}}\right) < 0 \tag{A.107}$$

For financial shocks, we have

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} = -\frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}}{\tilde{\sigma}} - \underline{\chi}} + \frac{1}{\underline{\chi} + \frac{\rho}{\tilde{\sigma}\sqrt{\delta^e}}}\right) < 0 \tag{A.108}$$

The value function of savers starting from steady state of bubble equilibrium,

$$\begin{aligned} \overline{V}^{s,b} &= \int_0^\infty e^{-\rho t}\left(\log\left(1 - \frac{\underline{\chi}\tilde{\sigma}(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)}\right) + \log aK_0 + \frac{g}{\rho}\right) dt \\ &= \frac{\log\left(1 - \frac{\underline{\chi}\tilde{\sigma}(\rho+\delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)}\right) + \log aK_0 + \frac{g}{\rho}}{\rho} \end{aligned} \tag{A.109}$$

The existence of bubble benefits savers since

$$\overline{V}^{s,b} > \overline{V}^{s,f} \tag{A.110}$$

Welfare responses to shocks also differ from the fundamental economy. We have savers' welfare changes with respect to parameters

$$\frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0; \quad \frac{\partial \overline{V}^{s,b}}{\partial \delta^e} > 0; \quad \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0 \tag{A.111}$$

For financial shock $\tilde{\sigma}$, we have

$$\frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} = \frac{1}{\rho}\left(\frac{1}{\frac{\sqrt{\delta^e}\rho}{(1-\underline{\chi})\delta^e - \underline{\chi}\rho} + \tilde{\sigma}} - \frac{1}{\tilde{\sigma} + \frac{\rho}{\sqrt{\delta^e}}}\right) < 0 \tag{A.112}$$

Comparing with fundamental economy,

$$\frac{\partial \overline{V}^{s,f}}{\partial \tilde{\sigma}} < \frac{\partial \overline{V}^{s,b}}{\partial \tilde{\sigma}} < 0 \tag{A.113}$$

For financial shock, we have

$$\frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} = -\frac{1}{\rho} \frac{\tilde{\sigma}(\rho + \delta^e)}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e} + \rho) - \underline{\chi}\tilde{\sigma}(\rho + \delta^e)} < 0 \tag{A.114}$$

Comparing with fundamental economy,

$$\frac{\partial \overline{V}^{s,f}}{\partial \underline{\chi}} < \frac{\partial \overline{V}^{s,b}}{\partial \underline{\chi}} < 0 \quad \text{if} \quad \frac{\tilde{\sigma}}{\delta^e}\underline{\chi}^2 - 2\underline{\chi} + 1 < 0 \tag{A.115}$$

### A.1.5 Transition dynamics

The evolution of wealth inequality is as follows,

$$\frac{d\eta_t^f}{\eta_t^f} = (1 - \eta_t^f)\left(-\delta^e + \left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right) dt \tag{A.116}$$

There is strong asymmetry in the transition dynamics due to *leverage* effect of entrepreneurs, as shown by figure A.1. The increase of wealth inequality is much faster than the decrease. This asymmetry due to leverage can help explain the rapid increase of top wealth inequality documented in the literature.

etatransitionf.png

Figure A.1: Transition dynamics

### A.1.6 Extension: CRRA utility

One can characterize the closed-form steady states with general CRRA utilities. Denote $\gamma^e$ and $\gamma$ as the coefficient of relative risk aversion for entrepreneurs and savers respectively.

At steady state of fundamental equilibrium, we have

$$\begin{aligned} \overline{r}^f &= \rho + \gamma g && \text{(A.117)} \\ \overline{\eta}^f &= \frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\frac{2(\delta^e + (\gamma^e - \gamma)g)}{\gamma^e(\gamma^e+1)}}} && \text{(A.118)} \\ \overline{q}^f &= \frac{a}{\rho + (\gamma - 1)g + \underline{\chi}\tilde{\sigma}\sqrt{\frac{2(\delta^e + (\gamma^e - \gamma)g)}{\gamma^e(\gamma^e+1)}}} && \text{(A.119)} \end{aligned}$$

At steady state (that is, constant investment opportunity), we have the risk-free rate

$$\begin{aligned} \overline{r}^f &= \rho^e + \gamma^e g - \frac{\gamma^e(\gamma^e+1)}{2}\left(\frac{\underline{\chi}\tilde{\sigma}}{\overline{\eta}^f}\right)^2 \\ &= \rho + \gamma g \end{aligned} \tag{A.120}$$

from which we can solve for $\overline{\eta}^f$ and $\overline{r}^f$.

And we have constant consumption to wealth ratios at steady state

$$\begin{aligned} \frac{\overline{c}^e}{\overline{W}^e} &= \rho + \frac{(\gamma+1)(\gamma^e-1)}{\gamma^e+1}g + \frac{2}{\gamma^e+1}\delta^e \\ \frac{\overline{c}^s}{\overline{W}^s} &= \rho + (\gamma-1)g \end{aligned} \tag{A.121}$$

We can also derive the evolution of entrepreneurs wealth share using Ito's lemma,

$$\frac{d\eta_t^f}{\eta_t^f} = \underbrace{(1-\eta_t^f)\left(-\left(\frac{c^e}{W^e} - \frac{c^s}{W^s}\right) + \gamma^e\left(\frac{\underline{\chi}\tilde{\sigma}}{\eta_t^f}\right)^2\right)}_{\equiv \mu_t^{\eta,f}} dt \tag{A.122}$$

Consumption good's market clearing condition follows as

$$\begin{aligned} aK_t &= C_t^e + C_t^s \\ &= \left(\rho + \frac{(\gamma+1)(\gamma^e-1)}{\gamma^e+1}g + \frac{2}{\gamma^e+1}\delta^e\right)W_t^e + (\rho + (\gamma-1)g)W_t^s \\ &= \left(\left(\rho + \frac{(\gamma+1)(\gamma^e-1)}{\gamma^e+1}g + \frac{2}{\gamma^e+1}\delta^e\right)\overline{\eta}^f + (\rho + (\gamma-1)g)(1-\overline{\eta}^f)\right)\overline{q}^f K_t \end{aligned} \tag{A.123}$$

from which we can solve for $\overline{q}^f$.

Similarly for bubble equilibrium steady state, we have

$$\begin{aligned}\overline{r}^f &= \rho^e + \gamma^e g - \frac{\gamma^e(\gamma^e+1)}{2}\left(\frac{\underline{\chi}\overline{q}\tilde{\sigma}}{\overline{\eta}_t^b[\underline{\chi}\overline{q} + (1-\underline{\chi})(\overline{p}+\overline{q})]}\right)^2 \\ &= \rho + \gamma g \\ &= \frac{a}{\overline{p}+\overline{q}} + g\end{aligned} \tag{A.124}$$

where the last equation comes from the fact that in equilibrium, outside equities earn the same return as the risk-free bond as they do not carry any risk.

And we have the consumption good market clearing condition

$$\begin{aligned}aK_t &= C_t^e + C_t^s \\ &= (\rho + \frac{(\gamma+1)(\gamma^e-1)}{\gamma^e+1}g + \frac{2}{\gamma^e+1}\delta^e)W_t^e + (\rho + (\gamma-1)g)W_t^s \\ &= \left((\rho + \frac{(\gamma+1)(\gamma^e-1)}{\gamma^e+1}g + \frac{2}{\gamma^e+1}\delta^e)\overline{\eta}^b + (\rho+(\gamma-1)g)(1-\overline{\eta}^b)\right)(\overline{q} + (1-\underline{\chi})\overline{p})K_t\end{aligned} \tag{A.125}$$

Combining equation (A.124) and (A.125), we have

$$\overline{r}^b = \rho + \gamma g \tag{A.126}$$

$$\overline{\eta}^b = \frac{\rho + (\gamma-1)g}{\rho + (\gamma-1)g + (1-\underline{\chi})\sqrt{\frac{2(\delta^e+(\gamma^e-\gamma)g)}{\gamma^e(\gamma^e+1)}}}\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\frac{2(\delta^e+(\gamma^e-\gamma)g)}{\gamma^e(\gamma^e+1)}}} \tag{A.127}$$

$$\overline{q}^b = \frac{a}{\rho + (\gamma-1)g + \tilde{\sigma}\sqrt{\frac{2(\delta^e+(\gamma^e-\gamma)g)}{\gamma^e(\gamma^e+1)}}} \tag{A.128}$$

$$\overline{p}^b = \frac{a}{\rho + (\gamma-1)g} - \overline{q}^b \tag{A.129}$$

In the case of $\gamma^e = \gamma > 1$, we have the main result still holds.

### A.1.7 Extension: investment and labor

In this extension, I characterize the steady state of the economy with investment and labor.

Assume the evolution of private capital is

$$\frac{dk_t^i}{k_t^i} = (\Psi(\iota_t) - \phi)dt + \tilde{\sigma}d\tilde{Z}_{i,t} \tag{A.130}$$

where $\Psi(\iota_t)$ is the investment function (increasing and concave in $\iota_t$) and $\phi$ is the depreciation rate of capital. Specify that

$$\Psi(\iota_t) = \frac{\log(\psi(\iota_t - \phi) + 1)}{\psi} + \phi \tag{A.131}$$

where $\psi$ is the adjustment cost of investment. We are back at the endowment economy as $\phi \to \infty$. At steady state, investment and depreciation of capital cancel each other.

Entrepreneurs have the same preference as before, however, savers also supply labor $l_{j,t}$. Saver $j$'s preference is

$$\int_0^\infty e^{-\rho t} \left( \log c_t^{s,j} - \frac{l_{j,t}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right) dt \tag{A.132}$$

where $\nu$ is the Frisch elasticity.

Production function of entrepreneur $i$ is given by

$$y_{it} = a k_{it}^{\alpha} l_{it}^{1-\alpha} \tag{A.133}$$

Since production function is constant return to scale, one can get the aggregate production function of the economy as

$$y_t = a k_t^{\alpha} l_t^{1-\alpha} \tag{A.134}$$

where $k_t$ and $l_t$ are aggregate capital and labor supply.

Savers' problem can still be mapped into a standard portfolio choice problem

$$\begin{aligned} \max_{\{c_t^{s,j}, l_{j,t}\}_{t=0}^\infty} \quad & \int_0^\infty e^{-\rho t} \left( \log c_t^{s,j} - \frac{l_{j,t}^{1+\frac{1}{\nu}}}{1+\frac{1}{\nu}} \right) dt \\ \text{s.t.} \quad & \int_0^\infty \xi_{j,t} c_t^{s,j}\, dt \le W_0^{s,j} + \int_0^\infty \xi_{j,t} w_t l_{j,t} dt \end{aligned} \tag{A.135}$$

where $\xi_{j,t}$ is the stochastic discount factor of saver $j$, and $W_0^{s,j}$ is the endowed wealth of saver $j$ at time $t = 0$.

Since labor does not affect capital market risk-taking and all entrepreneurs use the same wage, we can drop the subscripts and write wage as the marginal product of labor,

$$w_t = a(1-\alpha) k_t^{\alpha} l_t^{-\alpha} \tag{A.136}$$

From first-order conditions of savers, we have

$$l_t^{\frac{1}{\nu}} = \frac{w_t}{c_t^s} = \frac{1-\alpha}{(1-\eta_t^{c,f})l_t} \tag{A.137}$$

where $\eta_t^{c,f} = \frac{(\rho+\delta^e)\eta_t^f}{\delta^e\eta_t^f+\rho}$ is the consumption share of entrepreneurs.

Investment is maximized as entrepreneurs maximize the expected return of their private firm and it does not affect risk-taking,

$$\max_{\iota_t}\{\frac{y_t - \iota_t k_t}{q_t k_t} + \Psi(\iota) - \phi + \text{risk premium}\} \tag{A.138}$$

we have

$$\Psi'(\iota_t) = \frac{1}{q_t} \tag{A.139}$$

that is,

$$\iota_t = \frac{q_t - 1}{\psi} + \phi \tag{A.140}$$

The evolution of wealth inequality $\eta_t^f$ does not change since neither investment nor labor affect patience or risk-taking.

We also have the market clearing condition for consumption good,

$$(\delta^e\eta_t^f + \rho)q_t = ak_t^{\alpha-1}l_t^{1-\alpha} - (\frac{q_t-1}{\psi} + \phi) \tag{A.141}$$

Since $q_t = 1$ at steady state, we can solve for steady-state level of labor and capital

$$\bar{l}^f = \left(\frac{1-\alpha}{1-\overline{\eta}^{c,f}}\right)^{\frac{\nu}{1+\nu}} \tag{A.142}$$

where $\overline{\eta}^{c,f}$ is the same consumption share of savers in fundamental equilibrium steady state as in the previous sections.

$$\overline{k}^f = a^{\frac{1}{1-\alpha}}\left(\frac{1-\alpha}{1-\overline{\eta}^c}\right)^{\frac{\nu}{1+\nu}}(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi)^{\frac{-1}{1-\alpha}} = \left(\frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi}\right)^{\frac{1}{1-\alpha}}\bar{l}^f \tag{A.143}$$

Savers' consumption at steady state in fundamental equilibrium is given by

$$\begin{aligned}\overline{c}^{s,f} &= (1-\overline{\eta}^{c,f})a(\overline{k}^f)^\alpha(\overline{l}^f)^{1-\alpha} \\ &= a(1-\alpha)^{\frac{\nu}{1+\nu}}(1-\overline{\eta}^{c,f})^{\frac{1}{1+\nu}}\left(\frac{a}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi}\right)^{\frac{\alpha}{1-\alpha}} \\ &= a^{\frac{1}{1+\alpha}}(1-\alpha)^{\frac{\nu}{1+\nu}}\frac{(\rho(1-\frac{\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}}))^{\frac{1}{1+\nu}}}{(\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho)^{\frac{1+\alpha\nu}{(1+\nu)(1-\alpha)}}}\left(\frac{1}{1+\frac{\phi}{\underline{\chi}\tilde{\sigma}\sqrt{\delta^e}+\rho}}\right)^{\frac{\alpha}{1-\alpha}}\end{aligned} \tag{A.144}$$

A sufficient condition (given $\overline{\eta}^f > \frac{1}{2}$) for $\frac{\partial\overline{c}^{s,f}}{\partial\delta^e} > 0$ is

$$(2-\alpha)\underline{\chi}\tilde{\sigma} + (1-\alpha)\rho > \sqrt{\delta^e} \tag{A.145}$$

When entrepreneurs get more "patient" ($\delta^e \downarrow$), labor $l$ increases, capital $k$ increases, capital to labor ratio $\frac{k}{l}$ increases, and wage $w$ increases. Savers not only consume less, but also supply more labor.

One can show that with CRRA utilities, condition (A.145) is still a sufficient condition.

Similarly, we can solve for bubble equilibrium. The market clearing condition now is as follows

$$(\delta^e\eta_t^b + \rho)(q_t + (1-\underline{\chi})p_t) = ak_t^{\alpha-1}l_t^{1-\alpha} - (\frac{q_t-1}{\psi} + \phi) \tag{A.146}$$

and the return for the public equity is

$$\begin{aligned}dr_t^{mf} &= \frac{y_t - \iota_t k_t}{(p_t+q_t)k_t}dt + \frac{d((p_t+q_t)k_t)}{(p_t+q_t)k_t} \\ &= \left(\frac{ak_t^{\alpha-1}l_t^{1-\alpha} - \iota_t + p_t\mu_t^p + q_t\mu_t^q}{p_t+q_t} + \Psi(\iota_t) - \phi\right)dt\end{aligned} \tag{A.147}$$

In equilibrium, the return of public equity equals the risk-free rate, so we have

$$dr_t^{mf} = r_t^f dt \tag{A.148}$$

Solving for steady state, we have capital to labor ratio in bubble equilibrium

$$\frac{\overline{k}^b}{\overline{l}^b} = \left(\frac{a}{\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi}\right)^{\frac{1}{1-\alpha}} < \frac{\overline{k}^f}{\overline{l}^f} \tag{A.149}$$

and

$$\overline{p} = \frac{\tilde{\sigma}\sqrt{\delta^e}}{\rho} \tag{A.150}$$

Labor supply in bubble equilibrium is

$$\overline{l}^b = \left(\frac{1-\alpha}{1-\overline{\eta}^{c,b}}\right)^{\frac{\nu}{1+\nu}} < \overline{l}^f \tag{A.151}$$

as $\overline{\eta}^{c,b} < \overline{\eta}^{c,f}$. Labor supply is lower in bubble equilibrium than in fundamental equilibrium. Savers' consumption at steady state in bubble equilibrium is given by

$$\begin{aligned}
\overline{c}^{s,b} &= (1-\overline{\eta}^{c,b})a(\overline{k}^b)^{\alpha}(\overline{l}^b)^{1-\alpha} \\
&= a(1-\overline{\eta}^{c,b})^{\frac{1}{1+\nu}}\left(\frac{a}{\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi}\right)^{\frac{\alpha}{1-\alpha}} \\
&= a(1-\frac{(\rho+\delta^e)\underline{\chi}\tilde{\sigma}}{\sqrt{\delta^e}(\tilde{\sigma}\sqrt{\delta^e}+\rho)})^{\frac{1}{1+\nu}}\left(\frac{a}{\tilde{\sigma}\sqrt{\delta^e}+\rho+\phi}\right)^{\frac{\alpha}{1-\alpha}}
\end{aligned} \tag{A.152}$$

## A.2 Appendix B: Additional Proofs and Derivations for Chapter 2

Trade market: Notice that international trade is a static problem for both countries. Since there is no friction in the international trade market and homogeneous preferences, we have that

$$\frac{C_{HH,t}}{C_{HF,t}} = \frac{C_{FH,t}}{C_{FF,t}} = \frac{Y_{H,t}}{Y_{F,t}} \tag{A.153}$$

Using market clearing condition for home good and foreign good,

$$C_{HH,t} + C_{FH,t} = Y_{H,t}$$

$$C_{HF,t} + C_{FF,t} = Y_{F,t}$$

we have

$$C_{H,t} + C_{F,t} = \overline{Y}_t$$

where

$$C_{H,t} = \left[\alpha^{\frac{1}{\eta}} C_{HH,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{HF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$C_{F,t} = \left[\alpha^{\frac{1}{\eta}} C_{FH,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} C_{FF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

$$\overline{Y}_t = \left[\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}$$

As a result, the prices of the two goods produced by the two trees are given by

$$p_t^H = \left(\frac{Y_{H,t}}{\alpha \overline{Y_t}}\right)^{-\frac{1}{\eta}} \quad \text{and} \quad p_t^F = \left(\frac{Y_{F,t}}{(1-\alpha)\overline{Y_t}}\right)^{-\frac{1}{\eta}} \tag{A.154}$$

We have that

$$C_{H,t} = p_t^H C_{HH,t} + p_t^F C_{HF,t}$$

$$C_{F,t} = p_t^H C_{FH,t} + p_t^F C_{FF,t}$$

Recall that country size if defined as follows

$$s_t = \frac{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}}}{\alpha^{\frac{1}{\eta}} Y_{H,t}^{\frac{\eta-1}{\eta}} + (1-\alpha)^{\frac{1}{\eta}} Y_{F,t}^{\frac{\eta-1}{\eta}}} = \alpha^{\frac{1}{\eta}} \left(\frac{Y_{H,t}}{\overline{Y_t}}\right)^{\frac{\eta-1}{\eta}}$$

as home country's share of the world total output, i.e. the *country size* of home country, which will turn out to be an important state variable. We have that

$$p_t^H Y_{H,t} = s_t \overline{Y_t} \quad \text{and} \quad p_t^F Y_{F,t} = (1-s_t)\overline{Y_t}$$

The aggregate wealth of home country is

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F + p_t^H B_t^{H,H} + p_t^F B_t^{H,F}$$

The aggregate wealth of foreign country is

$$W_t^F = \chi_t^{F,H} S_t^H + \chi_t^{F,F} S_t^F + p_t^H B_t^{F,H} + p_t^F B_t^{F,F}$$

The optimization problem for home country is as follows:

$$\max_{\{C_{HH,t},C_{HF,t},\chi_t^{H,H},\chi_t^{H,f},\theta_t^{H,B^H},\theta_t^{H,B^F}\}_{t=0}^{\infty}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log\left(\left[\alpha^{\frac{1}{\eta}}C_{HH,t}^{\frac{\eta-1}{\eta}}+(1-\alpha)^{\frac{1}{\eta}}C_{HF,t}^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}\right)dt\right]$$

$$\begin{aligned}
\text{s.t.}\quad \frac{dW_t^H}{W_t^H} &= \frac{\chi_t^{H,H}S_t^H}{W_t^H}dr_t^{S^H}+\frac{\chi_t^{H,F}S_t^F}{W_t^H}dr_t^{S^F}+\theta_t^{H,B^H}dr_t^{B^H}+\theta_t^{H,B^F}dr_t^{B^F}\\
&\quad-\frac{p_t^H C_{HH,t}+p_t^F C_{FF,t}}{W_t^H}dt\\
1 &= \frac{\chi_t^{H,H}S_t^H}{W_t^H}+\frac{\chi_t^{H,F}S_t^F}{W_t^H}+\theta_t^{H,B^H}+\theta_t^{H,B^F}\\
0 &\le \chi_t^{H,F}\le\overline{\chi}^F
\end{aligned} \tag{A.155}$$

Define home country's wealth share as

$$n_t=\frac{W_t^H}{W_t^H+W_t^F} \tag{A.156}$$

**Lemma 2.** *Home country's wealth share* $n_t=\frac{W_t^H}{W_t^H+W_t^F}$ *is a function of country size* $s_t$*. And we always have that*

$$p_t^H B_t^{H,H}+p_t^F B_t^{H,F}=0 \tag{A.157}$$

*Proof.* From optimization problems, we have that

$$C_{H,t}=\rho W_t^H \quad\text{and}\quad C_{F,t}=\rho W_t^H$$

In the aggregate, we have that total consumption equals total output

$$C_{HH,t}+C_{FH,t}=Y_{H,t}$$
$$C_{HF,t}+C_{FF,t}=Y_{F,t}$$

And using the result from trade market optimization, we have

$$C_{H,t}+C_{F,t}=\overline{Y_t}$$

That is,

$$\rho W_t^H+\rho W_t^F=\overline{Y_t}$$

The total wealth in the world is given by

$$W_t^H + W_t^F = \frac{\overline{Y_t}}{\rho}$$

There are two state variables, wealth share $n_t$ and country size $s_t$. In equilibrium, all the prices and quantities must be functions of state variables $n_t$ and $s_t$. We can rewrite home country's wealth as

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = [n_t - (\chi_t^{H,H} q_t^H + \chi_t^{H,F} q_t^F)]\overline{Y_t}$$

where $q_t^H = \frac{S_t^H}{\overline{Y_t}}$ and $q_t^F = \frac{S_t^F}{\overline{Y_t}}$ are per unit price of home and foreign equity, respectively. Recall that

$$p_t^H Y_{H,t} = s_t \overline{Y_t} \quad \text{and} \quad p_t^F Y_{F,t} = (1 - s_t)\overline{Y_t}$$

we have

$$p_t^H = \alpha^{\frac{1}{\eta-1}} s_t^{-\frac{1}{\eta-1}} \quad \text{and} \quad p_t^F = (1-\alpha)^{\frac{1}{\eta-1}} (1 - s_t)^{-\frac{1}{\eta-1}}$$

And in equilibrium $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$, $\theta_t^{H,B^F} = \frac{p_t^F B_t^{H,F}}{W_t^H}$, $q_t^H$, $q_t^F$, $\chi_t^{H,H}$ and $\chi_t^{H,F}$ must be functions of state variables $n_t$ and $s_t$. We have that

$$n_t = \frac{\chi_t^{H,H} q_t^H + \chi_t^{H,F} q_t^F}{1 - (\theta_t^{H,B^H} + \theta_t^{H,B^F})} \equiv f(s_t, n_t) \tag{A.158}$$

Equation (A.158) is an implicit function and we can solve for $n_t$ as a function of $s_t$.

Now we have that in equilibrium $\theta_t^{H,B^H} = \frac{p_t^H B_t^{H,H}}{W_t^H}$, $\theta_t^{H,B^F} = \frac{p_t^F B_t^{H,F}}{W_t^H}$, $q_t^H$, $q_t^F$, $\chi_t^{H,H}$ and $\chi_t^{H,F}$ must be functions of the *only* state variable $s_t$.

Recall the dynamic budget constraint of home country,

$$\begin{aligned} \frac{dW_t^H}{W_t^H} = & \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ & - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \end{aligned}$$

And the asset return processes,

$$dr_t^{S^H} = \underbrace{\frac{p_t^H Y_{H,t}}{q_t^H \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^H \overline{Y}_t)}{q_t^H \overline{Y}_t}}_{\text{capital gain}} = \mu_t^{S^H}\, dt + (\frac{(q^H)'(s_t)s_t}{q_t^H}\sigma_{s_t} + \overline{\sigma}_t) dZ_t$$

and similarly

$$dr_t^{S^F} = \underbrace{\frac{p_t^F Y_{F,t}}{q_t^F \overline{Y}_t} dt}_{\text{dividend yield}} + \underbrace{\frac{d(q_t^F \overline{Y}_t)}{q_t^F \overline{Y}_t}}_{\text{capital gain}} = \mu_t^{S^F}\, dt + (\frac{(q^F)'(s_t)s_t}{q_t^F}\sigma_{s_t} + \overline{\sigma}_t) dZ_t$$

$$dr_t^{B^H} = \frac{d(p_t^H B_t^H)}{p_{H,t} B_t^H} = (\mu_{p^H,t} + r_t^H)\, dt + \sigma_{p^H,t}\, dZ_t$$

$$dr_t^{B^F} = \frac{d(p_t^F B_t^F)}{p_{F,t} B_t^F} = (\mu_{p^F,t} + r_t^F)\, dt + \sigma_{p^F,t}\, dZ_t$$

And we have

$$\sigma_{p_t^H} = -\frac{1}{\eta - 1}\sigma_{s_t}$$

$$\sigma_{p_t^F} = \frac{s_t}{(\eta-1)(1-s_t)}\sigma_{s_t}$$

Since $\rho W_t^H = n_t \overline{Y_t}$, we have

$$\frac{dW_t^H}{W_t^H} = \mu_t^{W^H} dt + (\frac{n'(s_t)s_t}{n_t}\sigma_{s_t} + \overline{\sigma}_t)\, dZ_t$$

Note that

$$\overline{\sigma}_t = [s_t \sigma^H, (1-s_t)\sigma^F]$$

$$\sigma_{s_t} = \frac{\eta - 1}{\eta}(1-s_t)[\sigma^H, -\sigma^F]$$

are *linearly independent* for non-degenerate $s_t$. Matching terms for $\overline{\sigma}_t$, we must have that

$$\frac{\chi_t^{H,H} S_t^H}{W_t^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} = 1 \tag{A.159}$$

That is

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \tag{A.160}$$

■

**Corollary 1.** *The total capital flow of home country induced by equity trade is given by*

$$\begin{aligned} dQ_t^H &= S_t^H d\chi_t^{H,H} - \chi_t^{H,H}(p_t^H Y_{H,t})dt + S_t^F d\chi_t^{H,F} - \chi_t^{H,F}(p_t^F Y_{F,t})dt \\ &\quad + d\chi_t^{H,H} dS^H + d\chi_t^{H,F} dS_t^F \end{aligned} \tag{A.161}$$

*And such capital flow must be financed and absorbed by trading in bonds and consumption goods*

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F})W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t})dt \tag{A.162}$$

*Similarly the total capital flow of foreign country induced by equity trade is given by*

$$\begin{aligned} dQ_t^F &= S_t^H d\chi_t^{F,H} - \chi_t^{F,H}(p_t^H Y_{H,t})dt + S_t^F d\chi_t^{F,F} - \chi_t^{F,F}(p_t^F Y_{F,t})dt \\ &\quad + d\chi_t^{F,H} dS^H + d\chi_t^{F,F} dS_t^F \end{aligned} \tag{A.163}$$

*and such capital flow must be financed and absorbed by trading in bonds and consumption goods*

$$dQ_t^F = (\theta_t^{B^{F,H}} dr_t^{B^H} + \theta_t^{B^{F,F}} dr_t^{B^F})W_t^F - (p_t^H C_{FH,t} + p_t^F C_{FF,t})dt \tag{A.164}$$

*Proof.* This follows from Lemma 1. Since we have

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F$$

Taking total differentiation on both sides, we have that

$$dW_t^H = \chi_t^{H,H} S_t^H \, dr_t^{S^H} + \chi_t^{H,F} S_t^F \, dr_t^{S^F} + dQ_t^H \tag{A.165}$$

Combining with dynamic budget constraint

$$\begin{aligned} \frac{dW_t^H}{W_t^H} &= \frac{\chi_t^{H,H} S_t^H}{W_t^H} dr_t^{S^H} + \frac{\chi_t^{H,F} S_t^F}{W_t^H} dr_t^{S^F} + \theta_t^{H,B^H} dr_t^{B^H} + \theta_t^{H,B^F} dr_t^{B^F} \\ &\quad - \frac{p_t^H C_{HH,t} + p_t^F C_{FF,t}}{W_t^H} dt \end{aligned}$$

We have that

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F})W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t})dt$$

Similar proof for foreign country. ■

**Lemma 3.** *In crisis region $s_t \in [0, s^u]$, we have that*

$$\chi_t^{H,H} = 1, \quad \chi_t^{H,F} = \overline{\chi}^F$$

$$p_t^H B_t^{H,H} = p_t^F B_t^{H,F} = 0$$

*Proof.* At the crisis region boundary $s_t = s^u$, we have $\chi_t^{H,H} = 1$ and $\chi_t^{H,F} = \overline{\chi}^F$,

$$dQ_t^H = S_t^H d\chi_t^{H,H} - (p_t^H Y_{H,t})dt + S_t^F d\chi_t^{H,F} - \overline{\chi}^F (p_t^F Y_{F,t})dt + d\chi_t^{H,H} dS^H + d\chi_t^{H,F} dS_t^F$$

For *any* realization of $ds_t < 0$ at $s_t = s^u$, it must be that $d\chi_t^{H,H} \leq 0$ and $d\chi_t^{H,F} \leq 0$. To satisfy this, we must have $d\chi_t^{H,H}$ and $d\chi_t^{H,F}$ are deterministic for any realization of $ds_t < 0$ (thus, any $s < s^u$) at $s_t = s^u$ .

Collecting terms for $\sigma_{s_t}$ and $\overline{\sigma}_t$, we have

$$dQ_t^H - \mathbb{E}[dQ_t^H] = [(S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F})\overline{\sigma}_t + (S_t^H d\chi_t^{H,H} \frac{(q^H)'(s_t)s_t}{q_t^H} + S_t^F d\chi_t^{H,F} \frac{(q^F)'(s_t)s_t}{q_t^F})\sigma_{s_t})]\, dZ_t$$

On the other side, we have

$$dQ_t^H = (\theta_t^{B^{H,H}} dr_t^{B^H} + \theta_t^{B^{H,F}} dr_t^{B^F})W_t^H - (p_t^H C_{HH,t} + p_t^F C_{HF,t})dt$$

which only consists of $\sigma_{s_t}$ risk. As a result of matching terms for $\overline{\sigma}_t$, we have

$$S_t^H d\chi_t^{H,H} + S_t^F d\chi_t^{H,F} = 0$$

That is, $d\chi_t^{H,H} = d\chi_t^{H,F} = 0$. It follows that matching terms for $\sigma_{s_t}$ on both sides should also be 0, and we have

$$-\theta_t^{B^{H,H}} + \frac{s_t}{1 - s_t}\theta_t^{B^{H,F}} = 0$$

Combining with

$$\theta_t^{B^{H,H}} + \theta_t^{B^{H,F}} = 0$$

We have that in crisis region,

$$\theta_t^{B^{H,H}} = \theta_t^{B^{H,F}} = 0$$

■

While in crisis when $s_t \in [0, s^U]$, $P_t^H = (1-\overline{\chi}^F)s_t + \overline{\chi}^F$ and $P_t^F = (1-\overline{\chi}^F)(1-s_t)$.

Because the two countries have different consumption prices which are non-degenerate stochastic processes, the same (real) bond corresponds to different return processes in the two countries. That is, real bond returns bear consumption price risks and real bonds can not help overcome consumption price deviations. As a result, any real bond will not be traded in the constrained equilibrium [1]. The equity holding constraints creates financial friction that can not be overcome by sovereign bonds.

**With one equity constraint** Recall the wealth of home country and its evolution

$$W_t^H = \chi_t^{H,H} S_t^H + \chi_t^{H,F} S_t^F + p_t^H B_t^{H,H} + p_t^F B_t^{F,F} \tag{A.166}$$

$$\frac{dW_t^H}{W_t^H} = \frac{\rho \chi_t^{H,H} q_t^H}{n_t} dr_t^{S^H} + \frac{\rho \chi_t^{H,F} q_t^F}{n_t} dr_t^{S^F} + \frac{p_t^H B_{1t}}{W_t^H} dr_t^{B_1} + \frac{p_t^F B_{2t}}{W_t^H} dr_t^{B_2} - \rho dt \tag{A.167}$$

Denote

$$\frac{dW_t^H}{W_t^H} = \mu_{W_t^H}\, dt + \sigma_{W_t^H}\, dZ_t \tag{A.168}$$

$$\frac{dW_t^F}{W_t^F} = \mu_{W_t^F}\, dt + \sigma_{W_t^F}\, dZ_t \tag{A.169}$$

$$\frac{dn_t}{n_t} = \mu_{n_t} dt + \sigma_{n_t} dZ_t \tag{A.170}$$

There are two risks in this world: the aggregate consumption risk, $\overline{\sigma}_t$, and the distribution risk, $\sigma_{s_t}$. Since there are four financial assets, there is some redundancy. With only one equity constraint, the two countries can still perfectly share consumption risk. So we have

$$\sigma_{n_t} = \sigma_{1-n_t} = 0 \tag{A.171}$$

and

$$\sigma_{W_t^H} = \sigma_{W_t^F} = \overline{\sigma}_t \tag{A.172}$$

[1]The symmetric setting in discount rate and preferences matters. (conjecture)

To find the portfolio weights on each asset, we have

$$\overline{\sigma}_t = \frac{\rho\chi_t^{H,H} q_t^H}{n_t}(\sigma_{q_t^H} + \overline{\sigma}_t) + \frac{\rho\chi_t^{H,F} q_t^F}{n_t}(\sigma_{q_t^F} + \overline{\sigma}_t) + \frac{p_t^H B_t^{H,H}}{W_t^H}\sigma_{p_t^H} + \frac{p_t^F B_t^{H,F}}{W_t^H}\sigma_{p_t^F} \tag{A.173}$$

Now we need to find $\sigma_{q_t^H}$. In the complete market case, we have

$$q_t^H = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau} s_\tau d\tau\Big] \tag{A.174}$$

Using Ito's lemma we have

$$\sigma_{q_t^H} = \frac{(q^H)'(s_t)s_t}{q_t^H}\sigma_{s_t} \tag{A.175}$$

and since

$$q_t^F = \frac{1}{\rho} - q_t^H \tag{A.176}$$

we have

$$\sigma_{q_t^F} = -\frac{q_t^H}{q_t^F}\sigma_{q_t^H} = -\frac{(q^H)'(s_t)s_t}{q_t^F}\sigma_{s_t} \tag{A.177}$$

And we have

$$\sigma_{p_t^H} = -\frac{1}{\eta - 1}\sigma_{s_t} \tag{A.178}$$

$$\sigma_{p_t^F} = \frac{s_t}{(\eta-1)(1-s_t)}\sigma_{s_t} \tag{A.179}$$

Note that

$$\overline{\sigma}_t^T = [s_t\sigma_1, (1-s_t)\sigma_2] \tag{A.180}$$

$$\sigma_{s_t}^T = \frac{\eta-1}{\eta}(1-s_t)[\sigma_1, -\sigma_2] \tag{A.181}$$

are *linearly independent*. Now coming back to the risk of home country's wealth (A.173) and matching $\overline{\sigma}_t$ term, we have

$$\frac{\rho\chi_t^{H,H} q_t^H}{n_t} + \frac{\rho\chi_t^{H,F} q_t^F}{n_t} = 1 \tag{A.182}$$

That is

$$W_t^H = \chi_t^{H,F} S_t^H + \chi_t^{H,F} S_t^F \tag{A.183}$$

and thus

$$p_t^H B_t^{H,H} + p_t^F B_t^{H,F} = 0 \tag{A.184}$$

To determine portfolio weights on bonds, we match $\sigma_{s_t}$ terms in home country's wealth

$$\frac{\rho\chi_t^{H,H}q_t^H}{n_t}\frac{(q^H)'(\frac{s_t}{n_t})s_t}{q_t^H}+\frac{\rho\chi_t^{H,F}q_t^F}{n_t}(-\frac{(q^H)'(s_t)s_t}{q_t^F})+\frac{p_t^H B_t^{H,H}}{W_t^H}(-\frac{1}{\eta-1}-\frac{s_t}{(\eta-1)(1-s_t)})=0 \tag{A.185}$$

Simplified to

$$\frac{\rho(q^H)'(s_t)s_t(\chi_t^{H,H}-\chi_t^{H,F})}{n_t}-\frac{p_t^H B_t^{H,H}}{W_t^H}\frac{1}{(1-s_t)(\eta-1)}=0 \tag{A.186}$$

So we have

$$\frac{p_t^H B_t^{H,H}}{W_t^H}=\frac{\rho\chi_t^{H,H}(q^H)'(s_t)s_t(1-s_t)(\chi_t^{H,H}-\chi_t^{H,F})(\eta-1)}{n_t}>0 \tag{A.187}$$

and

$$\frac{p_t^F B_t^{H,F}}{W_t^H}=-\frac{p_t^H B_t^{H,H}}{W_t^H}<0 \tag{A.188}$$

We can also find the drift of home country wealth share $n_t$ by looking at the drift term of the wealth. Using market clearing condition $B_t^{H,H}=-B_t^{F,H}$ and $B_t^{H,F}=-B_t^{F,F}$,

$$\mu_{W_t^F}=-\frac{n_t}{1-n_t}\mu_{n_t}+\overline{\mu}_t \tag{A.189}$$

$$=\frac{\mathbb{E}_t[dr_t^{S^F}]}{dt}-\frac{n_t}{1-n_t}\frac{p_t^H B_t^{H,H}}{W_t^H}m_t\sigma_{e_t}^T-\rho \tag{A.190}$$

and we have that

$$\mu_{q_t^H}=\rho-\frac{s_t}{q_t^H}+\mu_{n_t}-\sigma_{n_t}^2+\sigma_{q_t^H}\sigma_{n_t} \tag{A.191}$$

and $\sigma_{n_t}=0$,

$$\mu_{q_t^H}=\rho-\frac{s_t}{q_t^H}+\mu_{n_t} \tag{A.192}$$

and thus

$$\frac{\mathbb{E}_t[dr_t^{S^F}]}{dt}=\frac{1-s_t}{q_t^F}+(-\frac{q_t^H}{q_t^F}\mu_{q_t^H}+\overline{\mu}_t+\sigma_{q_t^F}\overline{\sigma}_t^T) \tag{A.193}$$

Substituting into (A.189), and solving for $\mu_{n_t}$, we have

$$\mu_{n_t}=\frac{\frac{q_t^H}{q_t^F}\sigma_{q_t^H}\overline{\sigma}_t^T+\frac{n_t\theta_t^{H,B^H}}{1-n_t}\sigma_{e_t}\overline{\sigma}_t^T}{\frac{n_t}{1-n_t}-\frac{q_t^H}{q_t^F}}=0 \tag{A.194}$$

as

$$\frac{q_t^H}{q_t^F}\sigma_{q_t^H}\overline{\sigma}_t^T = -\frac{(q^H)'(\frac{s_t}{n_t})s_t(1-s_t)(\eta-1)}{q_t^F}\sigma_{e_t}\overline{\sigma}_t^T = \frac{n_t\theta_t^{H,B^H}}{1-n_t}\sigma_{e_t}\overline{\sigma}_t^T \tag{A.195}$$

**With two equity holding constraints** The second step is to explore what will happen with two equity holding constraints. Now the two countries can not always perfectly share consumption risk and the wealth shares are not always constant.

Full model special case: with two equity holding constraints $0 \leq \chi_t^{H,F} \leq \overline{\chi}^F$, $0 \leq \chi_t^{F,H} \leq \overline{\chi}^H$, and symmetric parameters $\sigma_1 = \sigma_2 = \sigma$ there are three safety thresholds $s^c$, $s^u$ and $s^a$, $n_0 = \frac{1}{2}$:

$$s^c = \frac{1}{2} \tag{A.196}$$

$$q_1(s^u) = \frac{1-2\overline{\chi}^F}{2\rho(1-\overline{\chi}^F)} \tag{A.197}$$

and

$$0 < s^a < s^u \tag{A.198}$$

with parameter restrictions on $(\overline{\chi}^F, \eta)$. The equity shares are

$$\chi_{1t} = 1 \tag{A.199}$$

and

$$\chi_t^{H,F} = \begin{cases} \overline{\chi}^F & if \quad s_t < s^U(\overline{\chi}^F) \\ \frac{1-2\rho q_t^H}{2(1-\rho q_t^H)} & if \quad s^U(\overline{\chi}^F) < s_t < \frac{1}{2} \\ 0 & if \quad s_t > \frac{1}{2} \end{cases} \tag{A.200}$$

bond holdings are (in equilibrium)

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = \theta_t^{F,B^H} = \theta_t^{F,B^F} = 0 \tag{A.201}$$

where

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{A.202}$$

*Proof.* The equity holding constraint binds when $s_t < s^u$. There are no cross-border bond trading. The implied bond returns (within domestic country) are given as follows

$$\frac{\mathbb{E}_t[dr_t^{B^{H,H}}]}{dt} - r_t^{H,f} = m_{1t}^T \sigma_{p_t^H} = (\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^H} \tag{A.203}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{H,F}}]}{dt} - r_t^{H,f} = m_{1t}^T \sigma_{p_t^F} = (\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^F} \tag{A.204}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{F,H}}]}{dt} - r_t^{F,f} = m_{2t}^T \sigma_{p_t^H} = (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^H} \tag{A.205}$$

$$\frac{\mathbb{E}_t[dr_t^{B^{F,F}}]}{dt} - r_t^{F,f} = m_{2t}^T \sigma_{p_t^F} = (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)\sigma_{p_t^F} \tag{A.206}$$

where $-r_t^{H,f}$ and $-m_{1t}$ are the drift and volatility of home country's SDF and similarly $-r_t^{F,f}$ and $-m_{2t}$ are for foreign country.

$$r_t^{H,f} = \rho + \mu_{n_t} + \overline{\mu}_t + \sigma_{n_t}\overline{\sigma}_t - (\sigma_{n_t} + \overline{\sigma}_t)^2 \tag{A.207}$$

$$r_t^{F,f} = \rho - \frac{n_t}{1-n_t}\mu_{n_t} + \overline{\mu}_t - \frac{n_t}{1-n_t}\sigma_{n_t}\overline{\sigma}_t - (-\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t)^2 \tag{A.208}$$

$$m_{1t} = \sigma_{n_t} + \overline{\sigma}_t \tag{A.209}$$

$$m_{2t} = \sigma_{1-n_t} + \overline{\sigma}_t \tag{A.210}$$

When $s_t < s(\overline{\chi}^F)$, we can write out Country 1 and Country 2's wealth as

$$W_t^H = S_t^{H,F} + \overline{\chi}^F S_t^{H,F} \tag{A.211}$$

$$W_t^F = (1 - \overline{\chi}^F) S_t^{F,F} \tag{A.212}$$

And from the optimization of logarithmic utility, we have

$$\rho W_t^H = C_{H,t} = n_t \overline{Y}_t \tag{A.213}$$

$$\rho W_t^F = C_{F,t} = (1 - n_t)\overline{Y}_t \tag{A.214}$$

Looking at the volatility of $W_t^H$,

$$\sigma_{W_t^H} = \sigma_{n_t} + \overline{\sigma}_t \tag{A.215}$$
$$= \frac{\rho q_t^{H,H}}{n_t}(\sigma_{q_t^{H,H}} + \overline{\sigma}_t) + \frac{\rho \overline{\chi}^F q_t^{H,F}}{n_t}(\sigma_{q^{H,F}} + \overline{\sigma}_t) \tag{A.216}$$

and the volatility of $W_t^F$,

$$\sigma_{W_t^F} = -\frac{n_t}{1-n_t}\sigma_{n_t} + \overline{\sigma}_t \tag{A.217}$$
$$= \frac{\rho(1-\overline{\chi}^F)q_t^{F,F}}{1-n_t}(\sigma_{q_t^{F,F}} + \overline{\sigma}_t) \tag{A.218}$$

For both countries, portfolio weights add up to 1

$$\frac{\rho q_t^{H,H}}{n_t} + \frac{\rho \overline{\chi}^F q_t^{H,F}}{n_t} = 1 \tag{A.219}$$

$$\frac{\rho(1-\overline{\chi}^F)q_t^{F,F}}{1-n_t} = 1 \tag{A.220}$$

so we have

$$\frac{\rho q_t^H}{n_t} + \frac{\rho \overline{\chi}^F q_t^{H,F}}{n_t} + \frac{\rho(1-\overline{\chi}^F)q_t^{F,F}}{1-n_t} = 2 \tag{A.221}$$

where

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{A.222}$$

$$\frac{q_t^{H,F}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1-s_\tau}{n_\tau}d\tau\Big] \tag{A.223}$$

$$\frac{q_t^{F,F}}{1-n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1-s_\tau}{1-n_\tau}d\tau\Big] \tag{A.224}$$

That is,

$$\mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}(\frac{s_\tau + \overline{\chi}^F(1-s_\tau)}{n_\tau} + \frac{(1-\overline{\chi}^F)(1-s_\tau)}{1-n_\tau})d\tau\Big] = \frac{2}{\rho} \tag{A.225}$$

Using Feynman-Kac formula, we have

$$\frac{s_t + \overline{\chi}^F(1-s_t)}{n_t} + \frac{(1-\overline{\chi}^F)(1-s_t)}{1-n_t} = 2 \tag{A.226}$$

So $n_t$ is a function of the only state variable $s_t$ (the other solution $n_t = \frac{1}{2}$ is not achievable with equity constraint binding).

$$n_t = (1 - \overline{\chi}^F)s_t + \overline{\chi}^F \tag{A.227}$$

$$1 - n_t = (1 - \overline{\chi}^F)(1 - s_t) \tag{A.228}$$

$$\theta_t^{H,B^H} = \theta_t^{H,B^F} = 0 \tag{A.229}$$

And also we have

$$\frac{C_{H,t}}{C_{F,t}} = \frac{n_t}{1 - n_t} \tag{A.230}$$

Now we can solve for prices of risks,

$$m_{1t} = \sigma_{n_t} + \overline{\sigma}_t = \frac{(1 - \overline{\chi}^F)s_t}{(1 - \overline{\chi}^F)s_t + \overline{\chi}^F}\sigma_{s_t} + \overline{\sigma}_t \tag{A.231}$$

and solve for $s^a$ using

$$\frac{\mathbb{E}_t[dr_t^{B^{H,H}} - dr_t^{B^{H,F}}]}{dt} = m_{1t}(\sigma_{p_t^H} - \sigma_{p_t^F}) < 0 \tag{A.232}$$

we have

$$\frac{2(1 - \overline{\chi}^F)s^2 + (2\overline{\chi}^F\eta + (1 - \overline{\chi}^F)(\eta - 2))s - \eta\overline{\chi}^F}{(1 - \overline{\chi}^F)s_t + \overline{\chi}^F} > 0 \tag{A.233}$$

For $0 < s^a < s^u$, we need the right range for parameter pair $(\overline{\chi}^F, \eta)$. For example, if $\overline{\chi}^F = 0$, we have $s^u = \frac{1}{2}$ and $s^a = \frac{2-\eta}{2}$. And if $\eta = \infty$, we have $s^a = \frac{\overline{\chi}^F}{1+\overline{\chi}^F}$.

With a special parameter case, $\rho = (\frac{\eta-1}{\eta}\sigma)^2$, we can solve everything we need analytically. The price of equity 1 in unconstrained case is

$$q_t^H(s) = \frac{1}{2\rho}\Big(1 + \frac{1-s}{s}\ln(1-s) - \frac{s}{1-s}\ln(s)\Big) \tag{A.234}$$

$$q_{1t}'(s) = -\frac{1}{2\rho}\frac{1}{s(1-s)}\Big(1 + \frac{1-s}{s}\ln(1-s) + \frac{s}{1-s}\ln(s)\Big) \tag{A.235}$$

$$q_{1t}''(s) = -\frac{1}{2\rho}\frac{1}{s^2(1-s)^2}\Big((2s-1) - \frac{(1-s)^2}{s}\ln(1-s) + \frac{s^2}{1-s}\ln(s)\Big) \tag{A.236}$$

■

## A.3 Appendix C: Additional Graphs and Tables

Table A.1: Annualized UIP Premium of One-Year Government Bonds For G10 Currencies

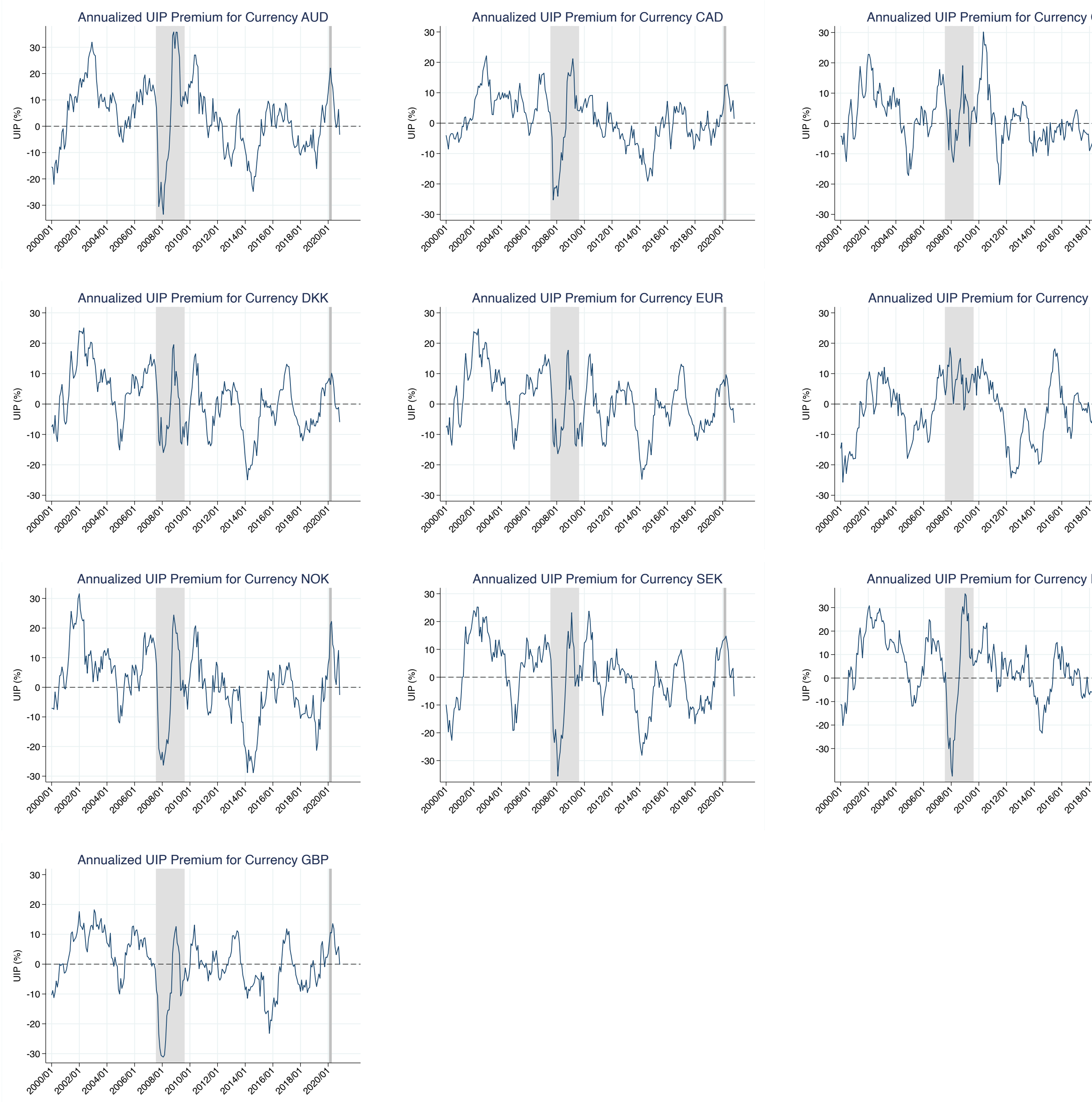

Table A.2: Annualized UIP Premium of One-Year Government Bonds for EME currencies

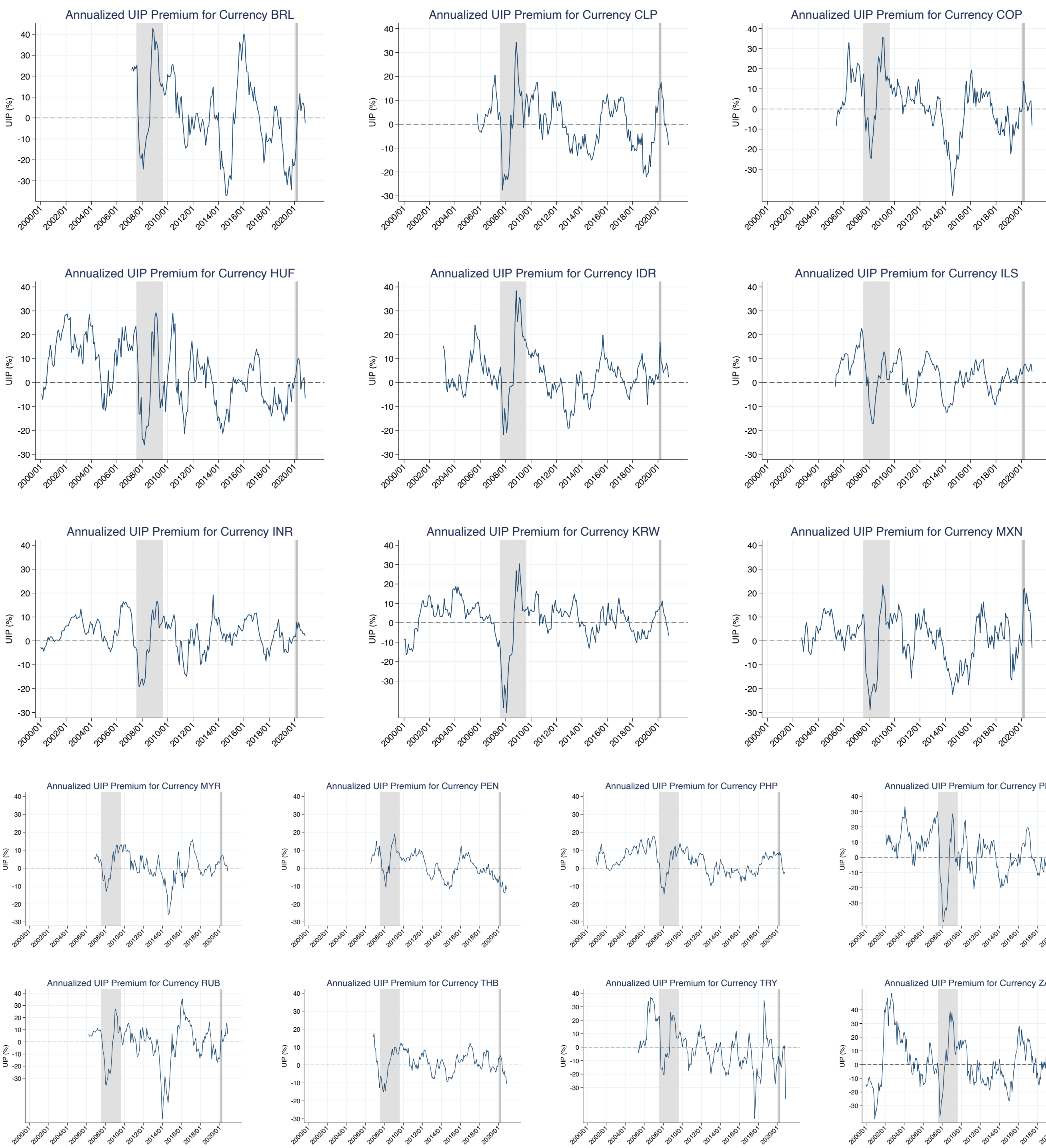

Table A.3: G10 Currency Monthly-Frequency (Annualized) UIP on GDP Share

| | (1) Full Sample | (2) Year < 2008 | (3) Year > 2008 |
|---|---|---|---|
| GDP Share (%) | -1.072*** | -0.665*** | -2.195*** |
| | (0.108) | (0.156) | (0.267) |
| Constant | 11.88*** | 11.32*** | 20.95*** |
| | (1.094) | (1.582) | (2.679) |
| Country + Year FE | yes | yes | yes |
| Observations | 2508 | 960 | 1548 |
| $R^2$ | 0.4408 | 0.4632 | 0.3796 |
| Adjusted $R^2$ | 0.434 | 0.453 | 0.371 |

Standard errors in parentheses

$^{*}$ $p < 0.1$, $^{**}$ $p < 0.05$, $^{***}$ $p < 0.01$

Table A.4: EME Currency Monthly-Frequency (Annualized) UIP on GDP Share

| | (1) Full Sample | (2) Year < 2008 | (3) Year > 2008 |
|---|---|---|---|
| GDP Share (%) | 0.0732 | -0.598 | 0.0400 |
| | (0.0865) | (0.377) | (0.117) |
| Constant | 1.374* | 13.18*** | 0.0162 |
| | (0.832) | (3.718) | (1.096) |
| Country + Year FE | yes | yes | yes |
| Observations | 2206 | 533 | 1673 |
| $R^2$ | 0.2977 | 0.2334 | 0.2906 |
| Adjusted $R^2$ | 0.288 | 0.207 | 0.281 |

Standard errors in parentheses

$^{*}$ $p < 0.1$, $^{**}$ $p < 0.05$, $^{***}$ $p < 0.01$

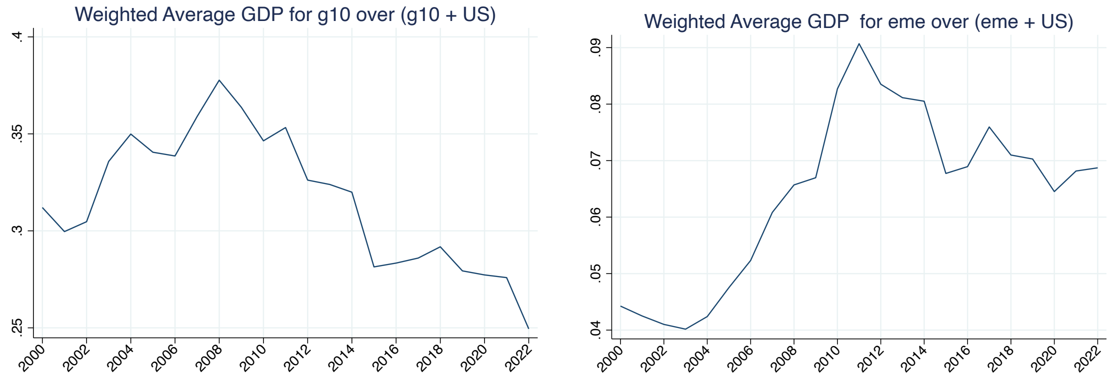


Table A.5: Relative GDP Ratio

**Note**: This panel of figures present weighted average GDP for G10 (left) or EME group (right) over the US GDP. The weights of its currency are its share of GDP within the G10 (or EME) group. The 1y EME currencies in our sample are BRL, CLP, COP, HUF, IDR, ILS, INR, MXN, MYR, PEN, PHP, PLN , RUB, THB, TRY, and ZAR.

## A.4 Appendix for chapter 3

This paper shares the technical appendix with C. He and Hui 2023 which includes model solutions. I briefly sketch proof here. As there is market segmentation in crisis region, asset prices differ in two countries. Denote $S_t^{H,H}$ as the home equity price in home country, $S_t^{F,H}$ as the home equity price in foreign country, $S_t^{H,F}$ as the foreign equity price in home country and $S_t^{F,F}$ as the foreign equity price in home country.

$$S_t^{H,H} = q_t^{H,H}\overline{Y}_t$$

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{A.237}$$

$$S_t^{F,H} = q_t^{F,H}\overline{Y}_t$$

$$\frac{q_t^{F,H}}{1-n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{1-n_\tau}d\tau\Big] \tag{A.238}$$

In normal regime, $n_t = n_0$, $S_t^{H,H} = S_t^{F,H} = S_t^H$ and $q_t^{H,H} = q_t^{F,H} = q_t^H$. Similarly for foreign equity. In normal region, we have

$$q_t^H = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}s_\tau d\tau\Big] \tag{A.239}$$

Using Ito's lemma we have

$$\sigma_{q_t^H} = \frac{(q^H)'(s_t)s_t}{q_t^H}\sigma_{s_t} \tag{A.240}$$

and since

$$q_t^F = \frac{1}{\rho} - q_t^H \tag{A.241}$$

we have

$$\sigma_{q_t^F} = -\frac{q_t^H}{q_t^F}\sigma_{q_t^H} = -\frac{(q^H)'(s_t)s_t}{q_t^F}\sigma_{s_t} \tag{A.242}$$

where

$$\overline{\sigma}_t^T = [s_t\sigma_1, (1-s_t)\sigma_2] \tag{A.243}$$

$$\sigma_{s_t}^T = \frac{\eta-1}{\eta}(1-s_t)[\sigma_1, -\sigma_2] \tag{A.244}$$

In crisis region, we have

$$\frac{q_t^{H,H}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{s_\tau}{n_\tau}d\tau\Big] \tag{A.245}$$

$$\frac{q_t^{H,F}}{n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1-s_\tau}{n_\tau}d\tau\Big] \tag{A.246}$$

$$\frac{q_t^{F,F}}{1-n_t} = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1-s_\tau}{1-n_\tau}d\tau\Big] \tag{A.247}$$

where $n_t = s_t + \overline{\chi}^F(1-s_t)$ and $1 - n_t = (1-\overline{\chi}^F)(1-s_t)$. It is easy to see that $q_t^{H,H}$ is increasing in $s_t$, $q_t^{F,F}$ is decreasing in $s_t$.

Home country's home equity portfolio in crisis region is given by

$$\theta_t^{H,H} = \frac{\rho q_t^{H,H}}{n_t} = \rho\mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1}{(1-\overline{\chi}^F) + \frac{\overline{\chi}^F}{s_\tau}}d\tau\Big] \tag{A.248}$$

and home country's foreign equity portfolio is given by

$$\theta_t^{H,F} = \frac{\rho q_t^{H,F}}{n_t} = \rho\mathbb{E}_t\Big[\int_0^\infty e^{-\rho\tau}\frac{1}{-(1-\overline{\chi}^F) + \frac{1}{1-s_\tau}}d\tau\Big] \tag{A.249}$$

and we have that

$$\frac{d\theta_t^{H,H}}{ds_t} > 0 \qquad \frac{d\theta_t^{H,F}}{ds_t} < 0 \tag{A.250}$$

**Equity home bias**

$$EHB_t^H = 1 - \frac{\chi_t^{H,F} S_t^F}{\theta_t^{F,optimal}} = 1 - \frac{\chi_t^{H,F}}{n_t} \tag{A.251}$$

$$EHB_t^F = 1 - \frac{\chi_t^{F,H} S_t^F}{\theta_t^{H,optimal}} = 1 - \frac{\chi_t^{F,H}}{1 - n_t} \tag{A.252}$$

For US, $\chi_t^{F,H} = 0$ in crisis, $1 = EHB_t^{F,crisis} \geq EHB_t^{F,nomal}$. For home country (G-10 currency group), the passive change upon crisis is given by $\frac{\chi_t^{H,F}}{n_t} - \frac{\chi_t^{H,F}}{n_0} > 0$. However, we argue that there is a shock that tightens the equity constraint facing home country, home investors might have to sell foreign equity if their foreign equity share were above the new constraint.

**US wealth and consumption share**

$$\frac{W_t^{F,crisis}}{W_t^{H,crisis} + W_t^{F,crisis}} = 1 - n_t > \frac{W_t^{F,nomral}}{W_t^{H,nomral} + W_t^{F,nomral}} = 1 - n_0 \tag{A.253}$$

$$\frac{C_{F,t}^{crisis}}{C_{H,t}^{crisis} + C_{F,t}^{crisis}} = 1 - n_t > \frac{C_{F,t}^{normal}}{C_{H,t}^{normal} + C_{F,t}^{normal}} = 1 - n_0 \tag{A.254}$$

## A.5 Appendix

### A.5.1 Unconstrained equilibrium

To solve for $q$, we have market clearing condition for consumption good:

$$\rho N = \rho \frac{qK_t}{\theta^K} = (a - \iota)K_t \tag{A.255}$$

So we get

$$q = \frac{a - \iota}{\sqrt{\rho}\tilde{\sigma}} \tag{A.256}$$

And let $\Phi(\iota) = \frac{1}{\kappa}\log(\kappa\iota + 1)$, we have

$$\kappa\iota = q - 1 \tag{A.257}$$

We now have $q$ and $\iota$ both constants in equilibrium. And solving for $q$, we have

$$q = \frac{\kappa a + 1}{\kappa\sqrt{\rho}\tilde{\sigma} + 1} \tag{A.258}$$

Parameter restrictions for the special case($\sigma = \sigma^* = \sigma^S$):

$$\mu^S + \delta = \frac{1}{\kappa}\log(\frac{\kappa a + 1}{\kappa\sqrt{\rho}\tilde{\sigma} + 1}) \tag{A.259}$$

And to make sure domestic bond has a positive value, we need:

$$\sigma^2 < \frac{\mu^S - r^*}{\eta\sqrt{\rho}}\tilde{\sigma} \tag{A.260}$$

Assume $\kappa \to 0$ then $\Phi(\iota) \to \iota$, we have

$$\mu^S + \delta \approx \frac{a - \sqrt{\rho}\tilde{\sigma}}{\kappa\sqrt{\rho}\tilde{\sigma} + 1} \tag{A.261}$$

For $\theta^* < 0$ when constraint is slack, we need $\sigma^2 - 2(\mu^S - r^*) < 0$, inequality gives

$$\sigma^2 < \frac{\mu^S - r^*}{\eta\sqrt{\rho}}\tilde{\sigma} < 2(\mu^S - r^*) \tag{A.262}$$

and we get

$$\eta > \frac{2\tilde{\sigma}}{\sqrt{\rho}} = \frac{1}{2\theta^K} \tag{A.263}$$

### A.5.2 Constrained equilibrium

Parameter restriction for always positive excess demand of safe asset:

$$3\sigma^2 - 2(\mu^K - r^* - \frac{\rho}{\theta^K}) < 0 \quad \text{or} \quad 3\sigma^2 - 2(\mu^K - r^* - \frac{\rho}{\theta^K}) > \frac{1}{2} \tag{A.264}$$

Risk premia for international risk in comparison (special case where $\theta^S = (2\eta - 1)\theta^K$):

$$\frac{\theta^K(1-\eta)\sigma^* + \theta^S\sigma^S}{1-\theta^P} = \frac{\eta\theta^K}{1-\theta^P} \tag{A.265}$$

$$\frac{\theta^K(1-\eta)\sigma^*}{1-\underline{\theta}^P} - \frac{\eta\theta^K}{1-\theta^P} \propto (1-\eta) - \eta^2 - (1-\eta)^2 = \eta(1-2\eta) > 0 \tag{A.266}$$

### A.5.3 A bit more general parameter space

Now let's keep $\sigma = \sigma^* = \sigma^S$ and drop $\theta^K(\mu^S - r^*) = \theta^K(\mu^K - r^*) - \rho$. To simplify notation, let $\theta^K(\mu^S - r^*) \equiv r^a$ and $\theta^K(\mu^K - r^*) - \rho \equiv r^b$. We have

$$\theta^S = \theta^K \cdot \frac{2r^a\eta^2 + (r^b - 2r^a)\eta + r^a - r^b}{r^a\eta - (r^a - r^b)} \tag{A.267}$$

When $\eta = 1$, $\theta^S = \frac{r^a}{r^b}\theta^K > 0$ and when $\eta = 0$, $\theta^S = -\theta^K < 0$.

If $r^b > 2r^a$, $\theta^S$ is an increasing function of $\eta$.

If $2r^a > r^b > r^a$, $\theta^S$ at first decreases with $\eta$ and then increases with $\eta$.

If $r^b < r^a$, then there is discontinuity for $\theta^S$ at $\eta = 1 - \frac{r^b}{r^a}$. When $0 < \eta < 1 - \frac{r^b}{r^a}$, $\theta^S$ is a decreasing function of $\eta$ if $1 - \sqrt{2r^b} < \frac{r^b}{r^a}$ and $\theta^S$ first increases (might or might not cross 0) and then decreases with $\eta$ otherwise. When $1 - \frac{r^b}{r^a} < \eta < 1$, $\theta^S$ first decreases and then increases with $\eta$ and remains positive for a reasonable range of parameter values.

(Graphs to be added ...)

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