The London School of Economics and Political Science

Essays on Wealth Inequality and Financial Economics

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

Statement of Conjoint Work. I confirm that Chapter 2 is jointly co-authored with Andrew Atkeson and has been published in *American Economic Review: Insights* (Atkeson and Irie (2022)). I contributed 50% of this work.

Statement of Use of Third Party for Editorial Help. I can confirm that an earlier draft of the first chapter of this thesis was copy edited for conventions of language, spelling and grammar by Wes Cowley of Words by Wes, LLC.

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Abstract

This thesis consists of three chapters on the role of finance in influencing the distribution of wealth.

In the first chapter, I study how improvements in entrepreneurial financing affect top wealth inequality. On the one hand, improved financing allows entrepreneurs to scale up, raising top inequality. Simultaneously, extreme wealth trajectories for entrepreneurs become less likely as better financing improves risk sharing, lowering top inequality. It turns out that which of these effects dominates depends on the amount of economic activity that is reallocated to entrepreneurs from elsewhere in the economy. Top wealth inequality rises provided that this reallocation is large enough.

In the second chapter, co-authored with Andrew Atkeson, we derive an analytical link between the fast dynamics of measured wealth inequality at the top on the one hand, and the prevalence of newly created fortunes on the other. Specifically, in the context of a random growth model of wealth accumulation, the shape of the top of the wealth distribution changes rapidly only if the pace with which new fortunes are created is fast.

In the final chapter, I study whether the rise in measured wealth inequality documented in the Distributional National Accounts (DINA) can be accounted for by the combination of changing asset prices on the one hand, and household heterogeneity in portfolio compositions on the other. In particular, I study the gap between the share of wealth held by individuals at the top of the wealth distribution, and those individuals' share of the associated capital income flows. I find that the size of this gap varies substantially over time. However, the steady rise in top wealth shares since the late 1970s is not primarily accounted for by a rise in the size of this gap, but by rising concentration in the associated capital income flows.

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Chapter 1

Innovations in Entrepreneurial Finance and Top Wealth Inequality

How does improved risk sharing through better entrepreneurial financing affect top wealth inequality? On the one hand, better entrepreneurial financing enables entrepreneurs to scale up, which tends to raise top wealth inequality. On the other hand, better risk sharing allows entrepreneurs to reduce the idiosyncratic volatility in their wealth portfolios. This risk reduction lowers wealth inequality by making extreme wealth trajectories less likely and by weakening entrepreneurs' precautionary savings motive. The novel insight in this paper is that which of these two effects dominates depends crucially on how much economic activity is reallocated to entrepreneurial firms from elsewhere in the economy when entrepreneurs try to scale up. When this reallocation is large, wealth inequality rises rapidly as equity financing improves, and the model makes sense of several empirical trends, most notably the dramatic rise of firms with a history of venture capital backing. The results suggest that improved risk sharing through better equity financing could have been a quantitatively important contributor to rising wealth concentration.

1.1 Introduction

A cursory glance at the names appearing on lists of wealthy Americans uncovers a striking fact: many of the richest individuals became wealthy through a risky investment in a single entrepreneurial firm. Prior work has emphasized the role of entrepreneurs with high exposures to idiosyncratic risk in explaining both the thick right tail of the wealth distribution and the prevalence of newly minted fortunes at the top. While improved entrepreneurial financing allows entrepreneurs to scale up, it also allows them to offload idiosyncratic risk. With lower levels of idiosyncratic risk, the extreme upward wealth trajectories that help account for the thick right tail of the wealth distribution become less likely. Moreover, with less idiosyncratic investment risk, entrepreneurs' precautionary savings motives are weaker, which slows their wealth accumulation. In addition, better risk sharing means that returns to successful firms are spread over a larger set of investors. Therefore, it is not immediately clear how wealth inequality is affected by better risk sharing. The question studied in this paper is therefore: how is top wealth inequality affected by improvements in financing for entrepreneurs?

I develop a tractable general equilibrium framework to answer this question. The framework concisely summarizes the impact of improved financing in three economic forces. Consider a hypothetical entrepreneur, Jeff. Suppose Jeff's financing constraints have just been relaxed. Specifically, he can now finance a larger fraction of his online bookstore startup by issuing equity to outsiders. Jeff could use the risk-sharing properties of improved equity financing to reduce his own idiosyncratic risk exposure. This *risk-reduction effect* would lower top wealth inequality in the long run by making extreme upward wealth trajectories for entrepreneurs less likely. However, Jeff could also use the improved financing to scale up and, with some luck, turn his online bookstore into a retail giant. If this *scaling-up effect* is strong enough, top wealth inequality rises. The existence of these two forces related to improvements in risk-sharing for entrepreneurs has been discussed in the literature.[1]

The tractability of the framework in this paper allows me to highlight a novel theoretical mechanism. Whether the risk-reduction or the scaling-up effect dominates

[1]See Bonfiglioli (2012) and Peter (2021).

depends on a third force: the *general equilibrium reallocation effect*. This measures the extent to which productive resources are reallocated to cutting-edge entrepreneurial firms from other firms in the economy when entrepreneurial financing improves. When every entrepreneur tries to scale up, competition among them reduces their equilibrium profitability. This reduces the attractiveness of scaling up. Why does entrepreneurs' equilibrium profitability fall when they all want to scale up? First, their profitability is reduced because their equilibrium cost of capital rises as they compete for financing. Second, their profitability is diminished because their labor costs rise, and the equilibrium prices of the goods they sell fall as they compete for labor and customers.

This downward pressure on entrepreneurial profitability is ameliorated if entrepreneurs as a group can poach customers, attract labor, and raise capital at the expense of other firms in the economy. A crucial feature of the model is that entrepreneurial firms compete not only with one another but also with traditional firms. These traditional firms produce goods that are imperfectly substitutable with those the entrepreneurial firms produce. The higher the elasticity of substitution between these goods, the stronger the reallocation effect: with high elasticity of substitution, entrepreneurs can expand by poaching demand and productive resources from traditional firms rather than competing down their equilibrium profitability. In this case, entrepreneurs' excess return remains relatively stable, while improvements in equity financing allow them to carry less risk per dollar invested, thus improving the risk-reward trade-off they face. Then, they choose to scale up so much that their *total* risk exposure increases even if improvements in equity financing allow them to carry a smaller *fraction* of the risk in their firm. Here, wealth inequality rises.

In contrast, when the elasticity of substitution between the goods is low, there is limited room for entrepreneurs to expand in equilibrium at the expense of the traditional firms. Hence, the downward pressure on their excess return is high. If the excess return falls enough for the risk-reward trade-off associated with entrepreneurial activity to deteriorate, entrepreneurs choose to reduce their idiosyncratic risk exposures, lowering wealth inequality in the long run.

As a second contribution, the framework makes sense of several other empirical trends documented in U.S. data, provided the elasticity of substitution between entrepreneurial firms' goods and traditional firms' goods is high. Most notable among these trends is the rapid growth in the share of economic activity associated with venture capital-backed firms. Other consistent trends include the stability of the accounting return to the aggregate capital stock despite these falling safe rates, and the fall in the aggregate labor share despite stable firm-level labor shares and a falling safe rate. The model exhibits these patterns precisely when the reallocation effect is strong.

Model overview. The extent to which entrepreneurs choose to bear idiosyncratic risk is an equilibrium outcome, so a comprehensive understanding of how improvements in entrepreneurial financing affect top wealth inequality requires an equilibrium model. To this end, I present a general equilibrium model where risks and expected returns associated with entrepreneurship are endogenously determined.

The immediate precursors to the model are the modified neoclassical growth models of Angeletos (2007), Brunnermeier and Sannikov (2017), and Di Tella and Hall (2021). The model features two sectors of production: an innovative entrepreneurial sector and a traditional sector. The firms in the innovative entrepreneurial sector are more productive than the traditional firms. However, a portion of each firm's idiosyncratic risk must be borne by the associated entrepreneur for incentive alignment purposes. Equity issuance is possible but limited. The traditional sector is less productive but has no idiosyncratic risk costs. The uninsurable risk of entrepreneurial production implies that entrepreneurs earn a positive idiosyncratic excess return. Entrepreneurs choose how much idiosyncratic risk to bear by weighing the excess return against the risk.

The allocation of capital to each type of firm is determined by the trade-off between the higher productivity of the entrepreneurial sector, the lower risk costs of the traditional sector, and the substitutability of the goods they produce. I model improvements in entrepreneurial financing as an increase in the fraction of the firm's risk that entrepreneurs can offload to financial markets. This greater offloading lowers the risk cost associated with entrepreneurial production, which, in turn, triggers a reallocation of economic activity from the traditional firms to the entrepreneurial firms.

The model makes stark predictions regarding the effect of improvements in equity financing on top wealth inequality, and the effect depends on the strength of this reallocation. When the degree of substitutability between the goods the two types of firms produce is high, even minor improvements in entrepreneurial equity financing cause a considerable reallocation of capital, labor, and sales to entrepreneurial firms. The large reallocation means the competitive pressure among entrepreneurial firms for financing, workers, and customers is less severe. Entrepreneurs can then expand more aggressively without their expected excess returns declining much. Moreover, better risk sharing reduces the risk per unit invested. If the risk-reward trade-off improves despite the slightly lower expected excess return, then entrepreneurs scale up not only by raising more capital from outsiders, but also by investing more of their personal wealth, thereby raising their idiosyncratic risk exposures. This raises top wealth inequality by making extreme upward wealth trajectories more likely, and also, albeit to a lesser extent, by strengthening entrepreneurs' precautionary savings motive.

The setup with two types of firms is essential to deliver these results. The presence of traditional firms from which entrepreneurial firms can draw productive resources and customers enables entrepreneurs, in the aggregate, to scale up without adversely affecting their returns. To the best of my knowledge, this aspect, that the extent to which entrepreneurs can attract economic activity from elsewhere in the economy is crucial for the effect of better risk-sharing on top inequality is, is new to the literature. Finally, I derive a closed-form solution for the model's steady state level of Pareto inequality. The formula reveals an intimate link between entrepreneurs' risk exposures, the share of wealth they hold in aggregate, and the thickness of the right tail of the overall cross-sectional distributional wealth.

Empirical overview. Although the main subject of this study is top wealth inequality, the framework can make sense of four other key empirical trends under the assumption of a high elasticity of substitution between the goods that entrepreneurial firms and traditional firms produce:

1. The dramatic growth in the fraction of firms with a history of venture capital-

backing among the largest publicly traded firms in the U.S.[2]

2. The fall in the aggregate labor share, despite relatively stable firm-level labor shares.[3]
3. The stable or slightly rising accounting return to the aggregate capital stock despite the falling real safe interest rate.[4]
4. The fall in real safe interest rates.[5]

The first trend is directly related to the mechanism at the heart of this paper. The three others have a more subtle connection to the main mechanism. Their significance arises from the fact that they are implied by the model precisely when the general equilibrium reallocation effect is so strong that the scaling-up effect dominates the risk-reduction effect. The model exhibits these trends under the same conditions under which improvements in entrepreneurial financing lead to higher top wealth inequality.

The growth of venture capital-backed firms. For improvements in entrepreneurial financing to be associated with rising top wealth inequality, the model requires that it causes a substantial reallocation of capital from traditional firms to cutting-edge entrepreneurial firms. Gornall and Strebulaev (2021) document precisely such a reallocation. For instance, they document that firms with a history of VC-backing constituted less than 5% of the market capitalization of publicly traded firms before 1980 but that this share has risen to around 41% in 2020. Since venture capital is explicitly aimed at providing financing for cutting-edge entrepreneurial firms, this suggests that there has been a significant reallocation to such firms over the past half-century.

Labor share. Improvements in entrepreneurial financing have two offsetting effects on the labor share in the model. First, the reallocation of production toward the low-labor-share entrepreneurial firms reduces the aggregate labor share via a composition effect. Conversely, it increases the labor share at the firm level for entrepreneurial firms, as they must raise wages to attract workers. The model displays the empirically observed pattern of a declining aggregate labor share alongside stable or weakly

[2] See Gornall and Strebulaev (2021) and Greenwood et al. (2022).
[3] See Autor et al. (2020) and Hartman-Glaser et al. (2019).
[4] See Reis (2022), Moll et al. (2022) and Farhi and Gourio (2018).
[5] See Holston et al. (2017), Auclert et al. (2021), Rachel and Summers (2019).

rising firm-level labor shares precisely when the reallocation effect is strong. The reason that the entrepreneurial firms have lower labor shares of income is that the entrepreneur's share of income comes out of both the pure capital share and the labor share.

Rates of return to capital. The same reasoning applies to the accounting return to the overall capital stock. Reallocating capital to high-return entrepreneurial firms raises the aggregate return to capital. On the other hand, diminishing returns within the entrepreneurial sector exert a counteracting downward pressure. Improvements in entrepreneurial financing only increase the return to the aggregate capital stock if the reallocation effect is strong.

Safe real interest rate. Improvements in entrepreneurial equity financing, combined with a strong general equilibrium reallocation force, lead entrepreneurs to take on more idiosyncratic risk. Higher idiosyncratic risk exposure increases entrepreneurs' precautionary savings motive, which depresses the equilibrium real safe interest rate.

Numerical assessment. To gauge the quantitative role played by improved equity financing, I study the model through numerical experiments. The tractability of the framework allows me to compute the model's transition dynamics straightforwardly. In this experiment, I feed in a decline in equity financing frictions that reproduces the rise in the average rate of equity issuance by firms associated with entrepreneurs at the top of the Forbes 400, as documented by Gomez and Gouin-Bonenfant (2024). The model can account for the fast transition dynamics of Pareto inequality in response to improved equity financing for entrepreneurs, provided that the general equilibrium reallocation effect is large enough. In particular, when the general equilibrium reallocation effect is strong enough to account for the rise in the market capitalization share of firms with a history of venture capital backing, the model can account for almost all of the rise in Pareto inequality.

Literature. This paper contributes to the literature on the consequences of idiosyncratic investment risk and entrepreneurship for wealth inequality. This literature was pioneered by Quadrini (2000) and further developed by Meh and Quadrini (2006) and Cagetti and De Nardi (2006), with recent contributions by Benhabib et al. (2014), Gabaix et al. (2016), Jones and Kim (2018), Peter (2021), Atkeson and Irie (2022), Hui

(2023), Gomez and Gouin-Bonenfant (2024) and Robinson (2023). In recent years, this literature has aimed at accounting for three stylized facts: the thick right tail of the wealth distribution, the rapid dynamics of the wealth distribution over time, and the prevalence of new fortunes at the top. These facts serve as important devices for disciplining models of top wealth inequality.

Because the literature has emphasized the role played by entrepreneurs with high idiosyncratic risk exposures to account for these facts, studies in this literature have concluded that less restrictive debt financing could raise top wealth inequality while better risk sharing would reduce wealth inequality. For instance, in contrast with the results presented in this paper, recent studies by Peter (2021), Hui (2023), and Robinson (2023) conclude that improved risk sharing for entrepreneurs lowers wealth inequality.[6] This is because the risk-sharing rather than the scaling-up force dominates in their settings. I discuss this in more detail in Section 1.3.3. In the context of a static model where agents choose between a risky project and a safe project, Bonfiglioli (2012) studies the impact of better risk-sharing on the extensive margin of risky entrepreneurship and how that relates to income inequality. In that context, improvements in risk-sharing *lowers* income dispersion for high-ability entrepreneurs, while at the same time inducing entry *into* risky entrepreneurship among agents with lower ability, thereby raising income dispersion among those agents. Average income inequality rises if the effect of the rise in income dispersion among lower ability agents is stronger than the effect of lower dispersion among high-ability entrepreneurs. However, because dispersion falls for high-ability entrepreneurs, improvements in risk-sharing do not generate rising inequality within the top.[7] A similar conclusion is reached by Heyerdahl-Larsen et al. (2023), who study how improvements in risk sharing impacts wealth and consumption inequality in a dynamic model where entrepreneurs differ in terms of their beliefs regarding the success probability of their firm, rather than in their ability. They also find that improvement in risk sharing lowers entrepreneurial risk, but stimulates entry on the extensive margin. The results in this paper complement those result by investigating the impact of improvements

[6] In Atkeson and Irie (2022), entrepreneurs' return and idiosyncratic risk are exogenous. In this paper, they are endogenized and shown to be tightly linked.

[7] Moreover, because the model is static, it is silent on the dynamics of inequality.

in risk-sharing on entrepreneurs decision of how much risk to take on the intensive margin, and how that impacts both the level of wealth inequality and its dynamics over time. In this context, when improvements in risk-sharing for entrepreneurs leads to more risk-taking, it does so for *all* entrepreneurs, which leads to rising wealth inequality and faster wealth dynamics within the top as well. That is, it changes the so-called Pareto shape of the wealth distribution. The results in this paper also complement those of İmrohoroğlu and Zhao (2022) and Gomez and Gouin-Bonenfant (2024), who study the impact of falling costs of capital for entrepreneurs on top wealth inequality. They emphasize that this may raise wealth inequality because entrepreneurs can more cheaply scale up their firms. The focus in the present paper is instead on improvements in risk-sharing, rather than a general fall in the cost of capital. Panageas and Gârleanu (2024) study how disruptive growth influences the demand for alternative asset classes such as private equity and venture capital, and how this interacts with the wealth accumulation process of innovators. Further, as in this paper, they study how entrepreneurs wealth accumulation process in turn shapes the top of the wealth distribution.

Compared to models in this literature that are based on the framework of Aiyagari (1994), the model in this paper is based on Angeletos (2007), so that the economy aggregates tractably despite the presence of idiosyncratic risk. The specification of the risk-sharing environment as one where equity issuance is possible but limited due to agency frictions as in Di Tella (2017) and Brunnermeier and Sannikov (2017). Risk-aversion, combined with the limits to risk-sharing, induces entrepreneurs to run smaller firms than under perfect risk-sharing as in Kihlstrom and Laffont (1979). Relative to the model of Angeletos (2007) (and Brunnermeier and Sannikov (2017)), I introduce imperfect substitutability between the goods produced by the two types of firms. I also consider a demographic setup where the between-type distribution of wealth and the cross-sectional distribution of wealth are stable in the long run despite the presence of idiosyncratic risk. These modifications are essential for considering the issues at the heart of this paper: top wealth inequality, the factor income distribution, and returns to wealth in the long run.

An ongoing discussion in the literature is the extent to which the rise in wealth inequality can be accounted for by changes in the *valuations* of the assets and liabilities in the portfolios of wealthy households relative to less wealthy households, without changes the distribution of the underlying assets and liabilities themselves, perhaps driven by falling interest rates (see Kuhn et al. (2020), Greenwald et al. (2021), Cioffi (2021), Gomez (2024), and others). Irie (2023b) points out that the increase in top wealth inequality measured in the Distributional National Accounts of Piketty et al. (2018) (which is a data set that has been used to argue that wealth inequality has risen substantially) is associated with more unequal distributions of the *income flows* that household wealth generates, suggesting that mere valuation effects do not entirely drive the rise in top wealth inequality. When studying declining interest rates, Gomez and Gouin-Bonenfant (2024) find that lower interest rates primarily raise Pareto inequality by lowering costs of capital for entrepreneurs, not through direct valuation effects.[8] Other such explanations include changes in taxation as emphasized by Aoki and Nirei (2017), Hubmer et al. (2021) and Kaymak and Poschke (2016); Moll et al. (2022), focusing on automation; and Jones and Kim (2018) and Atkeson and Irie (2022), who focus on entrepreneurs and business owners.[9] Aoki and Nirei (2017) attribute rising Pareto inequality to lower taxes, making entrepreneurs want to increase their exposure to their firms. In the present study, it is instead reduced equity financing frictions that make entrepreneurs want to scale up.

1.2 Scaling Up and Risk Reduction in a Simple Framework

In this section, I present a simplified partial equilibrium framework where improved risk sharing for entrepreneurs unambiguously leads to increases in risk-taking. In other words, the *scaling-up effect* dominates when risk sharing improves. I also show how moving from partial equilibrium to general equilibrium *can* turn this result on its head. This section thus serves as motivation for the full model presented in Section

[8]Interestingly, the long-run response of the model presented in this paper includes a fall in the cost of capital as well. Nevertheless, in contrast to Gomez and Gouin-Bonenfant (2024), this is an outcome rather than a driving force.

[9]Recent surveys of the determinants of wealth inequality in the context of macroeconomic models include for instance De Nardi and Fella (2017), Benhabib and Bisin (2018)

1.3. In that full model, the extent to which entrepreneurial firms can poach customers, raise capital, and attract labor at the expense of other firms in the economy is what determines precisely how strong the scaling-up effect ends up being in equilibrium.

1.2.1 Partial Equilibrium: The Scaling-Up Force Dominates

Consider a continuum $i \in [0,1]$ of entrepreneurs operating firms. Each firm produces an output flow using capital, labor, and a Cobb-Douglas production technology. Entrepreneur i accumulates capital subject to idiosyncratic risk:

$$\begin{aligned} y_{it} dt &= \bar{A} k_{it}^{\alpha} l_{it}^{1-\alpha} dt \\ dk_{it} &= (\iota_{it} - \delta)\, k_{it} dt + k_{it} \tilde{\sigma} dZ_{it} + d\Delta_{it}^{k} \end{aligned} \tag{1.1}$$

where ι_{it} is the investment rate, δ is the depreciation rate, $d\Delta_{it}^{k}$ is net purchases of capital, and $k_{it}\tilde{\sigma} dZ_{it}$ is the idiosyncratically risky part of capital accumulation. In particular, Z_{it} is an individual-specific Brownian motion. The entrepreneur finances the capital stock by investing their own wealth, by issuing risk-free securities d_{it}, and by issuing risky equity v_{it}^{out}. The capital structure of the firm is therefore

$$k_{it} = n_{it} + v_{it}^{\text{out}} + d_{it}.$$

The equity issued to outsiders carries the same risk as the risk in the firm's capital. Risk sharing through equity issuance is limited. In particular, the entrepreneur is subject to a skin-in-the-game constraint:

$$\frac{k_{it} - v_{it}^{\text{out}}}{k_{it}} \geq \chi \tag{1.2}$$

where χ is the fraction of the firm's risk that the entrepreneur must bear. The interest rate on risk-free debt is r_t. The required return on equity issued to outsiders is r_t^{out}. Because outsiders hold this equity as part of diversified portfolios, and since all risk is idiosyncratic and therefore washes away in such a portfolio, no-arbitrage implies that $r_t = r_t^{\text{out}}$. Although the two sources of financing have the same cost of capital, the critical difference between issuing risk-free debt and issuing outside equity is that the former raises the risk for the entrepreneur, whereas the latter reduces the risk. Because entrepreneurs are risk averse, they will be reluctant to issue too much risk-free debt. In this sense, it is the risk aversion of the entrepreneurs *together* with the limits to risk-sharing that limits the size of their firms, rather than an limit on borrowing. The wage rate is w_t. The entrepreneur consumes at rate c_{it} and has logarithmic utility. The entrepreneurs' problem is therefore

$$\max_{\{c_{it},k_{it},l_{it},v_{it}^{\text{out}},d_{it}\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_{it})\,dt\right]$$
$$dn_{it} = \left(y_{it} - w_t l_{it} - \delta k_{it} - (d_{it} + v_{it}^{\text{out}})r_t - c_{it}\right)dt + (k_{it} - v_{it}^{\text{out}})\tilde{\sigma}\, dZ_{it}$$
$$\text{subject to}\quad \frac{k_{it} - v_{it}^{\text{out}}}{k_{it}} \geq \chi, \quad \text{and} \quad n_{it} \geq 0.$$

To solve this, we first define the instantaneous return on the firm's capital as[10]

$$dR_t^k \equiv \underbrace{\left(\frac{y_{it} - w_t l_{it} - \delta k_{it}}{k_{it}}\right)}_{\text{expected return: } r_{it}^k} dt + \tilde{\sigma}\, dZ_{it}.$$

Next, we note that the firm's labor demand decision is static. In particular, the associated first-order condition pins down the labor-to-capital ratio as $(1-\alpha)\left(\frac{k_{it}}{l_{it}}\right)^\alpha = w_t \Rightarrow \frac{l_{it}}{k_{it}} = \left(\frac{1-\alpha}{w_t}\right)^{1/\alpha}$. Because the optimal labor-to-capital ratio does not depend on i, the expected return to capital r_{it}^k does not depend on i. Let r_t^k denote this common expected return (which will depend on the prevailing wage rate). We also note that the equity issuance constraint is always binding in optimum; if it were not, then the entrepreneur could issue more outside equity and invest the proceeds in the risk-free asset with the same expected return but no risk. This would reduce risk without affecting expected returns and, therefore, make the entrepreneur better off. Hence, $v_{it}^{\text{out}} = (1-\chi)k_{it}$. We can then redefine the entrepreneurs' problem as a Merton

[10]Note that purchases of capital do not contribute to the instantaneous return on capital because the cost of expanding the capital stock is equal to the instantaneous increase in value. Of course, the purchased capital contributes to returns going forward.

portfolio choice problem instead:

$$\max_{\left\{\frac{c_{it}}{n_{it}},\frac{k_{it}}{n_{it}}\right\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_{it})\,dt\right]$$
$$\frac{dn_{it}}{n_{it}} = \left(r_t + \frac{k_{it}}{n_{it}}(r_t^k - r_t) - \frac{c_{it}}{n_{it}}\right)dt + \frac{k_{it}}{n_{it}}\chi\tilde{\sigma}dZ_{it}.$$

Subject to the skin-in-the-game constraint and non-negative net worth. This problem has the following well-known solution for the optimal choice of investment in the risky asset, captured by the optimal firm size relative to wealth (see Merton (1992) or Brunnermeier and Sannikov (2017)):

$$\frac{k_{it}}{n_{it}} = \frac{r_t^k - r_t}{(\chi\tilde{\sigma})^2}. \tag{1.3}$$

The entrepreneurs' risk exposure, defined as the volatility of net worth, implied by this solution is

$$\tilde{\sigma}_{it}^E \equiv \frac{k_{it}}{n_{it}}\chi\tilde{\sigma} = \underbrace{\frac{r_t^k - r_t}{\chi\tilde{\sigma}}}_{\text{Sharpe ratio}} \tag{1.4}$$

where $\tilde{\sigma}_{it}^E$ is the resulting volatility of the entrepreneurs' net worth. In other words, entrepreneurs choose an exposure to the idiosyncratic risk equal to the Sharpe ratio associated with investing in entrepreneurial capital, taking into account that they only carry a fraction χ of the risk. Taking the wage rate w_t and the risk-free rate r_t as given, it is clear that improved risk sharing would induce entrepreneurs to increase their risk exposures: a fall in χ improves the risk-reward trade-off as measured by the Sharpe ratio. A higher Sharpe ratio means a higher optimal risk exposure. This is the *scaling-up force* in action. When risk sharing improves so that entrepreneurs can carry a smaller fraction of the risk in their firm, they scale up so much that their total risk exposure rises. The following lemma summarizes this discussion.

Lemma 1. *Keeping fixed expected returns, a fall in χ raises the Sharpe ratio $\frac{r_t^k - r_t}{\chi\tilde{\sigma}}$ and, therefore, entrepreneurs' optimal risk exposure.*

However, the expected excess return $r_t^k - r_t$ is an equilibrium object, and it is easy to see that the effect of improved risk sharing on entrepreneurs' risk exposure can easily go the other way around in equilibrium.

1.2.2 An Example of the Risk-Reduction Effect Dominating

As an example of a setting where the risk-reduction effect instead dominates, consider a framework where, in equilibrium, the aggregate capital stock of the economy K_t is limited by the aggregate net worth of the entrepreneurs $N_t^E \equiv \int_i n_{it} di$. An example of such a framework would be one where the entrepreneurs were the only capital owners in the economy, and the aggregate capital stock constituted aggregate wealth.[11] In such a setting, the optimal portfolio choice of entrepreneurs (3.9) combined with the condition $K_t = N_t^E$ would imply

$$\frac{r_t^k - r_t}{(\chi\tilde{\sigma})^2} = \frac{k_{it}}{n_{it}} = \frac{K_t}{N_t^E} = 1 \Rightarrow r_t^k - r_t = (\chi\tilde{\sigma})^2$$

so that the *equilibrium* excess return is in fact proportional to χ^2. In this economy, entrepreneurs' equilibrium risk exposure would then be

$$\tilde{\sigma}_{it}^E = \frac{r_t^k - r_t}{\chi\tilde{\sigma}} = \chi\tilde{\sigma}. \tag{1.5}$$

In this case, a looser inside equity constraint (a fall in χ) instead leads to a fall in the risk exposure. In other words, the partial equilibrium result in Lemma 1 is completely reversed. The intuition behind this result is straightforward. When risk sharing improves so that entrepreneurs can carry a smaller fraction of the risk in their firm, they attempt to scale up. However, in equilibrium, they cannot all scale up in the aggregate. To ensure that entrepreneurs are content with operating the existing capital stock, the excess return has to fall. Because firm sizes (relative to the entrepreneurs' net worth) are the same as before, but entrepreneurs now hold a smaller fraction of the risk, their total risk exposure is lower.

[11] Entrepreneurs could of course still share risks in such a setting by buying equity in each others firms.

1.2.3 What Determines Which Effect Dominates?

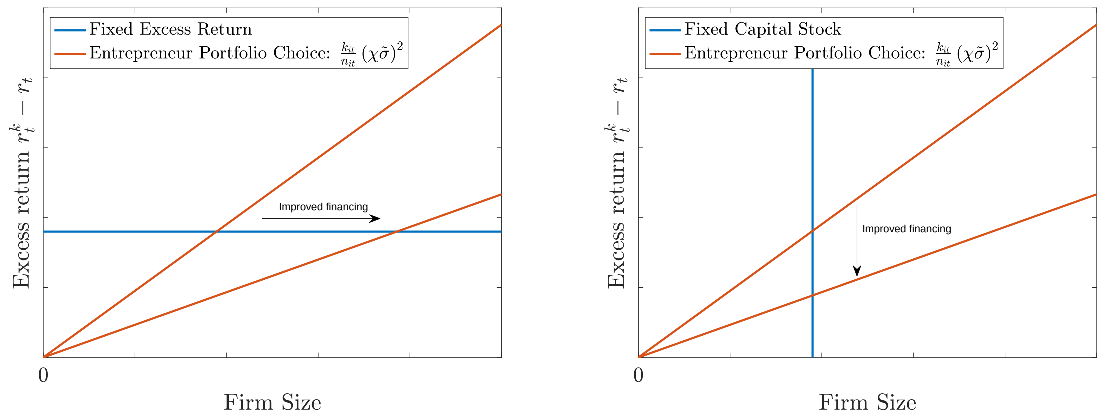
The intuition behind the contrasting results in the partial equilibrium case in Section 1.2.1, where excess returns are fixed, and the particular general equilibrium example in Section 1.2.2, where firm sizes relative to entrepreneurs' net worth are fixed, can be understood through Figure 1.1. The left panel, Figure 1.1a, represents the partial equilibrium framework of Section 1.2.1. We see an upward-sloping relationship between the excess return and entrepreneurs' choice of firm size. This upward-sloping curve represents the entrepreneurs' portfolio choice. When the excess return is high, entrepreneurs supply their firms with a lot of capital. The slope is determined by, among other things, the inside equity fraction χ. When χ falls, this supply schedule rotates outwards. In the left panel, where excess returns are fixed, the improvements in risk sharing lead to a substantial increase in optimal firm sizes.

In contrast, the right panel, Figure 1.1b, represents the framework of Section 1.2.2. Here firm size relative to entrepreneurs' net worth is fixed at $\frac{K_t}{N_t^E} = \frac{k_{it}}{n_{it}} = 1$ in equilibrium. Any improvement in risk sharing is, therefore, immediately accompanied by a reduction in the excess return.

In this paper, I will argue that both the partial equilibrium framework represented by Figure 1.1a and the particular general equilibrium framework represented by Figure 1.1b are too extreme. Specifically, I will develop a general equilibrium model where neither the excess return nor the amount of capital relative to entrepreneurs' net worth is fixed. The equilibrium response of entrepreneurs' choice of risk exposure will then depend on exactly how sensitive excess returns are when entrepreneurs try to scale up. When entrepreneurial firms are the only firms in the economy, they can not scale up at all in the aggregate. One way of avoiding this stark implication is, therefore, to introduce other types of firms into the economy.[12] The entrepreneurial firms will then be able to scale up in the aggregate at the expense of these other firms. The easier it is for the entrepreneurial firms to poach demand, raise capital, and attract labor from these other firms, the more closely this general equilibrium model will resemble the partial equilibrium framework represented by Figure 1.1a.

In Section 1.3, I present precisely such a model. In that model, the ease with

[12] Another way would be to allow capital inflow from abroad.



(a) Partial equilibrium response of better risk sharing (fall in χ). Excess return is stable, optimal firm size increases substantially. **(b)** One-sector general equilibrium response of better risk sharing (fall in χ). Excess return falls, no change in firm size.

Figure 1.1: Partial equilibrium versus one-sector general equilibrium response of excess return and optimal firm size k_{it} when risk sharing improves.

which entrepreneurial firms can attract economic activity from the other firms in the economy is governed by the elasticity of substitution between the goods that the entrepreneurial firms produce and those that these other firms produce. When the elasticity is high, entrepreneurial firms will be able to attract a lot of economic activity from the other firms in the economy as entrepreneurial financing improves, and the resulting equilibrium will resemble the partial equilibrium framework in Figure 1.1a, where the scaling-up effect dominates. When the elasticity is low, entrepreneurs will have a hard time attracting economic activity from these other firms, and the equilibrium will more closely resemble the one represented by Figure 1.1b, where the risk-reduction effect dominates.

1.3 Full Model

Relative to the simplified framework in the previous section, I now consider a model with three types of agents and two types of firms. In addition to entrepreneurs and hand-to-mouth workers, the model will also feature diversified investors. The model will now include a standard neoclassical firm as well as those operated by the entrepreneurs. The entrepreneur-operated firms will be more productive but will be constrained in their equity issuance, as in the previous section. The neoclassical firm, referred to as the traditional firm, will be less productive but will not face any financing constraints. The substitutability of the goods produced by the entrepreneurial firms and those produced by the traditional firm will be governed by a constant elasticity of substitution (CES) parameter ε. This makes the model into a modified version of Angeletos (2007). The key qualitative differences being that the entrepreneurial firms can also be publicly traded as in Brunnermeier and Sannikov (2017), that there is a class of diversified capitalists (which allows leverage in equilibrium even for wealthy entrepreneurs), that the substitutability of the goods produced by the firms is limited, and the details of the demographic setup discussed next.

Demographics. The demographics in the model are set up to allow the distribution of wealth to be stable in the long run. Specifically, the economy is populated by a continuum of hand-to-mouth workers endowed with L units of labor and a continuum $i \in [0, 1]$ of capitalists. The group of capitalists consists of two types: entrepreneurs and diversified capitalists, denoted by E and D, respectively. Entrepreneurs own a project and can choose to run a firm based on this project. Diversified capitalists do not have a viable project and instead passively invest their wealth. Entrepreneurs lose their ability to operate a firm at rate ϕ^l and then become diversified capitalists. Capitalists die at rate $\tilde{\delta}_d$. When this happens, the capitalist is replaced with offspring who either inherit the wealth and type of their parent, leaving the dynasty intact, or the dynasty breaks, and the new agent is reborn with the average wealth level of capitalists. The probability that the dynasty is broken conditional on death is π_0. We denote by $\delta_d = \tilde{\delta}_d \pi_0$, the rate at which dynasties are broken. When dynasties are broken, the newborn agent becomes an entrepreneur with probability ψ^0. Setting the initial fraction of entrepreneurs in the economy to $\bar{\psi} = \frac{\delta_d \psi^0}{\delta_d + \phi^l}$ ensures that the population structure remains intact over time.

Firms and technology. There are two types of intermediate goods-producing firms, namely (i) a representative traditional firm and (ii) a continuum of entrepreneurial firms. The representative traditional firm is entirely standard and owns and operates a capital stock K_t^T that evolves according to

$$\frac{dK_t^T}{K_t^T} = \left(\iota_t^T - \delta\right) dt + \sigma dZ_t \tag{1.6}$$

where ι_t^T and δ are the investment and depreciation rates respectively, and Z_t is an aggregate shock.[13] The firm finances this capital stock externally by issuing equity to the capitalists in the economy. The capital structure of the traditional firm is therefore $K_t = V_t^{T,\text{out}}$, where $V_t^{T,\text{out}}$ is the total amount of equity issued. The cost of this equity capital, its required return, is determined by competitive capital markets. In particular, this equity pays an expected return of r_t^T, to be determined in equilibrium, and has the same risk as the risk in the capital, so the return for investing in the equity of the traditional firm is

$$dR_t^T = r_t^T dt + \sigma dZ_t. \tag{1.7}$$

The firm hires labor from the workers at the wage rate w_t. The traditional firm uses a standard Cobb-Douglas technology to produce an output flow $Y_t^T dt = \underline{A}(K_t^T)^{\alpha}(L_t^T)^{1-\alpha} dt$. This is sold at price p_t^T. The traditional firm maximizes expected profit flows, $\pi_t^T = \max_{L_t^T, K_t^T} p_t^T Y_t^T - w_t L_t^T - \left(\delta + r_t^T\right) V_t^{T,\text{out}}$ subject to $K_T = V_t^{T,\text{out}}$.[14] This implies that wages and rates of returns are equated to the value of marginal products of the respective factors of production:

$$w_t = p_t^T (1-\alpha)\frac{Y_t^T}{L_t^T} \quad \text{and} \quad r_t^T + \delta = p_t^T \alpha \frac{Y_t^T}{K_t^T}. \tag{1.8}$$

Entrepreneurial firms produce the second type of intermediate goods. They also employ a Cobb-Douglas technology to produce an output flow $y_{it} dt = \bar{A} k_{it}^{\alpha} l_{it}^{1-\alpha} dt$, but where $\bar{A} > \underline{A}$ so that entrepreneurial firms have higher total factor productivity than does the traditional firm. Entrepreneurial firms hire labor at the wage rate w_t in the same competitive labor market as the traditional firm. The intermediate good entrepreneurial firms produce is sold to the final goods producer at a price p_t^E. The total quantity of this intermediate good is $Y_t^E = \int_{i \in E} y_{it}\, di$.

Each entrepreneurial capitalist manages the stock of capital used by their firm. The

[13] In Appendix A.2.2, I define the continuum of traditional firms that the representative traditional firm represents.

[14] Investment drops out of the optimization problem because investing one unit of capital decreases cash flows by one unit but instantaneously increases the value of the capital stock by one unit.

capital is subject to idiosyncratic risk. The stock of capital evolves according to

$$dk_{it} = (\iota_{it} - \delta)\, k_{it}dt + y_{it}\tilde{\sigma}dZ_{it} + k_{it}\sigma dZ_t + d\Delta_{it}^k \tag{1.9}$$

where ι_{it} and δ are the investment and depreciation rates, respectively, $d\Delta_{it}^k$ is net purchases of capital, Z_{it} is an idiosyncratic Brownian motion, Z_t is an aggregate Brownian motion, $\tilde{\sigma}$ and σ are scalars governing the loadings on these Brownian risks.

Note that the idiosyncratic shocks are proportional to output. This specification of the idiosyncratic risk is directly related to the risk specification in Di Tella and Hall (2021). One interpretation is that the idiosyncratic shocks become larger the more intensely the capital is used in production. This assumption has two implications. First, it makes the model *more* tractable because it will imply that the entrepreneurial firms and the traditional firms will choose the same labor-to-capital ratio. If the shocks were proportional to capital instead of output, the entrepreneurial firms would be less capital intensive than the traditional firms. The intuition for this is the following: for traditional firms, expanding production is associated with some marginal cost determined by the wage rate and required return on capital. For entrepreneurial firms, expanding production is also associated with higher risk. If the idiosyncratic shocks depended on capital alone, expanding by increasing capital would be risky on the margin, whereas expanding by hiring more labor would not. Hence, compared to the traditional firm, capital would be a relatively more costly factor of production when taking into account this risk cost. By having the idiosyncratic risk proportional to output, expanding by hiring more labor also becomes risky on the margin. This re-establishes the symmetry between capital and labor and ensures that the trade-off is not distorted by risk.[15] Secondly, this assumption also implies that the entrepreneurs' share of income will come at the expense of both the pure labor share and the pure capital share. This will imply that the entrepreneurial firms have lower labor shares and lower pure capital shares, providing the model with non-trivial

[15] Angeletos (2007) directly assumes that the production technologies are different in such a way that the two types of firms choose the same capital-labor ratio. In contrast, the fact that the capital-labor ratios are the same in this model is an *implication* of the fact that both labor and capital are risky.

testable implications for factor income shares.[16]

The return on capital for an entrepreneurial firm is

$$dR_{it}^{k} = \left(\frac{p_t^E y_{it} - w_t l_{it} - \delta k_{it}}{k_{it}} \right) dt + \tilde{\sigma}_{it}^{k} dZ_{it} + \sigma dZ_t \tag{1.10}$$

where $\tilde{\sigma}_{it}^{k} = \frac{y_{it}}{k_{it}} \tilde{\sigma}$ is the loading on the idiosyncratic Brownian.[17] Final output Y_t is produced by a representative firm using a CES technology and the two types of intermediate goods,

$$Y_t dt = \left[\nu \left(Y_t^E \right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu) \left(Y_t^T \right)^{\frac{\varepsilon-1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon-1}} dt$$

where ε is the elasticity of substitution between the intermediate goods. Note that this is not a model with monopolistic competition. Entrepreneurs are in perfect competition with each other. The limited substitutability is between entrepreneurs and traditional firms. The parameter ε governs the strength of the implicit competition between the sectors.[18] The final goods producer's first-order conditions are

$$p_t^E = \nu \left(\frac{Y_t^E}{Y_t} \right)^{-\frac{1}{\varepsilon}}, \quad p_t^T = (1-\nu) \left(\frac{Y_t^T}{Y_t} \right)^{-\frac{1}{\varepsilon}}. \tag{1.11}$$

Financial markets. Any capitalist can issue or invest in zero-net supply riskless debt at the riskless rate r_t. Entrepreneurial capitalists can also issue equity. However, this outside financing is constrained. In particular, the entrepreneur faces a skin-in-the-game constraint so that at least a fraction χ of the risk in the firm must be retained. Letting v_{it}^{out} denote the total value of the liabilities issued to outsiders by entrepreneur i, the constraint is

$$\frac{k_{it} - v_{it}^{\text{out}}}{k_{it}} \geq \chi. \tag{1.12}$$

[16]A growing literature emphasizes the role of risk for determining factor income shares. See for instance Hartman-Glaser et al. (2019) for idiosyncratic risk, and David et al. (2023) for aggregate risk.

[17]This is the instantaneous return to the existing capital stock and therefore does not include any references to capital purchases $d\Delta_{it}^{k}$.

[18]Relative to sector-specific capital adjustment costs, this imperfect substitutability assumption is more tractable. This is because sector-specific capital adjustment costs are both an intratemporal and intertemporal friction. Imperfect substitutability is solely intratemporal.

The risk in the liabilities issued to outsiders is determined by the riskiness of the productive assets of their firm, but the price of those liabilities, and hence their expected return, is determined in a competitive financial market. Outsiders hold the liabilities of firm i as part of a diversified portfolio of the liabilities of all firms and, therefore, do not require a risk premium for the idiosyncratic risk associated with firm i. Pricing by arbitrage then implies that the equilibrium expected return on the liabilities of firm i is $r_t^{\text{out}} = r_t + \varsigma_t \sigma = r_t^T$, where ς_t is the price of aggregate risk in the economy and r_t is the risk-free rate. Note in particular that the expected return on equity issued to outsiders is identical to the expected return to equity issued by traditional firms, r_t^T. This is because both carry the same amount of aggregate risk, and outsiders do not require compensation for idiosyncratic risk as they can diversify it away. The total return is therefore

$$dR_{it}^{\text{out}} = r_t^T dt + \tilde{\sigma}_{it}^k dZ_{it} + \sigma dZ_t. \tag{1.13}$$

From the perspective of households investing in the firms, it is without loss of generality to assume that they invest in a mutual fund consisting of the liabilities of all firms in the economy, traditional and entrepreneurial, with return

$$dR_t^{\text{fund}} = r_t^T dt + \sigma dZ_t. \tag{1.14}$$

I purposely model improvements in entrepreneurial financing in a stylized fashion rather than modeling it after the particularities of today's venture capital industry. Specifically, I model innovation in the financing of entrepreneurial firms as a fall in χ, the minimum inside equity financing fraction. This is motivated by two considerations. First, this paper focuses on the consequences improvements in entrepreneurial financing have for top wealth inequality rather than on the sources of those improvements. Second, although this study focuses on a particular historical episode, the framework is applicable more generally. Other contexts in which improvements in entrepreneurial financing have impacted top wealth inequality differ in the details while sharing the operational mechanism studied in this paper.

The valuation of entrepreneurial firms. The formulation of how entrepreneurial firms are financed in the model does not reference the number of shares the entrepreneurs issue or the prices of these shares. Instead, the financing of the entrepreneurial firms is expressed in terms of the amount of capital raised from outsiders and the expected return these outsiders receive. There is, of course, a link between the two ways of formulating the financing of these firms. Making this link explicit clarifies two things. First, it clarifies that the entrepreneurs' insider equity financing fraction χ should not be confused with their insider ownership fraction. Second, it demonstrates that the model produces deviations from Tobin's $q = 1$ using neither capital adjustment costs nor market power, which are the more common modeling devices that accomplish this.

An entrepreneur who has decided on operating a firm with total capital stock k_{it} must provide at least χk_{it} of the financing and can raise at most $(1 - \chi)k_{it}$ from outsiders. Let N_0 be the initial number of shares, all owned by the entrepreneur. The number of shares the entrepreneur has to issue to the outsider, Δ_{Nt}, is then defined by

$$v_{it}^{\text{out}} \equiv \Delta_{Nt} p_{it} = (1 - \chi)k_{it} \tag{1.15}$$

where p_{it} is the price per share issued. The equilibrium price per share issued, on the other hand, is pinned down by the condition that the equilibrium expected return on equity to outsiders is $r_t^T dt$. In other words,

$$\frac{\left(\frac{\Delta_{Nt}}{N_0 + \Delta_{Nt}}\right) k_{it}(1 + r_{it}^k dt)}{p_{it}\Delta_{Nt}} = 1 + r_t^T dt \tag{1.16}$$

where the numerator is the payoff to the outsider and the denominator is the amount invested by the outsider. Equations (1.15) and (1.16) jointly pin down the price and the number of shares issued in terms of the expected returns and the outside financing fraction $1 - \chi$:

$$\Delta_{Nt} = \frac{(1 + r_t^T dt)(1 - \chi)}{(r_{it}^k - r_t^T)dt + \chi(1 + r_t^T dt)} N_0$$

$$p_{it} = \left(\frac{(r_{it}^k - r_t^T)dt + \chi(1 + r_t^T dt)}{1 + r_t^T dt}\right) \frac{k_{it}}{N_0}.$$

Measuring outsiders' stake in the firm as $p_{it}\Delta_{Nt}$, the price per share times the number of shares they hold, coincides with the model notion of the value of their stake in the firm, since by construction $p_{it}\Delta_{Nt} = (1-\chi)k_{it}$. That is, however, not true for the entrepreneur if there is a risk premium associated with entrepreneurship so that $(r_t^k - r_t^T) > 0$. In particular, the valuation of the entrepreneur's shares is

$$p_{it}N_0 = \left(\frac{(r_{it}^k - r_t^T)dt + \chi(1 + r_t^T dt)}{1 + r_t^T dt}\right) k_{it} > \chi k_{it} \tag{1.17}$$

where the inequality follows from the fact that $(r_t^k - r_t^T) > 0$. This also illustrates that χ should not be confused with the entrepreneur's ownership share measured as the fraction of the outstanding shares the entrepreneur holds. Rather, χ is the insider financing share, the share of the financing that the entrepreneur provides.

The discrepancy stems from the fact that p_{it} is the price an investor with no exposure to the idiosyncratic risk in firm i is willing to pay for a share. This is more than what the entrepreneur associated with that firm is willing to pay for a share because the entrepreneur has to maintain a non-negligible exposure to the idiosyncratic risk in the firm and requires a risk premium for that.[19]

This has important implications for the measurement of the value of an entrepreneurial firm, both in the context of this model and in reality. First, there is a gap between the market cap of the firm, as measured as the price per share times the number of shares outstanding, and the value of the capital stock invested in the firm. In this sense, the entrepreneurial firms in the model have Tobin's Q's that differ from 1. Specifically, the deviation from $q = 1$ is

$$q_{it} - 1 = \frac{p_{it}\left(N_0 + \Delta_{Nt}\right)}{k_{it}} - 1 = \frac{1 + r_{it}^k dt}{1 + r_t^T dt} - 1, \tag{1.18}$$

which is the geometric excess return to entrepreneurship. In other words, the model produces deviations from $q = 1$ without adjustment costs to capital and without market power. This is but one of the dimensions along which idiosyncratic risk

[19] Although they study the valuation of human capital, the spirit is similar to that of Huggett and Kaplan (2016): valuing an entrepreneur's firm using the entrepreneur's own stochastic discount factor yields a substantially lower value than using the public market's stochastic discount factor.

and the payoff entrepreneurs earn from carrying it have similar implications as the presence of market power. Another such instance will be discussed when we examine the model's implications for the labor share of income. It will turn out that expected returns to capital r^k_{it} are the same for all entrepreneurs, and so $q_{it} = q_t$ will also be the same for all entrepreneurs. This means that the distributions of firm values will have the same shape in the tail as the distribution of k_{it}. Introducing heterogeneity in financial constraints χ_{it} may alter this implication so that the distributions of firm values (and therefore entrepreneurial wealth) displays a different shape in the tail compared to the distribution of k_{it}.[20]

Aggregates. The financial wealth in the economy is $N_t = N^E_t + N^D_t$, where $N^j_t = \int_{i \in j} n_{it}\, di$ is the financial wealth of capitalists of group $j \in \{E, D\}$. The share of financial wealth held by entrepreneurial capitalists is denoted by $\eta_t = \frac{N^E_t}{N_t}$. The financial wealth consists of claims on the productive assets of the economy, in other words, the real capital of the economy K_t. Since the financial wealth of the economy constitutes claims on the capital stock of the economy, we have $K_t = N^E_t + N^D_t$. The use of the capital stock is split between the traditional firms and the entrepreneurial firms. The share of the capital stock used by entrepreneurial firms is denoted $\kappa_t = \frac{K^E_t}{K_t}$. The labor-to-capital ratio is equalized across firms in equilibrium because the trade-off between labor and capital in production is the same for all firms. Therefore, the aggregate output can be written as

$$Y_t = A(\kappa_t) K^\alpha_t L^{1-\alpha} \tag{1.19}$$

where the aggregate total factor productivity (TFP) is

$$A(\kappa_t) = \left[\nu \left(\bar{A} \kappa_t \right)^{\frac{\varepsilon - 1}{\varepsilon}} + (1 - \nu) \left(\underline{A}(1 - \kappa_t) \right)^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon - 1}}, \tag{1.20}$$

which depends on the capital allocation. Aggregate investment in the economy is output less consumption. Therefore, the aggregate capital stock evolves according to

[20] With homothetic utility entrepreneurs' consumption will be proportional to k_{it} in this framework, hence if k_{it} is not proportional to the market value of their firm, the distribution of consumption will have a different shape in the tail than the distribution of wealth. This would potentially address the puzzle studied in Gaillard et al. (2023), without the need for non-homothetic utility. See also Ma and Toda (2021) for a different framework and mechanism studying this "puzzle".

$$dK_t = \left(Y_t - C_t^E - C_t^D - C_t^W - \delta K_t\right) dt + \sigma K_t dZ_t \tag{1.21}$$

where C_t^E, C_t^D, and C_t^W are the consumption of entrepreneurial capitalists, diversified capitalists, and workers, respectively.

Entrepreneurs' problem. In this section, I solve for the entrepreneurs' consumption and portfolio choices. In particular, I will solve for the entrepreneurs' choice of how much idiosyncratic risk to bear, which will be key for this paper's result on top wealth inequality because these choices determine the dynamics of the entrepreneurs' wealth accumulation process. The net worth of an individual entrepreneur can be written as

$$n_{it} = \underbrace{k_{it}}_{\text{capital}} - \underbrace{v_{it}^{\text{out}}}_{\text{outsiders' equity}} - \underbrace{d_{it}}_{\text{debt}} + \underbrace{v_{it}^{\text{fund}}}_{\text{diversified holdings}}. \tag{1.22}$$

Each of the components of an entrepreneur's net worth is associated with some expected excess return and some risk. Table 1.1 summarizes the returns and risk associated with each component.

	Expected return	Risk
k_{it}:	$\frac{p_t^E y_{it} - w_t l_{it} - \delta k_{it}}{k_{it}}$	$\tilde{\sigma}_{it}^k dZ_{it} + \sigma dZ_t$
v_{it}^{out}:	$r_t + \varsigma_t \sigma$	$\tilde{\sigma}_{it}^k dZ_{it} + \sigma dZ_t$
v_{it}^{fund}:	$r_t + \varsigma_t \sigma$	σdZ_t
d_{it}:	r_t	0

Table 1.1: Risk-return profiles for the various assets and liabilities that the entrepreneur can make use of. The level of idiosyncratic volatility is $\tilde{\sigma}_{it}^k \equiv \frac{y_{it}}{k_{it}} \tilde{\sigma}$

As in the simplified framework of Section 1.2, we can express the entrepreneurs' problem as a combination of a portfolio choice problem and a problem of choosing the optimal factor input mix. In particular, expressing each component of the firm's capital structure relative to the entrepreneur's financial wealth by letting $\theta_{it}^k = \frac{k_{it}}{n_{it}}$, $\theta_{it}^{\text{out}} = \frac{v_{it}^{\text{out}}}{n_{it}}$, $\theta_{it}^{\text{fund}} = \frac{v_{it}^{\text{fund}}}{n_{it}}$, $-\theta_{it}^d = 1 - \theta_{it}^k + \theta_{it}^{\text{out}} - \theta_{it}^{\text{fund}}$, and by letting $x_{it} = \frac{y_{it}}{k_{it}}$ denote the ratio of output to firm capital, we can write the entrepreneurs' problem as follows:[21]

$$\begin{aligned}
&\max_{\{c_{it}, x_{it}, \theta^k_{it}, \theta^{\text{out}}_{it}, \theta^{\text{fund}}_{it}\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\log(c_{it})\,dt\right] \\
&\frac{dn_{it}}{n_{it}} = \left(r_t + \theta^k_{it}\left(r^k_{it} - r_t\right) - \theta^{\text{out}}_{it}\varsigma_t\sigma + \theta^{\text{fund}}_{it}\varsigma_t\sigma - \frac{c_{it}}{n_{it}}\right)dt + \left(\theta^k_{it} - \theta^{\text{out}}_{it}\right)x_{it}\tilde{\sigma}\,dZ_{it} \\
&+ \left(\theta^k_{it} - \theta^{\text{out}}_{it} + \theta^{\text{fund}}_{it}\right)\sigma\,dZ_t, \\
&\text{where} \quad r_{it} = p^E_t x_{it} - w_t\left(\frac{x_{it}}{\bar{A}}\right)^{\frac{1}{1-\alpha}} - \delta \quad \text{and} \quad \frac{\theta^k_{it} - \theta^{\text{out}}_{it}}{\theta^k_{it}} \geq \chi.
\end{aligned} \tag{1.23}$$

and non-negative net worth $n_{it} \geq 0$. As shown in Section A.2.4 of the Appendix, solving this problem and expressing the solution in the unscaled variables implies

$$\begin{aligned}
&c_{it} = \rho n_{it}, \quad y_{it} = \bar{A}\left(\frac{1-\alpha}{\alpha}\frac{r^T_t + \delta}{w_t}\right)^{1-\alpha} k_{it}, \quad k_{it} = \frac{r^k_{it} - r^T_t}{(\chi\tilde{\sigma}^k_t)^2} n_{it} \\
&v^{\text{fund}}_{it} = \frac{r^T_t - r_t}{\sigma^2} n_{it} - \chi k_{it}, \quad v^{\text{out}}_{it} = (1-\chi)k_{it}, \quad d_{it} = n_{it} - k_{it} + v^{\text{out}}_{it} - v^{\text{fund}}_{it}
\end{aligned} \tag{1.24}$$

Note three important things. Firstly, all the decision rules are proportional to the entrepreneur's wealth, with the same proportionality for all entrepreneurs. This implies that the distribution of wealth within the group of entrepreneurs does not matter for aggregate quantities and prices. In particular, because $x_{it} \equiv y_{it}/k_{it}$ is identical for all entrepreneurs, the expected return to entrepreneurial capital is identical for all entrepreneurial firms so that we can write $r^k_{it} = r^k_t$. The same goes for the idiosyncratic risk exposure, $\tilde{\sigma}^k_{it} = \frac{y_{it}}{k_{it}}\tilde{\sigma} = \tilde{\sigma}^k_t$. Secondly, note that the skin-in-the-game constraint is always binding. This is because entrepreneurs have access to both issuing outside equity and investing in a diversified portfolio of other firms' equity. This diversified portfolio, or mutual fund, has the same expected return as issuing outside equity does, but it has lower risk. Hence, entrepreneurs will want to short (issue) as much outside equity as possible. Finally, the labor-to-capital ratio in each entrepreneurial firm is $\frac{l_{it}}{k_{it}} = \frac{1-\alpha}{\alpha}\frac{r^T_t+\delta}{w_t}$, which is the same as in the traditional sector. This means that

[21] One implication of writing the entrepreneurs' problem as a Merton optimal portfolio choice problem is that we view the entrepreneur as choosing how much capital to hold and supply to their firm k_{it}, instead of how much capital to purchase $d\Delta^k_{it}$. Hence, as in Brunnermeier and Sannikov (2017), we make no explicit reference to the capital purchase decision.

κ_t is not only the fraction of capital employed by the entrepreneurial sector but also the fraction of labor employed by the entrepreneurial sector, so aggregate supply of the intermediate good produced by entrepreneurial firms is $Y_t^E = \bar{A}\kappa_t K_t^\alpha L_t^{1-\alpha}$.

Diversified capitalists and workers. Diversified capitalists have wealth N_t^D in the aggregate. They invest this wealth in the mutual fund (diversified portfolio of equity in all firms) and riskless bonds. Diversified capitalists have log utility. Their consumption as a group is $C_t^D = \rho N_t^D$, and the fraction of their wealth invested in the mutual fund is $\theta_t^D = \frac{r_t^T - r_t}{\sigma^2}$. Workers supply labor inelastically and consume their labor income so that $C_t^W = w_t L$.

1.3.1 Characterizing The Equilibrium

In this section, I begin by characterizing the equilibrium of the model at a given point in time by considering the interactions between supply and demand for capital to entrepreneurial firms and traditional firms, respectively. I then characterize the dynamic equilibrium by describing how the economy's aggregate state variables evolve over time.

The equilibrium at a given point in time can be characterized in terms of the capital stock K_t and the share of wealth owned by entrepreneurs $\eta_t \equiv \frac{\int_{i\in E} n_{it}\, di}{N_t} = \frac{\int_{i\in E} n_{it}\, di}{K_t}$. Given values of these state variables, the equilibrium fraction of the capital stock operated by entrepreneurial firms $\kappa_t = \frac{K_t^E}{K_t}$ and the equilibrium excess return to entrepreneurial capital $r_t^k - r_t^T$ are jointly pinned down by the following system of equations:

$$\begin{aligned} \frac{\kappa_t}{\eta_t}\left(\chi\tilde{\sigma}_t^k\right)^2 &= r_t^k - r_t^T \\ r_t^k - r_t^T &= \left(\bar{A}p^E(\kappa_t) - \underline{A}p^T(\kappa_t)\right)\left(\frac{L}{K_t}\right)^{1-\alpha} \end{aligned} \tag{1.25}$$

where the prices are expressed as functions of κ_t as

$$p_t^E = \nu\left(\frac{\bar{A}\kappa_t}{A(\kappa_t)}\right)^{-1/\varepsilon}, \quad p_t^T = (1-\nu)\left(\frac{\underline{A}(1-\kappa_t)}{A(\kappa_t)}\right)^{-1/\varepsilon},$$

and aggregate TFP $A(\kappa_t)$ is defined in equation (1.20). The first of these equations

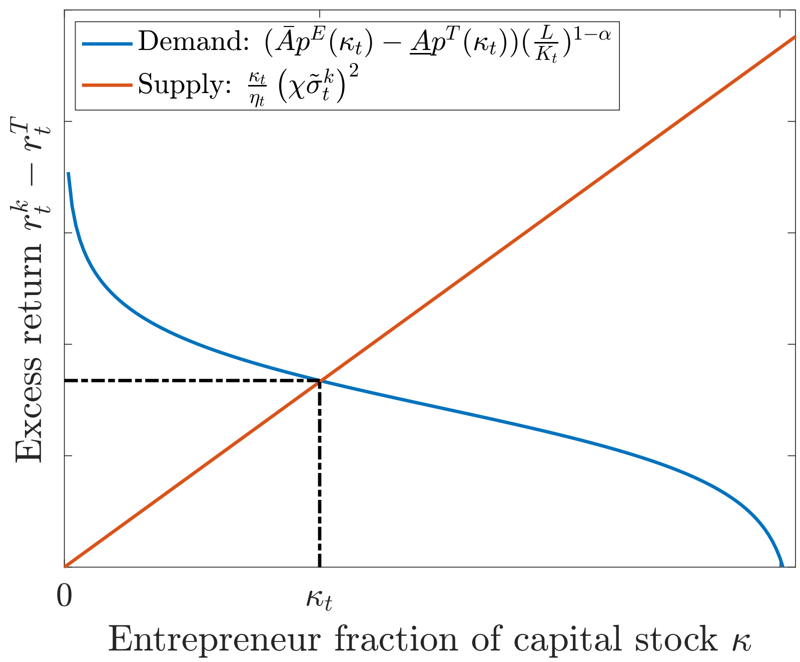


Figure 1.2: The equilibrium allocation of capital to entrepreneurs and excess return to entrepreneurial capital.

is the relative supply of capital to entrepreneurial firms. It is relative because the quantity variable is κ_t, the *fraction* of the aggregate capital stock operated by the entrepreneurial firms, and the price variable is the *excess return* $r_t^k - r_t^T$. This comes directly from the solution to the entrepreneurs' problem in (1.24), noting that the linearity of entrepreneurs' decision rules implies $\frac{k_{it}}{n_{it}} = \frac{K_t^E}{W_t^E} = \frac{\kappa_t}{\eta_t}$. The supply is upward sloping in the excess return to capital in the entrepreneurial sector as entrepreneurs are willing to invest larger amounts of capital in their firms when the excess return is high. From an asset pricing and portfolio choice perspective, this is commonly referred to as the entrepreneurs' risky asset *demand*, productive capital being the risky asset. However, of course, an entrepreneur's demand for capital as an investment vehicle constitutes the supply of capital to that entrepreneur's firm.

The second equation is instead the entrepreneurial firms' relative demand schedule, which can be derived by combining market clearing for capital, $K_t^E = K_t - K_t^T$, with the fact that the traditional sector's demand for capital is $K_t^T = \alpha\left(\frac{Y_t^T}{r_t^T+\delta}\right)$, according to (1.8).

In Section A.2.6 of the Appendix, I provide the definition of equilibrium. In Section A.2.7 of the Appendix, I show that there is a unique resource allocation κ_t that solves this system provided $\varepsilon > 0$. Given this equilibrium allocation of productive resources across the two sectors, all other prices and quantities are pinned down as well at the given point in time. These time-t prices and quantities determine the evolution of the state variables K_t and η_t going forward. These results are summarized in the following proposition.

Proposition 1. *Given values of K_t and η_t, and provided $\varepsilon > 0$, there is a unique solution $\kappa_t \in [0, 1]$ to the system of equations in (1.25). Given this solution, the relative prices of the intermediate goods are given by (1.11), while the other prices are given by*

$$\begin{aligned} r_t^T + \delta = p_t^T \alpha \underline{A} \left(\frac{L}{K_t}\right)^{1-\alpha}, \quad r_t^k = r_t^T + \frac{\kappa_t}{\eta_t}\left(\chi \tilde{\sigma}_t^k\right)^2 \\ r_t = r_t^T - \sigma^2, \quad w_t = (1-\alpha)\underline{A}\left(\frac{L}{K_t}\right)^{1-\alpha}. \end{aligned} \tag{1.26}$$

The evolution of an individual entrepreneur's wealth is

$$\frac{dn_{it}}{n_{it}} = \left(r_t^E - \rho\right) dt + \tilde{\sigma}_t^E dZ_{it} + \sigma dZ_t \tag{1.27}$$

where $\tilde{\sigma}_t^E = \frac{\kappa_t}{\eta_t}\chi\tilde{\sigma}_t^k$ is the entrepreneurs' idiosyncratic risk exposure and $r_t^E = r_t + \left(\tilde{\sigma}_t^E\right)^2 + \sigma^2$ is the expected return to the entrepreneurs' invested wealth. Finally, the system of stochastic differential equations that govern the evolution of the aggregate capital stock and entrepreneurs' wealth share is

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(\kappa_t r_t^k + (1-\kappa_t) r_t^T - \rho\right) dt + \sigma dZ_t \\ \frac{d\eta_t}{\eta_t} &= \left((1-\eta)\left(\tilde{\sigma}_t^E\right)^2 + \frac{(\bar{\psi} - \eta_t)}{\eta_t}(\delta_d + \phi^l)\right) dt. \end{aligned} \tag{1.28}$$

A brief comment is worth making regarding the consumption behaviour of the agents in equilibrium. The consumption rate out of net worth is the same for both types of capitalists, namely ρ. At the same time entrepreneurs earn a higher rate of return on their net worth in equilibrium, $r_t^E > r_t^D$. This means that entrepreneurs have a *higher* savings rate out of *income*. Because entrepreneurs will turn out to be overrepresented at the top wealth distribution, this means that the savings rate out of income is increase along the wealth distribution while the savings rate out of net worth is constant. This is inline with the observations of Fagereng et al. (2019).

The entrepreneurial appraisal ratio. A critical determinant of both the evolution of the wealth share of entrepreneurs as a group and the wealth accumulation process of individual entrepreneurs is their idiosyncratic risk exposure $\tilde{\sigma}_t^E$. This idiosyncratic risk exposure is key for understanding the level of top wealth inequality and the prevalence of "self-made" fortunes because it determines the likelihood of extreme upward wealth trajectories.

$\tilde{\sigma}_t^E$ appears directly as entrepreneurs' risk loading on their idiosyncratic risk process. However, because the equilibrium risk premium is determined by entrepreneurs' risk bearing, it also influences the return on entrepreneurs' invested wealth $r_t^E = r_t^T + \left(\tilde{\sigma}_t^E\right)^2$.

Closer inspection reveals that this idiosyncratic wealth exposure is, in fact, equal to the so-called appraisal ratio associated with investments in entrepreneurial capital. The appraisal ratio, sometimes called the information ratio, is a close cousin of the more well-known Sharpe ratio but measures instead the risk-reward trade-off associated with investing in an asset with idiosyncratic risk relative to one with the same systematic risk but no idiosyncratic risk. The fact that entrepreneurs choose an idiosyncratic risk exposure equal to the appraisal ratio is a special case of the solution to the standard optimal portfolio choice problem of Merton (1969). The fact that entrepreneurs have logarithmic utility greatly simplifies the analysis of the model, as the optimal risk exposure is unaffected by changes in the investment opportunity set.

In this model, the appraisal ratio is defined relative to the traditional firm return:

$$\text{appraisal ratio} = \frac{r_t^k - r_t^T}{\chi \tilde{\sigma}_t^k} = \frac{\kappa_t}{\eta_t} \chi \tilde{\sigma}_t^k = \tilde{\sigma}_t^E. \tag{1.29}$$

In other words, entrepreneurs choose an idiosyncratic risk exposure equal to the appraisal ratio associated with entrepreneurial capital. When the idiosyncratic risk-reward trade-off is more attractive, they choose a larger exposure, and their wealth grows faster on average at the individual level, as does the wealth share of entrepreneurs as a group. As shown below, this appraisal ratio will also determine top wealth inequality, and the effect of improved entrepreneurial financing on top wealth inequality will work through its effect on this appraisal ratio.

1.3.2 Steady State

In this section, I derive a closed-form formula for Pareto tail inequality in steady state as a function of the steady state wealth share of entrepreneurs. I thereby show a direct analytical link between the share of wealth entrepreneurs hold and the level of tail inequality. In particular, tail inequality will increase when entrepreneurs hold a larger fraction of wealth. Then, I describe how the steady state risk-reward trade-off that entrepreneurs face pins down the amount of risk they bear and how that, in turn, determines Pareto inequality.

A steady state of the economy is characterized by a pair of values for the capital stock and entrepreneurs' wealth share, K_{ss} and η_{ss}, such that $\frac{dK_t}{K_t} = 0$ and $\frac{d\eta_t}{\eta_t} = 0$. The presence of aggregate shocks to capital will, in general, prevent the economy from reaching, let alone staying in, any steady state. For this section, I study the economy's behavior along a path of zero realized aggregate shocks. In other words, I assume $dZ_t = 0$ for an indefinite time, which corresponds to studying the median path of the economy. This differs from shutting down aggregate shocks by setting $\sigma = 0$. In particular, we study the *realized* behavior of the economy in a setting where shocks are still possible but happen not to materialize.

Entrepreneurs' wealth share and Pareto inequality. In a steady state, the drift and volatility governing the wealth accumulation process of each entrepreneur is described by a geometric Brownian motion. In particular,

$$\frac{dn_{it}}{n_{it}} = \left(r_{ss} + \left(\tilde{\sigma}_{ss}^{E} \right)^2 + \sigma^2 - \rho \right) dt + \tilde{\sigma}_{ss}^{E} dZ_{it}. \tag{1.30}$$

The combination of individual wealth growing according to a geometric Brownian motion with entrepreneurial dynasties interrupted by death or type-switching implies that the steady state distribution of entrepreneurs' wealth follows a double Pareto distribution (Champernowne (1953), Reed (2001), Gabaix (2009)). The so-called Pareto tail coefficient describes the thickness of the right tail of this distribution.[22] This tail coefficient is determined by the drift and volatility of the wealth

[22]Because consumption and income are proportional to net worth, the distributions of these variables display the same limiting tail coefficient in this model. Gaillard et al. (2023) discuss how non-homothetic preferences and scale-dependent returns to wealth may address this counterfactual

accumulation process and the death and switching rates.[23] The key to the results in this paper is that the model implies a direct relationship between drift and the volatility of entrepreneurs' wealth accumulation process in a steady state equilibrium. In particular, in equation (1.28), $\frac{dK_t}{K_t} = 0$ implies that $\mu_{ss}^E = (1 - \eta_{ss}) \left(\tilde{\sigma}_{ss}^E\right)^2$, and $\frac{d\eta_t}{\eta_t} = 0$ implies that $(\tilde{\sigma}_{ss}^E)^2 = \frac{(1 - \frac{\bar{\psi}}{\eta_{ss}})(\delta_d + \phi^l)}{1 - \eta_{ss}}$. In other words, the drift and volatility are both directly related to the wealth share of entrepreneurs in a steady state equilibrium. This allows us to characterize the Pareto tail coefficient in terms of the steady state wealth share of entrepreneurs. The following proposition is proved in Appendix A.3.3.

Lemma 2. *The steady state right Pareto tail coefficient of entrepreneurs' wealth is*

$$\zeta = \eta_{ss} - \frac{1}{2} + \sqrt{\left(\eta_{ss} - \frac{1}{2}\right)^2 + \frac{2\eta_{ss}(1 - \eta_{ss})}{\eta_{ss} - \bar{\psi}}} \tag{1.32}$$

where η_{ss} is the steady state share of wealth entrepreneurs own. Moreover $\frac{\partial \zeta}{\partial \eta_{ss}} < 0$. Hence, keeping fixed the population fraction $\bar{\psi}$, tail inequality $1/\zeta$ will be higher when entrepreneurial capitalists hold a larger fraction of wealth in the economy.

Equation 1.32 is strictly decreasing in η_{ss} so that the tail is thicker the higher the share of wealth owned by entrepreneurs. This expression for the tail coefficient provides a direct analytical link between the cross-sectional distribution of wealth and the share of wealth held by entrepreneurs. Understanding how structural changes in the economy affect steady state top wealth inequality thus boils down to understanding how those structural changes affect the steady wealth share of entrepreneurs.

What determines the steady state wealth share of entrepreneurs? Looking at equation (1.28) with $\frac{d\eta_t}{\eta_t} = 0$, we see that the steady state value η_{ss} is pinned down by the exogenous demographic parameters, δ_d, ϕ^l, and $\bar{\psi}$, as well as the idiosyncratic volatility of entrepreneurs' wealth $\tilde{\sigma}_{ss}^E$, which is endogenous:

implication.

[23]Specifically, in Appendix A.3.3, I show that the stationary Kolmogorov forward equation that pins down the Pareto tail coefficient ζ is of the well-known form:

$$0 = \zeta \mu_{ss}^E + \frac{(\tilde{\sigma}_{ss}^E)^2}{2} \zeta (\zeta - 1) - (\delta_d + \phi^l) \tag{1.31}$$

where $\mu_{ss}^E = r_{ss} + \left(\tilde{\sigma}_{ss}^E\right)^2 - \rho$ is the drift of the entrepreneurs' wealth accumulation process.

$$(\tilde{\sigma}^E_{ss})^2 = \frac{(1 - \frac{\bar{\psi}}{\eta_{ss}})(\delta_d + \phi^I)}{1 - \eta_{ss}}. \tag{1.33}$$

From this equation, we see a strictly positive relationship between the steady state wealth share of entrepreneurs and their steady state idiosyncratic risk exposure. In other words, given the values of the demographic parameters, a steady state associated with a higher level of idiosyncratic risk exposure will be associated with a higher wealth share for entrepreneurs. This is because a higher idiosyncratic risk exposure will be associated with a larger idiosyncratic risk premium or, equivalently, a larger precautionary savings motive for entrepreneurs. That implies that their expected wealth growth rate will be higher than that of the diversified capitalists, which in turn implies a larger steady state wealth share.

No other endogenous objects appear in the steady state equation (1.33) for the entrepreneurs' wealth share and, consequently, in the steady state Pareto tail coefficient. In particular, apart from the entrepreneurs' idiosyncratic risk exposure $\tilde{\sigma}^E_{ss}$, which is endogenous, only exogenous demographic parameters appear in equation (1.33). Therefore, the response of the steady state Pareto tail coefficient to any non-demographic change in the economy must go via changes in the idiosyncratic risk exposure of entrepreneurs. Specifically, any change in the economy that increases the idiosyncratic risk exposure of entrepreneurs will increase entrepreneurs' share of wealth, lower the Pareto tail coefficient, and thereby increase top wealth inequality.

Finally, recall that entrepreneurs choose an idiosyncratic risk exposure equal to the appraisal ratio associated with entrepreneurial investment. These insights will allow us to thoroughly summarize the effect of improved entrepreneurial financing on top wealth inequality since we only need to determine how improved entrepreneurial financing affects the appraisal ratio associated with entrepreneurial investment.

Improved entrepreneurial financing and steady state Pareto inequality. The model features only one friction, the constraint on equity issuance that entrepreneurs face. Recall that the severity of this constraint is captured by the parameter χ, the fraction of the risk in the firm that must be borne by the entrepreneur themself. Improved entrepreneurial financing in this context thus refers to a fall in the parameter

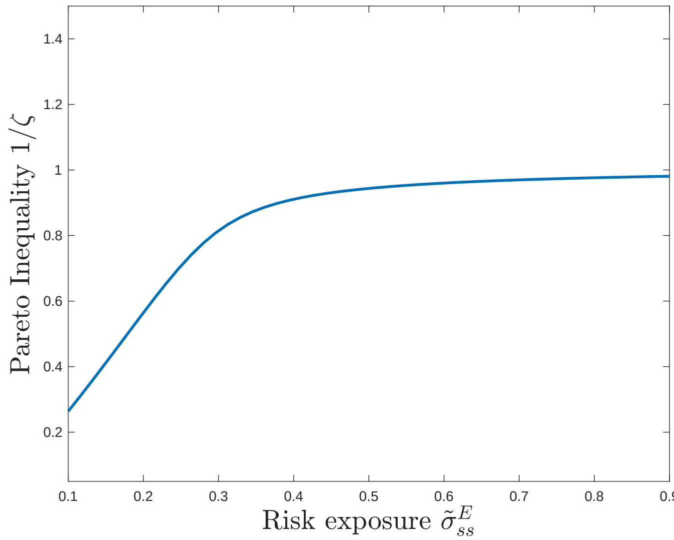


Figure 1.3: Pareto Tail Inequality $1/\zeta$ and Entrepreneurs' Idiosyncratic Risk Exposure $\tilde{\sigma}^E_{ss}$ Across Steady States.

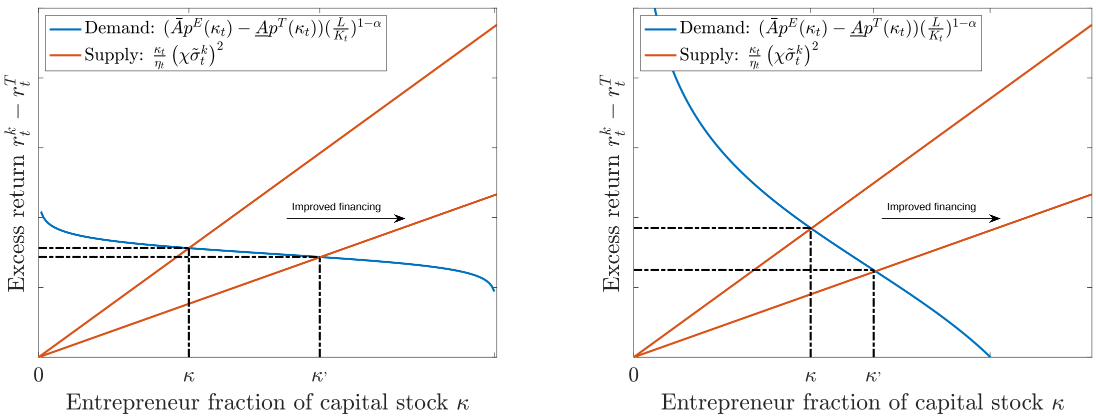
χ. Using the fact that any non-demographic change in the economy that affects the Pareto tail coefficient must operate via its effect on the amount of idiosyncratic risk that entrepreneurs choose to bear, we therefore have the following proposition:

Lemma 3. *Improvements in entrepreneurial financing, understood as a relaxation of the equity issuance constraint (a fall in χ), leads to a fall in the Pareto tail coefficient ζ (and, therefore a rise in Pareto inequality $1/\zeta$) if and only if it raises the idiosyncratic risk exposure of entrepreneurs, which is, in turn, equal to the appraisal ratio associated with entrepreneurship:*

$$\tilde{\sigma}^E_{ss} = \frac{r^k_{ss} - r^T_{ss}}{\chi \tilde{\sigma}^k_{ss}} \equiv \textit{appraisal ratio}. \tag{1.34}$$

In other words, improved financing for entrepreneurs leads to a rise in top wealth inequality if it makes the trade-off related to idiosyncratic risk bearing more attractive. Figure 1.3 depicts the relationship between the Pareto inequality $1/\zeta$ and the appraisal ratio associated with entrepreneurship, capturing the essence of the above proposition.

Examining the mechanism: the role of the elasticity of substitution ε. To understand the mechanism behind the effect of improved entrepreneurial financing, we consider the effect of a fall in χ, the inside equity constraint. This fall in χ induces a



(a) High elasticity **(b)** Low elasticity

Figure 1.4: The effect (on impact) of reduced financing frictions χ on capital allocation to entrepreneurs.

reallocation of capital from the traditional sector towards the entrepreneurial sector. This can be seen from (1.25), which we recall can be written as

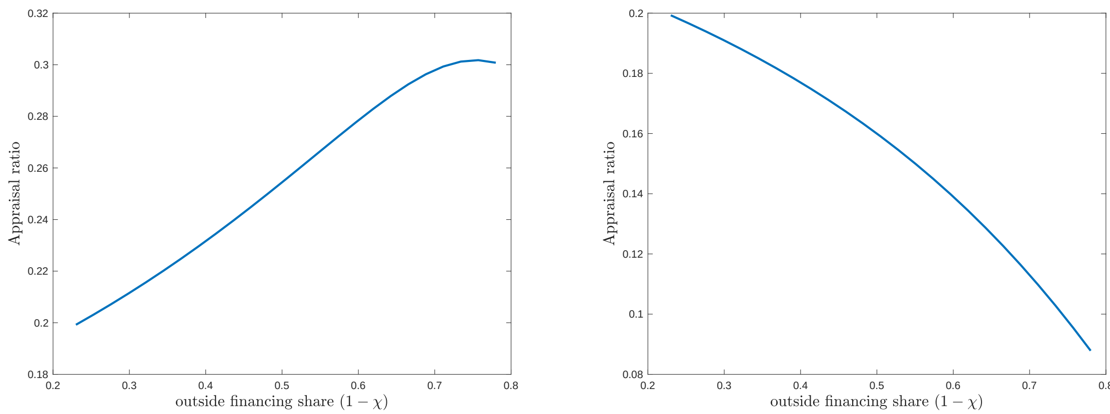
$$\frac{\kappa_t}{\eta_t}(\chi\tilde{\sigma}_t^k)^2 = \left(\bar{A}p^E(\kappa_t) - \underline{A}p^T(\kappa_t)\right)\left(\frac{L}{K_t}\right)^{1-\alpha} \tag{1.35}$$

where the prices are expressed in terms of κ_t as

$$p_t^E = \nu\left(\frac{\bar{A}\kappa_t}{A(\kappa_t)}\right)^{-1/\varepsilon}, \quad p_t^T = (1-\nu)\left(\frac{\underline{A}(1-\kappa_t)}{A(\kappa_t)}\right)^{-1/\varepsilon},$$

and $A(\kappa_t)$ is defined in equation (1.20). On impact, η_t and K_t, and therefore also $\tilde{\sigma}_t^k = \frac{Y_t^E}{K_t^E}\tilde{\sigma} = \bar{A}\tilde{\sigma}\left(\frac{L}{K_t}\right)^{1-\alpha}$, are fixed, so that the left-hand side is simply an increasing linear function of κ_t. In contrast, the right-hand side is a strictly decreasing function of κ_t. The fall in χ shifts the supply of capital to entrepreneurial firms, on the left-hand side, outward, leaving the demand schedule unaffected. The new equilibrium features a higher κ_t and a lower excess return than in the initial steady state. Figure 1.4 is a graphical representation of this for two different values of the elasticity ε.

What happens to the appraisal ratio, which we know determines entrepreneurs' risk exposure as well as Pareto inequality? Rewriting the above equation in terms of the appraisal ratio, we see that



(a) High elasticity **(b)** Low elasticity

Figure 1.5: Relationship between the steady state appraisal ratio and outside financing share $1-\chi$.

$$\tilde{\sigma}_t^E = \text{appraisal ratio} = \frac{\bar{A}p_t^E(\kappa_t) - \underline{A}p_t^T(\kappa_t)}{\chi\tilde{\sigma}\bar{A}} \tag{1.36}$$

so that the appraisal ratio may move up or down depending on how much of the fall in χ in the denominator is offset by a fall in the excess return in the numerator. The fall in the numerator is determined by how much the intermediate goods prices $p_t^E(\kappa_t)$ and $p_t^T(\kappa_t)$ change. The sensitivity of these prices is, because of the CES setup, determined by the constant elasticity of substitution parameter ε.

If the elasticity of substitution between the two sectors is high, the market adjusts primarily via quantities and not prices; that is, the excess return in the numerator is relatively stable. In this case, the appraisal ratio rises. If, on the other hand, ε is low, prices react strongly in response to any reallocation of capital. The fall in the excess return in the numerator will then be larger than the fall in the denominator, and the appraisal ratio will fall. Figure 1.5 displays the relationship between the steady state appraisal ratio and the outside financing fraction $1-\chi$. Figure 1.5a depicts the relationship when the elasticity of substitution is high, and 1.5b when this elasticity is low.

An interesting observation is that as the outside financing fraction becomes very large, the steady state appraisal ratio starts to decline. This happens because as the risk costs associated with production in the entrepreneurial sector decline, the entrepreneurial sector starts taking over all production in the economy. When this happens, the competitive pressure within the entrepreneurial sector becomes more severe since there is not much capital that can be squeezed out of the traditional sector. The increased competitive pressure between entrepreneurs for the existing capital stock then drives down the excess return to entrepreneurship so that the appraisal ratio falls.

The dynamic response of the economy after impact will also depend on whether the appraisal ratio rises or falls in the longer run as well. Recalling the equations for the evolution of the state variables in (1.28):

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(\kappa_t r_t^k + (1-\kappa_t) r_t^T - \rho\right) dt + \sigma dZ_t \\ \frac{d\eta_t}{\eta_t} &= \left((1-\eta)\left(\tilde{\sigma}_t^E\right)^2 + \frac{(\bar{\psi}-\eta_t)}{\eta_t}(\delta_d + \phi^l)\right) dt, \end{aligned} \tag{1.37}$$

we see that a rise in the appraisal ratio will raise the drift of η_t, which consequently starts to grow. The behavior of the capital stock also depends on the strength of the reallocation of capital relative to the reaction of prices to this reallocation. The expected accounting return to the capital stock is the weighted average expected return in the two sectors $\kappa_t r_t^k + (1-\kappa_t) r_t^T = r_t^T + \kappa_t (r_t^k - r_t^T)$. When capital is reallocated in response to the fall in χ, the capital stock will grow to the extent that the reallocation of capital to the higher-return entrepreneurial sector constitutes a stronger force than the excess return in that sector.

In this case, the resulting growth in the entrepreneurs' wealth share and the capital stock will induce entrepreneurs to scale up further and, therefore, increase κ_t over time. This increase in κ_t will lower the excess return and the appraisal ratio over time relative to the level reached on impact. What matters for the behavior of top inequality, in the long run, is whether the economy settles on an appraisal ratio that is higher or lower than in the initial steady state. Suppose the elasticity of substitution between the sectors is high enough. Then, the new steady state appraisal ratio will be higher than before, implying a larger share of wealth owned by entrepreneurs, faster wealth dynamics for entrepreneurs, and higher Pareto inequality. I summarize this discussion in the following proposition that says that inequality increases when χ falls, provided that the elasticity of substitution is high enough:

Proposition 2. *Suppose the economy is in an initial steady state $s_0 = (\eta_0, K_0, \kappa_0)$, where $\kappa_0 \in (0,1)$ and the initial inside equity constraint parameter is χ_0. Let $\zeta(\chi)$ denote the steady state Pareto tail coefficient as a function of χ. Then, there exists a $\varepsilon^*_{s_0}$ such that if the ε associated with the steady state s_0 is larger than $\varepsilon^*_{s_0}$, then $\frac{d\zeta}{d\chi}(\chi_0) > 0$.*

Proof. By Lemma 3, we need to show that the steady state appraisal ratio increases as χ falls for high values of ε. Let $\kappa(\chi)$ denote the steady state κ as a function of χ keeping all other parameters fixed. Then, we have by equation (1.36) evaluated in steady state:

$$\frac{d \log \tilde{\sigma}^E_{ss}}{d \log \chi} = \frac{d \log \left(\bar{A} p^E(\kappa(\chi)) - \underline{A} p^T(\kappa(\chi))\right)}{d \log \chi} - 1.$$

For the value ε associated with the steady state s_0, a change in χ will imply some change in $\kappa(\chi)$. Letting $\varepsilon^*_{s_0}$ be the value of the elasticity of substitution such that this change in $\kappa(\chi)$ leads to a small enough change in prices for the appraisal ratio to rise. In particular, since the price functions p^E and p^T can be made arbitrarily insensitive to changes in $\kappa(\chi)$ by picking a high enough value for the elasticity of substitution, we know that such a value exists. That is, there exists some $\varepsilon^*_{s_0}$ such that if $\varepsilon > \varepsilon^*_{s_0}$, we have $\frac{d \log\left(\bar{A} p^E(\kappa(\chi)) - \underline{A} p^T(\kappa(\chi))\right)}{d \log \chi} < 1$. This proves the result. □

Notice that this proposition is "local" as opposed to being uniform in the sense that the threshold value $\varepsilon^*_{s_0}$ depends on the steady state the economy starts in. This means that the relationship between χ and Pareto inequality can be non-monotonic. As we saw in Figure 1.5a, the relationship turns around when the outside financing fraction $1 - \chi$ becomes large. This extends the conclusion of Bonfiglioli (2012) that reduced financial frictions may have a non-monotonic relationship to wealth inequality, to a dynamic framework where risk-taking is an intensive margin decision. Hence, comparing steady state wealth inequality with and without financial frictions may not accurately reflect what would happen in an economy where financial frictions are reduced but not fully removed. In fact, for some initial steady states, even *infinite* elasticity of substitution is not enough to make top wealth inequality rise in response to reductions in financial frictions. As an example of this, we have the following proposition, which I prove in Appendix A.3.6:

Proposition 3. *Even with perfect substitutes, $\varepsilon = \infty$, starting in an initial steady state s_0,* *there is a value χ^* such that if χ is reduced to a value $\chi < \chi^*$, a further fall in χ reduces Pareto inequality.*

The intuition is that with perfect substitutes when χ becomes low enough, the entrepreneurial firms take over the entire economy so that $\kappa_{ss} = 1$. When κ_{ss} reaches this maximum value, entrepreneurs can no longer scale up at the expense of traditional firms. In this case, the model reduces to a one-sector model, and inequality falls when risk sharing improves because the scaling-up effect is mute. As I discuss in the next section, the absence of a sector from which entrepreneurs can attract resources is one of the reasons earlier work has found that improvements in risk sharing reduce inequality.

1.3.3 Why the Two-Sector Setup is Important

One key prediction of the model is that wealth inequality increases in response to improved entrepreneurial financing, provided that excess returns associated with entrepreneurship do not fall too much when entrepreneurs scale up. The degree to which the equilibrium excess return to entrepreneurial activity falls when entrepreneurs want to scale up depends on how fiercely they compete with each other for the fixed amount of labor and the existing capital stock. The presence of the traditional sector is critical for this model prediction. The traditional sector constitutes a source from which the entrepreneurial sector can attract labor, capital, and demand, relieving the competitive pressure. Instead of drawing economic activity from other entrepreneurial firms, which puts downward pressure on the excess return to entrepreneurship, they can draw this economic activity from the traditional sector.

Consider the extreme case where there is no sectoral reallocation possible. This would be the case in a one-sector model where all firms are entrepreneurial as in Section 1.2.2, or where the output produced by entrepreneurial firms and traditional firms are perfect complements so that they do not compete in the output market. In response to improved risk sharing, entrepreneurs want to scale up. However, because the factors of production are fixed in the short run, they cannot scale up in the aggregate.[24] Hence, the equilibrium expected excess return to entrepreneurial activity has to fall in order to render the entrepreneurs content with operating the existing capital stock and hire the avialable labor. Without the ability to scale up, entrepreneurs' risk exposure cannot rise. Because they operate a capital stock of the same size as before but carry a smaller fraction of the associated risk, their risk exposure unambiguously falls, as does wealth inequality. This mechanism is at the heart of why Hui (2023) finds that improved risk sharing for entrepreneurs lowers wealth inequality in a one-sector model. Peter (2021) also studies a model wherein production is in the hands of entrepreneurial firms. That model is rich and closer in spirit to Quadrini (2000) and Cagetti and De Nardi (2006). When calibrating this model to European data, Peter (2021) finds that improved risk sharing for entrepreneurs reduces steady state wealth inequality. The results in this paper complement those in Peter (2021) by showing that improved risk sharing can also raise top wealth inequality, provided entrepreneurs can scale up without adversely affecting the profitability of entrepreneurial activity. Furthermore, in Section 1.5, I show that at least part of the US experience of rising wealth inequality and an increasing preponderance of newly created fortunes at the top can be understood as a consequence of improved entrepreneurial equity financing. It should be said that because Peter (2021) considers a very rich model framework, it is slightly difficult to evaluate analytically precisely which of the model features produces the result that better equity financing for entrepreneurs lowers wealth inequality. The results of the present paper suggest that it may be that the equilibrium returns top entrepreneurial activity fall when entrepreneurs all try to scale up, so that the risk sharing effect ends up dominating the scaling up effect.

[24]This remains roughly true in the long run as well *despite* the fact that the capital supply is perfectly elastic in the long run in this model.

1.4 Improved Equity Financing and Top Wealth Inequality: Empirical Motivation

In the previous section, I established conditions under which improvements in risk sharing for entrepreneurs lead to higher top wealth inequality. In the next section, I will conduct a simple numerical exercise to understand whether improved risk sharing has any quantitative bite. In this section, I briefly describe the empirical motivation behind that numerical experiment. That experiment is motivated by two sets of observations. The first set relates to the characteristics of the wealthiest Americans today, how they became wealthy, and the rise of venture capital and venture capital-backed firms. Specifically, the wealthiest Americans of today are, to a larger extent than in decades past, founders or early investors in entrepreneurial firms, rather than inheritors of great fortunes.[25] These individuals were propelled to the top of the wealth distribution by raising substantial amounts of capital from outside investors, often venture capital funds. This allowed them to operate much larger firms than their wealth would have admitted. Relatedly, some evidence suggest that entrepreneurs' ability to scale up and share risks with the help of financial markets has improved over the past half-century, where a conspicuous example of this is the emergence and rapid growth of venture capital financing, which has undergone what has been referred to as a revolution.[26] The second set of facts relates to the evolution of measured top wealth inequality. Measured top wealth shares have risen substantially over the past half-century.[27] Especially noteworthy is that the observed rise in top wealth shares has been fractal, meaning that wealth inequality has risen *within* the top as well: not only has the top 1% wealth share risen, the top 0.01% share of the top 1% has risen as well. In other words, Pareto inequality has increased.[28]

The central proposition advanced by this paper is that these two sets of facts may be intimately related: improvements in the ability of entrepreneurs to raise outside

[25] See for instance Kaplan and Rauh (2013)

[26] See Gompers and Lerner (2001).

[27] There is some disagreement regarding the precise magnitudes of the increases in top wealth shares, see for instance Saez and Zucman (2016) and Smith et al. (2022b). Interestingly, both sets of authors document similar increases in Pareto inequality.

[28] See for instance Figure 4 in Gomez and Gouin-Bonenfant (2024), or figure 1.7b, based on data from Smith et al. (2022b) and Piketty et al. (2018), respectively.

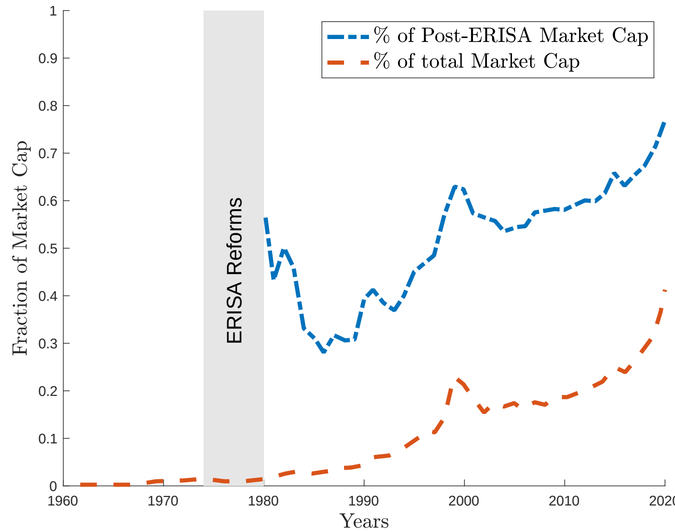


Figure 1.6: Evolution of market capitalization share of firms with history of venture capital backing. Data from Gornall and Strebulaev (2021).

equity capital and offload risk to financial markets, as exemplified by, but not limited to, the growth of the venture capital industry, may have contributed to the observed pattern of increased top wealth inequality. I now briefly discuss these motivating facts in more detail before presenting the parameterized model in Section 1.5.

The rise of venture capital-backed firms. The mechanism at the heart of this paper connects changes in the ability of innovative entrepreneurs to raise equity capital and offload risk to financial markets to the reallocation of economic activity to cutting-edge entrepreneurial firms and rising top inequality. Regarding the reallocation of economic activity, Gornall and Strebulaev (2021) document that firms with a history of VC-backing constituted around 0–5% of the total market capitalization before and up to 1980, rising to around 41% in 2020. Moreover, among firms founded after 1968, Gornall and Strebulaev (2021) document that firms with a history of VC-backing constituted around 50% of market cap in 1980, rising to 77% in 2020. Figure 1.6 from Gornall and Strebulaev (2021) summarizes the evolution of venture capital-backed firms. They argue that regulatory changes implemented through the 1974 Employee Retirement Income Security Act (ERISA) and its subsequent reinterpretation in 1979 created a substantial divergence in the creation rate of large successful companies between the U.S. and comparable countries. These reforms allowed a broader set of investors to invest in venture capital, investments that were previously regarded as too risky.[29] Many successful venture capital-backed firms and their associated founders are household names by now: Tesla, Amazon, Google, Uber, and Apple, to name but a few. It is important to note that venture capital is but a fraction of the outside financing that these firms receive. The purpose of highlighting the growth of firms with a history of venture capital backing is not to argue that venture capital was responsible for this growth. Rather, because venture capital is explicitly aimed at providing financing for cutting-edge entrepreneurial firms, the growth of venture capital is evidence of a reallocation of economic activity to these types of firms.

To highlight the role of firms with a history of venture capital backing for top wealth, Table 1.2 compares the ten wealthiest individuals on the Forbes 400 list in 2022 with the ten wealthiest individuals on the first edition of that list in 1982. As pointed out by Kaplan and Rauh (2013), and more recently by Gomez (2023), the table reflects the observation that the number of "self-made" entrepreneurs within the top 10 is markedly higher now. We also see that many of the wealthiest individuals in 2022 are associated with venture capital-backed firms.[30]

Pareto inequality. Saez and Zucman (2016) document a 13-percentage-point increase in the wealth share of the top 1%, from a low of 22% in 1978 to 35% in 2016. Similarly, Smith et al. (2022b) find an increase of 10 percentage points, to 33%, over the same period.

Interestingly, they also document substantial changes in the distribution of wealth *within* the top 1%. It is precisely these changes within the top 1% that are the subject of this paper. Figure (1.7a) depicts the evolution of the ratio of the top 0.1% to the top 1% and the top 0.01% to the top 0.1%. As in Figure 1.6, the grey area marks the period of the ERISA regulatory changes that Gornall and Strebulaev (2021) argue gave rise to the expansion of the venture capital industry. The similar level and evolution of these ratios indicate that the top of the wealth distribution roughly follows a Pareto distribution and that Pareto inequality, the inverse of the Pareto tail coefficient, has increased. Figure (1.7b) depicts an estimate of Pareto inequality based on these ratios

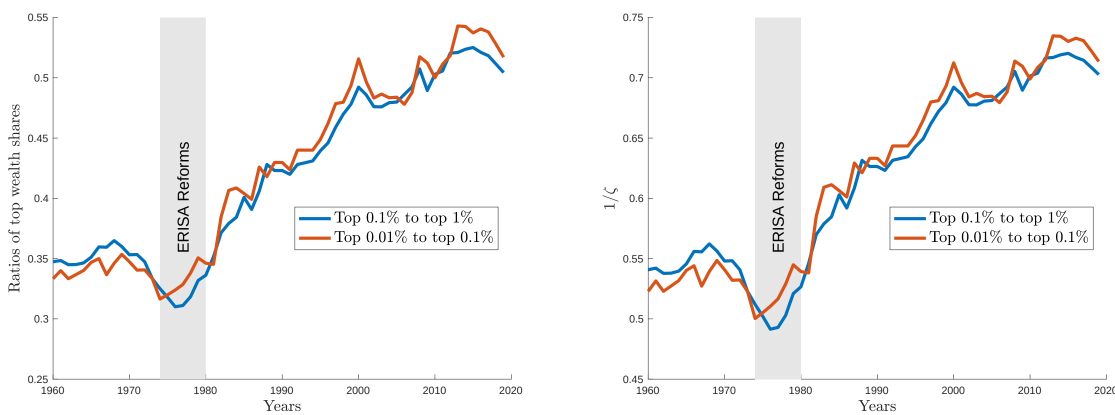
[29] See also Greenwood et al. (2022) for additional evidence.

[30] This is not to say that the VC connection is necessarily causally responsible for the rise of these firms. It could have been that they had just brief encounters with venture capitalists at some early stage. Instead, the point is that the fact that they have a VC connection suggests that their firms are the types of firms that correspond to the entrepreneurial firms in the model of this paper.

	2022				1982			
	Name	Firm	Self-made	VC connected	Name	Firm	Self-made	VC connected
1	Elon Musk	Tesla	✓	✓	1 Daniel K. Ludwig	Exportadora de Sal	✓	✗
2	Jeff Bezos	Amazon	✓	✓	2 Gordon Getty	Getty Oil	✗	✗
3	Bill Gates	Microsoft	✓	✓	3 Margaret Hunt Hill	-	✗	✗
4	Larry Ellison	Oracle	✓	✓	4 William H. Hunt	Halcon	✗	✗
5	Warren Buffet	Berkshire Hathaway	✓	✗	5 Marvin H. Davis	Davis Oil	✓	✗
6	Larry Page	Alphabet Inc.	✓	✓	6 David Packard	Hewlett-Packard	✓	✗
7	Sergey Brin	Alphabet Inc.	✓	✓	7 Lamar Hunt	-	✗	✗
8	Steve Ballmer	Microsoft	✗	✓	8 David Rockefeller Sr.	-	✗	✗
9	Michael Bloomberg	Bloomberg LP	✓	✗	9 Caroline R. Hunt	-	✗	✗
10	Jim Walton	Walmart	✗	✗	10 Nelson B. Hunt	Halcon	✗	✗

Table 1.2: Comparison of Forbes Top 10: 2022 and 1982. For 2022 the VC connection status is based on Gornall and Strebulaev (2021). For the 2022 cohort, the “Self-made” status is based on having a “Forbes Self-made score" of 8/10 or above.[31] All of the firms associated with the 1982 cohort were founded before the first VC fund was established. Gordon Getty inherited substantial wealth from J. Paul Getty, and the Hunt fortune was established by H. L. Hunt, whose children are prominent in the 1982 cohort.

of top wealth shares, using a formula provided by Jones and Kim (2018).



(a) Ratios of top wealth shares.

(b) Pareto inequality $1/\zeta$.

Figure 1.7: Ratios of top wealth shares and the Pareto tail coefficient. Data from Distributional National Accounts via Piketty et al. (2018)

These figures capture the essence of the stylized facts on which the literature on top wealth inequality has centered, accounting for the rise in the level of Pareto inequality as well as the speed with which this rise has occurred. Gabaix et al. (2016) point out that the speed of transition to higher Pareto inequality is not captured well by basic random growth models of wealth accumulation, rather the transition speed in these models is too slow. Atkeson and Irie (2022) point out a direct relationship between the ability of random growth models to match the speed of transition of top wealth inequality observed in the data on the one hand, and the existence of rapidly amassed “self-made" fortunes on the other. In particular, there is a direct relationship between the existence of a subset of extremely upwardly mobile agents and the speed of transition of the Pareto tail coefficient over time: in transitions between steady state, the time it takes for the Pareto shape of the wealth distribution is governed by the time it takes for an initially poor individual to get to the top. The delayed transition dynamics imply that the Pareto shape of the top of the wealth distribution at any given time is determined by the parameters that governed wealth accumulation a couple of decades earlier. The delayed transition dynamics of the Pareto shape of the wealth distribution are essential for understanding how the results in this paper relate to the observation by Decker et al. (2016) that business dynamism has declined in the United States. They argue that in the case of high-growth entrepreneurial firms, the decline in dynamism happened only after the year 2000. Before that, their measure of dynamism was actually rising for this group of firms. This decline in dynamism may not be visible at the top of the wealth distribution yet but might reveal itself in the coming decade.

The present paper incorporates one of the critical insights of Gabaix et al. (2016) and Atkeson and Irie (2022) in order to address the shortcomings of the basic random growth model. Namely, it includes a small minority of entrepreneurial capitalists with very high idiosyncratic risk exposures and higher expected returns to wealth than the other agents in the model. Importantly, and in contrast to Gabaix et al. (2016) and Atkeson and Irie (2022), entrepreneurs' high idiosyncratic risk exposures are endogenous outcomes of their optimal portfolio choice problems rather than exogenous parameters. Finally, because entrepreneurs are overrepresented at the top of the wealth distributions, the average rate of return on wealth will be positively correlated with wealth. This is in line with the empirical patterns documented by Bach et al. (2020) and Fagereng et al. (2020). I explore this model further in the next section.

1.5 The Quantitative Impact of the Reallocation Effect: A Numerical Approach

Section 1.3 described that the crucial determinant of how improved entrepreneurial financing affects top wealth inequality is how much economic activity is reallocated to entrepreneurs in equilibrium. When the reallocation is substantial, top wealth inequality rises, and when it is not, top wealth inequality falls. The size of this reallocation is, in turn, determined by the elasticity of substitution between the goods that entrepreneurial firms produce and those that traditional firms produce. When the substitutability is high, the economy reallocates much capital to the entrepreneurial firms in response to the reduced risk cost associated with production in that sector.

In this section, I examine the role played by the strength of the general equilibrium reallocation effect numerically. Specifically, I parameterize the model to be roughly consistent with key aspects of the data. Then, I investigate how the strength of the general equilibrium reallocation effect, as determined by the elasticity of substitution ε, impacts how reductions in equity financing constraints affect top wealth inequality. The tractability of the framework allows me to compute the transition dynamics of the model straightforwardly. This is important because we are interested in understanding how the strength of the equilibrium reallocation mechanism affects the speed of the dynamics of Pareto inequality. In particular, recall that the remarkable speed with which Pareto inequality has increased is one of the key stylized facts that Gabaix et al. (2016) argued that models of top wealth inequality should ideally be able to account for.

The takeaway from this exercise is that when the elasticity of substitution ε is set to a large enough value, the model produces a rapid rise in Pareto wealth inequality in response to improvements in entrepreneurial financing.

With this in mind, a natural follow-up question is whether there is additional empirical evidence consistent with this large reallocation effect. I answer this in the affirmative by pointing to the dramatic growth of venture capital-backed firms observed by, among others, Gornall and Strebulaev (2021), and by pointing to three additional well-documented macroeconomic trends, showing that the model captures these trends, at least qualitatively, precisely when the reallocation effect is large. These trends are (i) the fall in the aggregate labor share, despite relatively stable firm-level labor shares; (ii) falling safe real interest rates: and (iii) the stable or slightly rising accounting return to the aggregate capital stock, despite a falling real safe interest rate. I go through each of these trends in turn and explain how they are impacted by improvements in entrepreneurial financing qualitatively. The impact will depend on whether the general equilibrium reallocation effect is strong or weak. I conclude with a numerical examination of the model-implied transition dynamics for each trend. The model produces a smaller but still meaningful fraction of the fall in the aggregate labor share observed in the data, a temporarily elevated but long-run stable rate of return to the aggregate capital stock, and a sizeable fraction of the fall in the risk-free rate. One way of interpreting this is that the model requires the entrepreneurial and traditional firms to operate together, producing similar goods across a wide range of industries, rather than being isolated from one another in separate industries. However, it does not imply that entrepreneurial and traditional firms use the same production technologies. The entrepreneurial firms may use cutting-edge high-tech production technology but produce output that is highly substitutable with traditional firms' goods. Despite using various cutting-edge technologies in their production processes, Uber is in the taxi business, Amazon is in the retail business, and Google is in the advertising business.

Alternative explanations for the rise in wealth inequality. The purpose of the numerical exercise considered in this section is not to argue that the observed rise in wealth inequality is due to improvements in risk-sharing for innovative entrepreneurs alone. Instead, it is to point out that improvements in risk-sharing for innovative entrepreneurs are a quantitatively powerful mechanism, able to produce rapid rises in Pareto inequality. In addition, the mechanism also turns out to be consistent with a series of other well-documented macroeconomic trends: the fall in the aggregate labor share, despite relatively stable firm-level labor shares; falling safe real interest rates; and the stable or slightly rising accounting return to the aggregate capital stock, despite a falling real safe interest rate.

This does not mean that other explanations for rising wealth inequality are irrelevant. For instance, improvements in the technology operated by innovative entrepreneurial firms may generate rising Pareto wealth inequality without the need for improvements in entrepreneurial financing. However, improvements in technology and risk-sharing are not observationally equivalent: improvements in risk-sharing do not lead to declining labor shares at the firm level, whereas improvements in entrepreneurial technology do.[32] The empirical literature on the subject suggests that the measured fall in the aggregate labor share is not due to falling labor shares at the firm level but rather a reallocation to low labor share firms.

1.5.1 Parameterization

We want to numerically examine the effect of improved entrepreneurial financing, modeled as a reduction in equity frictions captured by the parameter χ, on top of wealth inequality. More specifically, we want to study how this effect is influenced by the strength of the general equilibrium reallocation of capital towards entrepreneurial firms, governed by ε. The focal parameters for this exercise are, therefore, χ and ε. What data can we use to discipline the way we parameterize the fall in χ that we feed in to the model, and what data can we use to discipline the parameterization of ε? The remaining parameters are set to roughly match relevant moments of the data on top wealth inequality, factor income shares, rates of return to business capital, the risk-free rate, the capital-output ratio, the average volatility of wealth growth at the top of the wealth distribution, and various facts regarding the share of economic activity accounted for by venture capital-backed firms. I will start by discussing how I parameterize the fall in χ.

Parameterizing the fall in χ. I parameterize the fall in χ by selecting an initial value χ_0 and a final value χ_1. I then let χ fall from χ_0 to χ_1 smoothly over time according to the sigmoid curve depicted in Figure 1.8. To emphasize the connection between the fall in χ and improvements in entrepreneurial financing, I let the lion's share of the fall occur in 1974–1979, which is the period of the ERISA regulatory reforms that Gompers and Lerner (2001), Greenwood et al. (2022), and Gornall and Strebulaev (2021) argue triggered the venture capital revolution.

[32] See section 1.5.4

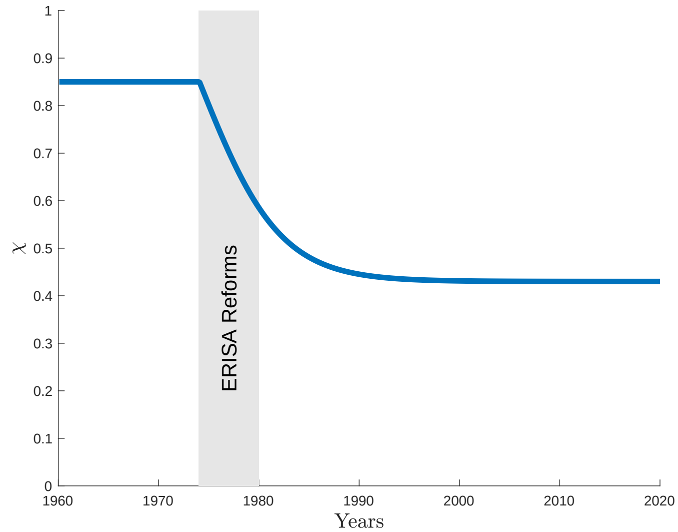


Figure 1.8: Baseline Fall in χ

I pick χ_0 and χ_1 by matching the average rate at which firms associated with entrepreneurs at the top of the Forbes 400 list issued equity. Specifically, Gomez and Gouin-Bonenfant (2024) document that the average lifetime growth rate of shares outstanding associated with entrepreneurs at the top of the Forbes 400 list has increased from 0.5% in 1985 to 2.9% in 2015. In Section A.4 of the Appendix, I show that the average lifetime rate of equity issuance of the entrepreneurial firms in the model is

$$\text{Lifetime equity issuance rate} = \left(1 + \frac{(1+r^T)(1-\chi)}{(r^k - r^T) + \chi(1+r^T)}\right)^{1/T_l} - 1 \tag{1.38}$$

where T_l is the number of years that the firm is considered to be associated with the entrepreneur. A few comments regarding this choice of calibrating χ are in order: The insider financing fraction χ should not be confused with the insider ownership fraction. As noted in Section 1.3, these are not the same. The constraint determines the financing fraction, while the ownership fraction is determined by competition in capital markets. Moreover, Brunnermeier et al. (2024) pursue a different way of calibrating χ. They look at the share of privately held business wealth in the economy and argue that this is the share of business capital that insiders hold. This approach would, however, be a problem in the present setting because we are specifically interested in firms that are not necessarily privately held. Finally, Gomez and Gouin-

Bonenfant (2024) measure the equity issuance rate in 1985, while the regulatory changes that motivate the fall in χ are prior to that. However, it seems reasonable to assume that many of the entrepreneurs at the top of the Forbes 400 list in 1985 had companies that were at the very least a decade old and so had operated mainly in the pre-ERISA era. This would mean that their estimate of the average *lifetime* equity issuance rate from 1985 reflects entrepreneurial financing conditions in the pre-ERISA era.

The exercise aims to examine the transition dynamics of top wealth inequality produced by the model. We want to understand how these are affected by the strength of the reallocation effect. To this end, I consider two values of the elasticity of substitution between the goods produced by the entrepreneurial firms and the traditional firms: $\varepsilon = 10$ and $\varepsilon = 100$. The high value of $\varepsilon = 100$ generates a rise in the fraction of the capital stock operated by the entrepreneurial firms in the model that roughly matches the rise in the share of U.S. market capitalization associated with firms with a history of venture capital-backing. The lower value of $\varepsilon = 10$ is to give us a sense of how the transition dynamics are affected quantitatively by a weaker reallocation effect. The remaining parameters are $\{\alpha, \rho, \delta, \sigma, \nu, \bar{A}, \tilde{\sigma}, \delta_d, \phi_I, \bar{\psi}, T_I\}$. I choose α, ρ, and σ to produce a steady state that matches the labor share, the rate of return to business capital, and the risk-free rate in 1960. These are important quantities for the trends we want to study. In addition, I use δ to target a (business) capital-output ratio of 2. This results in a value of $\delta = 0.1$, which is larger than in most standard calibrations. I choose $\bar{A}$, ν, and $\tilde{\sigma}$ to match an initial Pareto tail coefficient of $\zeta_0 = 1.85$, an initial fraction of the capital stock operated by innovative entrepreneurial firms of $\kappa_0 \approx 5\%$, and idiosyncratic volatility of stock returns of 30%.[33] The demographic parameters, the rate at which dynasties are broken δ_d, the rate at which innovative entrepreneurs become diversified capitalists ϕ_I, and the fraction of innovative entrepreneurs among capitalists $\bar{\psi}$, strongly influence the fraction of entrepreneurs found at various points in the wealth distribution. Kaplan and Rauh (2013) document that 69% of the Forbes 400 list in the 2011 edition were the first in their family to run their business, up from 40% in the first 1982 edition. I

[33] According to Herskovic et al. (2016), this was the average idiosyncratic volatility of stock returns in 1960.

therefore target a share of entrepreneurs in the initial steady state of 40%. Moreover, Gomez (2023) estimates that the average level of idiosyncratic volatility within the top 0.01% of the wealth distribution for the period 1960–1980 was 10%, slightly lower in 1960 than in 1980, so I target a level of 8% in the initial steady state. In the present model, this will be the weighted average of the volatility of entrepreneurs and diversified capitalists within this top quantile. I use $\bar{\psi}$ and ϕ_l to match these moments. I set the dynasty breaking rate to once a generation, $\delta_d = 1/30$, to reflect the risk of generational handover. I set the parameter T_l, the lifetime over which the model-implied average lifetime equity issuance rate is computed, to 30 years.

Finally, changing the value of ε while keeping all other parameters constant will, of course, alter most of the moments that the model produces in the initial steady state. In the extreme case, this could imply that each value of ε would need to be paired with a different parameterization of all the other variables. However, it turns out that changing the value of ε requires only a parsimonious re-parameterization of the other variables. In particular, different parameterizations of ε need to be coupled with different parameterizations of ν, but other than that, the model produces roughly the same moments across the two specifications.

1.5.2 Reallocation to Cutting-Edge Entrepreneurial Firms

In the model, a reduction in equity-issuance-related agency frictions increases the fraction of the capital stock operated by the entrepreneurial sector relative to the traditional sector. In other words, $\kappa_t = \frac{K_t^E}{K_t}$ rises. Exactly how much it rises depends on the elasticity of substitution between the goods that the two sectors produce. When the elasticity is high, the falling risk costs associated with entrepreneurial production motivate a substantial reallocation to that sector, and vice versa when the elasticity is low. This was illustrated in Figure 1.4. In this section, I study this question numerically. In particular, taking as a starting point the initial steady state associated with the baseline calibration in Table 1.3, I examine the transition dynamics of κ_t. Figure 1.9 illustrates the result of this exercise. We see that $\varepsilon = 100$ is associated with a rise in the relative size of the entrepreneurial sector. In contrast, we hardly see a budge with $\varepsilon = 10$.

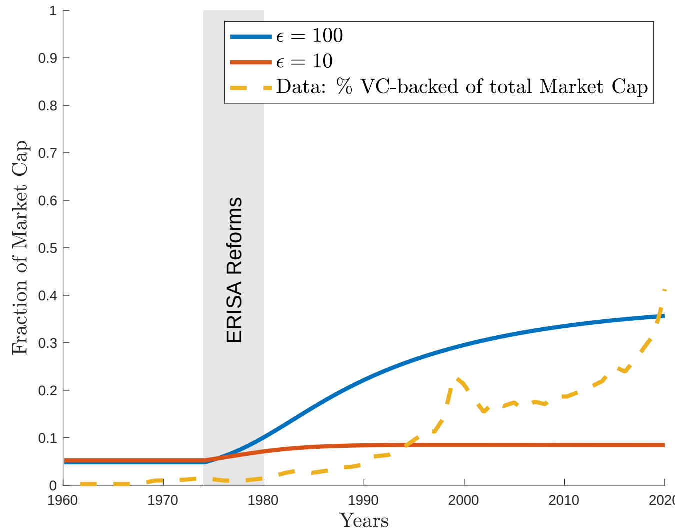
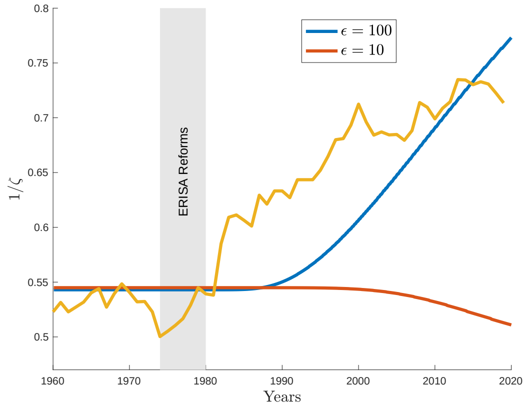


Figure 1.9: Transition of κ_t for $\varepsilon = 100$ and $\varepsilon = 10$. Data from Gornall and Strebulaev (2021).

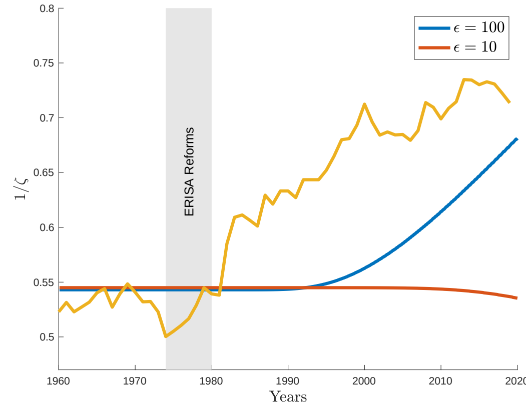
The reallocation in the data. If we interpret the entrepreneurial sector in the model as consisting of innovative entrepreneurial firms similar to venture capital-backed U.S. firms, we can compare the model-implied transitions with some relevant data. Specifically, in Section 1.4, I discussed the so-called "venture capital revolution." In Figure 1.6 from Gornall and Strebulaev (2021), we see that venture capital-backed firms constituted around 0–5% of the total market capitalization before and up to 1980, rising to around 41% of market cap in 2020.

1.5.3 Transition Dynamics of Wealth Inequality

In the previous section, we saw that the high value of $\varepsilon = 100$ was associated with a rise in the fraction of the capital stock operated by entrepreneurs in the model that is roughly in line with the rise in the market capitalization share of firms with a history of venture capital-backing in Gornall and Strebulaev (2021). In this section, I study how the value of ε affects the model-implied transition of Pareto inequality in response to ameliorated equity issuance frictions captured by the fall in χ depicted in Figure 1.8. That this high elasticity is indeed key is illustrated in Figure 1.10, where we examine the transition dynamics of Pareto inequality, the inverse of the Pareto tail coefficient, for the two different values $\varepsilon = 100$ and $\varepsilon = 10$. In Figure 1.10a, we examine the transition of tail inequality measured at the top 0.1%, and in Figure



(a) Tail inequality measured at top 0.1%



(b) Tail inequality measured at top 0.01%

Figure 1.10: Transition of Pareto inequality for $\varepsilon = 100$ and $\varepsilon = 10$. Pareto tails based on ratio of top 0.01% to 0.1% and top 0.1% to 1% wealth shares, respectively. Data (yellow) from Distributional National Accounts provided by Piketty et al. (2018).

1.10b, we examine the transition of tail inequality measured at the top 0.01%. The reason to look at tail inequality at two different points in the wealth distribution is that although tail inequality is the same throughout the wealth distribution in steady state, this is not the case in the transition. As pointed out by Gabaix et al. (2016), the transition speed is slower higher up in the wealth distribution. A comprehensive understanding of how well the model does with respect to the transition speed, therefore, requires us to look at various points along the wealth distribution. When $\varepsilon = 100$, Pareto inequality rises at a rate roughly consistent with the data. In contrast, when $\varepsilon = 10$, the downward pressure on entrepreneurial expected excess returns in response to the capital reallocation is so heavy that the risk-reward trade-off deteriorates: the risk falls as improved entrepreneurial financing enables more risk sharing, but the expected excess return declines even more so that the appraisal ratio falls. In this case, top wealth inequality declines slightly as entrepreneurs reduce their idiosyncratic risk exposure.

This exercise demonstrates that the model can account for a meaningful portion of the rapid transition dynamics of Pareto inequality, provided the elasticity of substitution is very high. In the following sections, I examine how other model predictions are affected by the strength of the general equilibrium reallocation effect, as captured by the value of ε. In particular, I focus on the model's predictions along three dimensions: the growing fraction of various measures of economic activity accounted for by innovative entrepreneurial firms (§1.5.2), factor income shares (§1.5.4), and rates of return to savings and investment (§1.5.5).

1.5.4 Factor Income Shares

In the model, improved entrepreneurial financing leads to a fall in the aggregate labor share despite stable or increasing labor shares at the firm level when the elasticity of substitution between the sectors is high. To see why, note first that the labor share in the traditional sector is $1-\alpha$. This is unaffected by changes in entrepreneurial financing.

Both the labor share and the pure capital share in the entrepreneurial sector are lower than in the traditional sector. This is because, following Di Tella and Hall (2021), the idiosyncratic risk in the firm renders the marginal product of each factor of production locally uncertain. They argue that this is a way of taking seriously the Knightian view (Knight, 1921) that entrepreneurs engage in risk-taking when renting capital and hiring labor because the marginal products of each are uncertain at the time that the cost of capital and wages are determined. As in David et al. (2023) and Hartman-Glaser et al. (2019), the fact that this uncertainty will be priced in equilibrium implies that risk-adjusted marginal products are lower than their unadjusted counterparts. More precisely, rental rates and wages are equal to their respective expected marginal products, less a risk premium. This risk premium constitutes the foundation for the entrepreneurial share of income.[34] Algebraically, the labor share in the entrepreneurial sector is

$$\frac{w_t L_t^E}{p^E(\kappa_t) Y_t^E} = (1-\alpha)\frac{p^T(\kappa_t)}{p^E(\kappa_t)}\frac{\underline{A}}{\bar{\bar{A}}} = (1-\alpha)\left(1 - \underbrace{\frac{(r_t^k - r^T)K_t^E}{p^E(\kappa_t)Y_t^E}}_{\text{“entrepreneurial" share}}\right) \tag{1.39}$$

and the pure capital share is analogously $\frac{r^T K_t^E}{p^E(\kappa_t)Y_t^E} = \alpha\left(1 - \frac{(r_t^k - r^T)K_t^E}{p^E(\kappa_t)Y_t^E}\right)$.

[34] I refer to it as the “entrepreneurial share” rather than the “entrepreneur’s share” because the entrepreneur also gets some pure capital income. The entrepreneur’s share is, therefore, the entrepreneurial share *plus* the entrepreneur’s pure capital income share.

The overall factor shares in the economy are the sales-weighted averages of the shares in each sector. There are two channels along which improved entrepreneurial financing affects these factor shares. Let us focus on the aggregate labor share, although the reasoning is identical for the pure capital share. Firstly, there is a composition effect coming from the reallocation of capital to low-labor-share entrepreneurial firms. This puts downward pressure on the aggregate labor share. The pressure is stronger when ε is higher because reallocation is more substantial. Secondly, the reallocation causes a rise in the labor share within the entrepreneurial sector. This is because entrepreneurs need to raise wages to attract labor in response to the reallocation of capital. Specifically, the price of the intermediate goods produced by the traditional sector $p^T(\kappa_t)$ rises as resources are allocated away from that sector. This raises the value of the marginal product of labor in that sector, which puts upward pressure on wages. Hence, the labor share within the entrepreneurial sector rises. This upward pressure on wages is higher if the elasticity of substitution ε is small because then the rise in $p^T(\kappa_t)$, and consequently the marginal product of labor in the traditional sector, is higher. The aggregate labor share only falls if the composition effect is stronger than the within-sector effect. Because the composition effect is larger than the within-sector effect when ε is large, this is, again, the key parameter for this prediction. Figure 1.11 depicts the evolution of the labor share in response to improved entrepreneurial financing for different values of ε in the baseline calibration.

We also note in equation 1.39 that an improvement in the technology used by entrepreneurial firms (an increase in $\bar{A}$) also leads to a fall in the labor share. However, in contrast to the fall in the aggregate labor share generated by a reallocation of economic activity to the entrepreneurial firms driven by improvements in risk-sharing for entrepreneurs, technological improvements lead to a falling labor share *at the firm level*. In the next section, I discuss how the empirical literature on the evolution of factor income shares seems to conclude that this is inconsistent with the data.

Evolution of factor income shares in the data. The debate on the precise cause and magnitude of the fall in the labor share of income since 1970 is ongoing (see Grossman and Oberfield (2022) for a review of this literature). Estimates range from,

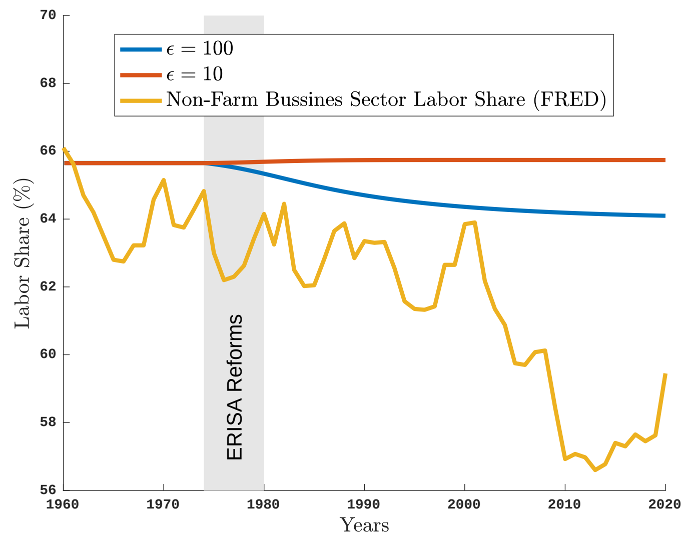


Figure 1.11: Transition dynamics of the aggregate labor share for $\varepsilon = 100$ and $\varepsilon = 10$. Data source: U.S. Bureau of Labor Statistics (2023)

on the high end, a fall of 13 percentage points according to Barkai (2020), to a fall of around 5–6 percentage points, found by Smith et al. (2022a). For the present purpose, a key aspect of the documented fall in the labor share is the observation that this fall has been driven by a reallocation of economic activity towards firms with low labor shares rather than by a general fall in the labor share at the firm level, which has remained relatively stable (Autor et al., 2020) or even increased (Hartman-Glaser et al., 2019). Qualitatively, the model presented in this study is very much in line with that observation.

Moreover, it has also been pointed out that the fall in the labor share has not been accompanied by a rise in the pure capital share of income. Instead, both the labor share and the capital share have fallen relative to what has been referred to as factorless income (Karabarbounis and Neiman, 2019). The nature and causes of this rise in factorless income have yet to be fully understood, and many studies have pointed out potential sources. Barkai (2020) emphasizes the role of pure profits, market power, and declining competition. Eisfeldt et al. (2022) and Smith et al. (2022a) instead focus on the role of human capital income of key employees and business owners. In the model presented in this paper, the rise in the factorless income share comes from the rise in innovative entrepreneurs' share of income. In this sense, the explanation is closer in spirit to Eisfeldt et al. (2022) and Smith et al. (2022a), focusing on idiosyncratic risk bearing as the source of this entrepreneurial share.

How much of such a fall can be accounted for quantitatively by the mechanism presented in this study depends on the value of ε. Looking at Figure 1.11, with $\varepsilon = 100$, around 20% of the fall is accounted for.

1.5.5 The Return to the Aggregate Capital Stock, and the Risk-Free Rate

A similar reasoning as for the labor share applies to the accounting return to the aggregate capital stock as well. In particular, the accounting return to the aggregate capital stock in the economy is given by

$$r_t^K \equiv \frac{Y_t - w_t L_t - \delta K_t}{K_t} = \kappa_t r_t^k + (1 - \kappa_t) r^T. \tag{1.40}$$

In other words, the aggregate return is the capital allocation weighted average of the return in each sector. As with the labor share, a reallocation of resources towards the entrepreneurial firm creates upward pressure on the aggregate return through a composition effect and downward pressure by lowering excess returns within the entrepreneurial sector. The composition effect is stronger than the within-sector effect when ε is high. The aggregate return rises if ε is high enough.

So far, the mechanism is analogous to that for the labor share. However, there are additional implications for returns to wealth in the long run. Recall that the basis for the model in this study is a version of the neoclassical growth model. This means that in the long-run steady state, the return to wealth settles down to the consumption rate out of wealth.[35] In other words, $r_{ss}^K = \rho$, because capital supply is perfectly elastic in the long run. This means that any movements in the return to the aggregate capital stock are temporary, and hence, the reallocation effect has no bite in the long run. However, the long-run stability of the return to aggregate capital is what makes the model's implication for the risk-free rate interesting.

Since the total wealth of the economy is the total capital stock, the return to cap-

[35]Again focusing on the median path of the economy where aggregate shocks dZ_t happen to be 0 for a long time.

ital has to be the wealth-weighted average return to wealth for entrepreneurs and diversified agents:

$$r_{ss}^K = \eta_{ss} r_{ss}^E + (1 - \eta_{ss}) r_{ss}^T. \tag{1.41}$$

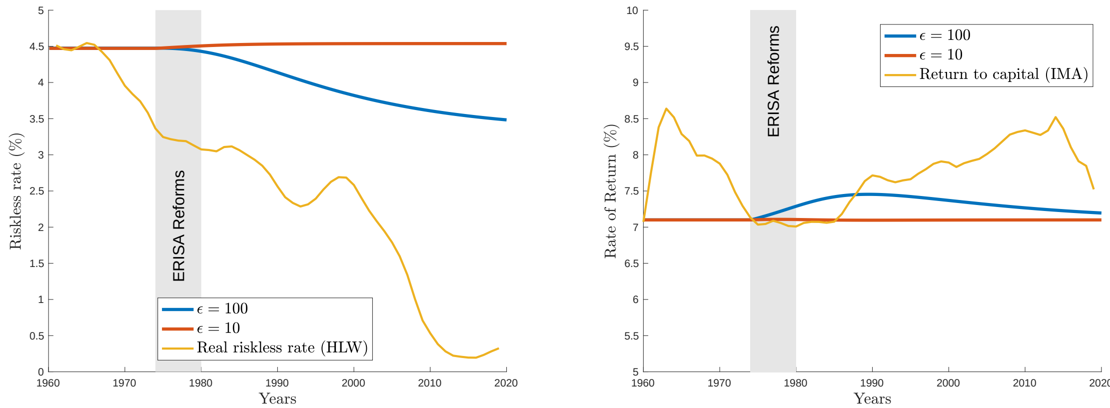
Noting that the difference in return between entrepreneurs and diversified capitalists is $r_{ss}^E - r_{ss}^T = \left(\tilde{\sigma}_{ss}^E\right)^2$, one obtains the following expression for the return to aggregate capital:

$$r_{ss}^K = r_{ss}^T + \eta_{ss} \left(\tilde{\sigma}_{ss}^E\right)^2, \tag{1.42}$$

which states that the return to the aggregate capital stock is the return in the traditional sector plus the risk premium received from investment in the entrepreneurial firms. More precisely, the risk premium that entrepreneurs receive per unit of wealth invested in their firms is $\left(\tilde{\sigma}_{ss}^E\right)^2$, and their fraction of all wealth is η_{ss} so that the risk premium for the economy as a whole is $\eta_{ss} \left(\tilde{\sigma}_{ss}^E\right)^2$. We know that when the general equilibrium capital reallocation effect is strong enough, entrepreneurs' risk exposure and share of wealth both increase, so that $\eta_{ss} \left(\tilde{\sigma}_{ss}^E\right)^2$ rises. However, the fact that $r_{ss}^K = \rho$ is fixed in the long run means that the rise in the risk premium $\eta_{ss} \left(\tilde{\sigma}_{ss}^E\right)^2$ must be associated with a fall in r_{ss}^T. Since the risk-free rate is $r_{ss} = r_{ss}^T - \sigma^2$, it also falls. An alternative way of interpreting the increase in $\eta_{ss} \left(\tilde{\sigma}_{ss}^E\right)^2$ and the resulting fall in the risk-free rate is as a more pronounced precautionary savings motive of entrepreneurs. As they take on more idiosyncratic risk, their precautionary savings motive rises, which puts downward pressure on the risk-free rate.[36] Figure 1.12 depicts the model-implied evolution of the rate of return to the aggregate capital stock and the risk-free rate for the two values of ε.

The rate of return to business capital and the risk-free rate in the data. Several recent studies document a relatively stable or slightly rising return to business capital in the U.S. (Farhi and Gourio, 2018; Gomme et al., 2011; Moll et al., 2022; Reis, 2022). With the different estimates these studies provide, one finds a return that

[36]Mian et al. (2021a) and Mian et al. (2021b) point to the savings behavior of wealthy households as a contributor to falling interest rates. Increased risk-taking by entrepreneurs is a different mechanism from those proposed by them, but because entrepreneurs are overrepresented among the wealthy, the flavors are similar.



(a) Riskless rate

(b) Return to aggregate capital stock

Figure 1.12: Transition dynamics of the riskless rate and the rate of return to capital for $\varepsilon = 100$ and $\varepsilon = 10$. Data on riskless rate from Holston et al. (2017). Data on return to capital from Moll et al. (2022) (smoothed).

hoovers around 7–10%. In contrast, estimates of the return on safe assets show a downward trend for the last half-century (Rachel and Summers, 2019). Holston et al. (2017) estimates a decline of 3–4 percentage points in the long-run return on safe assets between 1960 and 2020. Powerful forces, like demographic changes and the so-called international "savings glut" discussed in the literature on the secular stagnation hypothesis, can perhaps account for most of the fall in the risk-free rate (see Eichengreen (2015), Eggertsson et al. (2019), Rachel and Summers (2019) and Auclert et al. (2021)). Looking at Figure 1.12, we see, however, that for the larger values of ε, the mechanism discussed in this study also puts meaningful downward pressure on the risk-free rate, accounting for between around 30% of the drop when $\varepsilon = 100$.

1.6 Conclusion

This paper studies the effects of improvements in entrepreneurial equity financing on the level and dynamics of top wealth inequality. By developing a tractable general equilibrium model, I show that this impact is summarized by three key effects: the risk-reduction effect, the scaling-up effect, and the general equilibrium reallocation effect. First, improved financing enables entrepreneurs to offload more of their firms' risk to financial markets. This gives them the opportunity to reduce their idiosyncratic risk exposure, which would lower top wealth inequality by making extreme wealth trajectories less likely and by reducing entrepreneurs' precautionary savings motive. This is the risk-reduction effect. In contrast, improved financing also allows entrepreneurs to raise more capital and scale up, which raises top wealth inequality. This is the scaling-up effect.

The central theoretical contribution of the paper is the insight that a third general equilibrium effect determines the relative strengths of the risk-reduction and scaling-up effects: the reallocation effect. If entrepreneurs can attract substantial amounts of economic activity from other sectors of the economy without putting too much downward pressure on their equilibrium expected excess returns, the scaling-up effect dominates the risk-reduction effect, and wealth inequality rises. More generally, it illustrates that the relationships between top wealth inequality, entrepreneurial finance, and idiosyncratic risks and returns may be quite subtle.

The second contribution of the paper is to show that several well-documented trends in U.S. data point to the strength of the general equilibrium reallocation effect in practice. In particular, the dramatically growing fraction of venture capital-backed innovative entrepreneurial firms among the largest publicly traded firms in the U.S., the fall in the aggregate labor share despite relatively stable firm-level labor shares, and the stable or slightly rising accounting return to the aggregate capital stock despite falling safe rates are reflected by the model precisely when the general equilibrium capital reallocation effect is strong enough for the scaling-up effect to dominate the risk-reduction effect.

Table 1.3: Baseline parameterization and model fit. All rates are annualized.

Parameter	Value	Description
Macro parameters		
α	0.34	Output elas. of capital
ρ	0.071	Discount rate
δ	0.095	Depreciation
σ	0.15	Aggregate volatility
Distribution and allocation parameters		
ε	100, 10	Elas. of substitution
ν	0.5041, 0.43905	CES share parameter
$\bar{A}$	1.06	TFP of ent. firms
$\tilde{\sigma}$	0.3	Idiosyn. vol. scalar
δ_d	1/30	Dissipation rate
ϕ_l	1/15	Ent. switching rate
$\bar{\psi}$	1/25	Ent. capitalist frac.
χ_0	0.85	Ent. financing fraction
T_l	30	Top ent. firm lifetime

Moment	Target	Model
Pareto tail coefficient	1.85	1.85
Labor share	65%	65%
Average return to capital	7.41%	7.41%
Risk-free rate	4.51%	4.48%
Capital-output ratio	2	2.03
Equity issuance rate	0.5%	0.5%
Ent. share of capital	<5%	4.83%, 5.22%
Ent. firms idios. vol	30%	31%
Ent. share of Forbes 400	40%	40%
Average vol. wealth for top 0.01%	8%	8%

Chapter 2

Rapid Dynamics of Top Wealth Shares and Self-Made Fortunes: What Is the Role of Family Firms?

Co-authored with Andrew G. Atkeson

We derive an analytical link between the fast dynamics of inequality at the top of the wealth distribution and the prevalence of newly created fortunes. Specifically, in the context of a random growth model of wealth accumulation, the shape of the top of the wealth distribution changes rapidly only if the pace with which new fortunes are created is fast. Quantitatively, the decision of a few families to bear a large amount of idiosyncratic risk in the form of family firms is crucial in accounting for both the prevalence of new fortunes and the dynamics of top wealth inequality.

2.1 Introduction

This paper is motivated by two observations. First, many of the wealthiest families in the world got rich quickly.[1] Second, the concentration of wealth at the top of the wealth distribution in the United States has increased substantially over the course

[1]For example, *Forbes* magazine reports in 2020 that nearly 70% of those on its list of the 400 richest Americans have "self-made" fortunes. Bloomberg Magazine, reports a similar fraction of "self-made" fortunes in its list of top fortunes worldwide.

of the past 50 years.[2] In this paper, we use a canonical random growth model similar to that in Champernowne (1953) to show analytically that these two observations are directly linked. That is, that the shape of the very top of the wealth distribution can change rapidly over time only if there is rapid mobility of families from the bottom to the top of the distribution of wealth.

We then examine a quantitative version of our model calibrated to match data on innovations to wealth reported in Bach et al. (2020) to argue that the decision of a small minority of families to bear a great deal of idiosyncratic risk in their portfolios plays an key role in accounting both of the observations that motivate our study. We interpret this portfolio choice of some families as a decision to concentrate their wealth in a family firm. Certainly, one distinctive feature of capitalist economies worldwide is that many of the wealthiest families hold portfolios that are very concentrated in a single firm and hence are exposed to a high level of idiosyncratic risk in the returns to their wealth.[3]

At the same time, we note that this rapid mobility of families from the bottom of the wealth distribution to the top need apply only to the small minority of families with portfolios concentrated in a family firm. The vast majority of families in our model hold much more diversified portfolios and thus experience much less wealth mobility. Thus, in our model, the observation that most of those at the top of the wealth distribution are "self-made", while necessary to account for the rapid dynamics of the shape of the top tail of the wealth distribution, does not imply that wealth mobility is high for the typical family.[4]

Our paper is related to a large literature. Luttmer (2011) observed that a standard random growth model applied to firm dynamics in which every firm experienced the same distribution of idiosyncratic innovations to firm size failed to match the rapid

[2]See, for example, Saez and Zucman (2016), Smith et al. (2022b), Zheng (2020) and Gomez (2023).

[3]We focus on family firms over and above the traditional notion of entrepreneurship in shaping the distribution of wealth and its evolution (see, for example, Cagetti and De Nardi (2009) and Quadrini (2009)) because many of these families continue to hold these concentrated positions, and thus continue to bear this idiosyncratic risk to their wealth, long past the time that the family firm goes public and long past the time that the founding entrepreneur in the family has died. Evidence on the prevalence of concentrated holdings of equity in a single family firm is available in Goldsmith (1940), Anderson and Reeb (2003), Villalonga and Amit (2006), Klerk (2020), Peter (2021) and in the Ernst & Young University of St. Gallen Family Business Index at http://familybusinessindex.com/.

[4]See, e.g. Carroll and Hoffman (2017).

rise of young firms to the top of the firm size distribution when calibrated to match data on innovations to firm size. Relatedly, Luttmer (2016) and Gabaix et al. (2016) observed that the dynamics of the shape of the top of the distribution of firm size and/or family income implied by this standard model was too slow. Our analytical result shows that these two implications of a standard random growth model are necessarily linked. These papers, and others by Jones and Kim (2018), Benhabib et al. (2019), and Hubmer et al. (2021), have built more complex models with multiple types of agents experiencing type-specific distributions of idiosyncratic shocks to account for the dynamics of the distribution of income or wealth. We see our analytical results as clarifying that these models with multiple types of agents generate rapid dynamics of the shape of the top of the wealth distribution when they also generate rapid mobility of individuals from the bottom of the distribution to the top.

One question that arises out of this line of research is whether it is primarily heterogeneity across agents in the expected growth in the level of wealth or in the idiosyncratic volatility of innovations to wealth that is key in accounting for the dynamics of the top of the firm size, income, or wealth distribution. One might interpret this question as being about the relative importance of differences in skill or opportunities for investment versus luck in shaping the dynamics of the distribution of top wealth. Here, we argue that the answer to this question is likely to be different for wealth than it is for firm size or for income. It is clearly plausible to have wide heterogeneity across agents in the expected growth rates of firm size and/or annual income. That is less true both in theory and the data for wealth.

In theory, since wealth is the discounted present value of income, differences in the expected growth rate of wealth arise from differences in expected returns across families with different portfolios and/or different propensities for these families to consume out of wealth. Consider first differences in expected returns. Theory predicts that those families holding concentrated portfolios of publicly traded firms should not be compensated with higher expected returns for the idiosyncratic risk that they bear since those returns are available to any investor. In the data, many of the very richest families do in fact hold concentrated positions in publicly traded firms.[5] Moreover, many of the richest families in the Forbes 400 experienced very rapid realized growth in their wealth even after the firms that they hold went public. We see this evidence as pointing to an important role for substantial heterogeneity in families' exposure to idiosyncratic risk in the growth of their wealth.

We see the more systematic Scandinavian administrative data from Fagereng et al. (2020) and Bach et al. (2020) as also pointing to the importance of high levels of idiosyncratic risk for families with a family firm in shaping the dynamics of the distribution of wealth. These data show differences in expected returns for families at different points in the wealth distribution on the order of only a few percentage points.[6]. On the other hand, the data in Bach et al. (2020), and data on innovations to wealth for America's wealthiest households from the Forbes 400 in Zheng (2020) and Gomez (2023) indicate that the idiosyncratic volatilities of innovations to wealth at the top of the wealth distribution are very much higher than they are at the bottom.

When we calibrate our model to match data on the concentration of top wealth in Vermuelen (2018), Piketty et al. (2018) and Smith et al. (2021) and moments of innovations to wealth across the wealth distribution in Bach et al. (2020), we find that our model with family firms can account for much of the rapid mobility from the bottom to the top of the wealth distribution as measured by the prevalence of self-made fortunes and for much of the rapid dynamics of the shape of the top of the wealth distribution over the past 50 years. In contrast, when we consider alternative calibrations our model that do not include the very high idiosyncratic volatility of returns to wealth of those investing in family firms, we cannot account for the prevalence of self made fortunes at the top of the wealth distribution and thus cannot account for rapid dynamics of the shape of the distribution of top wealth even if we allow for differences in the expected growth of wealth across types well in excess of what is observed in the data.

In focusing on the role of volatility and wealth mobility in shaping the dynamics of the distribution of top wealth, our paper is most closely related to Zheng (2020), Gomez (2023), and Pugh (2021). We see our main result as complementary to their

[5]For example, in the top 100 fortunes on the Bloomberg list of billionaires, we found that 2/3 of the wealth of this group is in publicly traded equity and that 93% of that equity is concentrated in a single holding.

[6]See also Balloch and Richers (2021)

results on the role of displacement of incumbent top wealth holders in accounting for observed growth in the share of wealth held by the richest families.

The remainder of our paper is organized as follows. We present our model in Section 2.2. We present our analytical result in Section 2.3. We calibrate our model and explore the role of heterogeneity in expected growth in wealth and volatility of innovations to wealth in accounting for rapid mobility and dynamics of the distribution of top wealth in Section 2.4. In Section 2.5, we conclude. Technical results are included in the online appendix.

2.2 The Model

We present a discrete-time, trinomial model of the evolution of the distribution of wealth. Time is denoted by $t = 0, 1, 2, \ldots$, and the length of a time period in calendar time measured in fractions of a year is denoted by Δ_t.[7]

The economy is populated by a continuum of infinitely-lived families that we refer to as *dynasties*. We assume that there is no aggregate risk in this economy, so all shocks to the wealth of a dynasty are idiosyncratic. We assume that at each date t, each dynasty is one of two types $j \in \{D, F\}$, where the type j indexes the distribution of innovations to assets for that dynasty. Here D refers to dynasties that currently hold a diversified portfolio and F to dynasties with a concentrated portfolio in a family firm.

The wealth of individual dynasties of each type evolves in discrete time on a discrete grid of levels of wealth in a manner analogous to a continuous-time model in which dynastic log wealth follows Brownian motion with a type-dependent mean and standard deviation with a reflecting barrier at the bottom of the grid. Specifically, the grid of wealth levels is given by $W(n) = \exp(n\Delta)$ with $n = 0, 1, 2, \ldots, N$, where Δ is the step size of the grid for the logarithm of wealth and $N \leq \infty$. Each period, for dynasties with wealth $W(n)$ with $n > 0$, their wealth rises by one node on the grid with probability $p_{u,j}$, falls by one node on the grid with probability $p_{d,j}$, and remains at the current node with probability $1 - p_{u,j} - p_{d,j}$. For those dynasties of type j with

[7] We focus on a discrete-time, discrete-state version of such a model to simplify the mathematics needed for our analytical results.

wealth at the lowest node $n = 0$, wealth rises to node $n = 1$ with probability $p_{u,j}$ and remains at the lowest node with probability $1 - p_{u,j}$.

After these idiosyncratic innovations to dynastic wealth have been realized, each dynasty of type j experiences a shock to its type. It remains of the same type j with probability ϕ_j and transitions to the opposite type with probability $1 - \phi_j$. These transitions of types are independent over time and of dynastic wealth. We assume that at each date, the fraction of dynasties of type j is equal to the fraction ν_j corresponding to the stationary distribution induced by this Markov process over types.

To aid in the interpretation of these parameters, we use μ_j and σ_j^2 to denote the annualized expected value and variance of innovations to the logarithm of wealth for dynasties of type j and we use κ_j to denote the annualized rate at which dynasties of type j switch type.[8]

We interpret these idiosyncratic innovations to wealth for each dynasty as arising from idiosyncratic shocks to returns on wealth together with a constant, type-dependent propensity to consume out of wealth. Thus differences across types in the expected growth rate of wealth can arise from differences in type-specific expected returns or from differences in the propensity to consume out of wealth. We interpret differences in the volatility of innovations to wealth for different types of dynasties as arise from differences in the idiosyncratic volatility of returns of their portfolios.

We interpret the event of a dynasty switching from type D to type F as the founding of a new firm that is initially closely held by one dynasty. We interpret the event of a dynasty switching from type F to type D as the choice of a dynasty with a family firm to sell their interest in the firm and diversify its portfolio.

The fraction of dynasties of type j with wealth equal to $W(n)$ at time t is denoted by $g_{j,t}(n)$. The overall density of the distribution of assets across dynasties is given by the vector $g_t = \nu_F g_{F,t} + \nu_D g_{D,t}$. The evolution of the two densities of wealth by type from t to $t+1$ can be described by an operator $\mathbb{T}$ whose definition is straightforward but notationally tedious, so we put that definition in Appendix B.1.

To build intuition for the proofs of our analytical results, we consider the special

[8]See Appendix B.2 for the mapping between model parameters and these annualized moments.

case of our model in which dynasties do not switch type. In this simple case, the distribution of wealth for each type of dynasty evolves independently of the other as described by type-specific operators $\mathbb{T}_j$ for $j = F, D$ defined for nodes $n \geq 1$ of the grid of wealth levels by

$$g_{j,t+1}(n) = p_{j,u}g_{j,t}(n-1) + p_{j,d}g_{j,t}(n+1) + (1 - p_{j,u} - p_{j,d})g_{j,t}(n) \tag{2.1}$$

and for the bottom wealth level at $n = 0$ by

$$g_{j,t+1}(0) = p_{j,d}g_{j,t}(1) + (1 - p_{j,u})g_{j,t}(0). \tag{2.2}$$

2.2.1 The Steady-State Distribution of Wealth

The problem of finding the steady-state distribution implied by the operator $\mathbb{T}$ reduces to a problem of solving two linked second-order linear difference equations with constant coefficients. In Appendix B.4, we use standard results for solving such difference equations to show that the steady-state densities of log wealth by type are given by

$$g_{ss,j}(n) = a_j(1 - \lambda_a)\lambda_a^n + b_j(1 - \lambda_b)\lambda_b^n \tag{2.3}$$

for $j = F, D$, where $b_j = (1 - a_j)$. Here λ_a and λ_b are the two stable eigenvalues of the pair of the characteristic equations of the difference equations that define the operator $\mathbb{T}$. As a normalization, we label the larger of these two eigenvalues as λ_a, so $0 < \lambda_b < \lambda_a < 1$.

Note that equation 2.3 implies that the steady-state densities of the log of wealth by dynastic type are given as convex combinations of two geometric distributions over log wealth levels. We denote these geometric distributions compactly as vectors Λ_a and Λ_b, with $\Lambda_i(n) \equiv (1 - \lambda_i)\,\lambda_i^n$.

To gain intuition for this characterization of the steady-state distribution, consider the case in which dynasties do not switch type. In this case, one can solve the difference equations defining the operators $\mathbb{T}_j$ in equations 2.1 and 2.2 by hand for a stationary solution. In this case, each stationary distribution of log wealth by type j is a single Geometric distribution $g_{j,ss}(n) = (1-\lambda_j)\lambda_j^n$, whose shape is given analytically from the distributions of innovations to log wealth for each type: $\lambda_j = p_{u,j}/p_{d,j}$.

In the more general case in which dynasties do switch types, there is no simple analytical solution for the mapping between the parameters of the model and the parameters of the steady-state distribution of wealth by type in equation 2.3. We solve for the parameters of the steady-state distribution a_j, λ_a, and λ_b numerically as described in Appendix B.4.

Since the top tail of the distribution of wealth in the data appears Pareto, it is common to measure inequality at the top of the wealth distribution by the tail coefficient of that distribution, defined here as the negative of the slope of a graph with the logarithm of wealth on the x-axis and the logarithm of the fraction of families with wealth at or above this level on the y-axis. We denote this tail coefficient at node n of our grid of wealth by $\zeta_{ss}(n)$. If $\lambda_a > \lambda_b$ the limiting tail coefficient of the overall distribution of wealth for high levels of wealth approaches a constant $\zeta_{top} \equiv -\log(\lambda_a)/\Delta$.[9]

As noted by Jones and Kim (2018), if the top of the wealth distribution is Pareto, then there is a direct relationship between ratios of top wealth shares and the top Pareto tail coefficient. Specifically, let $x > y$ be two top percentiles of the distribution of wealth and $S(x)$ and $S(y)$ be the corresponding shares of aggregate wealth held by these two top percentiles. If the tail coefficient of the distribution of wealth at the top is constant at $\zeta(n) = \zeta_{top}$ for nodes n greater than those corresponding to wealth percentile x, then the log of the ratio of these two wealth shares is related to this top tail coefficient by

$$\frac{\zeta_{top} - 1}{\zeta_{top}} = \frac{\log S(y) - \log S(x)}{\log(y) - \log(x)} \tag{2.4}$$

We derive this formula and use it to document the dynamics of the shape of the top of the wealth distribution using estimates of top wealth shares from Piketty et al. (2018) and Smith et al. (2021) in Appendix B.5.1.

Note that in general there are dynasties of each type at the top of the wealth distribution. In particular, in steady-state, the fraction of dynasties with wealth equal

[9] In the case in which dynasties do not switch type, as we shrink the time interval to zero the tail coefficients for wealth for each type of dynasty approaches the standard formulas when log wealth follows a Brownian motion with a reflecting barrier at the bottom $\zeta_{ss,j} = -2\mu_j/\sigma_j^2$ for $j = F, D$. See Appendix B.2.

to $W = \exp(n\Delta)$ that own family firms is given by

$$\nu_F(n) = \frac{\nu_F(a_F(1-\lambda_a)\lambda_a^n + (1-a_F)(1-\lambda_b)\lambda_b^n)}{(\nu_F a_F + \nu_D a_D)(1-\lambda_a)\lambda_a^n + (1-\nu_F a_F - \nu_D a_D)(1-\lambda_b)\lambda_b^n} \quad (2.5)$$

Thus, with $\lambda_a > \lambda_b$ and $a_F > a_D$, the fraction of dynasties at any level of wealth that own family firms rises with the level of wealth. In our model, it is through this changing in the mix of dynastic types at each level of wealth that the moments of innovations to wealth change by wealth level.

2.3 Wealth Mobility and Top Wealth Dynamics

We now consider the determinants of the speed with which the shape of the distribution of top wealth converges to steady-state if we start from an initial distribution of wealth by dynastic type that does not correspond to the steady-state distribution. We show analytically, that if the initial distribution of log wealth by type has the same form as the steady-state distribution (as in equation 2.3), but with a different tail coefficient of top wealth, then the tail coefficient of top wealth changes over time only as dynasties transition from the bottom of the wealth distribution to the top. Thus, the dynamics of the tail coefficient of top wealth over time are tightly connected to the degree of mobility from the bottom to the top of the distribution of wealth.

Consider the dynamics of the distribution of wealth starting from initial distributions of log wealth by type defined as convex combinations of arbitrary pairs of geometric distributions of the form

$$g_{j,0} = a_{j,0}\Lambda_{a,0} + b_{j,0}\Lambda_{b,0} \quad (2.6)$$

with $a_{j,0} + b_{j,0} = 1$ for arbitrary nonnegative weights $a_{j,0}, b_{j,0}$ and arbitrary $\Lambda_{a,0}, \Lambda_{b,0}$ defined by $\lambda_{a,0} > \lambda_{b,0} \in [0, 1)$ with $\Lambda_{i,0}(n) \equiv (1 - \lambda_{i,0})\, \lambda_{i,0}^n$ for $i = a, b$. Over time, this pair of initial distributions of log wealth by type converges to the steady-state distributions $g_{j,ss}$ given in equation 2.3.

With this notation, we have the initial tail coefficient of the distribution of top wealth given by $\lim_{n\to\infty} \zeta_0(n) = -\log(\lambda_{a,0})/\Delta$ and the steady-state tail coefficient of the distribution of top wealth given by $\lim_{n\to\infty}\zeta_{ss}(n) = -\log(\lambda_{a,ss})/\Delta$.

To describe this evolution of the distribution of wealth, and more specifically the evolution the tail coefficient of top wealth, we now analyze the results of repeated application of the operator $\mathbb{T}$ to the initial distribution of wealth given by equation 2.6. We do so as follows. Let the vector $\mathbf{1}$ denote a distribution of wealth across nodes in our grid of wealth that places weight one on the lowest node $n = 0$ and zero on all other nodes. This distribution corresponds to the wealth distribution for a cohort of dynasties starting at the bottom of the wealth distribution. We then have, by direct calculations provided in Appendix B.3, the following result:

Main Proposition: The pair of densities of log wealth by dynastic type at date t of the transition to steady-state starting from initial densities in equation 2.6 are given by

$$\begin{bmatrix} g_{F,t} \\ g_{D,t} \end{bmatrix} = \begin{bmatrix} a_{F,t}\Lambda_{a,0} \\ a_{D,t}\Lambda_{a,0} \end{bmatrix} + \begin{bmatrix} b_{F,t}\Lambda_{b,0} \\ b_{D,t}\Lambda_{b,0} \end{bmatrix} + \sum_{k=0}^{t-1}\mathbb{T}^k \begin{bmatrix} c_{F,t-k}\mathbf{1} \\ c_{D,t-k}\mathbf{1} \end{bmatrix}. \tag{2.7}$$

where $a_{j,t}, b_{j,t}$ and $c_{j,t}$ are scalars that depend on parameters that govern the rates at which dynasties switch types and the initial weights $a_{F,0}, a_{D,0}$ as described in Appendix B.3.

This result in equation 2.7 implies that the distributions of wealth by type at time t of the transition to steady-state are each a convex combination of the two original geometric distributions $\Lambda_{a,0}$ and $\Lambda_{b,0}$ that define the initial distribution of wealth in equation 2.6, and distributions of wealth for cohorts that started at the bottom of the wealth distribution in each of the periods from t back to period 1 of the transition as captured by the final summation in equation 2.7. This result implies that the tail coefficient of the distribution of wealth at high levels of wealth at time t remains equal to its initial value of $\lim_{n\to\infty}\zeta_0(n) = \log(\lambda_{a,0})/\Delta$ until enough time has passed for the distribution of wealth for cohorts starting at the bottom has had time to reach those high levels of wealth. If this mobility from bottom to top wealth levels is slow, then this transition of the tail coefficient at the top of the wealth distribution is slow, while if this wealth mobility from the bottom to top wealth levels is fast, then it is possible to have fast transitions of the tail coefficient of the distribution of wealth at the top.

One can gain intuition for this result in equation 2.7 by considering the calculations involved in the following simple case.

Corollary: If dynasties do not switch types and the initial distributions of wealth by dynastic type j are each single geometric distributions $\Lambda_{j,0}(n) = (1-\lambda_{j,0})\lambda_{j,0}^{n}$, then we have that the density of log wealth for dynasties of type $j = F, D$ at time t is given by

$$g_{j,t} = A_j^t \Lambda_{j,0} + (1 - A_j) \sum_{k=0}^{t-1} A_j^{t-1-k} \mathbb{T}_j^k(\mathbf{1}) \tag{2.8}$$

where A_j is a scalar given by

$$A_j \equiv \left(p_{j,d}(1-\lambda_{j,0})(\frac{\lambda_{j,ss}}{\lambda_{j,0}} - 1) + 1 \right)$$

Here $\mathbb{T}_j$ defined by equations 2.1 and 2.2 and $\lambda_{j,ss} = p_{j,u}/p_{j,d}$ as discussed in section 2.2.1.

Proof: Direct calculation using equations 2.1 and 2.2 above gives that

$$\mathbb{T}_j(\Lambda_{j,0}) = A_j \Lambda_{j,0} + (1 - A_j)\mathbf{1} \tag{2.9}$$

The operator $\mathbb{T}_j$ is linear. Repeated application of this operator to calculate $g_{j,t+1} = \mathbb{T}_j(g_{j,t})$ starting from $g_{j,0} = \Lambda_{j,0}$ then gives the result.

The key insight to the proof of this corollary is that, if the initial distribution of log wealth has a geometric distribution and hence a top tail coefficient described by the parameter $\lambda_{j,0}$, then, as described in equation 2.9, the idiosyncratic innovations to log wealth described in equation 2.1 do not change the shape of this geometric distribution as measured by its tail coefficient away from the reflecting barrier at the bottom of the distribution of wealth. In the case that the initial distribution is equal to the steady-state distribution (so $\lambda_{j,0} = \lambda_{j,ss}$), this result is immediate. Our result follows from the observation that this same property holds for any initial geometric distribution of log wealth.[10]

[10] Aleh Tsyvinski kindly provided us with a proof of this corollary directly in continuous time when the log of wealth follows a Brownian motion with a reflecting barrier at the bottom. We reproduce this proof in Appendix B.3.2.

This main insight carries through to the case in which dynasties do switch types. In this case, the formulas for the evolution of the weights $a_{j,t}, b_{j,t}, c_{j,t}$ in equation 2.7 are more complex due to the impact of dynasties switching types on the evolution of the densities of wealth by type, but the main result that the shapes of the initial pair of geometric distributions in equation 2.6 is preserved by the operator $\mathbb{T}$ except at the reflecting barrier at the bottom of the distribution of wealth carries through to this case as in equation 2.7.

Gabaix et al. (2016) use an alternative approach to analyze the dynamics of the distribution of wealth to steady-state in the case in which dynasties do not switch types based on a spectral analysis of the continuous time analog of the operator $\mathbb{T}_j$. For the interested reader, in Appendix B.6, we provide direct analogs of their spectral analysis in our discrete-time discrete-state setting for this case with no switching of types. We do so for the case in which there is also a reflecting barrier at the top of the grid of wealth (so $N < \infty$). In this case, the operator $\mathbb{T}_j$ is an $N \times N$ dimensional Markov transition matrix whose (slowest) convergence dynamics to the steady-state distribution are characterized by the second largest eigenvalue of this matrix. In this special case, the eigenvalues of the matrix $\mathbb{T}_j$ have an analytical solution as a function of the mean and standard deviation of innovations to wealth for that one type of dynasty as presented in equation (13) of Gabaix et al. (2016).

In contrast, our result in equation 2.7 is not a calculation of the eigenvalues of the operator $\mathbb{T}$. Instead, it is a decomposition of the dynamics of distributions implied by that operator starting from a particular class of initial conditions. Moreover, our result 2.7 holds when dynasties do switch types. We see the principal difference between our result and theirs is that our analysis directly highlights the connection between the speed of wealth mobility from the bottom of the wealth distribution to the top and the dynamics of the shape of the top of the wealth distribution as it converges to steady state in way which is not readily apparent from the calculation of the eigenvalues and eigenvectors of a large Markov transition matrix.

2.4 Quantitative Implications of the Model

We now ask whether a version of our model calibrated to match data on the concentration of top wealth and moments of innovations to wealth across the wealth distribution can account for rapid mobility from the bottom to the top of the wealth distribution as measured by the prevalence of self-made top fortunes and for the rapid dynamics of the shape of the top of the wealth distribution in the US over the last 50 years.

For data on the dynamics of the shape of the top of the distribution of wealth in the US, we rely on recent estimates on the tail coefficient of top wealth from Vermuelen (2018) and of the evolution of that tail coefficient of top wealth over time using equation 2.4 and estimated top wealth shares from Piketty et al. (2018) and Smith et al. (2021). These sources give an estimate of the tail coefficient of top wealth close to 1.85 in the late 1960's and early 1970's corresponding to a ratio of the wealth share of the top 0.01% to that of the top 0.1% close to 0.35. The estimate of the tail coefficient of top wealth in recent years implied by these sources using equation 2.4 is between 1.4 and 1.5 corresponding to estimates of the ratio of the wealth share of the top 0.01% to that of the top 0.1% in the range of 0.46 to 0.52. We review these data on top wealth shares in greater detail in Appendix B.5.

For data on the moments of innovations to wealth by wealth level, we cite estimates using administrative data from Sweden as reported in Bach et al. (2020). We use these data as they are the most complete data of this kind available. These authors find that both the expected growth in wealth and the standard deviation of innovations to wealth rise with the level of wealth. Their findings on the standard deviation of innovations to wealth at the top of the wealth distribution in Sweden are similar to that reported in Gomez (2023) for the American households in the Forbes 400.

Our calibration strategy has two steps. Details are provided in Appendix B.5.

In the first step, we set the unconditional fraction of dynasties that have family firms and the rates at which dynasties switch types to match data on entrepreneurship and business dynamics. Cagetti and De Nardi (2009) find that the fraction of entrepreneurs in the U.S. population is 7.6%. Hurst and Pugsley (2009) argue that many of these entrepreneurs do not intend to grow their businesses. We choose to set $\nu_F = 5\%$ as a balance between these two papers. We set the rate at which families switch from the Family Firm type to the Diversified type to $\kappa_F = 1/15$. Thus, 6.66% of family firms diversify each year. This switching rate is roughly consistent with the data on the agent distribution of small business shown in Bhandari and McGrattan (2021). We choose a switching rate slightly slower than implied by that data so that the model will allow for the existence of family firms that are held by one family for multiple generations.

Second, we search for values of the mean and standard deviation of innovations to log wealth for the two types of dynasties to match the following four calibration targets:

(a) the tail coefficient of top wealth to $\zeta = 1.43$, corresponding to a ratio of wealth shares for the top 0.01% and the top 0.1% of 0.5,

(b) the difference in expected growth rates in the level of wealth of families at the top 0.1% and the bottom of the wealth distribution of 5.69%,

(c) the cross-sectional dispersion of innovations to log wealth for families at the bottom of the wealth distribution of 8.13%, and

(d) the cross-sectional dispersion of innovations to log wealth for families at the top 0.01% of the wealth distribution of 35.79%.

The moment (a) is estimated using data on ratios of wealth shares in recent years as described above. The moment (b) is taken from Bach et al. (2020) Table 1 column 1. The moments (c) and (d) are taken from Bach et al. (2020) Table 8, column 1.

The resulting model parameters are shown in row A of Table B.1.

We now consider several additional implications of our calibrated model.

The implied fraction of dynasties at the bottom of the wealth distribution that have family firms is only 0.3% while the fraction at the top it is 65%. We show how the moments of innovations to wealth in the model vary by the level of wealth in Appendix Figure B.4.

Our calibration entails a very high standard deviation of innovations to log wealth for family firms to match the overall dispersion of wealth growth at the top of the wealth distribution in moment (d). We argue as follows that the data in Gomez (2023) on the distribution of innovations to wealth for members of the Forbes 400 is consistent with our hypothesis that a significant portion of those at the top of the wealth distribution have a very large standard deviation of innovations to their wealth. Specifically, innovations to log wealth at the top of the wealth distribution in our model are given as a mixture of two normal distributions. Hence, our model implies that the distribution of such innovations to top wealth has fat tails. In Appendix B.5 we show that our calibrated model implies that this distribution of innovations to top wealth has an excess kurtosis that is actually conservative relative to the findings of Gomez (2023) regarding the large excess kurtosis of innovations to log wealth for members of the Forbes 400. Moreover, our model replicates the findings in Gomez (2023) that less than 10% of the families in the Forbes 400 in 1983 remain on this list today.

In Figure B.1, in the left panel (B.1a), we show the implications of this model for mobility of dynasties from the bottom of the distribution to the top. Specifically, we show the fraction of those dynasties above the percentile corresponding to the Forbes 400 who were at the bottom of the distribution of wealth k or fewer years ago, with k on the x-axis and the corresponding fraction of the most wealthy dynasties on the y-axis. We see that 63% of the Forbes 400 in the model were at the bottom of the wealth distribution within the last 50 years. In this sense, the model is nearly successful in reproducing the finding by *Forbes* magazine that, in 2021, 70% of those in the Forbes 400 are self made.

In the right panel of this figure (B.1b), we illustrate the model's implications for the speed of transition of the tail coefficient of wealth to steady-state starting from an initial distribution as in equation 2.6 in which $\Lambda_{a,0} = \Lambda_{b,0}$ with a common tail coefficient at all levels of wealth of $\zeta_0 = 1.85$. As described above, this estimated tail coefficient is consistent with a ratio of wealth shares for the top 0.01% to that for the top 0.1% of 0.35 as reported in Piketty et al. (2018) and Smith et al. (2021) for the late 1960's and early 1970's.

We show in this figure the convergence of the tail coefficient from its initial value of $\zeta_0 = 1.85$ towards its steady-state value of $\zeta_{ss} = 1.43$ at wealth levels corresponding to the top 0.1%, 0.01%, and the Forbes 400. We see in this figure that the convergence of this tail coefficient takes five to ten years to get started and proceeds more rapidly lower down in the distribution of wealth. That is, as indicated by our main proposition, the shape of the wealth distribution changes more slowly higher up in the distribution because this shape above any wealth level changes only as cohorts of dynasties starting at the bottom have had time reach that wealth level.

The central quantitative implication of the model presented in this paper is that the presence of a small minority of dynasties with portfolios subjected to high idiosyncratic volatility is crucial in accounting for the prevalence of large new fortunes as well as rapid changes in top wealth inequality. To illustrate this implication we consider two alternative calibrations of the model wherein we reduce σ_F, the volatility for the family firm type, to 75% and 50% of its baseline value. We implement this experiment by re-calibrating the model as follows. We maintain targets a)-c) while replacing target d), the dispersion in wealth growth rates at the top of the distribution, by directly setting σ_F to 75% and 50% of its baseline value, respectively. The resulting parameter values are reported in rows B and C of Table B.1.

In Figure B.2 we compare the baseline calibration of the model to the two alternative calibrations along two dimensions. Figure (B.2a) compares the transition of the tail coefficient measured at the node corresponding to the Forbes 400. Figure (B.2b) displays the transition of the ratio of the top 0.01% to the top 0.1% wealth shares. The relatively slower transitions in these alternative calibrations suggest that the absence of dynasties subjected to very large idiosyncratic volatility prevents the model from being able to account for rapid dynamics of top wealth inequality as measured by changes in the tail coefficient and in ratios of top wealth shares. As for the prevalence of new fortunes, these alternative calibrations also display lower values for the fractions of those dynasties above the percentile corresponding to the Forbes 400 who were at the bottom of the distribution within the last 50 years. In the baseline calibration this fraction is 63%. When σ_F is at 75% of its baseline value the fraction is 20%, and when σ_F is at 50% of its baseline value the fraction is close to 0.

In this paper, we focus specifically on a model in which innovations to wealth for dynasties depends on their type, rather than on their level of wealth. Hubmer et al. (2021) calibrate a rich quantitative model of wealth dynamics in which the innovations to wealth depend on the level of wealth. In that model, they consider a relatively small standard deviation of innovations to wealth at the top of the distribution of wealth. (See Figure 6 in that paper). Consistent with our findings in experiment C, they find that their model implies virtually no change in the shape of the top of the wealth distribution as measured by the model-implied ratios of wealth shares for the top 0.01% and 0.1% over 50 years as reported in Table 3 in their paper.

2.5 Directions for Future Research

We see several directions for future research suggested by our results.

One is positive. How have changes in the idiosyncratic volatilities of firm value over time (see Herskovic et al. (2016)) and/or differences in this idiosyncratic volatilities across countries (see Bekaert et al. (2023)) impacted differences in the distribution of top wealth? We explore this question in Atkeson and Irie (2020).

One is normative. What are the welfare implications of inequality if such inequality is driven by uninsured idiosyncratic risk? See, for example Lucas Jr. (1992). To give a satisfactory answer to this question, we must ask why do we families making undiversified investments in family firms for multiple generations? And what impact does policy have on this portfolio choice? See for example Bertrand and Schoar (2006), Aoki and Nirei (2017), Peter (2021), and Phelan (2019).

We see these as fruitful avenues for future research.

Chapter 3

Wealth Inequality and Changing Asset Valuations in the Distributional National Accounts

In this paper, I study whether the rise in measured wealth inequality in the Distributional National Accounts (DINA) provided by Piketty et al. (2018) can be accounted for by a combination of changing asset prices on the one hand, and household heterogeneity in portfolio compositions on the other. In particular, I study the gap between the share of wealth held by individuals in the top quantiles of the wealth distribution, and the same individuals' share of the capital income flows associated with that wealth. I find that the size of this gap varies a lot over time, being especially large after the financial crisis of 2008. However, the steady rise in top wealth shares since the late 1970s, is not primarily accounted for by a rise in the size of this gap. Rather, top wealth shares and shares of the associated cash flows rise together. I also examine whether the rise in measured wealth inequality is primarily associated with increasingly concentrated distributions of wealth *within* broad asset classes or with differences in performance *between* those asset classes. I find that the trend rise in measured wealth inequality is primarily associated with an increase in the concentration of wealth within asset classes.

3.1 Introduction

Measured wealth concentration has risen in the United States over the past half century. Among the more recent studies documenting this, Saez and Zucman (2016) and Smith et al. (2022b) estimate *wealth* from data on *income* by "capitalizing" income flows observed on tax returns (see also Giffen (1913), Stewart (1939), Saez and Zucman (2020), and Bricker et al. (2016)). Because direct data on individual-level wealth is scant, this method *estimates* wealth by multiplying the types of capital income observed on tax returns—dividends, interest income, business income, and so on—by time-varying and asset type-specific "capitalization factors". The end-product of this so-called capitalization method is an estimate of the *joint* distribution of wealth and its associated capital income flows.

A version of this estimated joint distribution is made available through the public-use Distributional National Accounts (DINA) microdata provided by Piketty et al. (2018). The asset type-specific capitalization factors used to estimate wealth from capital income change over time, year-by-year. As these valuation multiples for the different types of asset income move relative to one another over time, the estimated wealth distribution changes, even if the underlying distributions of the associated capital income flows remain stable. This raises the question: how much of the observed increase in wealth inequality is due to changes in capitalization factors rather than shifts in the distribution of underlying capital income flows?

In this paper, I study this question using the estimated joint distribution of wealth and its associated capital income flows in the DINA data. I first ask: to what extent has the rise in measured wealth concentration in the DINA data been accompanied by a corresponding rise in the concentration of the associated capital income flows generated by that wealth?

Why is this interesting? The asset pricing literature distinguishes between changes in the price of an asset that are driven by changes in the expected the cash flows generated by the asset, and changes in the market *valuation* of those cash flows. The former kind of asset price movements are sometimes referred to as cash flow induced asset price movements, or slightly misleadingly "fundamental". The latter kind of asset price movements are sometimes referred to as discount rate induced asset price movements, or slightly misleadingly changes in "valuations". This distinction has interesting implications for our understanding of the rise in wealth inequality. If the rise in wealth concentration is driven by the fact that wealthy households are more likely hold portfolios that are sensitive to changes in valuations, then it is entirely possible that the rise in wealth concentration has not been accompanied by a rise in the concentration of the associated capital income flows. This would cast doubt on theories of top wealth inequality wherein the rise in wealth concentration is primarily based on economic fundamentals.

To evaluate the role of changes in the distribution of capital income flows versus changes in valuations in accounting for the measured rise in wealth inequality in the DINA data, I rank individuals based on the measure of wealth in this data, identify individuals belonging to various top quantiles of this wealth distribution, and compute the share of the associated capital income flows that those same individuals receive. Note that this is not a question about the *marginal* distribution of capital income, but a question about the *joint* distribution of wealth and capital income.

I find that there is a gap between the share of wealth held by top quantiles of the wealth distribution, and the share of income received by these individuals. I also find that the size of this gap varies a lot over time, being especially large after the crisis of 2008. But I also find that over longer horizons the trend rise in top wealth shares documented in the DINA data is accounted for by a rise of similar magnitude in the shares of the associated cash flows, rather than a steady increase in the size of the aforementioned gap. If one takes the joint distribution of wealth and its associated cash flows estimated in DINA seriously, these findings suggest that theories relying solely on changing valuations of a stable distribution of income flows for explaining the rise in top wealth shares, are insufficient.

To interpret these results, it is important to understand how the capitalization method for estimating wealth works. This is because the result might partly be a mechanical artefact of this particular method of estimating the wealth distribution. As explained in both Saez and Zucman (2016) and Smith et al. (2022b), this method begins by breaking down the capital income observed on tax returns into different categories. This categorization is based on the source of the income: dividends, interest, business income etc. The asset holdings W_{ij} of individual i in asset category j, is estimated from the corresponding asset income I_{ij}, by positing that asset holdings are proportional to the associated asset income: $W_{ij} = \frac{1}{r_j} I_{ij}$. Here, $\frac{1}{r_j}$ is an asset category-specific "capitalization factor". These capitalization factors are based on estimates of the yield r_j, for each asset category. In this basic version of the capitalization method, wealth is proportional to income *within* each asset category, with the same proportionality for everyone, namely the asset category-specific capitalization factor. This means that wealth shares and shares of cash flows are identical on a category-by-category basis. However, this is not the case for *overall* wealth shares and the associated shares of overall capital income in the DINA data. In particular, the fact that capitalization factors are different for different asset categories, combined with the fact that portfolio compositions vary systematically along the wealth distribution, introduces a gap between wealth shares and shares of the associated capital income flows.[1]

Focusing in particular on the income flows associated with the assets that constitute wealth, the results in this paper agree with those of Kuhn et al. (2020) in that there are extended periods where there is a substantial disconnect between the distribution of wealth on the one hand and the distribution of income on the other.[2] As capitalization factors and the degree of portfolio heterogeneity along the wealth distribution varies over time, the gap between wealth shares and shares of the capital income flows associated with that wealth, also varies. Movements in top wealth shares can therefore be understood as movements in this gap, or movements in the distribution of the underlying cash flows. The results in this paper suggest that top wealth shares in the DINA data do move around a lot in response to movements in the gap, but that the trend rise in top wealth shares in the data is mostly due to a rise in the concentration of the underlying cash flows.

Additionally, a by-product of the capitalization method is the implied estimation of asset allocation decisions across broad asset categories within the wealth distribution.

[1]An additional reason why wealth shares and capital income shares are not identical in the DINA data is also that Piketty et al. (2018) make various adjustment to the basic capitalization method described above.

[2]In contrast to Kuhn et al. (2020), I focus on the capital income associated with the assets that constitute wealth, and are used to estimate wealth in the capitalization method, rather than overall all income (which also includes labor income).

Because asset allocations vary substantially along the wealth distribution, changes in the relative performance of the different asset classes lead to changes in measured top wealth shares, even if the distribution of wealth within each asset class is stable. For instance, top wealth shares rise when equity prices rise relative to house prices because wealthy households tend to invest more heavily in stocks. The second question I study in this paper is therefore: to what extent is the measured rise in top wealth shares in the DINA data accounted for by an out-performance of asset categories more prevalent in the portfolios of wealthy households relative to other households, as opposed to increasingly concentrated distributions of wealth within asset categories. I do this by means of a simple accounting decomposition.

Specifically, the share of wealth held by individuals belonging to a top quantile can be written as a weighted average those individuals' shares of wealth within each asset class, where the weights are the aggregate portfolio weights of each asset class. This invites a simple decomposition: how much of the increase in the wealth shares of top quantiles is accounted for by changes in the weights, and how much is accounted for by changes in the within-asset class wealth shares? Borrowing the language of Kuhn et al. (2020): should we understand variation in top wealth shares as a "race" between broad asset classes (housing versus equity for instance) that wealthy and less wealthy households make different asset allocation decisions about? I find that variation in the relative performance at the asset class level do induce meaningful swings in top wealth shares in the DINA data. However, I also find that the trend rise in top wealth shares measured in this data set since the late 1970s is overwhelmingly accounted for by increasingly concentrated distributions of wealth *within* asset classes.

3.1.1 Related Literature

A large and growing literature studies the evolution of wealth inequality in the United States (Kopczuk and Saez (2004), Saez and Zucman (2016), Bricker et al. (2016), Bricker et al. (2018), Batty et al. (2019), Catherine et al. (2020), Saez and Zucman (2020), Smith et al. (2022b)). Several recent papers highlight the potential connection between movements in asset prices and variations in top wealth shares (Gomez (2024), Bach et al. (2020), Greenwald et al. (2023), Kuhn et al. (2020)). Studies in this literature emphasize that not all sources of changes in asset prices have the same welfare implications, focusing particularly on the the distinction between discount rate driven versus cash flows driven asset price variation. This is highlighted in Cochrane (2020), explained in Moll (2020), and further studied in Fagereng et al. (2023) and Greenwald et al. (2023). A key implication of purely discount rate driven asset price variation is that it leads to variation in wealth inequality without corresponding rise in the concentration of cash flows generated by that wealth, or as expressed in Cochrane (2020): "In sum, much of the increase in wealth inequality reflects higher market values of the same income flows. Such increases indicate nothing about increases in lifetime consumption inequality, which better reflect individual command over resources".

I study whether the rise in measured top wealth shares does indeed reflect variation in the associated cash flow shares in the context of the DINA data.[3] The distinction between cash flow-induced and discount rate-induced asset price changes has been fruitfully used in the empirical asset pricing literature to understand the sources of asset price variation (Campbell and Shiller (1988), Cochrane (2011)). In reviewing this literature, Cochrane (2011) argues that the a lot of the variation in asset prices across a wide range of broad asset classes is due to discount rate variation rather than cash flow variation. This conclusion is questioned by Larrain and Yogo (2008), who emphasize the importance of expected cash flow variation. Similarly, looking at individual stocks rather than broad indices, Vuolteenaho (2002) also concludes that expected cash flow variation is more important. Other papers in this literature focus on the impacts of the combination of changing asset prices on the one hand, and heterogeneity in asset allocations along the wealth distribution on the other (see Fagereng et al. (2020), Bach et al. (2020), Kuhn et al. (2020), and Balloch and Richers (2021) for studies on return and asset allocation heterogeneity along the wealth distribution from a more empirical perspective, and Pástor and Veronesi (2016), Cioffi (2021) and Gomez (2024) for more theoretical perspectives). For instance, Kuhn et al. (2020) document that top wealth shares rise when stock prices rise, and fall when house prices rise. The

[3] I emphasize that I study the measured rise in wealth inequality according to the distribution of wealth estimated in the DINA data, critiques of the methodology behind these estimates are numerous, including Bricker et al. (2016), Smith et al. (2022b), Batty et al. (2019) and Bhandari et al. (2020).

present paper also studies how the relative performance of different asset classes impacts top wealth shares. I decompose the rise in top wealth shares in the DINA data in to a component related to changes in the relative performance between broad asset classes versus changes in wealth shares within asset classes. Like Kuhn et al. (2020) I find that top wealth shares rise between periods were asset classes primarily held by the wealthy perform well. However, I also find that the trend rise in top wealth shares since the late 1970s, documented in the DINA data, is more associated with an increase in the concentration of wealth within asset classes.

Finally, the results in this paper relate to theories of why top wealth shares have risen. Specifically, they suggest that theories of rising wealth inequality should be consistent with the observation that the cash flows associated with wealth have also become more concentrated in the hands of the wealthy. This includes a wide range of theories related to technology and automation (Moll et al. (2022)), entrepreneurship and the financial conditions of entrepreneurs (Gomez and Gouin-Bonenfant (2024), Jones and Kim (2018), Atkeson and Irie (2022), Irie (2023a)), and taxation and redistribution (Kaymak and Poschke (2016), Hubmer et al. (2021)). The results suggest that theories that are only based on changes in the valuations of a given distribution of income flows are insufficient for understanding the rise in wealth inequality.

3.2 Shares of Wealth and Shares of Income Flows in DINA.

The Distributional National Accounts provided by Piketty et al. (2018), the DINA consists of "a set of annual micro-files representative of the U.S. economy, where each line is a synthetic individual created by combining tax, survey, and national account data, and each column is a variable of the national accounts".[4] Because individual data on wealth is hard to come by, DINA relies heavily on the so-called "capitalization method". I will discuss this method in detail in the next section where I interpret the results presented in this section.

[4] I use the versions of the DINA files that are made available online by the authors, the so-called "external-use" files. The details of these data files can be found in the Codebook associated with Piketty et al. (2018).

The aim of this section is to study the joint distribution of wealth and the cash flows associated with that wealth. In particular, we are interested in understanding the extent to which the rise in measured top wealth shares in the DINA data is accounted for by a larger *gap* between the wealth shares and the associated cash flow shares, or if the rise is accounted for by a more concentrated distribution of the underlying cash flows.

Figure 3.1 depicts the share of wealth held by the top 10%, 1%, 0.1% and 0.01% of the wealth distribution. respectively, according to the DINA data, along with the share of the associated capital income flows received by these individuals.

Specifically, according to the measure of wealth in the DINA, wealth consists of equities, fixed income assets, housing, various forms of business assets, pension wealth, net of debt:

$$W_{it} = \sum_{j \in J} W_{it}^{j}. \tag{3.1}$$

Here, W_{it} is the overall wealth of individual i at time t, which is the sum across the aforementioned categories of wealth indexed by j, denoted W_{it}^{j}. I rank individuals based on this measure of wealth, identify the individuals belonging to various top quantiles, q, of the wealth distribution, and compute their share of wealth relative to the wealth of all individuals in the dataset:

$$s_t^{\text{top q\%}} = \frac{\sum_{i \in \text{top q\%}} W_{it}}{\sum_i W_{it}} \tag{3.2}$$

Moreover, each of the components of wealth are associated with a cash flow consisting of the associated capital income. Dividends, interest income, rents, various forms of private business income, and pension income (excluding social security). Summing all these forms of income, I compute the share of these income flows received by the individuals within the top q% of the *wealth* distribution

$$\tilde{s}_t^{\text{top q\%}} = \frac{\sum_{i \in \text{top q\%}} I_{it}}{\sum_i I_{it}} \tag{3.3}$$

Note that $\tilde{s}_t^{\text{top q\%}}$ is a statistic related to the joint distribution of wealth and capital income. It is not the share of capital income flows received by the top q% of the capital income distribution.

The left panel of Figure 3.1 depicts the evolution of $s_t^{\text{top q\%}}$ and $\tilde{s}_t^{\text{top q\%}}$ over time, and the right panel depicts the gap between the them. Three features stand out from Figure 3.1. Firstly, there is a gap for all top quantiles, sometimes positive, sometimes negative. Secondly, the size of this gap varies substantially over time. For the top 1% and above, the gap is especially large during the dotcom boom and in the 2010s. However, the third observation that stands out is how remarkably similar the evolution of these two shares are over longer time horizons. In summary, over shorter horizons, the share of wealth held by top quantiles of the wealth distribution may deviate from the share of the associated capital income flows they receive. However, the trend rise in top wealth shares documented in the DINA data is accounted for by a trend rise in the concentration of the associated cash flows. What determines the size of this gap? Why does it move over time? And looking at the very top wealth shares, why does it seem to be large during the dotcom boom and in the period after the financial crisis?

These questions are the topic of the next section, where I discuss how the method for estimating wealth in the DINA data influences the measured size of this gap and how it varies over time.

3.3 A Gap Between Wealth Shares and Shares of the Underlying Capital Income Flows: Analytical Framework

In this section, I use a simple analytical framework to illustrate precisely how the capitalization method for estimating wealth from income flows generates a gap between wealth shares and shares of the underlying income flows when household portfolios are heterogeneous along the wealth distribution. The purpose is to examine the determinants of the size of this gap, which is visible in the DINA data in Figure 3.1. I begin by briefly covering the basics of the capitalization method.[5] This crucially

[5] See Saez and Zucman (2016) and Smith et al. (2022b) for further discussions of this method.

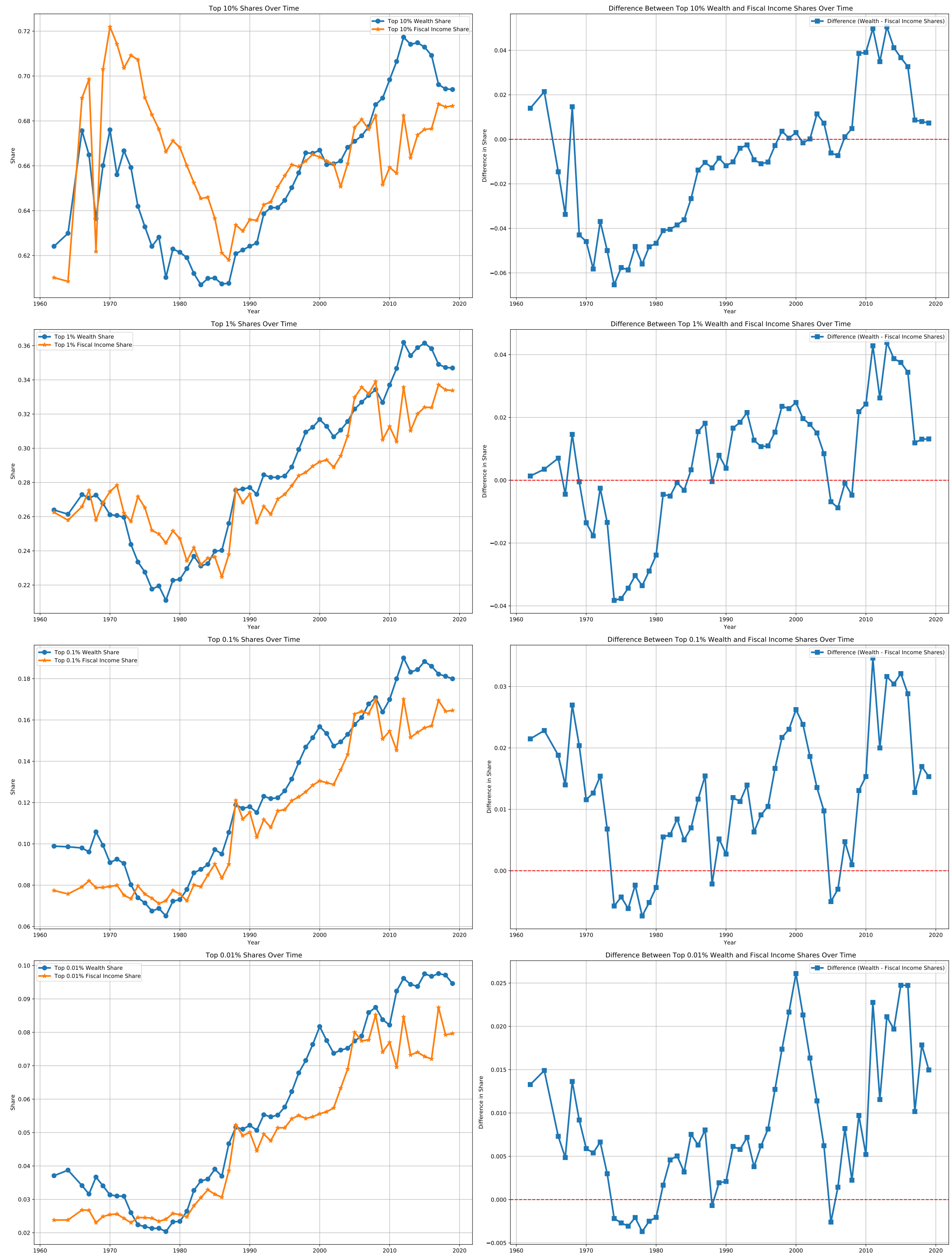


Figure 3.1: On the left is the share of wealth held by top quantiles (top 10%, 1%, 0.1% and 0.01%) of overall wealth distribution $s_t^{\text{top q\%}}$ defined in equation 3.2 along with their share of the associated capital income flows observed on tax returns (fiscal capital income) $\tilde{s}_t^{\text{top q\%}}$ defined in equation 3.3. On the right is the gap between these wealth shares and the associated fiscal capital income shares $s_t^{\text{top q \%}} - \tilde{s}_t^{\text{top q\%}}$. All computations using the DINA data provided by Piketty et al. (2018).

depends on the combination of heterogeneity in portfolio compositions along the estimated wealth distribution on the one hand, and heterogeneity in capitalization factors across asset categories on the other.

3.3.1 How Does the Capitalization Method Work?

In this section, I describe the capitalization method, which is at the heart of the wealth distribution estimates of both Saez and Zucman (2016) and Smith et al. (2022b). The most basic version of the capitalization method begins by breaking down asset income, observed on income tax returns, into different asset categories. Asset holdings for each category of asset is then estimated by multiplying the asset income by a time-varying and category specific valuation ratio, a "capitalization factor". The capitalization factor is the wealth-to-income rate for each specific asset category.

For instance, the capitalization factor for fixed income assets at time t, denoted ϕ_t^{FI}, is the ratio of the total value of fixed income assets in the economy, denoted W_t^{FI}, to the total interest income in the economy, denoted I_t^{FI}. In other words,

$$\phi_t^{\text{FI}} = \frac{W_t^{\text{FI}}}{I_t^{\text{FI}}}.$$

Note that $r_t^{\text{FI}} = \frac{1}{\phi_t^{\text{FI}}}$ can be viewed as an estimate of the yield on fixed income assets. Each single individual's interest income, denoted I_{it}^{FI}, is then multiplied by this *aggregate* capitalization factor to arrive at an estimate of that individuals fixed income wealth W_{it}^{FI}:

$$W_{it}^{\text{FI}} = \phi_t^{\text{FI}} I_{it}^{\text{FI}}.$$

As described by Saez and Zucman (2016): "For example, if the stock of fixed-income claims (bonds, deposits, etc.) recorded in the balance sheet of households is equal to 50 times the flow of interest income in tax data, we attribute $50,000 in fixed-income claims to a tax unit with $1,000 in interest". In this sense, the underlying distribution of asset holdings within each asset category is, in this sense, inferred from the distribution of asset incomes.

An individuals total wealth is measured as the sum of asset holdings across all asset categories:

$$W_{it} = W_{it}^{\text{FI}} + W_{it}^{\text{C-corporation equity}} + W_{it}^{\text{S-corporation equity}} + \dots \tag{3.4}$$

With these estimates, we can rank individuals based on their overall wealth, and let q denote the set of individuals belonging to the top quantile (like the top 1%) of the overall wealth distribution. The overall wealth of this top quantile q is then $W_t^q = \sum_{i \in q} W_{it}$, while amount of associated cash flows they receive are $I_t^q = \sum_{i \in q} I_{it}$. Their type j wealth and associated cash flows are similarly $W_t^{q,j} = \sum_{i \in q} W_{it}^j$, and $I_t^{q,j} = \sum_{i \in q} I_{it}^j$, respectively. The share of wealth held by this top quantile q is

$$s_t^q = \frac{W_t^q}{W_t}, \tag{3.5}$$

and their share of the associated cash flows is

$$\tilde{s}_t^q = \frac{I_t^q}{I_t}. \tag{3.6}$$

3.3.2 How Changing Capitalization Factors Impact Top Wealth Shares

In this simple framework, the we can show that the estimated share of overall wealth held by the top quantile q, defined in equation 3.5, is a weighted average of their shares of the underlying capital income flows, with weights that reflect the composition of the *aggregate* wealth portfolio. The following lemma summarizes this more precisely.

Lemma 4. *Let $\omega_t^j = \frac{W_t^j}{W_t}$ denote the weight of asset class j in the aggregate wealth portfolio, let $s_t^{q,j} = \frac{W_t^{q,j}}{W_t^j}$ denote the share of category j wealth held by the top quantile q, and let $\tilde{s}_t^{q,j} = \frac{I_t^{q,j}}{I_t^j}$ denote their share of the cash flows associated with that category of wealth. Then:*

- *the share of overall wealth held by the top quantile is*

$$s_t^q = \sum_{j \in J} \omega_t^j s_t^{q,j} = \sum_{j \in J} \omega_t^j \tilde{s}_t^{q,j} \tag{3.7}$$

- *the gap between the measured wealth share held by the top quantile s_t^q, and their share* *of the associated cash flows* $\tilde{s}_t^q$ *is*

$$\underbrace{s_t^q}_{\text{wealth share}} - \overbrace{\tilde{s}_t^q}^{\text{income share}} = \sum_j \underbrace{\frac{I_t^j}{I_t}}_{j\text{ income weight}} \cdot \overbrace{\left(\frac{\phi_t^j}{\phi_t} - 1\right)}^{\text{capitalization}} \cdot \underbrace{\tilde{s}_t^{q,j}}_{j\text{ income share}} \tag{3.8}$$

Proof. Consider first equation (3.7). The first equality in this equation is an accounting identity. In particular, we have

$$s_t^q = \frac{W_t^q}{W_t} = \sum_{j \in J} \frac{W_t^{q,j}}{W_t} = \sum_{j \in J} \frac{W_t^j s_t^{q,j}}{W_t} = \sum_{j \in J} \omega_t^j s_t^{q,j}.$$

The second equality follows because wealth shares $s_t^{q,j}$ and cash flow shares $\tilde{s}_t^{q,j}$ are identical within asset classes according to the capitalization method:

$$s_t^{q,j} \equiv \frac{W_t^{q,j}}{W_t^j} = \frac{\phi_t^j I_t^{q,j}}{\phi_t^j I_t^j} = \frac{I_t^{q,j}}{I_t^j} \equiv \tilde{s}_t^{q,j}.$$

Consider now equation (3.8). This follows from subtracting the cash flow share $\tilde{s}_t^q = \sum_{j \in J} \frac{I_t^{q,j}}{I_t} = \sum_{j \in J} \frac{I_t^j}{I_t} \tilde{s}_t^{q,j}$ from the weighted average in equation (3.7) and using the fact that aggregate portfolio weights can be written as

$$\omega_t^j = \left(\frac{\phi_t^j}{\phi_t}\right) \frac{I_t^j}{I_t}.$$

□

Lemma 4 makes it clear why changes in the capitalization factors might induce fluctuations in the measured top wealth share, even if all the underlying cash flow shares remain stable. They do so by shifting the weights of the different cash flow shares in the weighted average in equation (3.7). Moreover, it also clarifies that what matters is really *relative* capitalization factors ϕ_t^j/ϕ_t. Similarly, the composition of income flows I_t^j/I_t, also matters.[6]

[6]We also note that if all the cash flow shares $\tilde{s}_t^{q,j}$ are equal, then changes in relative capitalization

The next lemma clarifies that in order for changes in the aggregate portfolio weights (and therefore changes in capitalization factors) to have an impact on measured top wealth shares, portfolio compositions must vary along the wealth distribution.

Lemma 5. *Let $\omega_t^{q,j}$ denote the portfolio share devoted to asset category j in the wealth portfolio of the top quantile q. Keeping fixed the cash flow shares received by this quantile $\tilde{s}_t^{q,j}$ (or equivalently the within-asset class wealth share $s_t^{q,j}$), a change in the aggregate portfolio share ω_t^j changes the measured top wealth share if and only if $\omega_t^{q,j} \neq \omega_t^j$, i.e. if the top quantile has a different weight on asset category j in their wealth portfolio compared to the aggregate wealth portfolio.*

Proof. By definition, the ratio of the portfolio weight of asset category j in the portfolio of the top quantile, relative to the corresponding weight in the aggregate portfolio is

$$\frac{\omega_t^{q,j}}{\omega_t^j} \equiv \frac{\frac{W_t^{q,j}}{W_t^q}}{\frac{W_t^j}{W_t}} = \frac{s_t^{q,j}}{s_t^q}. \tag{3.9}$$

This means that $\omega_t^{q,j} \neq \omega_t^j$ if and only if $s_t^q \neq s_t^{q,j}$. But if $s_t^q \neq s_t^{q,j}$ then changing the weight ω_t^j in the weighted average in equation (3.7) will alter the top wealth share. However, if $s_t^q = s_t^{q,j}$ then changing the weight leaves the measured top wealth share unaltered. ☐

The above lemmas clarify that portfolio heterogeneity along the wealth distribution *combined* with different capitalization factors for different assets imply that the there is a gap between the wealth share held by the top quantile q and their share of the associated income flows. They also clarify that the size of this gap changes when relative capitalization factors change, even if the distributions of the underlying cash flows remain fixed. If either capitalization factors are the same across all asset categories, or if asset allocations do not vary with wealth, measured top wealth shares directly reflect shares of the underlying income flows. This would be the case if we viewed all wealth as belonging to one homogeneous asset category. In this

factors ϕ_t^j/ϕ_t (or changes in the composition of aggregate income flows I_t^j/I_t) do not affect this measure of top wealth shares.

case variation in capitalization factors have no impact on this measure of top wealth shares.

To summarize this, we can combine equation (3.8) from Lemma 4 and equation (3.9) from Lemma 5 and write the gap as a particular type of "covariance":

Proposition 4. *Let* $\tilde{\omega}_t^j = \frac{I_t^j}{I_t}$ *denote the weight of income of type j in the total income from wealth* I_t*. Then*

$$\frac{s_t^q - \tilde{s}_t^q}{s_t^q} = \widetilde{Cov}\left(\frac{\omega_t^{q,j}}{\omega_t^j}, \frac{\phi_t^j}{\phi_t}\right) \tag{3.10}$$

where the covariance $\widetilde{Cov}$ *is taken under the measure defined by the income composition weights* ω_t^j*.*

Proof. Combining equation (3.8) in Lemma 4 with equation (3.9) in Lemma 5 implies

$$s_t^q - \tilde{s}_t^q = \sum_j \frac{I_t^j}{I_t}\left(\frac{\phi_t^j}{\phi_t} - 1\right)\frac{\omega_t^{q,j}}{\omega_t^j} s_t^q \tag{3.11}$$

Under the measure defined by the income weights, this can clearly be written as

$$\frac{s_t^q - \tilde{s}_t^q}{s_t^q} = \widetilde{\mathbf{E}}\left[\left(\frac{\phi_t^j}{\phi_t} - 1\right)\frac{\omega_t^{q,j}}{\omega_t^j}\right]$$

which is equal to the covariance in (3.10) since

$$\widetilde{\mathbf{E}}\left[\frac{\phi_t^j}{\phi_t} - 1\right] = \sum_j \frac{I_t^j}{I_t}\frac{\phi_t^j}{\phi_t} - 1 = \sum_j \frac{W_t^j}{W_t} - 1 = \sum_j \omega_t^j - 1 = 0$$

where the last inequality follows from the fact that portfolio weights sum to 1. □

This proposition tells us that the gap is larger the larger is this covariance. In other words, the gap is larger if the top quantile wealth portfolio is tilted, relative to the aggregate portfolio, towards asset categories that have larger capitalization factors than the aggregate wealth-to-income ratio, for asset categories where the capital income flow from that asset category is a large fraction of all capital income flows. This proposition also mirrors the results in Greenwald et al. (2023) and Kuhn et al. (2020) regarding how capital gains affect top wealth shares when portfolio choices are heterogeneous across the wealth distribution: wealth inequality increases when capital gains are larger for the types of asset that wealthy individuals tend to hold.

In the context of wealth being estimated via the capitalization method, the proposition further characterizes the relationship between "asset valuations" (as captured by the capitalization factors) and the corresponding income flows.

3.3.3 The Role of Asset Category Delineations.

The previous subsection discusses what determines the gap between top wealth shares and the associated cash flow shares, *given* a particular delineation between asset categories. However, the choice of asset category delineations of course also matters for the size of this gap. In particular, equation (3.10) tells us that the gap is larger if there are large differences in capitalization factors *between* asset categories, and the top quantile invests more heavily in asset categories with higher capitalization factors. By assumption, there is no heterogeneity in capitalization factors *within* asset classes. In practice, any implementation of the capitalization method risks missing some heterogeneity within asset categories. As discussed extensively in Saez and Zucman (2016), Smith et al. (2022b) and Saez and Zucman (2020), the combination of heterogeneity within asset categories in capitalization factors, and systematic variation in portfolio choices correlating with this heterogeneity in capitalization factors, will lead to incorrect estimates of wealth.

I emphasize that the present study is *not* about correctly estimating wealth. However, it is important to understand how the choice of asset delineations affects the gap between measured top wealth shares and shares of the underlying cash flows. In the extreme case of only one asset category, top wealth shares are going to be identical to cash flow shares and any movements in the top wealth share will be associated with movements in cash flows shares.

Making a finer asset category delineation is in general going to affect both top wealth shares s_t^q and cash flows shares $\tilde{s}_t^q$. It affects the measure of top wealth shares s_t^q for all the reasons discussed in Saez and Zucman (2016), Smith et al. (2022b) and Saez and Zucman (2020). It affects the cash flows shares $\tilde{s}_t^q$, because this is a statistic based on the estimated joint distribution of cash flows and wealth: to compute the cash flow share received by the top percentile q of the wealth distribution, you must first identify those in the top quantile. To the extent that changing asset category delineations changes the ranking of individuals in the wealth distribution, $\tilde{s}_t^q$ also changes. Making other choice regarding asset class delineations may make the gap $s_t^q - \tilde{s}_t^q$ smaller or larger. It depends on if the new joint distribution of wealth and its associated cash flows features a stronger positive relationship between wealth-to-capital income ratios and wealth.

3.3.4 Deviations From the Baseline Capitalization Method in DINA

In addition to the fact that capitalization factors differ across asset categories, and portfolio choices vary systematically along the wealth distribution, there is an additional reason for why there is a gap between the measured top wealth shares and the associated shares of the underlying cash flows in the DINA data. This is simply that Piketty et al. (2018) make various adjustments to the baseline capitalization method. For instance, because they want their estimates of wealth to roughly agree with estimates from sources like the Forbes 400 list, they make upward adjustments of equity wealth for wealthy households relative to the basic capitalization method. Moreover, when capitalizing income from equity, they include realized capital gains instead of just capitalizing dividends. They also apply different capitalization factors to interest income depending on an individuals rank in the wealth distribution.[7]

All of these adjustments affect the size of the gap between the measured wealth share of various top quantiles of the wealth distribution, and their respective shares of the associated capital income flows in the Distributional National Accounts Data.

3.4 Top Wealth Shares: A Race Between Broad Asset Classes

In equation (3.7) of Section 3.3.2, we saw that the share of wealth held by a top quantile s_t^q, could, as a matter of pure accounting, be expressed as a weighted average

[7]The appropriate way of doing these adjustments is discussed extensively in Smith et al. (2022b) and Saez and Zucman (2020).

of within-asset category wealth shares:

$$s_t^q = \sum_{j \in J} \omega_t^j s_t^{q,j}.$$

This simply states that the overall share of wealth held by top quantile q is the weighted average of the shares of wealth that they hold in each asset category, weighted by the fraction of aggregate wealth that each category represents. This equality is an accounting identity and does not depend on the capitalization method.[8] With this, we can compare the top wealth share s_t^q at two different points in time, t and $t+k$

$$\Delta s_{t+k}^q \equiv \sum_{j \in J} \left(\omega_{t+k}^j s_{t+k}^{q,j} - \omega_t^j s_t^{q,j} \right) \tag{3.13}$$

and ask a very simple question: how much of the change in the top wealth share is accounted for by changes in the distribution of wealth *within* asset classes, and how much is accounted for by changes in the *weights*?

How should this question be understood? Lemma 5 tells us that when portfolio compositions vary systematically along the wealth distribution then changes in the aggregate portfolio weights ω_t^j will change the top wealth share, keeping fixed the within asset class wealth shares $s_t^{q,j}$. For instance, if the top quantile invests more heavily in equities (with the equity asset category being denoted $j = E$) compared to the aggregate portfolio, so that $\omega_t^{q,E} > \omega_t^E$, then the top quantile share of all equity wealth $s_t^{q,j}$, is larger than their share of overall wealth s_t^q. This means that when equities increase in value relative to aggregate wealth, the top wealth share will rise.[9]

Since the aggregate portfolio shares measure the total value of asset category j relative to aggregate wealth, changes in these weights over time measure the relative

[8]Under the capitalization method, we could go further, and replace the within-asset category wealth shares $s_t^{q,j}$ with the corresponding income shares $\tilde{s}_t^{q,j}$

$$s_t^q = \sum_{j \in J} \omega_t^j \tilde{s}_t^{q,j} \tag{3.12}$$

[9]The same flavor reasoning is in Meade (1964). In that context, the distribution of personal income can become more concentrated if either capital or labor income becomes more concentrated, or if the capital income share increases, since capital income is more concentrated than labor income.

performance of each asset class. This point is made in Kuhn et al. (2020), and borrowing their language, variation in the top wealth share can be understood partly as a "race" between broad asset classes. A tractable way of decomposing the change in the share of wealth held by the top quantile q is as follows

$$\Delta s^q_{t+k} = \underbrace{\sum_j s^{q,j}_t \Delta\omega^j_{t+k}}_{\text{asset class allocation}} + \underbrace{\sum_j \omega^j_t \Delta s^{q,j}_{t+k}}_{\text{within asset class}} + \underbrace{\sum_j \Delta\omega^j_{t+k} \Delta s^{q,j}_{t+k}}_{\text{interaction term}} . \quad (3.14)$$

This states that the change in the top wealth share can, as a matter of accounting, be decomposed in to three parts. The first term in equation (3.14) represents the fact that asset class allocations differ across the wealth distribution, and different asset classes perform differently on average. The second term represents the fact that the distribution of wealth changes within asset classes as well. The third part is the interaction between the two effects.[10]

3.4.1 The Race Between Asset Classes in the DINA Data

In this section, I study the decomposition in equation (3.14) in the DINA data. To compute this decomposition, the necessary ingredients are: (i) a delineation of asset categories, (ii) top quantile shares of wealth within each asset category, at time t and $t + k$, (iii) aggregate portfolio weights for each asset category at time t and $t + k$.

The delineation between asset categories is of course crucial for this exercise. This is obvious when considering the extreme case with only one asset category. By construction, any change in the top wealth share is going to be accounted for by changes within that asset category. I consider the following seven asset categories: C-corporation equity, Fixed Income Assets, Housing, Sole Proprietorships, Partnerships, S-Corporation equity, Pensions, all net of debt.

I begin by ranking individuals based on the DINA measure of wealth, identifying the individuals within the top 10%, top 1%, top 0.1% and 0.01%. I then compute the share of wealth that these individuals hold $s^{q,j}_t$ and $s^{q,j}_{t+k}$ in each of the seven asset

[10]A continuous time version of equation (3.14) is studied in Moll et al. (2022) in the context of the relationship between the personal income distribution and the factor income distribution. In this continuous time limit, the interaction term shrinks to zero.

Top quantile	Top 10%	Top 1%	Top 0.1%	Top 0.01%
Total measured increase	8.3	13.6	11.4	7.4
Asset class allocation	−1.8	−0.8	0.5	0.4
Within asset class	5.9	11.9	9.6	5.8
Interaction term	4.2	2.5	1.3	1.2

Table 3.1: Decomposition of the rise in various top wealth shares. All numbers in percentage points. All computations using the DINA data provided by Piketty et al. (2018).

categories at time t and time $t + k$. I choose time $t = 1978$ because this is the year that the top 1% wealth share reached its lowest value in the DINA data. I choose $t + k = 2019$ because this is the latest available year for the public use DINA-files. I compute the aggregate portfolio weights ω_t^j and ω_{t+k}^j as the ratio of category j wealth summed across all individuals, to overall wealth summed across all individuals. Finally, I compute each of the three terms of the decomposition in on the right-hand side of (3.14). Table 3.1 summarizes the results. Within each of the top percentiles, the vast majority of the measured increase in the corresponding top wealth share is accounted for by an increasingly concentrated distribution of wealth within each asset category. For example, for the 11.4 percentage point increase in the wealth share of the top 0.1%, 9.6 percentage points is accounted for by the top 0.1% holding a larger fraction of wealth *within* asset categories, while only 0.5 percentage points of the rise are accounted for by the top 0.1% holding portfolios that are tilted towards the asset classes that have performed well.

The race between the stock market and the housing market? How do these results square with the results of Kuhn et al. (2020), who find that changes in the relative performance of housing and equities, have substantial impact on top wealth shares? I will argue that the seeming contradiction is more apparent than real, and that the results in the present study are consistent with the evidence presented by Kuhn et al. (2020). What Kuhn et al. (2020) show is that the top 10% wealth share falls when house prices increase, and rises when stock prices increase. They do this by running regressions of changes in the wealth share of the top 10% on changes in house prices and stock prices, respectively. Mirroring the results from these regressions, they also conduct a counterfactual exercise showing that the top 10% wealth share would have been lower had stock prices remained constant over a long period of time, and vice versa if house prices had remained constant.

To interpret these results in the context of the decomposition in this paper, consider a simple example economy, where housing (H) and equity (E) are the only sources of wealth. Because $\omega_t^E + \omega_t^H = 1$ in this economy, we have that $\Delta\omega_{t+k}^E = -\Delta\omega_{t+k}^H$. That is, an increase in the relative value of equities is exactly the same thing as a fall in the relative value of housing.

To understand what happens to the top wealth share over time periods when $\Delta\omega_{t+k}^E > 0$, so that the total value of equities grows relative to the total value of housing, let us keep fixed the within-asset class wealth shares: $s_t^{q,j} = s_{t+k}^{q,j} = s^{q,j}$. In other words, consider the case when $\Delta s_{t+k}^{q,j} = 0$ in equation (3.14). Then substituting $\Delta\omega_{t+k}^E = -\Delta\omega_{t+k}^H$ into the decomposition yields

$$\Delta s_{t+k}^q = \left(s^{q,E} - s^{q,H}\right) \Delta\omega_{t+k}^E \tag{3.15}$$

where $s^{q,E}$ and $s^{q,H}$ are the fixed within-asset class wealth shares. In the data DINA data, the difference $s^{q,E} - s^{q,H}$ is large, around 0.4 on average (using C-corportation equity as the measure of equity). This means that if the aggregate portfolio share of equity wealth rises by 10 percentage points, then the wealth share of the top 10% rises by around 4 percentage points. This would account for about one third of the total increase in the top 10% share of wealth since 1978. Incidentally, the value of equity wealth relative to aggregate wealth rose by around 10% in the DINA data during the dot.com boom period 1990-1999, and the top wealth share rose by around 5 percentage points over this period. This suggests that changes in the relative performance of different asset classes do move top wealth shares around in the DINA data, just as Kuhn et al. (2020) document in the SCF+ data.

Note, however, that this does not necessarily mean that such changes account for a large part of steady rise in the top wealth shares over since the late 1970s. Partly because some of the movements in the aggregate portfolio compositions are reversed over time, as equities do not always outperform housing, and partly because there are countervailing movements in the portfolio shares of other asset classes, as equity and housing are not the only asset classes. In summary, Kuhn et al. (2020) uncover that *swings* in the relative valuations of housing and equity generate swings in the top wealth share, while the present study focuses on whether the observed *trends* in relative valuations can account for the trends in top wealth shares.

Figure 3.2 illustrates this by depicting year-by-year values of the between-asset allocation term's contribution to changes in the top wealth share in the decomposition in (3.14), for each year from 1978 to 2019. It is clear from this picture that changes in aggregate portfolio weights combined with the heterogeneity in asset allocations along the wealth distribution matter for changes in the wealth share of the top quantiles on a year-by-year basis. Note, in particular, that the spectacular movements in asset prices around the dotcom boom and the financial crisis of 2008 seem to have impacted top wealth shares through the heterogeneity in asset allocations along the wealth distribution. The race between asset classes is therefore likely important for understanding fluctuations in top wealth shares in the DINA data. However, the decomposition for the entire period from 1978 to 2019 displayed in Table 3.1, suggests that the documented *trend* rise in top wealth shares over this time period is not driven by the relative performance of broad asset classes.

3.5 Conclusion

This paper studied the joint distribution of wealth and its associated capital income flows in the Distributional National Accounts provided by Piketty et al. (2018). The gap between the share of wealth held by top quantiles of the wealth distribution and their share of the associated capital income flows, varies a lot in this data set. However, the half-century steady rise in top wealth shares as measured in this data set is not accounted for by steady growth in the size of this gap.

If one takes seriously the joint distribution of wealth and its associated cash flows in DINA, these findings suggest that theories relying solely on changing valuations of a stable distribution of income flows for explaining the rise in top wealth shares are insufficient, as the trend rise in wealth concentration is associated with a similar rise in the concentration of the associated capital income flows. However, an alternative

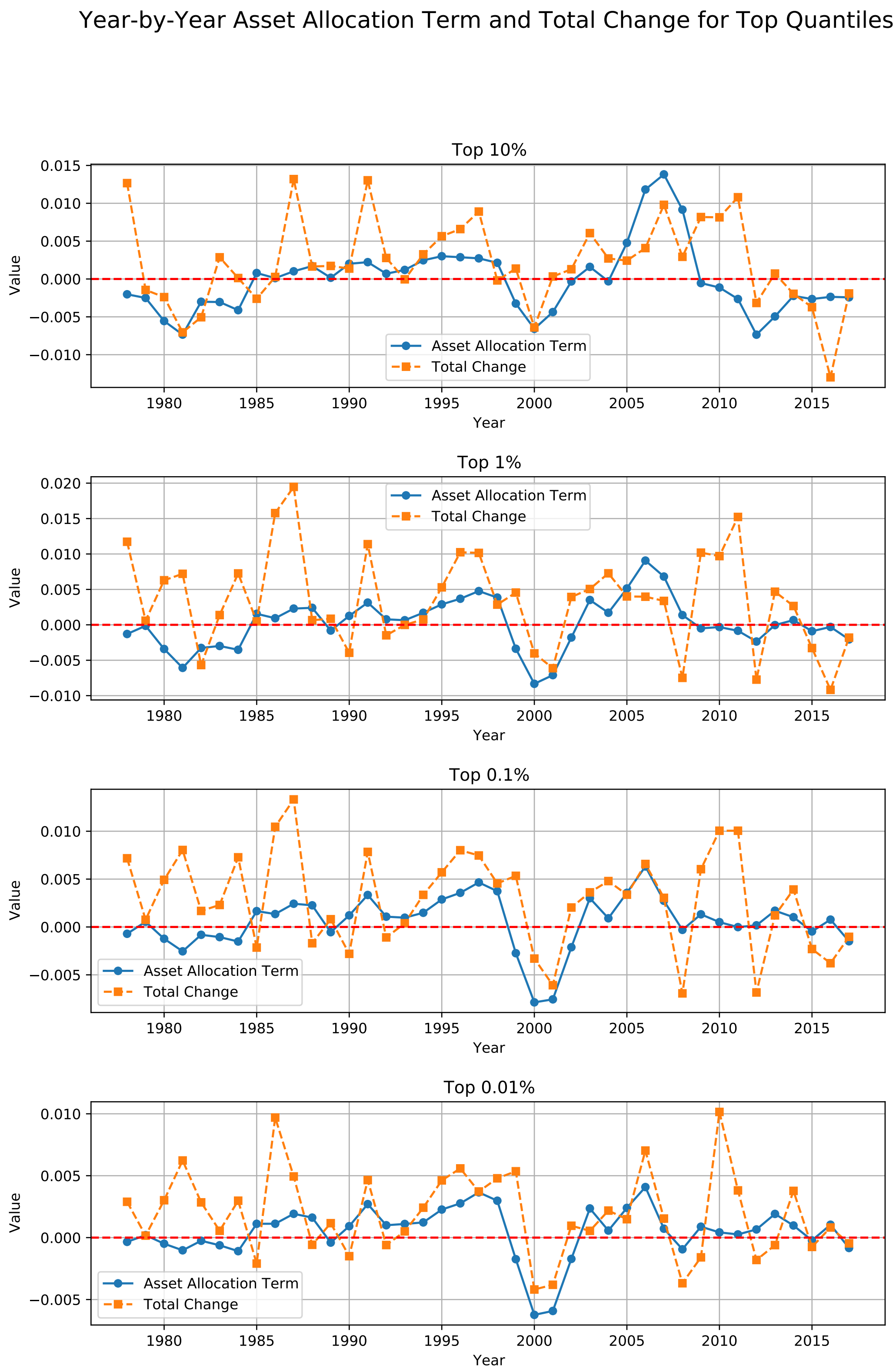


Figure 3.2: Year-by-year contribution of the asset allocation term in the decomposition in equation 3.14 (the first term in this equation) for various top quantiles. All computations using the DINA data provided by Piketty et al. (2018).

interpretation is that the joint rise in the share of wealth held by top quantiles of the wealth distribution and their share of the associated cash flows is a mechanical artefact of the capitalization method.

I also studied the relative contribution of between-asset class performance versus within-asset class inequality for the rise in top wealth shares. While the race between broad asset classes matters for fluctuations in top wealth shares over shorter horizons, the long-run trend is primarily accounted for by rising inequality within asset classes.

Appendices

Appendix A

Appendix for Chapter 1

A.1 Wealth and Demographics

A.1.1 Wealth.

The economy is populated by a representative worker endowed with L units of labor, and a continuum $i \in [0,1]$ of capitalists. The net worth of capitalist i is denoted by n_{it}. Workers have no net worth. For as long as capitalist i is alive, her net worth evolves according to

$$\frac{dn_{it}}{n_{it}} = \left(r_{it} - \frac{c_{it}}{n_{it}} \right) dt + \tilde{\sigma}_{it} dZ_{it} + \sigma_{it} dZ_t \tag{A.1}$$

where r_{it} is the expected return on the entrepreneurs' portfolio, c_{it} is consumption, $\tilde{\sigma}_{it}$ and σ_{it} are the exposures to the idiosyncratic Brownian motion Z_{it} and the aggregate Brownian motion Z_t, respectively. The expected return, consumption rate, and risk exposures are to be determined in the equilibrium of the model.

A.1.2 Demographics.

The group of capitalists consists of two types, entrepreneurial capitalists and fully diversified capitalists. These types are denoted by E and D respectively. Entrepreneurial capitalists are in possession of a viable entrepreneurial project and can choose to run a firm based on that project. Diversified capitalists do not have such a project and instead passively invest their wealth. Entrepreneurial capitalists lose their project at rate ϕ^l and then become fully diversified capitalists.

Capitalists die at rate $\tilde{\delta}_d$. When this happens, the capitalist is replaced with an agent who either inherits the wealth and type of their parent, leaving the dynasty intact, or the dynasty breaks and the new agent is reborn with the average wealth level of capitalists. The probability that the dynasty is broken is π_0. In other words, we can define $\delta_d = \tilde{\delta}_d \pi_0$ as the rate at which dynasties are broken. When dynasties are broken, the newborn agent becomes an entrepreneur with probability ψ^0.

The evolution of the fraction of capitalists that have a viable entrepreneurial project, denoted ψ_t, is therefore

$$d\psi_t = \left(-\delta_d \psi_t - \phi^l \psi_t + \delta_d \psi^0\right) dt \tag{A.2}$$

In steady state, $\psi_t = \bar{\psi} = \frac{\delta_d \psi^0}{\delta_d + \phi^l}$.

A.2 Firms and Technology

A.2.1 Final good

Final output Y_t is produced by a representative firm using a CES-technology and two types of intermediate goods Y_t^E and Y_t^T:

$$Y_t = \left[\nu \left(Y_t^E\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\left(Y_t^T\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.3}$$

where ε is the elasticity of substitution between the intermediate goods. The prices of the intermediate goods are p_t^E and p_t^T respectively. The first-order conditions associated with the final goods producer are

$$p_t^E = \nu \left(\frac{Y_t^E}{Y_t}\right)^{-\frac{1}{\varepsilon}}, \quad p_t^T = (1-\nu)\left(\frac{Y_t^T}{Y_t}\right)^{-\frac{1}{\varepsilon}} \tag{A.4}$$

A.2.2 Intermediate Goods-Producing Firms

Intermediate good Y_t^T is produced by a continuum of traditional firms indexed by $j \in [0,1]$. This sector will in the end be captured by a representative traditional firm. Intermediate good Y_t^E is produced by a continuum of entrepreneurial firms indexed by $i \in E$. In other words, the entrepreneurial firms are indexed by their associated entrepreneur. Both types of firm produce output using a Cobb-Douglas technology:

$$\begin{aligned} y_{it}dt &= \bar{A}\left(k_{it}\right)^{\alpha}\left(l_{it}\right)^{1-\alpha} \\ y_{jt}dt &= \underline{A}\left(k_{jt}\right)^{\alpha}\left(l_{jt}\right)^{1-\alpha} \end{aligned} \tag{A.5}$$

where $\bar{A} > \underline{A}$ so that entrepreneurial firms have higher total factor productivity. Both types of firm own and operate a capital stock.

$$\begin{aligned} dk_{it} &= k_{it}\left(\iota_{it}-\delta\right)dt + y_{it}\tilde{\sigma}dZ_{it} + k_{it}^{E}\sigma dZ_t + \Delta_{it}^{k} \\ dk_{jt} &= k_{jt}\left(\iota_{jt}-\delta\right)dt + y_{jt}\tilde{\sigma}Z_{jt} + k_{jt}\sigma dZ_t + \Delta_{jt}^{k} \end{aligned} \tag{A.6}$$

where ι_{it}, ι_{jt} are investment rates, δ is the depreciation rate, dZ_{it}, dZ_{jt} are idiosyncratic shocks, dZ_t is an aggregate shock, and $\Delta_{it}^k, \Delta_{jt}^k$ are net capital purchases from other firms. Note that both types of firms have the same level of risk exposures. Note that the risk associated with the firm is not in the form of a TFP shock, but rather a stockastic depreciation shock to capital (see Wälde (2011) for a review of different ways of adding risk to standard models of production in continuous time).

Traditional firms' problem and the representative traditional firm. Traditional firms are entirely externally financed. They finance their capital stock by issuing equity (to any capitalist) at the cost of capital $r_t^{\text{out}} = r_t + \varsigma_t\sigma$, where ς_t is the price of aggregate risk in the economy. This price of aggregate risk will be determined in equilibrium. Equity has the same risk (volatility) as the risk in the capital of the firm. In other words, holding the equity of a traditional firm gives the instantaneous return

$$dR_{jt}^{k,T} = r_t^{\text{out}}dt + \frac{y_{jt}}{k_{jt}}\tilde{\sigma}dZ_{it} + \sigma dZ_t$$

The cost of capital only depends on aggregate risk because the external financiers form diversified portfolios, and so do not care about the idiosyncratic risk in the firm. Traditional firms decide how much capital to raise, how much labor to hire, and how much to invest, to maximize expected profit flows:

$$\pi_{jt} = \max_{\{k_{jt}, l_{jt}, \iota_{jt}\}} p_t^T y_{jt} - w_t l_{jt} - \iota_{jt} k_{jt} + k_{jt} \left(\iota_{jt} - \delta\right) - r_t^{\text{out}} k_{jt}. \tag{A.7}$$

Profit maximization is consistent with any investment rate. The first-order conditions are

$$\begin{aligned} w_t &= p_t^T (1-\alpha) \frac{y_{jt}}{l_{jt}} \\ r_t^{\text{out}} &= p_t^T \alpha \frac{y_{jt}}{k_{jt}} - \delta. \end{aligned} \tag{A.8}$$

Maximized profits are $\pi_{jt} = 0$. This means that the expected return to capital in this sector, denoted r_t^T, is equal to the cost of capital in this sector, r_t^{out}. From the first-order conditions, we see that each traditional firm chooses the same production input mix (labor-to-capital ratio). Hence, it is without loss of generality to consider the traditional firms as being represented by a representative traditional firm that produces a flow output

$$Y_t^T dt = \underline{A} \left(K_t^T\right)^{\alpha} \left(L_t^T\right)^{1-\alpha}, \tag{A.9}$$

and finances a capital stock $K_t^T = \int_j k_{jt} dt$ that evolves according to

$$dK_t^T = \left(\iota_t^T - \delta\right) dt + \sigma dZ_t,$$

which it finances by issuing equity, that pays a return

$$dR_t^{k,T} = r_t^T dt + \sigma dZ_t$$

with first-order conditions

$$\begin{aligned} w_t &= p_t^T (1-\alpha) \frac{Y_t^T}{L_t^T} \\ r_t^{\text{out}} &= p_t^T \alpha \frac{Y_t^T}{K_t^T} - \delta. \end{aligned} \tag{A.10}$$

using $r_t^{out} = r_t^T$, we can write the labor-to-capital ratio as

$$\frac{L_t^T}{K_t^T} = \frac{1-\alpha}{\alpha}\frac{r_t^T + \delta}{w_t}$$

Entrepreneurial firms. The entrepreneurial firms hire labor on the same labor market as the traditional firm at wage rate w_t. The instantaneous return on the productive assets of an entrepreneurial firm is

$$dr_{it}^k = \left(\frac{p_t^E y_{it} - w_t l_{it} - \delta k_{it}}{k_{it}}\right) dt + \frac{y_{it}}{k_{it}}\tilde{\sigma} dZ_{it} + \sigma dZ_t \tag{A.11}$$

Entrepreneurial firms are partially financed internally by the associated entrepreneur, and partially financed externally by issuing equity to other capitalists. However, external financing is not unconstrained. In particular, the entrepreneur faces a skin-in-the-game constraint so that at least a fraction χ of the risk in the firm must be retained by the entrepreneur.[1] Letting v_{it}^{out} denote the total value of the liabilities issued to outsiders, the constraint on equity issuance is

$$\frac{k_{it} - v_{it}^{\text{out}}}{k_{it}} \geq \chi. \tag{A.12}$$

The risk in the liabilities issued to outsiders is the same as the risk in the productive assets of the firm, so the cost of external capital for entrepreneurs is the same as for the traditional firms $r_t^{\text{out}} = r_t + \varsigma_t \sigma = r_t^T$. The total return is therefore

$$dR_{it}^{\text{out}} = r_t^T dt + \frac{y_{it}}{k_{it}}\tilde{\sigma} dZ_{it} + \sigma dZ_t. \tag{A.13}$$

and the return on a diversified portfolio of the liabilities of all entrepreneurial firms is

$$dR_t^{\text{out}} = r_t^T dt + \sigma dZ_t. \tag{A.14}$$

Note that the return on investing in the traditional firms' equity, and investing in

[1]We technically allow χ to vary over time according to some Ito process, but suppress the dependence on time here.

entrepreneurial firms' equity, is identical from the perspective of an outsider. Capitalists are therefore indifferent between these investment opportunities. It is therefore without loss of generality to assume that capitalists hold shares in a mutual fund that buys the liabilities of all entrepreneurial firms and rents out capital to the established traditional firm. The return on this mutual fund is

$$dR_t^{\text{fund}} = r_t^T dt + \sigma dZ_t. \tag{A.15}$$

A.2.3 Aggregates

The total financial capital in the economy consists of the financial wealth of both types of capitalists, $N_t = N_t^E + N_t^D$. We let η_t denote the fraction of the financial capital in the economy held by capitalists with entrepreneurial projects:

$$\eta_t = \frac{N_t^E}{N_t}. \tag{A.16}$$

The financial wealth of the economy consists of claims on the productive assets of the economy, in other words the real capital of the economy K_t. Therefore, the balance sheet of the economy is

$$K_t = N_t^E + N_t^D. \tag{A.17}$$

Recalling that the aggregate capital stock is split between the established traditional firm and the entrepreneurial firms, we define κ_t as the fraction of the capital stock in the entrepreneurial sector:

$$\kappa_t = \frac{K_t^E}{K_t} \tag{A.18}$$

It will turn out to be the case that the labor-to-capital ratio in each firm is the same, and therefore aggregate output can be written as

$$Y_t = A(\kappa_t) K_t^\alpha L^{1-\alpha} \tag{A.19}$$

where aggregate TFP satisfies

	Expected return	Risk
k_{it}:	$\frac{p_t^E y_{it} - w_t l_{it} - \delta k_{it}}{k_{it}} - r_t$	$\frac{y_{it}}{k_{it}} \tilde{\sigma} dZ_{it} + \sigma dZ_t$
v_{it}^{out}:	$\varsigma_t \sigma$	$\frac{y_{it}}{k_{it}} \tilde{\sigma} dZ_{it} + \sigma dZ_t$
v_{it}^{fund}:	$\varsigma_t \sigma$	σdZ_t
b_{it}:	0	0

Table A.1: Risk-return profiles

$$A(\kappa_t) = \left[\nu \left(\bar{A} \kappa_t \right)^{\frac{\varepsilon - 1}{\varepsilon}} + (1 - \nu) \left(\underline{A} (1 - \kappa_t) \right)^{\frac{\varepsilon - 1}{\varepsilon}} \right]^{\frac{\varepsilon}{\varepsilon - 1}}.$$

The aggregate investment in the economy is the output less consumption so that the aggregate capital stock evolves over time according to

$$dK_t = \left(Y_t - C_t^E - C_t^D - C_t^W - \delta K_t \right) dt + \sigma K_t dZ_t. \tag{A.20}$$

Finally, since zero-net supply riskless bonds and aggregate risk can be traded without frictions, there is a unique riskless rate r_t and a unique price of aggregate risk ς_t.

A.2.4 Entrepreneur's Problem

The net worth of an individual entrepreneur can be written as

$$n_{it} = \underbrace{k_{it} - v_{it}^{\text{out}}}_{\text{stake in own firm}} + \underbrace{v_{it}^{\text{fund}}}_{\text{mutual fund holdings}} - \underbrace{d_{it}}_{\text{debt}}. \tag{A.21}$$

Each of the components of the net worth of an entrepreneur is associated with some expected excess return and some risk. Table A.1 summarizes the returns and risk associated with each of these components. Letting $\theta_{it}^k = \frac{k_{it}}{n_{it}}$, $\theta_{it}^{\text{out}} = \frac{v_{it}^{\text{out}}}{n_{it}}$, $\theta_{it}^{\text{fund}} = \frac{v_{it}^{\text{fund}}}{n_{it}}$ and $x_{it} = \frac{y_{it}}{k_{it}}$, and $-\theta_{it}^d = 1 - \theta_{it}^k + \theta_{it}^{\text{out}} - \theta_{it}^{\text{fund}}$ we can then write the entrepreneurs problem as a Merton optimal portfolio choice problem (See Merton (1992) and Brunnermeier and Sannikov (2017) for treatments of this type of problems):

$$\max_{\{c_{it}, x_{it}, \theta_{it}^k, \theta_{it}^{\text{out}}, \theta_{it}^{\text{fund}}\}} \mathbb{E} \left[\int_0^\infty e^{-\rho t} \log(c_{it}) dt \right] \tag{A.22}$$

subject to

$$\frac{dn_{it}}{n_{it}} = \left(r_t + \theta^k_{it}\left(r^k_{it} - r_t\right) - \theta^{\text{out}}_{it}\varsigma_t\sigma + \theta^{\text{fund}}_{it}\varsigma_t\sigma - \frac{c_{it}}{n_{it}}\right)dt$$
$$+ \left(\theta^k_{it} - \theta^{\text{out}}_{it}\right)x_{it}\tilde{\sigma}dZ_{it} + \left(\theta^k_{it} - \theta^{\text{out}}_{it} + \theta^{\text{fund}}_{it}\right)\sigma dZ_t$$
$$\text{where} \quad r^k_{it} = p^E_t x_{it} - w_t\left(\frac{x_{it}}{\bar{A}}\right)^{1/(1-\alpha)} - \delta$$
$$\text{and} \quad \frac{\theta^k_{it} - \theta^{\text{out}}_{it}}{\theta^k_{it}} \geq \chi$$

Let S_t denote a vector of all state variables except the individuals net worth. We allow χ to be such a state variable, and require that it follow an Ito process. The assumption that it follows an Ito process. Then, the HJB equation of this problem can be written as

$$\rho V(n,S) = \max_{\{c,x,\theta^k,\theta^{\text{out}},\theta^{\text{fund}}\}} \log(c) + V_n n\left(r + \theta^k\left(r^k - r\right) - \theta^{\text{out}}\varsigma\sigma + \theta^{\text{fund}}\varsigma\sigma - \frac{c}{n}\right)$$
$$+ \frac{1}{2}V_{nn}n^2\left(\left(\theta^k - \theta^{\text{out}}\right)^2(x\tilde{\sigma})^2 + \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)^2\sigma^2\right)$$
$$+ \sum_{s\in S} V_s\mu_s s + \frac{1}{2}V_{ss}s^2\sigma_s^2 + V_{ss'}ss'\sigma_s\sigma_{s'} + V_{sn}sn\sigma_s\sigma_n$$
$$+ \lambda((1-\chi)\theta^k - \theta^{\text{out}})$$

The first-order conditions of this problem are

$$c^{-1} = V_n$$
$$V_n n(r^k - r) = V_{nn}n^2\left(\left(\theta^k - \theta^{\text{out}}\right)(x\tilde{\sigma})^2 + \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2\right) - \lambda(1-\chi) + \sum_{s\in S} V_{sn}\frac{\partial\sigma_n}{\partial\theta^k}ns\sigma_s$$
$$V_n n\varsigma\sigma = V_{nn}n^2\left(\left(\theta^k - \theta^{\text{out}}\right)(x\tilde{\sigma})^2 + \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2\right) - \lambda + \sum_{s\in S} V_{sn}\frac{\partial\sigma_n}{\partial\theta^{\text{out}}}ns\sigma_s$$
$$V_n n\varsigma\sigma = V_{nn}n^2\left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2 + \sum_{s\in S} V_{sn}\frac{\partial\sigma_n}{\partial\theta^{\text{fund}}}sn\sigma_s$$
$$V_n n\theta^k\left(p^E - \frac{1}{1-\alpha}w\left(\frac{x}{\bar{A}}\right)^{\frac{\alpha}{1-\alpha}}\frac{1}{\bar{A}}\right) = V_{nn}n^2\left(\theta^k - \theta^{\text{out}}\right)^2 x\tilde{\sigma}^2 + \sum_{s\in S} V_{sn}\frac{\partial\sigma_n}{\partial x}sn\sigma_s$$

We can guess and verify that the value function takes the form $V(n,S) = \frac{1}{\rho}\log(n) +$

$g(S)$.[2] With this guess we have $V_n n = V_{nn} n^2 = \frac{1}{\rho}$ and all mixed derivatives are $V_{sn} = 0$. The first-order conditions can then be written as

$$
\begin{aligned}
r^k - r &= \left(\theta^k - \theta^{\text{out}}\right)(x\tilde{\sigma})^2 + \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2 - \rho\lambda(1-\chi) \\
\varsigma\sigma &= \left(\theta^k - \theta^{\text{out}}\right)(x\tilde{\sigma})^2 + \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2 - \rho\lambda \\
\varsigma\sigma &= \left(\theta^k - \theta^{\text{out}} + \theta^{\text{fund}}\right)\sigma^2 \\
\theta^k\left(p^E - \frac{1}{1-\alpha} w \left(\frac{x}{\bar{A}}\right)^{\frac{\alpha}{1-\alpha}} \frac{1}{\bar{A}}\right) &= \left(\theta^k - \theta^{\text{out}}\right)^2 x\tilde{\sigma}^2
\end{aligned}
$$

I look for solutions where entrepreneurs will actually want to invest some capital in their firm and so we consider the case $\theta^k > 0$. Combining the second and third first-order condition we obtain

$$
\rho\lambda = \left(\theta^k - \theta^{\text{out}}\right)(x\tilde{\sigma})^2 \tag{A.23}
$$

The skin-in-the-game constraint together with $\theta^k > 0$ ensures that the right-hand side of this equation is positive, which means $\lambda > 0$. Hence, the skin-in-the-game constraint is always binding. We then have $\theta^{\text{out}} = (1-\chi)\theta^k$ and the first-order conditions can be reduced to

$$
\begin{aligned}
r^k - r &= \chi\theta^k(x\tilde{\sigma})^2 + \left(\chi\theta^k + \theta^{\text{fund}}\right)\sigma^2 - \chi\theta^k(x\tilde{\sigma})^2(1-\chi) \\
\varsigma\sigma &= \chi\theta^k(x\tilde{\sigma})^2 + \left(\chi\theta^k + \theta^{\text{fund}}\right)\sigma^2 - \chi\theta^k(x\tilde{\sigma})^2 \\
\varsigma\sigma &= \left(\chi\theta^k + \theta^{\text{fund}}\right)\sigma^2 \\
p^E - \frac{1}{1-\alpha} w \left(\frac{x}{\bar{A}}\right)^{\frac{\alpha}{1-\alpha}} \frac{1}{\bar{A}} &= \theta^k\chi^2 x\tilde{\sigma}^2
\end{aligned} \tag{A.24}
$$

The second and the third first-order conditions are now identical. We can simplify this further to

[2]When utility is homothetic and budget constraints are linear, this is a standard guess for the form of the value function.

$$
\begin{aligned}
r^k - r &= \chi^2 \theta^k (x\tilde{\sigma})^2 + \varsigma\sigma \\
\varsigma &= \left(\chi\theta^k + \theta^{\text{fund}}\right)\sigma \\
p^E - \frac{1}{1-\alpha} w \left(\frac{x}{\bar{A}}\right)^{\frac{\alpha}{1-\alpha}} \frac{1}{\bar{A}} &= \theta^k \chi^2 x \tilde{\sigma}^2
\end{aligned}
\tag{A.25}
$$

we therefore have

$$
\begin{aligned}
\theta^k &= \frac{r^k - r^T}{(\chi x \tilde{\sigma})^2} \\
\theta^{\text{fund}} &= \frac{r^T - r}{\sigma^2} - \chi\theta^k \\
\theta^{\text{out}} &= (1-\chi)\theta^k
\end{aligned}
\tag{A.26}
$$

Multiplying the last first-order condition by x we obtain the following:

$$
p^E x - \frac{1}{1-\alpha} w \left(\frac{x}{\bar{A}}\right)^{\frac{1}{1-\alpha}} = \theta^k (\chi x \tilde{\sigma})^2 = r^k - r^T \tag{A.27}
$$

using the fact that $r^k = p^E x - w\left(\frac{x}{\bar{A}}\right)^{\frac{1}{1-\alpha}} - \delta$ we obtain

$$
\frac{1}{1-\alpha} w \left(\frac{x}{\bar{A}}\right)^{\frac{1}{1-\alpha}} = w \left(\frac{x}{\bar{A}}\right)^{\frac{1}{1-\alpha}} + r^T + \delta \tag{A.28}
$$

which implies

$$
x = \bar{A} \left(\frac{1-\alpha}{\alpha} \frac{r^T + \delta}{w}\right)^{1-\alpha} \tag{A.29}
$$

Or in terms of the labor-to-capital ratio

$$
\frac{l_{it}}{k_{it}} = \frac{1-\alpha}{\alpha} \frac{r^T + \delta}{w} = \frac{L^T}{K^T} \tag{A.30}
$$

confirming that every firm, including the representative traditional firm, has the same labor-to-capital ratio. In conclusion, the decision rules of any entrepreneur is

$$
\begin{aligned}
c_{it} &= \rho n_{it} \\
\frac{y_{it}}{k_{it}} &= \bar{A}\left(\frac{1-\alpha}{\alpha}\frac{R_t}{w_t}\right)^{1-\alpha} \\
\theta^k_{it} &= \frac{r^k_t - r^T_t}{(\chi \frac{y_{it}}{k_{it}}\tilde{\sigma})^2} \\
\theta^{\text{fund}}_{it} &= \frac{r^T_t - r_t}{\sigma} - \chi\theta^k_{it} \\
\theta^{\text{out}}_{it} &= (1-\chi)\theta^k_{it}
\end{aligned}
\tag{A.31}
$$

where $r^k_t = \frac{p^E_t Y^E_t - w_t L^E_t}{K^E_t} - \delta$. Also note that

$$
\theta^k_{it} - \theta^{\text{out}}_{it} + \theta^{\text{fund}}_{it} = \frac{\varsigma_t}{\sigma} \tag{A.32}
$$

which implies that their exposure to aggregate risk is $(\theta^k_{it} - \theta^{\text{out}}_{it} + \theta^{\text{fund}}_{it})\sigma = \varsigma_t$. Because each entrepreneurial firm chooses the same output-to-capital ratio, the aggregate supply of the intermediate good Y^E_t is

$$
Y^E_t = \bar{A}\left(K^E_t\right)^{\alpha}(L^E_t)^{1-\alpha} \tag{A.33}
$$

A.2.5 Diversified Capitalists and Workers

Diversified capitalists have wealth N^D_t that they invest in the mutual fund and riskless bonds. Diversified capitalists have log utility. Hence, their consumption as a group is $C^D_t = \rho N^D_t$ and the fraction of their wealth invested in the mutual fund is

$$
\theta^D_t = \frac{r_t + \varsigma_t\sigma - r_t}{\sigma^2} = \frac{\varsigma_t}{\sigma}. \tag{A.34}
$$

This implies that diversified capitalists net worth exposure to aggregate risk is $\theta^D_t\sigma = \varsigma_t$. Finally, workers supply labor inelastically and consume their labor income, so that $C^W_t = w_t L$.

A.2.6 Equilibrium

Given an initial capital stock K_0 and an initial share of wealth held by entrepreneurial capitalists η_0, an equilibrium is a map from histories of the Brownian shocks to price processes w_t, r_t, p_t^E, p_t^T and ς_t, and an allocation of capital between the established traditional firm and the entrepreneurial firms κ_t such that:

- All agents solve their respective problems given the prices.
- The markets for capital, labor, and financial assets clear.

$$\begin{aligned} &\int_{i\in E} k_{it} di + K_t^T = K_t = N_t^E + N_t^D \quad \int_{i\in E} l_{it} di + L_t^D = L \\ &\int_{i\in E} v_{it}^{out} di + K_t^T = \int_i v_{it}^{\text{fund}} di \end{aligned} \tag{A.35}$$

- The capital stock evolves according to

$$\frac{dK_t}{K_t} = \frac{(Y_t - C_t - \delta K_t)}{K_t} dt + \sigma dZ_t \tag{A.36}$$

- The share of wealth held by entrepreneurial capitalists evolves according to

$$\frac{d\eta_t}{\eta_t} = \frac{d\left(\frac{N_t^E}{N_t}\right)}{\frac{N_t^E}{N_t}} \tag{A.37}$$

  where N_t^E is the total wealth of entrepreneurial capitalists.

A.2.7 Characterizing the Equilibrium

The economy-wide state variables are η_t and K_t. To characterize equilibrium, we now derive an equation that pins down the allocation of capital to the entrepreneurial sector, $K_t^T = \kappa_t K_t$. All objects of interest in the model can then be expressed in terms of η_t, K_t, κ_t and exogenous parameters.

Combining the demand for intermediate goods in (1.11) with the supply of each intermediate good in equations (A.9) and (A.33) we obtain the following intermediate goods prices

$$p_t^T = (1-\nu)\left(\frac{\underline{A}}{A(\kappa_t)}(1-\kappa_t)\right)^{-\frac{1}{\varepsilon}}, \quad p_t^E = \nu\left(\frac{\bar{A}}{A(\kappa_t)}\kappa_t\right)^{-\frac{1}{\varepsilon}} \tag{A.38}$$

We can then derive an equation that pins down the fraction of the capital operated by the entrepreneurial firms, κ_t, by combining these expressions for the prices with the capital demand of entrepreneurial firms in equation (1.24). Specifically, using that $\frac{k_{it}}{n_{it}} = \frac{K_t^E}{N_t^E} = \frac{\kappa_t}{\eta_t}$ we obtain from this equation that

$$\frac{\kappa_t}{\eta_t}\left(\bar{A}\left(\frac{L}{K_t}\right)^{1-\alpha}\chi\tilde{\sigma}\right)^2 = r_t^k - r^T = p_t^E\bar{A}\left(\frac{L}{K_t}\right)^{1-\alpha} - w_t\left(\frac{L}{K_t}\right) - R_t \tag{A.39}$$

Combining this with the first-order conditions of the established traditional firm that provide expressions for w_t and R_t we obtain after some tedious algebra

$$\frac{\kappa_t}{\eta_t}\left(\bar{A}\chi\tilde{\sigma}\right)^2\left(\frac{L_t}{K_t}\right)^{1-\alpha} = p_t^E\bar{A} - p_t^T(1-\alpha)\underline{A} - p_t^T\alpha\underline{A} \tag{A.40}$$

which can be rewritten as

$$\frac{\kappa_t}{\eta_t}\left(\bar{A}\chi\tilde{\sigma}\right)^2\left(\frac{L_t}{K_t}\right)^{1-\alpha} = p_t^E\bar{A} - p_t^T\underline{A} \tag{A.41}$$

This equation pins down a unique $\kappa_t \in (0,1)$ if $\varepsilon > 0$. To see why, note that the left-hand side is a strictly increasing linear function of κ_t, given positive η_t and K_t. Moreover, using that $A(\kappa_t) = \left[\nu\left(\bar{A}\kappa_t\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\left(\underline{A}(1-\kappa_t)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}$ we can write the prices as

$$p_t^T = (1-\nu)\left[\nu\left(\frac{\bar{A}}{\underline{A}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{\kappa_t}{1-\kappa_t}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\right]^{\frac{1}{\varepsilon-1}} \tag{A.42}$$

and

$$p_t^E = \nu\left[\nu + (1-\nu)\left(\frac{\underline{A}}{\bar{A}}\right)^{\frac{\varepsilon-1}{\varepsilon}}\left(\frac{1-\kappa_t}{\kappa_t}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{1}{\varepsilon-1}} \tag{A.43}$$

We see that if $\varepsilon > 0$, then p_t^T is increasing in $\kappa_t \in (0,1)$ and p_t^E is decreasing. For κ_t close to 0 we will have $p_t^E\bar{A} - p_t^T\underline{A} > 0$, for κ_t close to 1 we will have $p_t^E\bar{A} - p_t^T\underline{A} < 0$. Hence, for some unique $\kappa_t \in (0,1)$, the left-hand side and right-hand side intersect.

A.2.8 The Entrepreneurial Appraisal Ratio.

In the model, financial innovation raises top wealth inequality only when it raises the so-called appraisal ratio associated with entrepreneurial activity. The appraisal ratio, which is sometimes called the "information ratio," is a close relative of the more widely known Sharpe ratio. In contrast to the Sharpe ratio, it compares the idiosyncratic risk premium of an investment relative to the idiosyncratic risk, instead of the total risk premium relative to the total risk. The appraisal ratio associated with entrepreneurship plays a crucial role because it determines how exposed an entrepreneur wants to be to her firm. Entrepreneurs choose a higher exposure to their firm if the appraisal ratio is high. I obtain tractable formulas for this choice in the model, using well-known tools from Merton (1969) and Angeletos (2007). In particular, entrepreneurs' choice of exposure is going to be proportional to the appraisal ratio associated with entrepreneurship.

A.2.9 Rates of Return

The idiosyncratic volatility of entrepreneurs' wealth is

$$\tilde{\sigma}_t^E = \frac{\kappa_t}{\eta_t}\bar{A}\left(\frac{L}{K_t}\right)^{1-\alpha}\chi\tilde{\sigma} = \frac{p_t^E - p_t^T\frac{A}{\bar{A}}}{\chi\tilde{\sigma}} \tag{A.44}$$

The expected return to entrepreneurs' wealth on the other hand is

$$r_t^E = r_t + \theta_t^k\left(r_t^k - r_t\right) - \theta_t^{\text{out}}\varsigma_t\sigma + \theta_t^{\text{fund}}\varsigma_t\sigma = r_t + \theta_t^k\left(r_t^k - (R_t - \delta)\right) - \chi\theta_t^k\varsigma_t\sigma + \theta_t^{\text{fund}}\varsigma_t\sigma \tag{A.45}$$

simplifying this using from entrepreneurs' capital demand that

$$\theta_t^k(r_t^k - (R_t - \delta)) = \left(\theta_t^k\chi\tilde{\sigma}\bar{A}\left(\frac{L}{K_t}\right)^{1-\alpha}\right)^2$$

and demand for the mutual fund gives us the following expression

$$r_t^E = r_t + \left(\tilde{\sigma}_t^E\right)^2 + \sigma^2 \tag{A.46}$$

Similarly, the expected return to diversified capitalists' wealth is

$$r_t^D = r_t + \sigma^2 \tag{A.47}$$

The overall return to wealth is then

$$r_t^K = \eta_t r_t^E + (1-\eta_t) r_t^D = r_t + \eta_t \left(\tilde{\sigma}_t^E\right)^2 + \sigma^2 \tag{A.48}$$

The return to the mutual fund and entrepreneurial capital is

$$r_t^T = r_t + \sigma^2, \quad r_t^k = r_t + \frac{\eta_t}{\kappa_t}\left(\tilde{\sigma}_t^E\right)^2 + \sigma^2 \tag{A.49}$$

Finally, the risk-free rate is pinned down by the first-order condition of the established traditional firms' capital demand:

$$r_t = R_t - \delta - \sigma^2 = p_t^T \alpha \underline{A} \left(\frac{L}{K_t}\right)^{1-\alpha} - \delta - \sigma^2 \tag{A.50}$$

A.2.10 Labor Share and Capital-Output Ratio

The labor share in the established traditional firm is

$$\frac{w_t L_t^T}{p_t^T Y_t^T} = 1 - \alpha \tag{A.51}$$

The labor share in the entrepreneurial firms is

$$\frac{w_t L_t^E}{p_t^E Y_t^E} = \frac{p_t^T (1-\alpha) \underline{A} \left(\frac{K_t}{L}\right)^{\alpha}}{p_t^E \bar{A} \left(\frac{K_t}{L}\right)^{\alpha}} = (1-\alpha) \frac{p_t^T}{p_t^E} \frac{\underline{A}}{\bar{A}} \tag{A.52}$$

using that $r_t^K - r_t^{\text{out}} = (p_t^E \bar{A} - p_t^T \underline{A}) \left(\frac{L}{K}\right)^{1-\alpha}$ This can be rewritten as

$$LS^E = (1-\alpha)\left(1 - \underbrace{\frac{(r_t^K - r_t^{\text{out}}) K_t^E}{p_t^E Y_t^E}}_{\text{“factorless" share}}\right) \tag{A.53}$$

The labor share in the overall economy is

$$\frac{w_t L}{Y_t} = (1-\alpha)\frac{p_t^T \underline{A}\left(\frac{K_t}{L}\right)^{\alpha} L}{A(\kappa_t)\left(\frac{K_t}{L}\right)^{\alpha} L} = (1-\alpha)p_t^T \frac{\underline{A}}{A(\kappa_t)} \tag{A.54}$$

We also derive the following expression for the labor share as the weighted average of the labor share in the two sectors. First, note that:

$$LS = \frac{wL}{Y} = \theta\frac{wL^E}{p_t^E Y_t^E} + (1-\theta)\frac{wL^T}{p_t^T Y_t^T} = \theta_t \frac{\kappa_t wL}{\nu_t Y_t} + (1-\theta_t)\frac{(1-\kappa_t)wL}{(1-\nu_t)Y_t}$$

$$= \theta_t \frac{\kappa_t}{\nu_t} LS + (1-\theta_t)\frac{1-\kappa_t}{1-\nu_t} LS$$

Which implies

$$1 = \theta_t \frac{\kappa_t}{\nu_t} + (1-\theta_t)\frac{1-\kappa_t}{1-\nu_t} \tag{A.55}$$

so that θ_t is

$$\theta_t = \frac{1 - \frac{1-\kappa_t}{1-\nu_t}}{\frac{\kappa_t}{\nu_t} - \frac{1-\kappa_t}{1-\nu_t}} \tag{A.56}$$

which can be rewritten as

$$\theta_t = \frac{1 - \nu_t - 1 + \kappa_t}{\frac{1-\nu_t}{\nu_t}\kappa_t - 1 + \kappa_t} \tag{A.57}$$

$$\theta_t = \frac{\nu_t(\kappa_t - \nu_t)}{(1-\nu_t)\kappa_t - \nu_t(1-\kappa_t)} \tag{A.58}$$

$$\theta_t = \nu_t \tag{A.59}$$

So the overall labor share is the sales weighted labor share between the two sectors. Using the previous expressions for the labor shares in the two sectors is

$$LS = \nu_t LS^E + (1-\nu_t)LS^T = LS^T - \nu_t\left(LS^T - LS^E\right) \tag{A.60}$$

which means

$$LS = (1-\alpha)\left[1 - \nu_t\left(1 - \frac{p_t^T}{p_t^E}\frac{\underline{A}}{\bar{\bar{A}}}\right)\right] \tag{A.61}$$

or differently

$$LS = (1-\alpha)\left[1 - \frac{\nu_t}{p_t^E}\left(p_t^E - p_t^T\frac{\underline{A}}{\bar{\bar{A}}}\right)\right] \tag{A.62}$$

Recall that $\tilde{\sigma}_t^E = \frac{p_t^E - p_t^T\frac{\underline{A}}{\bar{\bar{A}}}}{\chi\tilde{\sigma}}$. We can therefore write this expression of the labor share as

$$LS = (1-\alpha)\left[1 - \frac{\nu_t}{p_t^E}\tilde{\sigma}_t^E\chi\tilde{\sigma}\right] \tag{A.63}$$

We can go further by noting that

$$\frac{\nu_t}{p_t^E} = \frac{p_t^E Y_t^E}{p_t^E Y_t} = \frac{Y_t^E}{Y_t} = \frac{\bar{A}\left(\frac{L}{K}\right)^{1-\alpha}K_t^E}{A(\kappa_t)\left(\frac{L}{K}\right)^{1-\alpha}K_t} = \frac{\bar{A}\kappa_t}{A(\kappa_t)} \tag{A.64}$$

We therefore write the labor share as

$$LS = (1-\alpha)\left[1 - \frac{\bar{A}\kappa_t}{A(\kappa_t)}\chi\tilde{\sigma}\tilde{\sigma}_t^E\right] \tag{A.65}$$

We can use this expression to show that the behavior of the labor share is informative with regard to whether or not the supply of capital to entrepreneurs is high enough for financial innovation to increase the absolute risk exposure of entrepreneurs. To see this, suppose that we are in an economy where the elasticity is not high enough. In such a world, by definition, a fall in χ would lead to a fall in $\kappa_t\chi$ and therefore in $\tilde{\sigma}_t^E$. Noting that $A(\kappa_t)$ always increases when κ_t increases and $\varepsilon > 0$, we see that the expression $\frac{\bar{A}\kappa_t\chi}{A(\kappa_t)}\tilde{\sigma}_t^E\tilde{\sigma}$ must fall in this case. But then the aggregate labor share would go up. Hence, the aggregate labor share would go up in response to improvements in the ability of entrepreneurs to offload risk to financial markets if the supply elasticity was low.

Capital share

The pure capital share in the entrepreneurial firm is

$$\frac{R_t K_t^E}{p_t^E Y_t^E} = \alpha \frac{p_t^T}{p_t^E} \frac{\underline{\underline{A}}}{\bar{A}} \tag{A.66}$$

From this we see that the pure entrepreneurial share, or "factorless income" of the income in the entrepreneurial firms is

$$1 - \frac{w_t L_t^E}{p_t^E Y_t^E} - \frac{R_t K_t^E}{p_t^E Y_t^E} = 1 - \frac{p_t^T}{p_t^E} \frac{\underline{\underline{A}}}{\bar{A}} = \frac{\tilde{\sigma}\tilde{\sigma}^E}{p_t^E} \chi \tag{A.67}$$

Moreover, the capital-output-ratio in the economy is

$$\frac{K_t}{Y_t} = \frac{1}{A(\kappa_t)} \left(\frac{K_t}{L} \right)^{1-\alpha} \tag{A.68}$$

A.2.11 Evolution of State

The state variable K_t evolves according to

$$\frac{dK_t}{K_t} = \left(r_t^K - \rho \right) dt + \sigma dZ_t \tag{A.69}$$

and the state variable η_t evolves according to

$$\frac{d\eta_t}{\eta_t} = (1 - \eta_t) \left(\tilde{\sigma}_t^E \right)^2 dt + \left(\frac{\delta_d \psi^0 - \eta_t(\delta_d + \phi^I)}{\eta_t} \right) dt \tag{A.70}$$

A.2.12 Summary of Key Model Objects

Key model objects	
r_t:	$p_t^T \alpha \underline{A} \left(\frac{L}{K_t}\right)^{1-\alpha} - \delta - \sigma^2$
r_t^{out}:	$r_t + \sigma^2$
$R_t - \delta$:	$r_t + \sigma^2$
r_t^k:	$r_t + \sigma^2 + \frac{\eta_t}{\kappa_t} \left(\tilde{\sigma}^E\right)^2$
r_t^K:	$r_t + \sigma^2 + \eta_t \left(\tilde{\sigma}^E\right)^2$
r_t^E:	$r_t + \sigma^2 + \left(\tilde{\sigma}^E\right)^2$
r_t^D:	$r_t + \sigma^2$
LS^T:	$1 - \alpha$
LS^E:	$(1-\alpha)\frac{p_t^T}{p_t^E} \frac{\underline{A}}{\bar{A}}$
LS:	$(1-\alpha) p_t^T \frac{\underline{A}}{A(\kappa_t)}$
$\frac{K_t}{Y_t}$:	$\frac{1}{A(\kappa_t)} \left(\frac{K_t}{L}\right)^{1-\alpha}$

Table A.2: Summary of key model objects

The payout to owners of capital net of depreciation is

$$\frac{Y - wL - \delta K}{Y} = \frac{I + C^E + C^D - \delta K}{Y} = \frac{I + \rho K - \delta K}{Y} = \rho \frac{K}{Y} = \rho \frac{1}{A(\kappa)} \left(\frac{K}{L}\right)^{1-\alpha}$$

The share of external and internal financing can be expressed, respetively, as

$$\chi K_t^E = \text{internal financing} => \chi \frac{K_t^E}{N_t} = \chi \kappa_t \Rightarrow 1 - \chi \kappa_t = \text{external financing.}$$

A.3 Steady State Equilibrium and Transition Dynamics

A long-run steady state can be defined when setting aggregate shocks $dZ_t = 0$. In this case, a steady state equilibrium is an equilibrium where the state variables K_t and η_t are constant, i.e. where:

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(r_{ss}^K - \rho\right) dt = 0 \\ \frac{d\eta_t}{\eta_t} &= (1 - \eta_{ss}) \left(\tilde{\sigma}_{ss}^E\right)^2 dt + \left(\frac{\delta_d \psi^0 - \eta_{ss}(\delta_d + \phi^l)}{\eta_{ss}}\right) dt = 0 \end{aligned} \tag{A.71}$$

I will further assume throughout that $\varepsilon > 0$.

A.3.1 Analyzing the Steady State

Plugging in the expression for r^K_{ss} in equation (A.48) evaluated in steady state, the evolution of the economy is described by the following pair of ordinary differential equations:

$$\begin{aligned}\frac{dK_t}{K_t} &= \left(p^T \alpha \underline{A} \left(\frac{L}{K}\right)^{1-\alpha} + \eta \left(\tilde{\sigma}^E\right)^2 - \rho - \delta\right) dt \\ \frac{d\eta_t}{\eta_t} &= \left((1-\eta)\left(\tilde{\sigma}^E\right)^2 + \frac{(\bar{\psi}-\eta)\left(\delta_d + \phi^l\right)}{\eta}\right) dt\end{aligned} \tag{A.72}$$

where $\tilde{\sigma}^E_{ss} = \frac{\kappa}{\eta}\left(\chi\tilde{\sigma}\bar{A}\right)\left(\frac{L}{K}\right)^{1-\alpha}$, and the equilibrium condition for the allocation of the capital stock:

$$\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}} = \frac{\kappa}{\eta}\left(\chi\tilde{\sigma}\bar{A}\right)\left(\frac{L}{K}\right)^{1-\alpha} \tag{A.73}$$

where

$$\begin{aligned}p^E &= \nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}, p^T = (1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon} \\ A(\kappa) &= \left[\nu\left(\bar{A}\kappa\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\left(\underline{A}(1-\kappa)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}}\end{aligned} \tag{A.74}$$

The model admits a steady state if there is a solution to the following system of three equations in the three variables η, K and κ:

$$\begin{aligned}p^T\alpha\underline{A}\left(\frac{L}{K}\right)^{1-\alpha} + \eta\left(\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}}\right)^2 - (\rho+\delta) &= 0 \\ (1-\eta)\left(\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}}\right)^2 + \frac{(\bar{\psi}-\eta)\left(\delta_d+\phi^l\right)}{\eta} &= 0 \\ \frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}} - \frac{\kappa}{\eta}\left(\chi\tilde{\sigma}\bar{A}\right)\left(\frac{L}{K}\right)^{1-\alpha} &= 0\end{aligned} \tag{A.75}$$

Definition 1. *A non-degenerate steady state equilibrium is a triplet $s = (\eta, K, \kappa)$ that satisfies the equations (A.75) with $K, \eta > 0$ and $\kappa \in [0,1]$.*

I begin the analysis by proving the following lemmata:

Lemma 6. *If $\varepsilon > 1$ then $\lim_{\kappa\to 0} p^T = \underline{p}^T \equiv (1-\nu)^{\frac{\varepsilon}{\varepsilon-1}}$ and $\lim_{\kappa\to 0} p^E = \infty$. If $\varepsilon < 1$, then $\lim_{\kappa\to 0} p^T = 0$ and $\lim_{\kappa\to 0} p^E = \nu^{\frac{\varepsilon}{\varepsilon-1}} < \infty$. Symmetric limits apply to $\kappa \longrightarrow 1$.*

Proof.

$$p^T = (1-\nu)\left(\frac{A(\kappa)}{\underline{A}(1-\kappa)}\right)^{\frac{1}{\varepsilon}} = (1-\nu)\left(\nu\left(\frac{\bar{A}\kappa}{\underline{A}(1-\kappa)}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\right)^{\frac{1}{\varepsilon-1}}$$

If $\varepsilon > 1$, then the exponent on the ratio inside the bracket is positive and therefore

$$\lim_{\kappa\to 0} p^T = (1-\nu)\left(\nu\left(\frac{\bar{A}\cdot 0}{\underline{A}}\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\right)^{\frac{1}{\varepsilon-1}} = (1-\nu)^{\frac{\varepsilon}{\varepsilon-1}}.$$

If $0 < \varepsilon < 1$ then the exponent inside the bracket is negative, and so the argument inside the bracket goes to ∞ as $\kappa \to 0$. But the exponent outside the bracket is also negative and hence $\lim_{\kappa\to 0} p^T = 0$ in this case. For p^E we instead have

$$p^E = \nu\left(\frac{A(\kappa)}{\bar{A}\kappa}\right)^{\frac{1}{\varepsilon}} = \nu\left(\nu + (1-\nu)\left(\frac{\underline{A}(1-\kappa)}{\bar{A}\kappa}\right)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{1}{\varepsilon-1}}.$$

If $\varepsilon > 1$, the exponent inside the bracket is positive, and hence the expression inside the bracket goes to ∞ as $\kappa \to 0$. The exponent outside the brackets is positive and so $p^E \to \infty$ as $\kappa \to 0$. If $\varepsilon < 1$ then the exponents are both negative, and so $p^E \to \nu^{\frac{\varepsilon}{\varepsilon-1}}$. □

Lemma 7. *Steady state values of κ are bounded above by* $\bar{\kappa} \equiv \frac{\left(\frac{\underline{A}}{\bar{A}}\right)\left(\frac{\bar{A}\nu}{\underline{A}(1-\nu)}\right)^{\varepsilon}}{1+\left(\frac{\underline{A}}{\bar{A}}\right)\left(\frac{\bar{A}\nu}{\underline{A}(1-\nu)}\right)^{\varepsilon}} < 1$

Proof. From equation (A.73) we see that the numerator on the left-hand side cannot be negative, since all objects on the right-hand side are positive (risk exposure cannot be negative). Therefore

$$\bar{A}p^E - \underline{A}p^F = \bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon} - \underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon} > 0 \tag{A.76}$$

This implies

$$\frac{\bar{A}\nu}{\underline{A}(1-\nu)} > \left(\frac{\bar{A}\kappa}{\underline{A}(1-\kappa)}\right)^{1/\varepsilon} \Rightarrow \kappa < \bar{\kappa} \equiv \frac{\left(\frac{\underline{A}}{\bar{A}}\right)\left(\frac{\bar{A}\nu}{\underline{A}(1-\nu)}\right)^{\varepsilon}}{1+\left(\frac{\underline{A}}{\bar{A}}\right)\left(\frac{\bar{A}\nu}{\underline{A}(1-\nu)}\right)^{\varepsilon}} < 1 \tag{A.77}$$

□

Lemma 8. *Aggregate TFP $A(\kappa)$ is increasing in κ for $\kappa \leq \bar{\kappa}$.*

Proof. We can write

$$A(\kappa) = \left[\nu \left(\bar{A}\kappa\right)^{\frac{\varepsilon-1}{\varepsilon}} + (1-\nu)\left(\underline{A}(1-\kappa)\right)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{A.78}$$

Taking the derivative of this we have

$$\frac{dA(\kappa)}{d\kappa} = A(\kappa)^{\frac{1}{\varepsilon}}\left[\nu\bar{A}^{\frac{\varepsilon-1}{\varepsilon}}(\kappa)^{-\frac{1}{\varepsilon}} - (1-\nu)\underline{A}^{\frac{\varepsilon-1}{\varepsilon}}(1-\kappa)^{-\frac{1}{\varepsilon}}\right]$$

If we can show that $\nu\bar{A}^{\frac{\varepsilon-1}{\varepsilon}}(\kappa)^{-\frac{1}{\varepsilon}} \geq (1-\nu)\underline{A}^{\frac{\varepsilon-1}{\varepsilon}}(1-\kappa)^{-\frac{1}{\varepsilon}}$ for any $\kappa \leq \bar{\kappa}$, we are done. Using again the non-negativity of risk exposure, we have

$$\bar{A}p^E - \underline{A}p^F = \bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon} - \underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon} \geq 0$$

$$\Leftrightarrow \bar{A}\nu\left(\bar{A}\kappa\right)^{-1/\varepsilon} - \underline{A}(1-\nu)\left(\underline{A}(1-\kappa)\right)^{-1/\varepsilon} \geq 0$$

$$\nu\bar{A}^{\frac{\varepsilon-1}{\varepsilon}}(\kappa)^{-\frac{1}{\varepsilon}} \geq (1-\nu)\underline{A}^{\frac{\varepsilon-1}{\varepsilon}}(1-\kappa)^{-\frac{1}{\varepsilon}}$$

which is what we wanted to show. □

Lemma 9. *Steady state values of κ are bounded below by $\underline{\kappa} > 0$ if $\varepsilon > 1$.*

Proof. Through the second steady state equation in (A.75), we can implicitly define $\eta(\kappa)$ as the value of η that solves this equation for a given value of κ. $\eta(\kappa)$ is a decreasing function of κ for $\kappa < \bar{\kappa}$. This is because, when $\kappa < \bar{\kappa}$ we know by the earlier lemmas that p^E is strictly decreasing in κ, and p^T is strictly increasing in κ. This means that $\bar{A}p^E - \underline{A}p^T$ is strictly decreasing in κ. This implies that higher values of κ means a lower value for $\left(\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}}\right)^2$. With a lower value of $\left(\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}}\right)^2$, the value of η that solves (A.75) is also lower. Hence $\eta(\kappa)$ is decreasing in κ. Now, we show that there is a lowest admissible value for κ, denoted $\underline{\kappa} > 0$ when $\varepsilon > 1$. As $\kappa \to 0$, then p^E to ∞ when $\varepsilon > 1$. Looking at the equation

$$p^T\alpha\underline{A}\left(\frac{L}{K}\right)^{1-\alpha} + \eta\left(\frac{\bar{A}p^E - \underline{A}p^T}{\chi\tilde{\sigma}\bar{A}}\right)^2 - (\rho+\delta) = 0$$

we see that as $\kappa \to 0$, the second term increases without bound, and hence, it will surpass the value of $\rho + \delta$. Let $\bar{\kappa}$ be the value of for which $\eta \left(\frac{\bar{A} p^E - \underline{A} p^T}{\chi \tilde{\sigma} \bar{A}} \right)^2 - (\rho + \delta) = 0$, then there can be no steady states with $0 < \kappa \leq \bar{\kappa}$, because, then the first term $p^T \alpha \underline{A} \left(\frac{L}{K} \right)^{1-\alpha}$ has to be less than or equal to 0, which cannot happen for $\kappa > 0$. □

A.3.2 Existence of Steady State

We can now prove the existence of steady state for $\varepsilon > 1$. By substituting the expressions for the prices into the three steady state equations in (A.75), we obtain

$$
\begin{aligned}
(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon} \alpha \underline{A} \frac{1}{\eta} \left(\frac{L}{K} \right)^{1-\alpha} + \left(\frac{\kappa}{\eta} \left(\chi \tilde{\sigma} \bar{A} \right) \left(\frac{L}{K} \right)^{1-\alpha} \right)^2 - \frac{\rho + \delta}{\eta} &= 0 \\
(1-\eta) \left(\frac{\bar{A} \nu \left(\frac{\bar{A}\kappa}{A(\kappa)} \right)^{-1/\varepsilon} - \underline{A}(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon}}{\chi \tilde{\sigma} \bar{A}} \right)^2 + \frac{(\bar{\psi} - \eta)\left(\delta_d + \phi^l \right)}{\eta} &= 0 \qquad (A.79) \\
\frac{\bar{A} \nu \left(\frac{\bar{A}\kappa}{A(\kappa)} \right)^{-1/\varepsilon} - \underline{A}(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon}}{\chi \tilde{\sigma} \bar{A}} - \frac{\kappa}{\eta} \left(\chi \tilde{\sigma} \bar{A} \right) \left(\frac{L}{K} \right)^{1-\alpha} &= 0
\end{aligned}
$$

Note that the first equation does not depend K directly, but only on $\frac{1}{\eta} \left(\frac{L}{K} \right)^{1-\alpha}$, which can be solved for from the last equation in terms of κ. Specifically,

$$
\frac{1}{\eta} \left(\frac{L}{K} \right)^{1-\alpha} = \frac{\bar{A} \nu \left(\frac{\bar{A}\kappa}{A(\kappa)} \right)^{-1/\varepsilon} - \underline{A}(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon}}{\kappa (\chi \tilde{\sigma} \bar{A})^2} \qquad (A.80)
$$

Substituting this into the first equation gives

$$
\begin{aligned}
&(1-\nu) \alpha \underline{A} \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon} \frac{\bar{A} \nu \left(\frac{\bar{A}\kappa}{A(\kappa)} \right)^{-1/\varepsilon} - \underline{A}(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon}}{\kappa (\chi \tilde{\sigma} \bar{A})^2} \\
&+ \left(\frac{\bar{A} \nu \left(\frac{\bar{A}\kappa}{A(\kappa)} \right)^{-1/\varepsilon} - \underline{A}(1-\nu) \left(\frac{\underline{A}(1-\kappa)}{A(\kappa)} \right)^{-1/\varepsilon}}{\chi \tilde{\sigma} \bar{A}} \right)^2 - \frac{\rho + \delta}{\eta} = 0
\end{aligned} \qquad (A.81)
$$

This equation, together with

$$(1-\eta)\left(\frac{\bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}-\underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}}{\chi\tilde{\sigma}\bar{A}}\right)^2+\left(\frac{\bar{\psi}}{\eta}-1\right)\left(\delta_d+\phi^l\right)=0 \tag{A.82}$$

defines the steady states of the model. Given any κ, the second has exactly one solution $\eta(\kappa)$ on the interval $(0,1)$. To see this latter fact note that $\eta=0$ is not admissible, so we can multiply through by η without affecting the location of the roots. Then the right-hand side is a quadratic function of η. At $\eta=1$, the quadratic function is $(\bar{\psi}-1)\left(\delta_d+\phi^l\right)<0$ and at $\eta=0$, that quadratic is $\bar{\psi}(\delta_d+\phi^l)>0$. Hence, it crosses the x-axis once on the interval $\eta\in(0,1)$. Denote this value $\eta(\kappa)$. Note also that $\eta(\kappa)$ is strictly decreasing in κ. If κ rises, then the squared term in the brackets will fall, to maintain equality, η must also fall since the expression is strictly decreasing in η. In summary, candidate steady states are determined by the solutions to the following equation in κ

$$\begin{aligned}&(1-\nu)\alpha\underline{A}\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}\frac{\bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}-\underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}}{\kappa(\chi\tilde{\sigma}\bar{A})^2}\\&+\left(\frac{\bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}-\underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}}{\chi\tilde{\sigma}\bar{A}}\right)^2-\frac{\rho+\delta}{\eta(\kappa)}=0\end{aligned} \tag{A.83}$$

where $\eta(\kappa)$ is a strictly decreasing in κ. We can go further in narrowing down the candidate steady states. To prove that a steady state exists, we define the functions

$$\begin{aligned}h(\kappa)&=\frac{\rho+\delta}{\eta(\kappa)}-\left(\frac{\bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}-\underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}}{\chi\tilde{\sigma}\bar{A}}\right)^2\\f(\kappa)&=(1-\nu)\alpha\underline{A}\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}\frac{\bar{A}\nu\left(\frac{\bar{A}\kappa}{A(\kappa)}\right)^{-1/\varepsilon}-\underline{A}(1-\nu)\left(\frac{\underline{A}(1-\kappa)}{A(\kappa)}\right)^{-1/\varepsilon}}{\kappa(\chi\tilde{\sigma}\bar{A})^2}\end{aligned} \tag{A.84}$$

The steady state equation is then $f(\kappa)=h(\kappa)$. Note that $h(\underline{\kappa})=0$ by the definition of $\underline{\kappa}$. Note also that $h(\kappa)\to-\infty$ when $\kappa\to 0$, and $h(\kappa)\to\frac{\rho+\delta}{n(\kappa)}>0$ when $\kappa\to\bar{\kappa}$.

By definition of $\bar{\kappa}$ and $\underline{\kappa}$ we have $f(\bar{\kappa}) = 0$ and $f(\underline{\kappa}) > 0$, while $h(\bar{\kappa}) = \frac{\rho+\delta}{\eta(\underline{\kappa})} > 0$ and $h(\underline{\kappa}) = 0$. By the intermediate value theorem, these lines must cross at least once within the relevant interval, hence a steady state exists. Uniqueness of the steady state can be demonstrated analytically in the case when the goods are perfect substitutes, i.e. $\varepsilon = \infty$. If the goods are perfect substitutes, then note that $\eta(\kappa) = \bar{\eta}$ does not depend on κ since it is pinned down by the equation

$$(1-\eta)\left(\frac{\bar{A}\nu - \underline{A}(1-\nu)}{\chi\tilde{\sigma}\bar{A}}\right)^2 + \left(\frac{\bar{\psi}}{\eta} - 1\right)\left(\delta_d + \phi^l\right) = 0 \tag{A.85}$$

Given this, we have

$$h(\kappa) = \frac{\rho+\delta}{\bar{\eta}} - \left(\frac{\bar{A}\nu - \underline{A}(1-\nu)}{\chi\tilde{\sigma}\bar{A}}\right)^2 \tag{A.86}$$

so that $h(\kappa)$ also does not depend on κ. While at the same time

$$f(\kappa) = (1-\nu)\alpha\underline{A}\frac{\bar{A}\nu - \underline{A}(1-\nu)}{\kappa(\chi\tilde{\sigma}\bar{A})^2} \tag{A.87}$$

is strictly decreasing in κ as long as $\bar{A}\nu - \underline{A}(1-\nu) > 0$ (which ensures that entrepreneurs' idiosyncratic volatility is positive). There is one additional parameter restriction. In particular, the point at which $f(\kappa)$ and $h(\kappa)$ intersect must be such that $\kappa \in (0,1)$. This condition is

$$\kappa = \bar{\eta}\frac{(1-\nu)\alpha\underline{A}\frac{\bar{A}\nu-\underline{A}(1-\nu)}{(\chi\tilde{\sigma}\bar{A})^2}}{\rho+\delta-\bar{\eta}\left(\frac{\bar{A}\nu-\underline{A}(1-\nu)}{\chi\tilde{\sigma}\bar{A}}\right)^2} \in (0,1) \tag{A.88}$$

This ensures that entrepreneurs are not so much more productive than the traditional sector that they want to hold more than 100% of the capital stock, and that their precautionary savings motive is not so strong that the capital stock grows to without bound.

When the goods are not perfect substitutes, it is harder to see if the steady state is unique. However, one can examine this graphically by plotting the functions $f(\kappa)$ and $h(\kappa)$ for given parameter values and examining whether they have a unique intersection or not. Figure A.1 provides an illustrative example. I have been unable

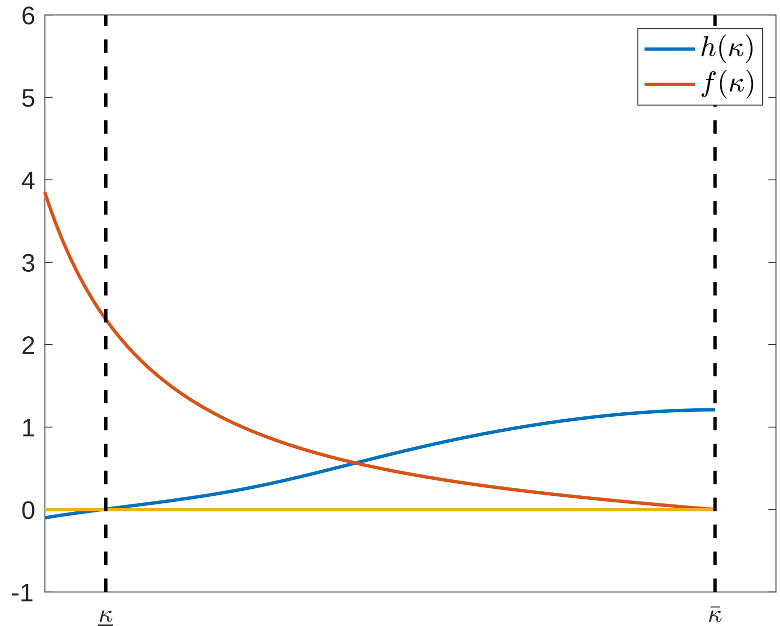


Figure A.1: An illustrative example of graphical determination of unique steady state at the intersection of $f(\kappa)$ and $h(\kappa)$.

to construct a numerical example with multiple steady states.

A.3.3 Steady State Tail Coefficient

In this section, I omit the subscript denoting steady state. All variables should be understood as being in steady state. First note that in steady state, the diffusion and drift of the wealth growth of entrepreneurs is

$$\tilde{\sigma}^E, \quad \text{and} \quad \mu^E = r^E - \rho = (1-\eta)\left(\tilde{\sigma}^E\right)^2 > 0. \tag{A.89}$$

For diversified capitalists, we instead have zero volatility and negative steady state mean growth rate of wealth $\mu^D = r^D - \rho < 0$. The style of proof in this section follows from those used in Gabaix (2009) and Gomez and Gouin-Bonenfant (2024), both of which consider random growth models of wealth accumulation. The relevant Kolmogorov forward equation for the distribution of entrepreneurs' wealth can be written as

$$0 = -\frac{\partial}{\partial n}\left[\mu^E n f_E(n)\right] + \frac{\partial^2}{\partial n^2}\left[\frac{1}{2}(\tilde{\sigma}^E)^2 n^2 f_E(n)\right] - (\delta_d + \phi^I) f_E(n). \tag{A.90}$$

We can guess and verify that the distribution takes the form of a double Pareto distribution

$$f_E(n) = A\left[n^{-\zeta_+ - 1}\mathbf{1}_{n>N} + n^{-\zeta_- - 1}\mathbf{1}_{n<N}\right] \tag{A.91}$$

where N is the steady state average wealth level (which is the wealth level at birth), $A = \frac{-\zeta_-\zeta_+}{(\zeta_+ - \zeta_-)N}$, and ζ_+ and ζ_- are the positive and negative roots, respectively, of the quadratic equation

$$0 = \zeta\mu^E + \frac{(\tilde{\sigma}^E)^2}{2}\zeta(\zeta - 1) - (\delta_d + \phi^I) \tag{A.92}$$

see Gabaix (2009) and references therein). For the diversified agents we guess that the distribution takes the following form:

$$f_D(n) = B\left[\omega_1 f_E(n) + \left(\omega_2 n^{-\zeta_- - 1} + (1 - \omega_1 - \omega_2) n^{-\tilde{\zeta}_- - 1}\right)\mathbf{1}_{n<N}\right]. \tag{A.93}$$

This guess is motivated by the fact that diversified capitalists consists of former entrepreneurs that have switched, but also of agents that were born diversified. Because $\mu^D < 0$, those born diversified will never have wealth above N, so all diversified agents with $n > N$ must be former entrepreneurs. The guess is verified if we can select the weights ω_1 and ω_2, and the tail-coefficient $\tilde{\zeta}$ to solve the Kolmogorov forward equation

$$0 = -\frac{\partial}{\partial n}\left[\mu^D n f_D(n)\right] - \delta_d f_D(n) + \frac{\phi^I \psi^E}{\psi^D} f_E(n). \tag{A.94}$$

Consider the case $n > N$. Then, the KFE is simply

$$\omega_1\mu^D\zeta_+ - \omega_1\delta_d + \frac{\phi^I\psi^E}{\psi^D} = 0 \tag{A.95}$$

This gives us $\omega_1 = \frac{\phi^I\psi^E}{\psi^D(\delta_d - \mu^D\zeta_+)}$, which pinns down the distribution of diversified agents' wealth above N. Next consider $n < N$. Let $g(n) = n^{-\tilde{\zeta}_- - 1}$. Then the KFE can in this case be written

$$0 = \left(-\frac{\partial}{\partial n}\left[\mu^D n\left(\omega_1 A + \omega_2\right) n^{-\zeta_- - 1}\right] - \delta_d\left(\omega_1 A + \omega_2\right) n^{-\zeta_- - 1} + \frac{\phi^I \psi^E}{\psi^D} A n^{-\zeta_- - 1}\right) + \left(-\frac{\partial}{\partial n}\left[\mu^D n g(n)\right] - \delta_d g(n)\right)(1 - \omega_1 - \omega_2). \tag{A.96}$$

Note that the expression in the second term $\left(-\frac{\partial}{\partial n}\left[\mu^D n g(n)\right] - \delta_d g(n)\right)$, is analogous to right-hand side of the KFE for a GBM with zero volatility, growth rate μ^D, constant death rate δ_D and rebirth at N. Specifically, this expression is 0 if we pick $\tilde{\zeta}_- = \frac{\delta_d}{\mu^D}$ (see Steindl (1965) and Gabaix et al. (2016)). Setting the first expression to 0 then gives an equation for ω_2, namely

$$0 = \mu^D\left(\omega_1 A + \omega_2\right)\zeta_- - \delta_d\left(\omega_1 A + \omega_2\right) + \frac{\phi^I \psi^E}{\psi^D} A \tag{A.97}$$

which implies

$$\omega_2 = \left[\frac{\omega_1\left(\delta_d - \mu^D\right) - \frac{\phi^I \psi^E}{\psi^D}}{\mu^D \zeta_+ - \delta_d}\right] A. \tag{A.98}$$

Finally, we pick B to ensure that the density integrates to 1. Integrating $f_D(n)$ and setting this to 1 gives

$$\begin{aligned} & B\left(\omega_1 + \omega_2 \frac{N^{-\zeta_-}}{\zeta_-} + (1 - \omega_1 - \omega_2)\frac{N^{-\tilde{\zeta}_-}}{\tilde{\zeta}_-}\right) \\ \Rightarrow\ & B = \frac{1}{\omega_1 + \omega_2 \frac{N^{-\zeta_-}}{\zeta_-} + (1 - \omega_1 - \omega_2)\frac{N^{-\tilde{\zeta}_-}}{\tilde{\zeta}_-}}. \end{aligned} \tag{A.99}$$

This verifies the guess. Note that the limiting right tail of the distribution of wealth for diversified agents is inherited from that of entrepreneurs, where as the limiting left tail depends on whether ζ_- is larger than or smaller than $\tilde{\zeta}_-$.

Characterizing the right tail

To further characterize the right tail we have the following quadratic equation from (A.92)

$$\zeta^2 + \zeta\left(\frac{2\mu^E}{\left(\tilde{\sigma}^E\right)^2} - 1\right) - \frac{2(\delta_d + \phi^l)}{\left(\tilde{\sigma}^E\right)^2} = 0 \tag{A.100}$$

using that $\frac{2\mu^E}{\left(\tilde{\sigma}^E\right)^2} = 2(1-\eta)$ we can write this as

$$\zeta^2 + \zeta\,(1 - 2\eta) - \frac{2(\delta_d + \phi^l)}{\left(\tilde{\sigma}^E\right)^2} = 0 \tag{A.101}$$

the positive solution to this is (which I from here on denote simply by ζ instead of ζ_+ to save on notation)

$$\zeta = \eta - \frac{1}{2} + \sqrt{\left(\eta - \frac{1}{2}\right)^2 + \frac{2(\delta_d + \phi^l)}{\left(\tilde{\sigma}^E\right)^2}} \tag{A.102}$$

We can go further in characterizing this since the steady state condition for η, $d\eta = 0$ implies

$$\eta\,(1-\eta)\,(\tilde{\sigma}^E)^2 = -\left(\delta_d\psi^0 - \delta_d\eta - \phi^l\eta\right) \tag{A.103}$$

using that in steady state $\delta_d\psi^0 = \bar{\psi}(\delta_d + \phi^l)$ this implies

$$\eta\,(1-\eta)\,(\tilde{\sigma}^E)^2 = (\eta - \bar{\psi})(\delta_d + \phi^l) \tag{A.104}$$

so that

$$\frac{2\eta(1-\eta)}{\eta - \bar{\psi}} = \frac{2(\delta_d + \phi^l)}{\left(\tilde{\sigma}^E\right)^2} \tag{A.105}$$

Note first that this tells us that $\bar{\psi} < \eta$ in steady state, otherwise the left-hand side is negative, which cannot happen since the right-hand side is strictly positive. Moreover, plugging this into the expression for ζ then gives

$$\zeta = \eta - \frac{1}{2} + \sqrt{\left(\eta - \frac{1}{2}\right)^2 + \frac{2\eta(1-\eta)}{\eta - \bar{\psi}}} \tag{A.106}$$

the left-tail coefficient can be solved similarly.

We now proceed to proving that ζ is strictly decreasing in η. In other words, inequality is increasing in η. First, note that the expression $\frac{2\eta(1-\eta)}{\eta-\bar{\psi}}$ is strictly decreasing in η. To see this, not that its derivative is

$$\frac{(2(1-\eta)-2\eta)\left(\eta-\bar{\psi}\right)-2\eta(1-\eta)}{(\eta-\bar{\psi})^2}=\frac{2\eta-4\eta^2-2\bar{\psi}+4\bar{\psi}\eta-2\eta+2\eta^2}{(\eta-\bar{\psi})^2}=$$

$$\frac{-2\eta^2+4\bar{\psi}\eta-2\psi^2+2\psi^2-2\bar{\psi}}{(\eta-\bar{\psi})^2}=\frac{-2(\eta-\bar{\psi})^2-2\bar{\psi}(1-\bar{\psi})}{(\eta-\bar{\psi})^2}=-2\left(1+\frac{\bar{\psi}(1-\bar{\psi}))}{(\eta-\bar{\psi})^2}\right)<0. \tag{A.107}$$

Moreover, clearly $\eta-\frac{1}{2}$ is increasing in η. The question is therefore if the slope of the second term under the bracket, $\frac{2\eta(1-\eta)}{\eta-\bar{\psi}}$, negative enough to counteract the positive slope coming from the terms $\eta-\frac{1}{2}$ and $(\eta-\frac{1}{2})^2$. To prove this, note that equation (A.107) implies that slope of $\frac{2\eta(1-\eta)}{\eta-\bar{\psi}}$ is *least* negative (smallest in magnitude) when $\bar{\psi}=0$. Moreover, for any other admissible value of $\bar{\psi}$ (that is $\bar{\psi}<\eta<1$) the slope of this term is more negative. Hence, if we can show that ζ is non-increasing in η when $\bar{\psi}=0$, then it must be the case that ζ is decreasing in η when $\bar{\psi}>0$. If we plug in $\bar{\psi}=0$ in the expression for η, we obtain

$$\eta-\frac{1}{2}+\sqrt{\left(\eta-\frac{1}{2}\right)^2+2(1-\eta)}=\eta-\frac{1}{2}+\sqrt{\eta^2-3\eta+\frac{9}{4}}=$$

$$=\eta-\frac{1}{2}+\sqrt{\left(\eta-\frac{3}{2}\right)^2}=\eta-\frac{1}{2}+|\eta-\frac{3}{2}|$$

Using that $\eta\in(0,1)$, we can write this as

$$=\eta-\frac{1}{2}+\sqrt{\left(\eta-\frac{3}{2}\right)^2}=\eta-\frac{1}{2}-\eta+\frac{3}{2}=1$$

The slope of this is 0. Hence, when $\bar{\psi}=0$, the slope of ζ with respect to η is 0, and we know that the slope is strictly smaller for all other admissible $\bar{\psi}$, hence, ζ is strictly decreasing in η.

Finally, we proceed by showing that ζ is strictly decreasing in $\tilde{\sigma}^E$. This follows from the fact that equation (A.105) implies that η is strictly increasing in $\tilde{\sigma}^E$, because the left-hand side is decreasing in η and the right-hand side is decreasing in $\tilde{\sigma}^E$. Since

ζ is decreasing in η, it must be then that ζ is decreasing in $\tilde{\sigma}^E$ as well.

A.3.4 Changes in Inequality After a Fall in χ

Proposition 2 discusses small changes in χ. In this section, I study what happens to inequality for larger changes in χ and we discuss transition dynamics. More specifically, we will consider the transition dynamics of the model in the following type of experiment. We let the initial values K_0, η_0, and κ_0 be associated with an initial steady state s_0 with skin-in-the-game parameter χ_0. We then examine the transition dynamics of the model in response to a change in χ to $\chi_1 < \chi_0$. For this exercise to make sense, I will assume that there is a unique (non-degenerate) steady state associated with the new value χ_1. I will also have to assume that the transition dynamics ensure that we converge to this new steady state.

Proposition 5. *Consider an initial (non-degenerate) steady state $s_0 = (K_0, \eta_0, \kappa_0)$ with $\kappa_0 \in (0,1)$, associated with skin-in-the-game parameter χ_0, and a different steady state $s_1 = (K_1, \eta_1, \kappa_1)$ with $\kappa_1 \in (0,1)$, associated with $\chi_1 < \chi_0$. Assume that the economy converges to s_1 from s_0 when changing χ_0 to χ_1. All other parameters are fixed, in particular, the parameter ε is the same for both steady states. Then, there exists a $\varepsilon^*_{s_0,s_1}$ such that if the ε associated with these two steady states is larger than $\varepsilon^*_{s_0,s_1}$, then $\eta_1 > \eta_0$ and Pareto inequality is higher in s_1.*

Proof. The initial value of the entrepreneurs' risk exposure is

$$\tilde{\sigma}^E_0 = \frac{\bar{A}p^E(\kappa_0) - \underline{A}p^T(\kappa_0)}{\chi_0 \tilde{\sigma} \bar{A}}.$$

Because the prices can be made arbitrarily insensitive to changes in $\kappa \in (0,1)$ by letting ε be large enough, we know that there exists some $\varepsilon^*_{s_0,s_1}$ such that if $\varepsilon > \varepsilon^*_{s_0,s_1}$ we have

$$\tilde{\sigma}^E_0 = \frac{\bar{A}p^E(\kappa_0) - \underline{A}p^T(\kappa_0)}{\chi_0 \tilde{\sigma} \bar{A}} < \frac{\bar{A}p^E(\kappa_1) - \underline{A}p^T(\kappa_1)}{\chi_1 \tilde{\sigma} \bar{A}} = \tilde{\sigma}^E_1.$$

Because $\tilde{\sigma}^E_1 > \tilde{\sigma}^E_0$, it must be that $\eta_1 > \eta_0$, which means Pareto inequality is higher in the new steady state. □

Now we examine the transition dynamics more closely. Recall that the transition dynamics of the model are determined by the evolution of the state variables

$$\begin{aligned}\frac{dK_t}{K_t} &= \left(p^T(\kappa_t)\alpha \underline{A}\left(\frac{L}{K_t}\right)^{1-\alpha} + \eta_t\left(\tilde{\sigma}_t^E\right)^2 - \rho - \delta\right)dt \\ \frac{d\eta_t}{\eta_t} &= \left((1-\eta_t)\left(\tilde{\sigma}_t^E\right)^2 + \frac{(\bar{\psi}-\eta_t)\left(\delta_d + \phi^l\right)}{\eta_t}\right)dt\end{aligned} \tag{A.108}$$

and the equilibrium condition for the allocation of the capital stock:

$$\frac{\bar{A}p(\kappa_t)^E - \underline{A}p^T(\kappa_t)}{\chi\tilde{\sigma}\bar{A}} = \frac{\kappa_t}{\eta_t}\left(\chi\tilde{\sigma}\bar{A}\right)\left(\frac{L}{K_t}\right)^{1-\alpha} \equiv \tilde{\sigma}_t^2 \tag{A.109}$$

To understand what happens, when χ falls, let's consider a transition between steady state when χ falls.

Lemma 10. *In this experiment, κ_t increases on impact.*

Proof. On impact, K_t and η_t are fixed, so (A.109) implies that κ_t must increase to maintain equilibrium if χ falls. □

The above lemma tells us what happens on impact. To examine what happens in the transition, we need to study how η_t and K_t evolve. Clearly, for high enough values ε, the idiosyncratic risk exposure of entrepreneurs $\tilde{\sigma}_t^E$ rises on impact. According to the equations describing the evolution of the state variables above, both K_t and η_t will start growing. Looking at the equilibrium condition for the capital allocation, as η_t and K_t start growing, κ_t rises further. The intuition is that as entrepreneurs become wealthier and the operational leverage of the economy (Y/K) becomes smaller, entrepreneurs are better equipped to bear risk and this scale up even more. However, looking at the left-hand side of the equilibrium capital allocation equation (A.109), we see that this scaling up *after* impact is going to be a reduction in the Sharpe ratio for entrepreneurs (compared to the initial upward jump). In other words, even if the Sharpe ratio jumps upwards on impact, this upward jump is moderated somewhat when η_t and K_t start to grow. However, to be consistent with a new steady state with higher risk exposure for entrepreneurs, $\tilde{\sigma}_t^E$ cannot come back down all the way to its initial value. Looking at the equation describing the evolution of η_t, we see that the growth rate of η_t slows down after impact because $\tilde{\sigma}_t^E$ declines and because a higher

η_t makes $d\eta_t$ smaller.

For the capital stock the dynamics are less clear. The increase in κ_t raises the risk-free rate on impact because $p^T(\kappa_t)$ rises. Moreover, the fact that both η_t and $\tilde{\sigma}_t^E$ rise, means the drift of the capital stock becomes even higher. However, as the capital stock rises, the marginal product of capital falls according to the standard neoclassical mechanism. Assuming that the economy actually converges to a new steady state it must be the case the effect from the decreasing marginal product of capital is more powerful than the increase in the price p_t^T in the long run. In particular, we know that there is a maximal κ consistent with steady state, $\bar{\kappa} < 1$, so that the price $p^T(\kappa)$ is bounded above by $p^T(\bar{\kappa})$.

Proving that the economy converges to a unique steady state in a multi-sector growth model is difficult. In fact, Boldrin and Deneckere (1990) shows that multi-sector growth models can display chaotic and cyclical behavior even without aggregate risk. Other than in the limit when $\varepsilon \to \infty$, so that p_t^T is constant, I study the convergence to steady state numerically.

A.3.5 Computing Transition Dynamics Numerically

Here is a brief outline of the numerical procedure for computing the transition dynamics of the model.

To explain the steps, I consider an experiment wherein only one parameter changes, at only one point in time, and that the transition experiment starts in an initial steady state. Generalizing this is straightforward.

1. Choose a duration of the transition dynamics in years, T, and a time period stepsize Δ_t. The number of time periods per year is $\frac{1}{\Delta_t}$.

2. Select values for the rest of the model's parameters.

3. Compute steady state values of the capital stock and entrepreneurs wealth share K_0, η_0 for these parameters. This is the initial steady state.

4. Compute the value of all equilibrium objects of interest in this initial steady state.

5. Change the value of a parameter of interest. Now, for each time period in the transition starting at $t = 0$:

   5.1 Given the current values for the capital stock and entrepreneurs wealth share K_t, η_t, compute the equilibrium fraction of capital operated by entrepreneurs in this new equilibrium, κ_t using equation A.41.

   5.2 Compute the new equilibrium values of all variables of interest for this time period.

   5.3 Use the following discretized version of the state transition dynamics equations in A.108 to compute next period values of the state variables:

$$\begin{aligned} K_{t+1} &= \left(p^T(\kappa_t)\alpha \underline{A} \left(\frac{L}{K_t} \right)^{1-\alpha} + \eta_t \left(\tilde{\sigma}_t^E \right)^2 - \rho - \delta \right) K_t \Delta_t + K_t \\ \eta_{t+1} &= \left((1-\eta_t) \left(\tilde{\sigma}_t^E \right)^2 + \frac{(\bar{\psi} - \eta_t)\left(\delta_d + \phi^l\right)}{\eta_t} \right) \eta_t \Delta_t + \eta_t \end{aligned} \tag{A.110}$$

   5.4 Go back to step 5.1 and repeat until reaching the last time period in the experiment.

Now you will have period-by-period values for the state variables K_t and η_t, from which all other equilibrium objects can be computed. To solve for the evolution of the distribution of wealth, I use the sequence of drifts and volatilities r_t^E, r_t^D, σ_t^E, to numerically solve the time-dependent Kolmogorov forward equation associated with the evolution of wealth for the respective type of agent using the numerical procedure in Brunnermeier et al. (2021).

A.3.6 Decreasing Inequality Even with Perfect Substitutes

The relationship between Pareto inequality and χ turns around even when $\varepsilon = \infty$, when χ becomes small enough. This does not contradict Proposition 2 or Proposition 5. These propositions tell us that starting in an initial steady state with interior κ, there exists large enough values of ε, so that inequality increases when χ falls. However, as χ is reduced further, and further the required ε becomes larger and larger. This is because as κ gets closer to 1, there is less room for entrepreneurs to scale up at the expense of the traditional firms. And even in the limit as $\varepsilon \to \infty$, there is a limit to the scaling-up effect coming from the fact that with perfects substitutes, small enough χ implies that in steady state $\kappa = \bar{\kappa} = 1$. Further falls in χ beyond this leads to less inequality. To prove this is straightforward for $\varepsilon = \infty$ because tedious the CES-algebra can be avoided. For $\varepsilon < \infty$, the traditional sector is never fully out competed because prices in that sector increase rapidly when κ gets close to 1. This means intuitively that the limits to the scaling-up effect occur even earlier than with $\varepsilon = \infty$. However, this is more challenging to prove analytically. I therefore produce the proof for $\varepsilon = \infty$ and confirm the intuition numerically.

Proposition 6. *Even with perfect substitutes, $\varepsilon = \infty$, there is a value χ^* such that if $\chi < \chi^*$, a further fall in χ reduces Pareto inequality.*

Proof. Note that with perfect substitutes, the condition in equation (A.88) must be satisfied for both sectors to be active. However, in that expression, we see that as $\chi \to 0$, this condition does not hold. This is because, with perfect substitutes and low χ, there is no steady state with $\kappa \in (0, 1)$. Instead, the entrepreneurs take over the entire economy. Hence, we let χ^* be the supremum of the values of $\chi \in (0, 1)$ for which (A.88) does *not* hold. We instead seek an equilibrium where only the entrepreneurs are active. In this economy entrepreneurs' optimal portfolio choice implies

$$\frac{1}{\eta}(\chi\tilde{\sigma}^k)^2 = r^k - r. \tag{A.111}$$

which follows from plugging in $\kappa = 1$ in entrepreneurs optimal portfolio choice. We recall that $\tilde{\sigma}^k_{ss} = \frac{Y}{K}\tilde{\sigma}$. Moreover, when the traditional firms are not active, the risk-free rate is no longer pinned down by the value of the marginal product in that sector. Instead, we have the following system jointly pinning down the wage rate and the risk-free rate:

$$\begin{aligned} \frac{Y}{K} &= \bar{A}\left(\frac{1-\alpha}{\alpha}\frac{r+\delta}{w}\right)^{1-\alpha} \\ \underbrace{\frac{Y}{K} - w\frac{L}{K} - \delta - r}_{r^k - r} &= \frac{1}{\eta}\left(\chi\frac{Y}{K}\tilde{\sigma}\right)^2 \end{aligned} \tag{A.112}$$

Solving this gives us

$$\begin{aligned} w &= (1-\alpha)\bar{A}\left(\frac{K}{L}\right)^{\alpha}\left[1-\frac{\chi\tilde{\sigma}}{\eta}\left(\chi\tilde{\sigma}\frac{Y}{K}\right)\right] \\ r &= \alpha\bar{A}\left(\frac{L}{K}\right)^{1-\alpha}\left[1-\frac{\chi\tilde{\sigma}}{\eta}\left(\chi\tilde{\sigma}\frac{Y}{K}\right)\right]-\delta \end{aligned} \tag{A.113}$$

Plugging this into the equations for the evolution of the state variables and setting these to zero we obtain

$$\begin{aligned} \frac{dK_t}{K_t} &= \left(\alpha\frac{Y}{K}+(1-\alpha)\frac{1}{\eta}\left(\chi\frac{Y}{K}\tilde{\sigma}\right)^2-\rho\right)dt = 0 \\ \frac{d\eta_t}{\eta_t} &= \left((1-\eta)\left(\frac{1}{\eta}\chi\frac{Y}{K}\tilde{\sigma}\right)^2+\frac{(\bar{\psi}-\eta)\left(\delta_d+\phi^l\right)}{\eta}\right)dt = 0 \end{aligned} \tag{A.114}$$

From the steady state for η we obtain

$$\frac{1}{\eta}\left(\chi\frac{Y}{K}\tilde{\sigma}\right)^2 = -\frac{(\bar{\psi}-\eta)(\delta_d+\phi^l)}{1-\eta}$$

We can then rewrite the steady state equations for K as

$$\alpha\frac{Y}{K}-\rho = (1-\alpha)\frac{(\bar{\psi}-\eta)(\delta_d+\phi^l)}{1-\eta} \tag{A.115}$$

The left-hand side is strictly decreasing in η (take derivative). This means that if a fall in χ leads to a fall in η, it must lead to a rise in $\frac{Y}{K}$ for this equation to hold, and vice versa. In other words, it must lead to a fall in K. If a fall in χ leads to a rise in η, then it must similarly lead to a rise in K, and vice versa. In other words, when χ falls, η and K must move in the same direction in steady state. To see that the direction is downward, we look at the steady state equation for η. When χ falls, this equation tells us that either η or K must fall. Since we know that they both move in the same direction, this means they must fall.

What happens to the idiosyncratic risk exposure of entrepreneurs? We have

$$\tilde{\sigma}^E = \frac{1}{\eta}\chi\frac{Y}{K}\tilde{\sigma}$$

which has to fall. To see this, note that when χ falls, $1/\eta$ and Y/K rise, but if they rise so much that this offsets the fall in χ in the sense that $\tilde{\sigma}^E$ does not fall, then this contradicts η falling, because if $\tilde{\sigma}^E$ falls, then the steady state equation for η tells us that η should not fall. So $\tilde{\sigma}^E$ must fall. Finally, Pareto inequality falls because $\tilde{\sigma}^E$ falls. This concludes the proof. □

A.4 Measuring Entrepreneurial Wealth

In this section I discuss how the model presented in this paper can help shed light on the proper measurement of the wealth of an entrepreneur. Clarifying how entrepreneurs' wealth is measured in the context of the model, and how it relates to common ways of measuring entrepreneurs' wealth in practice is also crucial for understanding the quantitative exercise in the next section.

Note that the formulation of how entrepreneurial firms are financed in the model makes no references to the number of shares that the entrepreneurs issue, or the prices of these shares. Instead, the financing of the entrepreneurial firms is expressed in terms of the amount of capital raised from outsiders and the expected return that these outsiders receive. There is of course a link between the two formulations of the financing of the firms. Making this link explicit clarifies the difference of how wealth is commonly measured in practice, and how it is measured in the model.

An entrepreneur who has decided on operating a firm with total capital stock k_{it} must provide χk_{it} of the financing herself, and raise $(1-\chi)k_{it}$ from outsiders. Letting N_0 be the initial number of shares, all owned by the entrepreneur, the number of shares that the entrepreneur has to issue, Δ_{Nt}, is defined by

$$\Delta_{Nt} p_{it} = (1-\chi)k_{it}$$

where p_{it} is the price per share issued. The price per share issued on the other hand is pinned down by the condition that the equilibrium expected return on equity to outsiders is $r_t^{\text{fund}} dt$. In other words,

$$\frac{\left(\frac{\Delta_{Nt}}{N_0+\Delta_{Nt}}\right) k_{it}(1+r_t^k dt)}{p_{it}\Delta_{Nt}} = 1 + r_t^{\text{fund}} dt$$

these equations jointly pin down the price and the number of shares issued in terms of the expected returns and the outside financing fraction $1-\chi$:

$$\Delta_{Nt} = \frac{(1+r_t^{\text{fund}}dt)(1-\chi)}{(r_t^k - r_t^{\text{fund}})dt + \chi(1+r_t^{\text{fund}}dt)} N_0$$

$$p_{it} = \left(\frac{(r_t^k - r_t^{\text{fund}})dt + \chi(1+r_t^{\text{fund}}dt)}{1+r_t^{\text{fund}}dt}\right)\frac{k_{it}}{N_0}$$

Note that measuring outsiders' stake in the firm as $p_{it}\Delta_{Nt}$, the price-per-share times the number of shares they hold, coincides with the model notion of the value of their stake in the firm: $(1-\chi)k_{it}$. That is however not true for the entrepreneur. In particular, the post-money valuation of the entrepreneurs' shares is

$$p_{it}N_0 = \left(\frac{(r_t^k - r_t^{\text{fund}})dt + \chi(1+r_t^{\text{fund}}dt)}{1+r_t^{\text{fund}}dt}\right) k_{it} > \chi k_{it} \tag{A.116}$$

where the inequality follows from the fact that $(r_t^k - r_t^{\text{fund}}) > 0$. This also illustrates that χ should not be confused with the entrepreneurs' ownership share measured as the fraction of the outstanding shares that the entrepreneur holds. Rather, χ is the insider financing share, the share of the financing that the entrepreneur provides.

The discrepancy stems from the fact that p_{it} is the price that an investor with no exposure to the idiosyncratic risk in firm i is willing to pay for a share. This is more than what the entrepreneur associated with that firm is willing to pay for a share. This discrepancy in valuation of a share means that the entrepreneur would like to issue additional shares, but cannot since the constraint is binding. The difference in the pre- and post-money valuations of the entrepreneur's shares reflects the fact that some of the entrepreneur's return from investing in the firm comes directly from selling shares. To see this, note that the expected return to the entrepreneur's stake in the firm coming purely from issuing shares is

$$\frac{p_{it}N_0}{\chi k_{it}} - 1 = \frac{r_t^k - r_t^{\text{fund}}}{\chi}dt > 0 \tag{A.117}$$

The overall expected return to the entrepreneur's stake, the insider equity return, is

$$r_t^{in}dt = \frac{\frac{N_0}{N_0+\Delta_{Nt}}(1 + r_t^k dt)k_{it}}{\chi k_{it}} - 1 = \left(r_t^{\text{fund}} + \frac{r_t^k - r_t^{\text{fund}}}{\chi}\right)dt \tag{A.118}$$

In other words, the insider return is the outsider return plus the return that the insider gets from issuing equity.

The fact that χ cannot be mapped to the insider ownership share of the entrepreneur, measured as the fraction of shares outstanding that the entrepreneur holds means that one must look for other sources of data that are informative about the value of χ. To this end, I map the value of χ to the rate at which entrepreneurs issue new shares. Specifically, the growth of the number of shares outstanding when the entrepreneur issues shares to outsiders is

$$\frac{\Delta_{Nt}}{N_0} = \frac{(1 + r_t^{\text{fund}}dt)(1 - \chi)}{(r_t^k - r_t^{\text{fund}})dt + \chi(1 + r_t^{\text{fund}}dt)}. \tag{A.119}$$

Note that this is the growth in the number of shares when the entrepreneur first issues equity to outsiders. It is not the steady growth rate of the number of shares outstanding over time. The growth rate of the total number of shares outstanding only grows after this initial equity issuance if the rates of return or χ change over time. In a steady state, the returns as well as χ are constant, and the annualized average growth rate of the number of shares outstanding over the time that a firm remains entrepreneurial is

$$\left(1 + \frac{\Delta_{Nt}}{N_0}\right)^{1/T_l} - 1 = \left(1 + \frac{(1 + r_t^{\text{fund}}dt)(1 - \chi)}{(r_t^k - r_t^{\text{fund}})dt + \chi(1 + r_t^{\text{fund}}dt)}\right)^{1/T_l} - 1 \tag{A.120}$$

where T_l is the average number of years that the firm remains entrepreneurial. The quantity $-\left(1 + \frac{\Delta_{Nt}}{N_0}\right)^{1/T_l}$ is the average lifetime buyback yield of an entrepreneurial firm. Gomez and Gouin-Bonenfant (2024) document that this has changed substantially over time for the entrepreneurial firms associated with the members of the

Forbes 400. In the quantitative exercise, I map the fall in the parameter χ to the change in this average lifetime buyback yield.

Appendix B

Appendix for Chapter 2

Table B.1: Calibrated Parameters

Parameters	g_F	g_D	σ_F	σ_D	ν_F	κ_F
A (Baseline)	0.0035	−0.0847	0.4409	0.0779	0.05	0.067
B (σ_F = 33%)	0.0216	−0.0721	0.3306	0.0791	0.05	0.067
C (σ_F = 22%)	0.0344	−0.0639	0.2204	0.0802	0.05	0.067
Targets	ζ_{ss}	$\bar{g} - \underline{g}$	$\bar{\sigma}_{bottom}$	$\bar{\sigma}_{n_{top0.01\%}}$	σ_F	
A (Baseline)	1.43	0.0569	0.0813	0.3579	N/A	
B (σ_F = 33%)	1.43	0.0569	0.0813	N/A	0.3306	
C (σ_F = 22%)	1.43	0.0569	0.0813	N/A	0.2204	

Rows labeled A are associated with the baseline calibration, rows B and C are associated with alternative calibrations where the volatility of the family firm type is set to 75% and 50% of its baseline value, respectively.

B.1 Appendix A: Definition of $\mathbb{T}$

The evolution of the distribution of wealth by type from t to $t+1$ can be described by an operator $\mathbb{T}$ that maps pairs of densities of wealth by type at t to pairs of densities of wealth by type at $t+1$. The definition of this operator $\mathbb{T}$ is given by two linked second-order differences equations characterizing the updating of the distribution of wealth for each type. For $n \geq 1$ the difference equations are for each type j

$$g_{j,t+1}(n) = \phi_j \left[p_{u,j} g_{j,t}(n-1) + p_{d,j} g_{j,t}(n+1) + (1 - p_{u,j} - p_{d,j}) g_{j,t}(n)\right] + \tag{B.1}$$

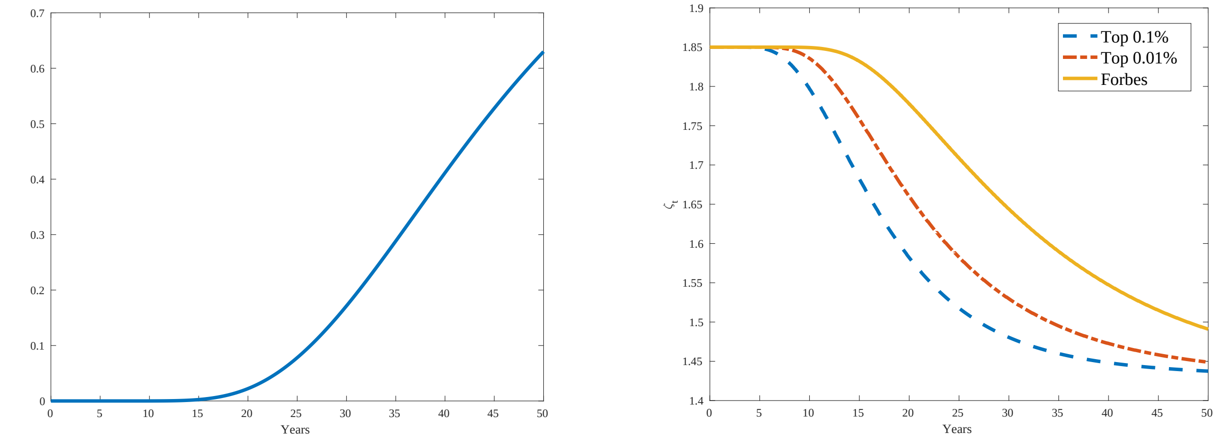
$$(1-\phi_j)\left[p_{u,-j} g_{-j,t}(n-1) + p_{d,-j} g_{-j,t}(n+1) + (1 - p_{u,-j} - p_{d,-j}) g_{-j,t}(n)\right],$$

where $-j$ denotes the type opposite to j. For $n = 0$, this evolution is given by

$$g_{j,t+1}(0) = \phi_j \left[p_{d,j} g_{j,t}(1) + (1 - p_{u,j}) g_{j,t}(0)\right] + \tag{B.2}$$

$$(1-\phi_j)\left[p_{d,-j} g_{-j,t}(1) + (1 - p_{u,-j}) g_{-j,t}(0)\right].$$

When the size of the grid is finite, we have the following additional equation describing the evolution at $n = N$

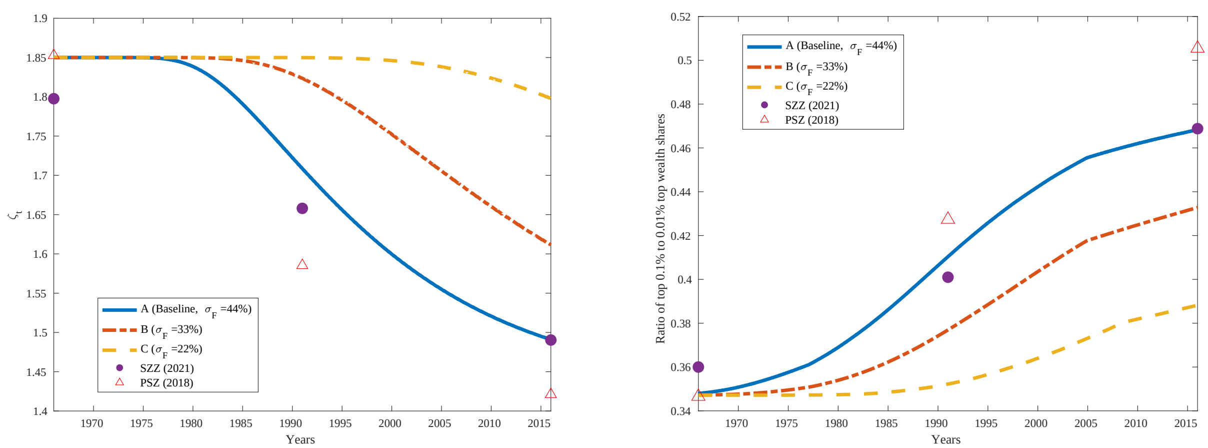


(a) Mobility of Forbes 400 from the bottom

(b) Transition of top tail coefficient

Figure B.1: Wealth Mobility and Dynamics of Top Wealth

The left panel (B.1a) shows the speed of wealth mobility from the bottom to the top of the wealth distribution, measured as the fraction of the Forbes 400 dynasties in steady-state that were last at the bottom of the wealth distribution within the prior k years. 63% of these wealthiest dynasties were at the bottom within the prior 50 years. The right panel (B.1b) shows the speed of transition of the tail coefficient of wealth at various top wealth levels starting from an initial distribution with tail coefficient equal to 1.85 everywhere converging to a steady-state tail coefficient of $\zeta_{ss} = 1.43$. The transition is slower when the tail coefficient is measured at higher wealth levels as implied by the proposition presented in 2.3. This also implies that the tail coefficient differs depending on where in the wealth distribution it is measured during the transition. As the distribution converges to steady state, all the tail coefficients converge to the same number provided that they are measured at sufficiently high wealth levels.



(a) Top Tail coefficient.

(b) Ratios of Top Wealth Shares

Figure B.2: Dynamics Implied by Baseline and Alternative Calibrations

This figure displays comparisons along two dimensions of the baseline calibration A with the alternative calibrations, B and C. In calibrations B and C the value of σ_F is set to 75% and 50% of its baseline value, respectively. Panel (B.2a) compares the transition of the tail coefficient measured at the cutoff for the Forbes 400, and Panel (B.2b) considers the transition of the ratio of the top 0.01% to the top 0.1% wealth shares. Along both dimensions, the presence of a minority of dynasties with very high idiosyncratic volatility is important for obtaining rapid transitions. The transition is computed for the years 1966-2016. Marked are also the data from Smith et al. (2021) (circles) and Piketty et al. (2018) (triangles). These are the ratios of the top 0.01% to the top 0.1% wealth shares and the implied tail coefficient using equation 2.4.

$$g_{j,t+1}(N) = \phi_j \left[p_{u,j} g_{j,t}(N-1) + (1-p_{d,j}) g_{j,t}(N)\right] + \tag{B.3}$$

$$(1-\phi_j) \left[p_{u,-j} g_{-j,t}(N-1) + (1-p_{d,-j}) g_{-j,t}(N)\right].$$

B.2 Appendix B: Setting parameters as $\Delta_t \to 0$

To compare results in our discrete time model with closely related results in continuous time versions of the model as presented in Luttmer (2016), Gabaix et al. (2016) and elsewhere, we use the following procedure to adjust the parameters of our model as we change the length of the time period Δ_t. This is done to consider the limiting implications of our model as the time period gets short. We set $p_{d,j}$ and $\frac{p_{u,j}}{p_{d,j}}$ to match annualized means μ_j and variances σ_j^2 of innovations to the logarithm of the idiosyncratic component of assets. Specifically, we set the grid step size Δ as a function of the length of a time period Δ_t as

$$\Delta = \sigma_{max}\sqrt{2\Delta_t},$$

where σ_{max} is the largest annualized standard deviation of innovations to the logarithm of assets that we consider.

Under the model assumptions regarding the evolution of wealth for each type, the expected value at t of the innovations to the logarithm of wealth for all dynasties of type j, except those at the lowest node on the grid, is given by

$$\mathbb{E}_t \left[\log W_{i,t+1} - \log W_{i,t}\right] = (p_{u,j} - p_{d,j})\Delta. \tag{B.4}$$

The uncentered second moment of these innovations to the logarithm of the idiosyncratic component of assets is given by

$$\mathbb{E}_t \left[\log W_{i,t+1} - \log W_{i,t}\right]^2 = (p_{u,j} + p_{d,j})\Delta^2. \tag{B.5}$$

We then choose the parameters $p_{d,j}$ and $\frac{p_{u,j}}{p_{d,j}}$ so that the expression in equation (B.4) is equal to the target per period mean $\Delta_t \mu_j$, and the expression in equation (B.5) is equal to the target per period uncentered second moment $\Delta_t \sigma_j^2 + \Delta_t^2 \mu_j^2$. We set the transition probabilities over types as $1 - \phi_j = \kappa_j \Delta_t$ for fixed target values of κ_j.

In the case in which dynasties do not switch type, as we shrink the time interval to zero the tail coefficients for wealth for each type of dynasty approaches the standard formulas when log wealth follows a Brownian motion with a reflecting barrier at the bottom, namely $\zeta_{ss,j} = -2\mu_j/\sigma_j^2$ for $j = F, D$. To see this, we use that the tail coefficient is $\zeta_{ss,j} = \log\left(\frac{p_{u,j}}{p_{d,j}}\right)/\Delta$ when the types do not switch. Moreover, equations B.4 and B.5 together with $\Delta = \sigma_{max}\sqrt{2\Delta_t}$, imply that

$$\begin{aligned}
\frac{\log\left(\frac{p_{u,j}}{p_{d,j}}\right)}{\Delta} &= \frac{1}{\Delta}\log\left(\frac{\sigma_j^2 + \mu_j^2\Delta_t + \mu_j\Delta}{\sigma_j^2 + \mu_j^2\Delta_t - \mu_j\Delta}\right) \\
&= \frac{1}{\Delta}\log\left(\frac{\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} + \mu_j\Delta}{\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} - \mu_j\Delta}\right) \\
&= \frac{1}{\Delta}\log\left(\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} + \mu_j\Delta\right) - \frac{1}{\Delta}\log\left(\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} - \mu_j\Delta\right)
\end{aligned} \tag{B.6}$$

Taking $\Delta_t \to 0$ implies taking $\Delta \to 0$, and applying L'Hôpital's rule to the above two terms separately gives us

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\log\left(\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} + \mu_j\Delta\right) = \lim_{\Delta\to 0}\frac{\Delta\frac{\mu_j^2}{\sigma_{max}^2} + \mu_j}{\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} + \mu_j\Delta} = \frac{\mu_j}{\sigma_j^2}$$

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\log\left(\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} - \mu_j\Delta\right) = \lim_{\Delta\to 0}\frac{\Delta\frac{\mu_j^2}{\sigma_{max}^2} - \mu_j}{\sigma_j^2 + \mu_j^2\frac{1}{2}\frac{\Delta^2}{\sigma_{max}^2} + \mu_j\Delta} = -\frac{\mu_j}{\sigma_j^2}$$

and hence

$$\lim_{\Delta_t\to 0}\zeta = \frac{\mu_j}{\sigma_j^2} - \left(-\frac{\mu_j}{\sigma_j^2}\right) = \frac{2\mu_j}{\sigma_j^2} \tag{B.7}$$

B.3 Appendix C: Analytical Solution for the Evolution of the Density of Wealth

In our main proposition, we provide an analytical solution for the evolution of the density of wealth in the transition to steady-state. We prove that proposition here.

B.3.1 One-type Model

We begin by providing an analytical expression for the evolution of the distribution of wealth in the context of the model with only one type, or, equivalently, as in the model in which dynasties do not switch types. In the one type model, the equations (B.1) and (B.2) can be written as

$$g_{t+1}(n) = p_u g_t(n-1) + p_d g_t(n+1) + (1 - p_u - p_d) g_t(n) \tag{B.8}$$

$$g_{t+1}(0) = p_d g_t(1) + (1 - p_u) g_t(0). \tag{B.9}$$

Champernowne (1953) showed that the stationary distribution implied by these equations is

$$g_{ss}(n) = (1 - \lambda_{ss}) \lambda_{ss}^n$$

where $\lambda_{ss} = \frac{p_u}{p_d}$. The stationary distribution exists provided that $p_u < p_d$. The proposition presented in this paper establishes an analytical expression for the distribution of wealth at each time period of the transitions from one steady state to another. Specifically, we consider initial distributions of wealth across dynasties that are of the same form as the steady-state distribution but with a different parameter, $\lambda_0 \neq \lambda_{ss}$. That is, we assume that the initial distribution is of the form

$$g_0(n) = (1 - \lambda_0) \lambda_0^n.$$

To develop our analytical formula in this case, we use the following notation. Let $\mathbb{T}$

be the operator mapping distributions over nodes n of our grid to new distributions defined by equations (B.8) and (B.9). Let Λ_0 be a vector corresponding to the initial distribution $g_0(n) = (1-\lambda_0)\lambda_0^n$. Let Λ_{ss} be the distribution to which the economy converges, $g_{ss}(n) = (1-\lambda_{ss})\lambda_{ss}^n$. Let $\mathbf{1}$ denote a distribution that places weight 1 on the node $n = 0$ and weight 0 on every node $n \geq 1$. That is, $\mathbf{1}$ corresponds to the distribution of assets for a cohort of dynasties all starting with the minimum level of assets. With this notation, we have the following result stated as a Corollary of our main proposition in the text.

Corollary Assume that the initial distribution at $t = 0$ of the idiosyncratic component of assets across dynasties is given by Λ_0 and that the transition probabilities in equations (B.8) and (B.9) are constant at p_d and $p_u = \lambda_{ss} p_d$ so that the stationary distribution of the idiosyncratic component of assets across dynasties is given by Λ_{ss}. Then the distribution at date t implied by equations (B.8) and (B.9) is given recursively by

$$(g_{t+1} - \Lambda_{ss}) = A\left(g_t - \Lambda_{ss}\right) + (1-A)\left(\mathbb{T}^t(\mathbf{1}) - \Lambda_{ss}\right), \tag{B.10}$$

so that the distribution at time t is given by

$$g_t = A^t \Lambda_0 + (1-A)\sum_{k=0}^{t-1} A^{t-1-k}\mathbb{T}^k(\mathbf{1}) \tag{B.11}$$

where A is a scalar given by

$$A \equiv \left(p_d(1-\lambda_0)(\frac{\lambda_{ss}}{\lambda_0} - 1) + 1\right),$$

Proof: Direct calculation gives that

$$\mathbb{T}(\Lambda_0) = A\Lambda_0 + (1-A)\mathbf{1}.$$

The operator $\mathbb{T}$ is linear, and $\mathbb{T}(\Lambda_{ss}) = \Lambda_{ss}$. Repeated application of this operator to $g_{t+1} = \mathbb{T}(g_t)$ starting from $g_0 = \Lambda_0$ then gives the result (B.10). Solving (B.10) forward then implies (B.11).

B.3.2 Continuous-Time Analogue

Aleh Tsyvinski generously provided the continuous-time result presented in this section. This result is analogous in the sense that it gives an analytical expression for the density of the logarithm of wealth at all times during the course of a transition to steady state from an initial distribution in which the logarithm of wealth is exponentially distributed (and hence, wealth is Pareto distributed).

In particular, let X_t be a Brownian motion with drift $\mu = -r < 0$ and diffusion σ, with a reflecting barrier at zero. The transition density $p_t(x, y)$ of the process X_t satisfies the following Kolmogorov backward equation

$$\frac{\partial p_t(x,y)}{\partial t} = -r\frac{\partial p_t(x,y)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p_t(x,y)}{\partial x^2} \tag{B.12}$$

and the Neumann boundary condition

$$\left.\frac{\partial p_t(x,y)}{\partial x}\right|_{x=0} = 0. \tag{B.13}$$

The stationary distribution for transition densities p_t is exponential with rate $2\frac{r}{\sigma^2}$. To see this, note that with $g(x) = 2\frac{r}{\sigma^2}e^{-2\frac{r}{\sigma^2}x}$, $x > 0$, we have

$$\frac{\partial}{\partial t}\int_0^\infty 2\frac{r}{\sigma^2}e^{-2\frac{r}{\sigma^2}x}p_t(x,y)dx = \int_0^\infty 2\frac{r}{\sigma^2}e^{-2\frac{r}{\sigma^2}x}\left(-r\frac{\partial p_t(x,y)}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p_t(x,y)}{\partial x^2}\right)dx =$$

$$\int_0^\infty \left(-\frac{\partial\left(re^{-2\frac{r}{\sigma^2}x}\right)}{\partial x} - 2\frac{r^2}{\sigma^2}e^{-2\frac{r}{\sigma^2}x}\right)\frac{\partial p_t(x,y)}{\partial x}dx = 0.$$

Where the second equality follows from integrating by parts. Consider now a transition experiment analogous to the one considered in our Corollary above. In other words, suppose that the initial distribution of the logarithm of wealth is given by $g_0(y) = \lambda e^{-\lambda y}$, $y > 0$. The distribution at time t, which we denote by $g_t(y)$ is then given by

$$g_t(y) = \int_0^\infty g_0(x)p_t(x,y)dx$$

Differentiating this (and using integration by parts) we obtain

$$\frac{\partial g_t(y)}{\partial t} = \int_0^\infty \lambda e^{-\lambda x} \frac{\partial p_t(x,y)}{\partial t} dx = \int_0^\infty \lambda e^{-\lambda x} \left(-r \frac{\partial p_t(x,y)}{\partial x} + \frac{\sigma^2}{2} \frac{\partial^2 p_t(x,y)}{\partial x^2} \right) dx =$$

$$\int_0^\infty \left(-\frac{\partial \left(\frac{\lambda \sigma^2}{2} e^{-\lambda x} \right)}{\partial x} - r\lambda e^{-\lambda x} \right) \frac{\partial p_t(x,y)}{\partial x} dx = \left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) \int_0^\infty e^{-\lambda x} \frac{\partial p_t(x,y)}{\partial x} dx$$

$$= \left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) \left(-p_t(0,y) + \int_0^\infty \lambda e^{-\lambda x} p_t(x,y) dx \right) =$$

$$\left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) (g_t(y) - p_t(0,y))$$

In other words, the distribution at time t satisfies the following non-homogeneous ordinary differential equation

$$\frac{\partial g_t(y)}{\partial t} = \left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) (g_t(y) - p_t(0,y))$$

Using the initial condition $g_0(y) = \lambda e^{-\lambda y}$ we obtain the solution

$$g_t(y) = e^{\left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) t} \lambda e^{-\lambda y} - \left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right) \int_0^t e^{\left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right)(t-s)} p_s(0,y) ds$$

This is analoguous to equation B.11 in that it shows that the distribution at time t is a linear combination of the initial distribution $\lambda e^{-\lambda y}$, and the distribution of agents coming up from the bottom, $\int_0^t e^{\left(\frac{\lambda^2 \sigma^2}{2} - r\lambda \right)(t-s)} p_s(0,y) ds$.

B.3.3 Two-type Model

In this section, we prove our main Proposition in the model with switching between the two types. We denote by Λ_i the distribution over nodes given by $\Lambda_i(n) = (1-\lambda_i)\lambda_i^n$ for any $\lambda_i \in (0,1)$ and for $n \geq 0$. We use $\mathbf{1}$ to denote a distribution that puts weight one on the node $n = 0$ and zero on every other node.

In the two-type model, the operator $\mathbb{T}$ defined by equations (B.1) and (B.2) maps a pair of distributions by type at t, $[g_{F,t}, g_{D,t}]'$ to a pair of distributions by type at $t+1$, $[g_{F,t+1}, g_{D,t+1}]'$. Define $\mathbb{T}_j$ to be the operator which maps pairs of distributions at t,

$[g_{F,t}, g_{D,t}]'$ to the distribution for type j at $t+1$. With these definitions

$$[g_{F,t+1}, g_{D,t+1}]' = \mathbb{T}\left[g_{F,t}, g_{D,t}\right]' = \left[\mathbb{T}_F\left[g_{F,t}, g_{D,t}\right]', \mathbb{T}_D\left[g_{F,t}, g_{D,t}\right]'\right]'$$

Our main proposition provides an analytical expression for the distribution of wealth at each time period in the transition between one steady state distribution and another. Specifically, fix the parameters of the operator $\mathbb{T}$ given by $\{p_{u,j}, p_{d,j}, \phi_j\}$. Let the initial distribution of assets by type be given by

$$g_{j,0} = a_{j,0}\Lambda_a + b_{j,0}\Lambda_b$$

with $a_{j,0} + b_{j,0} = 1$ for arbitrary non-negative weights $a_{j,0}, b_{j,0}$ and arbitrary Λ_a, Λ_b defined by $\lambda_a, \lambda_b \in [0,1)$. Then the following holds:

Main Proposition In the transition experiment described above the distributions of wealth by type at date t are given by

$$\begin{bmatrix} g_{F,t} \\ g_{D,t} \end{bmatrix} = \begin{bmatrix} a_{F,t}\Lambda_a \\ a_{D,t}\Lambda_a \end{bmatrix} + \begin{bmatrix} b_{F,t}\Lambda_b \\ b_{D,t}\Lambda_b \end{bmatrix} + \sum_{k=0}^{t-1} \mathbb{T}^k \begin{bmatrix} c_{F,t-k}\mathbf{1} \\ c_{D,t-k}\mathbf{1} \end{bmatrix}. \tag{B.14}$$

where $a_{j,0}, b_{j,0}$ are given by the initial distributions at $t = 0$,

$$\begin{bmatrix} a_{F,t+1} \\ a_{D,t+1} \end{bmatrix} = \begin{bmatrix} \phi_F A_F & (1-\phi_F)A_D \\ (1-\phi_D)A_F & \phi_D A_D \end{bmatrix} \begin{bmatrix} a_{F,t} \\ a_{D,t} \end{bmatrix} \tag{B.15}$$

and

$$\begin{bmatrix} b_{F,t+1} \\ b_{D,t+1} \end{bmatrix} = \begin{bmatrix} \phi_F B_F & (1-\phi_F)B_D \\ (1-\phi_D)B_F & \phi_D B_D \end{bmatrix} \begin{bmatrix} b_{F,t} \\ b_{D,t} \end{bmatrix} \tag{B.16}$$

where

$$A_j = \left[1 + p_{u,j}\frac{1-\lambda_a}{\lambda_a} - p_{d,j}(1-\lambda_a)\right] \tag{B.17}$$

$$B_j = \left[1 + p_{u,j}\frac{1-\lambda_b}{\lambda_b} - p_{d,j}(1-\lambda_b)\right] \tag{B.18}$$

and $c_{F,0} = c_{D,0} = 0$ and

$$c_{F,t+1} = \phi_F(a_{F,t} + b_{F,t}) + (1-\phi_F)(a_{D,t} + b_{D,t}) - (a_{F,t+1} + b_{F,t+1})$$

$$c_{D,t+1} = \phi_D(a_{D,t} + b_{D,t}) + (1-\phi_D)(a_{F,t} + b_{F,t}) - (a_{D,t+1} + b_{D,t+1})$$

Proof: Note that the operator $\mathbb{T}$ is linear in acting on pairs of distributions. Direct calculation gives that

$$\mathbb{T}_F \begin{bmatrix} a_{F,t}\Lambda_a \\ a_{D,t}\Lambda_a \end{bmatrix} = \left[\phi_F A_F a_{F,t} + (1-\phi_F)A_D a_{D,t}\right]\Lambda_a +$$

$$\left[\phi_F(1-A_F)a_{F,t} + (1-\phi_F)(1-A_D)a_{D,t}\right]\mathbf{1} =$$

$$a_{F,t+1}\Lambda_a + \left[\phi_F a_{F,t} + (1-\phi_F)a_{D,t} - a_{F,t+1}\right]\mathbf{1}$$

$$\mathbb{T}_F \begin{bmatrix} b_{F,t}\Lambda_b \\ b_{D,t}\Lambda_b \end{bmatrix} = b_{F,t+1}\Lambda_b + \left[\phi_F b_{F,t} + (1-\phi_F)b_{D,t} - b_{F,t+1}\right]\mathbf{1}$$

$$\mathbb{T}_D \begin{bmatrix} a_{F,t}\Lambda_a \\ a_{D,t}\Lambda_a \end{bmatrix} = a_{D,t+1}\Lambda_a + \left[\phi_D a_{D,t} + (1-\phi_D)a_{F,t} - a_{D,t+1}\right]\mathbf{1}$$

$$\mathbb{T}_D \begin{bmatrix} b_{F,t}\Lambda_b \\ b_{D,t}\Lambda_b \end{bmatrix} = b_{D,t+1}\Lambda_b + \left[\phi_D b_{D,t} + (1-\phi_D)b_{F,t} - b_{D,t+1}\right]\mathbf{1}$$

These results imply that when the operator $\mathbb{T}$ is applied to the initial distribution at $t = 0$, the pair of distributions that results at $t = 1$ is given by

$$\begin{bmatrix} g_{F,1} \\ g_{D,1} \end{bmatrix} = \begin{bmatrix} a_{F,1}\Lambda_a \\ a_{D,1}\Lambda_a \end{bmatrix} + \begin{bmatrix} b_{F,1}\Lambda_b \\ b_{D,1}\Lambda_b \end{bmatrix} + \begin{bmatrix} c_{F,1}\mathbf{1} \\ c_{D,1}\mathbf{1} \end{bmatrix}$$

Now consider applying the operator $\mathbb{T}$ to a pair of distributions at t of the form in equation (2.7). We get

$$\begin{bmatrix} g_{F,t+1} \\ g_{D,t+1} \end{bmatrix} = \mathbb{T}\begin{bmatrix} g_{F,t} \\ g_{D,t} \end{bmatrix} = \begin{bmatrix} a_{F,t+1}\Lambda_a \\ a_{D,t+1}\Lambda_a \end{bmatrix} + \begin{bmatrix} b_{F,t+1}\Lambda_b \\ b_{D,t+1}\Lambda_b \end{bmatrix} +$$

$$\begin{bmatrix} c_{F,t+1}\mathbf{1} \\ c_{D,t+1}\mathbf{1} \end{bmatrix} + \sum_{k=0}^{t-1} \mathbb{T}^{k+1} \begin{bmatrix} c_{F,t-k}\mathbf{1} \\ c_{D,t-k}\mathbf{1} \end{bmatrix} =$$

$$\begin{bmatrix} a_{F,t+1}\Lambda_a \\ a_{D,t+1}\Lambda_a \end{bmatrix} + \begin{bmatrix} b_{F,t+1}\Lambda_b \\ b_{D,t+1}\Lambda_b \end{bmatrix} + \sum_{k=0}^{t} \mathbb{T}^{k} \begin{bmatrix} c_{F,t+1-k}\mathbf{1} \\ c_{D,t+1-k}\mathbf{1} \end{bmatrix}$$

which proves the result.

B.3.4 Conditions That the Steady-State Distribution Must Satisfy

The following are necessary conditions of Steady-State that are useful in our calibration of the model.

We take as given the parameters of the two-type model $\phi_F, \phi_D, p_{u,F}, p_{d,F}, p_{u,D}, p_{d,D}$. Provided that these parameters are such that equation (2.7) converges to a steady state of the form

$$\begin{bmatrix} g_F \\ g_D \end{bmatrix} = \begin{bmatrix} a_F\Lambda_a + b_F\Lambda_b \\ a_D\Lambda_a + b_D\Lambda_b \end{bmatrix}$$

we can characterize the steady state as follows. The steady state distribution is given by six parameters: $\lambda_a, \lambda_b \in (0,1)$ and $a_F, a_D, b_F, b_D \in [0,1]$. These six parameters have to satisfy the following conditions. The weights a_F, a_D, b_F, b_D have to satisfy

$$a_F + b_F = 1$$

$$a_D + b_D = 1$$

and be a stationary solution to equations (B.15) and (B.16) with the coefficients A_j and B_j given by equations (B.17) and (B.18). These equations imply that

$$\frac{a_F}{a_D} = \frac{(1-\phi_F)A_D}{(1-\phi_F A_F)} = \frac{(1-\phi_D A_D)}{(1-\phi_D)A_F} \tag{B.19}$$

The second of these equations implies

$$0 = 1 - (1-\phi_F-\phi_D)A_F A_D - \phi_D A_D - \phi_F A_F \tag{B.20}$$

Since λA_F and λA_D are both quadratic in λ, we can multiply the left hand side of (B.20) and obtain a fourth order polynomial in λ when $(1 - \phi_F - \phi_D) \neq 0$. To have a unique stationary distribution, one must check that only two of the roots of this polynomial lie in the interval $(0, 1)$. By convention, λ_a is the largest root of this polynomial that lies in the interval $(0, 1)$ and λ_b is the smaller of the two roots in this interval. We have that b_F and b_D solve the analogous equation to (B.19) with λ_b being the smaller root in $(0, 1)$ of the analog to equation (B.20) defined by B_F and B_D in place of A_F and A_D.

B.4 Appendix D: The Steady State Distribution of Wealth

We previously provided necessary conditions that the steady-state distribution must satisfy if it is of a particular form. This appendix shows that the steady state of the two type model is of the form $g_j(n) = a_j(1 - \lambda_a)\lambda_a^n + b_j(1 - \lambda_b)\lambda_a^n$ for $j \in \{F, D\}$ provided that a steady state exists. We begin by writing the equations (B.1) and (B.2), that define the operator $\mathbb{T}$ in the form of matrix equations.

$$\begin{aligned} x_{t+1}(n+1) &= \Psi x_t(n+2) + \Gamma x_t(n+1) + \Theta x_t(n) \\ x_{t+1}(0) &= \Psi x_t(1) + \Xi x_t(0) \end{aligned} \tag{B.21}$$

where $x_t(n) = \begin{bmatrix} g_{t,F}(n) \\ g_{t,D}(n) \end{bmatrix}$ and the following matrices

$$\Psi = \begin{bmatrix} \phi_F p_{d,F} & (1-\phi_F)p_{d,D} \\ (1-\phi_D)p_{d,F} & \phi_D p_{d,D} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \phi_F p_{s,F} & (1-\phi_F)p_{s,D} \\ (1-\phi_D)p_{s,F} & \phi_D p_{s,D} \end{bmatrix}$$

$$\Theta = \begin{bmatrix} \phi_F p_{u,F} & (1-\phi_F)p_{u,D} \\ (1-\phi_D)p_{u,F} & \phi_D p_{u,D} \end{bmatrix}, \quad \Xi = \begin{bmatrix} \phi_F(1-p_{u,F}) & (1-\phi_F)(1-p_{u,D}) \\ (1-\phi_D)(1-p_{u,F}) & \phi_D(1-p_{u,D}) \end{bmatrix}$$

Since our goal is to find the stationary distribution, we consider these equations with time-subscripts removed. In particular, we want to solve the following second-order matrix difference equation

$$x(n+1) = \Psi x(n+2) + \Gamma x(n+1) + \Theta x(n) \tag{B.22}$$

with the initial condition $x(0) = \Psi x(1) + \Xi x(0)$. To solve this equation, we rewrite it as a first-order difference equation by letting $z(n) = \begin{bmatrix} x(n+1) \\ x(n) \end{bmatrix}$ and write the system as follows

$$\begin{aligned} z(n+1) &= Lz(n), \quad \text{for} \quad n \geq 1 \\ z(0) &= \begin{bmatrix} \Psi^{-1}(I_{2\times 2} - \Xi)x(0) \\ x(0) \end{bmatrix} \end{aligned}$$

with

$$L = \begin{bmatrix} \Psi^{-1}(I_{2\times 2} - \Gamma) & -\Psi^{-1}\Theta \\ I_{2\times 2} & 0_{2\times 2} \end{bmatrix} \tag{B.23}$$

The inverse of the matrix Ψ is given by $\Psi^{-1} = \frac{1}{\phi_D + \phi_F - 1} \begin{bmatrix} \frac{\phi_D}{p_{d,F}} & \frac{\phi_F - 1}{p_{d,F}} \\ \frac{\phi_D - 1}{p_{d,D}} & \frac{\phi_F}{p_{d,D}} \end{bmatrix}$. The inverse exists provided that $\phi_F + \phi_D \neq 1$ and the probability of moving down is positive for each type. Provided that L has four distinct eigenvalues we can diagonalize it and write

$$z(n) = L^n z(0) = P\Lambda^n P^{-1} z(0) \tag{B.24}$$

where P is the matrix with the eigenvectors of L as columns, and Λ is the diagonal matrix of eigenvalues. Moreover, we have

$$P^{-1} z(n) = \Lambda^n P^{-1} z(0) \tag{B.25}$$

so that to ensure that the $\sum_{n=0}^{\infty} z(n) < \infty$ holds we need to impose the condition that

$$\tilde{p}_i z(0) = 0, \text{ for every eigenvalue } |\lambda_i| \geq 1 \tag{B.26}$$

where $\tilde{p}_i$ is a row vector from $P^{-1} = [\tilde{p}_1, ..., \tilde{p}_4]'$. Let $\lambda_a, \lambda_b, \lambda_c$ and λ_d be the eigenvalues of L. It turns out that L has two eigenvalues that are stable, i.e., less than 1 in absolute value. Without loss of generality let λ_a and λ_b be the stable eigenvalues. Hence, for $i = c, d$ $|\lambda_i| \geq 1$. With $\tilde{p}_3 z(0) = \tilde{p}_4 z(0) = 0$ we can write equation (B.24) as

$$z(n) = P\Lambda^n \begin{bmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix} z(0) = P \begin{bmatrix} \lambda_a^n \tilde{p}_1 z(0) \\ \lambda_b^n \tilde{p}_2 z(0) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} p_{11}\lambda_a^n \tilde{p}_1 z(0) + p_{12}\lambda_b^n \tilde{p}_2 z(0) \\ p_{21}\lambda_a^n \tilde{p}_1 z(0) + p_{22}\lambda_b^n \tilde{p}_2 z(0) \\ p_{31}\lambda_a^n \tilde{p}_1 z(0) + p_{32}\lambda_b^n \tilde{p}_2 z(0) \\ p_{41}\lambda_a^n \tilde{p}_1 z(0) + p_{42}\lambda_b^n \tilde{p}_2 z(0) \end{bmatrix}$$

In other words, the pair of densities can be written on the form

$$g_F(n) = p_{31}\tilde{p}_1 z(0)\lambda_a^n + p_{32}\tilde{p}_2 z(0)\lambda_b^n$$
$$g_D(n) = p_{41}\tilde{p}_1 z(0)\lambda_a^n + p_{42}\tilde{p}_2 z(0)\lambda_b^n$$

By defining the weights a_F, b_F and a_D, b_D to solve the following system of equations

$$(1-\lambda_a)a_F = p_{31}\tilde{p}_1 \begin{bmatrix} (1-\lambda_a)a_F\lambda_a + (1-\lambda_b)b_F\lambda_b \\ (1-\lambda_a)a_F + (1-\lambda_b)b_F \end{bmatrix} \tag{B.27}$$

$$(1-\lambda_a)a_D = p_{41}\tilde{p}_1 \begin{bmatrix} (1-\lambda_a)a_F\lambda_a + (1-\lambda_b)b_F\lambda_b \\ (1-\lambda_a)a_F + (1-\lambda_b)b_F \end{bmatrix} \tag{B.28}$$

$$(1-\lambda_b)b_F = p_{32}\tilde{p}_2 \begin{bmatrix} (1-\lambda_a)a_F\lambda_a + (1-\lambda_b)b_F\lambda_b \\ (1-\lambda_a)a_F + (1-\lambda_b)b_F \end{bmatrix} \tag{B.29}$$

$$(1-\lambda_b)b_D = p_{42}\tilde{p}_2 \begin{bmatrix} (1-\lambda_a)a_F\lambda_a + (1-\lambda_b)b_F\lambda_b \\ (1-\lambda_a)a_F + (1-\lambda_b)b_F \end{bmatrix} \tag{B.30}$$

we have shown that the stationary distributions can be written as

$$g_F(n) = (1 - \lambda_a) a_F \lambda_a^n + (1 - \lambda_b) b_F \lambda_b^n$$
$$g_D(n) = (1 - \lambda_a) a_D \lambda_a^n + (1 - \lambda_b) b_D \lambda_b^n$$

which is what we wanted to show.

B.5 Appendix E: Calibration Details

This appendix details the procedure for implementing the baseline calibration of the model as well as the calibrations considered in various transition experiments.

The time step size Δ_t, the grid step size Δ, the size of of the grid N, the maximum standard deviation accomodated by the grid σ_{max} and the fraction of dynasties in the overall population that belong to the different types, ν_F and ν_D, as well as the rate at which family firms diversify, κ_F, are common to all calibrations. In particular, $\Delta_t = 1/15000$, $\sigma_{max} = 0.70$, $\Delta = \sigma_{max}\sqrt{2\Delta_t}$, $N = \frac{50}{\sqrt{\Delta_t}}$, $\nu_F = 0.05$, $\nu_D = 1 - \nu_F = 0.95$, and $\kappa_F = 1/15$. The relationship between Δ_t, σ_{max} and Δ ensures that the model is well behaved when $\Delta_t \to 0$, analogous to when one considers the continuous time limit of a binomial option pricing model.

With these parameters set directly, we set the remaining four parameters governing the first two moments of the innovations to log wealth for the two types of families as described next.

B.5.1 Baseline Calibration

The baseline calibration targets four data moments. These are (a) the steady state tail coefficient of top wealth which is set to a target of ζ =1.43, (b) the difference in expected growth rates in the level of wealth of families at the top 0.01% and the bottom of the wealth distribution which is set to a target of 5.69%, (c) the cross-sectional dispersion of innovations to log wealth for families at the bottom of the wealth distribution which is set to a target of 8.13%, and (d) the cross-sectional dispersion of innovations to log wealth for families at the top 0.01% of the wealth

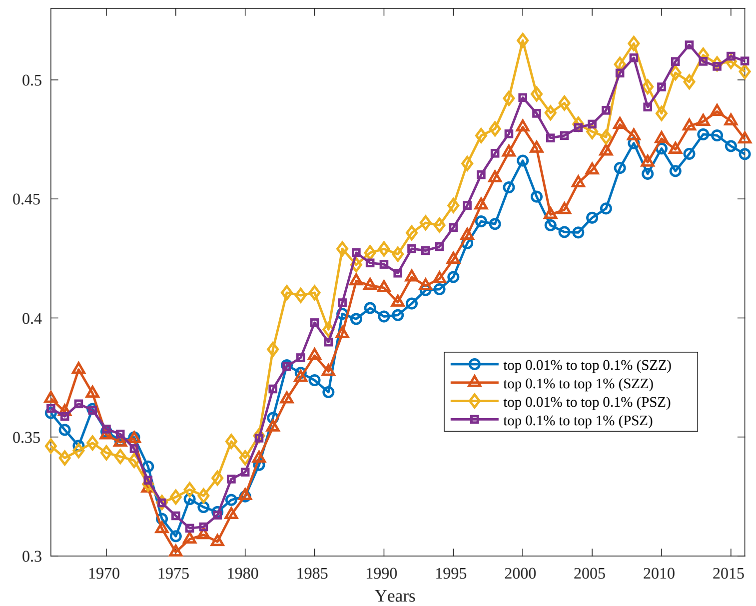


Figure B.3: Ratios of Top Wealth Shares for the Years 1966-2016.
Estimates of top wealth shares are from Piketty et al. (2018) (PSZ) and Smith et al. (2021) (SZZ).

distribution which is set to a target of 35.79%.

The moment (a) is estimated using equation 2.4 and data on ratios of wealth shares for the top 0.01% and 0.1% in 2016. This tail coefficient corresponds to a ratio of these top shares of 0.5. This lies in between the ratio estimated by Smith et al. (2021) and Piketty et al. (2018) that report ratios of 0.47 and 0.51 in the year 2016, respectively. To illustrate the ranges of values of that one could use for the ratios of wealth shares, which in turn imply a tail coefficients through equation 2.4, Figure B.3 displays the ratio of the top 0.01% to the top 0.1% wealth shares as well as the top 0.1% to the top 1%. The data comes from both Piketty et al. (2018) and Smith et al. (2021). Note that their findings in each paper that the ratio of the wealth shares of the top 0.01% to the 0.1% and of the top 0.1% to the top 1% are similar is consistent with the maintained assumption that the top of the wealth distribution above the top 1% has a Pareto density with a constant tail coefficient.

The moment (b) is taken from Bach et al. (2020) Table 1 column 1. The moments (c) and (d) are taken from Bach et al. (2020) Table 8, column 1.

Equation 2.4 can be derived as follows. Assume that the density of log wealth is geometric with parameter λ above some node $\bar{n}$ on our grid of wealth levels. That is, let $g(n) = \bar{g}\lambda^n$ for $n > \bar{n}$ for some constant $\bar{g}$. Let $H(n)$ be the complementary

CDF corresponding to this density. Then $H(n) = \bar{H}\lambda^n$ for $n > \bar{n}$ for some constant $\bar{H}$. With these assumptions, we have that the tail coefficient of wealth at nodes $n > \bar{n}$ is given by $\zeta(n) = \zeta_{top} = -\log(\lambda)/\Delta$.

Let $x > y$ be two top wealth percentiles (for example, the top 0.1% and 0.01%). Let $n(y) > n(x) > \bar{n}$ be the cutoff nodes for those percentiles. That is, let $n(x)$ solve

$$x = \bar{H}\lambda^{n(x)}$$

and likewise for $n(y)$. Assume that $\exp(\Delta)\lambda < 1$ so that top wealth shares are defined. Then the aggregate wealth held by the top x percentile is given by $W(x) = \bar{W}(\exp(\Delta)\lambda)^{n(x)}$ for some constant $\bar{W}$ and ratio of the share of wealth held by the top y to top x percentiles is given by

$$\frac{S(y)}{S(x)} = \left(\exp(\Delta)\lambda\right)^{n(y)-n(x)}$$

This implies that

$$\log S(y) - \log S(x) = (n(y) - n(x))(\Delta + \log \lambda) = \Delta(n(y) - n(x))(1 - \zeta)$$

Since

$$n(x) = (\log(x) - \log(H))/\log(\lambda)$$

and likewise for $n(y)$, we have

$$\log S(y) - \log S(x) = (\log(y) - \log(x))(1 - \frac{1}{\zeta})$$

which gives equation 2.4.

B.5.2 Calibration Procedure

To hit these four moments, we have 4 parameters: μ_F, σ_F, μ_D and σ_D. The subsequent steps of the calibration are the following

1. Guess values for μ_j and σ_j, $j \in \{F, D\}$.

2. Compute the corresponding probabilities $p_{u,j}$ and $p_{d,j}$.

3. Compute the stationary distribution implied by these probabilities

4. Check what the implied tail coefficient of the resulting stationary distribution and check if the difference in average growth rates between the top and the bottom as well as the target values for dispersion of wealth growth are obtained.

5. Update guess until targets are hit.

In step 2, we must translate the annualized moments μ_j and σ_j, $j \in \{F, D\}$ in to probabilities of moving up and down on the grid. The annualized moments of the innovations to log wealth for each type are related to the probabilities through the following equations for the first and second moments of growth in log wealth

$$\mu_j \Delta_t = (p_{u,j} - p_{d,j})\Delta$$

$$\sigma_j^2 \Delta_t + \mu_j^2 \Delta_t^2 = (p_{u,j} + p_{d,j})\Delta^2$$

Solving these equations for the probabilities, using $\Delta = \sigma_{max}\sqrt{2\Delta_t}$, gives

$$p_{u,j} = \frac{1}{2}\left[\sigma_j^2 \frac{\Delta_t}{\Delta^2} + \mu_j^2 \frac{\Delta_t^2}{\Delta^2} + \mu_j \frac{\Delta_t}{\Delta}\right] = \frac{1}{4\sigma_{max}^2}\left[\sigma_j^2 + \mu_j^2 \Delta_t + \mu_j \Delta\right]$$

$$p_{d,j} = \frac{1}{2}\left[\sigma_j^2 \frac{\Delta_t}{\Delta^2} + \mu_j^2 \frac{\Delta_t^2}{\Delta^2} - \mu_j \frac{\Delta_t}{\Delta}\right]$$

Therefore

$$p_{u,j} = \frac{1}{4\sigma_{max}^2}\left[\sigma_j^2 + \mu_j^2 \Delta_t + \mu_j \Delta\right] \tag{B.31}$$

$$p_{d,j} = \frac{1}{4\sigma_{max}^2}\left[\sigma_j^2 + \mu_j^2 \Delta_t - \mu_j \Delta\right] \tag{B.32}$$

In step 3, we must compute the stationary distribution. We do this by finding the two stable eigenvalues of the matrix L defined in equation (B.23) in Appendix B.4. We know that the steady-state distribution for each type is of the form

$$g_j(n) = a_j(1 - \lambda_a)\lambda_a^n + b_j(1 - \lambda_b)\lambda_b^n$$

so for high levels of wealth, the tail coefficient is given by $\zeta_{ss} = \frac{1}{\Delta}\log(\lambda_a)$, where λ_a is the larger of the two eigenvalues. This is the first of our targets. To fully specify the stationary distribution we also need to compute the weights a_j and b_j. Steady state implies that

$$\frac{a_F}{a_D} = \frac{(1-\phi_F)A_D}{(1-\phi_F A_F)} \tag{B.33}$$
$$\frac{b_F}{b_D} = \frac{(1-\phi_F)B_D}{(1-\phi_F B_F)} \tag{B.34}$$

where

$$A_j = \left[1 + p_{u,j}\frac{1-\lambda_a}{\lambda_a} - p_{d,j}(1-\lambda_a)\right] \tag{B.35}$$
$$B_j = \left[1 + p_{u,j}\frac{1-\lambda_b}{\lambda_b} - p_{d,j}(1-\lambda_b)\right] \tag{B.36}$$

Combining this with the fact that the steady state densities must sum to 1 also implies that $a_j + b_j = 1$, we obtain the system of equations

$$\frac{a_F}{a_D} = \frac{(1-\phi_F)A_D}{(1-\phi_F A_F)} \tag{B.37}$$
$$\frac{1-a_F}{1-a_D} = \frac{(1-\phi_F)B_D}{(1-\phi_F B_F)} \tag{B.38}$$

which implies

$$a_F = \frac{(1-\phi_F)A_D}{(1-\phi_F A_F)} \tag{B.39}$$
$$\frac{1-a_F}{1-a_D} = \frac{(1-\phi_F)B_D}{(1-\phi_F B_F)} \tag{B.40}$$

which can be solved for a_F and a_D, which in turn imply values for $b_F = 1 - a_F$ and $b_D = 1 - a_D$. The overall steady-state distribution of dynasties over nodes is therefore

$$\nu_F g_F(n) + \nu_D g_D(n) = (\nu_F a_F + \nu_D a_D)(1-\lambda_a)\lambda_a^n + (\nu_F b_F + \nu_D b_D)(1-\lambda_b)\lambda_b^n \tag{B.41}$$

and the fraction of family firm dynasties at node n is given by

$$\nu_F(n) = \frac{\nu_F(a_F(1-\lambda_a)\lambda_a^n + (1-a_F)(1-\lambda_b)\lambda_b^n)}{(\nu_F a_F + \nu_D a_D)(1-\lambda_a)\lambda_a^n + (1-\nu_F a_F - \nu_D a_D)(1-\lambda_b)\lambda_b^n} \tag{B.42}$$

which can be used to calculate node-specific moments. In particular, the average growth rate of wealth and the dispersion of log wealth growth at node n is given by

$$\bar{g}_n = \nu_F(n)(\mu_F + 0.5\sigma_F^2) + (1-\nu_F(n))(\mu_D + 0.5\sigma_D^2) \tag{B.43}$$

$$\bar{\sigma}_n^2 = ((\mu_F^2 + \sigma_F^2)\nu_F(n) + (1-\nu_F(n))(\mu_D^2 + \sigma_D^2) - (\mu_F \nu_F(n) + \mu_D(1-\nu_F(n)))^2 \tag{B.44}$$

These formulas are the formulas for the moments of a mixture of two normal distributions. Recall that target (b) is $\bar{g}_N - \underline{g}_0 = 0.0569$, target (c) is $\bar{\sigma}_0 = 0.0813$ and target (d) is $\bar{\sigma}_{n_{top0.01\%}} = 0.3579$. The node $n_{top0.01\%}$ is defined through the relationship

$$G(n_{top0.01\%}) \equiv \sum_{n_{top0.01\%}}^{N} g(n) = 0.0001$$

We use the MATLAB function 'fsolve' to find values of μ_j and σ_j that hit these targets. The resulting parameters are reported in row A of Table B.1. We can compute the excess kurtosis at node n implied by this calibration using the following formula

$$\text{ex kurtosis}(n) = \frac{\nu_F(n)\left(\mu_F^4 + 6\mu_F^2\sigma_F^2 + 3\sigma_F^4\right) + \nu_D(n)\left(\mu_D^4 + 6\mu_D^2\sigma_D^2 + 3\sigma_D^4\right)}{\left(\nu_F(n)\left(\mu_F^2 + \sigma_F^2\right) + \nu_D(n)\left(\mu_D^2 + \sigma_D^2\right)\right)^2} - 3 \tag{B.45}$$

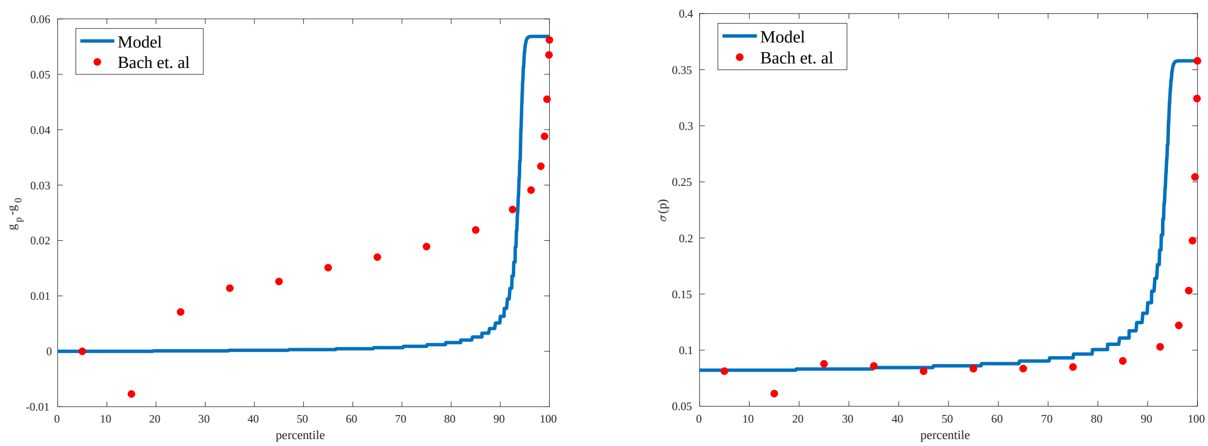
Table B.2 reports the excess kurtosis implied by the baseline calibration and compares it to the excess kurtosis reported by Gomez (2023).

Gomez (2023) also reports that less than 10% of the members of the 1983 cohort of the Forbes 400 list were still members in 2017. When we compute this measure of

Table B.2: Excess Kurtosis of Innovations to Top Wealth

Data	6.58
Baseline	1.31

The data on excess kurtosis for the Forbes 400 is from panel b) of Table 2 of Gomez (2023) for the period 1983-2017. The percentile used for the Forbes 400 in our model is the top 0.0003 percentile.



(a) Expected Growth

(b) Standard Deviation

Figure B.4: Moments of Innovations to Wealth Across the Distribution

The left panel (B.4a) shows the difference in the expected growth rate of the level of wealth for dynasties at different percentiles of the steady-state distribution of wealth relative to the bottom of the distribution. The right panel (B.4b) shows the corresponding dispersion of innovations to the logarithm of wealth. These moments of innovations to wealth differ across families at different percentiles of the wealth distribution because the mix of dynasties with family firms and with diversified portfolios varies with the level of wealth.

persistence in the membership of the Forbes 400 in the context of the steady state of our baseline calibration we obtain that about 7% of the members of the Forbes 400 are still members over a 34 year period.

We calibrated our model to match the differences in the expected growth rate of wealth and cross section dispersion of innovations to wealth at the top and the bottom of the wealth distribution. To evaluate how well our model fits the data at intermediate levels of wealth, in Figure B.4, in the left panel (B.4a), we show the expected growth in the level of wealth for dynasties at each wealth percentile, and in the right panel (B.4b), we show the corresponding cross section dispersion of growth rates of the logarithm of wealth at each wealth percentile implied by these changing fractions of dynasties of each type by wealth level. The red dots in these figures correspond to the data in Tables 1 and 8 of Bach et al. (2020).

B.5.3 Transition Experiments

Once we have the parameters governing the growth of wealth, we can compute the evolution over time of a given distribution of wealth. This is done by applying the $\mathbb{T}$ operator repeatedly to a given initial distribution. The $\mathbb{T}$ operator is defined by equation B.21. For instance, to compute the tail coefficient at node n at time $t+1$ given a vector of distributions of wealth by type at time t, x_t, we apply equations B.21 to obtain x_{t+1}. We then obtain the overal distribution of wealth as $g_{t+1}(n) = [\nu_F, \nu_D] \cdot x_{t+1}(n)$, which we use to compute the negative of the slope of the CCDF at the given node n. There is a question about what to do about at the last node of the grid. We impose a reflecting barrier at the top of the grid analogous to the one at the bottom. However, the grid size is so large that the mass at the top of the grid is very close to zero. In the numerical examples we compute, it does not seem to matter if one puts a reflecting barrier at the top or not. To understand this, consider the version of the model when types do not switch. As long as $p_{u,j}/p_{d,j} < 1$, the mass at the top of the grid is going to be negligible if the grid size is large enough since the mass is proportional to $\lambda^n_{j,ss} = \left(p_{u,j}/p_{d,j}\right)^n$.

B.5.4 Calibration of Alternative Experiments Presented in Section 2.4

We conduct a series of quantitative experiments. This appendix describes the calibration procedures of those experiments.

The first two counterfactual experiments are presented in Section 2.4 of the paper. Relative to the calibration procedure for the baseline, these two experiments replace the target for the dispersion of wealth growth at the top with directly setting the volatility of the F type. In particular, the first experiment sets $\sigma_F = 0.3306$, while the second sets $\sigma_F = 0.2204$. Recall that the baseline calibration does not set σ_F directly, but the the implied value for this parameter in the baseline calibration is $\sigma_F = 0.4409$. The values for the calibrated parameters are presented in rows B and C of Table B.1 of the paper.

In addition, when computing the persistence of membership in the Forbes 400, these alternative calibrations feature higher persistence than in the baseline and in the data reported by Gomez (2023). In particular, Gomez (2023) reports that less than 10% of the Forbes 400 cohort of 1983 were still on the list 33 years later. The corresponding number in the baseline calibration is around 7% while it is closer to 13% and 21% in the two alternative calibrations discussed here.

B.5.5 Additional Quantitative Experiments

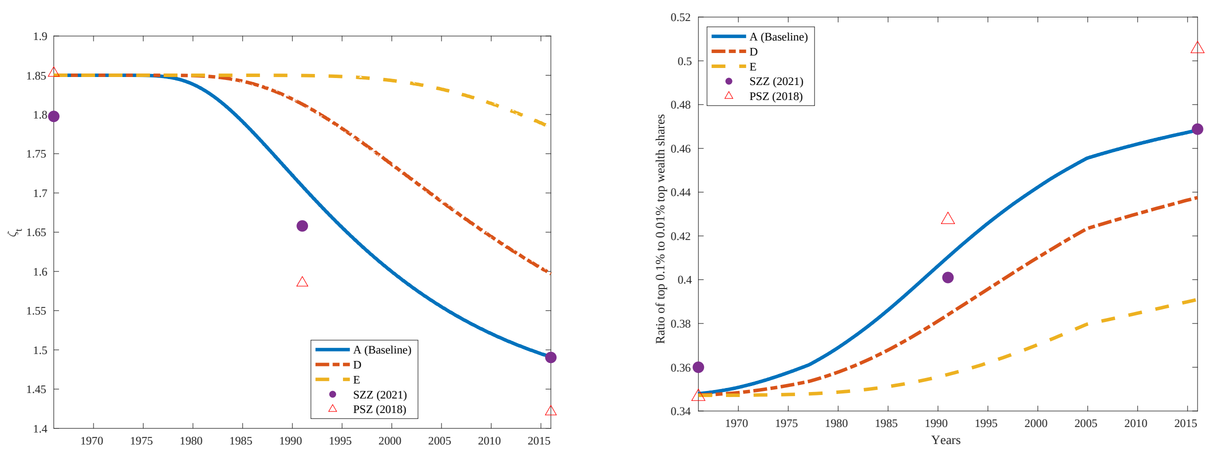
As robustness checks, we also consider three additional quantitative experiments in this appendix. In the first two additional experiments, we examine the results of calibrations wherein the volatility of the F type is reduced in the same manner as the two alternative calibrations presented in Section 2.4 of the paper, while the target for the difference in mean growth rates across the wealth distribution is simultaneously doubled. In other words, relative to the alternative calibrations considered in Section 2.4, we now also change the calibration target b) to $\bar{g}_N - \bar{g}_0 = 0.1138$. Increasing the target difference in mean growth rates is meant to gauge the extent to which larger differences in mean growth rates between the types rather than the very high volatility of the F type can account for the prevalence of new large fortunes and rapid transitions of top wealth inequality. The following Table B.3 presents the values of the calibrated parameters. Figure B.5 compares the fraction of the Forbes 400 members that where at the bottom within the last k years and the transition of ratios of top wealth shares between these alternative calibrations and the baseline calibration. We see that the presence of a substantially larger difference in mean growth rates across the wealth distribution is not enough to compensate for the absence of the high volatility of the F type that is characteristic of the baseline calibration.

The final alternative calibration we consider is one in which the target steady state wealth coefficient is set to $\zeta_{ss} = 1.4$ instead of the baseline value of $\zeta_{ss} = 1.43$. This is motivated by the following two reasons. First, there is some discrepancy between the ratios of top wealth shares reported by Piketty et al. (2018) and Smith et al. (2021). Second, the mapping between ratios of top wealth shares in equation 2.4 is a steady state relationship. It is entirely possible that the parameters governing wealth growth at a specific point in time are associated with a different steady state than what the

Table B.3: Calibrated Parameters in the Baseline and Alternative Calibrations

Parameters	g_F	g_D	σ_F	σ_D	ν_F	κ_F
A (Baseline)	0.0035	−0.0847	0.4409	0.0779	0.05	0.067
D	0.0223	−0.1319	0.3306	0.0798	0.05	0.067
E	0.0352	−0.1221	0.2204	0.0805	0.05	0.067
F	0.0068	−0.0828	0.4441	0.0778	0.05	0.067
Targets	ζ_{ss}	$\bar{g} - \underline{g}$	$\bar{\sigma}_{bottom}$	$\bar{\sigma}_{n_{top0.01\%}}$	σ_F	
A (Baseline)	1.43	0.0569	0.0813	0.3579	N/A	
D	1.43	0.1138	0.0813	N/A	0.3306	
E	1.43	0.1138	0.0813	N/A	0.2204	
F	1.4	0.0569	0.0813	N/A	0.3579	

Calibrated parameters in the baseline as well as the alternative calibrations D and E where the volatility of the F type is reduced to 75% and 50% of its baseline value, respectively, while the targeted difference in growth rates between the top and the bottom of the wealth distribution is doubled relative to the baseline. Alternative calibration F instead features a lower target for the steady state tail coefficient.

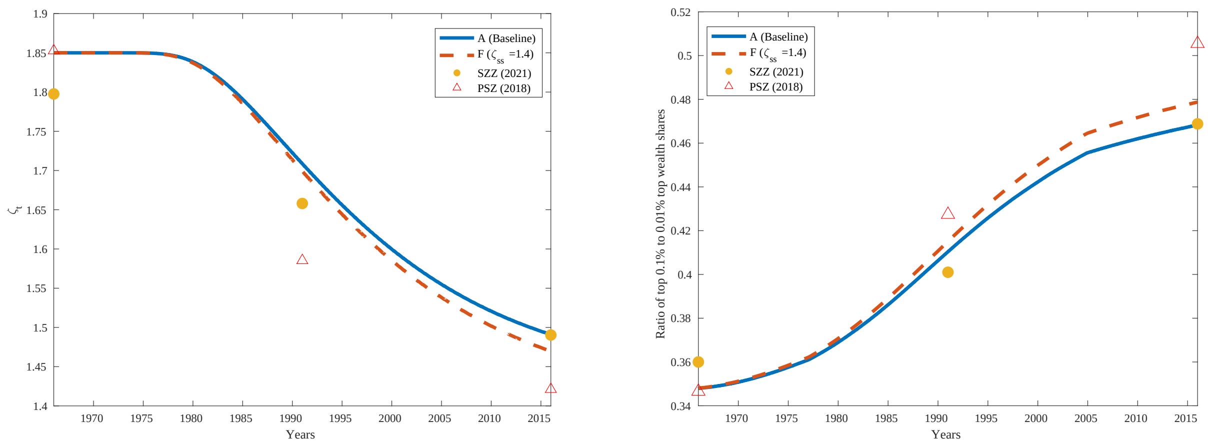


(a) Tail coefficient.

(b) Ratio of top wealth shares

Figure B.5: Transition Results from Baseline and Alternative Calibrations

This figure displays comparisons along two dimensions of the baseline calibration A with the alternative calibrations, D and E. In calibrations D and E the value of σ_F is set to 75% and 50% of its baseline value, respectively, while the target difference in mean growth rates across the wealth distribution is doubled. Figure (B.5a) compares the transition of the tail coefficient, and Figure (B.5b) considers the transition of the ratio of the top 0.01% to the top 0.1% wealth shares. Along both dimensions, the presence of a minority of dynasties with very high idiosyncratic volatility is important for obtaining rapid transitions. The transition is computed for the years 1966-2016. Marked are also the data from Smith et al. (2021) (circles) and Piketty et al. (2018) (triangles). These are the ratios of the top 0.01% to the top 0.1% wealth shares and the implied tail coefficient using equation 2.4.



(a) Tail coefficient

(b) Ratio of top wealth shares

Figure B.6: Transition Results from Final Alternative Calibration

This figure displays the transition of the tail coefficient as well as the ratio of the top 0.01% wealth share to the top 0.1% wealth share when the target steady state tail coefficient is $\zeta_{ss} = 1.4$ instead of the baseline value $\zeta_{ss} = 1.43$. The transition is computed for the years 1966-2016. Marked are also the data from Smith et al. (2021) (circles) and Piketty et al. (2018) (triangles).

current ratio of top wealth shares would imply. The resulting parameter values are reported in row F of Table B.3. Figure B.6 plots the transition of the tail coefficient as well as the evolution of the ratio of the top 0.01% to the top 0.1% wealth shares with this alternative target together with data from Piketty et al. (2018) and Smith et al. (2021). We see that the lower target value for the steady state distribution implies that the transition is somewhat faster.

B.6 Appendix F: A Spectral Analysis of the Dynamics of the Distribution

In this paper, we provide an analytical expression for the dynamics of the distribution of wealth over time as it converges to steady-state if the initial distribution of wealth is in a certain class of distributions. Gabaix et al. (2016) use an alternative approach to analyze the dynamics of the distribution of wealth to steady-state based on a spectral analysis of these dynamics in continuous time. In this appendix, we provide direct analogs of their spectral analysis in our discrete time - discrete state setting with the model restricted to have only one type by analyzing the eigenvalues and eigenvectors of our operator $\mathbb{T}$ in the version of our model with only one type of dynasty.

To prove their results, Gabaix et al. (2016) impose a boundedness assumption on tail coefficients of the distributions of wealth under consideration that is described in their Assumption 1. Here, we consider a related bound by computing the eigenvalues and eigenvectors of our operator $\mathbb{T}$ when the grid of wealth levels is finite (so $N < \infty$). In this case, this operator $\mathbb{T}$ is simply a square Markov transition matrix of size $(N+1) \times (N+1)$, so the calculation of eigenvalues and eigenvectors is standard. As is the case with finite Markov transition matrices, the largest eigenvalue is equal to one, and the speed of convergence of the distribution to steady-state is related to the size of the second largest eigenvalue, which is less than one. We are able to compute this second largest eigenvalue and consider its limiting value as $N \to \infty$. We find that this limiting value of the second largest eigenvalues of our finite Markov transition matrix $\mathbb{T}$ as N grows large corresponds to the formula found in Gabaix et al. (2016) Proposition 1.

We present this analysis for two reasons. First, it may be of interest to readers wishing to better understand spectral methods for analyzing dynamics of distributions. Second, it allows us to highlight two differences between the analytical characterization of the dynamics of the distribution of wealth that we present in our paper and those obtained using spectral methods. These are, first, that our analysis does not require that we impose a bound on the tail coefficient of the initial distribution under consideration. Second, and more important, our analysis directly highlights the connection between the speed of wealth mobility from the bottom of the wealth distribution to the top and the dynamics of the shape of the top of the wealth distribution as it converges to steady state.

B.6.1 The Eigenvalue Problem of $\mathbb{T}$

In the version of the model with one type, the operator $\mathbb{T}$ that maps a distribution g at time t to a distribution $\mathbb{T}(g)$ at time $t+1$ can be defined through the following equations

For $0 < n < N$,

$$\mathbb{T}(g)(n) = p_u g(n-1) + (1 - p_u - p_d)g(n) + p_d g(n+1) \tag{B.46}$$

for $n = 0$

$$\mathbb{T}(g)(0) = (1 - p_u)g(0) + p_d g(1) \tag{B.47}$$

and, if $N < \infty$, for $n = N$

$$\mathbb{T}(g)(N) = (1 - p_d)g(N) + p_u g(N-1) \tag{B.48}$$

The eigenvalue problem $\lambda g = \mathbb{T}(g)$ is therefore defined by the following equations: For $0 < n < N$,

$$\lambda g(n) = p_u g(n-1) + (1 - p_u - p_d)g(n) + p_d g(n+1), \tag{B.49}$$

for $n = 0$

$$\lambda g(0) = (1 - p_u)g(0) + p_d g(1) \tag{B.50}$$

and, if $N < \infty$, for $n = N$

$$\lambda g(N) = (1 - p_d)g(N) + p_u g(N-1) \tag{B.51}$$

Note that when $N < \infty$, $\mathbb{T}$ can be represented by an $N + 1 \times N + 1$ matrix P of the form

$$P = \begin{pmatrix} 1-p_u & p_d & 0 & \dots & \dots & \dots & 0 \\ p_u & 1-p_u-p_d & p_d & 0 & \dots & \dots & 0 \\ 0 & p_u & 1-p_u-p_d & p_d & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \dots & \dots & 0 & p_u & 1-p_d-p_u & p_d \\ 0 & \dots & \dots & 0 & 0 & p_u & 1-p_d \end{pmatrix} \tag{B.52}$$

So that for all g given by vectors of size $N + 1 \times 1$,

$$\mathbb{T}(g) = Pg$$

Thus, the eigenvalue problem for $\mathbb{T}$ corresponds to finding the eigenvalues of P.

Note that the matrix P is not symmetric. Similarly, when $N = \infty$, $\mathbb{T}$ is not self-adjoint. This prevents a direct application of the Spectral Theorem for analyzing the eigenvalue problem presented above.

Following Lemma 6 in Gabaix et al. (2016), we analyze a related operator $\mathbb{S}$ that is self-adjoint and which, under certain conditions discussed below, has the same eigenvalues as $\mathbb{T}$.

We define this related self-adjoint operator $\mathbb{S}$ as follows. For each n, scale the equations (B.46)-(B.48) that define the operator $\mathbb{T}$ by a factor $\left(\sqrt{p_d/p_u}\right)^n$. This gives the equations

$$\left(\frac{p_d}{p_u}\right)^{n/2}\mathbb{T}(g)(n) = p_u\left(\frac{p_d}{p_u}\right)^{n/2}g(n-1) + (1-p_u-p_d)\left(\frac{p_d}{p_u}\right)^{n/2}g(n) + p_d\left(\frac{p_d}{p_u}\right)^{n/2}g(n+1)$$

$$\mathbb{T}(g)(0) = (1-p_u)g(0) + p_d g(1)$$

$$\left(\frac{p_d}{p_u}\right)^{N/2}\mathbb{T}(g)(N) = (1-p_d)\left(\frac{p_d}{p_u}\right)^{N/2}g(N) + p_u\left(\frac{p_d}{p_u}\right)^{N/2}g(N-1)$$

For any vector g, let $h(n) = g(n)\left(\sqrt{p_d/p_u}\right)^n$. We will use the notation h_g refer to this vector. For $N < \infty$, define the operator $\mathbb{S}$ by

$$\mathbb{S}(h)(n) = \left(\frac{p_d}{p_u}\right)^{n/2}\mathbb{T}(g)(n) \tag{B.53}$$

In other words, $\mathbb{S}$ is defined by the following set of equations:

For $0 < n < N$,

$$\mathbb{S}(h)(n) = (\sqrt{p_u p_d})h(n-1) + (1-p_u-p_d)h(n) + (\sqrt{p_u p_d})h(n+1) \tag{B.54}$$

for $n = 0$,

$$\mathbb{S}(h)(0) = (1 - p_u)h(0) + (\sqrt{p_u p_d})h(1) \tag{B.55}$$

and, if $N < \infty$

$$\mathbb{S}(h)(N) = (1 - p_d)h(N) + (\sqrt{p_u p_d})h(N - 1) \tag{B.56}$$

As with the operator $\mathbb{T}$, for fixed $N < \infty$, the operator $\mathbb{S}$ can be represented as an $N + 1 \times N + 1$ matrix Q:

$$Q = \begin{pmatrix} 1-p_u & \sqrt{p_u p_d} & 0 & \dots & \dots & \dots & 0 \\ \sqrt{p_d p_u} & 1-p_u-p_d & \sqrt{p_u p_d} & 0 & \dots & \dots & 0 \\ 0 & \sqrt{p_u p_d} & 1-p_u-p_d & \sqrt{p_u p_d} & 0 & \dots & 0 \\ \vdots & 0 & \ddots & \ddots & \ddots & \dots & 0 \\ 0 & \dots & \dots & 0 & \sqrt{p_u p_d} & 1-p_u-p_d & \sqrt{p_u p_d} \\ 0 & \dots & \dots & 0 & 0 & \sqrt{p_u p_d} & 1-p_d \end{pmatrix} \tag{B.57}$$

That is, for all h given by vectors of size $N + 1 \times 1$,

$$\mathbb{S}(h) = Qh$$

Note that for any fixed $N \leq \infty$, we can recover the dynamics of g from the dynamics of h. That is, if we start from g_0, we construct $h_0(n) = g_0(n)\left(\frac{p_d}{p_u}\right)^{n/2}$. We then construct h_t by applying the operator $\mathbb{S}$, t times, or, equivalently, when $N < \infty$

$$h_t = Q^t h_0$$

We then can construct g_t from $g_t(n) = h_t(n)\left(\sqrt{p_u/p_d}\right)^n$.

Note as well that when $N < \infty$ the matrix Q is real valued and symmetric. That is

$$Q(i, j) = Q(j, i), \forall i, j$$

Thus, we have that when $N < \infty$ the eigenvalues of Q are real, that the eigenvectors are orthogonal, and that the Spectral Theorem for finite dimensional spaces applies.

That is, we can diagonalize Q and use that eigenvalue-eigenvector decomposition to characterize the dynamics of h_t.

The eigenvalue problem $\lambda h = \mathbb{S}(h)$ can be written

$$\lambda h(n) = (\sqrt{p_u p_d})h(n-1) + (1 - p_u - p_d)h(n) + (\sqrt{p_u p_d})h(n+1) \tag{B.58}$$

for $0 < n < N$ and for $n = 0$

$$\lambda h(0) = (1 - p_u)h(0) + (\sqrt{p_u p_d})h(1) \tag{B.59}$$

and, if $N < \infty$, for $n = N$

$$\lambda h(N) = (1 - p_d)h(N) + (\sqrt{p_u p_d})h(N-1) \tag{B.60}$$

Direct comparison of these two eigenvalue problems gives us our first proposition: **Proposition F1:** When $N < \infty$, the set of $N+1$ eigenvalues $\{\lambda_k\}_{k=1}^{N+1}$ of the two operators $\mathbb{T}$ and $\mathbb{S}$ are the same, and the eigenvectors of the two eigenvalue problems are related by $h(n;\lambda_k) = (\sqrt{p_d/p_u})^n g(n;\lambda_k)$.

To prove this proposition, observe that the operators satisfy $\mathbb{S}(h)(n) = \left(\frac{p_d}{p_u}\right)^{n/2} \mathbb{T}(g)(n)$ for any two vectors h and g such that $h(n) = (\sqrt{p_d/p_u})^n g(n)$. Suppose that λ_k is an eigenvalue of $\mathbb{S}$, and that $h(n;\lambda_k)$ is the corresponding eigenvector, then for all $0 \le n \le N$:

$$\mathbb{S}(h)(n) = \lambda_k h(n;\lambda_k)$$
$$\Leftrightarrow$$
$$\left(\frac{p_d}{p_u}\right)^{n/2} \mathbb{T}(g)(n) = \lambda_k h(n;\lambda_k)$$
$$\Leftrightarrow$$
$$\mathbb{T}(g)(n) = \lambda_k g(n;\lambda_k)$$

So that λ_k is also an eigenvalue of $\mathbb{T}$, with $g(n;\lambda_k)$ being the corresponding eigenvector. By the same argument, if λ_k is an eigenvalue of $\mathbb{T}$, with $g(n;\lambda_k)$ being the corresponding eigenvector, then λ_k is an eigenvalue of $\mathbb{S}$, with $h(n;\lambda_k)$ as the corresponding eigenvector.

Note that equations (B.49) and (B.58) in the eigenvalue problems for the operators $\mathbb{T}$ and $\mathbb{S}$ are both regular homogeneous second-order difference equation with constant coefficients. For a given eigenvalue, λ, the characteristic equations for these two difference equations are as follows

$$k(\lambda)^2 - \frac{(\lambda - 1 + p_u + p_d)}{p_d} k(\lambda) + \frac{p_u}{p_d} = 0 \tag{B.61}$$

$$m(\lambda)^2 - \frac{(\lambda - 1 + p_u + p_d)}{\sqrt{p_u p_d}} m(\lambda) + 1 = 0 \tag{B.62}$$

The characteristic equation for the eigenvalue problem for the operator $\mathbb{T}$ has the two solutions

$$k_1(\lambda) = \frac{(\lambda - 1 + p_u + p_d)}{2p_d} + \frac{\sqrt{(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d}}{2p_d} \tag{B.63}$$

$$k_2(\lambda) = \frac{(\lambda - 1 + p_u + p_d)}{2p_d} - \frac{\sqrt{(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d}}{2p_d} \tag{B.64}$$

while that for the operator $\mathbb{S}$ has the two solutions

$$m_1(\lambda) = \frac{(\lambda - 1 + p_u + p_d)}{2\sqrt{p_u p_d}} + \frac{\sqrt{(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d}}{2\sqrt{p_u p_d}} = \sqrt{p_d/p_u}\, k_1(\lambda) \tag{B.65}$$

$$m_2(\lambda) = \frac{(\lambda - 1 + p_u + p_d)}{2\sqrt{p_u p_d}} - \frac{\sqrt{(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d}}{2\sqrt{p_u p_d}} = \sqrt{p_d/p_u}\, k_2(\lambda) \tag{B.66}$$

Note that these roots of these characteristic equations are both real whenever

$$(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d \geq 0 \tag{B.67}$$

and are complex conjugates of each other whenever

$$(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d < 0 \tag{B.68}$$

Define the cutoffs $\bar{\lambda}$ and $\underline{\lambda}$ as the two solutions to

$$(\lambda - 1 + p_u + p_d)^2 - 4p_u p_d = 0$$

These are given by

$$\bar{\lambda} = 1 - (p_u + p_d) + 2\sqrt{p_u p_d} \tag{B.69}$$

and

$$\underline{\lambda} = 1 - (p_u + p_d) - 2\sqrt{p_u p_d} \tag{B.70}$$

we have $1 > \bar{\lambda} > \underline{\lambda}$. We distinguish between three cases surrounding the larger cutoff point $\bar{\lambda}$:

1. In the interval $(\bar{\lambda}, 1)$, the characteristic equations corresponding to the difference equations (B.49) and (B.58) have two distinct real roots, and the solutions to the difference equations are of the form

$$g(n; \lambda) = a_1(\lambda) k_1(\lambda)^n + a_2(\lambda) k_2(\lambda)^n \tag{B.71}$$

$$h(n; \lambda) = a_1(\lambda) m_1(\lambda)^n + a_2(\lambda) m_2(\lambda)^n \tag{B.72}$$

   respectively. Here the parameters $a_1(\lambda)$ and $a_2(\lambda)$ are to be chosen to match boundary conditions.

2. At $\bar{\lambda}$, the characteristic equations have one real root and the solutions to the difference equations are of the form

$$g(n; \lambda) = (a_1(\lambda) + n a_2(\lambda))\, k(\lambda)^n \tag{B.73}$$

$$h(n; \lambda) = (a_1(\lambda) + n a_2(\lambda))\, m_2(\lambda)^n \tag{B.74}$$

3. When $\lambda \in (\underline{\lambda}, \bar{\lambda})$, the roots of the two characteristic equations are complex and the solution to the difference equations can be written

$$g(n; \lambda) = \left(\sqrt{p_u / p_d}\right)^n a(\lambda) \cos(\theta(\lambda) n + \omega(\lambda)) \tag{B.75}$$

$$h(n;\lambda) = a(\lambda)\cos\left(\theta(\lambda)n + \omega(\lambda)\right) \tag{B.76}$$

where

$$\theta(\lambda) = \cos^{-1}\left(\frac{(\lambda - 1 + p_u + p_d)}{2\sqrt{p_d p_u}}\right) \tag{B.77}$$

and $a(\lambda)$ and $\omega(\lambda)$ are to be chosen to match boundary conditions

$$[a_1(\lambda)m_1(\lambda) + a_2(\lambda)m_2(\lambda)]$$

$$(a_1(\lambda)m_1(\lambda)^N + a_2(\lambda)m_2(\lambda)^N)(\lambda - 1 + p_d) = (\sqrt{p_u p_d})\left[a_1(\lambda)m_1(\lambda)^{N-1} + a_2(\lambda)m_2(\lambda)^{N-1}\right]$$

these in turn imply

$$a_1(\lambda) = -\left(\frac{\lambda - 1 + p_u - (\sqrt{p_d p_u})m_2(\lambda)}{\lambda - 1 + p_u - (\sqrt{p_d p_u})m_1(\lambda)}\right)a_2(\lambda)$$

$$a_1(\lambda) = -\left(\frac{m_2(\lambda)}{m_1(\lambda)}\right)^N\left(\frac{\lambda - 1 + p_d - (\sqrt{p_d p_u})m_2(\lambda)^{-1}}{\lambda - 1 + p_d - (\sqrt{p_d p_u})m_1(\lambda)^{-1}}\right)a_2(\lambda)$$

Hence, λ is an eigenvalue if and only if

$$\left(\frac{\lambda - 1 + p_u - (\sqrt{p_d p_u})m_2(\lambda)}{\lambda - 1 + p_u - (\sqrt{p_d p_u})m_1(\lambda)}\right) = \left(\frac{m_2(\lambda)}{m_1(\lambda)}\right)^N\left(\frac{\lambda - 1 + p_d - (\sqrt{p_d p_u})m_2(\lambda)^{-1}}{\lambda - 1 + p_d - (\sqrt{p_d p_u})m_1(\lambda)^{-1}}\right) \tag{B.78}$$

Recall that when $\lambda \in (\bar{\lambda}, 1)$, $m_1(\lambda) > 1 > m_2(\lambda)$. Hence, the left-hand side of (B.78) is larger than 1. But, the right-hand side is smaller than 1 for the same reason. Hence, there are no eigenvalues in the interval $\lambda \in (\bar{\lambda}, 1)$, when N is finite. A similar argument can be used to rule out eigenvalues smaller than $\underline{\lambda}$. We thus have the following proposition:

Proposition F2: For $N < \infty$, all eigenvalues of the operator $\mathbb{S}$ that are less than 1 lie in the interval $(\underline{\lambda}, \bar{\lambda})$.

Since $\mathbb{S}$ and $\mathbb{T}$ have the same eigenvalues when $N < \infty$, the proposition holds for the operator $\mathbb{T}$ as well.

Next, we show how to find the $N + 1$ eigenvalues. Since all eigenvalues lie in the interval $(\underline{\lambda}, \bar{\lambda})$, we know that for a given eigenvalue λ, the eigenvectors are of of the form (B.76) corresponding to complex roots of the characteristic equation associated with the difference equation defining $\mathbb{S}$:

$$h(n;\lambda) = a(\lambda)\cos(\theta(\lambda)n + \omega(\lambda))$$

To pin down $\omega(\lambda)$ for a given eigenvalue λ, we use the lower boundary condition (B.59) and the fact that $\cos(x+y) = \cos(y)\cos(x) - \sin(y)\sin(x)$. This gives us

$$0 = (1 - p_u - \lambda)h(0;\lambda) + (\sqrt{p_u p_d})h(1;\lambda)$$

$$\Leftrightarrow$$

$$0 = (1 - p_u - \lambda)\cos(\omega(\lambda)) + (\sqrt{p_u p_d})\left[\cos(\omega(\lambda))\cos(\theta(\lambda)) - \sin(\omega(\lambda))\sin(\theta(\lambda))\right]$$

This condition can in turn be written

$$\frac{\lambda - 1 + p_u - p_d}{2\sqrt{p_u p_d}} = -\sin(\theta(\lambda))\tan(\omega(\lambda))$$

by using (B.77). Moreover, note that

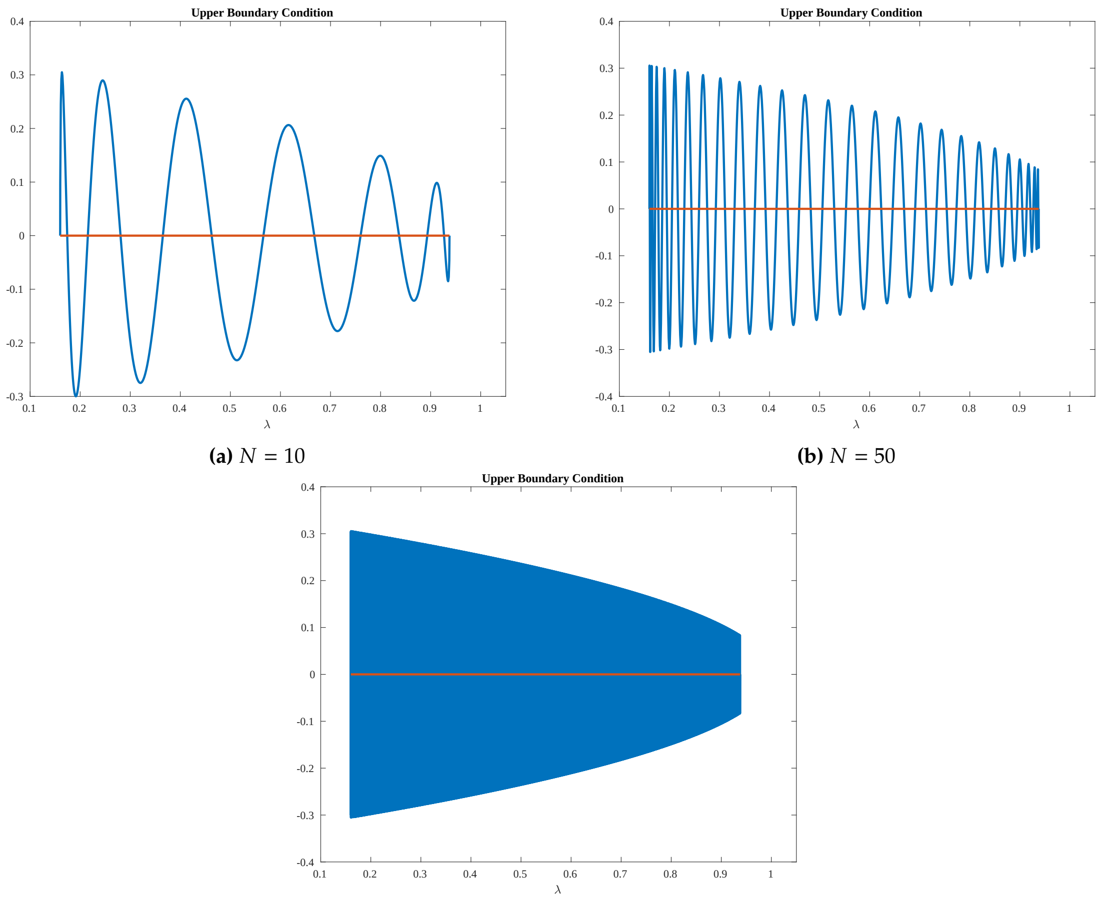
$$\sin(\theta(\lambda)) = \left(1 - \left(\frac{(\lambda - 1 + p_u + p_d)}{2\sqrt{p_d p_u}}\right)^2\right)^{1/2}$$

since $\theta(\lambda) = \cos^{-1}\left(\frac{(\lambda-1+p_u+p_d)}{2\sqrt{p_d p_u}}\right)$, so we can solve for $\omega(\lambda)$ as:

$$\omega(\lambda) = \arctan\left(-\frac{\left(\frac{\lambda-1+p_u-p_d}{2\sqrt{p_u p_d}}\right)}{\left(1-\left(\frac{(\lambda-1+p_u+p_d)}{2\sqrt{p_d p_u}}\right)^2\right)^{1/2}}\right) \tag{B.79}$$

We can then find all eigenvalues as solutions to the upper boundary condition (B.60) plugging in the above expressions for $\theta(\lambda)$ and $\omega(\lambda)$:

$$(\lambda - 1 + p_d)\cos(\theta(\lambda)N + \omega(\lambda)) - (\sqrt{p_u p_d})\cos(\theta(\lambda)(N-1) + \omega(\lambda)) = 0 \tag{B.80}$$



(a) $N = 10$

(b) $N = 50$

(c) $N = 2000$

Figure B.7: Roots of the Upper Boundary Condition

Roots of the upper boundary condition for $N = 10$, $N = 50$, and $N = 2000$. The red line indicates the interval $(\underline{\lambda}, \bar{\lambda})$

where $\theta(\lambda)$ and $\omega(\lambda)$ are given by (B.77) and (B.79), respectively. In figure B.7, we plot the left-hand side of (B.80) for increasing N. The eigenvalues are the points at which the left-hand side of (B.80) is equal to zero.

We see that as N grows, the eigenvalues successively fill out the entire interval $(\underline{\lambda}, \bar{\lambda})$. This shows that the second largest eigenvalue in a model with a finite grid approaches $\bar{\lambda}$ as the size of the grid grows.

In conclusion, an upper bound on the second largest eigenvalue of the operator $\mathbb{T}$ is given by $\bar{\lambda}$. To relate this to Gabaix et al. (2016) we compute the continuous time analogue of $\bar{\lambda}$:

$$\lim_{\Delta_t \to 0} -\frac{1}{\Delta_t} \log(\bar{\lambda})$$

and show that it is equal to $\frac{\mu^2}{2\sigma^2}$ which is the same value they obtain. To show this we first rewrite $\bar{\lambda}$ in terms of the annualized moments μ and σ using equations (B.31) and (B.32), and then apply L'Hôpital's rule. Specifically, we can rewrite $\bar{\lambda}$ as

$$\begin{aligned}
\bar{\lambda} &= 1 - p_u - p_d + 2\sqrt{p_d p_u} \\
&= 1 - \left(\frac{\Delta_t}{\Delta^2}\sigma^2 + \frac{\Delta_t^2}{\Delta^2}\mu^2\right) + 2\sqrt{\frac{1}{4}\left(\left(\frac{\Delta_t}{\Delta^2}\sigma^2 + \frac{\Delta_t^2}{\Delta^2}\mu^2\right) + \frac{\Delta_t}{\Delta}\mu\right)\left(\left(\frac{\Delta_t}{\Delta^2}\sigma^2 + \frac{\Delta_t^2}{\Delta^2}\mu^2\right) - \frac{\Delta_t}{\Delta}\mu\right)} \\
&= 1 - \left(\frac{\Delta_t}{\Delta^2}\sigma^2 + \frac{\Delta_t^2}{\Delta^2}\mu^2\right) + \sqrt{\left(\frac{\Delta_t}{\Delta^2}\sigma^2 + \frac{\Delta_t^2}{\Delta^2}\mu^2\right)^2 - \frac{\Delta_t^2}{\Delta^2}\mu^2} \\
&= 1 - \left(c\sigma^2 + c\Delta_t\mu^2\right) + \sqrt{(c\sigma^2 + c\Delta_t\mu^2)^2 - c\Delta_t\mu^2}
\end{aligned}$$

where $c = \frac{1}{2\sigma_{max}^2}$ is a constant. To use L'Hôpital's rule we need to compute $\frac{d}{d\Delta_t}\log(\bar{\lambda})$, which is given by

$$\frac{d}{d\Delta_t}\log(\bar{\lambda}) = \frac{-c\mu^2 + \frac{2(c\sigma^2 + c\Delta_t\mu^2)c\mu^2 - c\mu^2}{2\sqrt{(c\sigma^2 + c\Delta_t\mu^2)^2 - c\Delta_t\mu^2}}}{1 - (c\sigma^2 + c\Delta_t\mu^2) + \sqrt{(c\sigma^2 + c\Delta_t\mu^2)^2 - c\Delta_t\mu^2}}$$

Letting $\Delta_t \to 0$ we have

$$\frac{d}{d\Delta_t}\log(\bar{\lambda}) \to \frac{-c\mu^2 + \frac{2c^2\sigma^2\mu^2 - c\mu^2}{2c\sigma^2}}{1} = \frac{-\mu^2}{2\sigma^2}$$

So by L'Hôpital's

$$\lim_{\Delta_t \to \infty} -\frac{1}{\Delta_t}\log(\bar{\lambda}) = \lim_{\Delta_t \to \infty} -\frac{\frac{d}{d\Delta_t}\log(\bar{\lambda})}{1} = \frac{\mu^2}{2\sigma^2}$$

which is what we wanted to show.

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