**The London School of Economics and Political Science**

# *Essays in Asset Pricing and Institutional Investors*

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A thesis submitted to the Department of Finance of the London School of Economics and Political Science for the degree of Doctor of Philosophy

London, May 2012

## Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).



I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

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### Statement of Conjoint Work

I confirm that Chapter 3 'Institutional Asset Pricing with Heterogenous Belief' was jointly co-authored with Mr. Shiyang Huang, Dr. Zhigang Qiu and Dr. Ke Tang, and I contributed 30% of this work.

## Abstract

The thesis includes three papers:

### 1. Limited Arbitrage Analysis of CDS Basis Trading

By modeling time-varying funding costs and demand pressure as the limits to arbitrage, the paper shows that assets with identical cash-flows have not only different expected returns, but also different expected returns in excess of funding costs. I solve the model in closed-form to show that the arbitrage on the CDS and corporate bond market is a risky arbitrage. The sign of the expected excess return of the arbitrage is decided by the sign and size of market frictions rather than the observed price discrepancy. The size and risk of the arbitrage excess return are increasing in market friction levels and assets' maturities. High levels of market frictions also destruct the positive predictability of credit spread term structure on credit spread changes. Results from the empirical section support the above-mentioned model predictions.

### 2. General Equilibrium Analysis of Stochastic Benchmarking

This paper applies a closed-form continuous-time consumption-based general equilibrium model to analyze the equilibrium implications when some agents in the economy promise to beat a stochastic benchmark at an intermediate date. For very risky benchmark, these agents increase volatility and risk premium in the equilibrium. On the other hand, when they promise to beat less risky benchmark, they decrease volatility and risk premium in the equilibrium. In both cases, the degree of effect is state-dependent and stock price rises.

### 3. Institutional Asset Pricing with Heterogenous Belief (Co-authored)

We propose an equilibrium asset pricing model in which investors with heterogeneous beliefs care about relative performance. We find that the relative performance concern leads agents to trade more similarly, which has two effects. First, similar trading directly decreases volatility. Second, similar trading decreases the impact of the dominant agents. When the economy is extremely good or bad, the second effect is dominant so that the relative performance concern enlarges the excess volatility caused by heterogeneous beliefs. When the first effect is dominant, which corresponds to a normal economy, the volatility is lower than without the relative

performance concern. Moreover, this paper shows that the relative performance concern also influences investors' holdings, stock prices and risk premia.

## Acknowledgement

I would like to thank all people who have helped and inspired me during my doctoral study.

My deepest gratitude goes out to my main supervisor, Dimitri Vayanos, whose encouragement, guidance and support helped me greatly in the understanding and writing of my papers. I would like to thank Georgy Chabakauri, Mikhail Chernov, Amil Dasgupta, Stephane Guibaud, Dong Lou, Christopher Polk, David Webb, Hongjun Yan, Kathy Yuan and many others for their valuable comments and advice. I would also like to thank for the help offered by my co-authors, fellow Ph.D students, administration staffs and Ph.D programme directors, including the late Antoine Faure-Grimaud, who will always be remembered.

I owe special thanks to my wife, Amanda, and our parents, for their love and support throughout my life. This dissertation is simply impossible without them.

I'm also very grateful for the Ph.D in Finance Scholarship I received from the school and Mr. Paul Aylieff, and the Research Studentship from the Department of Finance.

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# 1 Limited Arbitrage Analysis of CDS Basis Trading


## Abstract

By modeling time-varying funding costs and demand pressure as the limits to arbitrage, the paper shows that assets with identical cash-flows have not only different expected returns, but also different expected returns in excess of funding costs. I solve the model in closed-form to show that the arbitrage on the CDS and corporate bond market is a risky arbitrage. The sign of the expected excess return of the arbitrage is decided by the sign and size of market frictions rather than the observed price discrepancy. The size and risk of the arbitrage excess return are increasing in market friction levels and assets' maturities. High levels of market frictions also destruct the positive predictability of credit spread term structure on credit spread changes. Results from the empirical section support the above-mentioned model predictions.


## 1.1 Introduction

As summarized in Gromb and Vayanos (2010), costs or demand shocks faced by arbitrageurs can prevent them from eliminating mis-pricings on the markets and therefore generate market anomalies. My model illustrates how the interaction of time-varying funding cost and demand pressure faced by arbitrageurs results in two assets with identical payoffs having different expected excess returns. This result implies that taking opposite positions on these two assets is a risky arbitrage that is expected to earn profit in excess of funding costs. I then use the arbitrage between CDS and corporate bonds as an example to show how funding costs and demand pressures determine the expected excess return of risky arbitrage and change the credit spreads term structure's predictability on future credit spread. Finally, the empirical section lends support to several major theoretical results.

I explain the riskiness of arbitraging on two defaultable bonds with identical cash-flows as the result of the interaction of funding illiquidity and market illiquidity. Under a continuous-time demand-based framework in which risk-averse arbitrageurs trade on two markets with identical defaultable bonds to meet demand pressure posed by local investors, the presence of demand pressure results in arbitrageurs requiring risk premium for the positions they take. Without other frictions, demand pressure alone doesn't generate pricing discrepancies between assets with identical cash-flows. Arbitrageurs also face time-varying funding costs on the two markets, which under certain conditions result in the two assets having different risk exposures and carrying different levels of risk premium. The co-existence of time-varying funding cost and demand pressure makes taking opposite positions on the two markets a risky arbitrage. As shown in the model, without any one of these two sources of friction, the arbitrage doesn't generate expected excess profit.

I solve the model in closed-form under two cases. The first case assumes constant demand pressure from local investors while the second case assumes off-setting stochastic demand pressure. In the latter case, even if the arbitrageurs can take the exact opposite positions on the two markets in equilibrium so that they are completely protected from the risk of defaults, they still have non-zero exposures to other risk factors. Applying the results to the CDS and corporate bond markets suggests CDS basis trading is in fact risky arbitrage and it is reasonable for the CDS basis to deviate from its theoretical frictionless value of zero under severe market frictions. The expected excess return of the risky arbitrage depends on market frictions rather than the level of the price discrepancy. The model also shows the arbitrageurs sometimes magnify rather than correct price distortion under market frictions and offers a number of results on term structure properties.

Empirical results support the model predictions. Using Markit CDX and iBoxx Indices data and corporate bond data from TRACE, I show that basis trading is exposed to systematic risk factors, while the interaction of funding cost and market liquidity have predictive power on abnormal basis trading returns. As predicted by the model, the predictability of credit spread term structure slope on future credit spread change may turn from positive to negative when market frictions are high. I also find the size and volatility of realized basis trading excess return is increasing in the degree of market frictions. Moreover, the size of realized basis trading excess return is increasing in underlying maturity.

In early theoretical literature, Tuckman and Vila (1990) show that exogenous price discrepancies between two assets with identical cash-flows do not necessarily create arbitrage opportunities if there's shorting-selling cost. As show in Gromb and Vayanos (2010), without other frictions, the discrepancies between the expected returns of the two assets should compensate exactly for the funding costs so that an arbitrageur is still expected to earn zero excess return. In contrary, my model suggests arbitrageur can earn a risky profit even after adjusting for the funding costs if the funding costs are time-varying and arbitrageur faces demand pressure.

The demand pressure faced by the arbitrageurs is another important source of market friction that contributes to the rise of market anomalies. Investors other than arbitrageurs can have endogenous demand shocks that arise via different channels,[1] but a number of papers have focused on the pricing implications given exogenous demand shocks as I do. These theoretical demand-based papers include Garleanu, et.al.(2009), Naranjo (2009) and Vayanos and Villa (2009). By assuming exogenous underlying asset prices, Garleanu, et.al.(2009) and Naranjo (2009) solve derivative prices endogenously to show that exogenous demand shocks and frictions in arbitrageurs' trading can drive derivative prices away from conventional prices implied by the no-arbitrage conditions. Naranjo (2009) considers extra funding cost faced by arbitrageurs as the friction limiting her ability to eliminate price discrepancies on the futures and underlying markets. However, he assumed the underlying asset's price as exogenously given, so the market frictions only have impact on the futures price but not on the underlying market, which is unrealistic as derivative markets' trading do have impact on underlying price.

My paper is the closest to Vayanos and Vila (2009) in that prices in both legs of the arbitrage are endogenized so frictions on one market affect all markets. Vayanos and Villa (2009) study arbitrage across different Treasury bonds maturities, i.e. assets with different cash-flows, and there's no other frictions than the demand shocks from other preferred habitat investors. However, my paper introduces an additional friction, which is the time-varying funding costs, to show that two assets with identical cash-flows can earn different expected excess return, which implies profitability yet risk for a cross-market arbitrage trade. The two assets with identical cash-flows are two defaultable bonds, one represents corporate bond, while the other represents a synthetic corporate bond position created by writing a CDS protection and default-free lending.[2] Trading corporate bonds through repo and reverse-repo generates additional funding costs that are sensitive to default intensity of the bonds. Therefore I model the funding costs as functions of default intensity. Although the assumption on funding costs are not fully endogenized, a clear motivation in **Appendix 1.A** along with empirical findings by Gorton and Metrick (2010) justify this assumption, which is further supported by a recent paper by Mitchell and Pulvino (2011), which illustrates the consequence of funding liquidity failure on arbitrage activities from practitioners' perspectives.

[1] See Gromb and Vayanos (2010) for details.

[2] The equivalence of synthetic corporate bond and writing CDS plus lending will be illustrated in the next section.

I apply the model's implications on CDS basis trading, which turned out to be a risky arbitrage activity in the 2007/08 crisis. Credit default swap (CDS) is an OTC contract in which one party pays the other party a periodical fee (the CDS premium) for the protection against credit events of an underlying bond, in which case the protection seller pays the protection buyer for the loss from the underlying bond. The CDS market offers hedgers and speculators to trade credit risks in a relatively easy way. It is a fast growing market with vast market volume. The exact way to trade CDS has experienced some significant changes in the recent years following hot debates over the role it plays in the 2007/08 financial crisis. In the past, the two parties of a CDS trade agree on the CDS premium that makes the CDS contract having zero value at origination. Then as conditions change in the life of this contract, it has a marked-to-market value that is not zero. Following the implementation of the so-called CDS Big-Bang regulations on the North American markets in April 2009, the CDS premia are fixed at either 100bps or 500bps, and the two parties exchange an amount of cash at origination to reflect the true value of the contract. For instance, if the reasonable CDS premium should be 200bps, then the protection buyer pays the protection writer a certain amount of money at the beginning, and then pays CDS premium at 100bps each period. This is certainly a change in order to make the market more standardized and more liquid, however, the CDS market is still very opaque and illiquid.

Theoretical papers such as Duffie (1999) and Hull and White (2000) show the parity between CDS price and credit spread under the no-arbitrage condition, i.e. investors who hold a defaultable bond and short a CDS (buy protection) on this bond is effectively holding a default-free bond and thus should earn the risk free return. In other words, the CDS basis, which is CDS premium minus the credit spread, should be zero if basis trading earns only risk free return. Empirical works including Hull, et. al.(2004), Blanco, et. al.(2005) and Zhu (2006) using relatively early data support this zero basis hypothesis.

However, practitioners observe positive or negative basis at times and many investors engage in basis trading. Buying a bond and CDS protection is known as negative basis trading, while shorting a bond and writing CDS protection is called positive basis trading. Meanwhile, recent work by Garleanu and Pedersen (2011) and Fontana (2010) document large negative basis persisted from the summer of 2007 to early 2009. **Figure 1.1**[3] shows that during the 2007/08 crisis the CDS

[3]All figures and tables for this chapter are listed in Appendix 1.C.

basis become very negative for a long period, which contradicts text-book arbitrage argument, yet negative basis trading still lost money even at quite negative CDS basis level. In this paper, I reveal the risky arbitrage nature of CDS basis trading in closed-form and show the expected excess return of basis trading depends on market frictions rather than the level of CDS basis.

To justify the deviation from the law of one price on the CDS and corporate bond markets, Garleanu and Pedersen (2011) attribute two otherwise identical assets' different exposures to risk factors to the different level of exogenous margin requirement on the two markets. Empirical work by Fontana (2010) finds that funding costs variables are important in explaining CDS basis changes, while Bai and Collin-Dufresne (2010) explain cross-sectional variations in CDS basis with funding liquidity risk, counterparty risk and collateral quality. However, their result concerns only the changes in CDS basis, which according to my model doesn't determine the expected excess return of basis trading in the presence of funding costs. My paper is also related to a number of empirical works in explaining credit spread and CDS price movements such as: Collin-Dufresne, et. al.(2001), Elton, et. al.(2001), Huang and Huang (2003), Blanco, et. al.(2005), Tang and Yan (2007), and Ellul, et. al.(2009) among others. A recent work of Giglio (2011) proposes a novel way of inferring implied joint distribution of financial institution's default risk from the CDS basis, which shows the role of counterparty risk in widening CDS basis.

The next section introduces the model set-up before Section 1.3 and Section 1.4 solves the model under two cases and provides a number of results on the expected excess return of basis trading and the term structure properties of credit spreads. Empirical study is carried out in Section 1.5 to support theoretical results. Finally, Section 1.6 concludes the paper.

## 1.2 The Model

### 1.2.1 The Markets

In a continuous time economy with a horizon from zero to infinity, there exists two defaultable bond markets, named C and D respectively. On each market, there's a continuum of zero-coupon[4] defaultable bonds with face value of 1 and time to maturity $\tau$, $\tau \in (0, T]$. Denote the time $t$ prices of these bonds by $P_t^c(\tau)$ and $P_t^d(\tau)$ respectively. When applying equilibrium results to explain real world phenomenon,

[4]The coupon rate is assumed to be zero so as to derive closed form solutions.

I regard market C as the cash market, which corresponds to the corporate bond market, and market D as the derivative market, which corresponds to the CDS market plus default-free lending.[5]

Assume all bonds on the two markets are issued by the same entity, which has an exogenous default time $T'$. Upon default, for all bonds on the two markets, a bond holder loses $L$ fraction of a bond's market value. To keep the model tractable, I assume $L$ is constant and the same across all bonds. The above specification ensures that a bond with time to maturity $\tau$ on market C has identical cash-flows as the bond with time to maturity $\tau$ on market D.[6]

Assume the default intensity (instantaneous default probability) of the bond entity is $\widetilde{\lambda} + \lambda_t$, the stochastic part $\lambda_t$ follows the Ornstein-Uhlenbeck process[7]:

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda dB_{\lambda,t} \tag{1.1}$$

Where $\kappa_\lambda$, $\bar{\lambda}$ and $\sigma_\lambda$ are positive constants, and $B_{\lambda,t}$ is a Brownian Motion. A detailed description of stochastic default probability and doubly-stochastic default time can be found in Duffie (2005).

Next, assume there's a money market account that generates instantaneous return at an exogenous short rate of $r_t$, which also follows an Ornstein-Uhlenbeck process:

$$dr_t = \kappa_r(\bar{r} - r_t)dt + \sigma_r dB_{r,t} \tag{1.2}$$

As mentioned earlier, the real world implication of the model refers positions on market C as the cash bond positions, and positions on market D as the deriva-

[5]The equivalence between CDS plus default-free lending and synthetic defaultable bond position is given at the end of this section.

[6]Although price discrepancies between similar assets can arise from small cash-flow discrepancies, e.g. difference contract specifications, this model only aims to derive price discrepancies between assets with identical cash-flows under market frictions. This aim is in line with other limited arbitrage models.

[7]Under the assumption that $\lambda_t$ follows the Ornstein-Uhlenbeck process, it is possible to have $\lambda_t < 0$. Adding a positive $\widetilde{\lambda}$ to the default intensity reduces the chance of having negative default intensity $\widetilde{\lambda} + \lambda_t$, but still cannot exclude negative default intensity completely. To avoid negative default intensity, I assume that $\lambda_t$ is mean-reverting with square-root diffusion term in Appendix 1.D., which shows that the model still has closed-form solution in the constant demand pressure case and the main conclusion on the expected excess return of basis trading still stands. However, in the stochastic demand pressure case, the model cannot be solved in closed-form under this alternative assumption. Therefore, I still present the model under the simple Ornstein-Uhlenbeck assumption for $\lambda_t$.

tive (synthetic bond) positions created by CDS positions and default-free borrowing/lending in practice. In early literature such as Duffie (1999), the equivalence between buying defaultable bond and writing CDS plus default-free lending is best illustrated using floating rate notes (FRN). For instance, the cash flow of a defaultable FRN is the same as the cash flow from writing CDS protection plus the cash flow of a default-free FRN.

Without using FRNs, the equivalence still holds in the following way: in the absence of default, the zero-coupon defaultable bond holder gets the face value of 1 dollar at maturity, and upon default, gets the default-free present value of this 1 dollar but loses $L$ fraction of the bond's pre-default market value; As for the synthetic defaultable bond position created by CDS and default-free lending, assume the CDS has the same maturity as the defaultable bond and the CDS premium is paid up-front so that unless there's a default, there's no cash-flow between the two parties except at the origination of the contract. Also assume that in the default-free lending, the lender lends the default-free present value of 1 dollar to the borrower, who pays back the face value of 1 dollar at the maturity of the CDS, or the default-free present value of 1 dollar at anytime before the maturity. Therefore, if there's no default, synthetic position's total payoff at maturity is the 1 dollar from the default-free lending at maturity. If there's default, the CDS position pays a fraction $L$ of the defaultable bond's pre-default market value, and the default-free lending position retrieves the present value of the 1 dollar lent. So the synthetic position's total payoff is the default-free present value of 1 dollar minus $L$ fraction of the bond's pre-default market value. Therefore, the cash-flow from the synthetic position is the same as that of a defaultable bond, whether there's default or not.

When applying model's predictions to discuss real world phenomenon, define negative basis trading as buying cash bond C and selling derivative bond D. This corresponds to buying corporate bond through borrowing and buying CDS protection in practice. In the contrary, positive basis trading is defined as buying derivative bond D and selling cash bond C. This corresponds to shorting corporate bond through reverse repo and writing CDS protection.

### 1.2.2 The Agents and Demand Pressure

The economy consists of two types of agents: local investors and arbitrageurs. Local investors are segmented into the two markets. Denote the cash market local investors' aggregate amount invested for maturity $\tau$ by $z_t^c(\tau)$, and the derivative market local investors' aggregate amount invested for maturity $\tau$ by $z_t^d(\tau)$. Local investors can only invest in their own markets.

In reality, large excess demand or supply from local investors exist on both the CDS market and the corporate bond market. Investors have different motives to trade on the CDS or the corporate bond markets, so it is reasonable to assume they are segmented.[8] The excess supply of corporate bonds could come from regulation restricted fire-sale of bonds by insurance companies or simply a flight to quality during crisis. The excess demand for corporate bonds can come from large inflows into bond market funds. On the other hand, excess demand or supply on the CDS markets can come from the hedging demand from banks who hold bonds/loans or those who gain exposure to default risk from other credit derivative market positions. Sometimes the local investors' demand on the two markets can be exactly the opposite. As documented by Mitchell and Pulvino (2011) and also in other practitioners' articles, some banks that hold corporate bonds and CDS protection on these underlying bonds unwound their positions after the Lehman Collapse in order to free up more cash. Their trades would have introduced negative $z_t^c$ and positive $z_t^d$ with the same absolute value in the model. In this scenario, the arbitrageurs face exactly the opposite demand pressures from the two markets. A special case of my model investigates this scenario in detail in the following sections.

There's an infinite number of risk-averse arbitraguers that form a continuum with measure 1 who can trade any amount on the two bond markets, and therefore renders the prices arbitrage free. At any time $t$, the continuum of arbitrageurs are born in time $t$ and die in $t + dt$, so arbitrageur's utility is to trade off the instantaneous mean and variance of their payoff. The arbitrageurs have zero wealth when they're born, i.e. an arbitrageur's time $t$ wealth $W_t = 0$. Denote the arbitrageur's amount invested in the cash market for maturity $\tau$ by $x_t^c(\tau)$ and the amount invested in the derivative market for maturity $\tau$ by $x_t^d(\tau)$.

Assume both markets have zero supply. At equilibrium, the arbitrageurs and local investors clear both markets, i.e. $x_t^c(\tau) + z_t^c(\tau) = 0$ and $x_t^d(\tau) + z_t^d(\tau) = 0$.

A positive $z_t^i$, $i = c, d$ corresponds to local investors' excess demand and suggests the arbitrageurs have a pressure to sell in equilibrium. On the other hand, a negative

[8]Like in Vayanos and Vila (2009), it is a simplification to assume that local investors are segmented, and to consider those who can trade on both markets as arbitrageurs.

$z_t^i$ corresponds to local investors' excess supply and the arbitrageurs have a pressure to buy in equilibrium.

In general, assume the aggregate demand from local investors are linear functions of a stochastic process $z_t$:

$$z_t^i(\tau) = \bar{\theta}^i(\tau) + \theta^i(\tau) z_t \tag{1.3}$$

$$dz_t = \kappa_z(\bar{z} - z_t)dt + \sigma_z dB_{z,t} \tag{1.4}$$

where $i = c, d$. $\bar{\theta}^i(\tau)$ and $\theta^i(\tau)$ are functions of $\tau$. The sign of $\bar{\theta}^i(\tau)$ and $\theta^i(\tau)$ will be specified later. So far there are three Brownian motions in the model, namely $B_{\lambda,t}$, $B_{r,t}$ and $B_{z,t}$. To derive closed form solutions, assume they are independent.

The excess demand or supply from local investors faced by the arbitrageur is often called demand pressure. The presence of demand pressure has implications on asset pricing because the arbitraguers, who provide liquidity by clearing the markets, need to be compensated for the risk exposure they get by doing so. As summarized by Gromb and Vayanos (2010), several kinds of demand pressure effects on treasury bonds, futures and options markets have been studied.[9] Without other frictions, demand pressure alone doesn't generate pricing discrepancies between assets with identical cash-flows. My model introduce the time-varying funding costs described in the next section as the source of friction that work together with demand pressure to cause pricing discrepancies. The inclusion of this funding cost also distinguishes my model from the above mentioned ones.

### 1.2.3 Funding Costs

In the real world, buying defaultable bonds through borrowing (using repo) and short-selling through reverse-repo incurs funding costs in excess of the short rate. If one buys defaultable bond through borrowing, she can only borrow at a rate higher than $r_t$. Similarly, if one shorts defaultable bond, she incurs short-selling cost. The additional borrowing cost and short-selling cost are called funding costs in this model. The presence of these funding costs represents a source of market friction. Assume that the funding costs $h_t^i(\tau)$, $i = c, d$, are exogenous linear functions of $\lambda_t$, the stochastic part of default intensity:

$$h_t^i(\tau) = \alpha^i(\tau)\lambda_t + \delta^i(\tau) \quad i = c, d \quad 0 < \alpha^i < L \tag{1.5}$$

[9] By Vayanos and Villa (2009), Naranjo (2008), and Garleanu, et.al. (2009) respectively

**Appendix 1.A** provides motivations for this assumption[10] by solving for the optimal hair-cuts applied in repo and reverse-repo transactions. Given exogenous interest rates, the cost of borrowing using the defaultable bond as collateral in a repo transaction is shown to be increasing in the default intensity risk $\lambda_t$. This is because the amount that can be borrowed using the bond as collateral is decreasing in $\lambda_t$. Therefore, when $\lambda_t$ is higher, the borrower has to borrow more at the uncollateralized rate, and therefore incur more borrowing costs. The short-selling cost is also shown to be increasing in $\lambda_t$. In a reverse-repo transaction, the short-seller of the bond will be asked to put more cash collateral to borrow the bond for sale when $\lambda_t$ is higher because the bond is more risky which makes the short-seller more likely to default on the obligation to return the bond. As a result, the short-seller lends more at collateralized rate, less at uncollateralized rate, therefore earns less interest from the proceeds of the short-selling (incurs more short-selling costs). The above rationale is supported by empirical evidence found by Gorton and Metrick (2010) that the repo hair-cut is increasing in the riskiness of collaterals. Mitchell and Pulvino (2011) also justify the above assumption from a practitioner's perspective. This funding cost models both the borrowing cost and short-selling cost, so it reduces arbitrageur's wealth regardless of the direction of her trades.

When making the analogy between derivative market D and the CDS market, if trading on the CDS market is frictionless, then $\alpha^d(\tau)$ and $\delta^d(\tau)$ can be set as zeros. However, although the CDS market used to have extremely low funding costs due to its low margin requirement, it is not frictionless. During the 2007/08 crisis, counterparty risk in CDS contract led to the raise in margin requirement on the CDS market. Counterparty risk refers to the possibility that protection writers may default on their obligation to pay the buyers upon the default of the underlying, or the possibility that protection buyers default on their obligation to pay the CDS premium. CDS protection writers were asked to put more collaterals than before, while protection buyers were asked to pay CDS premium up-front. These changes made CDS having comparable funding costs as trading corporate bonds. An alternative way for CDS protection writers to provide collateral that was widely used in practice is for them to buy a CDS on their own names for the protection buyer from a third party. In this way, if the CDS writer defaults, the buyer can still get paid by the third party. Therefore, the CDS protection writer incurs additional periodical cost that equals to the CDS premium on themselves. If the CDS writer's default

[10]The above specifications imply symmetric borrowing and short-selling costs. The model can be easily adapted to include asymmetric borrowing cost and short-selling costs.

can only be triggered by the underlying's default, then this additional cost should be increasing in the underlying's default intensity. So for the derivative market D, it's also reasonable to assume funding costs as an increasing function of $\lambda_t$.

### 1.2.4 The Arbitrageurs' Optimization Problem

An arbitrageur's optimization problem is:

$$\max_{x_t^c, x_t^d} [E_t(dW_t) - \frac{\gamma}{2} Var_t(dW_t)] \tag{1.6}$$

$$\begin{aligned} dW_t &= [W_t - \int_0^T x_t^c(\tau) d\tau - \int_0^T x_t^d(\tau) d\tau] r_t dt \\ &+ \int_0^T x_t^c(\tau) [\frac{dP_t^c(\tau)}{P_t^c(\tau)} - L dN_t] d\tau + \int_0^T x_t^d(\tau) [\frac{dP_t^d(\tau)}{P_t^d(\tau)} - L dN_t] d\tau \\ &- \int_0^T |x_t^c(\tau)| h_t^c(\tau) d\tau dt - \int_0^T |x_t^d(\tau)| h_t^d(\tau) d\tau dt \end{aligned} \tag{1.7}$$

The arbitrageur trades off instantaneous expected payoff and variance of payoff. In the above dynamic budget constraint, $W_t$ is the representative arbitrageur's wealth at time $t$ and is assumed to be zero. $\gamma$ is her risk aversion coefficient. Ignoring $\tau$, $x_t^c$ is the amount she invests into cash bond C and $x_t^d$ is the amount she invests into derivative bond D. The first term on the right hand side of the dynamic budget constraint gives the amount earned by her money market account. The arbitrageur's wealth is also affected by changes in the cash and derivative bond prices, as well as jumps upon default. Additionally, since the arbitrageur is born with zero wealth, she can only buy through borrowing or sell through short-selling. Therefore, trading on the cash and derivative markets incurs funding cost at the rate of $h_t^c(\tau)$ and $h_t^d(\tau)$. These costs reduce arbitrageur's wealth regardless of the direction of her trades, so the costs are multiplied by the absolute value of arbitrageur's trade.

Before presenting the equilibrium results, I first derive the arbitrageur's F.O.C.s so as to provide intuition and definition that facilitate the discussions in latter parts of this section. I then solve the equilibrium bond prices in closed form for two cases separately. Under each case, the closed form solutions clearly reveal the sign and size of the expected excess return of basis trading under market frictions, and also offer new properties of credit spreads.

### 1.2.5 The First Order Conditions

**Lemma 1.1.** *Ignoring $\tau$ where it doesn't cause confusion, the Arbitrageur's F.O.C.s are:*

$$\mu_t^c - r_t - h_t^c \frac{\partial |x_t^c|}{\partial x_t^c} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} + \sum_{j \in \{\lambda, r, z\}} \sigma_j \frac{1}{P_t^c} \frac{\partial P_t^c}{\partial j} \Phi_{j,t} \quad (1.8)$$

$$\mu_t^d - r_t - h_t^d \frac{\partial |x_t^d|}{\partial x_t^d} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} + \sum_{j \in \{\lambda, r, z\}} \sigma_j \frac{1}{P_t^d} \frac{\partial P_t^d}{\partial j} \Phi_{j,t} \quad (1.9)$$

*where $\mu_t^c$ and $\mu_t^d$ are the expected returns of bond C and D conditional on no default,*[11] *and*

$$\Phi_{J,t} = \gamma L \int_0^T [x_t^c(\tau) + x_t^d(\tau)] d\tau (\widetilde{\lambda} + \lambda_t) \quad (1.10)$$

$$\Phi_{j,t} = \gamma \sigma_j \int_0^T [x_t^c(\tau) \frac{1}{P_t^c} \frac{\partial P_t^c}{\partial j} + x_t^d(\tau) \frac{1}{P_t^d} \frac{\partial P_t^d}{\partial j}] d\tau \quad (1.11)$$

*are the market prices of risks, $j = \lambda, r, z$.*

*Proof.* see Appendix 1.B. □

The left hand side of the F.O.C.s is the instantaneous expected excess return, hereafter EER, and the right hand side is the risk premium. The EER is the expected return of a bond in excess of the short rate and the funding cost. The EER equals the risk premium which is given by the exposure to risk times the market price of risk. By construction, bond C and bond D have the same exposure to the jump risk of default, which carries market price of risk $\Phi_{J,t}$. They also have exposures to the other 3 risk factors, namely the default intensity risk factor $\lambda_t$, which carries market price of risk $\Phi_{\lambda,t}$, the short rate risk factor $r_t$, which carries market price of risk $\Phi_{r,t}$ and the demand shock risk factor $z_t$, which carries market price of risk $\Phi_{z,t}$. The magnitudes of these exposures depend on equilibrium terms $\frac{1}{P_t^i} \frac{\partial P_t^i}{\partial j}$, $i = c, d$, $j = \lambda, r, z$.

Subtracting the first F.O.C. from the second one gives the total expected excess return of the so-called negative basis trade. In contrary, subtracting the second F.O.C. from the first one gives the expected excess return of positive basis trade. Formally, I make the following definition.

**Definition 1.1.** *Negative basis trading (nbt) is defined as buying bond C and sell-*

[11] For definition of $\mu_t^c$ and $\mu_t^d$, see the Proof of Lemma 1 in Appendix B

*ing bond D for the same time-to-maturity, which corresponds to buying corporate bond through borrowing and buying CDS protection in the real world; Positive basis trading (pbt) is defined as selling bond C and buying bond D for the same time-to-maturity, which corresponds to shorting corporate bond and writing CDS protection plus lending in the real world.*

$$EER^{nbt} = \sum_{j\in\{\lambda,r,z\}} \sigma_j\left(\frac{1}{P_t^c}\frac{\partial P_t^c}{\partial j} - \frac{1}{P_t^d}\frac{\partial P_t^d}{\partial j}\right)\Phi_{j,t} \tag{1.12}$$

$$EER^{pbt} = \sum_{j\in\{\lambda,r,z\}} \sigma_j\left(\frac{1}{P_t^d}\frac{\partial P_t^d}{\partial j} - \frac{1}{P_t^c}\frac{\partial P_t^c}{\partial j}\right)\Phi_{j,t} \tag{1.13}$$

Based on the F.O.C.s, equilibrium results are solved using the market clearing condition. On each market, the optimally derived quantities of the arbitrageurs plus those from the local investors equals zero. To solve the model in closed-form, I make further assumptions on the local investors demand and derive equilibrium results in the following two cases.

## 1.3 Equilibrium Results 1: Under Constant Demand Pressure

In the first case, I assume that the local investors have constant demand on the two markets. Recall the specification for the local investors' demand $z_t^i(\tau)$ in Equation (1.3), this assumption implies $\theta^i(\tau) = 0$ and $z_t^i(\tau) = \bar{\theta}^i(\tau) = z^i(\tau)$, $i = c, d$. Assuming local investors having constant demand removes the demand shock factor $z_t$ from the model, I therefore conjecture that the bond prices to take the following form:

$$P_t^c(\tau) = e^{-[A_\lambda^c(\tau)\lambda_t + A_r^c(\tau)r_t + C^c(\tau)]} \tag{1.14}$$

$$P_t^d(\tau) = e^{-[A_\lambda^d(\tau)\lambda_t + A_r^d(\tau)r_t + C^d(\tau)]} \tag{1.15}$$

where $A_j^i(\tau)$ and $C^i(\tau)$, $i = c, d$, $j = \lambda, r$, are functions of $\tau$. To solve for the equilibrium, take the above conjectured prices into the arbitrageurs F.O.C.s in Equation (1.8) (1.9) and replace $x_t^i(\tau)$ with $-z^i(\tau)$ to reflect the market clearing condition in the equilibrium: $x_t^i(\tau) + z^i(\tau) = 0$, $i = c, d$.

### 1.3.1 Solutions

**Lemma 1.2.** *The $A_j^i(\tau)$, $i = c, d$, $j = \lambda, r, z$, in the conjectured price functions are solved as:*

$$
\begin{aligned}
A_\lambda^c(\tau) &= \{-\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} + L - \gamma L^2 \int_0^T [z^c(\tau) + z^d(\tau)]d\tau\}\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_\lambda^d(\tau) &= \{-\alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)} + L - \gamma L^2 \int_0^T [z^c(\tau) + z^d(\tau)]d\tau\}\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_r^c(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \\
A_r^d(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r}
\end{aligned}
\tag{1.16}
$$

*Proof.* see Appendix 1.B. □

Therefore, $A_r^c(\tau) = A_r^d(\tau)$, prices on the two markets have the same coefficients for the short rate risk. As for the coefficient for the default intensity risk, if local investors' demand on the two markets are in opposite directions, i.e. $sign[z^c(\tau)] = -sign[z^d(\tau)]$, then $A_\lambda^c(\tau) \neq A_\lambda^d(\tau)$, the prices on the two markets have different coefficients for the default intensity risk $\lambda_t$, as long as at least one market has funding cost that has non-zero sensitivity to the default intensity risk, i.e. $\alpha^c(\tau) \neq 0$ or $\alpha^d(\tau) \neq 0$; if local investors' demand on the two markets are in the same direction, then prices on the two markets have different coefficients for the default intensity risk if funding costs on the two markets have different sensitivity to the default intensity risk, i.e. $\alpha^c(\tau) \neq \alpha^d(\tau)$.

The prices' sensitivities to risk factors have important implications on the discussion of basis trading returns, which now can be derived by taking the solutions of $A_j^i(\tau)$ into Equation (1.12) (1.13).

### 1.3.2 The Expected Excess Returns of Basis Trading

Although conventional wisdom suggests market frictions such as funding cost may cause assets with identical cash-flows to have different prices, it is not clear whether arbitrageurs are expected to earn positive excess return if they try to take advantage from the price discrepancies. However, in this model, it is possible to derive the expected excess return of basis trading in closed form and analyze its sign and size in the following way:

**Proposition 1.1.** *The expected excess returns of basis trading are:*

$$
\begin{aligned}
EER^{nbt} &= \sigma_\lambda[\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} - \alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)}]\frac{1-e^{-\kappa_\lambda \tau}}{\kappa_\lambda}\Phi_{\lambda,t} && (1.17)\\
EER^{pbt} &= -EER^{nbt} && (1.18)\\
\Phi_{\lambda,t} &= \gamma\sigma_\lambda \int_0^T [z^c(\tau)A^c_\lambda(\tau) + z^d(\tau)A^d_\lambda(\tau)]d\tau && (1.19)
\end{aligned}
$$

*Proof.* see Appendix 1.B. □

**Corollary 1.1.** *The sign of the expected excess return of basis trading has the following properties:*

- *Basis trading earns non-zero expected excess return, except for the following three scenarios, : 1) $\alpha^c(\tau) = \alpha^d(\tau) = 0$; 2) $\alpha^c(\tau) = \alpha^d(\tau) \neq 0$ and $sign[z^c(\tau)] = sign[z^d(\tau)]$; and 3) $z^c(\tau) = z^d(\tau) = 0$.*
- *When local investors are selling on both markets, or only buying moderate amount on both markets, basis trading is expected to earn positive excess return by buying on the market whose funding cost has higher sensitivity to default intensity risk and selling on the other market.*
- *When local investors are buying in huge amount on both markets such that prices have positive sensitivities to default intensity risk, basis trading is profitable by buying on the market whose funding cost is less sensitive to default intensity risk and selling on the other market.*

*Proof.* see Appendix 1.B. □

The first point suggests that basis trading is a risky arbitrage that earns non-zero EER only when both funding cost friction and market liquidity friction are present. Under scenario 1), funding costs on the two markets have no sensitivities to $\lambda_t$; under scenario 2), they have the same sensitivities and arbitrageur takes the same side of trade on the two markets, then the two assets carry the same exposures to the default intensity risk factor, which results in them carrying the same risk premium. Therefore, under both scenarios 1) and 2), buying on one market and selling on the other results in zero aggregate exposure to risk factors and basis trading earns zero EER. Under scenario 3), even if assets on the two markets may have different risk exposures to default intensity risk, the market prices of risks are all zeros because the arbitrageur doesn't have to provide liquidity in equilibrium so all EERs should be zero. Other than in the above mentioned three scenarios, basis trading is expected to earn non-zero return in excess of the funding costs, i.e. basis trading is expected to be profitable.

Intuitively, basis trading can earn positive expected excess return, i.e. make profit after deducting funding costs, because the presence of demand pressure results in arbitrageurs requiring risk premium for the positions they take. The time-varying funding costs which are correlated with the underlying risk (default intensity risk) on the two markets cause the two assets to have different risk exposures. The aggregate risk exposure of buying on one market and selling the same quantity on the other market is thus non-zero, and therefore earns non-zero expected excess return.

However, in reality, one can do basis trading in two ways: Buying on market C while selling on market D, or buying on market D while selling on market C. The second and third points in **Corollary 1.1** concern the sign of basis trading EERs. When local investors on the two markets are all selling, bonds on both markets carry positive risk premia which compensates the arbitrageurs who are buying to provide liquidity. The arbitageur is exposed to more default intensity risk on the market with higher funding cost sensitivity to $\lambda_t$, therefore earns more default intensity risk premium on this market. So buying bond on this market and selling on the other generates positive expected excess return. For instance, if there's funding cost on corporate bond market but not on CDS market, i.e. $\alpha^c > \alpha^d = 0$, then if local investors are selling on both markets, the model predicts negative basis trading to be profitable even after deducting the funding costs. In the other case, when local investors on the two markets are all buying moderate amount, bonds on both markets carry negative risk premium. The market whose funding cost has higher sensitivity to $\lambda_t$ has lower sensitivity to the default intensity risk, therefore earns less negative risk premium. So buying bond on this market and selling on the other is expected to be profitable.

However, if local investors are buying too much on both markets, bonds also carry negative default intensity risk premium but the market whose funding cost has lower sensitivity to $\lambda_t$ now has lower sensitivity to the default intensity risk, therefore earns less negative risk premium. So buying bond on this market and selling on the other is expected to be profitable. The difference here with the moderately positive local investor demand case is that when local investors are buying too much, bond prices become increasing in default intensity. This is true due to the assumption that bond holders retain $(1 - L)$ fraction of bond's market value upon default. The compensation to arbitrageur for providing liquidity in this case is for price to be increasing in default intensity so that she's expected to gain more upon default when default intensity is higher.

**Corollary 1.2.** *The size of the risk exposure and expected excess return of basis trading has the following properties:*

- *When* $sign(z^c) = sign(z^d)$*, the size of basis trading's risk exposure is increasing in* $|\alpha^c - \alpha^d|$*.*
- *When* $sign(z^c) = -sign(z^d)$*, the size of basis trading's risk exposure is increasing in* $(\alpha^c + \alpha^d)$*.*
- *Size of basis trading EER is increasing in the volatility of default intensity.*

*Proof.* see Appendix 1.B. □

As mentioned above, the net exposure to default intensity risk depends on the funding costs' sensitivities to $\lambda_t$. When the local demands on the two markets are in the same direction, the net exposure is determined by the difference of $\alpha^c$ and $\alpha^d$. But when the local investors' demand are in opposite directions, the net exposure depends on the sum of $\alpha^c$ and $\alpha^d$. The size of EER of basis trading is also increasing in $\sigma_\lambda$, which is not surprising as $\sigma_\lambda$ is positively priced in both the basis trading's absolute net exposure to default intensity risk and the market price of default intensity risk.

### 1.3.3 Implications on Credit Spread Term Structure and the Predictability of Credit Spread

Earlier empirical works such as Bedendo et.al.(2007) found that the slope of the credit spread term structure positively predicts future changes in credit spread. My model supports this result when friction is moderate, but when there's high level of market friction, my model suggests that the positive predictability become negative. To see this point in details, I first make the following definitions:

**Definition 1.2.** *Credit spread is the difference between the yield to maturity of a defaultable bond and the yield to maturity of a default-free bond of the same maturity. Denote yields to maturity of the defaultable bond with time to maturity* $\tau$ *by* $Y_t^c(\tau)$*, and the credit spread by* $CS_t^c(\tau)$*.*

$$Y_t^c(\tau) = -\frac{logP_t^c(\tau)}{\tau} \tag{1.20}$$

There's no default-free bond market in my model, but it is safe to conjecture that a frictionless default-free bond market has default-free bond prices: $DF_t(\tau) = e^{-[A_r(\tau)r_t + C^{df}(\tau)]}$,[12] so that the yield to maturity of default-free bond can be denoted by $Y_t^{df}(\tau) = -\log DF_t(\tau)/\tau$. Because the defaultable bond and default-free bond have the same coefficient to short rate $r_t$, therefore the credit spread term structure $CS_t(\tau)$ only has one time-varying risk factor $\lambda_t$.

$$CS_t(\tau) = Y_t(\tau) - Y_t^{df}(\tau) = \frac{A_\lambda^c(\tau)}{\tau}\lambda_t + \frac{C^c(\tau)}{\tau} - \frac{C^{df}(\tau)}{\tau} \tag{1.21}$$

where $C^c(\tau)$ and $C^{df}(\tau)$ are functions of $\tau$ that do not matter in deriving the following results.

Bedendo et.al.(2007) run the following regression for credit spread of a certain maturity $\tau$ and found the coefficient estimate $\psi$ to be significantly positive, which suggests the slope of credit spread term structure positively predicts future credit spread changes.

$$CS_{t+\Delta\tau}(\tau - \Delta\tau) - CS_t(\tau) = \psi_0 + \psi[CS_t(\tau) - CS_t(\tau - \Delta\tau)] + \epsilon_{t+\Delta\tau} \tag{1.22}$$

According to my model, the coefficient $\psi$ is calculated in closed-form and analyzed in the following way:

**Proposition 1.2.** *When $\Delta\tau \to 0$, the regression coefficient $\psi(\tau) = \frac{F(\tau)}{1-F(\tau)e^{-\kappa_\lambda \tau}}$, where $F(\tau) = -\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} + L - \gamma L^2 \int_0^T [z^c(\tau) + z^d(\tau)]d\tau$*

- *When there's no friction, i.e. $\alpha^c(\tau) = 0$ and $z^c(\tau) = z^d(\tau) = 0$, then $\psi > 0$. The slope of credit spread term structure positively predicts future credit spread changes.*

- *When local investors are selling large quantities and funding cost is very sensitive to $\lambda_t$, i.e. $z^c(\tau) << 0$, $\alpha^c(\tau) >> 0$, then $\psi < 0$. The slope of credit spread term structure negatively predicts future credit spread changes.*

- *$\psi < 0$ is more likely to happen to bonds with short time-to-maturities.*

*Proof.* see Appendix 1.B. □

Under standard set-up without the funding cost and demand pressure frictions, the condition for $\psi > 0$ simplifies to $L < e^{\kappa_\lambda \tau}$, which is always satisfied as $L < 1$ by as-

[12] For example, this can be derived from Vayanos and Vila (2009) by assuming arbitrageurs in their model face no demand pressures from local investors.

sumption. Therefore, the credit spread slope should always positively predict future credit spread changes, which has been documented by Bedendo et.al.(2007) among others. But new features in this model suggests that market frictions can distort this predictability. The coefficient $\psi$ can be either positive or negative depending on market conditions. When local investors are selling large quantities on the corporate bond market and funding cost is very sensitive to default intensity risk, the credit spread slope may negatively predict future credit spread changes. In the empirical section, this point is supported by data during the crisis in 2008.

Intuitively, similar to the argument of the expectation hypothesis of the interest rate term structure, conventional thinking believes credit spread term structure contains information of the expectation of future credit spread changes, i.e. positive credit spread term structure slope implies that future default probability is likely to increase, hence credit spread will increase. However, in the presence of large market frictions, corporate bond prices reflect not only credit risk but also funding liquidity and market liquidity risk, so credit spread term structure becomes less informative about future credit spread changes.

To be more specific, the negative predictability of credit spread term structure slope on future credit spread changes can be explained in the following way: if market frictions are high and local bond market investors are selling, bonds with longer time-to-maturity may become less favorable to use as collateral for borrowing due to its higher sensitivity to risk factors. Therefore, although all bonds will be undervalued, bonds with longer time-to-maturity will have higher credit spread (lower price) than bonds with shorter time-to-maturity even if they carry the same default risk information. The credit spread term structure is thus upward sloping. Since the underpricing of bonds are caused by market frictions rather than default risk factor fluctuation, the credit spread in the future is likely to decrease as bond prices finally return to their fundamental level. Then empirical study can observe positive credit spread term structure followed by negative credit spread changes, i.e. negative predictability.

## 1.4 Equilibrium Results 2: Under Off-setting Stochastic Demand Pressure

Next, I relax the constant demand pressure assumption to allow local investors to have stochastic demand. To keep the model linear and thus allow closed form solutions, I make an additional assumption that the demand from local investors on the cash and derivative markets are exactly the opposite, i.e. $-z_t^d(\tau) = z_t^c(\tau)$. This is equivalent to assuming:

$$z_t^c(\tau) = \bar{\theta}(\tau) + \theta(\tau) z_t \tag{1.23}$$
$$z_t^d(\tau) = -\bar{\theta}(\tau) - \theta(\tau) z_t \tag{1.24}$$

The demand function is price in-elastic. If I assume the two demand pressures to be exactly the opposite and price-elastic, the equilibrium prices can still be solved in closed-form, but are very complicated that makes the discussions on assets returns unclear, so results for that case are not presented.

### 1.4.1 Solutions

Given the above assumptions on local investors' demand, the model has closed-form solutions for the following two scenarios: 1) $\bar{\theta}(\tau) << 0$ and $\theta(\tau) < 0$, and 2) $\bar{\theta}(\tau) >> 0$ and $\theta(\tau) > 0$. I conjecture that prices are exponential affine in the risk factors:

$$P_t^c(\tau) = e^{-[A_\lambda^c(\tau)\lambda_t + A_r^c(\tau) r_t + A_z^c(\tau) z_t + C^c(\tau)]} \tag{1.25}$$
$$P_t^d(\tau) = e^{-[A_\lambda^d(\tau)\lambda_t + A_r^d(\tau) r_t + A_z^d(\tau) z_t + C^d(\tau)]} \tag{1.26}$$

and solve for the coefficients for the two scenarios separately in the following Lemmas.

**Lemma 1.3.** *For very negative $\bar{\theta}(\tau)$ and negative $\theta(\tau)$:*

$$\begin{aligned}
A_\lambda^c(\tau) &= [\alpha^c(\tau) + L]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_\lambda^d(\tau) &= [-\alpha^d(\tau) + L]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_r^c(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \\
A_r^d(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \\
A_z^c(\tau) &= -\gamma \sigma_\lambda^2 A_\lambda^c(\tau) \int_0^T \theta(\tau)[\alpha^c(\tau) + \alpha^d(\tau)]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} d\tau \frac{1 - e^{-\kappa_z^* \tau}}{\kappa_z^*} \\
A_z^d(\tau) &= -\gamma \sigma_\lambda^2 A_\lambda^d(\tau) \int_0^T \theta(\tau)[\alpha^c(\tau) + \alpha^d(\tau)]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} d\tau \frac{1 - e^{-\kappa_z^* \tau}}{\kappa_z^*}
\end{aligned}$$

*where $\kappa_z^*$ is the unique solution to:*

$$\kappa_z^* = \kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.27}$$

*Proof.* see Appendix 1.B. □

**Lemma 1.4.** *For very positive $\bar{\theta}(\tau)$, positive $\theta(\tau)$ and very positive $\kappa_z$:*

$$\begin{aligned}
A_\lambda^c(\tau) &= [-\alpha^c(\tau) + L]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_\lambda^d(\tau) &= [\alpha^d(\tau) + L]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
A_r^c(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \\
A_r^d(\tau) &= \frac{1 - e^{-\kappa_r \tau}}{\kappa_r} \\
A_z^c(\tau) &= \gamma\sigma_\lambda^2 A_\lambda^c(\tau) \int_0^T \theta(\tau)[\alpha^c(\tau) + \alpha^d(\tau)]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} d\tau \frac{1 - e^{-\kappa_z^* \tau}}{\kappa_z^*} \\
A_z^d(\tau) &= \gamma\sigma_\lambda^2 A_\lambda^d(\tau) \int_0^T \theta(\tau)[\alpha^c(\tau) + \alpha^d(\tau)]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} d\tau \frac{1 - e^{-\kappa_z^* \tau}}{\kappa_z^*}
\end{aligned}$$

*where $\kappa_z^*$ is the unique solution to:*

$$\kappa_z^* = \kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.28}$$

*Proof.* see Appendix 1.B. □

Once again, prices on the two markets have the same coefficients for the short rate. Now, local investors' demand on the two markets are always in the opposite directions, so the prices on the two markets have different coefficients for $\lambda_t$ and $z_t$ even if the funding costs on the two markets have the same sensitivities to $\lambda_t$.

### 1.4.2 The Expected Excess Returns of Basis Trading and Corporate Bond

The calculation of expected excess return is complicated by the inclusion of $z_t$, as the EER of basis trading has exposure to two sources of risk premia now.

**Proposition 1.3.** *Under the two cases described in* ***Lemma 1.3*** *and* ***Lemma 1.4****, the expected excess returns of basis trading are:*

$$EER^{nbt} = -\gamma\sigma_\lambda^2 G^\lambda(\tau) \int_0^T G^\lambda(\tau) z_t^c(\tau)d\tau - \gamma\sigma_z^2 G^z(\tau) \int_0^T G^z(\tau) z_t^c(\tau)d\tau$$

$$
\begin{aligned}
EER^{pbt} &= -EER^{nbt} && (1.29)\\
G^{\lambda}(\tau) &= [\alpha^c(\tau) + \alpha^d(\tau)]\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda} \\
G^{z}(\tau) &= -\gamma\sigma_\lambda^2 G^{\lambda}(\tau)\int_0^T G^{\lambda}(\tau)\theta(\tau)d\tau\frac{1 - e^{-\kappa_z^* \tau}}{\kappa_z^*} && (1.30)
\end{aligned}
$$

*Proof.* see Appendix 1.B. □

**Corollary 1.3.** *Assume on each market, bonds of all time-to-maturities have the same funding cost sensitivity to default intensity risk, i.e.* $\alpha^i(\tau) = \alpha^i$. *Then, the sign and size of the expected excess return of basis trading has the following properties:*

- *EER of basis trading is non-zero unless* $\alpha^c = \alpha^d = 0$.
- $sign(EER^{nbt}) = -sign(EER^{pbt}) = -sign(z_t^c)$ *The market on which local investors are selling has higher EER, taking long position on this market and short position on the other market is expected to earn positive return in excess of funding costs.*
- *The size of expected excess return of basis trading is increasing in* $\alpha^i$, $\sigma_j$ *and* $|z_t^c|$.

*Proof.* see Appendix 1.B. □

Under the assumption about local investors' demand in this case, the arbitrageur is always taking opposite positions on the two markets. If the funding costs are constants that have no sensitivities to risk factors such as $\lambda_t$, the two assets carry exactly the same exposure to risk factors. Then the arbitrageur is left with zero aggregate exposure to any risk factors, therefore all market prices of risk will be zero. In that case, bonds on both markets and basis trading will all earn zero risk premia and hence zero EER. But as long as at least one funding cost has sensitivity to the default intensity risk factor $\lambda_t$, the two assets carry different exposures to default intensity risk and also different exposures to the demand shock risk. The arbitrageur captures non-zero aggregate exposures to these two risks, hence market prices of risks for these two factors are non-zero. Therefore, the two assets carry different level of risk premium and basis trading is expected to be profitable in excess of funding costs.

Moreover, the sign of basis trading EER is completely determined by the sign of local investors's demand on the cash market. Since the non-zero condition of basis trading EER is completely determined by the funding costs' sensitivities to default intensity risk, Corollary 3a suggests that the profitability of basis trading is completely determined by the properties of the two sources of limits to arbitrage. In the real-world, practitioners tend to see positive CDS basis as signal to do positive CDS basis trading and vice versa, but results here suggests that the timing and directional signals of CDS basis trading are current market frictions. The CDS basis which characterizes price discrepancy between the two markets is an endogenous term itself. Empirical proxies of the interaction of funding cost friction and demand pressure friction should be more reliable in predicting CDS basis trading excess returns than the CDS basis.

**Corollary 1.4.** *Assume that the funding costs sensitivities to default intensity risk are less than L, i.e.* $0 < \alpha^i < L$, $i = c, d$, *then the expected excess return of bond is positive no matter if local investors are buying or selling.*

- *When* $z_t^c < 0$*:* $EER^c > EER^d > 0$
- *When* $z_t^c > 0$*:* $EER^d > EER^c > 0$

*Proof.* see Appendix 1.B. □

The result that if $z_t^i > 0$ then $sign(EER^i) = sign(z_t^i)$ implies that the arbitrageur is providing liquidity at a loss, which is against conventional wisdom. The usual conclusion on the pricing implication of demand shocks is: when there's excess demand, those who provide the liquidity will only agree to sell at a high price in equilibrium so as to earn something extra, which is the compensation for providing liquidity. Therefore, the instantaneous expected excess return should have the opposite sign of the local investors demand. However, in the presence of the other market, the arbitrageur in this model is willing to provide liquidity at a loss on the market she shorts because she can earn more on the other market. By doing so, the arbitrageur doesn't correct but instead magnifies the price deviation and creates a bubble.

### 1.4.3 Implications on Basis

**Definition 1.3.** *Basis is the difference between the yield to maturity of bond on market D minus the yield to maturity of bond on market C with the same time to maturity.*

$$\begin{aligned} Basis(\tau) &= [-\frac{logP_t^d(\tau)}{\tau}] - [-\frac{logP_t^c(\tau)}{\tau}] \\ &= [\frac{A_\lambda^d(\tau)}{\tau} - \frac{A_\lambda^c(\tau)}{\tau}]\lambda_t + [\frac{A_z^d(\tau)}{\tau} - \frac{A_z^c(\tau)}{\tau}]z_t + \frac{C^d(\tau)}{\tau} - \frac{C^c(\tau)}{\tau} \end{aligned} \tag{1.31}$$

Under conventional wisdom, negative Basis implies doing negative basis trading is profitable while positive Basis implies doing positive basis trading is profitable. But according to the model, the value of Basis is not very meaningful in the presence of funding costs, as it is only an observed endogenous quantity. As shown in the previous subsection, the sign of basis trading EER is solely determined by the sign of local investors' demand. Deriving the value of Basis is very complicated, as it depends on the values of $\lambda_t$, $z_t$, $\tau$ and the solution of $C^i(\tau)$. However, the sensitivities of Basis to the risk factors can be derived clearly:

**Proposition 1.4.** *The sensitivities of Basis to risk factors are:*

- *When* $z_t^c < 0$*:* $\frac{\partial Basis}{\partial \lambda_t} < 0$*, and* $\frac{\partial Basis}{\partial z_t} > 0$*.*
- *When* $z_t^c > 0$*:* $\frac{\partial Basis}{\partial \lambda_t} > 0$*, and* $\frac{\partial Basis}{\partial z_t} > 0$*.*

*Proof.* see Appendix 1.B. □

For $z_t^c < 0$, which corresponds to the case when local investors are selling, corporate bond are more sensitive to default intensity risk than CDS. Therefore, increase of default intensity increases credit spread more than it increases CDS price, thus causing basis, which is CDS price minus credit spread, to decrease, vice versa. The result for the comparative statics for $z_t^c$ follows the same logic.

### 1.4.4 Implications on Term Structure

The coefficients for $\lambda_t$ and $z_t$ in the bond prices are increasing in the bond's time to maturity $\tau$. Assume the funding cost's sensitivity to $\lambda_t$ is the same across $\tau$, then basis trading on bonds with longer time to maturity has larger net exposure to the risk factors than basis trading on bonds with shorter time to maturity. Therefore, the size of basis trading EER is increasing in $\tau$. Together with the properties in **Corollary 1.3** regarding the sign of basis trading EER, I derive the following results:

**Proposition 1.5.** *Assume* $\alpha^c(\tau) = \alpha^c$ *and* $\alpha^d(\tau) = \alpha^d$ *for* $\tau \in (0, T]$*, then the term structures of basis trading expected excess returns are:*

- *When* $z_t^c < 0$*:* $\frac{\partial EER^{nbt}}{\partial \tau} > 0$*, and* $\frac{\partial EER^{pbt}}{\partial \tau} < 0$*. When local investors are selling corporate bonds, negative basis trading using longer maturity bonds have higher EER, while positive basis trading using shorter maturity bonds have higher EER.*
- *When* $z_t^c > 0$*:* $\frac{\partial EER^{nbt}}{\partial \tau} < 0$*, and* $\frac{\partial EER^{pbt}}{\partial \tau} > 0$*. When local investors are buying corporate bonds, negative basis trading using shorter maturity bonds*

*have higher EER, while positive basis trading using longer maturity bonds have higher EER.*

- *The absolute value of basis trading EER, $|EER^{nbt}|$ or $|EER^{pbt}|$, is increasing in the time to maturity of basis trading instruments.*

*Proof.* see Appendix 1.B. □

As in the previous case, $Y_t(\tau)$ denotes the time $t$ yield to maturity of a corporate bond with time to maturity $\tau$. The collection of $Y_t(\tau)$ gives the yield curve of corporate bonds. As shown below, the slope of corporate bond yield curve $\partial Y_t(\tau)/\partial\tau$ is decreasing in $\lambda_t$, but the degree of the decreasing relationship depends on local investors' demand.

**Proposition 1.6.** *Assume $0 < \alpha^i(\tau) < L$, $i = c, d$, then the sensitivity of yield curve slope to default intensity risk is:*

$$\frac{\partial^2 Y_t(\tau)}{\partial\tau\partial\lambda_t} = [\frac{A^c_\lambda(\tau)}{\tau}]' < 0 \tag{1.32}$$

.

- *The slope of corporate bond yield curve is decreasing in $\lambda_t$.*
- *The decreasing effect is stronger when $z^c_t < 0$ than when $z^c_t > 0$.*

*Proof.* see Appendix 1.B. □

The result that the slope of corporate bond yield curve is decreasing in $\lambda_t$ is not surprising as $\lambda_t$ is mean-reverting. But the level of the yield curve is higher in the case of $z^c_t < 0$, which corresponds to local investors selling, than in the case of $z^c_t > 0$, which corresponds to local investors buying. So the decreasing effect is stronger when $z^c_t < 0$.

## 1.5 Empirical Study

I focus on data between 2007 and 2009, a period which has not only persistently negative CDS basis, but also high funding costs and poor market liquidity. After describing the data and key variables, I run a set of regressions to show that the co-existence of frictions in funding cost and market liquidity turns basis trading into a risky arbitrage. Then, I show that the comparative statics results of the size and risk of realized excess returns of basis trading as consistent with the model. Moreover, I test the predictability of credit spread term structure on future credit spread changes to support the predictions made in earlier sections.

### 1.5.1 Data and Key Variables

I collect a number of CDS and corporate bond indices data from Markit, individual CDS data from Bloomberg and corporate bond data from TRACE. I also download several kinds of interest rates data from the Federal Reserve web-site and Fama-French three factor data from Kenneth French's web-site. For the majority of the data collected, daily observation starts as early as 01/01/2007 and ends at 31/12/2009.

A number of theoretical results to be tested concerns the expected excess return of basis trading. I use the realized excess return (hereafter RER) of basis trading between time $t$ and $t+k$ to proxy the expected excess return. Since my main results are on the instantaneous expected excess returns, I mainly focus on $k=1$ day and 1 week. The CDS basis was persistently negative for a large part of my sample period, which saw many practitioners engaged in negative basis trading. So I focus mainly on the RER of negative basis trading. The realized excess return of doing negative basis trading between time $t$ and $t+k$ is calculated as the return from holding a corporate bond index minus the return from holding a CDS index, and minus net funding costs. To be specific, I calculate the realized excess returns (RERs) of negative basis trading as:

$$RER^{nbt}_{t,t+k} = Return^{CBond}_{t,t+k} - Return^{CDS}_{t,t+k} - NetFundingCost_{t,t+k} \tag{1.33}$$

where $Return^{CBond}_{t,t+k}$ is the realized return of a value-weighted corporate bond index between time $t$ and $t+k$. I first collect the 1-10yrs Markit iBoxx USD Domestic Corporates AAA, AA, A and BBB indices[13], and then create the value-weighted corporate bond index using these 4 indices. Realized return of the corporate bond index is calculated as the change of the corporate bond index value between time $t$ and $t+k$ divided by the index value at time $t$. $Return^{CDS}_{t,t+k}$ is the realized return of the Markit CDX North America Investment Grade Excess Return Index, whose components have maturities of 5 years.

In the above calculation, $NetFundingCost_{t,t+k}$ is the funding cost of corporate bond minus the funding cost of writing CDS.[14] The funding costs are difficult to

[13] average maturity around 5-year

[14] The net funding cost is thus equivalent to the funding cost of buying corporate bond plus the funding cost of buying CDS protection. But as explained in the next paragraph, the funding cost of buying CDS protection is effectively negative, since it's bear by the writer.

observe directly, I proxy the funding cost of corporate bond as:

$$BondFundingCost_{t,t+k} = \Sigma_{s=t}^{t+k-1}[(1-haircut)*Tbill_s+haircut*LIBOR_s] \quad (1.34)$$

This assumption implies investors can fund the $1 - haircut$ fraction of their corporate bond purchase at the collateralized rate (TBill), and the *haircut* fraction at the un-collateralized rate (LIBOR). The funding cost is increasing in the hair-cut and the difference between collateralized and un-collateralized rate. This approximation for the funding cost of corporate bond is consistent with the model's assumption on the functional form of the funding costs. Motivation of this approximation can be found in **Appendix 1.A**. Ideally, the hair-cut input should be time-varying as well. However, daily data on hair-cut is very difficult to get. Therefore, I applied different values of hair-cut for different sub-periods in the sample based on the average hair-cut data described in Gorton and Metrick (2010). The empirical evidence is not very sensitive to different hair-cut assumptions. I calculate funding cost for each day and sum up to get the funding cost between time $t$ and $t + k$.

The funding cost of CDS is approximated by the average CDS premium on financial institutions. During the crisis, the main friction on CDS market is the counterparty risk, especially from the protection writers' side. A protection buyer would suffer from the joint default of the underlying entity and the protection writer. In order for the protection buyer to be willing to trade at the CDS premium without counterparty risk, the protection buyer requires the protection seller to buy a CDS on the seller herself for the buyer, so that in the event of joint default, the protection buyer can at least get paid from the CDS on the protection seller. Therefore, I calculate the average CDS premium on financial institutions using individual name CDS data from Bloomberg, and then adjust for the length of period $k$. As expected, the funding cost of CDS is close to zero before the crisis, but becomes very large during the crisis. After the Lehman collapse, the funding cost of CDS is even higher than the funding cost of corporate bond. This point is consistent with both empirical evidence found by others and observation made by practitioners.

Depending on market conditions, the realized return of negative basis trading can be positive or negative. The absolute value of the $RER^{nbt}$ measures the size of basis trading RERs, whose comparative statics properties are tested in the following sections. To test the term structure property of the size of basis trading RERs, I further calculated the RERs of negative basis trading on underlying with approximately 2 years, 5 years and 9 years using individual CDS and corporate bond data.

I also collect Markit iBoxx USD Domestic Corporate Rating Indices for 1-3 years, 3-5 years, 1-5 years, 5-7 years, 7-10 years and 5-10 years of maturities and use the asset swap spread of these indices to approximate the credit spread of these maturities and build the credit spread term structure.

Important parameters and variables in the model, such as $\alpha$, the funding cost's sensitivity to default intensity risk, $z^i$, the local investors' demand, and $\sigma_\lambda$, the volatility of default intensity risk are proxied in the following way: In the calculation of corporate bond's funding cost, *haircut* is multiplied by $LIBOR - Tbill$. Comparing with the model assumption on the functional form of funding costs, if hair-cut is linear in default intensity risk $\lambda_t$, then $\alpha$ is a multiple of $LIBOR - Tbill$, which is the TED spread. Therefore, $\alpha$ in the model is empirically proxied by the TED spread (hereafter $TED$), which is collected from Federal Reserve's web-site. As for $z^i$, the sign of local investors' demand is difficult to measure, but the absolute value of local investors' demand on the cash market can be proxied by the corporate bond market trading volume, hereafter $TV$, which is collected from TRACE. I also use the contemporaneous[15] volatility of asset swap spread (hereafter $ASW$) of Liquid Corporate Bond Index from Markit to proxy for $\sigma_\lambda$ since this particular asset swap rate is less affected by movements in interest rate and market liquidity. Other empirical proxies are introduced in the following sections when they emerge.

### 1.5.2 Rolling Window Time-series Tests on the Realized Excess Returns of Negative Basis Trading

**Hypothesis 1.1.** *Basis trading returns in excess of arbitrage costs contain time-varying exposures to systematic risk factors.*

The model implies that the expected excess return of basis trading contains compensation for the exposure to risk factors. Therefore, the realized excess return (or return) of basis trading might be explained by systematic risk factors. However, the model suggests that negative basis trading's exposures to risk factors are time-varying. Depending on the sign of local investors demand and relative sensitivity of funding cost to $\lambda_t$, negative basis trading can have positive or negative loadings on the risk premia. In other words, in a time series regression of basis trading RER on risk factor returns, the betas will be time-varying. Thus it is not appropriate to test the negative basis trading RERs on systematic factors over the entire sample period.

[15] 90-day period around day $t$

Instead, for each month starting from March 2007, I run times-series regressions on daily observations for the next quarter. For each type of regressions specified later, I do 30 regressions and obtain 30 sets of coefficient estimates and Newey-West t-stats. I report these estimates and t-stats in **Table 1.1** to **Table 1.3** to see the time-varying patterns of negative basis trading RER's exposures to risk factors. I also plot the coefficient estimates for certain factors to highlight the time-varying patterns that are consistent with the model's predictions. If basis trading RER represents compensation for taking systematic risk, then the regression results will have significant coefficient estimates for the systematic risk factors. I run the following two regressions to test the above **Hypothesis 1.1**:

**Regression A**

$$\begin{aligned} RER^{nbt}_{t,t+k} &= \beta_0 + \beta_1(LIBOR - FFR)_{t,t+k} + \beta_2 FFR_{t,t+k} + \beta_3 TED_{t,t+k} \\ &\quad + \beta_4(TED * ASW)_{t,t+k} + \beta_5 Basis_t + \epsilon_{A,t} \end{aligned} \tag{1.35}$$

**Regression B**

$$\begin{aligned} RER^{nbt}_{t,t+k} &= \beta_0 + \beta_1(LIBOR - FFR)_{t,t+k} + \beta_2 FFR_{t,t+k} + \beta_3 TED_{t,t+k} \\ &\quad + \beta_4(TED * ASW)_{t,t+k} + \beta_5 Basis_t + \beta_6 MKTRF_{t,t+k} + \beta_7 DEF_{t,t+k} \\ &\quad + \beta_8 TERM_{t,t+k} + \epsilon_{B,t} \end{aligned} \tag{1.36}$$

Because the funding costs in the calculation of basis trading RER may not be accurate enough, I include 4 additional variables to control for potential mis-measurement of funding costs in **Regression A** to see if funding costs can explain the RER of negative basis trading. The four control variables are $Libor - FFR$, $FFR$, $TED$ and $TED * ASW$. $FFR$ is the federal funds rate. The $TED * ASW$ term accounts for the effect from the mis-measurement of the hair-cut. I also include $Basis$ as an independent variable. $Basis$ is calculated as the average CDS basis[16] of a number of individual entities. Conventional thinking regards negative basis as signal for profit in doing negative basis trading. But the model doesn't suggest any relationship between the level of basis and basis trading returns. Therefore, I include this variable to test whether there's any significance relationship.

In **Regression B**, I add three of the Fama-French five factors. The $MKTRF$ factor is the excecss return of the market portfolio, the $DEF$ factor is the liquid

[16] CDS premium minus corporate bond credit spread

corporate bond index return minus the T-Note index return of similar maturity[17], and the $TERM$ factor is the T-Note return minus the T-Bill return from Kenneth French's web-site.[18] I use factor values that are contemporaneous to the dependent variable.

As can be seen from **Table 1.1**, for **Regression A**, the funding cost variables are significant during months 13-20, which correspond to the period from Bear Stearn's crisis to the end of 2008. This suggests there may be some mis-measurement in the funding costs when calculating realized returns, but also suggests funding costs have important roles in justifying the return of basis trading during the crisis. The $Basis$ factor is not significant for 27 out of 30 rolling windows. This is consistent with the model. It highlights the importance of not relying on observed price discrepancy when doing risky arbitrage because the signal given by price discrepancy is in fact an endogenous quantity affected by the existence of arbitrage cost such as funding costs.

However, funding costs variables alone are not good enough to explain the return of negative basis trading. **Regression A** has significant intercept estimates in most windows. Adding Fama-French factors doesn't reduce the significance of intercepts by much, but the Fama-French factors are indeed significant and the betas are indeed time-varying as predicted by the model. As shown in **Table 1.2** and **Figure 1.2**, the beta estimate for the $DEF$ factor is significantly negative for the early periods but not very significant in latter periods, while the beta estimate for the $TERM$ factor is not very significant for the early periods but is significantly positive in latter periods. These patterns in the betas are consistent with model prediction and empirical results from other papers. According to the model, when corporate bond market investors are selling, corporate bond returns are more sensitive to risk factors than CDS, therefore negative basis trading return's exposures to risk factors should have the same signs as corporate bond return's exposures to risk factors. Meanwhile, Kim et.al.(2010) show that corporate bond returns during the same period has negative exposure to the $DEF$ factor and positive exposure to the $TERM$ factor.

**Hypothesis 1.2.** *Interaction of funding liquidity and market liquidity predicts abnormal basis trading return.*

According to the model, the expected excess profit of basis trading is driven by the interaction of funding costs and local investors' demand. Therefore, I add two other

[17]calculated based on Markit iBoxx 1-10yrs USD Domestic Treasury Index

[18]Other Fama-French factors have been tested and removed due to their insignificance.

terms in **Regression C** in order to better explain the abnormal returns of basis trading. The first term $SignedTV$ is the time $t$ corporate bond market trading volume times the sign of corporate bond index values' change from $t-1$ to $t$. This term accounts for the momentum driven by market liquidity. The second term is TED spread squared times adjusted trading volume, where the adjusted trading volume is calculated as the corporate bond market trading volume minus its -45 days to +45 days median. This adjusted trading volume term is aimed to model the absolute value of demand shocks in local investors demand, i.e. $|z_t|$. This $TED_t^2 * AdjTV_t$ term is implied by the formula for negative basis trading return in the model.

**Regression C**

$$\begin{aligned} RER_{t,t+k}^{nbt} &= \beta_0 + \beta_1(LIBOR - FFR)_{t,t+k} + \beta_2 FFR_{t,t+k} + \beta_3 TED_{t,t+k} \\ &\quad + \beta_4(TED * ASW)_{t,t+k} + \beta_5 Basis_t + \beta_6 MKTRF_{t,t+k} + \beta_7 DEF_{t,t+k} \\ &\quad + \beta_8 TERM_{t,t+k} + \beta_9 SignedTV_t + \beta_{10} TED_t^2 * AdjTV_t + \epsilon_{C,t} \end{aligned} \tag{1.37}$$

As shown in **Table 1.3**, the coefficient estimates for the $SignedTV$ term are significantly positive for most windows. This is not surprising as negative basis trading consists of buying corporate bond, which is positively affected by the momentum in corporate bond returns. The coefficient estimates for the $TED_t^2 * AdjTV_t$ term are significantly negative during the latter half of 2008. This is highly consistent with the model's prediction. **Proposition 1.3** suggests that negative basis trading return is increasing in $\alpha$ and $-z_t$, which during the latter half of 2008 are well proxied by TED spread and the adjusted corporate bond trading volume respectively. More importantly, **Regression C** results in insignificant intercepts for most windows, and the $Basis$ factor is once again not significant for almost all windows.

To summarize, both **Hypothesis 1.1** and **Hypothesis 1.2** are supported by the data. Basis trading RERs contain time-varying exposures to systematic risk factors and the interaction of funding liquidity and market liquidity has predictive power on abnormal basis trading returns.

### 1.5.3 Term Structure of the Size of the Realized Excess Returns of Negative Basis Trading

**Proposition 1.5** predicts that the size of basis trading RER is increasing in the time to maturity of the underlying bond. I compared the absolute value of basis trading excess return on underlyings with 2 years, 5 years and 9 years time to maturity. I list the average 1-day, 1-week and 1-month absolute RERs for these maturities in **Table 1.4**, which shows clearly that basis trading on longer maturities have larger absolute RERs.

This result is also shown by the plot of cumulative 1-week absolute RER of basis trading on these maturities in **Figure 1.3**. Basis trading on longer maturities have higher cumulative absolute RERs, and the gaps between the lines are also increasing over time, which is consistent with the prediction that the size of basis trading RER is increasing in the time to maturity of the underlying bond.

### 1.5.4 Tests of Comparative Statics

The model yields several testable comparative statics results. The purpose here is to show that more severe market frictions make basis trading more risky, hence earning higher expected excess return. To be specific, I tested the following predictions:

**Hypothesis 1.3.**

- *The size of basis trading RER is increasing in $\alpha^c$ and $\sigma_\lambda$.*
- *Basis trading is risky, the size of basis trading RER is increasing in the volatility of basis trading RER.*
- *The volatility of basis trading RER is increasing in $\alpha^c$, which is proxied by TED spread.*

The volatility of basis trading RER is the next 90-day volatility of the negative basis trading $RER$. To test the comparative statics results, I sort the time-series of the absolute value and volatility of basis trading RERs based on their corresponding TED and default intensity volatility values. For instance, I assign each date in the sample period into 5 groups according to the TED spread value of each date, the first group contains dates with the lowest TED spread values, the 5th group contains dates with the highest TED spread values. Then I calculate the median value of the absolute value of negative basis trading RER for each group, and report in a table to see if the group with higher TED spread also has larger basis trading RER size. I also sort the sizes of basis trading RER into negative basis trading volatility groups. Results are summarized in **Table 1.5**, in which each panel provides result of each comparative statics. In general, the results support **Hypothesis 1.3** very well.

### 1.5.5 The Predictability of Credit Spread Term Structure Slope on Future Credit Spread Changes

**Hypothesis 1.4.**

- *Credit spread term structure slope positively predicts future credit spread changes when frictions are small.*
- *Credit spread term structure slope does not positively predict (and may negatively predict) future credit spread changes when investors are selling corporate bonds and funding cost has high sensitivity to default intensity risk. This scenario is more likely for shorter maturities.*

I carry out the following type of regression to test this hypothesis which corresponds to **Proposition 1.2**:

$$CS_{t+k} - CS_t = \psi_0 + \psi Slope_t + \epsilon_t \tag{1.38}$$

I use 4 sets of dependent variables and independent variables. For instance, for Set 1, I use the 1-5 years grade A corporate bond index asset swap spread as the CS variable. For this dependent variable, I use the 3-5 years grade A corporate bond index asset swap spread minus the 1-3 years grade A corporate bond index asset swap spread as the Slope variable. For this set, the dependent variable proxies the future change in credit spread of a 3-year corporate bond, while the independent variable proxies the difference in credit spreads of a 4 year corporate bond and a 2 year corporate bond issued by the same entity as the 3-year bond. The independent variable thus proxies the term structure of credit spread at 3 years. A full list of variables for other sets are listed in **Table 1.6**, which also reports regression results. I test for $k$ =5 days and 15 days respectively and run the regressions on three sub-periods: Sub-period 1 (07/2007 to 02/2008), Sub-period 2 (03/2008-03/2009) and Sub-period 3 (04/2009-09/2009) to account for different levels of market frictions.

The results support **Hypothesis 1.4**. For Sub-period 1 (07/2007 to 02/2008) and Sub-period 3 (04/2009-09/2009) that correspond to periods with low market frictions, the positive predictability of the independent variable on dependent variable is found in most cases. These results are consistent with Bedendo et.al.(2007). But in Sub-period 2 (03/2008-03/2009) when market frictions are high after the Bear Stern and Lehman Collapse, the predictability is lost for Set 1 and Set 3, which correspond to credit spread term structure at approximately 3 years, while the predictability is still significant for Set 2 and Set 4, which correspond to longer time to maturity at approximately 7.5 years. Such findings support the predictions from **Proposition 1.2**, which states that the regression coefficient $\psi$ may not be positive when friction is high, especially for short maturities. Therefore, it's not surprising to find that the predictability is lost during Sub-period 2 for Set 1 and 3. Using shorter sub-period windows, I find negative coefficient $\psi$ for the 3-year term structure as shown in **Figure 1.4**.

### 1.5.6 Robustness Checks

The main results are not affected when using alternative funding cost proxies. To test if the results are sensitive to the construction of the Markit indices, I also construct alternative negative basis trading RERs using individual CDS and corporate bond data. Regression results using these returns as dependent variables are consistent with results in the previous section. But individual CDS and corporate bond data are less reliable than the Markit indices, so I still report results using the Markit indices as the main results.

## 1.6 Conclusion

The paper tackles with the puzzle between CDS and corporate bond. Previous literatures show that buying CDS protection and buying corporate bond should earn risk free return and CDS-basis, which is CDS price minus credit spread, should be zero. However, the CDS basis was persistently non-zero during the 2007/2008 financial crisis, yet many arbitrageurs lost money trying to take advantage of this. The limited arbitrage literature suggests that no-arbitrage relationship can be violated when arbitrageurs face risk and costs to do the arbitrage. So it may be intuitive to think that market frictions such as funding costs cause CDS basis to be non-zero. But the CDS basis doesn't signal which market to long and which market to short so as to earn positive expected returns when there's funding costs. I use a limited arbitrage model to analyze the risky arbitrage nature of CDS-basis trading and derive the following three major results:

The first result tells us when CDS-basis trading is expected to earn positive return when arbitrageurs face funding costs that are increasing in default intensity of the bonds. To be more specific, there are two cases. In the first case, when local investors are trading towards the same directions on the two markets, one can earn positive expected excess return by taking long position on the market whose funding cost is more sensitive to default intensity; in the second case, when local investors are trading towards different directions on the two markets, one can earn positive expected excess return by taking long position on the market that local investors are selling. The rationale is: the presence of demand pressure results in arbitrageurs requiring risk premium for the positions they take. Without other frictions, demand pressure alone doesn't generate pricing discrepancies between assets with identical cash-flows. But since funding costs also contain systematic risk factor, the two assets have different loadings on systematic risk factors if they obtain different exposures to the risk factor through funding costs. Therefore, CDS-basis trading has non-zero loadings on systematic risk factors and is expected to earn non-zero excess return. As shown in the model, without any one of these two sources of frictions, the arbitrage only generates zero expected excess return. It may be intuitive to think that assets with identical cash-flows will have different expected returns when there's funding costs involved, but my model shows they can have different expected returns even after the deduction of funding costs, if funding costs are time-varying and correlated with assets' fundamental risk.

Secondly, the paper offers new properties of the term structure of credit spreads. Previous literatures found that the slope of credit spread term structure positively predicts future credit spread changes, but when market frictions are high, for instance, funding costs are very sensitive to default intensity and demand pressures are high, this positive predictability may turn negative. Empirically, I find positive predictability in periods other than the latter half of 2008 but negative predictability during the latter half of 2008 for short time-to-maturity bonds, which is consistent with model prediction.

Moreover, the model also offers closed form solution for defaultable bonds under market frictions. Defaultable bonds prices are solved as exponential affine functions of risk factors. As implied by the closed form solution of the expected excess return of CDS basis trading, an interaction term of funding liquidity and market liquidity predicts abnormal CDS-basis trading return, especially in the latter half of 2008.

As predicted by the model, empirical risk exposures of CDS-basis trading have strong time-varying patterns depending on market frictions. I therefore use rolling-window time-series regressions that are very useful considering the time-varying nature of betas. I show that basis trading is exposed to systematic risk factors, while the interaction of funding cost and market liquidity have predictive power on abnormal basis trading returns. In the cross-section, bonds with longer time-to-maturity realize larger yet more risky basis trading excess returns. In the time-series, periods with higher market friction realize larger yet more risky basis trading excess returns.

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## 1.8 Appendix 1.A: Motivation for the Funding Costs Function

### 1.A1. Borrowing Costs

When buying the defaultable bonds through borrowing, one can not borrow at the rate of $r_t$. The total cost is $r_t + h_t$, where $h_t$ reflects the borrowing cost in excess of $r_t$. The existence of $h_t$ represents a source of market friction. The model assumes that $h_t$ is an increasing linear function of the default intensity risk $\lambda_t$:

$$h_t^i(\tau) = \alpha^i(\tau)\lambda_t + \delta^i(\tau) \quad i = c, d \quad 0 < \alpha^i < L \tag{1.39}$$

This is a reasonable assumption with motivation illustrated in the following way:

Consider a simplified version of repo transaction, assume there's a continuum of competitive cash lenders who lend out $y_t$ amount of cash if the borrowers offer 1-dollar worth of defautable bond as collateral.[19] The borrower may default on the obligation to pay back, denote $\bar{N}_t$ as counting process for the borrower's default, assume the intensity of $\bar{N}_t$ is $f(\lambda_t)$, which is increasing in $\lambda_t$. This is a reasonable assumption in the real world, the borrowers are usually buyers of the defaultable bond who can only afford to buy through borrowing, and rely on the selling of the bond to pay back their borrowing. If the bond defaults, they are unlikely to pay back and would default. Denote the wealth of the lender by $w_t$, then the lender's optimization problem is:

$$\max_{y_t}[E_t(dw_t) - \frac{\bar{\gamma}}{2}Var_t(dw_t)] \tag{1.40}$$

$$dw_t = (w_t - y_t)r_t dt + y_t(r_t + s)dt + [(1 - L) - y_t]d\bar{N}_t \tag{1.41}$$

where $r_t + s$, $s > 0$, is the exogenous rate asked by the lender for lending against the defaultable bond as collateral.[20] The $w_t - y_t$ part of her wealth increases at the rate of $r_t$, while the $y_t$ part earns interest rate $r_t + s$. If the borrower defaults, the lender retains the bond, which is now worth $1 - L$ but loses the $y_t$ lent to the borrower. The optimal lending amount $y_t^*$ is solved as:

$$y_t^* = \frac{s}{\bar{\gamma} f(\lambda_t)} - \frac{1}{\bar{\gamma}} + (1 - L) \tag{1.42}$$

[19] The lender effectively buys the bond for $y_t$, while the borrower promises to buy back the bond at a price higher than $y_t$ at the end of the repo contract that reflects an interest rate higher than the short rate $r_t$.

[20] $r_t$ can be understood as the rate asked by the lender for lending against default-free bond as collateral.

Therefore, $y_t^*$ is decreasing in $f(\lambda_t)$, thus decreasing in $\lambda_t$. In other words, the amount a borrower can borrow using defaultable bond as collateral is decreasing in the riskiness of the bond. $1 - y_t^*$ is the so-called 'hair-cut'asked by the lender. The hair-cut is increasing in the riskiness of the bond. The borrower borrows $y_t^*$ fraction of her bond purchase at the $r_t + s$ rate charged by the lender, and the rest $1 - y_t^*$ fraction at an exogenous un-collateralized rate $r_t + u$ from somewhere else. Here, $u > s > 0$ since un-collateralized borrowing is riskier than collateralized borrowing. So the borrower's total borrowing cost in excess of $r_t$ is:

$$h_t = y_t^*(r_t + s) + (1 - y_t^*)(r_t + u) - r_t = u - (u - s)y_t^* \tag{1.43}$$

which is a decreasing function of $y_t^*$. Since $y_t^*$ is decreasing in $\lambda_t$, the borrowing cost $h_t$ is increasing in $\lambda_t$. With careful choice of the exogenous function $f(\lambda_t)$, the borrow cost has the linear functional form of $\alpha\lambda_t + \delta$ used in the model.

### 1.A2. Short-selling Costs

When short-selling the defaultable bond, one needs to borrow the bond from a bond lender. The bond lender asks for cash-collateral of a certain amount. Effectively, the short-seller buys the bond for that amount while the bond lender agrees to buy back the bond at the end of the reverse-repo contract for an amount higher than the initial cash-collateral. However, the end-of-the-day amount usually reflects an interest rate paid on the cash-collateral that is lower than the short-rate. The short-seller hence incur short-selling costs.

Assume there's a continuum of competitive bond lenders that require $Y_t$ amount of cash-collateral for 1-dollar worth of bond lent. The short-seller, i.e. the bond borrower, may default on the obligation to return the bond, denote $\widetilde{N}_t$ as counting process for the short-seller's default, assume the intensity of $\widetilde{N}_t$ is $g(\lambda_t)$ which is decreasing in $\lambda_t$. This is a reasonable assumption in the real world, when $\lambda_t$ is low, the bond price is high, the short-seller is more likely to suffer loss and hence default on the obligation to return the bond. Denote the wealth of the lender by $w_t$, then the lender's optimization problem is:

$$\max_{Y_t}[E_t(dw_t) - \frac{\widetilde{\gamma}}{2}Var_t(dw_t)] \tag{1.44}$$

$$dw_t = (w_t + Y_t)r_t dt - Y_t(r_t - S)dt + (Y_t - 1)d\widetilde{N}_t \tag{1.45}$$

where $r_t - S$, $S > 0$, is the exogenous special repo rate offered by the bond lender on the cash collateral. The $w_t + Y_t$ part of her wealth increases at the rate of $r_t$, while the $Y_t$ part pays interest rate $r_t - S$. If the short-seller defaults, the lender retains the cash $Y_t$, but loses the 1-dollar worth of bond lent to the short-seller.

$$Y_t^* = \frac{S}{\tilde{\gamma} g(\lambda_t)} + \frac{1}{\tilde{\tilde{\gamma}}} + 1 \tag{1.46}$$

Therefore $Y_t^*$ is decreasing in $g(\lambda_t)$, thus increasing in $\lambda_t$. The short-seller puts $Y_t^*$ fraction of her bond sales as cash-collateral which earns interest at the rate of $r_t - S$ and invests the rest $1 - y_t^*$ fraction at the short rate $r_t$. So her short selling cost is:

$$h_t = r_t - [Y_t^*(r_t - S) + (1 - Y_t^*)r_t] = SY_t^* \tag{1.47}$$

which is an increasing function of $Y_t^*$. Since $Y_t^*$ is also increasing in $\lambda_t$, the short-selling cost $h_t$ is increasing in $\lambda_t$. With careful choice of the exogenous function $g(\lambda_t)$, the short-selling cost has the linear functional form of $\alpha\lambda_t + \delta$ used in the model.

## 1.9 Appendix 1.B: Proofs of Lemmas and Propositions

### 1.B1. Proof of Lemma 1.1

In the most general set-up, there're 4 sources of uncertainties: the jump at default, characterized by $N_t$, and three Brownian Motions in the default intensity risk factor $\lambda_t$, short rate factor $r_t$ and local investors' demand shock factor $z_t$, i.e. $B_{\lambda,t}$, $B_{r,t}$ and $B_{z,t}$. The dynamic of $N_t$ only enters into play through $\lambda_t$, so I can rewrite $P_t^d(\tau)$ as $P_t^d(\tau, \lambda, r, z)$ and $P_t^c(\tau)$ as $P_t^c(\tau, \lambda, r, z)$. Assuming all three Brownian Motions $B_{\lambda,t}$, $B_{r,t}$ and $B_{z,t}$ are independent, I apply Ito's lemma to write $dP_t^c(\tau, \lambda, z)$ and $dP_t^d(\tau, \lambda, z)$ as:

$$dP_t^c(\tau, \lambda, r, z) = \mu_t^c(\tau) P_t^c dt + \sigma_\lambda \frac{\partial P^c}{\partial \lambda} dB_{\lambda,t} + \sigma_r \frac{\partial P^c}{\partial r} dB_{r,t} + \sigma_z \frac{\partial P^c}{\partial z} dB_{z,t} \tag{1.48}$$

$$dP_t^d(\tau, \lambda, r, z) = \mu_t^d(\tau) P_t^d dt + \sigma_\lambda \frac{\partial P^d}{\partial \lambda} dB_{\lambda,t} + \sigma_r \frac{\partial P^d}{\partial r} dB_{r,t} + \sigma_z \frac{\partial P^d}{\partial z} dB_{z,t} \tag{1.49}$$

where

$$\mu_t^i = \frac{1}{P_t^i} [\frac{\partial P^i}{\partial t} + \sum_{j \in \{\lambda, r, z\}} (\mu_j \frac{\partial P^i}{\partial j} + \frac{\sigma_j^2}{2} \frac{\partial^2 P^i}{\partial j^2})] \tag{1.50}$$

and $\mu_j = \kappa_j(\bar{j} - j_t)$, $i = c, d$, $j = \lambda, r, z$. $\mu_t^c$ and $\mu_t^d$ are the expected returns of bond C and bond D, conditional on no default. Dropping $\tau$, the $E(dW_t)$ and $Var(dW_t)$ terms in the optimization problem are:

$$\begin{aligned} E(dW_t) &= \{W_t - \int_0^T [x_t^c(\tau) + x_t^d(\tau)] d\tau\} r_t dt \\ &+ \sum_{i=c,d} \{\int_0^T x_t^i(\tau) [(\mu_t^i(\tau) - L(\widetilde{\lambda} + \lambda_t)] d\tau - \int_0^T |x_t^i(\tau)| h_t^i(\tau) d\tau\} dt \end{aligned} \tag{1.51}$$

$$\begin{aligned} Var(dW_t) &= \sum_{j \in \{\lambda, r, z\}} \{\int_0^T [x_t^c(\tau) \sigma_j \frac{1}{P_t^c(\tau)} \frac{\partial P_t^c(\tau)}{\partial j} + x_t^d(\tau) \sigma_j \frac{1}{P_t^d(\tau)} \frac{\partial P_t^d(\tau)}{\partial j}] d\tau\}^2 dt \\ &+ \{L \int_0^T [x_t^c(\tau) + x_t^d(\tau)] d\tau\}^2 (\widetilde{\lambda} + \lambda_t) dt \end{aligned} \tag{1.52}$$

Entering the above terms into the optimization problem, $\frac{\partial [E(dW_t) - \frac{\gamma}{2} Var(dW_t)]}{\partial x_t^i}$, $i = c, d$, gives the F.O.C.s in **Lemma 1.1**.

### 1.B2. Proof of Lemma 1.2

Apply Ito's lemma to the conjectured prices, the $dP_t^c/P_t^c$ and $dP_t^d/P_t^d$ terms are re-written as:

$$\frac{dP_t^c}{P_t^c} = \mu_t^c dt - A_\lambda^c \sigma_\lambda dB_{\lambda,t} - A_r^c \sigma_r dB_{r,t} \tag{1.53}$$

$$\frac{dP_t^d}{P_t^d} = \mu_t^d dt - A_\lambda^d \sigma_\lambda dB_{\lambda,t} - A_r^d \sigma_r dB_{r,t} \tag{1.54}$$

where, omitting $\tau$, the instantaneous expected returns conditional on no default are:

$$\mu_t^c = -A_\lambda^c \kappa_\lambda(\bar{\lambda} - \lambda_t) - A_z^c \kappa_r(\bar{r} - r_t) + A_\lambda^{c\prime}\lambda_t + A_r^{c\prime} r_t + C^{c\prime} + \frac{1}{2}A_\lambda^{c\,2}\sigma_\lambda^2 + \frac{1}{2}A_r^{c2}\sigma_r^2 \tag{1.55}$$

$$\mu_t^d = -A_\lambda^d \kappa_\lambda(\bar{\lambda} - \lambda_t) - A_r^d \kappa_r(\bar{r} - r_t) + A_\lambda^{d\prime}\lambda_t + A_r^{d\prime} r_t + C^{d\prime} + \frac{1}{2}A_\lambda^{d^2}\sigma_\lambda^2 + \frac{1}{2}A_r^{d^2}\sigma_r^2 \tag{1.56}$$

Substitute the above into the dynamic budget constraint, the arbitrageur's F.O.C.s are:

$$\mu_t^c(\tau) - r_t - h_t^c(\tau)\frac{|x_t^c(\tau)|}{x_t^c(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda A_\lambda^c(\tau)\Phi_{\lambda,t} - \sigma_r A_r^c(\tau)\Phi_{r,t} \tag{1.57}$$

$$\mu_t^d(\tau) - r_t - h_t^d(\tau)\frac{|x_t^d(\tau)|}{x_t^d(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda A_\lambda^d(\tau)\Phi_{\lambda,t} - \sigma_r A_r^d(\tau)\Phi_{r,t} \tag{1.58}$$

where

$$\Phi_{J,t} = \gamma L \int_0^T [x_t^d(\tau) + x_t^c(\tau)](\widetilde{\lambda} + \lambda_t)d\tau \tag{1.59}$$

$$\Phi_{\lambda,t} = \gamma\sigma_\lambda \int_0^T [-x_t^d(\tau)A_\lambda^d(\tau) - x_t^c(\tau)A_\lambda^c(\tau)]d\tau \tag{1.60}$$

$$\Phi_{r,t} = \gamma\sigma_r \int_0^T [-x_t^d(\tau)A_r^d(\tau) - x_t^c(\tau)A_r^c(\tau)]d\tau \tag{1.61}$$

are the market prices of risks to the default jump, default intensity and short rate factors.

In equilibrium, markets clear. So $x_t^i + z^i = 0$, $i = c, d$. Replace $x_t^i(\tau)$ by $-z^i(\tau)$, and replace $h_t^i$ by the functions defined in the funding cost section, then the F.O.C.s are affine equations in the risk factors $\lambda_t$ and $r_t$. Setting the linear terms in $\lambda_t$ and

$r_t$ to zeros implies that $A^i_j(\tau)$ are the solutions to a system of ODEs with initial conditions $A^i_j(0) = 0$, $i = c, d$ and $j = \lambda, r$:

$$A^{c\prime}_\lambda(\tau) + \kappa_\lambda A^c_\lambda(\tau) - [-\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} + L] = -\gamma L^2 \int_0^T [z^d(\tau) + z^c(\tau)]d\tau \tag{1.62}$$

$$A^{d\prime}_\lambda(\tau) + \kappa_\lambda A^d_\lambda(\tau) - [-\alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)} + L] = -\gamma L^2 \int_0^T [z^d(\tau) + z^c(\tau)]d\tau \tag{1.63}$$

$$A^{c\prime}_r(\tau) + \kappa_r A^c_r(\tau) - 1 = 0 \tag{1.64}$$

$$A^{d\prime}_r(\tau) + \kappa_r A^d_r(\tau) - 1 = 0 \tag{1.65}$$

Thus, $A^i_j(\tau)$ are solved as in **Lemma 1.2**.

### 1.B3. Proof of Proposition 1.1

Replace $\frac{1}{P^c_t}\frac{\partial P^c_t}{\partial \lambda}$, $\frac{1}{P^d_t}\frac{\partial P^d_t}{\partial \lambda}$, $\frac{1}{P^c_t}\frac{\partial P^c_t}{\partial r}$ and $\frac{1}{P^d_t}\frac{\partial P^d_t}{\partial r}$ with $-A^c_\lambda(\tau)$, $-A^d_\lambda(\tau)$, $-A^c_r(\tau)$, $-A^d_r(\tau)$ respectively in **Definition 1.1** gives **Proposition 1.1**. Because $-A^c_r(\tau)$ and $-A^d_r(\tau)$ are the same, so the two exposures to the short rate risk cancel out each other, only the exposure to the default intensity risk premium $\Phi_{\lambda,t}$ remains.

### 1.B4. Proof of Corollary 1.1

Point 1: 1) If $\alpha^c(\tau) = \alpha^d(\tau) = 0$, then $EER^{nbt} = EER^{pnt} = 0$. 2) If $\alpha^c(\tau) = \alpha^d(\tau) \neq 0$ and $sign[z^c(\tau)] = sign[z^d(\tau)]$, then the term inside the bracket in the EER formula equals zero, so $EER^{nbt} = EER^{pnt} = 0$. 3) If $z^c(\tau) = z^d(\tau) = 0$, then $\Phi_{\lambda,t} = 0$, so $EER^{nbt} = EER^{pnt} = 0$.

Point 2: If $z^c(\tau) < 0$ and $z^d(\tau) < 0$, then $\Phi_{\lambda,t} < 0$, and $sign(EER^{nbt}) = -sign(EER^{pbt}) = sign(\alpha^c(\tau) - \alpha^d(\tau)$. If $\alpha^c(\tau) > \alpha^d(\tau)$, then $EER^{nbt} > 0$, i.e. buying on market C and selling on market D earns positive expected excess return; if $\alpha^d(\tau) > \alpha^c(\tau)$, then $EER^{pbt} > 0$, i.e. buying on market D and selling on market C earns positive expected excess return.

If $z^c(\tau) > 0$ and $z^d(\tau) > 0$, but both not too large, then then $\Phi_{\lambda,t} > 0$, and $sign(EER^{nbt}) = -sign(EER^{pbt}) = sign(\alpha^c(\tau) - \alpha^d(\tau)$. The rest follows the above $z^c(\tau) < 0$ and $z^d(\tau) < 0$ case.

Point 3: If $z^c(\tau) >> 0$ and $z^d(\tau) >> 0$, then then $\Phi_{\lambda,t} < 0$, and $sign(EER^{nbt}) =$

$-sign(EER^{pbt}) = -sign(\alpha^c(\tau) - \alpha^d(\tau)$. If $\alpha^c(\tau) < \alpha^d(\tau)$, then $EER^{nbt} > 0$, i.e. buying on market C and selling on market D earns positive expected excess return; if $\alpha^d(\tau) < \alpha^c(\tau)$, then $EER^{pbt} > 0$, i.e. buying on market D and selling on market C earns positive expected excess return.

### 1.B5. Proof of Corollary 1.2

Point 1: If $sign(z^c) = sign(z^d)$, then the size of basis trading's exposure to the default intensity risk is: $|\pm\sigma_\lambda[\alpha^c(\tau) - \alpha^d(\tau)]\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda}|$. Because $\sigma_\lambda\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda} > 0$, it is increasing in $|\alpha^c - \alpha^d|$.

Point 2: If $sign(z^c) = -sign(z^d)$, then the size of basis trading's exposure to the default intensity risk is: $|\pm\sigma_\lambda[\alpha^c(\tau) + \alpha^d(\tau)]\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda}|$. Because $\sigma_\lambda\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda} > 0$, it is increasing in $\alpha^c + \alpha^d$.

Point 3: Size of basis trading EER is: $|\pm\sigma_\lambda[\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} - \alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)}]\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda}|$. It is equal to $\sigma_\lambda|[\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} - \alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)}]\frac{1-e^{-\kappa_\lambda\tau}}{\kappa_\lambda}|$, which is increasing in $\sigma_\lambda$.

### 1.B6. Proof of Proposition 1.2

The time-varying component in the dependent variable is: $\frac{A^c_\lambda(\tau-\Delta\tau)}{\tau-\Delta\tau}\lambda_{t+\Delta\tau} - \frac{A^c_\lambda(\tau)}{\tau}\lambda_t$, the time-varying componet in the independent variable is: $\frac{A^c_\lambda(\tau)}{\tau}\lambda_t - \frac{A^c_\lambda(\tau-\Delta\tau)}{\tau-\Delta\tau}\lambda_t$. So the regression coefficient is:

$$\psi = \frac{Cov[\frac{A^c_\lambda(\tau-\Delta\tau)}{\tau-\Delta\tau}\lambda_{t+\Delta\tau} - \frac{A^c_\lambda(\tau)}{\tau}\lambda_t, \frac{A^c_\lambda(\tau)}{\tau}\lambda_t - \frac{A^c_\lambda(\tau-\Delta\tau)}{\tau-\Delta\tau}\lambda_t]}{Var[\frac{A^c_\lambda(\tau)}{\tau}\lambda_t - \frac{A^c_\lambda(\tau-\Delta\tau)}{\tau-\Delta\tau}\lambda_t]} \tag{1.66}$$

$$= \frac{\tau A^c_\lambda(\tau-\Delta\tau)e^{-\kappa_\lambda\Delta\tau} - (\tau-\Delta\tau)A^c_\lambda(\tau)}{(\tau-\Delta\tau)A^c_\lambda(\tau) - \tau A^c_\lambda(\tau-\Delta\tau)} \tag{1.67}$$

when $\Delta\tau \to 0$,

$$\psi = -\frac{\tau[A^c_\lambda(\tau)' + \kappa_\lambda A^c_\lambda(\tau)]}{\tau[A^c_\lambda(\tau)' - 1]} \tag{1.68}$$

$$= \frac{F(\tau)}{1 - F(\tau)e^{-\kappa_\lambda\tau}} \tag{1.69}$$

where $F(\tau) = -\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} + L - \gamma L^2\int_0^T[z^c(\tau) + z^d(\tau)]d\tau$.

If $\alpha^c(\tau) = 0$ and $z^c(\tau) = z^d(\tau) = 0$, then $F = L$. $0 < F < 1$, so $\psi = \frac{F(\tau)}{1-F(\tau)e^{-\kappa_\lambda\tau}} > 0$.

If $z^c(\tau) << 0$, $\alpha^c(\tau) >> 0$, then $F >> 0$. So $\psi = \frac{F(\tau)}{1-F(\tau)e^{-\kappa_\lambda \tau}} < 0$.

When $F > 0$, the threshold for $\psi < 0$ is $F > e^{\kappa_\lambda \tau}$, which is easier to meet for small $\tau$ than for large $\tau$.

### 1.B7. Proof of Lemma 1.3 and Lemma 1.4

Apply Ito's lemma to the conjectured prices, the Arbitrageur's F.O.C.s are:

$$\mu_t^c(\tau) - r_t - h_t^c(\tau)\frac{|x_t^c(\tau)|}{x_t^c(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda A_\lambda^c(\tau)\Phi_{\lambda,t} - \sigma_r A_r^c(\tau)\Phi_{r,t} - \sigma_z A_z^c(\tau)\Phi_{z,t} \tag{1.70}$$

$$\mu_t^d(\tau) - r_t - h_t^d(\tau)\frac{|x_t^d(\tau)|}{x_t^d(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda A_\lambda^d(\tau)\Phi_{\lambda,t} - \sigma_r A_r^d(\tau)\Phi_{r,t} - \sigma_z A_z^d(\tau)\Phi_{z,t} \tag{1.71}$$

where

$$\Phi_{J,t} = \gamma L \int_0^T [x_t^d(\tau) + x_t^c(\tau)]d\tau(\widetilde{\lambda} + \lambda_t) \tag{1.72}$$

$$\Phi_{\lambda,t} = \gamma\sigma_\lambda \int_0^T [-x_t^d(\tau)A_\lambda^d(\tau) - x_t^c(\tau)A_\lambda^c(\tau)]d\tau \tag{1.73}$$

$$\Phi_{r,t} = \gamma\sigma_r \int_0^T [-x_t^d(\tau)A_r^d(\tau) - x_t^c(\tau)A_r^c(\tau)]d\tau \tag{1.74}$$

$$\Phi_{z,t} = \gamma\sigma_z \int_0^T [-x_t^d(\tau)A_z^d(\tau) - x_t^c(\tau)A_z^c(\tau)]d\tau \tag{1.75}$$

are the market prices of risks to the default jump, default intensity, short rate and demand shock factors.

In equilibrium, markets clear. So $x_t^i + z_t^i = 0$. Replace $x_t^i$ with $-z_t^i$ in the above F.O.C.s, then the F.O.C.s are affine equations in the risk factors $\lambda_t$, $r_t$ and $z_t$.[21] Setting the linear terms in $\lambda_t$, $r_t$ and $z_t$ to zero implies that $A_j^i(\tau)$ are the solutions to the a system of ODEs with initial conditions $A_j^i(0) = 0$, $i = c, d$ and $j = \lambda, r, z$.

If $\bar{\theta}(\tau) << 0$ and $\theta(\tau) < 0$, then $z_t^c(\tau) < 0$, $z_t^d(\tau) > 0$. So $x_t^c(\tau) > 0$ and $x_t^d(\tau) < 0$. The system of ODEs becomes:

$$A_\lambda^{c\prime}(\tau) + \kappa_\lambda A_\lambda^c(\tau) - [\alpha^c(\tau) + L] = 0 \tag{1.76}$$

$$A_\lambda^{d\prime}(\tau) + \kappa_\lambda A_\lambda^d(\tau) - [-\alpha^d(\tau) + L] = 0 \tag{1.77}$$

[21] By assuming that the two markets have exact opposite demand pressure, the market price of risk for the default jump factor becomes zero as the arbitrageur's aggregate positions are free of default jump risk.

$$A_r^{c\prime}(\tau) + \kappa_r A_r^c(\tau) - 1 = 0 \tag{1.78}$$

$$A_r^{d\prime}(\tau) + \kappa_r A_r^d(\tau) - 1 = 0 \tag{1.79}$$

$$A_z^{c\prime}(\tau) + \kappa_z A_z^c(\tau) = -\gamma\sigma_\lambda^2 A_\lambda^c(\tau) \int_0^T \theta(\tau)[A_\lambda^c(\tau) - A_\lambda^d(\tau)]d\tau - \gamma\sigma_z^2 A_z^c(\tau) \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.80}$$

$$A_z^{d\prime}(\tau) + \kappa_z A_z^d(\tau) = -\gamma\sigma_\lambda^2 A_\lambda^d(\tau) \int_0^T \theta(\tau)[A_\lambda^c(\tau) - A_\lambda^d(\tau)]d\tau - \gamma\sigma_z^2 A_z^d(\tau) \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.81}$$

The first 4 equations can be solved independently. The solutions are as summarized in **Lemma 1.3**. For the last 2 equations, the solutions are as in **Lemma 1.3** because the equation:

$$\kappa_z^* = \kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.82}$$

has a unique solution in the region $(0, +\infty)$, as $\kappa_z^*$ is increasing from the origin in the region $(0, +\infty)$ and $\kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau$ is positive decreasing in the region $(0, +\infty)$.

If $\bar{\theta}(\tau) >> 0$ and $\theta(\tau) > 0$, then $z_t^c(\tau) > 0$, $z_t^d(\tau) < 0$. So $x_t^c(\tau) < 0$ and $x_t^d(\tau) > 0$. The system of ODEs becomes:

$$A_\lambda^{c\prime}(\tau) + \kappa_\lambda A_\lambda^c(\tau) - [-\alpha^c(\tau) + L] = 0 \tag{1.83}$$

$$A_\lambda^{d\prime}(\tau) + \kappa_\lambda A_\lambda^d(\tau) - [\alpha^d(\tau) + L] = 0 \tag{1.84}$$

$$A_r^{c\prime}(\tau) + \kappa_r A_r^c(\tau) - 1 = 0 \tag{1.85}$$

$$A_r^{d\prime}(\tau) + \kappa_r A_r^d(\tau) - 1 = 0 \tag{1.86}$$

$$A_z^{c\prime}(\tau) + \kappa_z A_z^c(\tau) = -\gamma\sigma_\lambda^2 A_\lambda^c(\tau) \int_0^T \theta(\tau)[A_\lambda^c(\tau) - A_\lambda^d(\tau)]d\tau - \gamma\sigma_z^2 A_z^c(\tau) \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.87}$$

$$A_z^{d\prime}(\tau) + \kappa_z A_z^d(\tau) = -\gamma\sigma_\lambda^2 A_\lambda^d(\tau) \int_0^T \theta(\tau)[A_\lambda^c(\tau) - A_\lambda^d(\tau)]d\tau - \gamma\sigma_z^2 A_z^d(\tau) \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.88}$$

The first 4 equations can be solved independently. The solutions are as summarized in **Lemma 1.4**. For the last 2 equations, the solutions are as in **Lemma 1.4** if $\kappa_z >> 0$. $\kappa_z^*$ is still increasing from the origin in the region $(0, +\infty)$, but $\gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau$ is negative and decreasing in the region $(0, +\infty)$. However, if $\kappa_z >> 0$, then $\kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau$ is decreasing from a positive value in the region $(0, +\infty)$. Then the equation has unique solution in $(0, +\infty)$.

$$\kappa_z^* = \kappa_z + \gamma\sigma_z^2 \int_0^T \theta(\tau)[A_z^c(\tau) - A_z^d(\tau)]d\tau \tag{1.89}$$

### 1.B8. Proof of Proposition 1.3

Replace $\frac{1}{P_t^c}\frac{\partial P_t^c}{\partial \lambda}$, $\frac{1}{P_t^d}\frac{\partial P_t^d}{\partial \lambda}$, $\frac{1}{P_t^c}\frac{\partial P_t^c}{\partial r}$, $\frac{1}{P_t^d}\frac{\partial P_t^d}{\partial r}$, $\frac{1}{P_t^c}\frac{\partial P_t^c}{\partial z}$ and $\frac{1}{P_t^d}\frac{\partial P_t^d}{\partial z}$ with $-A_\lambda^c(\tau)$, $-A_\lambda^d(\tau)$, $-A_r^c(\tau)$, $-A_r^d(\tau)$, $-A_z^c(\tau)$ and $-A_z^d(\tau)$ respectively in **Definition 1.1** gives **Proposition 1.3**. Because $-A_r^c(\tau)$ and $-A_r^d(\tau)$ are the same, so the two exposures to the short rate risk cancel out each other, only the exposure to the default intensity risk premium $\Phi_{\lambda,t}$ and the exposure to the demand shock risk premium $\Phi_{z,t}$ remain.

### 1.B9. Proof of Corollary 1.3

Point 1: If $\alpha^c = \alpha^d = 0$, then $G^\lambda = 0$ and $G^z = 0$, so $EER^{nbt} = 0$ and $EER^{pbt} = 0$. Otherwise, $G^\lambda \neq 0$ and $G^z \neq 0$, so $G^\lambda(\tau) \int_0^T G^\lambda(\tau) z_t^c(\tau) d\tau \neq 0$ and $G^z(\tau) \int_0^T G^z(\tau) z_t^c(\tau) d\tau \neq 0$, and

$$sign[G^\lambda(\tau) \int_0^T G^\lambda(\tau) z_t^c(\tau) d\tau] = sign[G^z(\tau) \int_0^T G^z(\tau) z_t^c(\tau) d\tau] = sign[z_t^c(\tau)] \tag{1.90}$$

so $EER^{nbt} \neq 0$ and $EER^{pbt} \neq 0$.

Point 2: Directly derived from Equation (1.90). Point 3: Directly derived from the first order derivatives of $|EER^{nbt}|$.

## 1.B10. Proof of Corollary 4

The EERs of defautable bonds on market C and market D are:

$$EER^c = -\sigma_\lambda^2 A_\lambda^c(\tau) \int_0^T [A_\lambda^c(\tau) - A_\lambda^d(\tau)] z_t^c(\tau) d\tau - \sigma_z^2 A_z^c(\tau) \int_0^T [A_z^c(\tau) - A_z^d(\tau)] z_t^c(\tau) d\tau \tag{1.91}$$

$$EER^d = -\sigma_\lambda^2 A_\lambda^d(\tau) \int_0^T [A_\lambda^c(\tau) - A_\lambda^d(\tau)] z_t^c(\tau) d\tau - \sigma_z^2 A_z^d(\tau) \int_0^T [A_z^c(\tau) - A_z^d(\tau)] z_t^c(\tau) d\tau \tag{1.92}$$

When $z_t^c < 0$, according to **Lemma 1.3**, $A_\lambda^c(\tau) - A_\lambda^d(\tau) > 0$, $A_\lambda^i(\tau) > 0$, $A_z^c(\tau) - A_z^d(\tau) < 0$ and $A_z^i(\tau) < 0$. So $EER^c > 0$ and $EER^d > 0$. According to **Corollary 1.3**, when $z_t^c < 0$, $EER^{nbt} = EER^c - EER^d > 0$, therefore, $EER^c > EER^d > 0$.

When $z_t^c > 0$, according to **Lemma 1.4**, $A_\lambda^c(\tau) - A_\lambda^d(\tau) < 0$, $A_\lambda^i(\tau) > 0$, $A_z^c(\tau) - A_z^d(\tau) < 0$ and $A_z^i(\tau) > 0$. So $EER^c > 0$ and $EER^d > 0$. According to **Corollary 1.3**, when $z_t^c > 0$, $EER^{pbt} = EER^d - EER^c > 0$, therefore, $EER^d > EER^c > 0$.

### 1.B11. Proof of Proposition 1.4

$$\frac{\partial Basis}{\partial \lambda_t} = \frac{A_\lambda^d(\tau)}{\tau} - \frac{A_\lambda^c(\tau)}{\tau} \tag{1.93}$$

$$\frac{\partial Basis}{\partial z_t} = \frac{A_z^d(\tau)}{\tau} - \frac{A_z^c(\tau)}{\tau} \tag{1.94}$$

Therefore, $sign(\frac{\partial Basis}{\partial \lambda_t}) = sign(A_\lambda^d - A_\lambda^c)$ and $sign(\frac{\partial Basis}{\partial z_t}) = sign(A_z^d - A_z^c)$. When $z_t^c < 0$, according to **Lemma 1.3**, $A_\lambda^c(\tau) > A_\lambda^d(\tau)$, so $\frac{\partial Basis}{\partial \lambda_t} < 0$; $A_z^c(\tau) < A_\lambda^d(\tau)$, so $\frac{\partial Basis}{\partial z_t} > 0$. When $z_t^c > 0$, according to **Lemma 1.4**, $A_\lambda^c(\tau) < A_\lambda^d(\tau)$, so $\frac{\partial Basis}{\partial \lambda_t} > 0$; $A_z^c(\tau) < A_\lambda^d(\tau)$, so $\frac{\partial Basis}{\partial z_t} > 0$.

### 1.B12. Proof of Proposition 1.5

$$\begin{aligned}
\frac{\partial EER^{nbt}}{\partial \tau} &= -\frac{\kappa_\lambda e^{-\kappa_\lambda \tau}}{1 - e^{-\kappa_\lambda \tau}} \gamma \sigma_\lambda^2 G^\lambda(\tau) \int_0^T G^\lambda(\tau) z_t^c(\tau) d\tau \\
&\quad - \frac{\kappa_z e^{-\kappa_z \tau}}{1 - e^{-\kappa_z \tau}} \gamma \sigma_z^2 G^z(\tau) \int_0^T G^z(\tau) z_t^c(\tau) d\tau \\
\frac{\partial EER^{pbt}}{\partial \tau} &= -\frac{\partial EER^{nbt}}{\partial \tau}
\end{aligned} \tag{1.95}$$

Then, the proof is similar to that in the **Proof of Corollary 1.3**.

### 1.B13. Proof of Proposition 1.6

$\frac{\partial^2 Y_t(\tau)}{\partial \tau} = [\frac{A_\lambda^c(\tau)}{\tau}]' \lambda_t + constants$, where $\frac{A_\lambda^c(\tau)}{\tau} = [\alpha^c + L] \frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda \tau}$ when $z_t^c < 0$ and $\frac{A_\lambda^c(\tau)}{\tau} = [-\alpha^c + L] \frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda \tau}$ when $z_t^c > 0$. $\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda \tau}$ is a decreasing function of $\tau$. Also note that $0 < \alpha^i(\tau) < L$, therefore $[\alpha^c + L](\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda \tau})' < [-\alpha^c + L](\frac{1 - e^{-\kappa_\lambda \tau}}{\kappa_\lambda \tau})' < 0$, So $\frac{\partial^2 Y_t(\tau)}{\partial \tau \partial \lambda_t} < 0$. And it is more negative when $z_t^c < 0$.

## 1.10 Appendix 1.C: Figures and Tables

Figure 1.1: *CDS Basis and 1-week Negative Basis Trading RER*

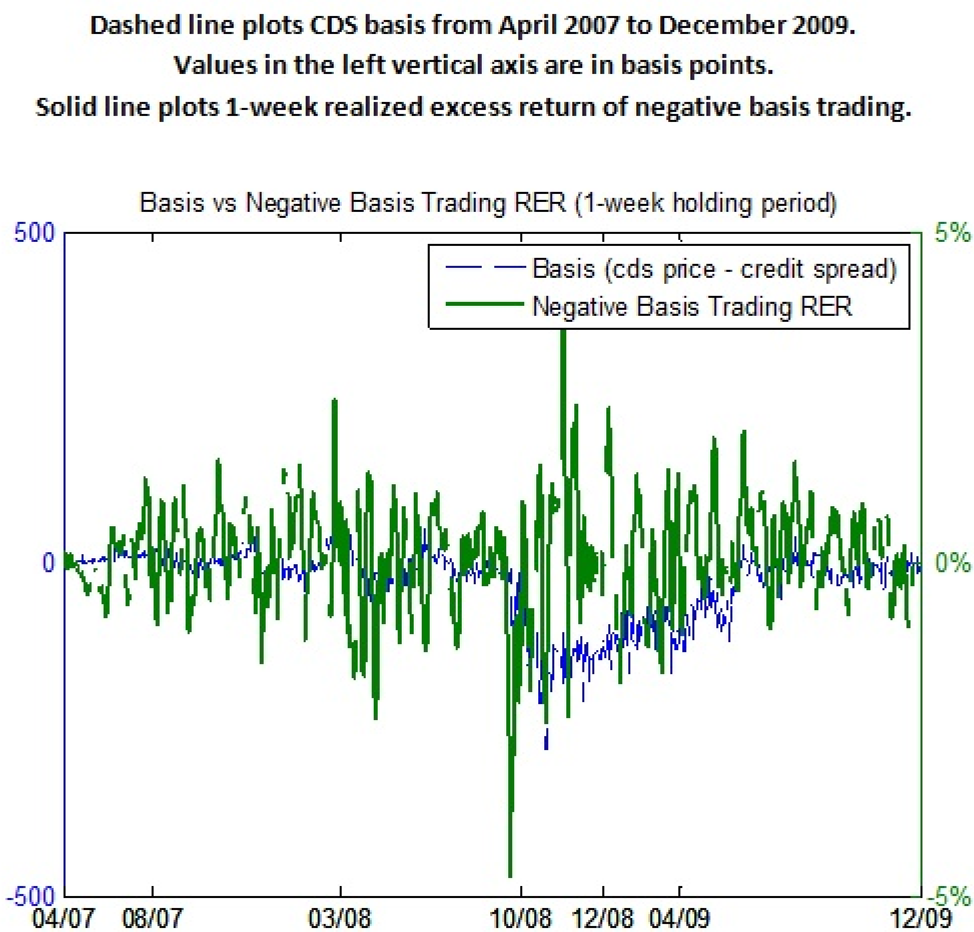

Figure 1.2: *Negative Basis Trading RER's Time-varying Exposure to Systematic Factors*

**The solid line plots rolling window beta of the DEF factor in Regression B.**
**The dashed line plots rolling window beta of the TERM factor in Regression B.**
**Dependent variable: 1-day realized excess return of negative basis trading.**
**As predicted by the model, betas are highly time-varying.**

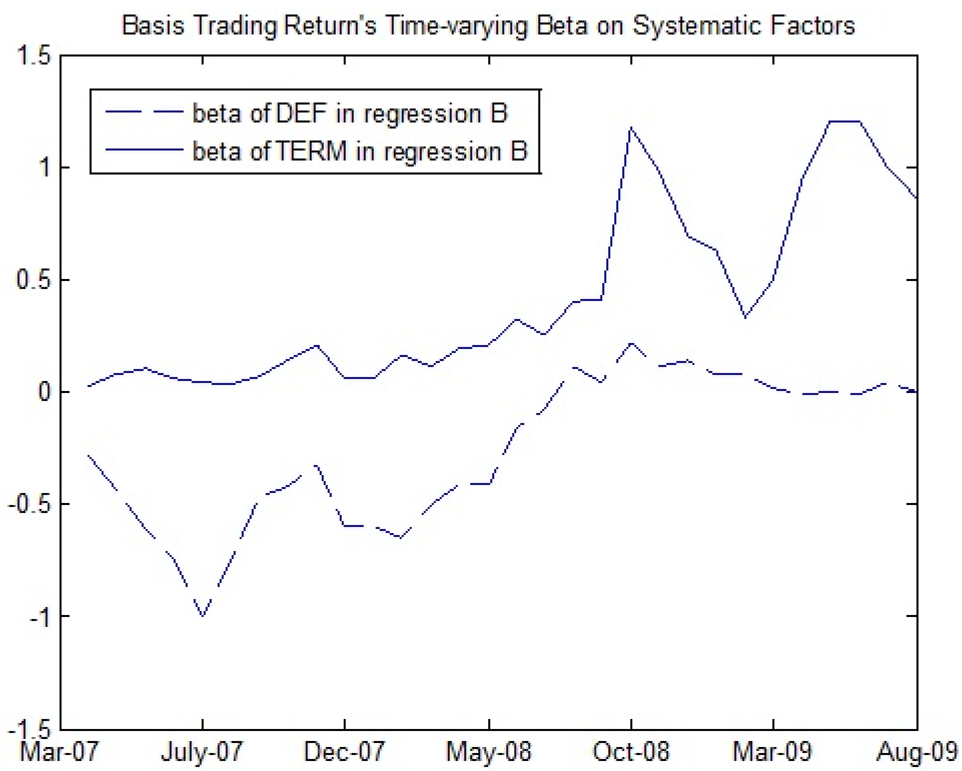

Figure 1.3: *Cumulative Absolute RERs of Negative Basis*

**The three lines plot the cumulative sum of the absolute value of negative CDS basis trading RER on underlying bonds of three different maturities.**
**This figure shows that the size of basis trading RER is larger for underlying bond with longer time to maturity.**

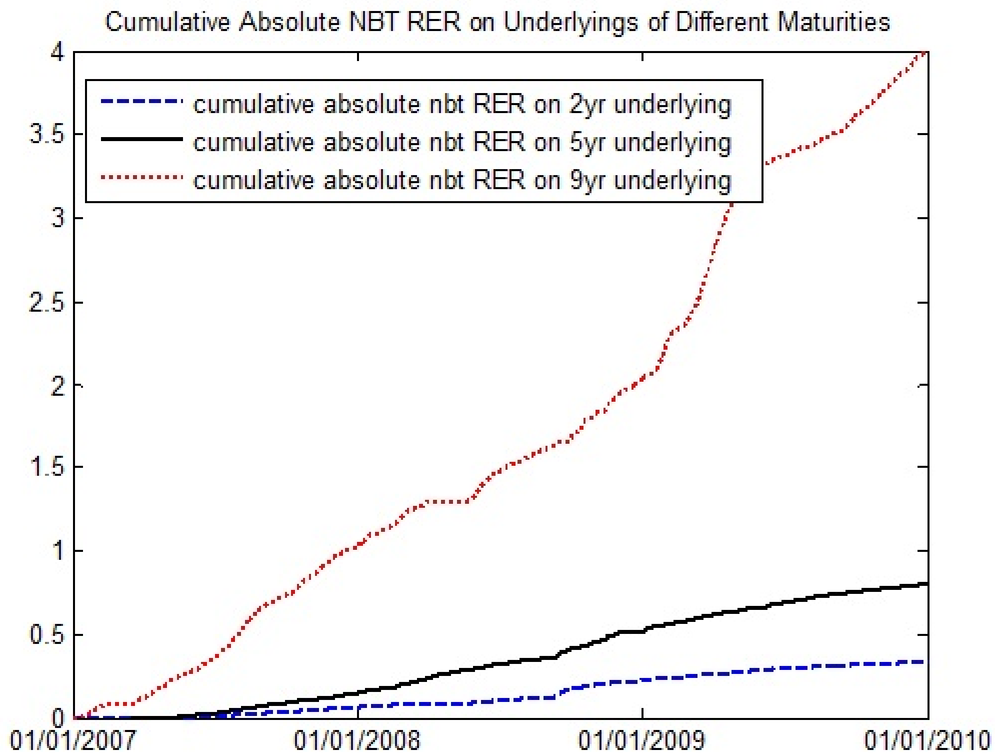

Figure 1.4: *Time-varying Coefficient Estimate of $\psi$ for 3yr Grade A Corporate Bond Index*

**The solid line plots rolling window coefficient estimate of the 3yr Grade A credit spread term structure in regression for Hypothesis 1.4.**
**The dashed line plots the corresponding Newey-West T-stats.**
**Dependent variable: future 15-day change in 3yr Grade A credit spread.**
**Independent variable: slope of Grade A credit spread term structure at 3yr.**
**psi was significantly positive before/after the crisis but significantly negative during crisis.**

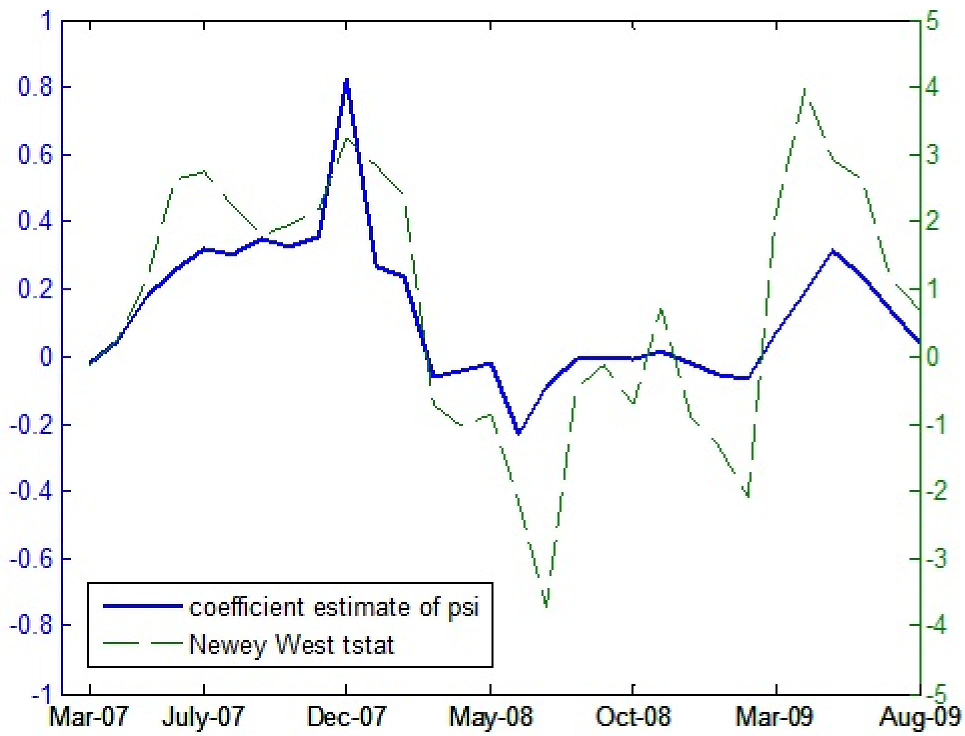

Table 1.1: *Beta and NW-Tstats of Rolling-Window Regression A*

**Table 1.1: Beta and NW-Tstats of Rolling-Window Regression A**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel A: Beta**

| Start | End | constant | Libor-FFR | FFR | TED | TED*ASW | Basis |
|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | 0.001 | 0.544 | -0.064 | -0.739 | 0.605 | 0.506 |
| Apr-07 | Jun-07 | 0.040 | -2.401 | -2.752 | -1.596 | 3.298 | 0.703 |
| May-07 | Jul-07 | -0.049 | 3.389 | 3.314 | 0.110 | -0.872 | 0.713 |
| Jun-07 | Aug-07 | -0.025 | 2.370 | 1.792 | 0.180 | -1.170 | 0.159 |
| Jul-07 | Sep-07 | 0.006 | 0.480 | -0.315 | 0.295 | -0.796 | 0.099 |
| Aug-07 | Oct-07 | 0.005 | -0.008 | -0.321 | 0.049 | -0.222 | 0.380 |
| Sep-07 | Nov-07 | 0.001 | -0.057 | -0.110 | 0.884 | -0.698 | 0.290 |
| Oct-07 | Dec-07 | 0.007 | -0.773 | -0.527 | 0.503 | -0.180 | 0.044 |
| Nov-07 | Jan-08 | 0.001 | -0.064 | 0.036 | 0.367 | -0.420 | 0.075 |
| Dec-07 | Feb-08 | -0.002 | 0.572 | 0.275 | 0.474 | -0.378 | 0.118 |
| Jan-08 | Mar-08 | -0.007 | 2.252 | 0.917 | -0.212 | -0.214 | 0.217 |
| Feb-08 | Apr-08 | -0.008 | 2.777 | 1.339 | -1.726 | 0.349 | 0.261 |
| Mar-08 | May-08 | 0.011 | 1.640 | -2.049 | -2.549 | 1.182 | -0.314 |
| Apr-08 | Jun-08 | -0.018 | 5.203 | 3.304 | -8.051 | 3.159 | -0.141 |
| May-08 | Jul-08 | -0.019 | 4.961 | 3.399 | -4.758 | 1.362 | -0.470 |
| Jun-08 | Aug-08 | 0.018 | 0.099 | -2.994 | 1.141 | -0.910 | -0.275 |
| Jul-08 | Sep-08 | 0.010 | 0.744 | -0.929 | -2.635 | 0.596 | 0.422 |
| Aug-08 | Oct-08 | 0.008 | 0.275 | -1.250 | -1.188 | 0.331 | 0.456 |
| Sep-08 | Nov-08 | 0.002 | -0.204 | -1.322 | 0.085 | 0.178 | 0.334 |
| Oct-08 | Dec-08 | 0.001 | -0.680 | -1.958 | 1.878 | -0.200 | 0.303 |
| Nov-08 | Jan-09 | -0.002 | 0.906 | 0.692 | 0.623 | -0.280 | -0.102 |
| Dec-08 | Feb-09 | 0.007 | 1.539 | -13.012 | 4.885 | -1.609 | -0.069 |
| Jan-09 | Mar-09 | 0.004 | 1.677 | -7.665 | 3.424 | -1.181 | -0.030 |
| Feb-09 | Apr-09 | 0.004 | 0.679 | -9.861 | 0.800 | -0.180 | 0.149 |
| Mar-09 | May-09 | 0.005 | 1.720 | -9.133 | 0.439 | -0.275 | 0.259 |
| Apr-09 | Jun-09 | -0.002 | -0.302 | 4.188 | 0.513 | 0.023 | 0.155 |
| May-09 | Jul-09 | -0.005 | 3.115 | 6.883 | 5.158 | -2.234 | 0.220 |
| Jun-09 | Aug-09 | -0.008 | 5.581 | 18.200 | 0.867 | -1.487 | 0.099 |
| Jul-09 | Sep-09 | 0.001 | 3.351 | -4.239 | 3.337 | -2.016 | 0.315 |
| Aug-09 | Oct-09 | 0.002 | 5.987 | -3.484 | 5.294 | -4.361 | 0.027 |

**Table 1.1 Continued: Beta and NW-Tstats of Rolling-Window Regression A**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel B: Newey West Tstat**

| Start | End | constant | Libor-FFR | FFR | TED | TED*ASW | Basis |
|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | 0.022 | 0.153 | -0.018 | -1.338 | 1.210 | 1.503 |
| Apr-07 | Jun-07 | 0.504 | -0.501 | -0.505 | -1.581 | 1.750 | 1.965 |
| May-07 | Jul-07 | -1.755 | 2.169 | 1.765 | 0.111 | -0.684 | 1.224 |
| Jun-07 | Aug-07 | -2.641 | 3.988 | 2.655 | 0.192 | -1.332 | 0.385 |
| Jul-07 | Sep-07 | 0.550 | 0.779 | -0.429 | 0.302 | -0.773 | 0.248 |
| Aug-07 | Oct-07 | 0.879 | -0.012 | -0.650 | 0.050 | -0.247 | 1.647 |
| Sep-07 | Nov-07 | 0.126 | -0.084 | -0.132 | 1.846 | -1.640 | 1.367 |
| Oct-07 | Dec-07 | 1.616 | -1.891 | -1.407 | 0.985 | -0.561 | 0.251 |
| Nov-07 | Jan-08 | 0.334 | -0.127 | 0.128 | 0.308 | -0.501 | 0.274 |
| Dec-07 | Feb-08 | -0.590 | 0.941 | 0.763 | 0.398 | -0.612 | 0.392 |
| Jan-08 | Mar-08 | -0.993 | 1.849 | 0.974 | -0.082 | -0.167 | 0.698 |
| Feb-08 | Apr-08 | -0.838 | 1.833 | 0.799 | -0.715 | 0.359 | 0.597 |
| Mar-08 | May-08 | 0.373 | 0.454 | -0.388 | -0.878 | 1.870 | -1.066 |
| Apr-08 | Jun-08 | -1.985 | 3.651 | 2.078 | -2.765 | 2.139 | -0.515 |
| May-08 | Jul-08 | -2.055 | 3.395 | 1.991 | -2.464 | 1.538 | -2.239 |
| Jun-08 | Aug-08 | 1.370 | 0.089 | -1.186 | 0.758 | -1.798 | -1.524 |
| Jul-08 | Sep-08 | 2.900 | 1.581 | -1.585 | -2.629 | 2.225 | 1.802 |
| Aug-08 | Oct-08 | 2.884 | 0.430 | -2.405 | -0.868 | 1.004 | 2.506 |
| Sep-08 | Nov-08 | 1.298 | -0.254 | -2.975 | 0.059 | 0.590 | 1.835 |
| Oct-08 | Dec-08 | 0.845 | -0.671 | -1.512 | 1.021 | -0.507 | 1.704 |
| Nov-08 | Jan-09 | -1.115 | 0.883 | 0.337 | 0.297 | -0.514 | -0.575 |
| Dec-08 | Feb-09 | 3.081 | 1.067 | -3.909 | 2.042 | -2.520 | -0.456 |
| Jan-09 | Mar-09 | 1.153 | 1.635 | -1.315 | 1.345 | -1.682 | -0.218 |
| Feb-09 | Apr-09 | 1.258 | 0.519 | -1.319 | 0.855 | -1.218 | 0.892 |
| Mar-09 | May-09 | 2.264 | 1.288 | -1.807 | 0.462 | -1.004 | 1.675 |
| Apr-09 | Jun-09 | -0.695 | -0.084 | 0.620 | 0.232 | 0.085 | 0.763 |
| May-09 | Jul-09 | -1.641 | 1.009 | 0.914 | 1.370 | -1.897 | 0.725 |
| Jun-09 | Aug-09 | -1.957 | 1.735 | 1.855 | 0.310 | -1.699 | 0.420 |
| Jul-09 | Sep-09 | 0.601 | 1.069 | -0.722 | 0.514 | -0.532 | 1.200 |
| Aug-09 | Oct-09 | 0.976 | 1.535 | -0.676 | 0.973 | -1.239 | 0.126 |

Table 1.2: *Beta and NW-Tstats of Rolling-Window Regression B*

**Table 1.2: Beta and NW-Tstats of Rolling-Window Regression B**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel A: Beta**

| Start | End | constant | Libor-FFR | FFR | TED | TED*ASW | Basis | MKTRF | DEF | TERM |
|---|---|---|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | 0.068 | -3.273 | -4.633 | -0.484 | 0.709 | 0.270 | 0.016 | -0.282 | 0.020 |
| Apr-07 | Jun-07 | 0.021 | -2.775 | -1.433 | -3.512 | 8.881 | 0.696 | -0.030 | -0.436 | 0.079 |
| May-07 | Jul-07 | -0.057 | 3.635 | 3.935 | 0.505 | -1.418 | 0.798 | -0.029 | -0.607 | 0.104 |
| Jun-07 | Aug-07 | -0.013 | 1.515 | 1.024 | -0.960 | 0.273 | 0.237 | -0.051 | -0.745 | 0.056 |
| Jul-07 | Sep-07 | 0.018 | 0.051 | -1.145 | -2.355 | 2.169 | 0.376 | -0.067 | -1.007 | 0.040 |
| Aug-07 | Oct-07 | 0.002 | 0.360 | -0.063 | -0.565 | 0.367 | 0.272 | -0.032 | -0.757 | 0.030 |
| Sep-07 | Nov-07 | -0.006 | 0.110 | 0.494 | 0.401 | -0.276 | 0.369 | -0.019 | -0.466 | 0.062 |
| Oct-07 | Dec-07 | 0.001 | -0.543 | -0.055 | 0.599 | -0.167 | 0.027 | -0.041 | -0.424 | 0.132 |
| Nov-07 | Jan-08 | 0.000 | -0.258 | 0.040 | 0.448 | 0.009 | -0.208 | -0.059 | -0.330 | 0.209 |
| Dec-07 | Feb-08 | 0.000 | 0.646 | 0.048 | 0.587 | -0.413 | -0.213 | -0.048 | -0.603 | 0.056 |
| Jan-08 | Mar-08 | -0.011 | 2.635 | 1.402 | -1.024 | 0.253 | -0.034 | -0.057 | -0.605 | 0.060 |
| Feb-08 | Apr-08 | -0.013 | 4.292 | 2.066 | -3.766 | 1.316 | 0.328 | -0.103 | -0.650 | 0.166 |
| Mar-08 | May-08 | 0.010 | 1.730 | -1.818 | -3.473 | 1.722 | -0.248 | -0.142 | -0.502 | 0.106 |
| Apr-08 | Jun-08 | -0.019 | 5.671 | 3.676 | -9.383 | 3.915 | 0.111 | -0.036 | -0.413 | 0.187 |
| May-08 | Jul-08 | -0.023 | 5.239 | 4.147 | -5.518 | 1.835 | -0.218 | -0.046 | -0.413 | 0.209 |
| Jun-08 | Aug-08 | 0.006 | 1.868 | -0.575 | 1.885 | -1.786 | -0.679 | -0.065 | -0.161 | 0.319 |
| Jul-08 | Sep-08 | 0.008 | 1.558 | -0.061 | -3.650 | 0.676 | 0.341 | -0.041 | -0.088 | 0.251 |
| Aug-08 | Oct-08 | 0.006 | 0.479 | -0.642 | -1.354 | 0.338 | 0.403 | -0.033 | 0.110 | 0.395 |
| Sep-08 | Nov-08 | 0.002 | 0.141 | -0.666 | -0.647 | 0.317 | 0.368 | -0.039 | 0.041 | 0.410 |
| Oct-08 | Dec-08 | 0.001 | -0.168 | -1.424 | 0.875 | -0.048 | 0.214 | -0.042 | 0.213 | 1.172 |
| Nov-08 | Jan-09 | 0.000 | 1.597 | -0.474 | 1.450 | -0.518 | 0.003 | -0.035 | 0.106 | 0.980 |
| Dec-08 | Feb-09 | 0.006 | 0.531 | -11.117 | 5.893 | -1.694 | -0.009 | 0.002 | 0.140 | 0.691 |
| Jan-09 | Mar-09 | 0.002 | 1.948 | -4.978 | 4.822 | -1.629 | -0.050 | -0.010 | 0.072 | 0.628 |
| Feb-09 | Apr-09 | 0.004 | 0.151 | -9.468 | 0.844 | -0.139 | 0.124 | 0.000 | 0.079 | 0.328 |
| Mar-09 | May-09 | 0.004 | 1.066 | -7.447 | 0.398 | -0.141 | 0.245 | -0.032 | 0.009 | 0.492 |
| Apr-09 | Jun-09 | -0.001 | -4.554 | 1.482 | 2.018 | 0.222 | 0.229 | -0.020 | -0.015 | 0.941 |
| May-09 | Jul-09 | -0.003 | -2.891 | 1.308 | 7.599 | -2.122 | 0.175 | -0.046 | -0.003 | 1.195 |
| Jun-09 | Aug-09 | -0.006 | 2.394 | 11.082 | 2.999 | -1.610 | -0.078 | -0.029 | -0.011 | 1.199 |
| Jul-09 | Sep-09 | 0.001 | 2.117 | -4.418 | 11.093 | -5.735 | -0.020 | 0.011 | 0.042 | 0.996 |
| Aug-09 | Oct-09 | 0.002 | 8.331 | -4.068 | 10.783 | -8.135 | -0.186 | 0.005 | -0.003 | 0.858 |

**Table 1.2 Continued: Beta and NW-Tstats of Rolling-Window Regression B**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel B: Newey West Tstat**

| Start | End | constant | Libor-FFR | FFR | TED | TED*ASW | Basis | MKTRF | DEF | TERM |
|---|---|---|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | 1.040 | -0.878 | -1.038 | -0.867 | 1.006 | 0.939 | 0.640 | -4.756 | 1.105 |
| Apr-07 | Jun-07 | 0.276 | -0.554 | -0.276 | -3.322 | 4.422 | 3.065 | -1.079 | -6.020 | 1.787 |
| May-07 | Jul-07 | -2.823 | 3.272 | 2.884 | 0.511 | -1.250 | 1.658 | -1.323 | -3.341 | 1.856 |
| Jun-07 | Aug-07 | -0.790 | 1.982 | 0.918 | -0.945 | 0.274 | 0.692 | -2.478 | -4.520 | 0.925 |
| Jul-07 | Sep-07 | 1.831 | 0.105 | -1.637 | -2.130 | 2.075 | 1.046 | -3.405 | -5.428 | 1.080 |
| Aug-07 | Oct-07 | 0.394 | 0.894 | -0.143 | -0.678 | 0.490 | 1.019 | -1.571 | -7.699 | 0.686 |
| Sep-07 | Nov-07 | -0.466 | 0.143 | 0.520 | 0.510 | -0.586 | 1.378 | -0.980 | -3.154 | 0.805 |
| Oct-07 | Dec-07 | 0.365 | -1.281 | -0.151 | 0.575 | -0.258 | 0.126 | -2.083 | -3.228 | 1.277 |
| Nov-07 | Jan-08 | -0.080 | -0.548 | 0.154 | 0.279 | 0.008 | -0.849 | -2.813 | -4.444 | 2.364 |
| Dec-07 | Feb-08 | -0.029 | 1.009 | 0.174 | 0.471 | -0.677 | -0.884 | -1.798 | -5.866 | 0.509 |
| Jan-08 | Mar-08 | -1.420 | 1.870 | 1.446 | -0.423 | 0.220 | -0.127 | -1.457 | -5.918 | 0.610 |
| Feb-08 | Apr-08 | -1.545 | 2.860 | 1.471 | -1.833 | 1.600 | 0.883 | -2.289 | -5.915 | 1.490 |
| Mar-08 | May-08 | 0.456 | 0.602 | -0.467 | -1.419 | 2.244 | -0.827 | -2.325 | -5.526 | 1.342 |
| Apr-08 | Jun-08 | -1.835 | 3.613 | 1.940 | -3.137 | 2.736 | 0.547 | -1.647 | -6.488 | 1.875 |
| May-08 | Jul-08 | -2.048 | 3.145 | 2.058 | -2.823 | 2.264 | -1.131 | -3.010 | -4.561 | 1.986 |
| Jun-08 | Aug-08 | 0.587 | 1.928 | -0.322 | 1.661 | -4.954 | -3.058 | -4.014 | -1.619 | 3.000 |
| Jul-08 | Sep-08 | 2.688 | 2.015 | -0.095 | -2.680 | 2.000 | 1.220 | -1.717 | -0.906 | 1.595 |
| Aug-08 | Oct-08 | 2.557 | 0.674 | -1.304 | -0.933 | 1.035 | 2.076 | -1.658 | 0.925 | 1.948 |
| Sep-08 | Nov-08 | 1.541 | 0.181 | -1.316 | -0.489 | 1.167 | 2.164 | -1.969 | 0.294 | 1.657 |
| Oct-08 | Dec-08 | 0.945 | -0.192 | -1.444 | 0.553 | -0.142 | 1.399 | -2.082 | 1.915 | 3.010 |
| Nov-08 | Jan-09 | -0.519 | 1.748 | -0.331 | 0.748 | -1.031 | 0.016 | -1.575 | 1.076 | 3.834 |
| Dec-08 | Feb-09 | 3.582 | 0.325 | -4.750 | 3.427 | -2.968 | -0.063 | 0.158 | 1.622 | 4.773 |
| Jan-09 | Mar-09 | 0.814 | 1.660 | -0.868 | 2.556 | -3.002 | -0.318 | -0.620 | 0.802 | 4.131 |
| Feb-09 | Apr-09 | 1.328 | 0.094 | -1.369 | 0.844 | -0.864 | 0.736 | -0.002 | 0.979 | 1.576 |
| Mar-09 | May-09 | 1.575 | 0.749 | -1.374 | 0.424 | -0.505 | 1.801 | -1.401 | 0.201 | 1.909 |
| Apr-09 | Jun-09 | -0.350 | -1.476 | 0.304 | 1.140 | 0.954 | 1.699 | -0.717 | -0.390 | 3.744 |
| May-09 | Jul-09 | -0.783 | -0.908 | 0.203 | 2.546 | -2.413 | 1.120 | -2.092 | -0.128 | 8.719 |
| Jun-09 | Aug-09 | -2.505 | 0.964 | 1.977 | 1.467 | -2.317 | -0.527 | -1.384 | -0.289 | 8.388 |
| Jul-09 | Sep-09 | 0.590 | 0.727 | -1.016 | 1.676 | -1.478 | -0.133 | 0.426 | 0.518 | 5.990 |
| Aug-09 | Oct-09 | 1.235 | 2.794 | -0.980 | 2.005 | -2.431 | -1.166 | 0.267 | -0.031 | 4.087 |

Table 1.3: *Beta and NW-Tstats of Rolling-Window Regression C*

**Table 1.3: Beta and NW-Tstats of Rolling-Window Regression C**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel A: Beta**

| Start | End | constant | libor-ffr | ffr | ted | ted*asw | Basis | MKTRF | DEF | TERM | signedtv | ted2*tv |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | -0.007 | 1.140 | 0.446 | -1.035 | 1.043 | 0.216 | 0.002 | -0.420 | -0.009 | 0.033 | 0.412 |
| Apr-07 | Jun-07 | -0.011 | -0.243 | 0.790 | -1.874 | 5.106 | 0.460 | -0.014 | -0.503 | 0.065 | 0.046 | -0.110 |
| May-07 | Jul-07 | -0.067 | 4.641 | 4.663 | 0.115 | -2.412 | 0.631 | -0.016 | -0.704 | 0.025 | 0.057 | 0.275 |
| Jun-07 | Aug-07 | -0.020 | 1.906 | 1.511 | -1.508 | 0.659 | 0.178 | -0.024 | -0.900 | 0.035 | 0.066 | 0.016 |
| Jul-07 | Sep-07 | 0.011 | 0.423 | -0.580 | -2.578 | 2.301 | 0.254 | -0.041 | -1.131 | 0.050 | 0.064 | -0.010 |
| Aug-07 | Oct-07 | -0.001 | 0.777 | 0.202 | -1.127 | 0.778 | 0.276 | -0.018 | -1.026 | 0.016 | 0.090 | -0.018 |
| Sep-07 | Nov-07 | 0.007 | -0.503 | -0.464 | 0.095 | -0.185 | -0.059 | 0.002 | -0.666 | 0.015 | 0.100 | 0.070 |
| Oct-07 | Dec-07 | 0.001 | -0.382 | -0.050 | -0.039 | -0.002 | -0.003 | -0.029 | -0.663 | 0.028 | 0.088 | 0.059 |
| Nov-07 | Jan-08 | 0.000 | -0.257 | -0.037 | -0.483 | 0.485 | -0.195 | -0.042 | -0.528 | 0.139 | 0.075 | 0.075 |
| Dec-07 | Feb-08 | -0.001 | 1.386 | 0.112 | 1.501 | -0.863 | -0.224 | -0.015 | -0.744 | -0.003 | 0.090 | -0.126 |
| Jan-08 | Mar-08 | -0.010 | 2.310 | 1.278 | -0.559 | 0.099 | -0.111 | -0.035 | -0.781 | 0.011 | 0.113 | -0.103 |
| Feb-08 | Apr-08 | -0.012 | 3.895 | 1.667 | -2.581 | 0.933 | 0.322 | -0.072 | -0.784 | 0.081 | 0.108 | -0.120 |
| Mar-08 | May-08 | 0.008 | 1.496 | -1.569 | -2.732 | 1.444 | -0.129 | -0.117 | -0.623 | 0.092 | 0.068 | -0.030 |
| Apr-08 | Jun-08 | -0.012 | 4.305 | 2.255 | -6.985 | 3.087 | 0.256 | -0.020 | -0.533 | 0.210 | 0.059 | -0.116 |
| May-08 | Jul-08 | -0.020 | 4.501 | 3.625 | -5.248 | 2.066 | -0.050 | -0.026 | -0.571 | 0.255 | 0.064 | -0.125 |
| Jun-08 | Aug-08 | 0.001 | 2.238 | 0.303 | 1.509 | -1.660 | -0.663 | -0.043 | -0.131 | 0.320 | 0.016 | -0.033 |
| Jul-08 | Sep-08 | 0.005 | 1.422 | 0.139 | -3.258 | 0.674 | 0.368 | -0.011 | -0.089 | 0.234 | 0.094 | -0.023 |
| Aug-08 | Oct-08 | 0.004 | 0.535 | -0.241 | -1.031 | 0.321 | 0.343 | 0.007 | 0.072 | 0.337 | 0.156 | -0.051 |
| Sep-08 | Nov-08 | 0.001 | 0.230 | -0.530 | -0.706 | 0.298 | 0.333 | -0.025 | -0.082 | 0.263 | 0.099 | 0.002 |
| Oct-08 | Dec-08 | 0.001 | 0.002 | -1.290 | 1.019 | -0.153 | 0.217 | -0.034 | 0.126 | 0.938 | 0.067 | 0.032 |
| Nov-08 | Jan-09 | 0.000 | 1.937 | -0.720 | 1.098 | -0.633 | -0.089 | -0.030 | -0.058 | 0.611 | 0.086 | 0.083 |
| Dec-08 | Feb-09 | 0.004 | 0.596 | -8.394 | 5.413 | -1.574 | 0.008 | 0.003 | 0.061 | 0.648 | 0.049 | 0.023 |
| Jan-09 | Mar-09 | 0.002 | 1.355 | -4.665 | 5.818 | -1.725 | -0.060 | -0.006 | -0.008 | 0.525 | 0.055 | -0.273 |
| Feb-09 | Apr-09 | 0.005 | -0.431 | -11.309 | 1.872 | -0.245 | 0.133 | -0.001 | -0.009 | 0.207 | 0.033 | -0.177 |
| Mar-09 | May-09 | 0.005 | 0.379 | -9.181 | 1.407 | -0.243 | 0.219 | -0.029 | 0.009 | 0.394 | 0.003 | -0.143 |
| Apr-09 | Jun-09 | -0.001 | -4.500 | 2.013 | 1.850 | 0.225 | 0.234 | -0.022 | -0.002 | 0.990 | -0.014 | 0.031 |
| May-09 | Jul-09 | -0.003 | -2.506 | 1.751 | 7.430 | -2.176 | 0.181 | -0.051 | 0.004 | 1.220 | -0.012 | 0.048 |
| Jun-09 | Aug-09 | -0.006 | 2.327 | 10.628 | 2.993 | -1.576 | -0.046 | -0.022 | -0.037 | 1.159 | 0.020 | 0.011 |
| Jul-09 | Sep-09 | 0.001 | 1.944 | -4.402 | 10.927 | -5.568 | -0.040 | 0.011 | 0.038 | 0.994 | 0.001 | -0.094 |
| Aug-09 | Oct-09 | 0.002 | 8.687 | -4.225 | 11.403 | -8.536 | -0.210 | 0.008 | -0.044 | 0.826 | 0.009 | -0.122 |

**Table 1.3 Continued: Beta and NW-Tstats of Rolling-Window Regression C**

**Dependent varible: 1-day realized excess return of negative basis trading**

**Panel B: Newey West Tstat**

| Start | End | constant | libor-ffr | ffr | ted | ted*asw | Basis | MKTRF | DEF | TERM | signedtv | ted2*tv |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Mar-07 | May-07 | -0.102 | 0.289 | 0.100 | -1.681 | 1.631 | 0.857 | 0.079 | -5.302 | -0.483 | 3.762 | 1.693 |
| Apr-07 | Jun-07 | -0.216 | -0.073 | 0.216 | -1.741 | 2.487 | 2.045 | -0.639 | -6.056 | 1.864 | 3.362 | -0.518 |
| May-07 | Jul-07 | -2.720 | 2.971 | 2.743 | 0.142 | -2.003 | 1.510 | -0.726 | -4.053 | 0.442 | 2.510 | 1.268 |
| Jun-07 | Aug-07 | -1.612 | 3.229 | 1.757 | -1.431 | 0.722 | 0.587 | -1.110 | -5.878 | 0.640 | 2.461 | 0.305 |
| Jul-07 | Sep-07 | 0.851 | 0.862 | -0.674 | -2.028 | 2.088 | 0.798 | -1.654 | -6.825 | 1.565 | 2.301 | -0.216 |
| Aug-07 | Oct-07 | -0.346 | 2.235 | 0.636 | -1.445 | 1.260 | 1.403 | -1.214 | -11.036 | 0.397 | 6.066 | -0.517 |
| Sep-07 | Nov-07 | 0.711 | -0.757 | -0.650 | 0.143 | -0.429 | -0.191 | 0.118 | -4.593 | 0.223 | 4.106 | 1.646 |
| Oct-07 | Dec-07 | 0.312 | -0.944 | -0.184 | -0.040 | -0.004 | -0.010 | -1.485 | -3.904 | 0.268 | 3.143 | 1.572 |
| Nov-07 | Jan-08 | 0.136 | -0.623 | -0.176 | -0.339 | 0.485 | -0.919 | -1.850 | -5.728 | 1.714 | 2.949 | 2.247 |
| Dec-07 | Feb-08 | -0.531 | 1.780 | 0.394 | 1.450 | -1.640 | -0.943 | -0.430 | -6.909 | -0.031 | 2.552 | -1.608 |
| Jan-08 | Mar-08 | -1.537 | 1.705 | 1.588 | -0.297 | 0.115 | -0.432 | -0.839 | -7.537 | 0.132 | 3.733 | -0.895 |
| Feb-08 | Apr-08 | -1.639 | 2.907 | 1.357 | -1.406 | 1.329 | 0.926 | -1.326 | -7.163 | 0.784 | 3.103 | -0.998 |
| Mar-08 | May-08 | 0.394 | 0.566 | -0.432 | -1.246 | 2.260 | -0.432 | -1.571 | -5.666 | 1.143 | 2.003 | -0.246 |
| Apr-08 | Jun-08 | -1.134 | 2.644 | 1.159 | -2.350 | 2.263 | 1.406 | -0.740 | -5.942 | 2.294 | 1.535 | -1.873 |
| May-08 | Jul-08 | -1.932 | 2.712 | 1.952 | -3.131 | 3.193 | -0.239 | -1.646 | -4.980 | 2.457 | 1.908 | -2.405 |
| Jun-08 | Aug-08 | 0.092 | 2.194 | 0.164 | 1.446 | -5.559 | -3.368 | -2.540 | -1.073 | 3.158 | 0.425 | -2.218 |
| Jul-08 | Sep-08 | 2.078 | 2.118 | 0.239 | -2.950 | 2.120 | 1.296 | -0.612 | -0.822 | 1.589 | 1.901 | -1.233 |
| Aug-08 | Oct-08 | 1.989 | 0.964 | -0.615 | -0.963 | 1.096 | 1.635 | 0.408 | 0.683 | 1.724 | 3.411 | -2.614 |
| Sep-08 | Nov-08 | 1.034 | 0.290 | -1.391 | -0.545 | 1.182 | 1.927 | -1.062 | -0.647 | 0.872 | 1.670 | 0.075 |
| Oct-08 | Dec-08 | 0.943 | 0.002 | -1.459 | 0.668 | -0.464 | 1.379 | -1.471 | 1.096 | 2.168 | 1.864 | 0.725 |
| Nov-08 | Jan-09 | 0.101 | 2.771 | -0.587 | 0.708 | -1.573 | -0.463 | -1.519 | -0.506 | 3.498 | 2.488 | 3.802 |
| Dec-08 | Feb-09 | 2.708 | 0.368 | -3.036 | 3.129 | -2.665 | 0.054 | 0.196 | 0.670 | 4.773 | 1.803 | 0.835 |
| Jan-09 | Mar-09 | 0.821 | 1.268 | -0.815 | 2.899 | -2.912 | -0.383 | -0.383 | -0.089 | 3.560 | 2.724 | -2.442 |
| Feb-09 | Apr-09 | 1.508 | -0.278 | -1.587 | 1.557 | -1.200 | 0.816 | -0.055 | -0.079 | 1.047 | 1.177 | -1.512 |
| Mar-09 | May-09 | 1.903 | 0.234 | -1.793 | 1.131 | -0.877 | 1.623 | -1.322 | 0.172 | 1.380 | 0.121 | -1.295 |
| Apr-09 | Jun-09 | -0.321 | -1.248 | 0.365 | 0.808 | 0.921 | 1.720 | -0.770 | -0.055 | 4.691 | -0.833 | 0.285 |
| May-09 | Jul-09 | -0.780 | -0.695 | 0.271 | 2.571 | -2.293 | 1.285 | -2.135 | 0.167 | 9.519 | -0.785 | 0.357 |
| Jun-09 | Aug-09 | -2.505 | 0.943 | 1.906 | 1.371 | -1.932 | -0.333 | -0.985 | -0.745 | 6.681 | 1.191 | 0.040 |
| Jul-09 | Sep-09 | 0.592 | 0.633 | -1.025 | 1.649 | -1.414 | -0.263 | 0.419 | 0.371 | 5.147 | 0.066 | -0.212 |
| Aug-09 | Oct-09 | 1.331 | 2.888 | -1.051 | 2.096 | -2.530 | -1.438 | 0.379 | -0.308 | 3.559 | 0.574 | -0.267 |

Table 1.4: *Term Structure of the Size of Negative Basis Trading RER*

**Table 1.4: Term Structure of the Size of Negative Basis Trading RER**

| Average Absolute RER of NBT on Underlyings of Different Maturity | | | | |
|---|---|---|---|---|
| | | 2year | 5year | 9year |
| | 1day | 0.08% | 0.19% | 2.90% |
| \|RER_nbt\| | 1week | 0.22% | 0.46% | 3.12% |
| | 1month | 0.79% | 1.24% | 4.21% |

Table 1.5: *Comparative Statics Results on Basis Trading*

**Table 1.5: Comparative Statics Results on Basis Trading**

1.5A

Relationship between Size of Basis Trading RER and TED

| | | TED | | | | |
|---|---|---|---|---|---|---|
| | | low | | medium | | high |
| | 1day | 0.172 | 0.110 | 0.173 | 0.204 | 0.281 |
| \|RER_nbt\| | 1week | 0.362 | 0.271 | 0.450 | 0.463 | 0.707 |
| | 1month | 1.181 | 0.693 | 1.221 | 1.326 | 1.487 |
| | 2month | 1.951 | 1.657 | 1.860 | 2.813 | 2.317 |

1.5B

Relationship between Size of Basis Trading RER and Vol of Default Intensity

| | | Vol of Lambda | | | | |
|---|---|---|---|---|---|---|
| | | low | | medium | | high |
| | 1day | 0.192 | 0.186 | 0.158 | 0.224 | 0.259 |
| \|RER_nbt\| | 1week | 0.537 | 0.384 | 0.397 | 0.472 | 0.707 |
| | 1month | 0.764 | 0.929 | 1.275 | 1.061 | 1.979 |
| | 2month | 1.413 | 1.673 | 1.623 | 3.247 | 3.813 |

1.5C

Relationship between Size and Volatility of Basis Trading RER

| | | Vol_nbt | | | | |
|---|---|---|---|---|---|---|
| | | low | | medium | | high |
| \|RER_nbt\| | 1day | 0.117 | 0.167 | 0.181 | 0.262 | 0.255 |
| | 1week | 0.191 | 0.464 | 0.433 | 0.524 | 0.608 |

1.5D

Relationship between Volatility of Basis Trading RER and TED

| | | TED | | | | |
|---|---|---|---|---|---|---|
| | | low | | medium | | high |
| Vol_nbt | 1day | 0.072 | 0.138 | 0.177 | 0.243 | 0.263 |
| | 1week | 0.179 | 0.271 | 0.338 | 0.495 | 0.539 |

Table 1.6: *Predictability of Credit Spread Term Structure on Credit Spread Changes*

**Table 1.6: Predictability of Credit Spread Term Structure on Credit Spread Changes**

Set 1 Y variable: changes in 1-5yr Grade A ASW spread
Set 1 X variable: 3-5yr Grade A ASW spread minus 1-5yr Grade A ASW spread
Set 2 Y variable: changes in 1-5yr Grade AA ASW spread
Set 2 X variable: 3-5yr Grade AA ASW spread minus 1-5yr Grade AA ASW spread
Set 3 Y variable: changes in 5-10yr Grade A ASW spread
Set 3 X variable: 7-10yr Grade A ASW spread minus 5-7yr Grade A ASW spread
Set 4 Y variable: changes in 5-10yr Grade AA ASW spread
Set 4 X variable: 7-10yr Grade AA ASW spread minus 5-7yr Grade AA ASW spread

1.6A

| Regression Results of 15-day Credit Spread Change on Slope of CS Term Structure | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Set 1 | | Set 2 | | Set 3 | | Set 4 | |
| Period 1: 07/2007-02/2008 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.09 | (2.27) | 0.39 | (3.82) | -0.14 | (-1.83) | -0.04 | (-0.62) |
| slope | 0.00 | (1.75) | -0.01 | (-1.05) | 0.00 | (0.05) | 0.02 | (2.86) |
| adj.R^2 | 0.03 | | 0.01 | | -0.01 | | 0.09 | |
| Period 2: 03/2008-03/2009 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.13 | (1.42) | 0.01 | (0.17) | -0.92 | (-7.91) | -1.01 | (-11.92) |
| slope | 0.00 | (-0.04) | 0.01 | (2.30) | 0.00 | (0.99) | 0.02 | (6.98) |
| adj.R^2 | 0.00 | | 0.09 | | 0.02 | | 0.39 | |
| Period 3: 04/2009-09/2009 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.02 | (0.76) | 0.05 | (1.24) | -0.98 | (-14.16) | -1.25 | (-5.53) |
| slope | 0.00 | (7.07) | 0.02 | (10.81) | 0.03 | (6.55) | 0.03 | (4.29) |
| adj.R^2 | 0.60 | | 0.60 | | 0.38 | | 0.46 | |

1.6B

| Regression Results of 5-day Credit Spread Change on Slope of CS Term Structure | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Set 1 | | Set 2 | | Set 3 | | Set 4 | |
| Period 1: 07/2007-02/2008 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.00 | (0.29) | 0.17 | (2.13) | -0.22 | (-3.20) | -0.01 | (-0.11) |
| slope | 0.00 | (2.87) | 0.00 | (0.37) | 0.00 | (0.37) | 0.00 | (0.39) |
| adj.R^2 | 0.14 | | 0.00 | | 0.00 | | 0.00 | |
| Period 2: 03/2008-03/2009 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.03 | (0.84) | -0.05 | (-1.03) | -0.91 | (-8.63) | -0.94 | (-17.79) |
| slope | 0.00 | (-0.55) | 0.01 | (3.40) | 0.00 | (0.44) | 0.01 | (7.89) |
| adj.R^2 | 0.00 | | 0.24 | | 0.00 | | 0.32 | |
| Period 3: 04/2009-09/2009 | | | | | | | | |
| | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat | estimate | Nwtstat |
| constant | 0.00 | (-0.24) | 0.20 | (9.91) | -0.88 | (-16.29) | -1.05 | (-7.20) |
| slope | 0.00 | (3.16) | 0.01 | (14.11) | 0.02 | (5.61) | 0.02 | (5.89) |
| adj.R^2 | 0.22 | | 0.72 | | 0.29 | | 0.54 | |

## 1.11 Appendix 1.D: Alternative Assumption for $\lambda_t$

To avoid the problem of having negative default intensity, assume the default intensity is still $\widetilde{\lambda} + \lambda_t$, where $\lambda_t$ follows:

$$d\lambda_t = \kappa_\lambda(\bar{\lambda} - \lambda_t)dt + \sigma_\lambda\sqrt{\lambda_t}dB_{\lambda,t} \tag{1.96}$$

Where $\kappa_\lambda$, $\bar{\lambda}$ and $\sigma_\lambda$ are positive constants, and $B_{\lambda,t}$ is a Brownian Motion. The drift term of the bond prices then change to:

$$\mu_t^c = -A_\lambda^c\kappa_\lambda(\bar{\lambda} - \lambda_t) - A_z^c\kappa_r(\bar{r} - r_t) + A_\lambda^{c\prime}\lambda_t + A_r^{c\prime}r_t + C^{c\prime} + \frac{1}{2}A_\lambda^{c\,2}\sigma_\lambda^2\lambda_t + \frac{1}{2}A_r^{c2}\sigma_r^2 \tag{1.97}$$

$$\mu_t^d = -A_\lambda^d\kappa_\lambda(\bar{\lambda} - \lambda_t) - A_r^d\kappa_r(\bar{r} - r_t) + A_\lambda^{d\prime}\lambda_t + A_r^{d\prime}r_t + C^{d\prime} + \frac{1}{2}A_\lambda^{d2}\sigma_\lambda^2\lambda_t + \frac{1}{2}A_r^{d2}\sigma_r^2 \tag{1.98}$$

Substitute the above into the dynamic budget constraint, the F.O.C.s are:

$$\mu_t^c(\tau) - r_t - h_t^c(\tau)\frac{|x_t^c(\tau)|}{x_t^c(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda\sqrt{\lambda_t}A_\lambda^c(\tau)\Phi_{\lambda,t} - \sigma_r A_r^c(\tau)\Phi_{r,t} \tag{1.99}$$

$$\mu_t^d(\tau) - r_t - h_t^d(\tau)\frac{|x_t^d(\tau)|}{x_t^d(\tau)} - L(\widetilde{\lambda} + \lambda_t) = L\Phi_{J,t} - \sigma_\lambda\sqrt{\lambda_t}A_\lambda^d(\tau)\Phi_{\lambda,t} - \sigma_r A_r^d(\tau)\Phi_{r,t} \tag{1.100}$$

where

$$\Phi_{J,t} = \gamma L\int_0^T [x_t^d(\tau) + x_t^c(\tau)](\widetilde{\lambda} + \lambda_t)d\tau \tag{1.101}$$

$$\Phi_{\lambda,t} = \gamma\sigma_\lambda\sqrt{\lambda_t}\int_0^T [-x_t^d(\tau)A_\lambda^d(\tau) - x_t^c(\tau)A_\lambda^c(\tau)]d\tau \tag{1.102}$$

$$\Phi_{r,t} = \gamma\sigma_r\int_0^T [-x_t^d(\tau)A_r^d(\tau) - x_t^c(\tau)A_r^c(\tau)]d\tau \tag{1.103}$$

are the market prices of risks to the default jump, default intensity and short rate factors.

Under constant demand pressure assumption, in equilibrium, markets clear. So $x_t^i + z^i = 0$, $i = c, d$. Replace $x_t^i(\tau)$ by $-z^i(\tau)$, and replace $h_t^i$ by the functions defined in the funding cost section, then the F.O.C.s are affine equations in the risk factors $\lambda_t$ and $r_t$. Setting the linear terms in $\lambda_t$ and $r_t$ to zeros implies that the parameters $A_j^i(\tau)$ in the conjectured bond prices are the solutions to a system of

ODEs with initial conditions $A_j^i(0) = 0$, $i = c, d$ and $j = \lambda, r$:

$$
\begin{aligned}
& A_\lambda^{c\prime}(\tau) + \frac{1}{2}\sigma_\lambda^2 A_\lambda^c(\tau)^2 + \{\kappa_\lambda + \gamma\sigma_\lambda^2 \int_0^T [z^d(\tau)A_\lambda^d(\tau) + z^c(\tau)A_\lambda^c(\tau)]d\tau\}A_\lambda^c(\tau) \\
= \quad & -\alpha^c(\tau)\frac{|z^c(\tau)|}{z^c(\tau)} + L - \gamma L^2 \int_0^T [z^d(\tau) + z^c(\tau)]d\tau \qquad (1.104) \\
& A_\lambda^{d\prime}(\tau) + \frac{1}{2}\sigma_\lambda^2 A_\lambda^d(\tau)^2 + \{\kappa_\lambda + \gamma\sigma_\lambda^2 \int_0^T [z^d(\tau)A_\lambda^d(\tau) + z^c(\tau)A_\lambda^c(\tau)]d\tau\}A_\lambda^d(\tau) \\
= \quad & -\alpha^d(\tau)\frac{|z^d(\tau)|}{z^d(\tau)} + L - \gamma L^2 \int_0^T [z^d(\tau) + z^c(\tau)]d\tau \qquad (1.105)
\end{aligned}
$$

$$
\begin{aligned}
A_r^{c\prime}(\tau) + \kappa_r A_r^c(\tau) - 1 &= 0 \qquad (1.106) \\
A_r^{d\prime}(\tau) + \kappa_r A_r^d(\tau) - 1 &= 0 \qquad (1.107)
\end{aligned}
$$

Comparing to the simple mean-reverting $\lambda_t$ used in the main text, the additional square-root assumption adds a quadratic term of $A_\lambda^i$ to the coefficient of $\lambda_t$ in the drift term of the bond prices. It also adds a linear term of $A_\lambda^i$ to the coefficient of $\lambda_t$ on the right hand side of the F.O.C.. These changes the equation for $A_\lambda^i$ from simple linear ODE to Riccati equation, which can still be solved in closed-form easily. Note that the equations for $A_\lambda^c$ and $A_\lambda^d$ are symmetric but for the difference in parameter $\alpha^c$ and $\alpha^d$ on the right hand side. Therefore, the solutions are symmetric but for the difference in parameter $\alpha^c$ and $\alpha^d$, just like under the simple mean-reverting assumption in the main text. So the main conclusions on the expected excess return of basis trading in **Corollary 1.1** still hold.

However, the above results are derived under the assumption of constant demand pressure. Under the stochastic demand pressure in **Case 2**, the right hand side of the F.O.C.s will emerge a $z_t\lambda_t$ term that makes the system non-linear. In the constant demand pressure case, the market price of risk contains square-root of the variable, and the exposure to the risk factor also contains square-root of the variable, so the risk premium is linear in the variable. However, under stochastic demand pressure, the exposure to the risk factor still contains square-root of the variable, but the market price of risk contains the square-root of one variable multiplied by the linear term of the other variable, so the risk premium is no longer linear in the variables. This problem cannot be resolved even if the conjectured price function changes to include non-linear terms of the variables. Therefore, results in the main text are still presented under the simple mean-reverting assumption for $\lambda_t$, even though it has the drawback of creating negative default intensity.

# 2 General Equilibrium Analysis of Stochastic Benchmarking


## Abstract

This paper applies a closed-form continuous-time consumption-based general equilibrium model to analyze the equilibrium implications when some agents in the economy promise to beat a stochastic benchmark at an intermediate date. For very risky benchmark, these agents increase volatility and risk premium in the equilibrium. On the other hand, when they promise to beat less risky benchmark, they decrease volatility and risk premium in the equilibrium. In both cases, the degree of effect is state-dependent and stock price rises.


## 2.1 Introduction

This paper studies the equilibrium implications when some agents in the economy face stochastic benchmarking constraint. These constrained benchmarking agents characterize fund managers who promise to beat a stochastic benchmark at an intermediate date. Using the martingale approach, I solve a continuous-time consumption-based general equilibrium model explicitly to get the equilibrium assets prices, risk premium, volatility and optimal strategies in this benchmarking economy featuring both normal unconstrained agents and constrained benchmarking agents. I also compare these equilibrium quantities with those in a normal economy featuring only normal unconstrained agents so as to highlight the impact of the benchmarking constraint on the economy. To my knowledge, this is the first paper to investigate the equilibrium effects of this type of benchmarking constraint.

The benchmarking constraint in this paper is a requirement that an agent's wealth at a pre-specified intermediate date is no less than a stochastic benchmark index. The problem of beating a constant floor has been studied by the portfolio insurance literature, e.g. the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). However, the portfolio insurance constraint only ensures the agent doesn't lose more than a certain level without asking for a higher return when the economy is good. Fund manager's performance is often evaluated against a benchmark index. Facing this benchmarking constraint is equivalent to promising to beat the performance of the benchmark index at a certain evaluation date. The economy consists of two assets, a risk-free money market account and a risky stock. The stochastic benchmark index is a replication portfolio using the money market account and the stock. Therefore, it is achievable through passive management and the riskiness of the benchmark is measured by the positions in the risk-free money market account.

After describing the finite horizon standard Lucas (1978) economy with an intermediate constraint date, I first characterize the portfolio choice problems for the normal agents and the constrained agents. The approach is the martingale representation approach as in Cox and Huang (1989) and Basak (1995). Then I get explicit solutions for the equilibrium market dynamics under log utility using the martingale approach for option pricing and Ito's lemma. This paper adopts similar set-ups as in Basak (1995), but the alternative consideration of the stochastic benchmark complicates the calculation of equilibrium quantities and the discussion of equilibrium effects.

I find that in the benchmarking economy before the constraint date, the stock price is higher than that in the normal economy because the constrained agents consume less than if they're not constrained. Since the money market account is in zero net supply, the extra investment from the constrained agents goes into the stock market and drives up the stock price before the constraint date. The increase in stock price also reflects the constrained agents' preferences for consumption and dividend after the constraint date.

If the benchmark is risky, which means the replication portfolio of the benchmark index has a short position in the money market account, then the risk premium and volatility are higher than those in the normal economy. Moreover, the optimal fraction of wealth invested in the stock by the constrained agent is higher than that of the unconstrained normal agent. While if the benchmark is safe, which means the replication portfolio of the benchmark index has positive position in the money market account, then the risk premium and volatility are lower than those in the normal economy, and the optimal fraction of wealth invested in the risky asset by the constrained agent is lower than that of the unconstrained normal agent. In both cases, the degree of the effect is state-dependent.

The rationale behind these findings is that the presence of the benchmarking constraint results in more (or less) demand for the stock from the constrained agent, then the volatility has to increase (or decrease) so as to induce unconstrained normal agents to change their demand and clear the market. It is also because that the stock price in this economy is equal to the normal economy price plus the present value of an option-like payoff that represents the effects from the benchmarking constraint. For the risky benchmark case, the extra payoff is like a call option payoff and the effect is thus volatility increasing; but for the safe benchmark case, the extra payoff is like a put option payoff and the effect becomes volatility decreasing. From the modeling aspect, the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the use of the consumption good as numeraire and the existence of intermediate dividend for consumption, which are not true in some other papers that have contrary conclusions for certain parts of the model, for instance Grossman and Zhou (1996). However, the conclusion of this paper is consistent with that for the portfolio insurance model of Basak (1995), which has the similar set-up and approach with this paper.

The closest literatures are the equilibrium analysis of portfolio insurance by Basak (1995) and Grossman and Zhou (1996). Basak (1995) builds a similar consumption-based general equilibrium model and compares the explicit expressions for equilibrium market dynamics in the portfolio insurance economy with those in the normal economy. The portfolio insurers' strategies are similar to the synthetic put approach and the presence of the intermediate portfolio insurance constraints decreases the risk premium, volatility and optimal fraction of wealth invested in the risky asset. The use of log utility ensures the SPDs are not affected by the constraints since they are derived by market clearing of intermediate consumption. In contrast, Grossman and Zhou (1996) adopts a different set-up in which the portfolio insurance constraint is on the final date and there's no intermediate consumption so agents only care about consumption at the final date which is financed by a lump-sum of dividend. Therefore, the pricing kernels before the final date are directly affected by the constraint and that makes the overall effect of portfolio insurance to be increasing risk premium and volatility. However, the use of bond price as the numeraire results in different predictions with Basak (1995) and makes the model impossible to be solved explicitly. As mentioned above, these two papers only consider the case of portfolio insurance which is benchmarking on a constant floor while Tepla (2001) studies the optimal portfolio choice of an agent who performs against a stochastic benchmark similar to the one considered here but doesn't derive the equilibrium results.

The general topic of optimal strategy and asset pricing implications of constrained or benchmarking institutional investors have been explored by a number of papers. Basak and Shapiro (2001) reveal in a general equilibrium model that VaR risk managers amplify volatility in poor market and attenuate volatility in good market. Basak and Chabakauri (2012) provide a new framework that derives the optimal strategy of portfolio managers who care about their tracking error to a benchmark. Basak and Pavlova (2012) show that institutional investors favor stocks that comprise their benchmark index and amplify the index stock volatilities and aggregate stock market volatility.

In the rest of the paper, Section 2.2 presents the model and characterizes the optimization problems of agents. Section 2.3 solves for the equilibrium and shows the main effects of the benchmarking constraint on the economy. Then Section 2.4 presents more discussion on the equilibrium effects before Section 2.5 concludes.

## 2.2 The Model

### 2.2.1 The Economy

In a finite horizon $[0, T']$ pure-exchange economy, all quantities are in units of a consumption good. Let $B$ denote a Brownian Motion on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\{\mathcal{F}_t; t \in [0, T']\}$ be the augmentation by null sets of the filtration generated by $B$, which represents all uncertainties in the economy. Prior to $T'$, some agents face a constraint at the constraint date $T$, which will be specified later.

### 2.2.2 Securities

The economy consists of two assets. $S^0$ is a risk-less money market account in zero net supply that pays interest at rate $r_t$, which is to be determined in the equilibrium, and $S$ is a risky stock in constant net supply of 1 and pays dividend at an exogenous rate of $\delta_t$ in $[0, T']$. Assume that the dividend process follows a Geometric Brownian Motion.

$$d\delta_t = \delta_t(\mu_\delta dt + \sigma_\delta dB_t), t \in [0, T'] \tag{2.1}$$

where $\mu_\delta$ and $\sigma_\delta$ are both constants. Similar to Basak (1995), I anticipate a price discontinuity in equilibrium around the intermediate constraint date $T$.[22] Therefore, I model the stock price as a diffusion process with an $\mathcal{F}_T$-measurable jump at time $T$:

$$dS_t + \delta_t dt = S_t(\mu_t dt + \sigma_t dB_t + q dA_t), t \in [0, T'] \tag{2.2}$$

[22]For more intuition, see the section for equilibrium

And $S_{T'} = 0$. Here, $A_t$ is the right-continuous step function defined by $A_t \equiv 1_{t \geq T}$, and the $\mathcal{F}_T$-measurable random variable $q$ is the jump size parameter, defined as $q = ln(S_T/S_{T-})$, where $S_{T-}$ denotes the left limit of $S_T$. As explained in Basak (1995), the above specifications ensure that the stock price has continuous local martingale part and discontinuous bounded variation part that contains an $\mathcal{F}_T$-measurable jump. The money market account's value also has a jump:

$$dS_t^0 = S_t^0(r_t dt + q^0 dA_t), t \in [0, T'] \tag{2.3}$$

By construction, the jump sizes are revealed immediately before the jumps occur. So to rule out arbitrage, $q = q^0$.

### 2.2.3 State Price Density

Given market completeness, define the state price density process (SPD) as:

$$\pi_t = \frac{1}{S_0^0} exp(-\int_o^t r_s ds - \int_o^t \theta_s dB_s - \frac{1}{2}\int_o^t \theta_s^2 ds - qA_t) \tag{2.4}$$

where $\theta_t = (\mu_t - r_t)/\sigma_t$ is the market price of risk. The SPD process which represents the price of consumption also contains a jump, which is of the opposite direction of jumps in asset prices.[23] Apply Ito's lemma to $\pi_t$ gives:

$$d\pi_t = -\pi_t(r_t dt + \theta_t dB_t + q dA_t), t \in [0, T'] \tag{2.5}$$

and using the SPD process, the relationship between the stock price and future dividends is

$$S_t = \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \delta_s ds \mid \mathcal{F}_t], t \in [0, T'] \tag{2.6}$$

### 2.2.4 Agents

The economy has two types of agents, i.e. agent type $n$ and agent type $m$. Type $n$ agent is the normal agent, while type $m$ is referred to as the constrained agent later on. Each type has infinite number of agents that form a continuum with measure 1. Each agent belonging to type $n$ or type $m$ is endowed with initial wealth $x_{n0}$ or $x_{m0}$ at time zero respectively.[24] Let $X_{it}$ denote the wealth of an agent of type $i$ at

[23] Because asset prices are in units of consumption goods, when asset prices jump downward, SPD as the price of consumption jumps upward, and the product of $\pi S$ remains continuous.

[24] The endowment can be in the form of shares of the stock, each type $n$ and type $m$ agents' endowed shares of stock worth $x_{n0}$ and $x_{m0}$, which add up to the stock price at time 0.

time $t$, $i = n, m$, then $X_{it}$ follows:

$$dX_{it} = (1-\Phi_{it})X_{it}(r_t dt + qdAt) + \Phi_{it}X_{it}(\mu_t dt + \sigma_t dB_t + qdA_t) - c_{it}dt, t \in [0, T'] \quad (2.7)$$

where $\Phi_{it}$ denotes the fraction of wealth invested in the risky asset and $c_{it}$ is the consumption process for agent $i$. Both type of agents have time-additive state-independent utility function over consumption. The function $u(c_{it})$ is the same for all agents and is continuous with continuous first derivatives, strictly increasing, strictly concave. Hereafter, denote all optimal quantities with a caret ( $\hat{}$ ). The optimal wealth of an agent satisfies:

$$\hat{X}_{it} = \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \hat{c}_{is} ds \mid \mathcal{F}_t], t \in [0, T'] \quad (2.8)$$

In this model, a type $n$ agent is the normal agent. But a type $m$ agent is the constrained agent who faces a stochastic benchmarking constraint at time $T$ so that she has to maintain her time $T$ wealth above a stochastic benchmark index $aS_T + b$, where $a$ and $b$ are exogenously given constants. For a type $m$ agent, the constraint is $X_{mT-} \geq aS_T + b$, where $X_{mT-}$ is the left limit of $X_{mT}$.[25]

Although the parameters $a$ and $b$ are exogenous, the benchmark index value $aS_T + b$ is endogenous. Type $m$ agents characterize fund managers who promise investors to beat a benchmark index at a given evaluation date (time $T$). The value of $b$ characterizes the riskiness of the benchmark. As will be explained in more details in latter sections, an unconstrained investor with log utility will optimally choose to invest all wealth in the stock, so $b < 0$ suggests the benchmark is more risky than the agent's original strategy; and $b > 0$ means the benchmark is relatively safe as it contains positive position in the money market account while the agent's original strategy doesn't. The parameter values $a$ and $b$ have important implications on the equilibrium properties.

### 2.2.5 The Optimization Problems

A normal agent solves the following standard problem:

$$\max_{c_n} E[\int_0^{T'} u(c_{ns}) ds] \quad (2.9)$$

[25]The idea of comparing time $T-$ wealth against time $T$ benchmark value may be confusing at first glance, but in equilibrium $S_T = (T' - T)\delta_{T-}$, so the constraint is effectively requiring time $T-$ wealth to be no less than a stochastic time $T-$ value.

$$subject\ to\ E[\int_0^{T'} \pi_s c_{ns} ds] \leq \pi_0 x_{n0} \tag{2.10}$$

Assuming a solution exists, the optimal consumption is simply:

$$\hat{c}_{nt} = I(\lambda_n \pi_t) \quad t \in [0, T'] \tag{2.11}$$

where $I(\cdot)$ is the inverse of $u'(\cdot)$ and $\lambda_n$ satisfies:

$$E[\int_0^{T'} \pi_s I(\lambda_n \pi_s) ds] = \pi_0 x_{n0} \tag{2.12}$$

Alternatively, a constrained agent faces an additional benchmarking constraint as well as the normal budget constraint which is now written in two parts:

$$\max_{c_m, X_{mT-}} E[\int_0^{T'} u(c_{ms}) ds] \tag{2.13}$$

$$subject\ to\ E[\int_0^{T} \pi_s c_{ms} ds + \pi_{T-} X_{mT-}] \leq \pi_0 x_{n0} \tag{2.14}$$

$$E[\int_T^{T'} \pi_s c_{ms} ds \mid \mathcal{F}_T] \leq \pi_{T-} x_{mT-}\ almost\ surely, \tag{2.15}$$

$$X_{mT-} \geq aS_T + b\ almost\ surely \tag{2.16}$$

**Lemma 2.1.** *Assuming a solution exists, a constrained agent's optimal consumption is:*

$$\begin{aligned} \hat{c}_{mt} &= I(\lambda_{m1} \pi_t) \quad t \in [0, T) & (2.17) \\ \hat{c}_{mt} &= I(\lambda_{m2} \pi_t) \quad t \in [T, T'] & (2.18) \end{aligned}$$

*where $\lambda_{m1}$ and $\lambda_{m2}$ satisfy:*

$$E[\int_0^{T} \pi_s I(\lambda_{m1} \pi_s) ds + \pi_{T-} \max\{aS_T + b, \frac{1}{\pi_{T-}} E[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds | \mathcal{F}_T]\}] = \pi_0 x_{m0} \tag{2.19}$$

$$E[\int_T^{T'} \pi_s I(\lambda_{m2} \pi_s) ds | \mathcal{F}_T] = \pi_{T-} \max\{aS_T + b, \frac{1}{\pi_{T-}} E[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds | \mathcal{F}_T]\} \tag{2.20}$$

*N.B. (1) $\lambda_n, \lambda_{m1}$ are constants, $\lambda_{m2}$ is an $\mathcal{F}_T$-measurable random variable. (2) If $x_{n0} = x_{m0}$, then $\lambda_{m1} \geq \lambda_n$. (3) $\lambda_{m1} = \lambda_{m2}$ if the benchmarking constraint is not binding; $\lambda_{m1} > \lambda_{m2}$ if it is binding.*

*Proof.* see Appendix 2. □

The above results suggest if a constrained agent is equally endowed as a normal agent, then the constrained agent consumes less before the constraint date than the normal agent so as to ensure her wealth at the constraint date is higher than the benchmark value. If the constraint is binding, then the constrained agent no longer restricts her consumption after the constraint date so it may rise.

## 2.3 The Equilibrium

### 2.3.1 Market Clearing Conditions

The equilibrium conditions are the market clearing of consumption goods, the market clearing of the money market account and the market clearing of the stock. These conditions imply that the aggregate optimal consumption from all agents adds up to the dividend:

$$\delta_t = \hat{c}_{nt} + \hat{c}_{mt} \quad t \in [0, T'] \tag{2.21}$$

Following this equilibrium condition, the SPD satisfies the following equations:

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m1} \pi_t) \quad t \in [0, T) \tag{2.22}$$

$$\delta_t = I(\lambda_n \pi_t) + I(\lambda_{m2} \pi_t) \quad t \in [T, T'] \tag{2.23}$$

For $u(c) = log(c)$,[26] the solution for the SPD $\pi_t$ is:

$$\pi_t = (\frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}})\frac{1}{\delta_t} \quad t \in [0, T) \tag{2.24}$$

$$\pi_t = (\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}})\frac{1}{\delta_t} \quad t \in [T, T'] \tag{2.25}$$

Applying Ito's lemma on $\pi_t$ which follows the dynamics in equation (2.5), I solve for the following equilibrium quantities as:

$$r_t = \mu_\delta - \sigma_\delta^2 \tag{2.26}$$

$$\theta_t = \sigma_\delta \tag{2.27}$$

$$q = ln(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m1}}) - ln(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}}) \leq 0 \tag{2.28}$$

The short-rate $r_t$ and market price of risk $\theta_t$ are constants. Hereafter, denote $r_t = \mu_\delta - \sigma_\delta^2 = r$ and $\theta_t = \sigma_\delta = \theta$.

[26]For simplicity, I only consider log-utility while similar approach can be applied to power and negative exponential utilities.

The SPD has an upward jump at time $T$. This jump is necessary for the equilibrium condition to hold. If the SPD is continuous, the normal agent's demand for consumption will also be continuous. But when type $m$ agent's benchmarking constraint is binding, since $\lambda_{m1} > \lambda_{m2}$ and $\hat{c}_{mT-} = 1/\lambda_{m1}\pi_{T-}, \hat{c}_{mT} = 1/\lambda_{m2}\pi_T$, her demand for consumption will jump upwards immediately after time $T$, resulting in the aggregate demand for consumption jumping upwards, which is impossible since the dividend process is continuous. Hence, there must be a jump in the SPD to smooth the constrained agent's demand for consumption. And the jump in SPD is upward because the constrained agent values consumption after the constraint date more than before the date.

As mentioned above, the other two equilibrium conditions are the market clearing of the money market account and the market clearing of the stock. The money market account is in zero net supply, so the aggregate optimal wealth invested into the money market account adds up to zero in the equilibrium. On the other hand, the stock is in a supply of 1, so the aggregate optimal wealth invested into the stock adds up to the stock price. Recall that $\hat{X}_{nt}$ and $\hat{X}_{mt}$ denote the optimal wealth of agents and $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$ denote the optimal fraction of wealth invested in the stock, the above two equilibrium conditions imply:

$$0 = \hat{X}_{nt}(1 - \hat{\Phi}_{nt}) + \hat{X}_{mt}(1 - \hat{\Phi}_{mt}) \tag{2.29}$$

$$S_t = \hat{X}_{nt}\hat{\Phi}_{nt} + \hat{X}_{mt}\hat{\Phi}_{mt} \tag{2.30}$$

Combining the above two equations implies that the stock price is the sum of all agents' optimal wealth.

$$S_t = \hat{X}_{nt} + \hat{X}_{mt} \tag{2.31}$$

### 2.3.2 Asset Prices

In this section, I derive explicit solutions for the equilibrium stock price. Then in the following sections, I provide explicit solutions to the risk premium, volatility and agents' optimal fraction of wealth invested into stock. For each equilibrium quantity, I also provide results under a normal economy consisting only normal agents to compare with the results under this benchmarking economy with both normal and constrained agents. Hereafter, denote equilibrium quantities under the normal economy with a bar ($\bar{\ }$).

**Lemma 2.2.** *In the normal economy, the equilibrium stock price is:*

$$\bar{S}_t = (T' - t)\delta_t \quad t \in [0, T'] \tag{2.32}$$

*In the benchmarking economy, after the constraint date ($t \in [T, T']$), the stock price $S_t$ is the same as that in the normal economy:*

$$S_t = (T' - t)\delta_t \quad t \in [T, T'] \tag{2.33}$$

*But before the constraint date ($t \in [0, T)$), the stock price is no less than that in the normal economy:*

$$S_t = \bar{S}_t + \frac{1}{\pi_t}E[\pi_{T-}\max\{aS_T + b - \frac{\lambda_n}{\lambda_n + \lambda_{m1}}(T' - T)\delta_{T-}, 0\}|\mathcal{F}_t] \tag{2.34}$$

$$\geq \bar{S}_t \qquad t \in [0, T) \tag{2.35}$$

*Proof.* see Appendix 2. □

In the benchmarking economy before the constraint date, the stock price could be higher than that in the normal economy because the stock price is the aggregate amount of invested wealth, which reflects the present value of aggregate future consumptions. In this benchmarking economy, agents value consumption after the constraint date time $T$ more than before the constraint date. This concern is reflected in the SPDs and then may results in a higher stock price than in the normal economy.

Now focus on the explicit solution of $S_t$ for $t \in [0, T)$. From the equation (2.49), $S_T = (T' - T)\delta_T$, and by the continuity of $\delta_t$, $\delta_{T-} = \delta_T$. So, replace $S_T$ by $(T' - T)\delta_{T-}$, equation (2.53) becomes:

$$S_t = (T' - t)\delta_t + \frac{1}{\pi_t}E[\pi_{T-}\max\{(a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)\delta_{T-} + b, 0\}|\mathcal{F}_t] \tag{2.36}$$

Since the process $\delta_t$ is a Geometric Brownian Motion, I apply the martingale approach to take the expectation in equation (2.36). Hereafter, assume $(a - \frac{\lambda_n}{\lambda_n+\lambda_{m1}})b < 0$, which means there's always uncertainty over the relative performance of the unconstrained strategy against the benchmark index $aS_T + b$. This assumption excludes the cases where either the constraint will never bind ($b < 0$ *and* $a - \frac{\lambda_n}{\lambda_n+\lambda_{m1}} < 0$), in which case the constrained agent always behaves the same as the unconstrained one, or the constraint will always bind ($b > 0$ *and* $a - \frac{\lambda_n}{\lambda_n+\lambda_{m1}} > 0$), in which case the constrained agent will just hold the benchmark index. Now, the stock price $S_t$

is solved from equation (2.36) as:

**Proposition 2.1.** *In the benchmarking economy, for $t \in [0, t)$:*

$$S_t = (T' - t)\delta_t + (a - \frac{\lambda_n}{\lambda_n + \lambda_{m1}})(T' - T)N[-\frac{b}{|b|}z_1]\delta_t + be^{-r(T-t)}N[-\frac{b}{|b|}z_2] \quad (2.37)$$

*where*

$$\begin{aligned} z_1 &= z_2 + \sigma_\delta\sqrt{T-t} \\ z_2 &= \frac{ln[(a - \frac{\lambda_n}{\lambda_n+\lambda_{m1}})(T' - T)\delta_t / - b] + (r - \frac{1}{2}\sigma_\delta^2)(T-t)}{\sigma_\delta\sqrt{T-t}} \end{aligned}$$

*$N(\cdot)$ is the distribution function of normal random variable. And the optimal invested wealth of agent $m$ at time $t$ is:*

$$\hat{X}_{mt} = S_t - \hat{X}_{nt} = S_t - \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}}(T' - t)\delta_t \quad (2.38)$$

*Proof.* see Appendix 2. □

### 2.3.3 Volatility and Risk Premium

Applying Ito's lemma on $S_t$ provides explicit solutions for the volatility and risk premium in both the normal and benchmarking economy.

**Proposition 2.2.** *In the normal economy, for $t \in [0, T']$, the stock return volatility $\bar{\sigma}_t$ and risk premium $\bar{\mu}_t - r$ are:*

$$\begin{aligned} \bar{\sigma}_t &= \sigma_\delta & (2.39) \\ \bar{\mu}_t - r &= \sigma_\delta^2 & (2.40) \end{aligned}$$

*In the benchmarking economy, after the constraint date ($t \in [T, T']$), the stock return volatility $\sigma_t$ and risk premium $\mu_t - r$ are the same as those in the normal economy:*

$$\begin{aligned} \sigma_t &= \sigma_\delta & (2.41) \\ \mu_t - r &= \sigma_\delta^2 & (2.42) \end{aligned}$$

*But before the constraint date ($t \in [0, T)$), the stock return volatility and risk premium are different from those in the normal economy:*

$$\sigma_t = [1 - \frac{be^{-r(T-t)}N[-\frac{b}{|b|}z_2]}{S_t}]\sigma_\delta \quad (2.43)$$

$$\mu_t - r \quad = \quad [1 - \frac{be^{-r(T-t)}N[-\frac{b}{|b|}z_2]}{S_t}]\sigma_\delta^2 \tag{2.44}$$

*Proof.* see Appendix 2. □

Before the constraint date, whether the volatility and risk premium in the benchmarking economy are higher or lower than those in the normal economy only depends on the sign of the fraction term inside the bracket in the above set of equations.

**Corollary 2.1.** *Before the constraint date ($t \in [0, T)$): for $b < 0$ ($b > 0$), the volatility and risk premium in the benchmarking economy are higher (lower) than those in the normal economy.*

*Proof.* see Appendix 2. □

The volatility and risk premium in the normal economy are constant. However, now they become stochastic in the presence of the stochastic benchmarking constraint. As will be explained in more details in latter sections, $b < 0$ means the benchmark index is risky while $b > 0$ means the benchmark index is safe. The above corollary suggests that when some agents in the economy promise to beat risky benchmark, they increase volatility and risk premium in the equilibrium. On the other hand, when they promise to beat safe benchmark, they decrease volatility and risk premium in the equilibrium. In both cases, the degree of the increase or decrease is state-dependent as the $N(z_2)$ and $S_t$ terms both depend on $\delta_t$. The value of volatility and risk premium are also affected by the parameter $a$ in the stochastic index, as both $z_2$ and $S_t$ contains $a$. To better understand the results for volatility, I further investigate agents' optimal strategies.

### 2.3.4 The Optimal Strategy

Denote the optimal fraction of wealth invested in the risky asset by $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$ for agent of type $n$ and agent of type $m$. Applying Ito's lemma on $\pi_t \hat{X}_{nt}$ and $\pi_t \hat{X}_{mt}$ provides explicit solutions for $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$ in the benchmarking economy. I also provide results in the normal economy for comparison.

**Proposition 2.3.** *In the normal economy, the normal agent optimally invest all her wealth in the stock.*

*In the benchmarking economy, for the normal agent $n$, the optimal fraction of wealth invested in the stock is:*

$$\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 \tag{2.45}$$

*While for the constrained agent m, after the constraint date ($t \in [T, T']$), the optimal fraction of wealth invested in the stock is the same as that of the normal agent:*

$$\hat{\Phi}_{mt} = \hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 \tag{2.46}$$

*Before the constraint date ($t \in [0, T)$), the optimal fraction of wealth invested in the stock is different from that of the normal agent:*

$$\hat{\Phi}_{mt} = [1 - \frac{be^{-r(T-t)}N[-\frac{b}{|b|}z_2]}{\hat{X}_{mt}}]\frac{\mu_t - r}{\sigma_t^2} \tag{2.47}$$

*Proof.* see Appendix 2. □

**Corollary 2.2.** *In the benchmarking economy before the constraint date ($t \in [0, T)$): for $b < 0$ ($b > 0$), the constrained agent invests more (less) fraction of wealth in the risky asset than the normal agent. And both agents' optimal fraction of wealth invested in the risky asset are now stochastic.*

*Proof.* see Appendix 2. □

## 2.4 Discussion of the Equilibrium

In the normal economy, agents invest all their wealth into the stock. In the benchamrking economy, the benchmark faced by some agents is $aS_T + b$, so $a = 1$ and $b = 0$ corresponds to a strategy that replicates the normal agent's behavior in the normal economy. If the benchmark has parameter $b < 0$, this is equivalent to a strategy involving borrowing in the money market account to invest in stock. Such a strategy is riskier than the normal agent's behavior in the normal economy. Therefore, such a benchmark is regarded as a risky benchmark. On the other hand, if $b > 0$, this is equivalent to investing less in the stock but more in the money market account, comparing with the normal agent's behavior in the normal economy. Such a benchmark is regarded as safe.

The existence of risky benchmark increases risk premium and volatility conditionally, while a safe benchmark decreases these terms. In both cases, the risky asset price is higher in the presence of the benchmarking constraint than in the normal economy. The conclusion here is consistent with that of Basak (1995), in which the portfolio insurance constraint can be viewed as a special case of the benchmarking constraint studied here.

Such findings may be contrary to conventional wisdom, e.g. Grossman and Zhou (1996), which says the inclusion of the portfolio insurance constrained agent should increase volatility and risk premium. However, the results here are understandable in the following ways:

Firstly, seen from **Proposition 2.2** and **Corollary 2.2**, the effect of volatility can be explained by the optimal fractions of wealth invested in the stock by both agents. In the normal economy, both agents optimally invest all their wealth into the stock. In the benchmarking economy, with safe benchmark, the constrained agents have to invest less in the stock so as to hold some risk-free asset. To clear the market, the normal agents have to buy more stock than they would in the normal economy. Since $\hat{\Phi}_{nt} = (\mu_t - r)/\sigma_t^2 = \theta/\sigma_t$, the only way to make the normal agent hold more stock is to decrease $\sigma_t$. If the benchmark is risky, the constrained agents have to hold more stock. So the volatility has to increase to induce the unconstrained agents to hold less stock than they would in the normal economy. The above mentioned rationale suggests that in the benchmarking economy which has a constant market price of risk ($\theta$), the volatility of the stock has to decrease (or increase) so as to make the stock more (or less) attractive to the normal agent. Since the constrained agents have to adjust their demand for the stock conditionally, the degree to which the volatility is increased or decreased is therefore state-dependent.

For $b < 0$, agent $m$ is more constrained in good states, so the effects on equilibrium dynamics are stronger in good states. While for $b > 0$, the agent is more constrained in bad states, so the effects are stronger in bad states.

In another attempt to explain the results on volatility, recall the expression for $S_t$ before $T$ in equation (2.36). The stock price in the benchmarking economy is the normal economy price plus an expectation term which is the present value of an option-like payoff, which is a call option payoff when the benchmark is risky, and a put option payoff when the benchmark is safe. In the risky benchmark case, when the market goes down the call option value goes down as well so the stock price falls even further. Therefore, the constraint destabilizes price and increases volatility. However, for the safe benchmark case, when the market goes down, the put option value goes up, which helps stabilize stock price. So the volatility is decreased in the presence of the constraint.

The results for the safe benchmark case is consistent with Basak (1995) which has the similar set-up in using consumption as numeraire and having intermediate consumption, dividend and constraint date. These features result in a constant market price of risk as compared to the price of consumption. But in Grossman and Zhou (1996), the market price of risk is time-varying as they used bond price as numeraire and there's no intermediate consumption so the pre-constraint pricing kernels are conditional expectations of the final one and therefore directly affected by the constraint. Therefore, the volatility is affected by both the change in market price of risk and the change in agent's risk aversion, and they show that the overall effect is a higher volatility.

## 2.5 Concluding Remarks

This paper studies the equilibrium effect of a stochastic benchmarking constraint. The economy has normal agents with log utility over continuous consumption and constrained agents whose wealth at an intermediate date must lie above a stochastic benchmark index. Using the martingale approach, I solve this consumption-based general equilibrium model in closed form to get the equilibrium assets prices, risk premium, volatility, optimal strategy and compare them with those in a normal economy. The problem of the equilibrium effects of portfolio insurance studied by Basak (1995) and Grossman and Zhou (1996) is a special case of the problem studied here.

The constrained agents can be understood as fund managers who promise investors to beat a benchmark index that is achievable through passive management. When they promise to beat risky benchmark, they increase volatility and risk premium in the equilibrium. On the other hand, when they promise to beat safe benchmark, they decrease volatility and risk premium in the equilibrium. In both cases, the degree of the increase or decrease is state-dependent.

The rationale behind these findings is that when there's more (less) demand for the stock from the constrained agent, the volatility has to increase (decrease) so as to clear the market. This is true because in this model the market price of risk is constant and the SPDs before the constraint date are not directly affected by the existence of the constraint due to the existence of intermediate dividend for consumption. The model has consistent results with the portfolio insurance model of Basak (1995) that has the similar set-up and approach with this paper.

Further development of this paper may include numerically analyzing the equilibrium effect of the benchmarking constraint under the Grossman and Zhou (1996) set up, in which the constraint directly affect the SPDs, and studying more realistic variation of the benchmarking constraint allowing for tracking errors.

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## 2.7 Appendix 2: Proofs of Lemmas and Propositions

### 2.1. Proof of Lemma 2.1

Replacing $K$ in Basak (1995)'s proof of Lemma 3 by $aS_T + b$ proves Lemma 2.1.

### 2.2. Proof of Lemma 2.2

In the normal economy, the equilibrium stock price is:

$$\begin{aligned} \bar{S}_t &= \frac{1}{\bar{\pi}_t} E[\int_t^{T'} \bar{\pi}_s \frac{1}{\lambda_n \bar{\pi}_s} ds \mid \mathcal{F}_t] \\ &= (T' - t)\delta_t \quad t \in [0, T'] \end{aligned} \tag{2.48}$$

In the benchmarking economy, the stock price $S_t$ is derived for the two horizons $t \in [T, T']$ and $t \in [0, T)$ separately. For $t \in [T, T']$, taking the optimal consumption solutions into equation (2.8) gives the stock price $S_t$ as:

$$\begin{aligned} S_t &= \hat{X}_{nt} + \hat{X}_{mt} \\ &= \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \frac{1}{\lambda_n \pi_s} ds \mid \mathcal{F}_t] + \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \frac{1}{\lambda_{m2} \pi_s} ds \mid \mathcal{F}_t] \\ &= \frac{1}{(\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}})\frac{1}{\delta_t}} (\frac{1}{\lambda_n} + \frac{1}{\lambda_{m2}})(T' - t) \\ &= (T' - t)\delta_t \\ &= \bar{S}_t \end{aligned} \tag{2.49}$$

In the benchmarking economy, for $t \in [0, T)$:

$$\begin{aligned} S_t &= \hat{X}_{nt} + \hat{X}_{mt} \\ &= \frac{1}{\pi_t} E[\int_t^{T'} \pi_s \frac{1}{\lambda_n \pi_s} ds | \mathcal{F}_t] + \frac{1}{\pi_t} E[\int_t^{T} \pi_s \frac{1}{\lambda_{m1} \pi_s} ds + \pi_{T-} \hat{X}_{mT-} | \mathcal{F}_t] \end{aligned} \tag{2.50}$$

Taking the optimal consumption solutions into equation (2.8) gives:

$$\hat{X}_{nT-} = \frac{1}{\pi_{T-}} E[\int_T^{T'} \pi_s I(\lambda_n \pi_s) ds \mid \mathcal{F}_T] \tag{2.51}$$

$$\hat{X}_{mT-} = \max\{aS_T + b, \frac{1}{\pi_{T-}} E[\int_T^{T'} \pi_s I(\lambda_{m1} \pi_s) ds \mid \mathcal{F}_T]\} \tag{2.52}$$

Therefore,

$$S_t = \frac{\lambda_{m1}}{\lambda_n + \lambda_{m1}} (T' - t)\delta_t + \frac{\lambda_n}{\lambda_n + \lambda_{m1}} (T - t)\delta_t$$

$$
\begin{aligned}
&+\frac{1}{\pi_t}E[\pi_{T-}\max\{aS_T+b,\frac{1}{\pi_{T-}}E[\int_T^{T'}\pi_s I(\lambda_{m1}\pi_s)ds|\mathcal{F}_T]\}|\mathcal{F}_t] \\
= \quad & \frac{\lambda_{m1}}{\lambda_n+\lambda_{m1}}(T'-t)\delta_t+\frac{\lambda_n}{\lambda_n+\lambda_{m1}}(T-t)\delta_t+\frac{\lambda_n}{\lambda_n+\lambda_{m1}}(T'-T)\delta_t \\
&+\frac{1}{\pi_t}E[\pi_{T-}\max\{aS_T+b-\frac{\lambda_n}{\lambda_n+\lambda_{m1}}(T'-T)\delta_{T-},0\}|\mathcal{F}_t] \\
= \quad & \bar{S}_t+\frac{1}{\pi_t}E[\pi_{T-}\max\{aS_T+b-\frac{\lambda_n}{\lambda_n+\lambda_{m1}}(T'-T)\delta_{T-},0\}|\mathcal{F}_t] \qquad (2.53) \\
\geq \quad & \bar{S}_t
\end{aligned}
$$

### 2.3. Proof of Proposition 2.1

The expectation term in equation (2.36) is:

$$
\frac{1}{\pi_t}E[\pi_{T-}\max\{(a-\frac{\lambda_n}{\lambda_n+\lambda_{m1}})(T'-T)\delta_{T-}+b,0\}|\mathcal{F}_t] \qquad (2.54)
$$

Define an equivalent martingale $Q(A)\equiv E[z(T')1_A],\ A\in\mathcal{F}_{T'}$, where

$$
z_t \equiv exp\{-\int_0^t \theta_s dB_s-\frac{1}{2}\int_0^t \theta_s^2 ds\}=\pi_t S_t^0 \qquad (2.55)
$$

Therefore, the expectation term in equation (2.36) under measure $Q$ can be written as:

$$
\frac{S_t^0}{S_{T-}^0}E^Q[\max\{(a-\frac{\lambda_n}{\lambda_n+\lambda_{m1}})(T'-T)\delta_{T-}+b,0\}|\mathcal{F}_t] \qquad (2.56)
$$

Since the process $\delta_t$ is a Geometric Brownian Motion, the above term can be calculated similarly as Black-Scholes option prices. When $b<0$ *and* $a-\frac{\lambda_n}{\lambda_n+\lambda_{m1}}>0$, it is equivalent to Black-Scholes European Call Option Price; when $b>0$ *and* $a-\frac{\lambda_n}{\lambda_n+\lambda_{m1}}<0$, it is equivalent to Black-Scholes European Put Option Price. Summarizing the above two cases, it can be integrated explicitly as $(a-\frac{\lambda_n}{\lambda_n+\lambda_{m1}})(T'-T)N[-\frac{b}{|b|}z_1]\delta_t+be^{-r(T-t)}N[-\frac{b}{|b|}z_2]$ where $z_1$ and $z_2$ are as in Proposition 2.1.

### 2.4. Proof of Proposition 2.2

By definition $d\bar{S}_t+\delta_t dt=\bar{S}_t(\bar{\mu}_t dt+\bar{\sigma}_t dB_t)$ and $dS_t+\delta_t dt=S_t(\mu_t dt+\sigma_t dB_t), t\in[0,T)$ *and* $t\in[T,T']$. Apply Ito's lemma on the explicit solution of $\bar{S}_t$ and $S_t$, take the diffusion terms into the above equations. Matching the diffusion terms solves for the volatility $\bar{\sigma}_t$ and $\sigma_t$ respectively. Then use the relationship that $(\bar{\mu}_t-r)/\bar{\sigma}_t=\bar{\theta}_t=\sigma_\delta$ and $(\mu_t-r)/\sigma_t=\theta_t=\sigma_\delta$ to derive the risk premium $\bar{\mu}_t-r$ and $\mu_t-r$.

## 2.5. Proof of Corollary 2.1

$-e^{-r(T-t)}N[-\frac{b}{|b|}z_2] < 0$, so for $b < 0$, $[1 - \frac{be^{-r(T-t)}N[-\frac{b}{|b|}z_2]}{S_t}] > 1$ and vice versa.

## 2.6. Proof of Proposition 2.3

Applying Ito's lemma on the product $\pi_t \hat{X}_t$ gives

$$d(\pi_t \hat{X}_{it}) + \pi_t \hat{c}_{it} dt = \pi_t \hat{X}_{it}(\hat{\Phi}_{it}\sigma_t - \theta_t)dB_t \tag{2.57}$$

$i = n, m$. Then apply Ito's lemma on the products of $\pi_t \hat{X}_{nt}$ and $\pi_t \hat{X}_{mt}$, take the results into the above equation, then equaling the diffusion terms solves for $\hat{\Phi}_{nt}$ and $\hat{\Phi}_{mt}$.

## 2.7. Proof of Corollary 2.2

Same as the proof of Corollary 2.1.

# 3 Institutional Asset Pricing with Heterogenous Belief


## Abstract

We propose an equilibrium asset pricing model in which investors with heterogeneous beliefs care about relative performance. We find that the relative performance concern leads agents to trade more similarly, which has two effects. First, similar trading directly decreases volatility. Second, similar trading decreases the impact of the dominant agents. When the economy is extremely good or bad, the second effect is dominant so that the relative performance concern enlarges the excess volatility caused by heterogeneous beliefs. When the first effect is dominant, which corresponds to a normal economy, the volatility is lower than without the relative performance concern. Moreover, this paper shows that the relative performance concern also influences investors' holdings, stock prices and risk premia. [27]


## 3.1 Introduction

Fund managers care about their relative performance compared to their peer group. In the fund management industry, the compensation for the money managers could be a fixed proportion of the assets under management, or a fixed proportion plus performance-based rewards. Under fixed contract, managers care not only about the trading profit but also about the fund flows. Empirical evidence, such as in Chevalier and Ellison (1997), Sirri and Tufano (1998) and Huang, Wei and Yan (2007), shows the positive and convex relationship between the fund flows and the relative performance. In reality, fund of fund investors' decisions depend on fund managers' rankings. Under performance based contracts, peer group performance is often used as benchmark in evaluating manager performance.

In the literature of delegated portfolio management, most people focus on how the relative performance affects risk taking behaviors and the equilibrium implications of the asset prices (as discussed below). However, how relative performance could affect the trading generated from difference of opinions remains uncertain. It is difficult to solve for the asset pricing implications with endogenous fund flows. We instead assume that managers receive a bonus/penalty based on their relative performance comparing to their peers.[28] In a dynamic general equilibrium model with

[27] This paper was jointly co-authored with Mr. Shiyang Huang, Dr. Zhigang Qiu and Dr. Ke Tang. Huang is from LSE, Qiu and Tang are from Hanqing Advanced Institute of Economics and Finance, Renmin University of China.

[28] Our assumption is consistent with the observation of mutual fund managers' compensation by Ma, Tang and Gomez (2012), and takes both fixed (AUM related) and performance based contracts

heterogeneous beliefs, we analyze the effects of the relative performance concern on equilibrium quantities.

We consider a continuous time, finite horizon economy with two assets, interpreted as the risky stock and the risk-free bond, respectively. There are two groups of risk-averse agents, interpreted as fund managers, who optimally allocate their wealth between two assets to maximize their utility at the final date. Each manager in a certain group is identical; she has a CRRA utility function over both the final wealth and the relative performance compared to managers in the other group. We adopt the standard exchange economy with the Lucas (1978) type of aggregate dividends, which follow the geometric Brownian motion. The heterogeneous beliefs come from two groups of agents' different opinions about the drift process of the dividend.

We solve the model in closed form by assuming that the risk aversion coefficient is an integer. To illustrate our result and compare with the benchmark case with only heterogeneous belief but not relative performance, we focus on a special case in which the risk aversion coefficient is equal to 2. We first analyze the stock holdings, specifically, the relative performance leads agents to trade more similarly. When the relative performance is infinitely strong, both groups of agents submit the same demand. The result is the same as the economy with one representative agent whose beliefs are the average for the economy. The relative performance affects the way that two groups of agents share the final dividend, and hence affects their expectations for the final wealth. Note that the expectations are conditional on the current state of the world. When both groups of agents believe that the economy is very good, on expectation, the pessimistic group of agents with relative performance holds more shares than the agents without relative performance, while the optimistic group of agents holds fewer shares. Thus, in this case, the pessimistic group of agents has more impact compared to the benchmark case. Perceiving this, the optimistic group of agents tends to demand less relative to the benchmark case, and the pessimistic group of agents also demands less. When both groups of agents believe that they are in a very bad economy, the opposite is true. In some cases, the two groups of agents can disagree with each other regarding the status of the economy, thus the optimistic group tends to demand fewer stocks and the pessimistic demands more.

Regarding the market price of risk, we show that when the economy is good, the

into consideration.

optimistic group of agents possesses less wealth with relative performance than they do without relative performance. Therefore, although the optimistic group of agents still dominates the market, the stock is less overvalued with relative performance. Hence, the market price of risk is higher with relative performance than it is without relative performance. When the economy is bad, by a similar logic, the market price of risk is lower than it is without relative performance. Moreover, the model also indicates that the market price of risks is counter-cyclical for both groups of agents.

The stock price is also affected by the relative performance. When the economy is very good (bad), the stock price is lower (higher) with relative performance than it is without relative performance. This result is the aggregate of the stock holdings. When both groups of agents believe that the economy is good, both groups hold fewer shares relative to the benchmark case, and hence, the aggregate demand is less and the stock price is lower. When both groups of agents believe that they are in a bad economy, the opposite is true. When the two groups disagree regarding whether it is a good or a bad economy, the stock price could either be higher or lower than the price without relative performance.

The relative performance also affects the stock volatility. When the economy is normal, the volatility is smaller relative to the benchmark case; however, in the extreme economy, it is larger. Relative performance leads agents to trade similarly, which has two effects. On the one hand, it makes the agents trade more similarly, which has the direct effect of decreasing the stock volatility, and this effect is dominant on normal days; on the other hand, it decreases the impact of the dominant group of agents,[29] which is dominant in the extreme economy. As a result, the volatility is larger with relative performance than it is without relative performance in the extreme economy. This result is consistent with the scenario of financial crisis.

One application of our model regards price impact and the survival of irrational traders. This issue can be analyzed by assuming that one group of agents are rational and correct in their belief, and the other group of agents are irrational and with the wrong belief. The case without relative performance is analyzed by Kogan, Ross, Wang and Westerfield (2006) who demonstrate the range where the irrational traders can survive. In our paper, the irrational traders have a higher survival probability in the presence of relative performance because they trade more similarly to

[29] It decreases the fraction of wealth held by the optimistic (pessimistic) agent in the good (bad) economy.

the rational traders.

Our paper is closely related to the asset pricing literature with heterogeneous beliefs and delegated portfolio management. For asset pricing with heterogeneous beliefs, the general framework is by Basak (2001, 2005), in which two agents disagree with the drift of the dividend's process. Other researchers consider the framework in which one agent has the correct belief, and the other has the incorrect one, for example, Kogan, Ross, Wang and Westerfield (2006) and Yan (2008). Those papers examine the mis-pricing caused by the agent with the incorrect belief. Moreover, Scheinkman and Xiong (2003) combine the heterogeneous beliefs and the short sales constraints and show that this combination can create bubbles. Our paper combines Basak's framework with the relative performance and examines the equilibrium asset prices.

Delegated portfolio management literature is a growing field of research, which is reasonable because a large fraction of the financial assets are held by institutional investors (Allen, 2000). Therefore, it is important for us to consider how the behavior of institutions affects asset prices. In the literature, most of people consider models that have a single representative fund manager. For example, Vayanos (2004), Vayanos and Woolley (2008), and He and Krishnamurthy (2009, 2010) belong to this category. Because there is only one agent, the relative performance does not matter.

For the investigation of relative performance, researchers either use the relative performance compared to some exogenous benchmark, or the relative performance within the peer group, which is the same as is our paper. For example, Cuoco and Kaniel (2010), Shang (2008) and Basak and Pavlova (2010) consider the relative performance compared to a passive benchmark, e.g., the S&P 500. On the other hand, Kapur and Timmermann (2005), Basak and Makarov (2009, 2010) and Kaniel and Kondor (2009) consider the relative performance within a peer group of managers. All of these papers, however, consider only how the relative performance might affect the risk taking behaviors of investors. To the best of our knowledge, this paper is the first to investigate how the relative performance affects the trading behavior generated by a difference of opinions.

Some papers study the asset pricing model with asymmetric information in which the agents either know or do not know. For example, Dasgupta and Prat (2006, 2008) show that career concerns can increase uninformed trading and slow down the information revelation process, and Guerreri and Kondor (2010) show that career concerns can generate a 'reputation premium' for the bond return and hence increase the volatility of bond prices. In some sense, the relative performance concerns are similar to reputation concerns. Our paper, which is different from those, considers the case that agents either agree or disagree with their observations (i.e., heterogeneous beliefs).

Generally speaking, our paper is also related to the literature of 'social status', which considers the asset pricing implications when the investors care about the status of their wealth relative to the average of the society. For example, Bakshi and Chen (1996) examine the impact of social status on portfolio and consumption choices. In our model, two groups of agents want to beat the average (or each other) which is, in some sense, very similar to the concerns regarding social status. In Bakshi and Chen (1996), the average wealth level of the society is exogenously given, but in our model the average level is endogenous. Thus, our model can be thought of as a special case of 'social status' if we relax the assumption that the agents are fund managers. Moreover, some papers consider 'catching up with the Joneses', for example Chan and Kogan (2002), which has a similar interpretation regarding social status. Our paper thus captures some of the futures of those models.

The rest of the paper is organized as follows. We first introduce the model setup in Section 3.2, and develop a benchmark case without relative performance in Section 3.3. Section 3.4 presents a general model with relative performance. Section 3.5 shows a special case when the risk aversion coefficient equals two, and analyzes the characteristics of volatility, portfolio choices, stock prices and market prices of risks. In Section 3.6, we numerically consider more special cases as a robustness check. As an extension of this paper, Section 3.7 discusses the survivalship of irrational traders when they care about relative performance. Section 3.8 concludes.

## 3.2 The Model Setup

In this section, we first present the model setup for the economy including heterogeneous beliefs and the relative performance.

### 3.2.1 Economy

We consider a continuous time, finite horizon [0,T] economy with two assets that are risky and risk-free, respectively. We interpret the risky asset as a stock that has the following dynamics

$$\frac{dS_t}{S_t} = \mu_{s,t}dt + \sigma_{s,t}dB_t \tag{3.1}$$

where $\sigma_{s,t} > 0$ and $B_t$ is the standard Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$. Note that the Brownian motion $B_t$ is the *only* source of uncertainty in this economy. The drift $\mu_{s,t}$ and diffusion $\sigma_{s,t}$ are determined in equilibrium. The stock is in positive net supply and pays the liquidating dividend $D_T$ at time $T$. We assume $D_t$ follows a geometric Brownian motion

$$\frac{dD_t}{D_t} = \mu_D dt + \sigma_D dB_t \tag{3.2}$$

where $\mu_D$ and $\sigma_D$ are positive constants. The risk-free asset, interpreted as a bond, is in zero net supply and has a constant return $r$. For simplicity, we assume $r = 0$.

There are two groups of agents, interpreted as fund managers, in the market who optimally allocate their fund between the risky and the risk-free assets. Each group has infinite number of managers that form a continuum with measure 1. Because there are two groups, with a little abuse of notations, we use subscript of $i$ to denote the manager in group $i$, $i \in (1, 2)$[30]. Each manager $i$ invests a fraction, $\theta_{i,t}$, of her investment wealth $W_{i,t}$ on the stock. Hence, $W_{i,t}$ follows

$$dW_{i,t} = \theta_{i,t}W_{i,t}\left(\mu_{s,t}dt + \sigma_{s,t}dB_t\right) \tag{3.3}$$

We assume that the managers have the same initial endowment, which means that each manager has $W_{i,0} = \frac{S_0}{2}$ initial wealth.

### 3.2.2 Relative Performance and Objective Function

In reality, two types of compensation contracts exist for fund managers: fixed (proportional to Asset Under Management), or performance based. Under fixed contract, managers care about fund flows, which according to empirical evidence[31], depend on a fund's relative performance compared to peers. Under performance based contract, a manager's performance is compared to benchmarks that also include peer performance. Consistent with the compensation contracts in the fund management industry observed by Ma, Tang and Gomez (2012), we assume that the managers receive a time $T$ bonus or penalty that is related to their relative performance comparing to their peers. We assume that a manager receives bonus if her own type

[30] Thus, manager $i$ means an individual manager who belongs to group $i$.
[31] Such as Chevalier and Ellison (1997), Sirri and Tufano (1998) and Huang, Wei and Yan (2007).

beats the other type of managers, and suffers penalty if the other type beats her own type of managers.

We define the functional form of the bonus/penalty in the following way: first denote $W_{i,T}$ as the worth of manager $i$'s portfolio at time $T$, and $R_{i,T}$ as the aggregate return of all managers in group $i$ relative to the aggregate return of all managers in group $j$, where $i = 1$, $j = 2$ or vice versa. Manager $i$'s relative performance, $R_{i,T}$, is defined as:

$$R_{i,T} = \frac{W_{i,T}/W_{i,0}}{W_{j,T}/W_{j,0}} \tag{3.4}$$

From our previous assumption, $W_{1,0} = W_{2,0} = \frac{S_0}{2}$. Then $R_{i,T} = \frac{W_{i,T}}{W_{j,T}}$ depends only on the ratio of their performance because they start with the same initial wealth. Then define the bonus/penalty as:

$$BP_{i,T} = W_{i,T}(R_{i,T}^k - 1) \tag{3.5}$$

where $k > 1$. If $W_{i,T} > W_{j,T}$, then $R_{i,T} > 1$ and $BP_{i,T} > 0$, so that manager $i$ receives a bonus. If $W_{i,T} < W_{j,T}$, then $R_{i,T} < 1$ and $BP_{i,T} < 0$, so that manager $i$ receives a penalty. The assumption $k > 1$ ensures the bonus/penalty is increasing and convex in manager $i$'s relative performance, which is consistent with empirical results. Following this assumption, manager $i$'s wealth from investment plus bonus/penalty adds up to $W_{i,T} + BP_{i,T} = W_{i,T}R_{i,T}^k$, which is the objective in her optimization problem. Denote $f_{i,T} := (R_{i,T})^k$, then a CRRA manager $i$'s objective function[32] is then:

$$v_{i,T} = \frac{(W_{i,T}f_{i,T})^{1-\gamma}}{1-\gamma} \tag{3.6}$$

### 3.2.3 Heterogeneous Beliefs

Manager $i$ has the probability space $\left(\Omega, \mathcal{F}^i, \left\{\mathcal{F}_t^i\right\}, \mathcal{P}^i\right)$. Following the standard filtering theorem, the dividend process under fund manager $i$'s belief follows

$$\frac{dD_t}{D_t} = \mu_{i,D}dt + \sigma_D dB_{i,t} \tag{3.7}$$

By Girsanov's theorem, $dB_{i,t} = dB_t + \eta_i dt$ is the Brownian motion in manager $i$'s probability space , and $\eta_i = \frac{\mu_D - \mu_{i,D}}{\sigma_D}$. For two groups of agents, 1 and 2, equation (3.7) implies

$$dB_{2,t} = dB_{1,t} + \overline{\mu}dt \tag{3.8}$$

[32]The objective function is consistent with the catch-up with Jones utility function.

where

$$\overline{\mu} = \frac{\mu_{1,D} - \mu_{2,D}}{\sigma_D} \tag{3.9}$$

(3.9) represents the investors' disagreement on the drift of the dividend process, normalized by its diffusion term. $\overline{\mu} > 0$ implies that the agents in group 1 are more optimistic and vice versa. Given the priors of agents, $\overline{\mu}$ is an exogenous parameter.[33] Under the subjective measures of groups 1 and 2, the stock has the dynamics

$$\begin{aligned} dS_t &= S_t[\mu_{s,t}dt + \sigma_{s,t}dB_t] \\ &= S_t[\mu_{i,t}dt + \sigma_{s,t}dB_{i,t}], \quad for\ i = 1, 2 \end{aligned} \tag{3.10}$$

The two groups of agents must agree with the price, so we have the relationship between the perceived means

$$\mu_{1,t} - \mu_{2,t} = \sigma_{s,t}\overline{\mu} \tag{3.11}$$

Because the market is complete, there exists a unique state price density process, $\pi_i$, for each manager $i$

$$\frac{d\pi_{i,t}}{\pi_{i,t}} = -\kappa_{i,t}dB_{i,t} \tag{3.12}$$

where

$$\kappa_{i,t} = \frac{\mu_{i,t}}{\sigma_{s,t}} \tag{3.13}$$

is the *perceived* market price of risk (Sharpe ratio) for group 1 and 2 respectively. We also have $\kappa_1 - \kappa_2 = \overline{\mu}$ which is the measure of the disagreement between the agents' perceived market price of risk.

## 3.3 The Benchmark Case: No Relative Performance ($k = 0$)

In this section, we analyze a benchmark case model as if there is no relative performance; that is, $k = 0$. When $k = 0$, the indirect utility function, (3.6), becomes a standard CRRA utility function, so that the problem for manager $i$ becomes

$$\max E^i \left[ \frac{{W_{i,T}}^{1-\gamma}}{1-\gamma} \right] \tag{3.14}$$

$$s.t.\ \ dW_{i,t} = \theta_{i,t} W_{i,t} \left( \mu_{i,t}dt + \sigma_{s,t}dB_{i,t} \right) \tag{3.15}$$

[33] Details can be found from Basak (2004).

This problem becomes the standard model with heterogeneous beliefs (e.g., Basak 2005).[34] Solving the above problem, we show the optimal consumption and state prices at time $T$ in the following Lemma.

**Lemma 3.1.** *When $k = 0$, the final wealth for the two agents are*

$$W_{1,T}^0 = \frac{D_T}{1 + \lambda(T)^{\frac{1}{\gamma}}} \quad ; \qquad W_{2,T}^0 = \frac{\lambda(T)^{\frac{1}{\gamma}} D_T}{1 + \lambda(T)^{\frac{1}{\gamma}}} \tag{3.16}$$

*The state prices at time $T$ are*

$$\pi_{1,T}^0 = \frac{(1 + \lambda(T)^{\frac{1}{\gamma}})^\gamma}{y_1 D_T^\gamma} \quad ; \quad \pi_{2,T}^0 = \frac{(1 + \lambda(T)^{\frac{1}{\gamma}})^\gamma}{y_2 D_T^\gamma \lambda(T)} \tag{3.17}$$

*The process $\lambda(t)$ is*

$$\lambda(t) = \frac{y_1 \pi_{1,t}}{y_2 \pi_{2,t}} \tag{3.18}$$

*where $y_i$ is the Lagrange multiplier for manager $i$'s optimization problem, and $\pi_{i,t}$ is the perceived state price density for manager $i$, and $i = 1, 2$.*

*Proof.* see Appendix 3. □

The superscript 0 means no relative performance ($k = 0$). (3.16) shows that two groups of agents share the final dividend $D_T$, and the sharing rule depends on $\lambda(T)^{\frac{1}{\gamma}}$. (3.17) gives the state prices at time $T$. (3.18) shows the dynamics of $\lambda(t)$ which is the stochastic weight for the central planner's problem (Basak (2005))[35]. By Ito's Lemma, we can obtain the dynamics of $\lambda(t)$:

$$\frac{d\lambda(t)}{\lambda(t)} = -\overline{\mu} dB_{1,t} \tag{3.19}$$

$$d\frac{1}{\lambda(t)} = \frac{1}{\lambda(t)} \overline{\mu} dB_{2,t} \tag{3.20}$$

Given that the priors of two groups of agents, $\overline{\mu}$ is exogenous, (3.19) and (3.20) indicate that $\lambda(t)$ is an exogenous process. Note that there is only one uncertainty $B_t$ in the economy; from (3.19) one can see that $\lambda(t)$ has a one-to-one relationship with $B_t$, and hence $\lambda(t)$ can represent the status of the economy. In particular, $\lambda(t)$ is the opposite of the status of economy; for example, when the economy is good, $B_t$ has a large positive value (i.e., the stock price is high), while $\lambda(t)$ has a rather

[34] However, in the model, agents only consume at time $T$ which is different to Basak (2005) in which agents consume continuously.

[35] The central planner's problem is $\max_{c_1 + c_2 = c} u_1(c_1) + \lambda(t) u_2(c_2)$.

small value.

Calibrating the equilibrium requires the explicit expression of state price density, $\pi_{i,t}$, which can be calculated as $\pi_{i,t} = E_t^i(\pi_{i,T})$ by its martingale property. However, the difficulty in calculating the expectation is the term $(1+\lambda(T)^{\frac{1}{\gamma}})^{\gamma}$, which can be solved as

$$\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(T)^{\frac{i}{\gamma}} \tag{3.21}$$

when $\gamma$ is an integer. Thus, we assume that $\gamma$ is an integer and solve the equilibrium in the following Proposition.

**Proposition 3.1.** *When $\gamma$ is an integer, the state prices are*

$$\begin{aligned} \pi_{1,t}^0 &= \frac{1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i}{\gamma}} e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i}{\gamma} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i}{\gamma}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)} \\ \pi_{2,t}^0 &= \frac{1}{y_2 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i-\gamma}{\gamma}} e^{\left[\frac{\overline{\mu}^2}{2}\frac{i-\gamma}{\gamma} - \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i-\gamma}{\gamma}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)} \end{aligned}$$

*The market prices of risk are*

$$\begin{aligned} \kappa_{1,t}^0 &= \gamma\sigma_D + \delta_{1,t}^0\overline{\mu} \\ \kappa_{2,t}^0 &= \gamma\sigma_D - \delta_{2,t}^0\overline{\mu} \end{aligned}$$

*where $\delta_{1,t}^0$ and $\delta_{2,t}^0$ are two functions of $\lambda(t)$.*
*The stock shares for the two groups of agents at $t$ are:*

$$\begin{aligned} \theta_{1,t}^0 &= \frac{\mu_{1,t}}{\gamma\sigma_{s,t}^2} + \frac{1 - \frac{1}{\gamma}\delta_{1,t}^0}{\sigma_{s,t}}\overline{\mu} - \frac{\beta_{1,t}^0}{\sigma_{s,t}}\overline{\mu} \\ \theta_{2,t}^0 &= \frac{\mu_{2,t}}{\gamma\sigma_{s,t}^2} - \frac{1 - \frac{1}{\gamma}\delta_{2,t}^0}{\sigma_{s,t}}\overline{\mu} + \frac{\beta_{2,t}^0}{\sigma_{s,t}}\overline{\mu} \end{aligned}$$

*where $\beta_{1,t}^0$ and $\beta_{2,t}^0$ are two functions of $\lambda(t)$..*

This proposition gives us the benchmark case without concerns about relative performance, and all of our results will be compared to this benchmark. Given the state prices, we can easily calculate the stock price, $S_t^0$, and the volatility, $\sigma_{s,t}^0$, which can be found in Appendix 3 (when $k = 0$).

## 3.4 The Model with Relative Performance

In this section, we solve the model with relative performance ($k > 1$) and compare the equilibrium to the benchmark case. Given the indirect utility function, (3.6) , for each agent $i$, the optimization problem is

$$\max E^i \left[ \frac{(W_{i,T} f_{i,T})^{1-\gamma}}{1-\gamma} \right] \tag{3.22}$$

$$s.t. \;\; dW_{i,t} = \theta_{i,t} W_{i,t} \left( \mu_{i,t} dt + \sigma_{s,t} dB_{i,t} \right) \tag{3.23}$$

By the standard martingale approach (Cox and Huang 1989), manager $i$'s optimization problem is static

$$\max E^i \left[ \frac{(W_{i,T} f_{i,T})^{1-\gamma}}{1-\gamma} \right] \tag{3.24}$$

$$s.t. E^i \left[ \pi_{i,T} W_{i,T} \right] = \frac{S_0}{2} \tag{3.25}$$

Solving (3.25), we have the following Lemma.

**Lemma 3.2.** *There is an unique equilibrium, where*

$$\widehat{W_{i,T}} = (y_i \pi_{i,T})^{-\frac{1}{\gamma}} (f_{i,T})^{\frac{1-\gamma}{\gamma}} \tag{3.26}$$

*where* $\widehat{W_{i,T}}$ *is the optimal final wealth for individual manager* $i$.

*Proof.* see Appendix 3. □

Note that because there is infinite number of managers in group $i$, each manager group $i$ takes $f_{i,T}$ as given; hence, this equilibrium belongs to a competitive equilibrium. By the market clearing condition, $W_{1,T} + W_{2,T} = D_T$, we can solve the final wealth of each agent in **Lemma 3.3**.

**Lemma 3.3.** *At time* $T$*, two agents share the final dividend* $D_T$

$$W_{1,T} = \frac{D_T}{1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}}} \quad ; \qquad W_{2,T} = \frac{\lambda(T)^{\frac{1}{\widehat{\gamma}}} D_T}{1 + \lambda(T)^{\frac{1}{\widehat{\gamma}}}} \tag{3.27}$$

*where* $\widehat{\gamma} = \gamma + 2k(\gamma - 1)$.

*Proof.* see Appendix 3. □

Compared to the results in Lemma 1, two groups of managers still share the final dividend $D_T$. However, the sharing rule now depends on $\lambda(T)^{\frac{1}{\widehat{\gamma}}}$ instead of $\lambda(T)^{\frac{1}{\gamma}}$.

$\widehat{\gamma}$ is a function of $k$ so that the relative performance affects the fraction of the final dividend that is shared by the two groups. By choosing different values of $\gamma$, we have the following Lemma.

**Lemma 3.4.** *When $\gamma = 1$ and $\widehat{\gamma} = \gamma$, the relative performance has no effect; for $\gamma > 1$, $\widehat{\gamma} > \gamma$, we then have*
*when $\lambda(T)$ is small enough, $W_{1,T} < W_{1,T}^0$ and $W_{2,T} > W_{2,T}^0$;*
*when $\lambda(T)$ is large enough, $W_{1,T} > W_{1,T}^0$ and $W_{2,T} < W_{2,T}^0$.*

The case of $\gamma = 1$ refers to the log utility and the relative performance does not matter in this case. When $\gamma > 1$, we have two scenarios that are conditional on the realizations of $\lambda(T)$. As mentioned before, a small $\lambda(T)$ corresponds to a good economy and a large $\lambda(T)$ corresponds to a bad economy. The results show that in a very good economy, the wealth of the optimistic group is lower than it is in the benchmark case, and in the very bad economy, the opposite is true. Note that the optimistic group of agents is dominant in the very good economy, and the pessimistic group of agents is dominant in the very bad economy. We can then draw the conclusion that the relative performance decreases the impact of the dominant group of agents in the extreme economy.

**Lemma 3.5.** *The state price densities at time $T$ are*

$$\pi_{1,T} = \frac{(k+1)}{y_1} \frac{(1+\lambda(T)^{\frac{1}{\widehat{\gamma}}})^{\gamma}}{D_T^{\gamma}} \lambda(T)^{\frac{\theta(\overline{\gamma}-1)}{\widehat{\gamma}}} \tag{3.28}$$

$$\pi_{2,T} = \frac{(k+1)}{y_2} \frac{(1+\lambda(T)^{\frac{1}{\widehat{\gamma}}})^{\gamma}}{D_T^{\gamma}} \lambda(T)^{-\frac{\theta(\overline{\gamma}-1)+\gamma}{\widehat{\gamma}}} \tag{3.29}$$

*Proof.* see Appendix 3. □

Similar to the benchmark case in the last section, the term $(1+\lambda(T)^{\frac{1}{\widehat{\gamma}}})^{\gamma}$ can be expressed as $\sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(T)^{\frac{i}{\widehat{\gamma}}}$ when $\gamma$ is an integer. We then provide the equilibrium in the following Proposition.

**Proposition 3.2.** *When $\gamma$ is an integer, the state prices are:*

$$\pi_{1,t} = \frac{k+1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)}$$

$$\pi_{2,t} = \frac{k+1}{y_2 D_t^{\gamma}} \sum_{i=0}^{\gamma} \binom{\gamma}{i} \lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}} e^{\left[\frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) + \frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2\right](T-t)}$$

*The market prices of risk are*

$$\begin{aligned}\kappa_{1,t} &= \gamma\sigma_D + \delta_{1,t}\overline{\mu} \\ \kappa_{2,t} &= \gamma\sigma_D - \delta_{2,t}\overline{\mu}.\end{aligned}$$

*The stock holdings of the two groups of agents at t are*

$$\begin{aligned}\theta_{1,t} &= \frac{\mu_{1,t}}{\gamma\sigma_{s,t}^2} + \frac{1-\frac{1}{\gamma}\delta_{1,t}}{\sigma_{s,t}}\overline{\mu} - \frac{\beta_{1,t}}{\sigma_{s,t}}\overline{\mu} \\ \theta_{2,t} &= \frac{\mu_{2,t}}{\gamma\sigma_{s,t}^2} - \frac{1-\frac{1}{\gamma}\delta_{2,t}}{\sigma_{s,t}}\overline{\mu} + \frac{\beta_{2,t}}{\sigma_{s,t}}\overline{\mu}\end{aligned}$$

*The stock price is*

$$S_t = \frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma \\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\right](T-t)} D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma \\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left\{-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\right\}(T-t)}} \tag{3.30}$$

*and the volatility is*

$$\sigma_{s,t} = \sigma_D + K_t\overline{\mu} \tag{3.31}$$

*Note that $\delta_{1,t}$, $\delta_{2,t}$, $\beta_{1,t}$, $\beta_{2,t}$ and $K_t$ are shown in Appendix 3.*

*Proof.* see Appendix 3. □

Compared to the results in **Proposition 3.1**, we can see that all of the equilibrium quantities are affected by the relative performance $k$. The relative performance affects $\beta_{1,t}$, $\beta_{2,t}$, $\delta_{1,t}$ , $\delta_{2,t}$ and $K_t$ in the stock holdings, the Sharpe ratio and the volatility. These parameters are all at play through the disagreement parameter, $\overline{\mu}$. Thus, the relative performance affects those quantities that are generated by the difference of opinions.

To analyze the effects of the relative performance, we consider a special case with $\gamma = 2$ as an example, where the equilibrium can be analyzed in more detail.[36] For a robustness check, in Section 3.6, we also analyze those cases when $\gamma = 3, 4$.

[36]The approach of using an integer for the risk aversion coefficient is the same as Yan (2008) who uses numerical simulation to analyze the equilibrium. Rather than doing the numerical study, we choose a special case with $\gamma = 2$.

## 3.5 Special Case: $\gamma = 2$

In this section, we solve the equilibrium by choosing $\gamma = 2$. The purpose of this section is to compare the equilibrium with relative performance to that without relative performance. Because we can solve everything in closed form, the comparative statics are also analyzed in this section.

### 3.5.1 Stock Holdings

The following proposition shows the portfolio choices for the managers.

**Proposition 3.3.** *When $\gamma = 2$, the stock holdings of the two groups of agents at $t$ are*

$$\theta_{1,t} = \frac{\mu_{1,t}}{2\sigma_{s,t}^2} + \frac{1 - \frac{1}{2}\delta_{1,t}}{\sigma_{s,t}}\overline{\mu} - \frac{\beta_{1,t}}{\sigma_{s,t}}\overline{\mu} \tag{3.32}$$

$$\theta_{2,t} = \frac{\mu_{2,t}}{2\sigma_{s,t}^2} - \frac{1 - \frac{1}{2}\delta_{2,t}}{\sigma_{s,t}}\overline{\mu} + \frac{\beta_{2,t}}{\sigma_{s,t}}\overline{\mu} \tag{3.33}$$

*$\beta_{1,t}$ and $\beta_{2,t}$ are functions of $k$, which can be found in Appendix 3.*

*Proof.* The proof is in see Appendix 3. □

From (3.32) and (3.33), the optimal stock holdings consist of three terms. The first terms is the traditional Merton (1971) myopic demand without heterogeneous beliefs.[37] The second and the third terms are both hedging demand, but the $\frac{1-\frac{1}{\gamma}\delta_{i,t}}{\sigma_{s,t}}\overline{\mu}$ term is for hedging against variation in market price of risk $\kappa_{i,t}$ caused by heterogeneous belief, hereafter called the variation hedging demand; while the $\frac{\beta_{i,t}}{\sigma_{s,t}}\overline{\mu}$ term is for hedging against the heterogeneous belief itself, hereafter called the heterogeneity hedging demand. From (3.27), we can see that the two groups of agents share the final dividend $D_T$, and the fraction depends on $\lambda(T)^{\frac{1}{\gamma}}$. Given the realization of different states, the agents have state dependent shares of wealth. For example, the optimistic group of agents has larger fraction of wealth than the pessimistic group when the economy is good. For this reason, the additional uncertainty originating from different opinions generates heterogeneity demand. Note that the parameter of the relative performance, $k$, affects $\delta_{i,t}$, $\beta_{i,t}$ and $\sigma_{s,t}$, hence affects both the variation hedging demand and the heterogeneity hedging demand.

[37] Without heterogeneous demand, $\overline{\mu} = 0$, then risk premium and volatility are both constants, so is the myopic demand.

### 3.5.2 Comparison to the Benchmark Case

To analyze the effect of relative performance, we need to compare $\beta_{i,t}$ to the benchmark case.

**Proposition 3.4.** *The following relationships hold:*

$$\theta_{1,t} - \theta_{2,t} = \overline{\mu}\left[1 - (\beta_{1,t} + \beta_{2,t})\right] \tag{3.34}$$

$$\beta_{1,t} + \beta_{2,t} - (\beta^0_{1,t} + \beta^0_{2,t}) > 0 \tag{3.35}$$

*moreover,*

$$\frac{d\left(\beta_{1,t} + \beta_{2,t}\right)}{dk} > 0 \tag{3.36}$$

*Proof.* see Appendix 3. □

(3.34) shows that the difference in the two groups of agents' stock holdings only depends on the *beta*s, and is decreasing in $\beta_{1,t}+\beta_{2,t}$, and (3.35) shows that $\beta_{1,t}+\beta_{2,t}$ is greater than in the benchmark case. Thus, with relative performance, two agents trade more *similarly* than they do without relative performance. (3.36) shows that the more important the relative performance is, the more similarly the managers trade. The following corollary shows the case when the relative performance is infinitely strong ($k \to \infty$).

**Corollary 3.1.** *The difference between two demands goes to zero when $k \to \infty$.*

*Proof.* One can show that both $\beta_{1,t}$ and $\beta_{2,t}$ are smaller than $\frac{1}{2}$. Thus, given (3.36), we have the above corollary. □

Intuitively, when concerns of the relative performance are infinitely strong, the difference of opinions goes to zero; hence, the two groups of agents trade like one group. We also show how the heterogeneity hedging demand of each manager changes with respect to the relative performance in the following proposition.

**Proposition 3.5.** *For both agents, there exist cutoffs, $g_{c1} < g_{c2}$*

$$\begin{aligned}
&Case1 \;: when \quad \lambda(t) < g_{c1} \;;\; \beta_{1,t} > \beta^0_{1,t} \;,\; \beta_{2,t} < \beta^0_{2,t} \\
&Case2 \;: when \quad \lambda(t) > g_{c2} \;;\; \beta_{1,t} < \beta^0_{1,t} \;,\; \beta_{2,t} > \beta^0_{2,t} \\
&Case3 \;: when \quad g_{c1} < \lambda(t) < g_{c2} \;;\; \beta_{1,t} > \beta^0_{1,t} \;,\; \beta_{2,t} > \beta^0_{2,t}
\end{aligned}$$

*Proof.* see Appendix 3. □

We compare the heterogeneity hedging demand with and without the relative performance conditional on $\lambda(t)$. The results are intuitive because the realization of $\lambda(T)$ determines the fraction of wealth allocated to each agent, which is shown by (3.27). We use the following figure to illustrate the three cases in the proposition.

Figure 3.1: *The difference of $\beta$ with relative performance to that without relative performance.*

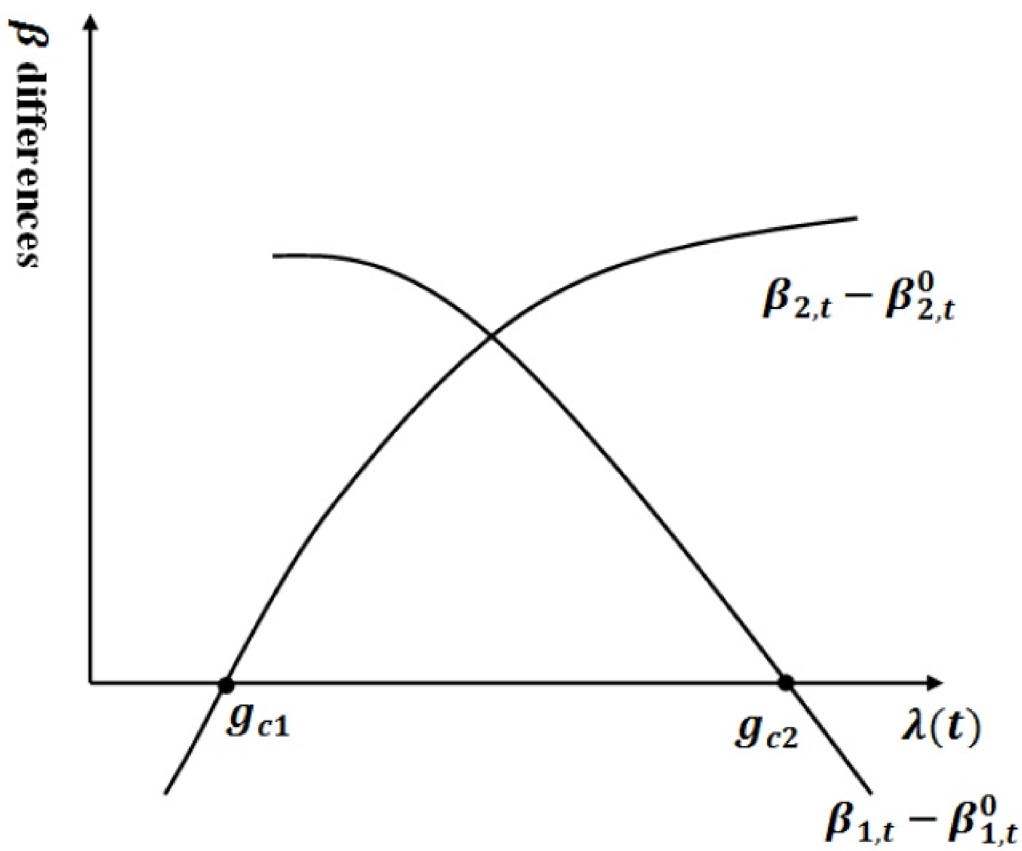



**Figure 3.1** gives the graphical illustration of the proposition. It shows how the difference of heterogeneity hedging demands with and without relative performance changes with respect to $\lambda(t)$. We discuss each case separately.

Case 1 indicates the situation in which both groups of agents believe that the economy is good. The reason is shown in the following. Compared to the case without relative performance, the heterogeneity hedging demand of the optimistic (pessimistic) agent is higher (lower). From the results of **Lemma 3.5**, when $\lambda(T)$ is small, $W_{1,T} < W^0_{1,T}$ and $W_{2,T} > W^0_{2,T}$. Given that $\lambda(t)$ is small, the possibility that $W_{1,T} < W^0_{1,T}$ and $W_{2,T} > W^0_{2,T}$ is high. On expectation, the pessimistic group of agents will have the larger share of wealth than in the case without relative performance. Consequently, the optimistic (pessimistic) group of agents will have a lesser (greater) fraction of wealth so that she needs to have a larger heterogeneity hedging demand.

Case 2 indicates the situation in which both groups of agents believe that the economy is bad. Following the same logic as in case 1, given that $\lambda(t)$ is large, the possibility that $W_{1,T} > W^0_{1,T}$ and $W_{2,T} < W^0_{2,T}$ is high. The optimistic group of agents will end up with higher fraction of wealth and will hence have less heterogeneity hedging demand.

Note that the definition of a 'good economy' and a 'bad economy'is subjective considering the two types of investors' beliefs. In case 3, the optimistic group of agents believes that the economy is 'good', and the pessimistic group of agents believes that it is 'bad'.[38] Thus, regarding the expectations over the subjective belief, the possibility that $W_{1,T} < W^0_{1,T}$ and $W_{2,T} < W^0_{2,T}$ is high for the optimistic and the pessimistic groups of agents, respectively. In this range, both groups of agents have a higher heterogeneity hedging demand.

### 3.5.3 Market Price of Risk (Sharpe Ratio)

The following proposition shows the Sharpe ratios with/without relative performance.

**Proposition 3.6.** *When $\gamma = 2$, the market prices of risk are*

$$\kappa_{1,t} = 2\sigma_D + a_k\overline{\mu} \;\; ; \kappa_{2,t} = 2\sigma_D - (1 - a_k)\overline{\mu} \tag{3.37}$$

*where $a_k$ is a function of $k$ and is shown in Appendix 3. Moreover, the market prices of risk in the benchmark case are:*

$$\kappa^0_{1,t} = 2\sigma_D + a_0\overline{\mu} \;\; ; \;\; \kappa^0_{2,t} = 2\sigma_D - (1 - a_0)\overline{\mu} \tag{3.38}$$

[38] There is no other possibility (e.g. the pessimistic agent believes the economy is good, and optimistic one believes bad.) given the priors of two agents because the optimistic agent, by definition, is always more 'optimistic' than the pessimistic one.

*Then we have*

$$\begin{aligned} &Case1 \ \ : when \qquad \lambda(t) < \exp[-2\overline{\mu}\sigma_D\,(T-t)] \ ,a_k > a_0 \ , and \ \frac{\partial a_k}{\partial k} > 0 \\ &Case2 \ \ : when \qquad \lambda(t) > \exp[-2\overline{\mu}\sigma_D\,(T-t)] \ ,a_k < a_0 \ , and \ \frac{\partial a_k}{\partial k} < 0 \end{aligned}$$

*Proof.* see Appendix 3. □

(3.37) shows that some risk is actually transferred from the pessimistic agents to the optimistic agents because $\kappa_{1,t} - \kappa_{2,t} = \overline{\mu}$. This result is standard for asset pricing with heterogeneous beliefs. Given that $a_k$ is a function of $k$, we know that the transferred risk is affected by the relative performance. (3.38) gives Sharpe ratios without relative performance ($k = 0$), so the analysis depends on the comparison between $a_k$ and $a_0$, which is shown in the two cases of the proposition.

Similar to the analysis of the stock holdings, case 1 corresponds to a good economy. We show that the market price of risk with relative performance is higher than without relative performance, and the more important the relative performance, the higher the market price of risk. In case 2 (a bad economy), the market price of risk with relative performance is smaller than it is without relative performance, and the more important the relative performance, the smaller the market price of risk. We have above results because: when the economy is good, the optimistic group of agents possesses less wealth with relative performance than they do without relative performance. Although the optimistic group of agents still dominates the market, the stock is less overvalued with relative performance. Hence, the market price of risk is higher with relative performance than it is without relative performance. When the economy is bad, by a similar logic, the market price of risk is smaller with relative performance than it is without relative performance.

**Proposition 3.7.** *The first order derivative of $\kappa_i$ on $\lambda$ is positive for both investors, i.e.*

$$\frac{d\kappa_{i,t}}{d\lambda(t)} > 0 \quad , \qquad i = 1, 2 \tag{3.39}$$

*Proof.* see Appendix 3. □

(3.39) shows that Sharpe ratios of both groups of agents are *counter-cyclical.* Intuitively, when the market is good, the optimistic agents dominate the market so that the stock is overvalued. The excess return is lower, hence the Sharpe ratio is lower.

### 3.5.4 Stock Price and Volatility

In this section, we solve the equilibrium price and the volatility and compare these to the benchmark case.

**Proposition 3.8.** *The stock price is*

$$S_t = \frac{\left\{1 + \lambda(t)^{\frac{1}{2+2k}} 2e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)} + \lambda(t)^{\frac{2}{2+2k}} e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)}\right\} D_t e^{\left(\mu_1 - 2\sigma_D^2\right)(T-t)}}{\left\{1 + 2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D \frac{1}{1+k}\right)(T-t)} + \lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D \frac{2}{1+k}\right)(T-t)}\right\} e^{\left(\frac{1}{2}\overline{\mu}\sigma_D \frac{k}{1+k}\right)(T-t)}} \tag{3.40}$$

*Denote $S_t^0$ as the stock price when $k = 0$; then we have*

$$\begin{aligned} Case1 &: \quad when\ \lambda(t) \to 0,\ S_t < S_t^0 \\ Case2 &: \quad when\ \lambda(t) \to \infty,\ S_t > S_t^0 \end{aligned}$$

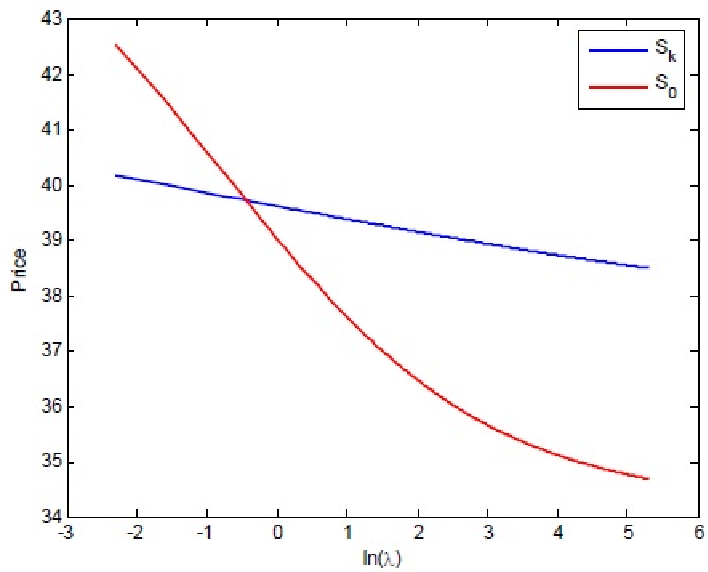
*Proof.* see Appendix 3. □

(3.40) shows the expression of the stock price, and we compare it to the benchmark case price, $S_t^0$, in two extreme cases. Case 1 and case 2 depend on the process $\lambda(t)$, so we have a similar interpretation to that of the stock holdings. In case 1, $\lambda(t) \to 0$, so we interpret it as the extremely good economy. We show that the stock price is lower with the relative performance than without the relative performance. Case 2 is interpreted as the extremely bad economy, and the stock price is higher with the relative performance than without.

When $\lambda(t)$ is small, the aggregate demands are lower than in the benchmark case. When $\lambda(t)$ is large, the aggregate demands are higher. Given that the stock has a fixed supply, the price is lower in case 1 and higher in case 2 relative to the benchmark case. **Figure 3.2** explains this proposition.

From the graph below and the proposition, we can see that when $\lambda(t)$ is very large (a very bad economy), the stock price is higher than the benchmark case. When $\lambda(t)$ is very small (a very good economy), the stock price is lower. For the middle range of $\lambda(t)$, the stock price can be either higher or lower than it is in the benchmark case. Note that the middle range corresponds to case 3 in proposition 5 in which both agents disagree with the 'good' or the 'bad' economies.

Figure 3.2: *The comparison of stock prices with and without relative performance for different* $\lambda$*. The model parameters are* $\sigma_D = 0.3$*,* $\overline{\mu} = 1$*,* $k = 1.5$




**Corollary 3.2.** *When* $k \to \infty$*,* $S_t = D_t e^{\left(\frac{\mu_1+\mu_2}{2} - 2\sigma_D^2\right)(T-t)}$*.*

This corollary is an extension of **Corollary 3.1**. When $k \to \infty$, we know that, from **Corollary 3.1**, both groups of agents submit the same demand so that the economy is the same as the economy with one representative investor. Moreover, this investor has the average belief, $\frac{\mu_1+\mu_2}{2}$, regarding the dividend process. As a result, we have the stock price in the corollary.

**Proposition 3.9.** *The volatility is*

$$\sigma_{s,t} = \sigma_D + \frac{1}{1+k} \left\{ \begin{array}{l} \left[ \dfrac{e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D \frac{1}{1+k}\right)(T-t)} \lambda(t)^{\frac{1}{2}\frac{1}{1+k}} + e^{\left(\overline{\mu}\sigma_D \frac{2}{1+k}\right)(T-t)} \lambda(t)^{\frac{1}{1+k}}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D \frac{1}{1+k}\right)(T-t)} + e^{\left(\overline{\mu}\sigma_D \frac{2}{1+k}\right)(T-t)} \lambda(t)^{\frac{1}{1+k}}} \right] \\ - \left[ \dfrac{e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)} \lambda(t)^{\frac{1}{2}\frac{1}{1+k}} + e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)} \lambda(t)^{\frac{1}{1+k}}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2} + \frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)} + e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)} \lambda(t)^{\frac{1}{1+k}}} \right] \end{array} \right\} \overline{\mu} \tag{3.41}$$

*Comparing the volatility with relative performance,* $\sigma_{s,t}$*, to the benchmark case* $\sigma^0_{s,t}$*, there exist two cutoffs,* $d_{c1}$ *and* $d_{c2}$*, where* $d_{c1} < d_{c2}$*.*

$$Case1 \ : When \quad d_{c1} < \lambda(t) < d_{c2} \ , \ \sigma_{s,t} < \sigma^0_{s,t}$$

$$Case2 \ : When \qquad \lambda(t) < d_{c1} \ \ or \ \lambda(t) > d_{c2} \ , \ \sigma_{s,t} > \sigma^0_{s,t}$$

*Moreover, the larger the $k$ is, the larger the $d_{c2}-\ d_{c1}$ is.*

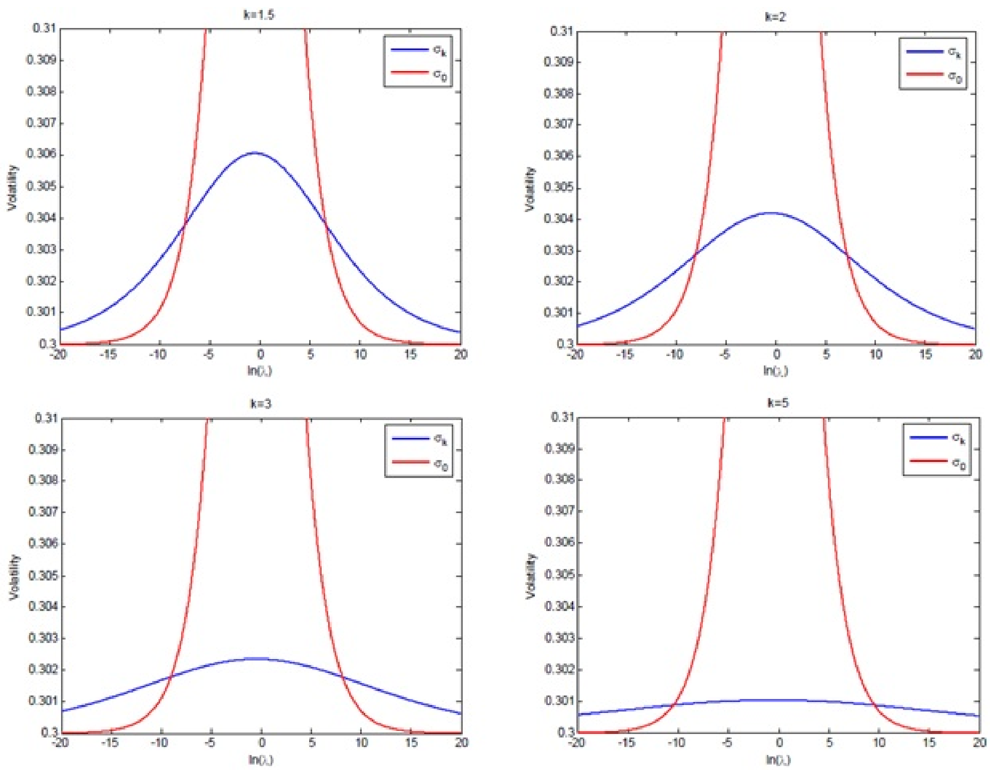
*Proof.* see Appendix 3. □

From (3.41), the volatility with relative performance is greater than $\sigma_D$. That is, the difference of opinions generates excess volatility. In case 1, $\lambda(t)$ has upper and lower bounds so that we interpret it as normal days (the economy is not very good or very bad). We show that the volatility is smaller with relative performance than the volatility without relative performance. Case 2 indicates an extreme economy (very good or very bad); we show that the volatility is larger with relative performance than that without relative performance. Moreover, the stronger the relative performance is (the large $k$), the wider the range of the normal days is. **Figure 3.3** depicts the numerical simulations.

The relative performance leads the two groups of agents to trade more similarly, which has two effects on the volatility. First, it makes managers trade more similarly to each other, hence, it has the direct effect of decreasing the volatility. This effect is dominant in the middle range of the economy. Second, it decreases the wealth (and impact) of the dominant group of agents, which in turn increases the volatility. For example, when the economy is extremely good, the optimistic agents dominate the market. We can imagine that the pessimistic agents are driven out of the market when they lose a lot of money. Thus, only the optimistic agents survive in the market. As a result, the difference in opinions is not reflected on the market, hence no excess volatility. However, with concerns regarding relative performance, the pessimistic agents trade more like the optimistic agents, so they can stay in the market even when the economy is extremely good[39]. The existence of pessimistic agents exaggerates the effects caused by the difference of opinions and hence results in excess volatility.

[39]The similar story holds for the extremely bad economy and the pessimistic agent.

Figure 3.3: *Comparison of volatilities with and without relative performance. Volatility vs. the value of* $\lambda(t)$. $\sigma_k$ *and* $\sigma_0$ *denote the volatility with and without relative performance, respectively. Choose different* $k$ *for different graphs. Other model parameters are* $\sigma_D = 0.3$, $\overline{\mu} = 1$




Overall, in normal days, the first effect dominates the second one so that the volatility with relative performance is smaller. In extreme cases, the second effect dominates the first so that the volatility is larger. The stronger the concerns regarding relative performance, the more similarly the agents trade; as a result, the range of normal days becomes wider. This result is shown by the changes in the middle range (increasing) from the first to the fourth graph in **Figure 3.3**. However, when $k$ goes to infinity, we have the one agent economy again which is shown in the following corollary.

**Corollary 3.3.** *When* $k \to \infty$, $\sigma_{s,t} \to \sigma_D$.

It is easy to see that, in (3.41), the expression in the bracket is between 0 and 1 so that when $k \to \infty$, $\sigma_{s,t} \to \sigma_D$. The intuition is similar to those in **Corollary 3.1** and **Corollary 3.2**. When the relative performance is infinitely strong, we have the representative agent economy.

### 3.6 More Special Cases

In Section 3.5, we use one special case with $\gamma = 2$ to illustrate our general model. However, to show that our general model works for more cases, we do some numerical studies using different risk aversion parameters. In particular, we use the results in **Proposition 3.2** (the general case) by choosing $\gamma = 3$ and 4 and simulating the volatilities in the different cases.

Note that we use the volatility as the embodiment of more cases because it can best illustrate the theory in a normal and an extreme economy. Similar to the special case when $\gamma = 2$, the volatility with relative performance is smaller than the volatility without relative performance on normal days. The result reflects the smaller difference of opinions that is dominant on normal days. However, in the extreme economy, the wealth decrease for the dominant agents is substantial.

Figure 3.4: *Comparison of volatilities with and without relative performance for $\gamma = 3$. $\sigma_k$ and $\sigma_0$ denote the volatility with and without relative performance, respectively. Other model parameters are $k = 1.5$, $\sigma_D = 0.3$, $\overline{\mu} = 1$.*

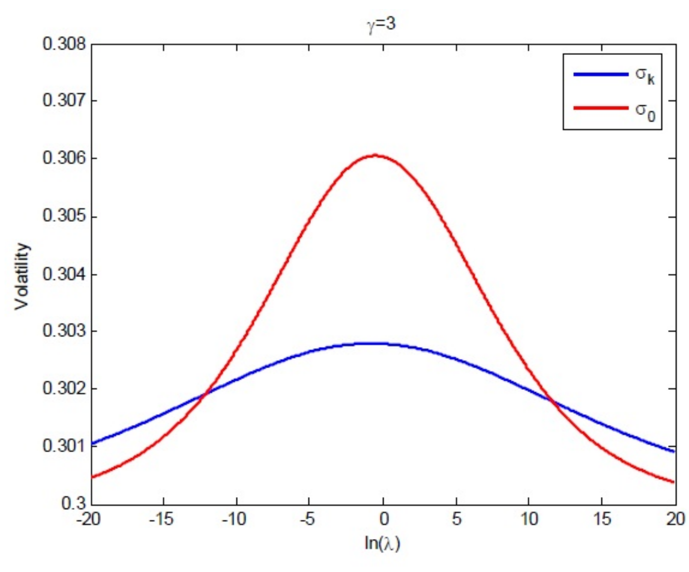
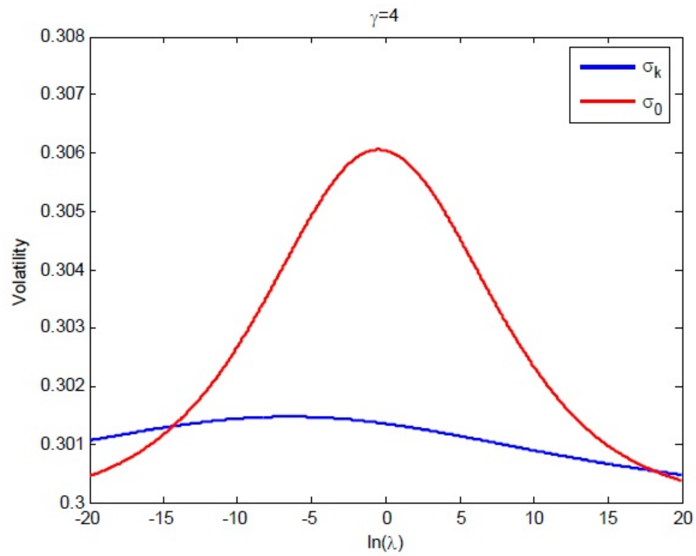


Figure 3.5: *Comparison of volatilities with and without relative performance for* $\gamma = 4$. $\sigma_k$ *and* $\sigma_0$ *denote the volatility with and without relative performance, respectively. Other model parameters are* $k = 1.5$, $\sigma_D = 0.3$, $\overline{\mu} = 1$.




### 3.7 The Irrational Traders' Survivalship

In this section, we discuss one important implication for our model, i.e., the survival of irrational traders in the long run. Without a loss of generality, we suppose that the first group of agents has a rational belief and is always right about the economy. The second group has an irrational belief, but both groups of traders care about their performance relative to each other. We relax our assumption that $\overline{\mu} > 0$ so that the irrational traders can be either optimistic or pessimistic. Importantly, $\overline{\mu}$ represents the *extent of the wrong opinions* that the irrational traders hold. Intuitively, the larger the absolute value of $\overline{\mu}$, the more unlikely that the irrational traders will survive in the long run. In this paper, we define the extinction of a group of traders in the long run if

$$\lim_{T \to \infty} \frac{W_{2,T}}{W_{1,T}} = 0.^{40} \tag{3.42}$$

From the competitive equilibrium derived in Section 3.4, we have the following Proposition.

[40]For further discussion of this definition, please refer to Kogan, Ross, Wang and Westerfield (2006).

**Proposition 3.10.** *Define* $\overline{\mu}^* := -2\sigma_D(\widehat{\gamma}-1)$*, where* $\widehat{\gamma} = \gamma + 2k(\gamma - 1)$*. For* $\gamma > 1$ *and* $\overline{\mu} \neq \overline{\mu}^*$*, only one of the traders survives in the long run. In particular, we have:*
$\overline{\mu} > 0$*, pessimistic irrational trader* $\Rightarrow$*Rational trader survives*
$\overline{\mu}^* < \overline{\mu} < 0$*, moderately optimistic irrational trader* $\Rightarrow$*Irrational trader survives*
$\overline{\mu} < \overline{\mu}^*$*, strongly optimistic irrational trader* $\Rightarrow$*Rational trader survives*

*Proof.* see Appendix 3. □

Note that for $\overline{\mu} = \overline{\mu}^*$, both the rational and the irrational traders survive. This proposition identifies three distinct regions regarding the extent of the wrong opinions $\overline{\mu}$. In particular, the range in which the irrational trader survives depends on $\overline{\mu}^*$. Note that in the benchmark case ($k = 0$ ), $\overline{\mu}^* = -2\sigma_D(\gamma - 1)$.

**Corollary 3.4.** *The range of* $\overline{\mu}$ *for the irrational trader to survive* $(\overline{\mu}^*, 0)$ *is larger in the case of relative performance than it is without relative performance.*

*Proof.* By the expression of $\widehat{\gamma}$ and **Proposition 3.10**, we can easily get the result. □

The above Corollary is consistent with our results, i.e., because both types of traders care about their relative performance and hence they trade more similarly, the group of the irrational traders has a higher probability of survival.

## 3.8 Conclusion

This paper studies an equilibrium asset pricing model in which institutional investors with heterogeneous beliefs care about their relative performance. We focus on the investor's stock holdings, the asset prices, the volatility and the market price of risk. Relative performance leads the agents to trade more similarly. On the one hand, this similarity lowers the stock volatility. On the other hand, relative performance decreases the wealth of the dominant agents and increases the stock volatility. Combining the two effects, in this paper, we show that the volatility is smaller with relative performance than that without relative performance in normal economy; and larger in the extreme economy. The asset price is lower with relative performance than without relative performance when the economy is extremely good; it is higher when the economy is extremely bad. The model shows that the stock holdings for both groups of agents can be decomposed into myopic demand, variation hedging demand and heterogeneity hedging demand. The heterogeneity hedging demand is influenced by the relative performance. When the economy is good, the market risk premium is higher relative to the case without relative performance; it is lower when the economy is bad. As an application of our model, we show that irrational traders tend to have a higher survival probability with relative performance concerns because they tend to trade similarly to the rational traders.

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## 3.10 Appendix 3: Proofs of Lemmas and Propositions

### 3.1. Proof of Lemma 3.1, 3.2, 3.3, 3.5

The Lagrangian for (3.25) is:

$$E^i\left[\frac{[W_{i,T}f_{i,T}]^{1-\gamma}}{1-\gamma}\right]+y_i\left[\frac{S_0}{2}-E^i\left(\pi_{i,t}W_{i,T}\right)\right] \tag{3.43}$$

By FOC, we have $W_{i,T}^{-\gamma}(f_{i,T})^{1-\gamma}=(y_i\pi_{i,T})$, so in equilibrium, we will have:

$$\widehat{W_{1,T}}=(y_1\pi_{1,T})^{-\frac{1}{\gamma}}(f_{1,T})^{\frac{1-\gamma}{\gamma}} \tag{3.44}$$

$$\widehat{W_{2,T}}=(y_2\pi_{2,T})^{-\frac{1}{\gamma}}(f_{2,T})^{\frac{1-\gamma}{\gamma}} \tag{3.45}$$

in equilibrium, we should have:

$$f_{1,T}=(\frac{\widehat{W_{1,T}}}{\widehat{W_{2,T}}})^k \tag{3.46}$$

$$f_{2,T}=(\frac{\widehat{W_{2,T}}}{\widehat{W_{1,T}}})^k \tag{3.47}$$

now,we will have

$$\frac{\widehat{W_{2,T}}}{\widehat{W_{1,T}}}=\lambda(T)^{\frac{1}{\widehat{\gamma}}} \tag{3.48}$$

where $\lambda(T)=\frac{y_1\pi_{1,T}}{y_2\pi_{2,T}}$ and $\widehat{\gamma}=\gamma+2k(\gamma-1)$. Together with the market clearing clearing conditions, $W_{1,T}+W_{2,T}=D_T$, we get the (3.27) in **Lemma 3.3**. (3.16) in **Lemma 3.1** is just a special case of (3.27). Together with the FOC (which shows the relationship between $W_{i,T}$ and $\pi_{i,T}$), we can solve $\pi_{i,T}$ in **Lemma 3.5**.

### 3.2. Proof of Proposition 3.2 and 3.9:

### State Prices

When $\gamma$ is an integer, (3.28) and (3.29) become:

$$\pi_{1,T} = \frac{(k+1)}{y_1}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}\lambda(T)^{\frac{k(\gamma-1)}{\gamma+2k(\gamma-1)}}=\frac{(k+1)}{y_1}\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(T)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}$$

$$\pi_{2,T} = \frac{(k+1)}{y_2} \frac{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(T)^{\frac{i}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}} \lambda(T)^{-\frac{k(\gamma-1)+\gamma}{\gamma+2k(\gamma-1)}} = \frac{(k+1)}{y_2} \frac{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(T)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma}}$$

Given the dynamics of $\lambda(t)$ and $D_t$ w.r.t. $B_{1,t}$, we have

$$\lambda(T) = \lambda(t) \exp\left[-\frac{\overline{\mu}^2}{2}(T-t) - \overline{\mu}(B_{1,T} - B_{1,t})\right] \tag{3.49}$$

$$D_T = D_t \exp\left[\left(\mu_1 - \frac{\sigma_D^2}{2}\right)(T-t) + \sigma_D (B_{1,T} - B_{1,t})\right] \tag{3.50}$$

Then we can rewrite $\pi_{1,T}$ as:

$$\pi_{1,T} = \frac{k+1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left\{ \begin{array}{c} \left[-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right)\right](T-t) \\ -\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right](B_{1,T} - B_{1,t}) \end{array} \right\}} \tag{3.51}$$

by $\pi_{1,t} = E_t^1(\pi_{1,T})$, we have

$$\pi_{1,t} = \frac{k+1}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right](T-t)} \tag{3.52}$$

Similarly, for the dynamics w.r.t $B_{2,t}$, we have

$$\lambda(T) = \lambda(t) \exp\left[\frac{\overline{\mu}^2}{2}(T-t) - \overline{\mu}(B_{2,T} - B_{2,t})\right] \tag{3.53}$$

$$D_T = D_t \exp\left[\left(\mu_2 - \frac{\sigma_D^2}{2}\right)(T-t) + \sigma_D (B_{2,T} - B_{2,t})\right] \tag{3.54}$$

Following the similar procedure, we have

$$\pi_{2,t} = \frac{k+1}{y_2 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}} e^{\left[ \begin{array}{c} \frac{\overline{\mu}^2}{2}\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i-k(\gamma-1)+\gamma}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right](T-t)} \tag{3.55}$$

**Market Price of Risk**

By Ito's lemma on $\pi_{1,t}$ and matching the diffusion terms, we can get Market price of risk

$$
\begin{aligned}
-\pi_{1,t}\kappa_{1,t} &= \frac{(k+1)}{y_1 D_t^{\gamma}} \sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right] (T-t) \right) \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \\
&\quad \times \left( -\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} - \gamma\sigma_D \right)
\end{aligned}
$$

Then, we can get $\kappa_{1,t}$ in the proposition with

$$
\delta_{1,t} = \frac{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right] (T-t) \right) \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_1 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right] (T-t) \right)}
$$

Similarly, we can get $\kappa_{2,t}$ with

$$
\delta_{2,t} = \frac{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right] (T-t) \right) \frac{k(\gamma-1)+\gamma-i}{\gamma+2k(\gamma-1)}}{\sum_{i=0}^{\gamma} \begin{pmatrix} \gamma \\ i \end{pmatrix} \lambda(t)^{\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)} - \\ \gamma\left(\mu_2 - \frac{\sigma_D^2}{2}\right) + \\ \frac{1}{2}\left[\frac{i-k(\gamma-1)-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu} + \gamma\sigma_D\right]^2 \end{array} \right] (T-t) \right)}
$$

**Portfolio Choices**

For agent 1, we have $\pi_{1,t}W_{1,t} = E_t^1\left(W_{1,T}\pi_{1,T}\right)$. By some manipulation, it can be written as

$$
\pi_{1,t}W_{1,t} = \frac{k+1}{y_1 D_t^{\gamma-1}} \sum_{i=0}^{\gamma-1} \begin{pmatrix} \gamma-1 \\ i \end{pmatrix} \lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} \exp\left( \left[ \begin{array}{c} -\frac{\overline{\mu}^2}{2} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} - (\gamma-1)\left(\mu_1 - \frac{\sigma_D^2}{2}\right) \\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu} + (\gamma-1)\sigma_D\right]^2 \end{array} \right] (T-t) \right) \tag{3.56}
$$

by Ito's lemma and matching the diffusion terms

$$\pi_{1,t}W_{1,t}\left(\theta_{1,t}\sigma_{s,t}-\kappa_{1,t}\right)dB_{1,t}=\frac{(k+1)}{y_1D_t^{\gamma-1}}\left\{\begin{array}{c}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}-\\(\gamma-1)\left(\mu_1-\frac{\sigma_D^2}{2}\right)+\\\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\end{array}\right](T-t)}\\\times\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\left(-(\gamma-1)\sigma_D-\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}\right)\end{array}\right\}dB_{1,t}\tag{3.57}$$

then $\theta_{1,t}=\frac{\mu_{1,t}}{\gamma\sigma_{s,t}^2}+\frac{1-\frac{1}{\gamma}\delta_{1,t}}{\sigma_{s,t}}\overline{\mu}-\frac{\beta_{1,t}}{\sigma_{s,t}}\overline{\mu}$ with

$$\beta_{1,t}=\frac{\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\exp\left(\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\\+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\end{array}\right](T-t)\right)\times\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\overline{\mu}}{\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\\+\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\end{array}\right](T-t)}}$$

Similarly,

$\theta_{2,t}=\frac{\mu_{2,t}}{\gamma\sigma_{s,t}^2}-\frac{1-\frac{1}{\gamma}\delta_{2,t}}{\sigma_{s,t}}\overline{\mu}+\frac{\beta_{2,t}}{\sigma_{s,t}}\overline{\mu}$ with

$$\beta_{2,t}=\frac{\begin{array}{c}\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}e^{\left[\begin{array}{c}\frac{\overline{\mu}^2}{2}\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}+\\\frac{1}{2}\left(\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right)^2\end{array}\right](T-t)}\times\\\frac{k(\gamma-1)1+\gamma-1-i}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}}\overline{\mu}\end{array}}{\sum_{i=0}^{\gamma-1}\binom{\gamma-1}{i}\lambda(t)^{\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}\frac{\overline{\mu}^2}{2}\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}+\\\frac{1}{2}\left(\frac{1+i-k(\gamma-1)1-\gamma}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right)^2\end{array}\right](T-t)}}$$

**Stock Price**

By the martingale property,

$$S_t=\frac{E_t^1(\pi_{1,T}D_T)}{\pi_{1,t}}S_t=E_t^1\left[\frac{\frac{(k+1)}{y_1}\frac{\sum_{i=0}^{\gamma}\binom{\gamma}{i}\lambda(T)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{D_T^{\gamma-1}}}{\pi_{1,t}}\right]\tag{3.58}$$

$$
= \frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\end{array}\right](T-t)} D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}}
\tag{3.59}
$$

**Volatility**

Denote the stock price as

$$
S_t = \frac{X_t}{Y_t} \tag{3.60}
$$

set

$$
X_t = \sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\exp\left\{\left[\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\begin{array}{c}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\\ (\gamma-1)\sigma_D\end{array}\right]^2\end{array}\right](T-t)\right]\right\} D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}
$$

and

$Y_t = \sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\exp\left\{\left[\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)\right]\right\}$, then for diffusion term, we only need to consider $d\frac{X_t}{Y_t} = \frac{Y_t dX_t - X_t dY_t}{Y_t^2}$. Apply Ito's Lemma to calculate diffusion terms of $dX_t$ and $dY_t$.

The diffusion term for $dX_t$ is

$$
e^{\left(\mu_1-\sigma_D^2\right)(T-t)}\left\{\begin{array}{c}\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\sigma_D\right]^2\end{array}\right](T-t)}\\ \times\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\begin{pmatrix}D_t\sigma_D\\ -\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}D_t\overline{\mu}\end{pmatrix}\end{array}\right\}dB_t
\tag{3.61}
$$

The diffusion term for $dY_t$ is

$$
\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)} dB_t \tag{3.62}
$$

by matching the diffusion terms, we have $\frac{X_t}{Y_t}\sigma_{s,t} = \frac{Y_t\sigma_X - X_t\sigma_Y}{Y_t^2}$ and $\sigma_{s,t} = \frac{Y_t\sigma_X - X_t\sigma_Y}{Y_t X_t}$.

$$\sigma_{s,t} = \sigma_D - \frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix} e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)} \frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\frac{\sigma_D^2}{2}\\ +\frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+(\gamma-1)\,\sigma_D\right]^2\end{array}\right](T-t)} D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}}\overline{\mu}$$

$$+\frac{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)} e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}}{\sum_{i=0}^{\gamma}\begin{pmatrix}\gamma\\ i\end{pmatrix}\lambda(t)^{\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}} e^{\left[\begin{array}{c}-\frac{\overline{\mu}^2}{2}\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}+\\ \frac{1}{2}\left[\frac{i+k(\gamma-1)}{\gamma+2k(\gamma-1)}\overline{\mu}+\gamma\sigma_D\right]^2\end{array}\right](T-t)}}\overline{\mu}$$

then, we can get the expression in the proposition.
Similarly, by setting $\gamma = 2$, we can get (3.41). Thus, **Proposition 3.9** is proved.

### 3.3. Proof of Proposition 3.3

Similar to the proof of **Proposition 3.2** by setting $\gamma = 2$, we can get ( 3.32) and (3.33) where:

$$\begin{aligned}
\beta_{1,t} &= \frac{\frac{k}{2+2k}+\frac{1}{2}\lambda(t)^{\frac{1}{2(k+1)}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}}{1+\lambda(t)^{\frac{1}{2(k+1)}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}} \\
\beta_{2,t} &= \frac{\frac{k}{2+2k}\lambda(t)^{\frac{1}{2(1+k)}} e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+\frac{1}{2}}{\lambda(t)^{\frac{1}{2(1+k)}} e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+1}
\end{aligned} \tag{3.63}$$

### 3.4. Proof of Proposition 3.4

Define $\exp(a) = \lambda(t)^{\frac{1}{2}} \exp\left[\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)\right]$ and $b = \frac{\overline{\mu}^2}{8}$, then we have:

$$\beta_{1,t} = \frac{\frac{k}{2+2k}+\frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}} \quad ; \quad \beta_{2,t} = \frac{\frac{1}{2}+\frac{k}{2+2k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}}{1+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}} \tag{3.64}$$

when $k=0$, we have $\beta^0_{1,t}=\frac{\frac{1}{2}e^{-b+a}}{1+e^{-b+a}}$ ; $\beta^0_{2,t}=\frac{\frac{1}{2}}{1+e^{b+a}}$. By some manipulation, we can write the difference between two investors' portfolio choices as

$$\begin{aligned}\theta_{1,t}-\theta_{2,t} &= \frac{\overline{\mu}}{\sigma_{s,t}}\left[1-(\beta^k_{1,t}+\beta^k_{2,t})\right]\\ where &: \ \beta^k_{1,t}<\frac{1}{2} \ , \beta^k_{2,t}<\frac{1}{2}\end{aligned}$$

For this reason, relative performance's effect on portfolio choice depends on $\beta^k_{1,t}+\beta^k_{2,t}-(\beta^0_{1,t}+\beta^0_{2,t})$ which can be calculated as

$$\frac{\frac{1+2k}{2(1+k)}+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{k}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1+2k}{2(1+k)}e^{\frac{2a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{2a}{(k+1)}}}-\frac{\frac{1}{2}+e^{-b+a}+e^{2a}}{1+e^{-b+a}+e^{b+a}+e^{2a}} \tag{3.65}$$

If we use notation $\frac{A}{B}-\frac{C}{D}$ for above expression, its sign depends on $AD-CB$, which can be calculated as:

$$\begin{aligned}&\frac{k}{2(1+k)}-\frac{1}{2(1+k)}e^{-b+a}+\frac{1+2k}{2(1+k)}e^{b+a}+\frac{k}{2(1+k)}e^{2a}+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\left(\frac{1}{2}+e^{-b+a}+e^{2a}\right)\\ &+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\left[\frac{k}{1+k}-\frac{1}{2}-\frac{1}{(1+k)}e^{-b+a}+\frac{k}{1+k}e^{b+a}+\left(\frac{k}{1+k}-\frac{1}{2}\right)e^{2a}\right]\\ &+e^{\frac{2a}{(k+1)}}\left[\frac{k}{2(1+k)}-\frac{1}{2(1+k)}e^{-b+a}+\frac{1+2k}{2(1+k)}e^{b+a}+\frac{k}{2(1+k)}e^{2a}\right]\end{aligned}$$

Then we can conclude

$$\beta^k_{1,t}+\beta^k_{2,t}-(\beta^0_{1,t}+\beta^0_{2,t})>0 \ \ when\ k>1, \ <0 \ \ when\ k<1 \ \ and\ a \ \ is\ large\ enough \tag{3.66}$$

Moreover, we have:

$$\frac{d\left(\beta^k_{1,t}+\beta^k_{2,t}\right)}{dk}=\frac{d\left[\frac{\frac{1+2k}{2(1+k)}+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{k}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1+2k}{2(1+k)}e^{\frac{2a}{(k+1)}}}{1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{2a}{(k+1)}}}\right]}{dk} \tag{3.67}$$

After some manipulation, $\frac{d\left(\beta^k_{1,t}+\beta^k_{2,t}\right)}{dk}$ can be expressed as:

$$\begin{aligned}&\left[\frac{1}{2(1+k)^2}+\frac{1}{(1+k)^2}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)^2}e^{\frac{2a}{(k+1)}}\right]\left[1+e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+e^{\frac{2a}{(k+1)}}\right]\\ &+\frac{2b}{(1+k)^3}\left[\frac{1}{2(1+k)}+\frac{1}{1+k}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)}e^{\frac{2a}{(k+1)}}\right]e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\end{aligned}$$

$$+e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}\frac{2b}{(1+k)^3}\left[\frac{1}{2(1+k)}+\frac{1}{1+k}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{1}{2(1+k)}e^{\frac{2a}{(k+1)}}\right]$$
$$-\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{-\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}e^{\frac{2a}{(k+1)}}$$
$$-\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}+\frac{a}{(k+1)^2}\frac{1}{2(1+k)}e^{\frac{b}{(1+k)^2}+\frac{a}{(k+1)}}e^{\frac{2a}{(k+1)}}$$

It is easy to see that the above expression is greater than 0.

### 3.5. Proof of Proposition 3.5

Given $\beta_{i,t}$ in the proof of **Proposition 3.3**, we can easily get $\beta^0_{i,t}$ by setting k=0. Then for optimistic agent, the sign of $\beta_{1,t}-\beta^0_{1,t}$ depends on

$$\frac{\frac{k}{1+k}+\lambda(t)^{\frac{1}{2(k+1)}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}}{1+\lambda(t)^{\frac{1}{2(k+1)}}e}-\frac{\lambda(t)^{\frac{1}{2(}}e^{\left(-\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}}{1+\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}} \tag{3.68}$$

by some manipulation, the sign depends on

$$\frac{k}{1+k}+\lambda(t)^{\frac{1}{2(k+1)}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2(k+1)}\overline{\mu}\sigma_D\right)(T-t)}-\frac{1}{1+k}\lambda(t)^{\frac{1}{2(}}e^{\left(-\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)} \tag{3.69}$$

Let $x\equiv\lambda(t))^{\frac{1}{2}}e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}$, define $F(x)$ as

$$F(x)=\frac{k}{1+k}+e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}x^{\frac{1}{1+k}}-\frac{1}{1+k}x \tag{3.70}$$

When

$$F^{'}(x)=\frac{1}{1+k}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}x^{\frac{1}{1+k}-1}-\frac{1}{1+k}=0 \tag{3.71}$$

We have $x=e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$.When $x>e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$, $F^{'}(x)<0$, $F(x)$ is a monotonically decreasing function of $x$; when $x<e^{\frac{-\overline{\mu}^2}{8(1+k)k}(T-t)}$, $F^{'}(x)>0$, $F(x)$ is a monotonically increasing function of $x$. For this reason, there exist one cutoff $x_{c1}$, when $x>x_{c1}$, $F(x)<0$, when $x<x_{c1}$, $F(x)>0$

For pessimistic agent, the sign of $\beta_{2,t}-\beta^0_{2,t}$ depends on

$$\frac{\frac{k}{1+k}\lambda(t)^{\frac{1}{2(1+k)}}e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+1}{\lambda(t)^{\frac{1}{2(1+k)}}e^{\left(\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2}\frac{1}{1+k}\overline{\mu}\sigma_D\right)(T-t)}+1}-\frac{1}{\lambda(t)^{\frac{1}{2}}e^{\left(\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}+1} \tag{3.72}$$

by some manipulation, the sign depends on

$$\frac{k}{1+k}+e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left[\lambda(t)^{\frac{1}{2}}e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}\right]^{-\frac{1}{1+k}}-\frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left[(\lambda(t))^{\frac{1}{2}}e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}\right]^{-1} \tag{3.73}$$

Let $x := \lambda(t)^{\frac{1}{2}}e^{\left(\frac{1}{2}\overline{\mu}\sigma_D\right)(T-t)}$, define $G(x)$ as

$$G(x) = \frac{k}{1+k} + e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}x^{-\frac{1}{1+k}} - \frac{1}{1+k}x^{-1} \tag{3.74}$$

When

$$G^{'}(x) = -\frac{1}{1+k}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}x^{-\frac{1}{1+k}-1} + \frac{1}{1+k}x^{-2} = 0 \tag{3.75}$$

We have $x = e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$. When : $x > e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$, $G^{'}(x) < 0$, $G(x)$ is a monotonically decreasing function of $x$; when $x < e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}$, $G^{'}(x) > 0$, $G(x)$ is a monotonically increasing function of $x$. Consequently, there exist one cutoff $x_{c2}$, when $x > x_{c2}$, $G(x) > 0$, when $x < x_{c2}$, $G(x) > 0$.

Now we study the $x_{c1}$ and $x_{c2}$. We have $F(1) = G(1) = \frac{k}{1+k}+e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}-\frac{1}{1+k} > 0$. Because $0 = G(x_{c2}) < G(1) < G\left[e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}\right]$, we have: $x_{c2} < 1$. In addition, because $0 = F(x_{c1}) < F(1) < F\left[e^{\left(\frac{\overline{\mu}^2}{8(1+k)k}\right)(T-t)}\right]$, we have: $x_{c1} > 1$. To sum up, $x_{c1} > x_{c2}$. Let $g_{c1} := x_{c1}^2 e^{(-\overline{\mu}\sigma_D)(T-t)}$ and $g_{c2} := x_{c2}^2 e^{(-\overline{\mu}\sigma_D)(T-t)}$, **Proposition 3.5** is proved.

### 3.6. Proof of Proposition 3.6

Similar to the proof of **Proposition 3.2** by setting $\gamma = 2$, we can get (3.37) and (3.38) where

$$a_k = \frac{\frac{k}{2+2k} + \lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} + \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} + \lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}. \tag{3.76}$$

$$a_0 = \frac{\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)} + \lambda(t)e^{(2\overline{\mu}\sigma_D)(T-t)}}{1+2\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)} + \lambda(t)e^{(2\overline{\mu}\sigma_D)(T-t)}} \tag{3.77}$$

If $a_k \geq (\leq) a_0$,then $\frac{a_k}{1-a_k} \geq (\leq) \frac{a_0}{1-a_0}$. The sign of $\frac{a_k}{1-a_k} - \frac{a_0}{1-a_0}$ depends on

$$\frac{\frac{k}{2+2k} + \lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D \frac{1}{1+k}\right)(T-t)} + \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D \frac{2}{1+k}\right)(T-t)}}{\frac{2+k}{2+2k} + \lambda(t)^{\frac{1}{2}\frac{1}{1+k}} e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2} + \overline{\mu}\sigma_D \frac{1}{1+k}\right)(T-t)} + \frac{k}{2+2k}\lambda(t)^{\frac{1}{1+k}} e^{\left(\overline{\mu}\sigma_D \frac{2}{1+k}\right)(T-t)}}$$
$$- \frac{\lambda(t)^{\frac{1}{2}} e^{\left(-\frac{\overline{\mu}^2}{8} + \overline{\mu}\sigma_D\right)(T-t)} + \lambda(t) e^{(2\overline{\mu}\sigma_D)(T-t)}}{1 + \lambda(t)^{\frac{1}{2}} e^{\left(-\frac{\overline{\mu}^2}{8} + \overline{\mu}\sigma_D\right)(T-t)}}$$

After some manipulation, we can show that the sign depends on

$$\frac{k}{2+2k}\left(1 - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+2}\right) + e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}+2}\right)$$
$$+\frac{2+k}{2+2k}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{2}\right)$$
$$+\frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1} - \lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right)$$

If $\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)} < 1$, we have $\frac{k}{2+2k}\left(1 - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+2}\right) > 0$ and $\frac{2+k}{2+2k}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{2}\right) > 0$. Moreover,

$$\begin{aligned}
& e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{1+k}+2}\right) \\
> \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right) \\
> \; & e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right) \\
> \; & \frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left(\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)} - \left\{\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{1+k}+1}\right)
\end{aligned}$$

We have

$$e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}} \\ -\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\} + \frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1} \\ -\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\end{array}\right\} > 0 \tag{3.78}$$

Hence, in this case, $\frac{a_k}{1-a_k} > \frac{a_0}{1-a_0}$, and it is easy to show that $a_k > a_0$.
If $\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)} > 1$, we have $\frac{k}{2+2k}\left\{1 - \left[\lambda(t)^{\frac{1}{2}} e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+2}\right\} < 0$ and

$\frac{2+k}{2+2k}\left\{\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}}-\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]\right\}^2<0$. Then,

$$\begin{aligned}
& e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}- \\ \left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\}+\frac{1}{1+k}e^{-\frac{\overline{\mu}^2}{8}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1}- \\ \left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]\end{array}\right\} \\
< \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}- \\ \left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\}+e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1}- \\ \left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]\end{array}\right\} \\
= \; & e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left\{\begin{array}{c}\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}}-\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}+\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{2}{1+k}+1} \\ -\left[\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}\right]^{\frac{1}{1+k}+2}\end{array}\right\} \\
< \; & 0
\end{aligned}$$

In this case $\frac{a_k}{1-a_k}-\frac{a_0}{1-a_0}<0$, it is easy to show $a_k<a_0$.
For comparative statics, we consider how the ratio $\frac{a_k}{1-a_k}$ changes w.r.t. $k$.

$$\frac{a_k}{1-a_k}=\frac{\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}}{\frac{2+k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\frac{k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}} \tag{3.79}$$

Let $a=\overline{\mu}\sigma_D\,(T-t)$, $b=\frac{\overline{\mu}^2}{8}\,(T-t)$, we calculate $\frac{\partial\frac{a_k}{1-a_k}}{\partial k}$. After some calculation, we find that the sign depends on

$$\begin{aligned}
&-\left[\frac{1}{(1+k)^2}+\frac{2a}{(1+k)^3}\right]\left[e^{\frac{2a}{1+k}}-1\right]-\frac{2a}{(1+k)^3} \\
&-e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}}\left(e^{\frac{2a}{1+k}}-1\right)\left[\frac{1}{(1+k)^2}+\frac{a}{(1+k)^3}+\frac{2b}{(1+k)^4}\right] \\
&-\frac{2a}{(1+k)^3}e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}}-\frac{1}{2(1+k)^3}\left[e^{\frac{2a}{1+k}}-1\right]^2
\end{aligned}$$

If $e^{\frac{a}{1+k}}-1>0$ , then $a>0$, and the above expression is smaller than zero. If $e^{\frac{a}{1+k}}-1<0$ , then $a<0$, and we can show it is greater than zero. Since $\frac{\partial\frac{a_k}{1-a_k}}{\partial k}$ has the same sign with $\frac{\partial a_k}{\partial k}$, the proposition is thus proved.

### 3.7. Proof of Proposition 3.7

$$\frac{d\kappa_{1,t}}{d\lambda(t)} = \overline{\mu}\frac{\left\{\frac{1}{2}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} + \frac{2+k}{2+2k}\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}}{\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}} - \tag{3.80}$$

$$\overline{\mu}\frac{\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+ \\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} \\ +\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}}{\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\}^2} \tag{3.81}$$

The numerator is:

$$\begin{aligned} = & \left\{\begin{array}{c}\frac{1}{2}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} \\ +\frac{2+k}{2+2k}\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+ \\ \lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\} \\ & -\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+ \\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{\begin{array}{c}\frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)} \\ +\frac{1}{1+k}\lambda(t)^{\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\end{aligned}$$

then $\frac{d\kappa_{1,t}}{d\lambda(t)}$ is:

$$\begin{aligned} = & \frac{1}{1+k}\lambda(t)^{\frac{1}{2}\frac{1}{1+k}-1}e^{\left(\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}\left[\frac{1}{2}e^{(-\frac{\overline{\mu}^2}{8(1+k)^2})(T-t)}+\frac{1}{2}\frac{2+k}{1+k}[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}}\right] \\ & \times\left\{1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\right\} \\ & -\left\{\begin{array}{c}\frac{k}{2+2k}+\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+ \\ \frac{2+k}{2+2k}\lambda(t)^{\frac{1}{1+k}}e^{\left(\overline{\mu}\sigma_D\frac{2}{1+k}\right)(T-t)}\end{array}\right\}\left\{e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}+\left[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}\right]^{\frac{1}{1+k}}\right\}\end{aligned}$$

After some manipulation, it becomes:

$$\frac{1}{2(1+k)}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}+\frac{1}{(1+k)}\left[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}\right]^{\frac{1}{1+k}}+\frac{1}{2(1+k)}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}\left[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}\right]^{\frac{2}{1+k}} \tag{3.82}$$

$$\frac{d\kappa_{1,t}}{d\lambda(t)} = \lambda(t)\overline{\mu}\frac{1}{1+k}[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}}\left\{\begin{array}{c}\frac{1}{2(1+k)}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}+\frac{1}{(1+k)}[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}]^{\frac{1}{1+k}} \\ +\frac{1}{2(1+k)}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}[\lambda(t)^{\frac{1}{2}}e^{(\overline{\mu}\sigma_D)(T-t)}]^{\frac{2}{1+k}}\end{array}\right\}$$

$$= \lambda(t)\overline{\mu}\frac{1}{(1+k)^2}\left[\frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{a}{1+k}} + e^{\frac{2a}{1+k}} + \frac{1}{2}e^{-\frac{b}{(1+k)^2}+\frac{3a}{1+k}}\right] > 0$$

where $\exp(a) = \lambda(t)^{\frac{1}{2}}\exp(\overline{\mu}\sigma_D)(T-t)$, $b = \frac{\overline{\mu}^2}{8}$.
Since $\kappa_{1,t} - \kappa_{2,t}$ is a constant, it is easy to show $\frac{d\kappa_{2,t}}{d\lambda(t)} > 0$.

### 3.8. Proof of Proposition 3.8

Similar to the proof of **Proposition 3.2** by setting $\gamma = 2$, we can get (3.40). When $k = 0$, we have $S_t^0 = \frac{\left\{\begin{array}{c} 1+\lambda(t)^{\frac{1}{2}}2e^{\left[-\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right](T-t)} \\ +\lambda(t)e^{\overline{\mu}\sigma_D(T-t)}\end{array}\right\}D_t e^{\left(\mu_1-\sigma_D^2\right)(T-t)}}{\left\{\begin{array}{c} 1+2\lambda(t)^{\frac{1}{2}}e^{\left(-\frac{\overline{\mu}^2}{8}+\overline{\mu}\sigma_D\right)(T-t)}+ \\ \lambda(t)e^{2\overline{\mu}\sigma_D(T-t)}\end{array}\right\}e^{\sigma_D^2(T-t)}}$. Denote $S_t = \frac{X_t^k}{Y_t^k}D_t\exp\left(\mu_1-\sigma_D^2\right)(T-t)$ and $S_t^0 = \frac{X_t^0}{Y_t^0}D_t\exp\left(\mu_1-2\sigma_D^2\right)(T-t)$. For $S_t^k \leq (\geq) S_t^0$, we need $\frac{X_t^k}{Y_t^k} \leq (\geq)\frac{X_t^0}{Y_t^0}$, we need to look at the sign of $X_t^kY_t^0 - Y_t^kX_t^0$

$$\begin{aligned} X_t^k &= 1+\lambda(t)^{\frac{1}{2+2k}}2e^{\left[-\frac{\overline{\mu}^2}{8(1+k)^2}+\frac{1}{2+2k}\overline{\mu}\sigma_D\right](T-t)} + \lambda(t)^{\frac{2}{2+2k}}e^{\frac{2}{2+2k}\overline{\mu}\sigma_D(T-t)} \\ &= 1+2\left\{\lambda(t)e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{2+2k}}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)} + \left\{\lambda(t)e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{2}{2+2k}} \\ &= 1+2e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}K^{\frac{1}{2+2k}} + K^{\frac{2}{2+2k}} \end{aligned}$$

$$\begin{aligned} Y_t^k &= \left\{\begin{array}{c} 1+2\lambda(t)^{\frac{1}{2}\frac{1}{1+k}}e^{\left(-\frac{\overline{\mu}^2}{8(1+k)^2}+\overline{\mu}\sigma_D\frac{1}{1+k}\right)(T-t)}+ \\ \lambda(t)^{\frac{1}{1+k}}e^{\overline{\mu}\sigma_D\frac{2}{1+k}(T-t)}\end{array}\right\}e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)} \\ &= e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)} + 2K^{\frac{1}{2}\frac{1}{1+k}}e^{\frac{1}{2}\overline{\mu}\sigma_D(T-t)}e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)} + K^{\frac{1}{1+k}}e^{\frac{2+k}{2+2k}\overline{\mu}\sigma_D(T-t)} \end{aligned}$$

$$\begin{aligned} X_t^0 &= 1+\lambda(t)^{\frac{1}{2}}2e^{\left[-\frac{\overline{\mu}^2}{8}+\frac{1}{2}\overline{\mu}\sigma_D\right](T-t)} + \lambda(t)e^{\overline{\mu}\sigma_D(T-t)} \\ &= 1+2\left\{\lambda(t)e^{\overline{\mu}\sigma_D(T-t)}\right\}^{\frac{1}{2}}e^{-\frac{\overline{\mu}^2}{8}(T-t)} + \lambda(t)e^{\overline{\mu}\sigma_D(T-t)} = 1+2e^{-\frac{\overline{\mu}^2}{8}(T-t)}K^{\frac{1}{2}}+K \end{aligned}$$

$$Y_t^0 = 1+2\lambda(t)^{\frac{1}{2}}e^{\overline{\mu}\sigma_D(T-t)}e^{-\frac{\overline{\mu}^2}{8}(T-t)}+\lambda(t)e^{2\overline{\mu}\sigma_D(T-t)} = 1+2K^{\frac{1}{2}}e^{\frac{1}{2}\overline{\mu}\sigma_D(T-t)}e^{-\frac{\overline{\mu}^2}{8}(T-t)}+Ke^{\overline{\mu}\sigma_D(T-t)} \tag{3.83}$$

where $K = \lambda(t)e^{\overline{\mu}\sigma_D(T-t)}$. After some calculation, the sign of $X_t^kY_t^0 - Y_t^kX_t^0$ depends on

$$1-e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)}+2\left[e^{-\frac{\overline{\mu}^2}{8(1+k)^2}(T-t)}-e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}\right]K^{\frac{1}{2+2k}}$$

$$+K^{\frac{1}{1+k}}\left[1-e^{\frac{2+k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]+2\left[e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}-e^{\left(\frac{k}{2+2k}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}\right]K^{\frac{1}{2}}$$

$$+\left[e^{\overline{\mu}\sigma_D(T-t)}-e^{\frac{2+k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]K^{\frac{2+k}{1+k}}+2\left[e^{\left(\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}-e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8(1+k)^2}\right)(T-t)}\right]K^{\frac{3+2k}{2+2k}}$$

$$+\left[e^{\overline{\mu}\sigma_D(T-t)}-e^{\frac{k}{2+2k}\overline{\mu}\sigma_D(T-t)}\right]K+2\left[e^{\left(\frac{1}{2}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}-e^{\left(\frac{2+k}{2+2k}\overline{\mu}\sigma_D-\frac{\overline{\mu}^2}{8}\right)(T-t)}\right]K^{\frac{3+k}{2+2k}}$$

Then when $K\to\infty$, $S_t>S_t^0$. When $K\to 0$, $S_t<S_t^0$

### 3.9. Proof of Proposition 3.10

$W_{1,T}=\frac{D_T}{1+\lambda(T)^{\frac{1}{\widehat{\gamma}}}}$ ;$W_{2,T}=\frac{\lambda(T)^{\frac{1}{\widehat{\gamma}}}D_T}{1+\lambda(T)^{\frac{1}{\widehat{\gamma}}}}$, where $\widehat{\gamma}=\gamma+2k(\gamma-1)$. We thus obtain

$$\frac{W_{2,T}}{W_{1,T}}=\lambda(T)^{-\frac{1}{\widehat{\gamma}}}=\exp\left[\frac{1}{\widehat{\gamma}}\left(-\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)\right)T-\frac{1}{\widehat{\gamma}}\overline{\mu}B_T\right] \tag{3.84}$$

Using the strong Law of Large Numbers for Brownian motion (see Karatzas and Shreve (1991), for any value of $\sigma$, we have:

$$\lim_{T\to\infty}\exp(aT+\sigma B_T)=\left\{\begin{array}{c}0, a<0\\ \infty, a>0\end{array}\right\} \tag{3.85}$$

so we have:

$$\lim_{T\to\infty}\frac{W_{2,T}}{W_{1,T}}=\left\{\begin{array}{c}0,-\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)<0\\ \infty,-\frac{1}{2}\overline{\mu}^2-\overline{\mu}\sigma_D\left(\widehat{\gamma}-1\right)>0\end{array}\right\} \tag{3.86}$$

then we can easily get the result in the proposition.