Directional High Frequency Trading in the Kyle-Back Model

A thesis presented for the degree of Doctor of Philosophy

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it). The copyright of this thesis rests with the author. Quotation from it is permitted, provided that full acknowledgement is made. This thesis may not be reproduced without my prior written consent. I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

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Statement of co-authored work

I confirm that a version of all chapters was co-authored jointly with Professor Umut Çetin.

To my father, Professor Paulo Roberto Silveira Gomes

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Abstract

In traditional Kyle-Back models, the only source of information is a static or dynamic signal about the price of a risky asset received by the insider. We consider a more realistic version of the Kyle-Back model with a private and a public signal. The insider builds a linear combination of the public and private signals to make their valuation about the risky asset. The market maker uses the public signal and the total demand to set a linear pricing rule that is a martingale on their filtration. We show that any optimal strategy in equilibrium is such that the mispricing, the difference between the price process and the insider's valuation, converges to zero almost surely in the insider's probability measure. We introduce a particular linear admissible trading strategy to show the existence of equilibrium in the economy. Moreover, we use numerical analysis to show that the insider's ex-ante expected profit relies on the public signal and how this setting is able to explain a high-frequency trading pattern at the end of the trading period.

Contents

Introduction

In this thesis, we develop a Kyle-Back model with a public signal. In the traditional literature of Kyle-Back models, the only source of exogenous information is the insider. The seminal work Back (1992) presents the original Kyle-Back model (an extension for continuous time of Kyle (1985)) considering an insider who knows in advance the value, which will only be made public later, of a risky asset. Later, Back, Pedersen (1998) considers a setting in which the insider receives a dynamic signal throughout the trading period such that the uncertainty about the price of the asset will be known by the end of the trading period. However, in all those cases and all the Kyle-Back models so far, the market maker only learns about the price of the asset by trading with an insider (or several insiders as in Holden, Subrahmanyam (1992), Foster, Viswanathan (1996), and Back et al. (2000)). If it were not for the existence of someone with legal or illegal information, the market makers would not know anything about the asset that is being traded.

Therefore, we understand that we should fill this gap in the literature. The idea behind the setting of this model, that is introduced in Section 1.2, is that the market maker does receive exogenous information about the price of the asset throughout the trading period. In fact, as actual market makers, they have access to trading technologies, research divisions, and other relevant sources such that they are able to process a lot of information coming from outside the model that will be taken into account while they make their valuation of the asset. As a consequence of that, we suppose that instead of having all the information released at the end of the trading period, there is a signal that is public to all market participants so that the uncertainty about the true value of the asset considering only the public signal will be zero at the end of the trading period. That is, if we call X^M the public signal, as we do in Equation (1.1), we shall have $Var(V|\mathscr{F}_t^{X^M})$, where V is the price of the asset at the end of the trading period and $\eta \sim N(\mu, \sigma_V^2)$, to be such that $\lim_{t\to 1} Var(V|\mathscr{F}_t^{X^M}) = 0$.

There are quite a few interesting mathematical questions we had to deal with while expanding the Kyle-Back model to introduce a public signal. The first one is the issue of the insider's valuation of the asset. In the previous literature, the insider's valuation of the risky asset was either given by her dynamic private signal as in Danilova (2010) or completely inexistent[1], as in the original Back (1992), since the insider already knew the final value of the asset since the beginning of the trading period. Conversely, now the insider must take into account both her private signal and the public signal to make her valuation of the risky asset.

In Chapter 2 we study the insider's valuation of the risky asset. Since we make a lot of use of the theory of stochastic filtering, we take the opportunity to review some of its basic aspects; but, most importantly, we present some theoretical results that will be used throughout the thesis in section 2.1 to make it more accessible to the reader. In section 2.2, we conjecture that the linear combination of two Markov bridges with the same terminal condition could be written as a Markov bridge with the same terminal condition itself and show what the coefficients of such a linear combination should be for it to be true. This conjecture will greatly help us in proving the main theorem of the chapter, Theorem 2.3, in section 2.3. In this section, we discuss that this result is slightly more general than we need, but it is fundamental to us as it provides that $Z_t := \mathbb{E}[\eta|\mathscr{F}_t^I]$ can be written as $Z_t = \lambda_0(t)X_t^I + \lambda_1(t)X_t^M$, where $\lambda_0(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}$, $\lambda_1(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}$, $\Sigma_Z(t) = \int_0^t \sigma_Z^2(s)ds = \int_0^t \left(\left(\frac{1-\Sigma_Z(s)}{1-\Sigma_I(s)}\right)^2 \sigma_I^2(s) + \left(\frac{1-\Sigma_Z(s)}{1-\Sigma_M(s)}\right)^2 \sigma_M^2(s)\right) ds$, and Σ_M and Σ_I are given by equations (1.1) and (1.2) respectively.

In Chapter 3 we study the insider's optimisation problem. According to Definition 1.3, the insider's objective is to maximize her expected profit at the end of the trading period. We applied the dynamic programming principle to the value function of the insider, so we find the related Hamilton–Jacobi–Bellman equation. From that procedure, we are able to derive the value function for the insider under some assumptions on the price function. As a consequence, we can prove Theorem 3.1, which establishes the conditions for any admissible trading strategy to be optimal. The main condition turns out to be such that $\lim_{t\to1}(S_t - \sigma_V Z_t - \mu)^2 = 0$, $\mathbb{P}^z$-a.s..

That is in fact in line with the literature, as one can see from condition (ii) of Theorem

[1] Inexistent in the sense that one does not need to estimate the value of something they already know the value of. Alternatively, it is possible to call it trivial valuation

6.1 in Çetin, Danilova (2018). This condition is often referred to as saying that the insider makes a bridge so that the price converges to the true price of the asset at the end of the trading period. Let us consider the static case for a moment to ease the explanation. The idea is that the insider always buys when the price is below the actual price of the asset (i.e., the price that they know the price process is going to converge to) and sells every time the price is above it. Every time the insider buys, she drives the price up, and every time she sells, she drives the price down. Therefore, if by the end of the trading period the price is below the final value, it means that the insider could have bought more stocks at a cheap price during the trading period, and the other way around if the price of the asset reaches the end of the trading period and is above the final value. Therefore, it is in the insider's best interest - hence its optimal strategy - to drive the price to the final value of the asset.

However, we cannot say the same when dealing with a public signal. Indeed, condition $\lim_{t\to 1}(S_t - \sigma_V Z_t - \mu)^2 = 0$, $\mathbb{P}^z$-a.s. means that in equilibrium the price will be driven by the final value of the asset. But by whom? Now, if the insider did not trade, the price would still converge to the true value of the asset since the public signal X^I will be enough to do so. Therefore, in a way, the interpretation changes a lot as now what we can say is that it is optimal for the insider to not drive the price away from the true value of the asset at the end of the trading period.

Although we have proven in Theorem 3.1 for admissible trading strategies are such that the price would be driven to the final value of the asset, we restrict ourselves to a smaller class of trading strategies. As we do not claim that any equilibrium is unique, we allowed ourselves to consider only linear trading strategies as described in equation (4.4). Hence, from Chapter 4 onwards we consider only trading strategies of the form given by (4.4). We do not add this restriction to the class of admissible trading strategies in Definition 1.2 as we do for the linear pricing rules in Definition 1.2 because we wanted to make Theorem 3.1 more general.

In Chapter 4 we address the issue of the rationality condition for the market maker. As we compare the rationality condition in other Kyle-Back models, it becomes clear that our task is much more complex. In particular, it is interesting to note that as there was no non-trivial projection of η into the insider's filtration. Traditional Kyle-Back models did not have to concern themselves with the matter of how to project η into the market maker's filtration. Now we need to handle the fact that there is a projection of η directly into the market maker's filtration, and there is the projection of Z, the projection of η into the insider's filtration, into the market maker's filtration. In Chapter 4 not only do we calculate these projections, but we need to worry about how they can be combined together. The consequence of this is a much more complex system of ODEs for the variance of the price process than what is usually seen in the literature. As we point out in section 1.3, we need to develop substantial machinery to prove the existence and uniqueness of the system.

In Chapter 5 we show that for the particular pair of admissible trading strategy and rational pricing rule, we have that the condition that $\lim_{t \to 1}(S_t - \sigma_V Z_t - \mu)^2 = 0$, $\mathbb{P}^z$-a.s. is satisfied. The major tool we have used in this proof was the Doob's h-transform. Once again, in order to keep this thesis more self-contained, we present a very brief review of this tool in section 5.1.

Once we understand the conditions for an admissible pricing rule to be rational and have shown that our candidate trading strategy is optimal, we can add those solutions together and prove the existence of an equilibrium for our economy. That is done in Chapter 6.

We believe that the addition of a public signal will be able to expand substantially the application of the Klye-Back framework for a variety of situations. In particular, we believe that variations of the current model we are presenting will be useful to deal with toxic arbitrage (see Foucault et al. (2017)) and multiple insiders (see Back et al. (2000)). However, this initial version is already substantially robust to deal with high-frequency trading as presented by Foucault et al. (2016).

Motivated by a series of empirical findings from several authors, Foucault et al. (2016) concludes that directional high-frequency trading is done based on soon-to-be-released information. However, as the authors mention, the findings of Carrion (2013) suggest that "directional HFTs realize a large fraction of their profits on aggressive orders over relatively long horizons" (Foucault et al. (2016), p. 336). We believe our model can reconcile these two empirical findings.

As we explain in Section 1.1, directional high-frequency trading means that the insider's strategy is based on the long-term value of the asset, as it is in our and other Kyle-Back models. Therefore, the insider's strategy is applied over the entire trading period. Nevertheless, we are able to integrate that with the fact that the insider also trades on information that is about to be revealed.

Let us now consider what happens when time approaches the end of the trading period, which in our model means that t approaches one. The price process as given by (4.7) is

$$dS_t = \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)},$$

where $N^{(1)}$ is the innovation process in the filtration of the market maker related to the demand process and $N^{(2)}$ is the one related to the public signal. In Proposition 4.2 we know that ${}_{t\to 1} w(t) = 0$ and the numerical analysis developed in Chapter 7 show that ${}_{t\to 1}\beta_2(t) = 1$.

This means that insider trading becomes less influential in the price process when the end of the trading period is approaching. Hence, the market becomes more and more liquid towards the end of the trading period, converging to perfect liquidity at the end of it.

It is important to recall that one of the main features of the Kyle and Back framework is the so-called *feedback effect*. This means that the insider can affect prices when trading. Indeed, even in Kyle (1985) if the volume of the insider's trading was unable to affect the price, the insider's profit would be infinite. Obviously, that would not be a reasonable hypothesis; as for the insider to have finite but high enough profits, it would require a volume of trading that would affect the price process. However, some of the intuition remains in our case. As the market becomes more liquid the *feedback effect* declines, the insider is able to trade aggressively, in a high-frequency manner, towards the end of the trading period when the information about the asset is about to become public.

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Chapter 1

The Model

The study of Market Microstructure is dedicated to understanding, according to O'Hara (1998), "the desire to know how prices are formed in the economy" (p. 1). Therefore, the aim of this research field is to go beyond the traditional asset pricing theories that assume no liquidity risk and perfect competition and study more realistic models that describe the stylised facts that emerge from the real world in which bubbles and crashes seem to be more the rule than the exception. Embedded in the assumption of perfect competition is the assumption of symmetry of information. However, if one asks practitioners in the City of London or Wall Street, they most likely are going to say that asymmetry of information would be a much more realistic modelling choice.

The seminal work Kyle (1985) is a milestone in the development of the study of asymmetric information using market microstructure. Kyle develops a model in discrete time in which there is a single asset being traded by three different types of agents: market makers, noise traders, and an insider. The latter agent has perfect information about the value of the asset at the beginning of the trading period and exploits this advantage to maximize their expected profits as they are risk neutral. On the other end of the spectrum, the noise traders buy and sell their stocks non-strategically based on exogenous reasons. In his original model, noise traders' demand follows a random walk.

The market makers - like the insider - trade strategically. They observe the total demand and clear the market under a fair-price rule, considering that the uncertainty about the final value of the asset follows a normal distribution. These agents cannot observe directly the insider's demand as they cannot distinguish between the trades of the insider and the non-strategic trades of the noise traders.

The work of Kyle was extended to a more realistic continuous-time version by Back (1992). As should be expected, the discrete-time random walk trades of the noise traders become a Brownian motion, and the initial distribution of the asset can be extended to other than Gaussian distributions. The relevance of this extension is so great that since its publication, this paper, the research field is known as the Kyle-Back framework; hence the title of the present work.

Therefore, it is worth mentioning the equilibrium of the basic Kyle-Back framework aforementioned. The first one is that equilibrium is defined in a Nash-type equilibrium in which a pair of a trading strategy and pricing rule is optimal if, given the fair-pricing rule, the strategy is optimal and, given the pricing rule, the strategy is the one that maximises the insider's profit. It is shown that equilibrium is reached for a pair of a fair-pricing rule and a strategy that drives the price to be a Markov bridge converging to the value of the asset known in advance by the insider.

The price process is a martingale in the market maker's filtration since $\mathbb{E}(V|\mathscr{F}^M)$, where V is the price of the asset at the end of the trading period and $\mathscr{F}^M$ is the market maker's filtration, is a martingale. Furthermore, the demand in equilibrium is a Brownian motion which makes the insider's trading inconspicuous. As a consequence, it is not possible to know in equilibrium what the insiders' orders are or the noise traders' orders are.

So far, the motivation behind these models was the intuitive meaning of insider trading: a market participant who has information inaccessible to others via lawful means trading with the goal of taking advantage of this illegal advantage. However, in Back, Pedersen (1998) and later in Danilova (2010) we are presented with models in which the insider is fed their private information through a continuous Gaussian signal. This version became known as the dynamic information version of the Kyle-Back model. Besides the mathematical relevance of those improvements in the original Kyle-Back model, there is a relevant change in the interpretation of what an insider may be. Not only may the insider receive privileged information in a continuous fashion, but it could also be interpreted as genuine participants in the market that have either or both better access to information - in a lawful way - or can process the information publicly available faster than others. Therefore, insiders in this context could be seen as large investment banks with research divisions that update their valuation of the asset based on a large flow of information. Another advantage of this approach under a modelling perspective is the fact that we can consider a larger time horizon, as with a continuous stream of information, we allow the insider to use information that did not exist in the beginning of the trading period.

The takeaways of the model are quite similar. Taking into consideration Danilova (2010), all the considerations made above are contemplated, but one. The only major difference is that now the static Markov bridge is now a dynamic bridge (See Campi et al. (2011) and Çetin, Danilova (2018) for a comprehensive approach on dynamic Markov bridges). Danilova (2010) manages to find very mild conditions for the construction of the bridge. If V defines the cumulative volatility of the dynamic signal of the insider, the only major condition is that the insider's informational advantage,$V(t) - t$, to be positive. Therefore, as long as the insider has some informational advantage to share, she will exploit it.

In the following section, we are going to describe the model proposed by Foucault et al. (2016) which was a previous development of high frequency trading modelling using the Kyle-Back framework. In Section 1.2 we shall present the setting of our model, considering both a public and private signal that will be used throughout the thesis.

In section 1.3 we summarize the main challenges of adding a public signal to the Kyle-Back model introduced to our model and how we addressed those issues mathematically.

1.1 High Frequency in Kyle-Back

To the best of our knowledge, the greatest attempt to model high frequency trading so far was made by Foucault et al. (2016).

In their work, the insider receives a private signal, as is the norm in the literature, about the fundamental value of the asset. The market maker, on the other hand, receives a noisy version of the private signal that the authors named "news". As is required for a model from the Kyle-Back framework, there are also noise or liquidity traders. They buy and sell for liquidity reasons, and their cumulative demand is given by a Brownian motion.

Their analysis consists in comparing two flow-execution models: the fast and slow models. In an infinitesimal time interval $[t, t + dt]$ the market maker receives two signals about the value of the asset: the news flow and the insider order flow. The market maker knows that the total order flow is given by the sum of the insider's demand and the noise trader's demand. Therefore, it is a noisy version of the insider's demand hence it is informative of the insider's strategy and ultimately of their private signal.

A market maker is said to be fast if she updates her valuation of the asset before executing the order flow, and she is said to be slow if she updates her knowledge after executing the order flow. As the insider is the focal agent of this model, the model is said to be fast if the market maker is slow, and it is said to be slow if the market maker is fast. Therefore, the model is said to be fast if the insider is able to trade ahead of the news.

As a consequence, we get the concept of directional and non-directional trading in high-frequency models. Let us take a moment to introduce a toy example to understand the concepts of directional and non-directional trading. Suppose that the insider knows the true value of the asset and it is \$100. If the asset is being traded at \$110 the long-term knowledge of the insider would tell that she should reduce her inventory of the asset. However, if she knows that before converging to \$100 in the long term the price of the asset will go up in the short term - lets say to \$120 - the insider will trade non-directionally using her short-term information to increase her inventory now and than to sell in the near future when the price of the stock reaches \$120.

In traditional Kyle-Back models, the insider can only trade directionally as the only source of information is either their knowledge about the true value of the asset, as in Back (1992), or they receive a signal about the asset, as in Back, Pedersen (1998). However, in our model, all market participants also receive a signal about the value of the asset simultaneously that plays an analogous role to the news in the slow model of Foucault et al. (2016).

Returning to Foucault et al. (2016), in the slow model, there is only directional trading as the insider is unable to anticipate the news. Indeed, that is their Proposition 1. In the fast model, on the other hand, the insider is able to profit from both the news and the long-term information about the asset.

The conclusions of the paper claim that the increase in the trading volume as a consequence of the more aggressive trading of the insider derives from the fact that she is able to trade ahead of news. Therefore, they conclude that, in a purely directional model, there would not be an increase in the trading volume at any point.

Indeed, one should consider that the trading strategy we have found in equilibrium by Theorem 6.1 is a directional one, as it is neither correlated with the Brownian motion of equation (1.1) nor takes advantage of short deviations of X^M. Note that the optimal strategy we have in equilibrium is such that only considers the mispricing of the asset - that is, it is possible to write θ^* having as the only stochastic term $\sigma_V Z_t + \mu - S_t$, which is the difference between the insider's valuation of the asset, $\sigma_V Z_t + \mu$, and the market maker's valuation of the asset, S_t, at time t.

However, our findings show that there is an increase in the market's liquidity toward the end of the trading period, even though we have an equilibrium that is a purely directional one.

In Chapter 4, the price process in the filtration of the market maker is given by equation (4.7), reproduced below:

$$dS_t = \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)},$$

where $N^{(i)}$s are the innovations processes given by (4.2). In particular, $N^{(1)}$ is the innovation process that comes from the demand component. Furthermore, in Proposition 4.2 we show that w is such that $w > 0$ in $[0, 1)$ and $\lim_{t\to 1} w(t) = 0$. Therefore, as times approach one, the liquidity of the market increases such that it converges to a perfect liquid one at the end of the trading period.

The liquidity of the market is a keystone in Kyle-Back models. As we mentioned earlier, if the market were completely liquid at some point in $[0, 1)$, the insider would be able to make infinite profits. That is, if the insider was able to trade any amount of stocks without affecting the price, they would trade an infinite amount of it, making any mispricing enough to provide infinite profits. That is actually one of the reasons why it is so important for our model to show that $w > 0$ in $[0, 1)$. The amount of insider influence on the price is the *feedback effect*.

Therefore, the findings of this thesis that we mentioned above show that the *feedback effect* decreases as the end of the trading period approaches, allowing the insider to trade more aggressively.

One of the main stylised facts that the authors of Foucault et al. (2016) were trying to model is that high-frequency trading often occurs in soon-to-be-released news. As discussed above and much further in detail in Chapters 4 and 7, our model is a directional one that models this soon-to-be-released-news phenomenon as we see the increase in the volatility of the trading when the information is about to be released, which happens in time one.

1.2 Setting of the Model

Before proceeding to the description of the model, we establish that we are working in the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions of right continuity and $\mathbb{P}$ completeness. This probability space must contain three Brownian motions: B, B^M, and B^I.

In this model, we are considering a risky asset that is traded between moments 0 and 1 in an economy where the risk-free interest rate is set to be 0. At the end of the trading period, information about the true value of the asset, V, will be made public to market participants. V is supposed to be a Gaussian random variable, hence we write $V = \sigma_V \eta + \mu$ such that η is a standard Gaussian random variable. η and hence V is assumed to be independent of all Brownian motions B, B^M, B^I.

The hypothesis that V will be made public at time 1 is aligned with the theory since the early works of Kyle (1985) and Back (1992). The main innovation of this thesis is how information is distributed to the public. Instead of being presented at all at once at the end of the trading period, the following continuous bridge gives a public signal that converges to η:

$$X_t^M = \int_0^t \sigma_M(s) dB_s^M + \int_0^t \sigma_M^2(s) \frac{\eta - X_s^M}{1 - \Sigma_M(s)} ds \tag{1.1}$$

such that $\Sigma_M(t) = \int_0^t \sigma_M^2(s)ds$ and $\Sigma_M(1) = 1$. In the sake of generality one may be tempted to allow $\Sigma_M(t^\star) = 1$ for some $t^\star < 1$. However, note that if we allowed that, we would allow to have the full disclosure of the information about V before the moment we set to be the one of the release of the information $t = 1$. Hence, we require that $\Sigma_M(t) < 1 \; \forall \; t \in [0, 1)$. Furthermore, the precision of the public signal is given by $\frac{1}{1-\Sigma_M(t)}$. As discussed before, it will become perfectly accurate at the end of the trading period.

The insider, on the other hand, has access not only to the public information but also to their own stream of information. Since Back, Pedersen (1998), it is traditional in the literature to allow for a dynamic private signal. Not only could the extension of the model developed by Danilova (2010) be cited, but also Campi et al. (2013) that extends the model of a private dynamic signal for the trading of a risky asset that may default show that the seminal work of Back, Pedersen (1998) established the dynamic signal as the benchmark for Kyle and Back models. As discussed previously, the dynamic signal is also fundamental for the high-frequency behaviour we aim to model.

Therefore, we can define the private signal, X^I, given by the continuous bridge also converging to η as follows:

$$X_t^I = X_0^I + \int_0^t \sigma_I(s)dB_s^I + \int_0^t \sigma_I^2(s)\frac{\eta - X_s^I}{1-\Sigma_I(s)}ds \tag{1.2}$$

where $\Sigma_I(t) = c^2 + \int_0^t \sigma_I^2(s)ds$, $\Sigma_I(1) = 1$. It is also important to point out that B^I is a Brownian motion that is independent of B^M. Moreover, we require that $\eta \sim N(0,1)$ can be expressed as the sum of two independent random variables such that $\eta = X_0^I + \eta_1$ where X_0^I is a $\mathscr{F}_0$-measurable random variable such that $X_0^I \sim N(0, c^2)$.

Often in the literature, we may want to separate the static and dynamic cases as is done in Çetin, Danilova (2018). The so-called static case being the one in which the insider has perfect information about the asset that is being traded, and the dynamic case being the one in which the insider receives the information throughout the trading period (see Chapter 7 of Çetin, Danilova (2018)). One of the roles of c in this setting is to allow the static case to be a particular case of our model when $c = 1$. All the results we present in the thesis hold without any problem in the case when $\Sigma_I(0) = 1$. Furthermore, we also allow the private signal to converge before one, even if $c < 1$. There is nothing that prevents us from having an explosion in precision of the private signal $\frac{1}{\Sigma_I(t)}$ at any point in the trading period $t \in [0,1]$. On the other hand, we ask c not to be set at zero, as it would create a situation at $t = 0$ in which the insider does not have a better signal than the insider.

We shall make some regularity assumptions about the functions σ_I and σ_M:

Assumption 1.1. *The functions* $\sigma_I : [0,1] \to \mathbb{R}_+$ *and* $\sigma_M : [0,1] \to (0,\infty)$ *are continuous.*

It is important to realise that X^I is not the insider's signal. If that were the case, it would be natural to demand $\Sigma_I(t) \geq \Sigma_M(t)$ $\forall t \in [0,1]$. In fact, as will be discussed in Chapter 2, we are able to find the optional projection of η into the insider's filtration. In fact, the main theorem of the chapter, Theorem 2.3, shows that it is given by a linear combination of both signals. The precision of the insider's signal will be given by $\frac{1}{\Sigma_Z(t)}$ where, as expected, as long as $c > 0$, we have $\Sigma_Z(t) \geq \Sigma_M(t)$ $\forall t \in [0,1]$ as given by Corollary 2.1.

We can now proceed to describe the microstructure of the market, that is comprised of three different types of agents: noise/liquidity traders, market makers and the informed trader:

- *Noise/liquidity trade* trade for reasons that are exogenous to the model. Their cumulative demand is given by a standard $(\mathscr{F}_t)$-Brownian motion B independent of both B^M and B^I.

- *Market makers:* observe the total demand given by

$$Y = \theta + B, \tag{1.3}$$

where θ is the demand of the informed trader, and also observe the public signal given by (1.1). As it is the tradition in the literature of Kyle-Back model, see for example Back (1992), Campi, Cetin (2007), and Campi et al. (2013), the market maker uses the information from the demand process to set their price process, but as they have access to a public stream of information they also incorporate it to the price process, S, such that

$$S(Y_{[0,t]}, X^M_{[0,t]}, t) = H(t, X_t, X^M_t) \quad \forall\, t \in [0,1) \tag{1.4}$$

where X is the unique strong solution to

$$dX_t = w(t)dY_t + \left(r_0(t) + r_1(t)X_t + r_2(t)X^M_t\right)dt \quad \forall\, t \in [0,1) \tag{1.5}$$

for some deterministic functions w and $(r_i)_{i=0}^2$ that will satisfy the admissibility conditions specified in Definition 1.1. Furthermore, we shall require that, in equilibrium, we must have a pricing rule that given the optimal trading strategy for the insider will be a so-called rational pricing rule, i.e. $H(t, X_t, X^M_t) = \mathbb{E}[V|\mathscr{F}^M_t]$.

It is interesting to notice that the structure of X allows the market makers to incorporate not only the contemporary levels of the demand and the public signal, but also the cumulative level of the demand without losing its Markov property. In a modelling point of view, one should consider that the market maker must take into consideration two signals. The cumulative level of any of those signals must be relevant to weight the level of each of the signals as their own uncertainty about the final value of the asset vanishes.

Furthermore, the above definition of X allows more flexibility to the model so we can address two issues separately. One should note that equations (1.8) and (1.5) could be combined so at the end of the day we shall have a pricing rule that indeed satisfies equation (1.4). However, such separation allow us to first solve the HJB equation of the insider setting $(r_i)_{i=0}^2$ for any set of $(\beta_i)_{i=0}^2$ according to Assumption 3.2. After that, we can both find the values of $(\beta_i)_{i=0}^2$ that satisfy the rationality condition Chapter 4 henceforth the correspondent values of $(r_i)_{i=0}^2$.

Moreover, a pricing rule H must also be admissible in the sense of definition 1.1. This pricing rule is enforced via a *Bertrand competition:* Market makers are willing to buy or sell any quantity offered to them under their pricing rule. Furthermore, one can note that the market maker's filtration, which shall be denoted by $\mathscr{F}^M$, the minimal right-continuous and complete filtration generated by the demand, Y, and the public signal X^M.

- *The informed investor:* observes the price process S, given by equation (1.4), the public signal, X^M, given by equation (1.1), and a private signal, X^I, given by equation (1.2). As we assume the insider to be risk-neutral, her objective is to maximize her expected final wealth

$$\sup_{\theta \in \mathcal{A}(H)} \mathbb{E}^z[W_1^\theta] \tag{1.6}$$

where

$$W_1^\theta = (V - S_{1^-})\theta_{1^-} + \int_0^{1^-} \theta_{s^-} dS_s, \tag{1.7}$$

$\mathcal{A}(H)$ is the set of admissible strategies for the given pricing rule H as defined in Definition 1.1. $\mathbb{E}^z$ is the expectation with respect to $\mathbb{P}^z$, which is the regular conditional distribution of $(B_s, X_s^I, X_s^M; s \leq 1)$ given $B_0 = X_0^M = 0$ and $X_0^I = z$.

The existence of the probability measure $\mathbb{P}^z$ is ensured by Theorem 44.3 in Bauer (1996). Note that $\mathscr{F}$ must be rich enough to contain all the null sets of $\mathscr{F}^I$. The insider filtration $\mathscr{F}^I$ is the universal completion of the filtration generated by X^I, X^M and S. Hence, the cited theorem guarantees the existence and uniqueness of $\mathbb{P}|\mathscr{F}^I$. Section 2 of Cetin, Danilova (2021) provides a detailed discussion of the expectation operator in this case and the filtration $\mathscr{F}^I$.

We can now proceed to set the requirements for equilibrium. Hence, as mentioned above, we must define both the set of admissible pricing rules and the set of admissible trading strategies. The first will be defined in Definition 1.1 and the latter in Definition 1.2. When considering the price process in equilibrium, one should note that $H(t, X_t, X_t^M) = \mathbb{E}[V|\mathscr{F}_t^M]$ is a projection of a Gaussian random variable into the market maker's filtration that is comprised of Gaussian processes X and X^I (provided that Y is Gaussian, which will be true due to the fact that in equilibrium α will be defined by (4.4) as a linear equation of X, X^M and Z). As will be clarified in Chapter 2, such a projection is a linear one, ultimately leading to the fact that $\mathbb{E}[V|\mathscr{F}_t^M]$ must be a Gaussian process itself.

In a mathematical point of view, it is important to point out that we do not claim that there would not be a more general set of pricing rules and strategies that would also lead to other equilibria. The main focus of our research is to show the existence of an equilibrium.

Definition 1.1. *An* admissible pricing rule *is any quintuple* (H, w, r_0, r_1, r_2) *fulfilling the following conditions:*

1. *w and $(r_i)_{i=0}^2$ are continuously differentiable real valued functions defined on* $[0, 1]$. *Moreover, w takes values in* $(0, \infty)$.

2. $$H(t, x, x_1) = \beta_0(t) + \beta_1(t)x + \beta_2(t)x_1, \tag{1.8}$$

   where $\beta_i \in C^1([0, 1))$ *for* $i = 0, 1$ *and* 2.

3. *$x \mapsto H(t, x, u)$ is strictly increasing for every $t \in [0, 1)$ and $u \in \mathbb{R}$. That is, $\beta_1 > 0$ on* $[0, 1)$.

We will write (H, w, r) in short to denote (H, w, r_0, r_1, r_2).

One of the key features in the literature regarding the insider's optimal strategy is that she can invert the pricing rule to know exactly the value of the demand Y at it, which is explained in Section 6.2 of Çetin, Danilova (2018). That will also be the case in our model. As a consequence of Proposition 4.2, the value of w never vanishes (note that according to the previous definition w takes positive values). Therefore, since we also require β_1 to be positive, the insider can invert the price process to know exactly the value of Y_t at any given moment $t \in [0, 1]$. As given by the sum of the insider strategy and the demand process for noise traders, the insider has perfect knowledge of $(B_s)_{s \leq t}$ at any $t \in [0, 1]$.

Since Back (1992) it is very well established in the literature (see, e.g. Campi et al. (2013), Cho (2003), and Wu (1999)) that the optimal strategy for the insider should be one that is absolutely continuous with respect to the Lebesgue measure, as martingale and jump components in trading strategies are known to be suboptimal. There is no reason to believe that the insider would profit from being able to perform jumps in their strategy, as is clear from the proof of Theorem 6.1 in Çetin, Danilova (2018). However, with the introduction of a public signal X^M it is not possible to be sure that a nontrivial martingale part is still suboptimal. Indeed, one could conjecture that by correlating their strategy to the public signal (or the private one), the insider would be able to filter the flux of information that the market maker gets. Therefore, in the admissible trading strategies, we have strategies of the following form:

$$d\theta_t = \alpha_t dt + \gamma_0(t) d\beta_t^I + \gamma_1(t) d\beta_t^M$$

Note that the Brownian motions above are not the ones we define in equations (1.2) and (1.1). That is because the bridges are defined in a filtration that considers the knowledge of η. Indeed, each of the bridges is a martingale in their own filtration. For the insider, as long as $c < 1$, they do not observe η. As a consequence, they also do not observe neither B^I nor B^M (i.e., B^I and B^M are not $\mathscr{F}^I$-adapted) that are defined in a filtration enlarged with $\sigma(\eta)$. Therefore, the insider's strategy must be defined with respect to the innovation processes in the insider's filtration, namely β^I nor β^M. As will be clear in Chapter 2, β^I is the innovation process with respect to B^I and β^M is the one with respect to β^M.

Furthermore, we write α_t to keep the notation simple. In fact, α should not only be considered a function of time. We allow the insider to use all the information available to her in order to build her strategy. Therefore, we should consider that each admissible strategy α is potentially a function of t, X, X^I, X^M and Z. As we shall see in Theorem 3.1, any strategy that drives the mispricing to zero at time one is an optimal strategy. In Chapter 4, we restrict ourselves to a particular class of linear trading strategies given by Equation (4.4). Since Z is a linear combination of X^M and X^I, we write α as a linear combination of t, X, X^M and Z with coefficients that depend on time. At the expense of being repetitive, our main theorem, Theorem 6.1, shows the existence of an equilibrium that does not claim uniqueness.

Definition 1.2. *A continuous $\mathscr{F}^I$-semimartingale θ is said to be an admissible trading strat-*

egy for a given pricing rule (H, w, r) if it satisfies the following:

1. *It has an $\mathscr{F}^I$-Doob-Meyer decomposition given by*

$$d\theta_t = \alpha_t dt + \gamma_0(t) d\beta^I_t + \gamma_1(t) d\beta^M_t, \tag{1.9}$$

   where the first term is the finite variation component, β^I and β^M are the Brownian motions from Theorem 2.3 , and γ_0 and γ_1 are $\mathscr{F}^I$-predictable processes.

2. *There exists a unique strong solution to (1.5) on $[0, 1)$.*

3. *The following integrability conditions hold to rule out doubling strategies:*

$$\mathbb{E}^z \int_0^1 H^2(s, X_s, X^M_s) ds < \infty, \tag{1.10}$$

$$\mathbb{E}^z \int_0^1 (H^{-1}(s, \sigma_V Z_s + \mu, X^M_s) - X_s)^2 \frac{\beta^2_2(s)\sigma^2_M(s) + \sigma^2_Z(s)}{w^2(s)} ds < \infty, \tag{1.11}$$

   where $H^{-1}(t, z, u)$ is the unique solution of $H(t, x, u) = z$ and X is the unique strong solution of (1.5).

The set of admissible trading strategies for a given (H, w, r) is denoted by $\mathcal{A}(H, w, r)$.

Once we have established what are the requirements for both a price rule and an admissible strategy, we can now define the equilibrium. That will be a Nash-type of equilibrium in which a pair of admissible strategy and an admissible pricing rule is an equilibrium if at the same time given the pricing rule the insider's strategy is optimal and given that strategy the pricing rule satisfies a rationality rule. The rationality rule given by equation (1.12) guarantees that the market makers are trading at a price that is exactly their valuation of the risky asset. Obviously, one should recall that the fairness is of the market maker's on making. By the Bertrand competition hypothesis we have set previously, implies that the perfect competition among market makers is such that if anyone is willing to sell at a discounted price, the demand will be infinite for the risky asset as well as if someone is willing to buy at a premium price. Therefore, one can understand the Bertrand competition hypothesis as the output of perfect competition among indefinitely many equally informed risk-neutral market makers.

We decided to label the equilibrium as a Nash-type equilibrium because in the original paper Nash (1951) a (Nash) equilibrium is defined as one in which all agents maximise their payoffs. Therefore, an equilibrium is reached if given the other players' best strategies, every player is maximising their own payoff. As a consequence, we should not call the equilibrium in our model a proper Nash equilibrium. In this game, on the other hand, only the insider is maximising their payoff while the market maker's requirement is only that the price follow their valuation of the price of the asset. The justification for naming it a Nash-type equilibrium comes from the fact that there is no incentive for any of the players to deviate from their strategy. From the point of view of the insider, she is maximising her payoff and, regarding the market maker's perspective, they should not deviate from their strategy as they are trading the risky asset at a fair price.

A final remark about the rational pricing rule is that as the price is enforced in the market by the market makers at that level, the liquidity traders also buy and sell the risky asset at a fair price. By that, we mean that they always trade the asset at the proper valuation of the asset given the information that is available to the public. That is the reason why they are known as liquidity traders: they do not trade strategically; instead, they buy and sell the risky asset as a consequence of their consumption and investment decisions. Furthermore, they are also known as noise traders, as if it weren't for their existence, the market makers would trivially identify who the insider is since they would be the only other agent they would be trading with. As can be easily observed from equation (1.3), if there was no demand coming from the noise traders, the market makers would be able to observe θ directly. As is the case in the literature, we shall see that the insider is able to trade inconspicuously because the market maker cannot identify whether the orders come from the insider or noise traders.

Without further remarks,, we are now able to define the equilibrium in our economy:

Definition 1.3. *A couple* $((H^*, w^*, r^*), \theta^*)$ *is said to form an equilibrium if* (H^*, w^*, r^*) *is an admissible pricing rule,* $\theta^* \in \mathcal{A}(H^*, w^*, r^*)$*, and the following conditions are satisfied:*

1. Market efficiency condition: *given* θ^**,* (H^*, w^*, r^*) *is a rational pricing rule, i.e.*

$$H^*(t, X_t, X_t^M) = \mathbb{E}[V|\mathscr{F}_t^M]. \tag{1.12}$$

2. Insider optimality condition: *given* (H^*, w^*, r^*)*,* θ^* *solves the insider optimisation problem for all* z*:*

$$\mathbb{E}^z[W_1^{\theta^*}] = \sup_{\theta \in \mathcal{A}(H^*, w^*, r^*)} \mathbb{E}^z[W_1^\theta] < \infty.$$

It is worth pointing out the language of Definition 1.3 that only defines what is a possible equilibrium. In the main theorem of the thesis, Theorem 6.1, we show that there exists an equilibrium, but do not claim the uniqueness for it.

1.3 Challenges and Contributions

Adding a public signal to the Kyle-Back model introduced several additional challenges to it. In this section, we will make a summary of the main obstacles and the innovations we present to deal with them. As a summary, we expect it to be supplemented by the remarks presented throughout this thesis, in particular, at the beginning of each chapter.

In Chapter 2, we introduce the insider's valuation of the risky asset. In the literature on Kyle-Back models, we have two types of information delivery: static and dynamic. In the static case, as presented in Kyle (1985) and Back (1992), the insider observes the value of the asset at the beginning of the trading period. Therefore, there is no point in talking about the insider's valuation of the risky asset, as she already knows the value of it from $t = 0$.

The dynamic case was first presented by Back, Pedersen (1998) and later extended by Danilova (2010). In this case, the insider receives a dynamic signal as described by X^I in Equation 1.2. However, since there is obviously no public signal, we can interpret η solely as the distribution of X_1^I. In our case, we have two bridges given by 1.1 and 1.2 converging to the same final condition; hence, there is a bridge component taking place here. However, if there were no public signal, all we would be saying is 1.2 would be converging to some final condition X_1^I that we are calling η. Hence, if there was no public signal 1.1 could have been written as

$$X_t^I = X_0^I + \int_0^t \sigma_I(s)dB_s^I + \int_0^t \sigma_I^2(s)\frac{X_1^I - X_s^I}{1 - \Sigma_I(s)}ds$$

which in turn is just a bridge representation of the process

$$X_t^I = X_0^I + \int_0^t \sigma_I(s)dB_s^I.$$

Therefore, if there were no public signals, we would have a special case of the model of Danilova (2010) in which the only signal the observer inside is the process X^I as described above.

Furthermore, the main theorem of Chapter 2, Theorem 2.3, is more general than the application we present in this context. In fact, it works for every two martingales that share the same final condition. It is interesting to note that the $\mathscr{F}^I$-decompositions of X^I and of X^M are dynamic Markov bridges (see Chapter 5 of Çetin, Danilova (2018)).

In our particular case, Theorem 2.3 gives us that $Z_t := \mathbb{E}^z[\eta|\mathscr{F}_t^I]$ is given by $Z_t = \lambda_0(t)X_t^I + \lambda_1(t)X_t^M$, where $\lambda_0(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}, \quad \lambda_1(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}$. Hence, the insider's valuation of the risky asset that is being traded is a linear combination of both the public signal and the private signal instead of just the private signal as in the previous literature.

Chapter 3 follows the literature on Kyle-Back models more closely. We have a more complex task, but we do not use technology that is far from what has been done previously. The structure of the value function given by (3.53) is not far from what is usually done in the literature, as can be seen from equation (6.30) of Çetin, Danilova (2018). Obviously, there are additional challenges to the model while incorporating an extra signal. Firstly, as we allow the insider to correlate her strategies with the Brownian motions coming from the signals, the insider's optimisation problem now involves not only the α component as it is the norm in the literature (as, for example, in condition (ii) of Theorem 7.1 of Çetin, Danilova (2018)), but also the γ_0 and γ_1 components. That particular task is simplified by the fact that we find it suboptimal for the insider to correlate their strategies. Furthermore, the value function for the insider that used to be a function of time, the demand, and either the value of the asset, in the static case, or the insider's signal, in the dynamic case, now is a function of time, the demand, the public signal, and Z, the linear combination of the public and private signals. As Z is a linear combination of X^I and X^M, knowing any two of the three is equivalent to knowing all three.

One can note that in the model presented in Çetin, Danilova (2018), there is an analytical solution to the value function, as both H and w in there do not follow very restrictive functional forms. From equation (3.53) it may seem that we have a very clear closed form for our value function as well, but it depends on w which will have its form discussed later in Chapter 4. Indeed, after quite some work we are able to prove the existence and uniqueness of w in Theorem 4.4.

The main theorem of Chapter 3, Theorem 3.1, shows the conditions for the optimality of a given trading strategy. From Chapter 4 forward, we concentrate on showing the existence of an equilibrium for the model in terms of Definition 1.3. In particular, we concentrate our efforts on showing the existence of equilibrium for a particular functional form of α. We have left Theorem 3.1 in a more general form because it could be used for other models in this research agenda in generalizations of this model.

Chapter 4 has imposed several mathematical difficulties. As we mentioned previously in this section, even in the dynamic information setting, there is no projection of η into the insider's filtration. The insider would observe X^I and the realization of η at the end of the trading period would be just X^I_1. Therefore, while dealing with the projection of η into the filtration of the market maker, we are left with two projections: the projection of η into the filtration of the market maker and the projection of Z_t, which is itself the projection of η into the insider's filtration into the filtration of the market maker. Obviously, those two projections must coincide in some sense. Therefore, we had to go deeper into the theory of stochastic filtering than is usually necessary in the literature. Furthermore, the ODE system given by (4.29) resulting from the combination of both projections was much more complex than what we would normally encounter; see, for example, equation (3.30) in Campi et al. (2011).

In fact, a lot of effort was necessary to prove the existence and uniqueness of the system given by (4.29). Firstly, we had to show the existence and uniqueness of a modified version of the system given by (4.1), which would be equivalent to the original system if a particular condition of the initial condition of the function w was satisfied. This condition, given by equation (4.33), would be satisfied if equation (4.37) held. However, since we do not have a closed form for w, we had to use a fixed-point algorithm to prove the existence of a solution (4.29) for an initial condition given by (4.37). The use of all this technology mentioned in this paragraph is necessary because we have a public signal, and none of it has been used in the previous literature.

The use of Doob's h-transform is not new in the literature to prove that a given trading strategy is indeed optimal. As it was the case in Chapter 3, the fact that the market maker observes a public signal makes this task much more complex. Even if we did not have to use different mathematical tools, we certainly had to use the same ones more heavily.

Once we have found all the conditions for a pair of strategies to be optimal, all that is left to do is to add them together to prove the equilibrium. That is basically what we do in Chapter 6. However, some of the findings of the model need additional research beyond the analytical results.

The fact that we only had existence and uniqueness results, but not a closed form for v

and w as defined by (4.29) did not allow us to know some things that were quite important for our analysis. The first was about the ex-ante value of information, i.e., the $\mathbb{E}(W_1)$ in equilibrium. An interesting fact about this quantity is that, as is the case in previous models in the literature, it does not depend on the signal X^I, not even if there is a signal X^I or we are dealing with the so-called static case. Therefore, we are interested to know if that is also the case for X^M. Furthermore, for the high-frequency trading that motivates the name of this thesis, we need to understand the behaviour of the price process, that is given by equation (4.7). As a consequence, we need to understand the behaviour of β_2, which itself depends on v as given by equation (4.26). Therefore, we had to use an ODE solver to find a numerical approximation of the functions v and w and, hence, of the quantities we were interested in studying. In order to do so, we developed a code in `R`. Although the code developed was neither computationally nor mathematically difficult, since the purpose of the code is to find approximations for the functions given by (4.29), it does not rely on the previous literature.

Chapter 2

Stochastic Filtering

This chapter has two main purposes. The first is to introduce the technology used in the thesis regarding the Theory of Stochastic Filtering. Such task is developed in Section 2.1. In this section, we briefly explain the main ideas behind the Theory of Stochastic Filtering and summarise the main results of Khalil (2002) that are most commonly used throughout the thesis. That is particularly important for us because we deal with quite a few projections of η (the stochastic part of V). In Section 2.3 we do the projection of η into the insider's filtration. We define such projection to be called Z. In Chapter 4, we both project η directly into the market maker's projection and project Z into the same filtration. As a consequence, we see the need to expose the reader to the collection of results in stochastic filtering so that we can clearly show the innovations we develop in this thesis.

The second purpose is to prove our main contribution we have made in this chapter, Theorem 2.3. It is a well-known fact that for Markov bridges such as the ones represented in equations (1.1) and (1.2), we have $X_t^M = \mathbb{E}\left(\eta|\mathcal{F}_t^{X^M}\right)$ and $X_t^I = \mathbb{E}\left(\eta|\mathcal{F}_t^{X^I}\right)$. One could find the projection of η into the filtration containing both X^M and X^I using the theory that we present in Section 2.1. Our contribution is to show that this projection $\mathbb{E}\left(\eta|\mathcal{F}_t^{X^M,X^I}\right)$ is a linear combination Z, as given by the equation (2.11) of X^M and X^I. Furthermore, the structure of this linear combination is such that $Z_t = \mathbb{E}\left(\eta|\mathcal{F}_t^{X^M,X^I}\right) = \mathbb{E}\left(\eta|\mathcal{F}_t^{Z}\right)$. The first step we take is to find the coefficients of the linear structure that return a Markov bridge. Hence, in section 2.2 we find the coefficients λ_0 and λ_1 such that $Z_t = \lambda_0(t)X_t^I + \lambda_1(t)X_t^M$ is a Markov bridge. The linear structure of the optional projection given by Theorem 2.1 will ensure to us that $Z_t = \mathbb{E}\left(\eta|\mathcal{F}_t^{X^M,X^I}\right)$ which is developed in section 2.3. Indeed, from a mathematical point of view, one could claim that the role of section 2.2 is almost a motivating one except for the form of Σ_Z that must coincide with the variance of the projection found in the following section. Apart from that, one could skip it and only read 2.3 ignoring where the values of λ_0 and λ_1 are coming from.

As it is the norm in applied mathematics, Theorem 2.3 is true for any two Markov bridges with the same terminal condition, but our motivation is to find the insider's valuation of the risky asset. Once we are able to do it under the conditions explained here, $\sigma_V Z + \mu$ will be equivalent to the insider's signal about the final value of the asset.

Unlike most chapters in this thesis, there is no counterpart to this chapter in the literature. As one can see in Chapters 6 and 7 of Çetin, Danilova (2018), the traditional Kyle-Back models with dynamic information are such that the insider observes a martingale such that its state at $t = 1$ will determine the price of the final value of the asset. In order to keep the same notation we are using here, it would be the case that there is no public signal so X^I would be a martingale $dX_t^I = \sigma_I(t)dW_t^I$ such that the final price of the asset would be given by $f(X_1^I)$ for some function f. Therefore, since X^I would be the only source of information for the insider, it would be obvious that $X_t^I = \mathbb{E}\left(X_1^I|\mathcal{F}_t^{X^I}\right) = \mathbb{E}\left(X_1^I|\mathcal{F}_t^I\right)$ where $\mathcal{F}^I$ stands for the insider's filtration so there would not be a point in talking about the insider's projection of an η that would be just X_1^I. On the other hand, once the insider has two sources of information, it makes sense to talk about the insider's filtration and her valuation about the final value of the asset that is being traded.

It is also interesting to note that often in the literature there are significant differences between the settings of the case when the insider has perfect foreknowledge of the value of the asset or when she receives a signal that converges to it, as explained in Çetin, Danilova (2018). However, as it is also the case in the literature, one can understand the static case as a particular case of the dynamic signal. In fact, the very last corollary of this section, Corollary 2.2, states that if any of the signals converge to η also will Z, the signal given by the linear combination of the signals. Obviously, the interesting case is when the private signal converges to η before one being the static case when $Z_t = \eta$ for all t.

2.1 Review of Stochastic Filtering

Before anything else, it is necessary to understand the problem it is addressing. Suppose that one is working in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and wants to study a process Y that is adapted to the filtration $\mathcal{F}_t$ under a smaller filtration that is generated by some $\mathcal{F}_t$-adapted processes. Such processes are denoted by X and are known as the observational processes. The intuition here is quite obvious: we consider agents that do not have access to $\mathcal{F}_t$ and therefore must estimate Y with the information available to them.

The process of interest Y is called the signal process. Therefore, our aim is to find $\mathbb{E}\left(f(Y_t)|\mathcal{F}_t^X\right)$, where $\mathcal{F}_t^X$ is the minimal filtration generated by X and f is an $\mathcal{F}_t^X$-measurable function.

Before proceeding with any calculation, it is important to address the issues of existence and uniqueness. Beginning with the latter, as an immediate consequence of Kolmogorov's theorem on conditional expectations (see Williams (1991) Theorem 9.2) is defined almost surely. However, it is possible to define the optional projection of $(f(Y_t))_{t\in[0,T]}$:

Definition 2.1. *Let Y be an $(\mathcal{F}_t)$-adapted integrable process. The $(\mathcal{F}_t^X)$-optional projection of Y is an $(\mathcal{F}_t^X)$-optional process, ${}^oY$, such that for any $(\mathcal{F}_t^X)$-stopping time τ,*

$$\mathbb{E}\left(Y_\tau|\mathcal{F}_\tau^X\right) =^o Y_\tau.$$

Theorem IV.5.5 of Revuz, Yor (2004) guarantees uniqueness up to indistinguishability. The role of indistinguishability is very-well discussed by Karatzas et al. (1988) in page its section 1.1. The proof of the existence of the optional projection can be found in Theorem VI.7.1 of Rogers, Williams (2000). As is standard, in the literature, we denote the optional projection of a process Y as $\hat{Y}$. Furthermore, throughout this thesis, every time we refer to the conditional expectation of a process given the minimal filtration of an observation process, we are referring to its optional projection. Hence,

$$\mathbb{E}\left(f(Y_t)|\mathcal{F}_t^X\right) =^o f(Y_t) = f(\hat{Y}_t)$$

for any measurable function f.

We can now introduce the so-called innovation approach to filtering. We shall now assume that the observation process is of the form

$$X_t = \int_0^t h_s ds + W_t,$$

where W is an $\{\mathcal{F}_t\}$-Brownian motion in $\mathbb{R}^n$, $W_0 = 0$, and h is an $\{\mathcal{F}_t\}$-adapted process with values in $\mathbb{R}^n$ such that

$$\mathbb{E}\int_0^1 |h_s|^2 ds < \infty.$$

The innovation approach is named this way because of the definition of innovation process, presented here in Definition 2.2 that is a consequence of Theorem 2.1 first published at Fujisaki et al. (1972). A proof of the following theorem can be found in Rogers, Williams (2000) as Theorem VI.8.4.

Theorem 2.1 (Fujisaki, Kallianpur and Kunita)**.**

- *The process*

$$N_t \equiv X_t - \int_0^t \hat{h}_s ds \tag{2.1}$$

  is a $\mathcal{F}_t^X$-Brownian motion in $\mathbb{R}^n$.

- *If Z is an L^2-bounded $\mathcal{F}_t^X$-martingale, $Z_0 = 0$, then there exists a $\mathcal{F}_t^X$-previsible process $C = (C^1, \ldots, C^n)$ such that:*

$$\mathbb{E}\left[\int_0^1 \sum_{i=1}^n (C_s^i)^2 ds\right] < \infty,$$

  and such that:

$$Z_t = \int_0^t (C_s, dN_s) \equiv \int_0^t \sum_{i=1}^n C_s^i dN_s^i.$$

We are now ready to define the innovation process as follows:

Definition 2.2. *The $\mathcal{F}_t^X$-Brownian motion N defined by the equation* (2.1) *is called the innovation process.*

It is interesting to note that in our model we are going to work with two Markov bridges. For example, the public signal is a bridge given by:

$$X_t^M = \int_0^t \sigma_M(s) dW_s^M + \int_0^t \sigma_M^2(s) \frac{\eta - X_s^M}{1 - \Sigma_M(s)} ds$$

where W is an $\mathcal{F}_t$-Brownian motion. All agents observe X_t^M, but not all observe W and η. Hence, Theorem 2.1 says that

$$X_t^M = \int_0^t \sigma_M(s) dN_s^M + \int_0^t \sigma_M^2(s) \frac{\hat{\eta}_t - X_s^M}{1 - \Sigma_M(s)} ds$$

where now N is an $\mathcal{F}^{\mathcal{X}}{}_t$-Brownian motion, the innovation process, and $\hat{\eta}_t$ is the optional projection of η into $\mathcal{F}_t^X$. Once we are motivated to understand why it is relevant to understand the optional projection of the observation process, we can now proceed to the more obvious one: the optional projection of the signal process. The connection between Markov bridges and the enlargement of flirtations is explored at length in Çetin, Danilova (2018).

We take the same approach as Wu (1999) and summarise the main results of Kallianpur (2013) used in this thesis in a single theorem:

Theorem 2.2. *(Modified from Kallianpur (2013)) Let the m-dimensional signal process $(Y_t)_{t\geq 0}$ and the n-dimensional observation process $(X_t)_{t\geq 0}$ be given by the stochastic differential equations:*

$$\begin{aligned} dY_t &= [A_0(t) + A_1(t)Y_t + A_2 X_t]dt + B(t)dW_t && (2.2) \\ dX_t &= [C_0(t) + C_1(t)Y_t + C_2 X_t]dt + D(t)dW_t && (2.3) \end{aligned}$$

with initial randon variable X_0 independent of $(\mathcal{F}_t^X)$, and $X_0 = 0$ a.s.. Besides that, (W_t) is a q-dimensional standard Brownian motion and A_i, C_i, U and V $(i = 0, 1, 2)$ are nonrandom matrices of appropriate dimensions. The entries in coefficients A_i and C_i $(i = 0, 1, 2)$ are integrable and those in B and D are square-integrable. Then,

$$\begin{aligned} d\hat{Y}_t &= [A_0(t) + A_1(t)\hat{Y}_t + A_2 X_t]dt \\ &\quad + [v(t)C_1^*(t) + B(t)D^*(t)][D(t)D^*(t)]^{-\frac{1}{2}} dN_t && (2.4) \end{aligned}$$

where $\hat{Y}_t$ is the optional projection of Y with into $(\mathcal{F}_t^X)$. Furthermore, the innovation process N_t defined as following:

$$N_t := X_t - \int_0^t \left(C_0(s) + C_1(s)\hat{Y}_s + C_2(u)X_s \right) ds$$

is an $(\mathcal{F}_t^X)$*-martingale.*

Moreover, let $v(t) = \mathbb{E}\left[Y_t^2 - \hat{Y}_t^2 | \mathscr{F}_t^X\right]$. *Then,*

$$\begin{aligned}\frac{dv(t)}{dt} &= A_1(t)v(t) + v(t)A_1^*(t) + B(t)B^*(t) - [v(t)C_1^*(t) \\ &+B(t)D^*(t)][D(t)D^*(t)]^{-1}[C_1(t)v(t) + D(t)B^*(t)].\end{aligned} \tag{2.5}$$

Note that the equation (2.4) above is Equation (10.3.11a) and equation (2.5) is Equation (10.3.15) of Kallianpur (2013).

2.2 Linear Combination of Markov Bridges

We are considering a Kyle-Back model with private and public information. The flow of information delivered to all participants in the market is denoted by $(X_t^M)_{t\in[0,1]}$, while the information that is provided exclusively to the insider is denoted by $(X_t^I)_{t\in[0,1]}$.

Both $(X_t^M)_{t\in[0,1]}$ and $(X_t^I)_{t\in[0,1]}$ are Markov bridges converging to the true value of the asset η. We should take a moment to discuss what we mean by a Markov bridge in this context. If we take either of the signals X^M or X^I on their own, there is not much to say. If we take into account the public signal, for example, a martingale $dX_t^M = \sigma(t)dW_s^I$ with initial condition $X_0^M = 0$, $\Sigma_M(t) = \int_0^t \sigma_M^2(s)ds$ and $\Sigma_M(1) = 1$, one would get $X_1^M \sim N(0,1)$. Therefore, considering only equation (1.1), what we have is just a Markov bridge representation of a martingale labelling X_1^M as η. The real bridge component here is when we consider both signals given by equations (1.1) and (1.2) together as their final value is the same. One would be tempted to say that what we have here for say X^M is a dynamic Markov bridge with final value given by another stochastic process X^I, but after reading this chapter it should be clear that it is not the case, even though the rationale would not be so far off. As a sufficient condition, the first requirement in Assumption 5.1 of Çetin, Danilova (2018) for dynamic bridges is not here and can be broken. Indeed, once we are done presenting the theory in this chapter, we shall have two dynamic Markov bridges given by equations (2.34) and (2.35). The $\mathscr{F}^I$-decomposition of X^I and of X^M are dynamic Brownian bridges with the same bridge condition Z.

The purpose of this section is to show that a linear combination of two Markov bridges that share the same final condition is a Markov bridge converging to such a terminal condition. However, one could say that this is an optional section. In fact, when combined with the other sections of this chapter, what we are really doing is showing how we have found the linear coefficients in Theorem 2.3.

It is interesting to note that $(X_t^M)_{t\in[0,1]}$ is not a delayed version of $(X_t^I)_{t\in[0,1]}$, so we are not dealing with an insider who is able to receive the public information in advance. Instead, we consider that the insider has access to a stream of information of their own. As a consequence, Corollary 2.1 shows that the insider's uncertainty with respect to the final value of the asset is always smaller or equal to the market maker's uncertainty.

We sake generality when presenting the setting of the main theorem of this chapter, Theorem 2.3. Suppose that we have two static Markov bridges converging to the same final value. We show the equivalence between the optimal projection of the minimal filtration of the two bridges and the optional projection of a particular linear combination of the two of them.

We can summarise the basic hypotheses of the Markov bridges in the above equations. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space and

$$
\begin{aligned}
\eta &= \, = \eta_0 + \eta_1 && (2.6)\\
\eta_0 &= X_0^I \sim N(0, c^2), && (2.7)\\
\eta &\sim N(0,1), && (2.8)\\
X_t^I &= X_0^I + \int_0^t \sigma_I(s) dW_s^I + \int_0^t \sigma_I^2(s) \frac{\eta - X_s^I}{1 - \Sigma_I(s)} ds, \text{ s. t.}\\
\Sigma_I(t) &= c^2 + \int_0^t \sigma_I^2(s) ds, \;\; \Sigma_I(1) = 1,\\
X_t^M &= \int_0^t \sigma(s) dW_s^M + \int_o^t \sigma_M^2(s) \frac{\eta - X_s^M}{1 - \Sigma_M(s)} ds, \text{ s. t.}\\
\Sigma_M(t) &= \int_0^t \sigma_M^2(s) ds \text{ and } \Sigma_M(1) = 1
\end{aligned}
$$

where W_t^I and W_t^W are independent $\mathcal{F}_t$-Brownian motions.

Now, we want to find a process Z that satisfies two conditions. The first is that Z is a Markov bridge with terminal condition η itself. Hence, Z must be written as follows:

$$\begin{aligned} Z_t &= Z_0 + \int_0^t \sigma_Z(s)dW_s + \int_0^t \sigma_Z^2(s)\frac{\eta - Z_s}{1-\Sigma_Z(s)}ds && (2.9)\\ \Sigma_Z(t) &= c^2 + \int_0^t \sigma_Z^2(s)ds \quad \text{and} \quad \Sigma_Z(1) = 1, && (2.10) \end{aligned}$$

where W is a $\mathcal{F}_t$-Brownian motions and η is as described in equation (2.8).

Furthermore, the second aim is that Z be a linear combination of the two Markov bridges with terminal condition η. As a consequence, we would like that Z_t be written in the following form:

$$Z_t = \lambda_0(t)X_t^I + \lambda_1(t)X_t^M \tag{2.11}$$

for some time-dependent functions λ_0 and λ_1.

In order to do so, we shall start applying the Itô formula to the above equation to find the functional forms of λ_0 and λ_1 such that we can write Z as in equation (2.9):

$$\begin{aligned} dZ_t &= \lambda_0'(t)X_t^I dt + \lambda_0(t)dX_t^I + \lambda_1'(t)X_t^M dt + \lambda_1(t)dX_t^M && (2.12)\\ &= \lambda_0'(t)X_t^I dt + \lambda_0(t)\left(\sigma_I(t)dW_t^I + \sigma_I^2(t)\frac{\eta - X_t^I}{1-\Sigma_I(t)}dt\right)\\ &\quad +\lambda_1'(t)X_t^M dt + \lambda_1(t)\left(\sigma_M(t)dW_t^M + \sigma_M^2(t)\frac{\eta - X_t^M}{1-\Sigma_M(t)}dt\right)\\ &= \lambda_0'(t)X_t^I dt + \lambda_0(t)\left(\sigma_I^2(t)\frac{\eta - X_t^I}{1-\Sigma_I(t)}dt\right)\\ &\quad +\lambda_1'(t)X_t^M dt + \lambda_1(t)\left(\sigma_M^2(t)\frac{\eta - X_t^M}{1-\Sigma_M(t)}dt\right)\\ &\quad +\lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M\\ &= \lambda_0'(t)X_t^I dt + \lambda_0(t)\sigma_I^2(t)\frac{\eta}{1-\Sigma_I(t)}dt - \lambda_0(t)\sigma_I^2(t)\frac{X_t^I}{1-\Sigma_I(t)}dt\\ &\quad +\lambda_1'(t)X_t^M dt + \lambda_1(t)\sigma_M^2(t)\frac{\eta}{1-\Sigma_M(t)}dt - \lambda_1(t)\sigma_M^2(t)\frac{X_t^M}{1-\Sigma_M(t)}dt\\ &\quad +\lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M \end{aligned}$$

(2.13)

Hence,

$$\begin{aligned}
dZ_t &= \lambda_0'(t)X_t^I dt + \frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)}\eta dt - \frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)}X_t^I dt \\
&\quad +\lambda_1'(t)X_t^M dt + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\eta dt - \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}X_t^M dt \\
&\quad +\lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M \\
&= \left(\lambda_0'(t) - \frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)}\right)X_t^I dt + \left(\lambda_1'(t) - \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\right)X_t^M dt \\
&\quad + \left(\frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\eta dt \\
&\quad +\lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M. \\
&= \left(\frac{\lambda_0'(t)}{\lambda_0(t)} - \frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\right)\lambda_0(t)X_t^I dt + \left(\frac{\lambda_1'(t)}{\lambda_1(t)} - \frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\lambda_1(t)X_t^M dt \quad (2.14) \\
&\quad + \left(\frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\eta dt \quad (2.15) \\
&\quad +\lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M. \quad (2.16)
\end{aligned}$$

Our strategy is to match equations (2.14), (2.15), and (2.16) with their respective parts in equations (2.9) and (2.10). We may start by aggregating both Brownian motions into one as it must be in (2.9). Hence, we have the following:

$$\begin{aligned}
\sigma_Z(t)dW_t &= \lambda_0(t)\sigma_I(t)dW_t^I + \lambda_1(t)\sigma_M(t)dW_t^M \\
&= \sqrt{\lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t)}dW_t \quad (2.17)
\end{aligned}$$

As a consequence, we get that for any functions λ_0 and λ_1 we must have that

$$\sigma_Z^2(t) = \lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t). \quad (2.18)$$

In the final part of this section, we show that for the functions λ_0 and λ_1 we will find shortly, they will be such that the conditions in equation (2.10) will be satisfied.

We can now begin to find the functional form of λ_0. One should note that in order to the linear form we proposed (2.14) be a Markov bridge, we must have

$$\begin{aligned}\frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)}(\lambda_0(t)X_t^I+\lambda_1(t)X_t^M) &= \left(\frac{\lambda_0'(t)}{\lambda_0(t)}-\frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\right)\lambda_0(t)X_t^I \\ &\quad+\left(\frac{\lambda_1'(t)}{\lambda_1(t)}-\frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\lambda_1(t)X_t^M \qquad (2.19)\end{aligned}$$

In particular, taking into consideration only the parts that multiply X_t^I we find that we need to have

$$\left(\frac{\lambda_0'(t)}{\lambda_0(t)}-\frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\right)=\frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)}. \qquad (2.20)$$

Therefore, we can rewrite the above equation as

$$\ln\left(\frac{\lambda_0(t)}{\lambda_0(0)}\right)+\ln\left(\frac{1-\Sigma_I(t)}{1-\Sigma_I(0)}\right)=\ln\left(\frac{1-\Sigma_Z(t)}{1-\Sigma_Z(0)}\right).$$

Therefore, we may rewrite it as follows:

$$\frac{\lambda_0(t)}{\lambda_0(0)}=\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}\frac{1-\Sigma_I(0)}{1-\Sigma_Z(0)}.$$

If we set $k_0=\lambda_0(0)\frac{1-\Sigma_I(0)}{1-\Sigma_Z(0)}$, we can write the above equation as

$$\lambda_0(t)=\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}k_0. \qquad (2.21)$$

Analogously, doing the same calculations for λ_1, we get

$$\frac{\lambda_1(t)}{\lambda_1(0)}=\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\frac{1-\Sigma_M(0)}{1-\Sigma_Z(0)}.$$

Therefore,

$$\lambda_1(t)=\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}k_1. \qquad (2.22)$$

where $k_1=\lambda_1(0)\frac{1-\Sigma_M(0)}{1-\Sigma_Z(0)}$.

There are two possible ways to proceed now. We could either find the initial conditions of λ_0 and λ_1 by considering what should be the initial conditions of the Markov bridge of Z and check that the final restriction involving (2.15) matches, or the other way around. The latter has a simpler way of presenting.

Therefore, we shall analyse what the values of k_0 and k_1 should be in order to have (2.15) that match the part that multiplies η in the equation (2.9). I.e.,

$$\frac{\sigma_Z^2(t)}{1-\Sigma_Z(t)} = \frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}$$

Combining the above equation with equations (2.21) and (2.22), we get the following:

$$\sigma_Z^2(t) = \frac{\lambda_0^2(t)\sigma_I^2(t)}{k_0} + \frac{\lambda_1^2(t)\sigma_M^2(t)}{k_1}.$$

However, by equation (2.18) we must have that

$$\lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t) = \frac{\lambda_0^2(t)\sigma_I^2(t)}{k_0} + \frac{\lambda_1^2(t)\sigma_M^2(t)}{k_1}.$$

Matching the denominators of the coefficients on the right-hand side with those on the left-hand side, we get $k_0 = k_1 = 1$.

Therefore, we find that λ_0 must be such that:

$$\lambda_0(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)} \tag{2.23}$$

and

$$\lambda_1(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}. \tag{2.24}$$

Indeed, one can double-check the solution by noticing that :

$$\lambda_1'(t) \quad = \quad \frac{(1-\Sigma_Z(t))\sigma_M^2(t) - (1-\Sigma_M(t))\sigma_Z^2(t)}{(1-\Sigma_M(t))^2}$$

Therefore,

$$\begin{aligned}
\frac{\lambda_1'(t)}{\lambda_1(t)} - \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} &= \frac{(1-\Sigma_Z(t))\sigma_M^2(t) - (1-\Sigma_M(t))\sigma_Z^2(t)}{\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}(1-\Sigma_M(t))^2} - \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} \\
&= \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} - \frac{\sigma_Z^2(t)}{1-\Sigma_Z(t)} - \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} \\
&= \frac{-\sigma_Z^2(t)}{1-\Sigma_M(t)}. \qquad (2.25)
\end{aligned}$$

Hence, doing the analogous calculations for λ_1, we can rewrite (2.14) as

$$\begin{aligned}
&\left(\frac{\lambda_0'(t)}{\lambda_0(t)} - \frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\right)\lambda_0(t)X_t^I dt \\
+&\left(\frac{\lambda_1'(t)}{\lambda_0(t)} - \frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\lambda_1(t)X_t^M dt = \frac{-\sigma_Z^2(t)}{1-\Sigma_I(t)}X_t^I dt + \frac{-\sigma_Z^2(t)}{1-\Sigma_M(t)}X_t^M dt \\
&= \frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)}\left(\lambda_0(t)X_t^I dt + \lambda_1(t)X_t^M dt\right) \\
&= \frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)}Z_t dt. \qquad (2.26)
\end{aligned}$$

We can now proceed to show that our proposed solution also satisfies that (2.15) is equal to $\frac{\sigma_Z^2(t)}{1-\Sigma_Z(t)}\eta dt$.

$$\begin{aligned}
\left(\frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\eta dt &= \frac{1-\Sigma_Z(t)}{1-\Sigma_Z(t)}\left(\frac{\lambda_0(t)\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1(t)\sigma_M^2(t)}{1-\Sigma_M(t)}\right)\eta dt \\
&= \frac{1}{1-\Sigma_Z(t)}\left(\lambda_0(t)\sigma_I^2(t)\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)} + \lambda_1(t)\sigma_M^2(t)\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\right)\eta dt \\
&= \frac{1}{1-\Sigma_Z(t)}\left(\lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t)\right)\eta dt \\
&= \frac{1}{1-\Sigma_Z(t)}\sigma_Z^2(t)\eta dt \qquad (2.27)
\end{aligned}$$

Therefore, we can rewrite (2.12) the following way:

$$
\begin{aligned}
dZ_t &= \frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)} Z_t dt \\
&\quad + \frac{1}{1-\Sigma_Z(t)} \sigma_Z^2(t) \eta dt \\
&\quad + \sigma_Z(t) dW_t \\
&= \frac{-\sigma_Z^2(t)}{1-\Sigma_Z(t)} Z_t dt + \frac{1}{1-\Sigma_Z(t)} \sigma_Z^2(t) \eta dt + \sigma_Z(t) dW_t \\
&= \sigma_Z^2(t) \frac{\eta - Z_t}{1-\Sigma_Z(t)} dt + \sigma_Z(t) dW_t
\end{aligned}
$$

Therefore, combining the solutions (2.23) and (2.24) with (2.11) leads to the conclusion that Z must be

$$
Z_t = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)} X_t^I + \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)} X_t^M. \tag{2.28}
$$

As we mentioned above, while setting the values for k_0 and k_1 we have implicitly chosen a particular initial condition for our Z. It should be clear from equation (2.28) that given X_0^I and X_0^M for this particular choice of λ's we have an initial condition Z_0 for Z. As we aim for Z to be the optimal projection of η into the filtration consisting of the processes of X^I and X^M, by the calculations we will make in equations (2.45) and (2.46), we must have $Var\left(\eta|\mathscr{F}_0^{X^M,X^I}\right) = Var(\eta|X_0^I, X_0^M)) = 1 - \Sigma_Z(0) = 1 - c^2$. The rationale behind it is that since the only information available considering both X^I and X^M is that coming from X_0^I that must be fully used to reduce the uncertainty about η when considering both filtrations.

Therefore, now one must show that if one assumes that $\Sigma_Z(0) = c^2$ the final condition of Σ_Z is such that it does not violate the hypothesis that $\Sigma_Z(1) = 1$ given by Equation (2.10) which in turn is necessary to have a Markov bridge.

We can begin by rewriting Equation (2.18) as

$$
\frac{\Sigma_Z'(t)}{(1-\Sigma_Z(t))^2} = \frac{\Sigma_I'(t)}{(1-\Sigma_I(t))^2} + \frac{\Sigma_M'(t)}{(1-\Sigma_M(t))^2}.
$$

Or equivalently in integral form:

$$
\frac{\Sigma_Z(t)}{1-\Sigma_Z(t)} = \frac{\Sigma_I(t)}{1-\Sigma_I(t)} + \frac{\Sigma_M(t)}{1-\Sigma_M(t)} + k. \tag{2.29}
$$

Since $\Sigma_I(0) = c^2$ and $\Sigma_M(0) = 0$, if $\Sigma_Z(0) = c^2$ then $k = 0$.

If we set

$$\frac{\Sigma_Z(x)}{1 - \Sigma_Z(x)} = a,$$

we have that

$$\Sigma_Z(x) = a(1 - \Sigma_Z(x)) = a - a\Sigma_Z(x) \Rightarrow \Sigma_Z(x)(1 + a) = a \Rightarrow \Sigma_Z(x) = \frac{a}{1 + a} \tag{2.30}$$

In this case, we would have that,

$$\begin{aligned}
a &= \frac{\Sigma_I(x) + \Sigma_M(x) - 2\Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))} \qquad (2.31)\\
1 + a &= 1 + \frac{\Sigma_I(x) + \Sigma_M(x) - 2\Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))}\\
&= 1 + \frac{\Sigma_I(x) + \Sigma_M(x) - 2\Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))}\\
&= \frac{(1 - \Sigma_I(x))(1 - \Sigma_M(x)) + \Sigma_I(x) + \Sigma_M(x) - 2\Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))}\\
&= \frac{1 - \Sigma_I(x) - \Sigma_M(x) + \Sigma_I(x)\Sigma_M(x) + \Sigma_I(x) + \Sigma_M(x) - 2\Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))}\\
&= \frac{1 - \Sigma_I(x)\Sigma_M(x)}{(1 - \Sigma_I(x))(1 - \Sigma_M(x))} \qquad (2.32)
\end{aligned}$$

Therefore, we can replace equations (2.31) and (2.32) in equation (2.30)

$$\Sigma_Z(t) = \frac{\Sigma_I(t) + \Sigma_M(t) - 2\Sigma_I(t)\Sigma_M(t)}{1 - \Sigma_I(t)\Sigma_M(t)} \tag{2.33}$$

As $\Sigma_I(1) = \Sigma_M(1) = 1$, we have that

$$\Sigma_Z(1) = \frac{\Sigma_I(1) + \Sigma_M(1) - 2\Sigma_I(1)\Sigma_M(1)}{1 - \Sigma_I(1)\Sigma_M(1)} = \frac{0}{0}$$

Hence, we can apply L'Hôpital to check the limit

$$\lim_{t\to 1} \frac{\Sigma_I'(t)+\Sigma_M'(t)-2\Sigma_I'(t)\Sigma_M(t)-2\Sigma_M'(t)\Sigma_I(t)}{-\Sigma_I'(t)\Sigma_M(t)-\Sigma_M'(t)\Sigma_I(t)} = \frac{\Sigma_I'(1)+\Sigma_M'(1)-2\Sigma_I'(1)-2\Sigma_M'(1)}{-\Sigma_I'(1)-\Sigma_M'(1)}$$
$$= \frac{-\Sigma_I'(1)-\Sigma_M'(1)}{-\Sigma_I'(1)-\Sigma_M'(1)} = 1$$

ultimately leading to the conclusion that $\Sigma_Z(1) = 1$. Therefore, equation (2.33) is a solution to equation (2.18) with initial condition $\Sigma_Z(0) = c^2$ and final condition $\Sigma_Z(1) = 1$.

2.3 Main Result

Once we managed to show that the sum of two Markov bridges as of equation (2.28) is a Markov bridge itself, we can proceed to show the main result of this chapter: to show that the sum of (2.28) is such that it is equivalent to both signals.

Our innovation is to show that $Z_t = \mathbb{E}(\eta|\mathscr{F}_t^Z) = \mathbb{E}\left(\eta|\mathscr{F}_t^{X^M,X^I}\right)$ where Z is as described in equations (2.9) and (2.10). Furthermore, we also show that $\eta|\mathscr{F}_t^{X^M,X^I}$ has the same distribution as $\eta|\mathscr{F}_t^Z$.

By Corollary 3.1 of Çetin, Danilova (2018) it is clear that Z_t is a martingale in its own filtration and since the final condition of Z is η, $Z_t = \mathbb{E}(\eta|\mathscr{F}_t^Z)$. By Theorem 3.4 also of Çetin, Danilova (2018) given $\mathscr{F}_t^Z$, η is normally distributed with mean Z_t and variance $1 - \Sigma_Z(t)$ for each $t \in [0, 1]$. A more pedagogical explanation can be found in section 4 of Çetin (2018).

Our motivation to prove this theorem is to find the insider's valuation of the risky asset, i.e. $\sigma_V \mathbb{E}\left(\eta|\mathscr{F}^I\right) + \mu$. As the insider observes both public and private signals and there is no other external information, it is obvious that $\mathscr{F}^I = \mathscr{F}^{X^I,X^M}$. As a consequence, we are ultimately proving that Z as given by equation (2.28), is the insider's valuation of the asset. This will be very helpful in the following chapters of this thesis as we will be able to consider $\sigma_V Z + \mu$ as equivalent to the insider's signal about the value of the asset.

Therefore, we can proceed to the main theorem of this chapter showing that if Z is as in (2.28), then $Z_t = \mathbb{E}\left(\eta|\mathscr{F}_t^{X^M,X^I}\right)$ and $\eta|\mathscr{F}_t^{X^M,X^I} \sim N(Z_t, 1 - \Sigma_Z(t))$:

Theorem 2.3. *The $\mathscr{F}^I$-decomposition of X^I and of X^M are given by*

$$X_t^I = X_0^I + \int_0^t \sigma_I(s)d\beta_s^I + \int_0^t \sigma_I^2(s)\frac{Z_s - X_s^I}{1-\Sigma_I(s)}ds, \tag{2.34}$$

$$X_t^M = \int_0^t \sigma_M(s)d\beta_s^M + \int_0^t \sigma_M^2(s)\frac{Z_s - X_s^M}{1-\Sigma_M(s)}ds, \tag{2.35}$$

where β^I and β^M are independent $\mathscr{F}^I$-Brownian motions and $Z_t := \mathbb{E}^z[\eta|\mathscr{F}_t^I]$.

Moreover, $Z_t = \lambda_0(t)X_t^I + \lambda_1(t)X_t^M$, where

$$\lambda_0(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}, \quad \lambda_1(t) = \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}.$$

Furthermore, $[Z, Z] = \Sigma_Z$, $\Sigma_Z(t) = c^2 + \int_0^t \sigma_Z^2(s)ds$, $\lim_{t\to 1}\Sigma_Z(t) = 1$, where

$$\sigma_Z^2(t) = \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_I(t)}\right)^2 \sigma_I^2(t) + \left(\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\right)^2 \sigma_M^2(t), \tag{2.36}$$

Finally, given $\mathscr{F}_t^I$, η is normally distributed with mean Z_t and variance $1-\Sigma_Z(t)$ for each $t \in [0, 1]$.

Proof. Let us begin identifying the observation processes and the signal process. In the notation of Theorem 2.2 in which the signal process is given by Y and the observation process is given by X, we see that

$$\begin{aligned} Y_t &= \eta \\ X_t &= \begin{bmatrix} X_t^I \\ X_t^M \end{bmatrix} \end{aligned}$$

Therefore, following the notation of the theorem, the coefficients are as follows:

$$
\begin{aligned}
A_0(t) &= A_1(t) = A_1(t) = A_2(t) = B(t) = 0 && (2.37)\\
C_1(t) &= \begin{bmatrix} \frac{\sigma_I^2(t)}{1-\Sigma_I(t)} \\ \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} \end{bmatrix} && (2.38)\\
C_2(t) &= \begin{bmatrix} \frac{-\sigma_I^2(t)}{1-\Sigma_I(t)} & 0 \\ 0 & \frac{-\sigma_M^2(t)}{1-\Sigma_M(t)} \end{bmatrix} && (2.39)\\
D(t) &= \begin{bmatrix} 0 & \sigma_I(t) & 0 \\ 0 & 0 & \sigma_M(t) \end{bmatrix} && (2.40)
\end{aligned}
$$

We can find the optimal projection of the signal process applying equation (2.4) of Theorem 2.2:

$$
\begin{aligned}
d\hat{Y}_t &= v(t) \begin{bmatrix} \frac{\sigma_I^2(t)}{1-\Sigma_I(t)} & \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_I(t)} & 0 \\ 0 & \frac{1}{\sigma_M(t)} \end{bmatrix} dN_t^{I,M} \\
&= v(t) \begin{bmatrix} \frac{\sigma_I(t)}{1-\Sigma_I(t)} & \frac{\sigma_M(t)}{1-\Sigma_M(t)} \end{bmatrix} dN_t^{I,M} \\
&= v(t) \left(\frac{\sigma_I(t)}{1-\Sigma_I(t)} dN_t^{I,M(1)} + \frac{\sigma_M(t)}{1-\Sigma_M(t)} dN_t^{I,M(2)} \right) \\
&= v(t) \frac{1-\Sigma_Z(t)}{1-\Sigma_Z(t)} \left(\frac{\sigma_I(t)}{1-\Sigma_I(t)} dN_t^{I,M(1)} + \frac{\sigma_M(t)}{1-\Sigma_M(t)} dN_t^{I,M(2)} \right) \\
&= v(t) \frac{1}{1-\Sigma_Z(t)} \left(\lambda_0(t)\sigma_I(t) dN_t^{I,M(1)} + \lambda_1(t)\sigma_M(t) dN_t^{I,M(2)} \right)
\end{aligned}
$$

where $N^{I,M}$ is a two-dimensional $\mathscr{F}_t^{X^I,X^M}$-Brownian motion, the innovation process, with coordinates $N_t^{I,M(1)}$ and $N_t^{I,M(2)}$. Moreover, $v(t) = \mathbb{E}\left[Y_t^2 - \hat{Y}_t^2 | \mathscr{F}_t^{X^I,X^M}\right]$ and * stand for the transpose matrix.

Just as in (2.17), we can rewrite the linear combination of Brownian motions with coefficients $\lambda_0(t)\sigma_I(t)$ and $\lambda_1(t)\sigma_M(t)$ in the following way:

$$
\begin{aligned}
\sigma_Z(t) d\bar{N}_t &= \lambda_0(t)\sigma_I(t) dN_t^{I,M(1)} + \lambda_1(t)\sigma_M(t) dN_t^{I,M(2)} && (2.41)\\
&= \sqrt{\lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t)} d\bar{N}_t
\end{aligned}
$$

where $\bar{N}_t$ is a $\mathscr{F}_t^{X^I,X^M}$-Brownian motion. Consequently, we have that

$$d\hat{Y}_t = v(t)\frac{1}{1-\Sigma_Z(t)}\sigma_Z(t)d\bar{N}_t \tag{2.42}$$

Theorem 2.2 also provides a formula for the variance of the process $v(t)$ as defined above through equation (2.5):

$$\frac{dv(t)}{dt} = -(v(t)^2)[C_1^*(t)][D(t)D^*(t)]^{-1}[C_1(t)]$$

If we consider equations (2.37)-(2.40) and the fact that $v(t)$ is unidimensional, we can apply equation (2.5) for our particular setting as follows:

$$\begin{aligned}
\frac{dv(t)}{dt} &= -(v(t)^2)\begin{bmatrix}\frac{\sigma_I^2(t)}{1-\Sigma_I(t)} & \frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\end{bmatrix}\begin{bmatrix}\frac{1}{\sigma_I^2(t)} & 0\\ 0 & \frac{1}{\sigma_M^2(t)}\end{bmatrix}\begin{bmatrix}\frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\\ \frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\end{bmatrix}\\
&= -(v(t)^2)\begin{bmatrix}\frac{1}{1-\Sigma_I(t)} & \frac{1}{1-\Sigma_M(t)}\end{bmatrix}\begin{bmatrix}\frac{\sigma_I^2(t)}{1-\Sigma_I(t)}\\ \frac{\sigma_M^2(t)}{1-\Sigma_M(t)}\end{bmatrix}\\
&= -(v(t)^2)\left[\left(\frac{\sigma_I(t)}{1-\Sigma_I(t)}\right)^2+\left(\frac{\sigma_M(t)}{1-\Sigma_M(t)}\right)^2\right]\\
&= -(v(t)^2)\frac{(1-\Sigma_Z(t))^2}{(1-\Sigma_Z(t))^2}\left[\left(\frac{\sigma_I(t)}{1-\Sigma_I(t)}\right)^2+\left(\frac{\sigma_M(t)}{1-\Sigma_M(t)}\right)^2\right]\\
&= -(v(t)^2)\frac{1}{(1-\Sigma_Z(t))^2}\left[\lambda_0^2(t)\sigma_I^2(t)+\lambda_1^2(t)\sigma_M^2(t)\right]
\end{aligned}$$

Hence, by equation (2.36), we have that

$$\frac{v(t)}{dt} = -(v(t)^2)\frac{\sigma_Z^2(t)}{(1-\Sigma_Z(t))^2} = -\left(\frac{v(t)\sigma_Z(t)}{1-\Sigma_Z(t)}\right)^2 \tag{2.43}$$

Thus, $v(t)$ must satisfy the following differential equation:

$$v(t) = v(0) - \int_0^t \left(\frac{v(s)\sigma_Z(s)}{1-\Sigma_Z(s)}\right)^2 ds. \tag{2.44}$$

We may now study the initial condition of the ODE. Since all the information available at time $t=0$ is X_0^I, $\mathscr{F}_0^{X^I,X^M} = \sigma(X_0^I)$, by equations (2.6), (2.7), and (2.8), we know that

$$\begin{aligned} v(0) &= Var(\eta | X_0^I, X_0^M) \\ &= Var(\eta_0 + \eta_1 | X_0^I, X_0^M) \\ &= Var(X_0^I + \eta_1 | X_0^I, X_0^M) \\ &= Var(\eta_1) \end{aligned} \tag{2.45}$$

the last equality being true because we assume that $\eta_0 \perp\!\!\!\perp \eta_1$. Since $\eta \sim N(0,1)$ and $\eta_0 \sim N(0, c^2)$, we have that:

$$\begin{aligned} Var(\eta) &= Var(\eta_0 + \eta_1) \\ &= Var(\eta_0) + Var(\eta_1) \\ 1 &= c^2 + Var(\eta_1) \\ 1 - c^2 &= Var(\eta_1) \end{aligned} \tag{2.46}$$

Therefore, we find that since $1 - \Sigma_Z(0)$ is $v(t)$, it must be equal to $1 - c^2$. Indeed, one can realise that $1 - \Sigma_Z(t) = v(t)$ is in fact a solution applying it to equation (2.43):

$$-\sigma_Z^2(t) = v'(t) = -\left(\frac{(1 - \Sigma_Z(t))\,\sigma_Z(t)}{1 - \Sigma_Z(t)}\right)^2 = -\sigma_Z^2(t).$$

Therefore, we can rewrite Equation (2.44) as follows:

$$v(t) = 1 - c^2 - \int_0^t \sigma_Z^2(s) ds.$$

Once we have found that $1 - \Sigma_Z(t)$ is a solution for $v(t)$, it is possible to replace it in Equation (2.42) as follows:

$$\hat{Y}_t = \hat{Y}_0 + \int_0^t \sigma_Z(s) \left(\frac{1 - \Sigma_Z(s)}{1 - \Sigma_Z(s)}\right) dB_s = \hat{Y}_0 + \int_0^t \sigma_Z(s) d\bar{N}_s$$

By Theorem 2.1 we know that the innovation process, $N_t^{I,M}$, is the difference between the observation process and its projection. Therefore, we have the following:

$$
\begin{aligned}
dN_t^{I,M(1)} &= d\frac{X_t^I}{\sigma_I(t)} - \left(\frac{-\sigma_I'(t)}{\sigma_I^2(t)}X_t^I + \sigma_I(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}\right)dt \\
&= \frac{-\sigma_I'(t)}{\sigma_I^2(t)}X_t^I dt + \frac{1}{\sigma_I(t)}dX_t^I + \frac{\sigma_I'(t)}{\sigma_I^2(t)}X_t^I dt - \sigma_I(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}d_t \\
&= \frac{1}{\sigma_I(t)}dX_t^I - \sigma_I(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}d_t
\end{aligned}
$$

Hence,

$$
\lambda_0(t)\sigma_I(t)dN_t^{I,M(1)} = \lambda_0(t)dX_t^I - \lambda_0(t)\sigma_I^2(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}dt \tag{2.47}
$$

Analogously,

$$
\lambda_1(t)\sigma_M(t)dN_t^{I,M(2)} = \lambda_1(t)dX_t^M - \lambda_1(t)\sigma_I^2(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}dt \tag{2.48}
$$

If we sum (2.47) and (2.48) and replace it using (2.41), we have the following:

$$
\begin{aligned}
\sigma_Z(t)d\bar{N}_t &= \lambda_0 dX_t^I - \lambda_0\sigma_I^2(t)\frac{\hat{Y}_t - X_t^I}{1-\Sigma_I(t)}dt \\
&\quad + \lambda_1 dX_t^M - \lambda_1\sigma_M^2(t)\frac{\hat{Y}_t - X_t^M}{1-\Sigma_M(t)}dt \\
&= \lambda_0 dX_t^I + \lambda_1 dX_t^M \\
&\quad - \hat{Y}_t\left(\frac{\lambda_0\sigma_I^2(t)}{1-\Sigma_I(t)} + \frac{\lambda_1\sigma_M^2(t)}{1-\Sigma_M(t)}\right)dt \\
&\quad + \frac{\lambda_0\sigma_I^2(t)X_t^I}{1-\Sigma_I(t)}dt + \frac{\lambda_1\sigma_M^2(t)X_t^M}{1-\Sigma_I(t)}dt \\
&= \lambda_0 dX_t^I + \lambda_1 dX_t^M \\
&\quad - \hat{Y}_t\frac{\sigma_I^Z(t)}{1-\Sigma_Z(t)}dt \\
&\quad + \frac{\lambda_0\sigma_I^2(t)X_t^I}{1-\Sigma_I(t)}dt + \frac{\lambda_1\sigma_M^2(t)X_t^M}{1-\Sigma_I(t)}dt
\end{aligned}
$$

by equation (2.27). Replacing equation (2.12) into the above equation one gets that

$$\begin{aligned}
\sigma_Z(t)d\bar{N}_t &= dZ_t - \lambda_0'(t)X_t^I dt - \lambda_1'(t)X_t^M dt \\
&\quad -\hat{Y}_t\frac{\sigma_I^Z(t)}{1-\Sigma_Z(t)}dt \\
&\quad +\frac{\lambda_0\sigma_I^2(t)X_t^I}{1-\Sigma_I(t)}dt + \frac{\lambda_1\sigma_M^2(t)X_t^M}{1-\Sigma_I(t)}dt \\
&= dZ_t - \hat{Y}_t\frac{\sigma_I^Z(t)}{1-\Sigma_Z(t)}dt \\
&\quad -\left(\frac{\lambda_0\sigma_I^2(t)}{1-\Sigma_I(t)} - \lambda_0'(t)\right)X_t^I dt \\
&\quad -\left(\frac{\lambda_1\sigma_M^2(t)}{1-\Sigma_I(t)}dt - \lambda_1'(t)\right)X_t^M dt \\
&= dZ_t - \hat{Y}_t\frac{\sigma_Z^2(t)}{1-\Sigma_Z(t)}dt \\
&\quad -\frac{-\sigma_Z^2(t)}{1-\Sigma_I(t)}X_t^I dt - \frac{-\sigma_Z^2(t)}{1-\Sigma_M(t)}X_t^M dt
\end{aligned}$$

by equations (2.20) and (2.25).Consequently, placing equation (2.26) into it, we have that

$$\sigma_Z(t)d\bar{N}_t = dZ_t - \sigma_Z^2(t)\frac{\hat{Y}_t - \sigma_Z^2(t)}{1-\Sigma_Z(t)}dt$$

Since by equation (2.42) we have that $d\hat{Y}_t = \sigma_Z(t)d\bar{N}_t$, we may write

$$d\hat{Y}_t = dZ_t - \sigma_Z^2(t)\left(\frac{\hat{Y}_t - Z_t}{1-\Sigma_Z(t)}\right)dt$$

Notice that $\hat{Y}_t$ is a solution to equation (2.49) given X^I and X^M. As (2.49) is a linear SDE, $\hat{Y} = Z$ is the unique solution. Hence Z is a $\mathscr{F}^{X^I,X^M}$-martingale given by

$$Z_t = Z_0 + \int_0^t \sigma_Z^2(s)d\bar{N}_s$$

Furthermore, equations (2.34) and (2.35) become an immediate consequence of Theorem 2.1 since $\hat{Y} = Z$.

Also, recall that η is Gaussian by equation (2.8) so the claims about the mean and variance are a consequence of the definition of optional projection.

□

2.3.1 Properties of Σ_Z

This subsection brings two very relevant corollaries to understanding Theorem 2.3. The first one, Corollary 2.1, shows that the remaining variance of the linear combination of Markov bridges is always at least as small as either of them. Therefore, our application makes sense as the insider must always have at least as much information about the true value of the asset as the market maker. Furthermore, due to 2.1 we also have that both (2.34) and (2.35) are such that $\Sigma_Z(t) > \Sigma_M(t)$ and $\Sigma_Z(t) > \Sigma_M(t)$ for all $t \in (0, 1)$.

Corollary 2.2 shows that if one of the signals is such that $Var(\eta|\mathscr{F}_t^{X^I}) = 0$ or $Var(\eta|\mathscr{F}_t^{X^M}) = 0$ then $Var(\eta|\mathscr{F}_t^Z) = 0$. That is, if any of the signals is such that there is a disclosure of the true value of the asset, the insider will know about it. By hypothesis, we know that the asset will have its value made public at $t = 1$, so we are not really interested in the possibility that $\Sigma_M(t) = 1$ for some $t < 1$, but with the possibility that $\Sigma_I(t^\star) = 1$ for some $t^\star < 1$.

With respect to our model, there is a particular setting that is very interesting to us. If $\Sigma_I(0) = 1$ we have the so-called static case. In this case, the insider knows the true value of the asset at the beginning of the trading period, as was the case with Back (1992). Obviously, the latter does not consider a public signal.

It is also worth mentioning that both corollaries are indeed just immediate consequences of equation (2.33).

Corollary 2.1. $\Sigma_Z(t) \geq \Sigma_M(t)\ \forall t\ \in [0, 1]$.

Proof. By Equation (2.29):

$$\frac{\Sigma_Z(t)}{1 - \Sigma_Z(t)} \geq \frac{\Sigma_M(t)}{1 - \Sigma_M(t)}\ \forall t\ \in [0, 1]$$

Now suppose that there exist some $t^\star\ \in [0, 1]$ such that $\Sigma_M(t^\star) > \Sigma_Z(t^\star)$, then

$$\frac{\Sigma_Z(t^\star)}{1 - \Sigma_Z(t^\star)} < \frac{\Sigma_M(t^\star)}{1 - \Sigma_M(t^\star)} \tag{2.49}$$

which leads to a contradiction. □

Corollary 2.2. *If* $\Sigma_I(t) = 1$ *or* $\Sigma_M(t) = 1$ *for any* $t \in [0, 1]$, *then* $\Sigma_Z(t) = 1$.

Proof. Since the case of the existence of a $t^\star$ in which $\Sigma_I(t^\star) = 1$ and $\Sigma_M(t^\star) = 1$, but $\Sigma_I(t^{\star-}) < 1$ and $\Sigma_M(t^{\star-}) < 1$ is already covered, consider $\Sigma_I(t) = 1$ and $\Sigma_M(t) < 1$, then by

(2.33):

$$\Sigma_Z(t) = \frac{1 + \Sigma_M(t) - 2\Sigma_M(t)}{1 - \Sigma_M(t)} = \frac{1 - \Sigma_M(t)}{1 - \Sigma_M(t)} = 1 \tag{2.50}$$

□

Chapter 3

Insider's Optimisation Problem

In Chapter 2 we have found the insider's valuation of the price of the risky asset we are studying as it is given by $\sigma_V Z_t + \mu$. It is a Markov bridge that converging to $V = \sigma_V \eta + \mu$ in $t = 1$, or whenever $t^\star$ is such that $\Sigma_Z(t^\star) = 1$, as described by equation (2.9) and (2.10).

We can now go back to Chapter 1 in order to analyse the insider's maximisation problem. Recall that in order to have an equilibrium we must find a pair of a strategy in which the insider is maximising their final wealth for a given pricing rule and a pricing rule that is a rational one.

In this chapter, we devote our attention to the insider's maximisation problem. The main theorem of this chapter is Theorem 3.1. It gives the conditions for an admissible strategy to be optimal given a pricing rule. Besides the fact that it is suboptimal for the insider to correlate their strategy with the Brownian motions driving X^I and X^M, the second condition is that the difference between the price process, S_t, and the insider's valuation about the price of the asset, $\sigma_V Z_t + \mu$, goes to zero at some rate almost surely. The conditions under which we are able to prove that such a requirement is fulfilled are developed in Chapter 4 and the final proof of the almost sure convergence of the mispricing is developed in Chapter 5.

It is interesting to mention that in Chapter 4 we shall restrict ourselves to a particular form of trading strategies given by equation (4.4). However, Theorem 3.1 is more general than we will later use.

The path to Theorem 3.1 begins in Section 3.1. In this section, we find the expected wealth of the insider at the end of the trading period. From there, in Section 3.2 we establish the value function of the insider associated with the optimal insider trading strategy. We are able to apply the dynamic programming principle to obtain the associated Hamilton-Jacobi-Bellman equation. The HJB equation provides a series of conditions both for the value function of the insider and for the optimality condition of her controls. Upon finding these conditions, we face a problem of overparameterization due to the structure of X given by equation (1.5) combined with the price process given by equation (1.8). This issue is addressed in Section 3.3 and is unravelled by Assumption 3.1.

The linear structure of the price process given by equation (1.8) combined with the fact that there is a closed form for the derivative of the value function with respect to X as one realises by equation (3.11) leads to a closed form for the value function for the insider. The form of equation (3.11) is not particular to our model. It is analogous, for example, to equation (6.31) of Çetin, Danilova (2018) when dealing with the dynamic version of the Kyle-Back model and equivalent to the static case. On the other hand, the closed form of the value function comes from the closed form of the price process. As one may lose in generality, one gains in interpretability, which will be tackled as the thesis progresses to other chapters. In the final section of the chapter, Section 3.4, we find this value function as described in Equation (3.53) that provides all the results we need to prove Theorem 3.1.

3.1 Wealth

As we have discussed in Chapter 1, the insider's goal as a risk-neutral agent is to maximise her expected profit. All sections of this chapter, except this, are devoted to finding the conditions under which we have an optimal strategy. However, before we are able to proceed with such a task, we must find the expected wealth of the insider as a function of her controls α, γ_0, and γ_1.

The final wealth of the insider is given by (1.7) where it becomes clear that the insider has two sources of profit: from the trading in the time interval $[0, 1)$ and from the gain from the price jump from S_{1-} to V times her holding θ_{1-} of the risky asset at time 1.

We shall begin our derivation by noting that the final condition of Z is such that $Z_1 = \eta$, hence $\sigma_V Z_1 + \mu = \sigma_V \eta + \mu = V$, and applying integration by parts to the processes θS and

θZ. By doing so, it is possible to rewrite equation (1.7) as:

$$\begin{aligned}
W_1 &= \int_0^1 \theta_t dS_t + \theta_1(V - S_1) \\
&= \theta_1 V - \int_0^1 S_t d\theta_t - [\theta, S]_1 \\
&= \sigma_V \int_0^1 \theta_t dZ_t + \int_0^1 (\sigma_V Z_t + \mu) d\theta_t + \sigma_V [\theta, Z]_1 - \int_0^1 S_t d\theta_t - [\theta, S]_1 \\
&= \sigma_V \int_0^1 \theta_t dZ_t + \int_0^1 (\sigma_V Z_t + \mu - S_t) d\theta_t + \sigma_V [\theta, Z]_1 - [\theta, S]_1 \qquad (3.1)
\end{aligned}$$

Applying the expectation leads to:

$$\begin{aligned}
\mathbb{E}^z(W_1) &= \mathbb{E}^z\bigg(\int_0^1 (\sigma_V Z_t + \mu - S_t)\alpha_t dt + \int_0^1 (\sigma_V Z_t + \mu - S_t)(\gamma_0(t) dW_t^I + \gamma_1(t) dW_t^M) \\
&\quad + \sigma_V \int_0^1 \theta_t dZ_t + \sigma_V [\theta, Z]_1 - [\theta, S]_1\bigg) \\
&= \mathbb{E}^z\bigg(\int_0^1 (\sigma_V Z_t + \mu - S_t)\alpha_t dt + \sigma_V [\theta, Z]_1 - [\theta, S]_1\bigg) \qquad (3.2)
\end{aligned}$$

From Theorem 2.3, we have that $N^{(1)}$ and $N^{(2)}$ are the innovation processes respectively related to B^I and B^M, hence the projection of θ into the insider's filtration is be given by

$$d\theta_t = \alpha_t dt + \gamma_0(t) dN_t^{(1)} + \gamma_1(t) dBN_t^{(2)}. \qquad (3.3)$$

It is now possible to calculate both $[\theta, Z]_1$ and $[\theta, S]_1$. From equation (3.3) and the projected version of equation (1.8) into the insider's filtration,

$$\begin{aligned}
d[\theta, S]_t &= \beta_1(t) w(t) d[\theta, \theta]_t + \beta_2(t)\sigma_M(t)\gamma_1(t) dt \\
&= \beta_1(t) w(t)(\gamma_0^2(t) + \gamma_1^2(t)) dt + \beta_2(t)\sigma_M(t)\gamma_1(t) dt \qquad (3.4)
\end{aligned}$$

and because of the martingale representation of Z, given by equation (2.17), and equation (3.2),

$$d[\theta, Z]_t = \lambda_0(t)\gamma_0(t)\sigma_I(t) dt + \lambda_1(t)\gamma_1(t)\sigma_M(t) dt \qquad (3.5)$$

As both quadratic covariations are only functions of time, we can rewrite equation (3.2) as

$$\begin{aligned}\mathbb{E}(W_1) &= \mathbb{E}\left(\int_0^1 (\sigma_V Z_t + \mu - S_t)\alpha_t dt\right.\\ &\quad +\sigma_V \int_0^1 (\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t))\, dt\\ &\quad \left.-\int_0^1 \left(\beta_1(t)w(t)(\gamma_0^2(t)+\gamma_1^2(t)) + \beta_2(t)\sigma_M(t)\gamma_1(t)\right) dt\right) \qquad (3.6)\end{aligned}$$

3.2 HJB Equation

In this section, we begin the study of the value function of the insider. It will be key in proving Theorem 3.1 not only because it will provide the insider's expected wealth at the end of the trading period, but we show that any strategy that achieves this threshold given by the value function is optimal. In fact, the mispricing condition mentioned in the beginning of this chapter derives from the fact that $\lim_{T\to 1} J(T, X_T, X_T^M, Z_T) = 0$ a.s., where J is the value function of the insider.

We can now start our task by setting the insider's problem. Recall that α, γ_0 and γ_1 are the controls of the insider. Hence, the insider's problem is to maximise the function (3.6) with respect to her controls as defined below:

$$\begin{aligned}\sup_{\alpha,\gamma_0,\gamma_1} \mathbb{E}^z(W_1) &= \sup_{\alpha,\gamma_0,\gamma_1}\left[\mathbb{E}\left(\int_0^1 (\sigma_V Z_t + \mu - S_t)\alpha_t dt\right.\right.\\ &\quad +\sigma_V \int_0^1 (\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t))\, dt\\ &\quad \left.\left.-\int_0^1 \left(\beta_1(t)w(t)(\gamma_0^2(t)+\gamma_1^2(t)) + \beta_2(t)\sigma_M(t)\gamma_1(t)\right) dt\right)\right] \qquad (3.7)\end{aligned}$$

The value function of the insider, J, is the expected profit of the insider under her optimal controls. Therefore, J must be equal to the r.h.s of equation (3.8). However, one may be asking what J should be a function of. At any given moment between 0 and 1 the value function must provide the final expected wealth given the states of the processes the insider is able to observe. Taking into account insider filtration, these processes should be X, X^M,

X^I, and Z. However, we recall from Theorem 2.3 that any one of the processes Z, X^M, and X^I is just a linear combination of the other two. As a consequence, one may select two of them to be incorporated in the value function. Therefore, we establish J to be a function of time, X, X^M, and Z. Hence, one should interpret $J(t, X_t, X_t^M, Z_t)$ as the wealth obtained by the insider from trading between time t and 1 given that the state of the processes X, X^M, and Z are respectively X_t, X_t^M, and Z_t. Moreover, under this interpretation of J, the requirement that $\lim_{T\to 1} J(T, X_T, X_T^M, Z_T) = 0$ a.s. becomes quite intuitive as one would expect that the profit in the remaining trading period from t to 1 should go to zero as t approaches 1.

Once we have established the parameters J should be a function of, we can write it as follows:

$$
\begin{aligned}
J(t,x,u,z) \quad = \quad & \operatorname*{ess\,sup}_{\alpha,\gamma_0,\gamma_1} \Bigg[\mathbb{E}\Bigg(\int_t^1 (\sigma_V Z_t + \mu - S_t)\alpha_t ds \\
& + \sigma_V \int_t^1 \left(\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t)\right) ds \\
& - \int_t^1 \left(\beta_1(t)w(t)(\gamma_0^2(t)+\gamma_1^2(t)) + \beta_2(t)\sigma_M(t)\gamma_1(t)\right) ds \Bigg| X_t = x, X_t^M = u, Z_t = z\Bigg)\Bigg]
\end{aligned} \tag{3.8}
$$

The application of the dynamic programming principle leads to the following Hamilton–Jacobi–Bellman equation:

$$
\begin{aligned}
0 \quad = \quad & \sup_{\alpha,\gamma_0,\gamma_1} \big[(J_x w(t) + \sigma_V z + \mu - H(t,x,u))\alpha \\
& + J_{xz} w(t)(\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t)) \\
& + J_{xu} w(t)\lambda_1(t)\gamma_1(t)\sigma_M(t) + \frac{1}{2}J_{xx}(\gamma_0^2(t)+\gamma_1^2(t)) \\
& + \sigma_V(\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t)) \\
& - \left(\beta_1(t)w(t)(\gamma_0^2(t)+\gamma_1^2(t)) + \beta_2(t)\sigma_M(t)\gamma_1(t)\right)\big] \\
& + J_t + J_x\left(r_0(t) + r_1(t)x + r_2(t)u\right) \\
& + J_u \sigma_M^2(t)\frac{z-u}{1-\Sigma_M(t)} + J_{zu}\lambda_1\sigma_M^2(t) \\
& + \frac{1}{2}J_{xx}w^2(t) + \frac{1}{2}J_{uu}\sigma_M^2(t) + \frac{1}{2}J_{zz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right).
\end{aligned} \tag{3.9}
$$

In order to simplify our analysis, we may define the following differential operator:

$$\mathcal{L} = (r_0 + r_1 x + r_2 u)\frac{\partial}{\partial x} + \sigma_M^2 \frac{z-u}{1-\Sigma_M}\frac{\partial}{\partial u} + \frac{\beta_1^2 w^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\sigma_M^2}{2}\frac{\partial^2}{\partial u^2} + \frac{\sigma_Z^2}{2}\frac{\partial^2}{\partial z^2} + \lambda_1 \sigma_M^2 \frac{\partial^2}{\partial z \partial u}.$$

As one may note from the equation above, we can establish such operator as it does not contain any of the insider's controls so it does not provide any information about them. As a consequence, we can rewrite Equation (3.11) as

$$\begin{aligned} 0 = J_t + \mathcal{L}J + \sup_{\alpha,\gamma_0,\gamma_1} \big\{ &(J_x w + \sigma_V z + \mu - H)\alpha + J_{xz} w(\lambda_0 \gamma_0 \sigma_I + \lambda_1 \gamma_1 \sigma_M) \\ &+ J_{xu} w \lambda_1 \gamma_1 \sigma_M + \frac{1}{2} J_{xx} w^2 (\gamma_0^2 + \gamma_1^2 + \sigma_V(\lambda_0 \gamma_0 \sigma_I + \lambda_1 \gamma_1 \sigma_M) \\ &- \big(\sigma_V w(\gamma_0^2 + \gamma_1^2) + \beta_2 \sigma_M \gamma_1\big) \big\} \end{aligned} \tag{3.10}$$

As it is recurrent in the literature of Kyle-Back models (see, for example, Equation (6.18) of Çetin, Danilova (2018)), in order to guarantee the existence of an optimal α and the finiteness of the value function, we must have that the following equation must be satisfied:

$$0 = J_x w(t) + \sigma_V z + \mu - H(t, x, u). \tag{3.11}$$

One should note from equation (3.7) that if it were not for the equation above, the maximisation problem would not have a solution. Indeed, suppose that one claims that there is an optimal strategy $\alpha^\star$. It would always be possible to find another strategy $\alpha^{\star\star} > \alpha^\star$ that would be more profitable for the insider[1].

Furthermore, the first order condition of the maximisation of γ_0 gives

$$0 = J_{xz} w(t)\lambda_0(t)\sigma_I(t) + J_{xx} w^2(t)\gamma_0(t) + \sigma_V \lambda_0(t)\sigma_I(t) - 2\beta_1(t) w(t)\gamma_0(t) \tag{3.12}$$

[1] The reader should be remainder that α's are functions so $\alpha^{\star\star} > \alpha^\star$ has the ordinary meaning of pointwise dominance in the domain of the functions.

and the first order condition of the maximisation of γ_1 is

$$\begin{aligned} 0 &= J_{xz}w(t)\lambda_1(t)\sigma_M(t) + J_{xu}w(t)\sigma_M(t) + J_{xx}w^2(t)\gamma_0(t) \\ &\quad +\sigma_V\lambda_1(t)\sigma_M(t) - 2\beta_1(t)w(t)\gamma_1(t)) - \beta_2(t)\sigma_M(t) \end{aligned} \tag{3.13}$$

From equation (3.11), we have that:

$$J_x w(t) = \beta_0(t) + \beta_1(t)x + \beta_2(t)u - z \tag{3.14}$$

$$J_{xz} w(t) = -\sigma_V \tag{3.15}$$

$$J_{xx} w(t) = \beta_1(t) \tag{3.16}$$

$$J_{xu} w(t) = \beta_2(t). \tag{3.17}$$

As a consequence, we can rewrite Equation (3.12) as

$$0 = -\lambda_0(t)\sigma_I(t) + \beta_1(t)w(t)\gamma_0(t) + \sigma_V\lambda_0(t)\sigma_I(t) - 2\beta_1(t)w(t)\gamma_0(t) \tag{3.18}$$

hence, $-\beta_1(t)w(t)\gamma_0(t) = 0$. Since $\beta_1(t)w(t) > 0$ for all $t \in (0,1)$ we have that $\gamma_0(t) = 0$. Moreover, we have that the second order condition for the maximisation is $-\beta_1(t)w(t) < 0$. Likewise, we can rewrite equation (3.13) as

$$\begin{aligned} 0 &= -\lambda_1(t)\sigma_M(t) + \beta_2(t)\sigma_M(t) + \beta_2(t)w(t)\gamma_0(t) + \sigma_V\lambda_1(t)\sigma_M(t) \\ &\quad -2\beta_1(t)w(t)\gamma_1(t)) - \beta_2(t)\sigma_M(t). \end{aligned} \tag{3.19}$$

Like the previous case, $-\beta_1(t)w(t)\gamma_1(t) = 0$ and $-\beta_1(t)w(t) < 0$ guarantee that $\gamma_1(t) = 0$ is the maximum.

Once we have proven the validity of equations (3.11), (3.12), and (3.13), it is possible to replace them into equation (3.11) to rewrite it as

$$
\begin{aligned}
0 \quad = \quad & J_t + J_x\left(r_0(t) + r_1(t)x + r_2(t)u\right) + J_u\sigma_M^2(t)\frac{z-u}{1-\Sigma_M(t)} + J_{zu}\lambda_1(t)\sigma_M^2(t) \\
& + \frac{1}{2}J_{xx}w^2(t) + \frac{1}{2}J_{uu}\sigma_M^2(t) + \frac{1}{2}J_{zz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right)
\end{aligned} \tag{3.20}
$$

That in the sense of the operator defined above means that

$$
J_t + \mathcal{L}J = 0 \tag{3.21}
$$

One can note that equation (3.11) shows a relationship between the derivative of the value function with respect to x and the coefficients of H and w. In equation (3.22) we differentiate (3.20) with respect to x so we can do two things at the same time: first we find a relationship between the coefficients of J and those given by (3.11) and secondly, as long as the function f defined in equation (3.34) does not depend on x, satisfying equation (3.22) will also satisfy (3.20).

Therefore, we can take the derivative (3.20) with respect to x again to find that

$$
\begin{aligned}
0 \quad = \quad & J_{xt} + J_{xx}\left(r_0(t) + r_1(t)x + r_2(t)u\right) + J_x r_1(t)x + J_{xu}\sigma_M^2(t)\frac{z-u}{1-\Sigma_M(t)} \\
& + J_{xzu}\lambda_1\sigma_M^2(t) + \frac{1}{2}J_{xxx}w^2(t) + \frac{1}{2}J_{xuu}\sigma_I^2(t) + \frac{1}{2}J_{xzz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right)
\end{aligned} \tag{3.22}
$$

The linear structure of H is inherited by J_x, as we can see in Equation (3.14). As a consequence, $J_{xzu} = J_{xxx} = J_{xuu} = J_{xzz} = 0$, hence,

$$
J_{xzu}\lambda_1\sigma_M^2(t) + \frac{1}{2}J_{xxx}w^2(t) + \frac{1}{2}J_{xuu}\sigma_I^2(t) + \frac{1}{2}J_{xzz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right) = 0.
$$

Furthermore, one can find the coefficients of J_{xt} by its relationship with J_x as developed here:

$$
J_{xt} = \frac{\beta_0'(t) + \sigma_V\beta_1'(t)x + \beta_2'(t)u}{w(t)} - \frac{\beta_0(t) + \beta_1(t)x + \beta_2(t)u - z}{w^2(t)}w'(t) \tag{3.23}
$$

The same can be done with the other coefficients of (3.22) considering equation (3.11) and (1.8):

$$\begin{aligned} J_{xx} &= \frac{\sigma_V \beta_1(t)}{w(t)} \\ J_{xz} &= \frac{\beta_2(t)}{w(t)} \\ J_{xxx} = J_{xzz} = J_{xzu} = J_{xuu} &= 0 \end{aligned}$$

As a consequence, we can rewrite Equation (3.22) as

$$\begin{aligned} 0 &= \frac{\beta_0'(t) + \beta_1'(t)x + \beta_2'(t)u}{w(t)} - (\beta_0(t) + \beta_1(t)x + \beta_2(t)u - z\sigma_V - \mu)\frac{w'(t)}{w^2(t)} + \\ &+ r_1(t)\left(\frac{\beta_0(t) + \beta_1(t)x + \beta_2(t)u - z\sigma_V - \mu}{w(t)}\right) + \frac{\beta_1(t)}{w(t)}(r_0(t) + r_1(t)x + r_2(t)u) \\ &+ \beta_2(t)\frac{\sigma_M^2(t)}{1 - \Sigma_M(t)}\frac{z-u}{w(t)} \end{aligned} \tag{3.24}$$

One way to be sure that regardless of the values of x, u, and z equation (3.24) is satisfied is to make sure that all coefficients of the above equation are set to be zero. Therefore, we would like to have

$$0 = \frac{\beta_0'(t)}{w(t)} - (\beta_0(t) - \mu)\frac{w'(t)}{w^2(t)} + r_1(t)\frac{\beta_0(t) - \mu}{w(t)} + r_0(t)\frac{\beta_1(t)}{w(t)} \tag{3.25}$$

$$0 = \frac{\beta_1'(t)}{w(t)} - \beta_1(t)\frac{w'(t)}{w^2(t)} + 2r_1(t)\frac{\beta_1(t)}{w(t)} \tag{3.26}$$

$$0 = \frac{w'(t)}{w^2(t)} - \frac{r_1(t)}{w(t)} + \frac{\beta_2(t)}{w(t)}\frac{\sigma_M^2(t)}{1 - \Sigma_M(t)} \tag{3.27}$$

$$\begin{aligned} 0 &= \frac{\beta_2'(t)}{w(t)} - \beta_2(t)\frac{w'(t)}{w^2(t)} + r_1(t)\frac{\beta_2(t)}{w(t)} \\ &+ r_2(t)\frac{\beta_1(t)}{w(t)} - \frac{\beta_2(t)}{w(t)}\frac{\sigma_M^2(t)}{1 - \Sigma_M(t)} \end{aligned} \tag{3.28}$$

which leads to equation (3.24) being satisfied. Rearranging the terms of the equations above one gets that

$$r_0(t) = \frac{1}{\beta_1(t)}\left((\beta_0(t)-\mu)\frac{w'(t)}{w(t)} - \beta_0'(t) - (\beta_0(t)-\mu)r_1(t)\right), \tag{3.29}$$

$$r_1(t) = \frac{1}{2}\left(\frac{w'(t)}{w(t)} - \frac{\beta_1'(t)}{\beta_1(t)}\right) = \frac{w'(t)}{w(t)} + \frac{\beta_2(t)}{\sigma_V}\frac{\sigma_M^2(t)}{1-\Sigma_M(t)}, \tag{3.30}$$

$$r_2(t) = \frac{\beta_2(t)}{\beta_1}\left(\left(1-\frac{\beta_2(t)}{\sigma_V}\right)\frac{\sigma_M^2(t)}{1-\Sigma_M(t)} - \frac{\beta_2'(t)}{\beta_2(t)}\right). \tag{3.31}$$

3.3 Overparametrization

We can take a moment to realise that we are dealing with an overparameterization. Indeed, the first step one should take to note that is applying Itô formula to the price process:

$$\begin{aligned} dS_t &= (\beta_0'(t) + \beta_1'(t)X_t + \beta_2'(t)X_t^M)dt \\ &\quad +\beta_1(t)w(t)\hat{\alpha}_t dt \\ &\quad +\beta_1(t)\left(r_0(t) + r_1(t)X_t + r_2(t)X_t^M\right)dt \\ &\quad +\beta_2(t)\sigma_M^2(t)\frac{\frac{S_t-\mu}{\sigma_V} - X_t^M}{1-\Sigma_M(t)}dt \\ &\quad +\beta_1(t)w(t)dN_t^{(1)} + \beta_2(t)\sigma_M(t)dN_t^{(2)} \end{aligned} \tag{3.32}$$

where $N^{(1)}$ and $N^{(2)}$ are innovation processes given by equations (4.2) and $\hat{\alpha}$ the projection of α into the market maker's filtration also given by (4.2). Those innovation processes are relevant as the price process is defined by the market maker following a rational pricing rule. Now the overparameterization becomes clear when substituting equation (3.24) into the one above to get the following price process:

$$dS_t = \beta_1(t)w(t)\hat{\alpha}_t dt + \beta_1(t)w(t)dN_t^{(1)} + \beta_2(t)\sigma_M(t)dN_t^{(2)}. \tag{3.33}$$

One can note that w and β_1 are not distinguishable for the market maker regarding the price process. The market maker aims to follow a rational pricing rule, so she will set the values of the βs in order to do so. Hence, for any function w there will be an β_1 such that the above equation follows a rational pricing rule. However, it should be noted that the process X is also built by the market maker. As both w and β_1 work analogously as controls for the market maker[2], it is irrelevant to her what values each one of the functions are taking at any given moment as long as the pricing rule is rational. Therefore, instead of allowing w to be almost anything, we shall set the value of β_1 to be equal to a constant σ_V and ask w to do the job of ensuring that the market maker's goal is achieved.

Therefore, we can, without loss of generality, make the following assumption:

Assumption 3.1. *$\beta_1 \equiv \sigma_V$, where β_1 is the function appearing in the representation of the pricing rule as defined in Definition 1.1.*

The above assumption becomes quite relevant to us at this point because we may now simplify the conditions on the r's to have Equation (3.24) satisfied. We can rewrite equations (3.29), (3.30), and (3.31) under the conditions of the above assumption, as of Assumption 3.2 below:

Assumption 3.2. *The functions $(r_i)_{i=0}^2$ and $(\beta_i)_{i=0}^2$ satisfy the following relationships for all $t \in [0, 1)$:*

$$\begin{aligned}
r_0(t) &= \sigma_V^{-1}(\beta_0(t) - \mu)\frac{w'(t)}{w(t)} - \beta_0'(t)\sigma_V^{-1} - \sigma_V^{-1}(\beta_0(t) - \mu)r_1(t),\\
r_1(t) &= \frac{w'(t)}{2w(t)} = \frac{w'(t)}{w(t)} + \frac{\beta_2(t)}{\sigma_V}\frac{\sigma_M^2(t)}{1 - \Sigma_M(t)},\\
r_2(t) &= \frac{\beta_2(t)}{\sigma_V}\left(\left(1 - \frac{\beta_2(t)}{\sigma_V}\right)\frac{\sigma_M^2(t)}{1 - \Sigma_M(t)} - \frac{\beta_2'(t)}{\beta_2(t)}\right).
\end{aligned}$$

3.4 The Value Function

Once we know the conditions the value function must satisfy, we can now show what functional form it should take. From the previous section, we know that the γ's optimal controls are such that $\gamma_0 = \gamma_1 = 0$. Furthermore, equation (3.11) ensures that the derivative of J with respect to x must be such that $J_x = \frac{H(t,x,u) - \sigma_V z + \mu}{w(t)}$ and we also have a form for the derivative of J with respect to t given by equation (3.21).

This structure allows us to wonder about the actual form of J as a function of time, X, X^M, and Z. Equation (3.11) ensures that J is such that there should be an integral as in the r.h.s of (3.34). Furthermore, the choice of the lower limit of integration comes from the fact

[2] Note that it does not make sense to say that they are controls for the market maker as they are not solving any maximisation problem. They work analogously as controls because the market maker sets those functions so that they can satisfy the rationality condition

that J must be such that $\lim_{T\to 1} J(T, X_T, X_T^M, Z_T) = 0$ a.s.. Furthermore, from equation (3.24) we know that equation (3.37) below is satisfied for φ defined as the integral in (3.34). As a consequence, f in Equation (3.34) must be a function that depends only on time. We go further in this section and show the functional form of f.

Therefore, we can now conjecture that J must be of the form below for some function f that will be found also in this section:

$$J(t,x,u,z) = \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H(t,y,u) - \sigma_V z - \mu,}{w(t)} dy + f(t), \tag{3.34}$$

for some function f to be determinate where H follows equation (1.8) and $H^{-1}(t,y,u)$ is such that $H(t, H^{-1}(t,y,u) = y$, hence $H(t, H^{-1}(t, \sigma_V z + \mu, u) = \sigma_V z + \mu$. Therefore,

$$H^{-1}(t,y,u) = \frac{z - \beta_0(t) - \beta_2(t)y}{\sigma_V}. \tag{3.35}$$

Let us begin by defining

$$\varphi(t,x,u,z) = \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H(t,y,u) - \sigma_V z - \mu}{w(t)} dy. \tag{3.36}$$

Note that by the previous calculations, we must have that φ is such that

$$\varphi_t + \varphi_x \left(r_0(t) + r_1(t)x + r_2(t)u\right) + \varphi_u \sigma_M^2(t) \frac{z-u}{1-\Sigma_M(t)} = 0. \tag{3.37}$$

which is equivalent to satisfying the equation (3.22). As a consequence, the role of f is to make sure that once (3.22) is satisfied, it also will be (3.20).

Let us start by recalling that, by the structure of (3.36), it is trivial that:

$$\varphi_x(t,x,u,z) = \frac{H(t,x,u) - \sigma_V z - \mu}{w(t)} \tag{3.38}$$

hence we can see that equation (3.11) is fulfilled. Furthermore, we have that

$$\varphi_{xx}(t,x,u,z) = \frac{H_x(t,x,u) - \sigma_V z - \mu}{w(t)} = \frac{\sigma_V}{w(t)} \tag{3.39}$$

We may now proceed by calculating φ_u:

$$\begin{aligned}
\varphi_u(t,x,u,z) &= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H_u(t,y,u)}{w(t)}dy \\
&\quad -H_u^{-1}(t,\sigma_V z+\mu,u)\left(\frac{H(t,H^{-1}(t,\sigma_V z+\mu,u),u)-\sigma_V z-\mu}{w(t)}\right) \\
&= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H_u(t,y,u)}{w(t)}dy \\
&\quad -H_u^{-1}(t,\sigma_V z+\mu,u)\left(\frac{\sigma_V z+\mu-\sigma_V z-\mu}{w(t)}\right) \\
&= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H_u(t,y,u)}{w(t)}dy \qquad (3.40)\\
&= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{\beta_2(t)}{w(t)}dy \\
&= \frac{\beta_2(t)}{w(t)}\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right) \\
&= \frac{\beta_2(t)}{w(t)}\left(x-\frac{\sigma_V z+\mu-\beta_0(t)-\beta_2(t)u}{\sigma_V}.\right) \qquad (3.41)
\end{aligned}$$

This also allows us to calculate φ_{uu}:

$$\begin{aligned}
\varphi_{uu}(t,x,u) &= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H_{uu}(t,y,u)}{w(t)}dy \\
&\quad -H_u^{-1}(t,\sigma_V z+\mu,u)\left(\frac{H_u(t,H^{-1}(t,\sigma_V z+\mu,u),u)}{w(t)}\right) \\
&= 0-\left(\frac{-\beta_2(t)}{\sigma_V}\right)\frac{\beta_2(t)}{w(t)} = \frac{\beta_2^2(t)}{w(t)\sigma_V} \qquad (3.42)
\end{aligned}$$

We can finish calculating all the coefficients that do not depend on t by calculating φ_z and its second derivatives:

$$
\begin{aligned}
\varphi_z(t,x,u,z) &= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{\partial}{\partial z}\left(\frac{H(t,y,u)-\sigma_V z-\mu}{w(t)}\right)dy \\
&\quad -H_z^{-1}(t,\sigma_V z+\mu,u)\left(\frac{H(t,H^{-1}(t,\sigma_V z+\mu,u),u)-\sigma_V z-\mu}{w(t)}\right) \\
&= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} -\frac{\sigma_V}{w(t)}dy \\
&\quad -H_z^{-1}(t,\sigma_V z+\mu,u)\left(\frac{\sigma_V z+\mu-\sigma_V z-\mu}{w(t)}\right) \\
&= -\sigma_V\frac{\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)}{w(t)} \\
&= -\left(\sigma_V x-\frac{\sigma_V z+\mu-\beta_0(t)-\beta_2(t)u}{w(t)}\right) \qquad (3.43)
\end{aligned}
$$

Which also allow us to the calculate φ_{zu}:

$$
\varphi_{zu}(t,x,u,z) = -\frac{\partial}{\partial u}\left(\sigma_V\frac{\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)}{w(t)}\right)=\frac{-\beta_2(t)}{w(t)} \qquad (3.44)
$$

and φ_{zz}:

$$
\varphi_{zz}(t,x,u,z) = -\frac{\partial}{\partial z}\left(\frac{\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)}{w(t)}\right)=\frac{\sigma_V}{w(t)} \qquad (3.45)
$$

We must also have the calculations of the derivative of φ with respect to t. First note that

$$
\begin{aligned}
\varphi_t(t,x,u,z) &= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{\partial}{\partial t}\left(\frac{H(t,y,u)}{w(t)}\right)dy \\
&\quad -H_t^{-1}(t,\sigma_V z+\mu,u)\left(\frac{H(t,H^{-1}(t,\sigma_V z+\mu,u),u)-\sigma_V z-\mu}{w(t)}\right) \\
&= \int_{H^{-1}(t,z,u)}^{x} \frac{\partial}{\partial t}\left(\frac{H(t,y,u)}{w(t)}\right)dy \\
&\quad -H_t^{-1}(t,z,u)\left(\frac{\sigma_V z+\mu-\sigma_V z-\mu}{w(t)}\right) \\
&= \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{\partial}{\partial t}\left(\frac{H(t,y,u)}{w(t)}\right)dy \qquad (3.46)
\end{aligned}
$$

so we can now perform the calculations on the derivative inside the integral above:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{H(t,y,u)}{w(t)}\right) &= \frac{H_t(t,y,u)w(t)-(H(t,y,u))w'(t)}{w^2(t)} \\ &= \frac{H_t(t,y,u)w(t)-(H(t,y,u))w'(t)}{w^2(t)}\end{aligned} \tag{3.47}$$

which can be written as:

$$\begin{aligned}\frac{H_t(t,y,u)w(t)-H(t,y,u)w'(t)}{w^2(t)} &= \frac{\beta_0'(t)w(t)-\beta_0(t)w'(t)}{w^2(t)} \\ &+\left(\frac{-\sigma_V w'(t)}{w^2(t)}\right)y \\ &+\left(\frac{\beta_2'(t)w(t)-\beta_2(t)w'(t)}{w^2(t)}\right)u\end{aligned}$$

Therefore,

$$\begin{aligned}\varphi_t(t,x,u,z) &= \frac{\beta_0'(t)w(t)-\beta_0(t)w'(t)}{w^2(t)}\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right) \\ &+\left(\frac{-\sigma_V w'(t)}{w^2(t)}\right)\frac{1}{2}\left(x^2-(H^{-1}(t,\sigma_V z+\mu,u))^2\right) \\ &+\left(\frac{\beta_2'(t)w(t)-\beta_2(t)w'(t)}{w^2(t)}\right)u\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right) \\ &= \left[\frac{\beta_0'(t)w(t)-\beta_0(t)w'(t)}{w^2(t)}\right. \\ &+\frac{1}{2}\left(\frac{-\sigma_V w'(t)}{w^2(t)}\right)\left(x+H^{-1}(t,\sigma_V z+\mu,u)\right) \\ &+\left.\left(\frac{\beta_2'(t)w(t)-\beta_2(t)w'(t)}{w^2(t)}\right)u\right]\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right).\end{aligned}$$

Now, one can note that

$$\begin{aligned}\left(x+H^{-1}(t,\sigma_V z+\mu,u)\right) &= \left(x+\frac{\sigma_V z+\mu-\beta_0(t)-\beta_2(t)u}{\sigma_V}\right) \\ &= \left(2x+\frac{\sigma_V z+\mu-\beta_0(t)-\sigma_V x-\beta_2(t)u}{\sigma_V}\right)\end{aligned}$$

As a consequence,

$$\begin{aligned}\varphi_t(t,x,u,z) &= \left[\frac{\beta_0'(t)w(t)-\beta_0(t)w'(t)}{w^2(t)}\right.\\ &+\left(\frac{-\sigma_V w'(t)}{w^2(t)}\right)x+\left(\frac{\beta_2'(t)w(t)-\beta_2(t)w'(t)}{w^2(t)}\right)u\\ &\left.+\frac{1}{2}\frac{-w'(t)\sigma_V}{w^2(t)}\left(\frac{\sigma_V z+\mu-\beta_0(t)-\sigma_V x-\beta_2(t)u}{\sigma_V}\right)\right]\\ &\times\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)\end{aligned}$$

We can replace the equation for r_1 in Assumption 3.2 into the equation above so we have that

$$\begin{aligned}\varphi_t(t,x,u,z) &= \left[\frac{\beta_0'(t)w(t)-\beta_0(t)w'(t)}{w^2(t)}\right.\\ &+\left(\frac{-\sigma_V w'(t)}{w^2(t)}\right)x+\left(\frac{\beta_2'(t)w(t)-\beta_2(t)w'(t)}{w^2(t)}\right)u\\ &\left.+r_1(t)\left(\frac{\sigma_V z+\mu-\beta_0(t)-\sigma_V x-\beta_2(t)u}{w(t)}\right)\right]\\ &\times\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)\end{aligned}$$

Moreover, note that

$$\frac{H(t,x,u)-\sigma_V z-\mu}{w(t)}=\frac{\sigma_V}{w(t)}\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right) \tag{3.48}$$

Hence, if we define $\tilde{\varphi}(t)=\varphi_t+\varphi_x\left(r_0(t)+r_1(t)x+r_2(t)u\right)+\varphi_u\sigma_M^2(t)\frac{z-u}{1-\Sigma_M(t)}$

$$\begin{aligned}\tilde{\varphi}(t) &= \left[\frac{\beta_0'(t)+\beta_2'(t)u}{w(t)}-\left(\beta_0(t)+\sigma_V)x+\beta_2(t)u-\sigma_V z-\mu\right)\frac{w'(t)}{w^2(t)}+\right.\\ &+r_1(t)\left(\frac{\beta_0(t)+\sigma_V x+\beta_2(t)u-\sigma_V z-\mu}{w(t)}\right)\\ &+\frac{\sigma_V}{w(t)}\left(r_0(t)+r_1(t)x+r_2(t)u\right)\\ &\left.+\frac{\beta_2(t)}{w(t)}\frac{\sigma_M^2(t)}{1-\Sigma_M(t)}(z-u)\right]\left(x-H^{-1}(t,\sigma_V z+\mu,u)\right)\end{aligned}$$

As a consequence of equation (3.24), we have that $\tilde{\varphi}(t)=0$.

Hence, in order to make equation (3.20) true we need to set

$$-f'(t) = \varphi_{zu}\lambda_1(t)\sigma_M^2(t) + \frac{1}{2}\varphi_{xx}w^2(t) + \frac{1}{2}\varphi_{uu}\sigma_M^2(t) + \frac{1}{2}\varphi_{zz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right) \tag{3.49}$$

Thus,

$$-f'(t) = \frac{-\beta_2(t)}{\sigma_V w(t)}\frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)}\sigma_M^2(t) + \frac{1}{2}\sigma_V w(t) + \frac{1}{2}\frac{\beta_2^2(t)}{\sigma_V w(t)}\sigma_M^2(t) + \frac{1}{2}\frac{1}{\sigma_V w(t)}\sigma_Z^2(t) \tag{3.50}$$

As a consequence, we may define the integral:

$$f(t) = \frac{1}{2}\int_t^1 \sigma_V w(s) + \frac{\beta_2^2(s)\sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_Z^2(s)}{\sigma_V w(s)} - \frac{2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{\sigma_V w(s)}ds \tag{3.51}$$

Therefore, we can rewrite equation (3.34) as

$$\begin{aligned} J(t,x,u,z) &= \int_{H^{-1}(t,z,u)}^{x} \frac{H(t,y,u)-\sigma_V z-\mu}{w(t)}dy \\ &\quad + \frac{1}{2}\int_t^1 \sigma_V w(s) + \frac{\beta_2^2(s)\sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_Z^2(s)}{\sigma_V w(s)} - \frac{2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{\sigma_V w(s)}ds \end{aligned} \tag{3.52}$$

The calculations above prove the following proposition:

Proposition 3.1. *Suppose that the integral*

$$\int_0^1 \sigma_V w(s) + \frac{\beta_2^2(s)\sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_V\sigma_Z^2(s)}{w(s)} - \frac{2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{w(s)}ds$$

exists and is finite. Then, the function $J : [0,1)\times\mathbb{R}^3 \to \mathbb{R}$ *defined by*

$$\begin{aligned} J(t,x,u,z) = &\int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H(t,y,u)-\sigma_V z-\mu}{w(t)}dy \\ &+ \frac{1}{2}\int_t^1 \sigma_V w(s) + \frac{\beta_2^2(s)\sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_V\sigma_Z^2(s)}{w(s)} - \frac{2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{w(s)}ds \end{aligned} \tag{3.53}$$

solves (3.21) *with* $J_x = \frac{H(t,x,u)-\sigma_V z-\mu}{w(t)}$.

We can now proceed to prove the main theorem of this chapter. Equation (3.53) is equivalent to ψ in Equation (6.30) of Çetin, Danilova (2018). In the Kyle-Back models without a public signal, we still have the same structure for the derivative of the value function with respect to the process X, so some version of equation (3.11) is present. Therefore, the first integral in the above equation is quite equivalent to what we have in our case. With some subtleties, the same could be said about the second integral. However, it is true that a lot of care was taken to ensure that the second integral would depend only on time in our case. Furthermore, as is traditional in the literature, the form of the value function depends on the price process. In both Equation (6.30) of Çetin, Danilova (2018) and Equation (3.53) in our case, the form of the pricing rule H provides the form of the value function. Therefore, the main innovation of the theorem below is not how it is different from the previous Kyle-Back models but how we are able to adapt a public signal without completely changing the structure of the value function.

There is also an interesting debate about the fact that in the literature it is said that the insider drives the price to the final value of the asset almost surely. Indeed, the price goes to V when time goes to one, as we need that $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.. However, it is no longer clear whether that is insider-made. Since now the market maker has her own signal X^M about the final value of the asset, she no longer depends on the insider driving the price to V at the end of the trading period. If the insider did not trade, the price would also go to the same value. In Chapters 5 to 7 we address this issue a few times. In those chapters, we discuss whether at the end of the trading period the market maker is relying on her own (public) signal or if she is being driven by the information coming from the process X.

Theorem 3.1. *Suppose that the hypothesis of Proposition 3.1 is valid. Let (H, w, r) be an admissible pricing rule and define*

$$\phi(t, x, u, z) = \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H(t, y, u) - \sigma_V z - \mu}{w(t)} dy$$

If θ is an admissible trading strategy for this pricing rule with $\gamma_0 = \gamma_1 \equiv 0$ and $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s., then it is optimal. Moreover, the expected terminal wealth associated with any op-

timal strategy is given by

$$J(0,0,0,z) = \frac{(\beta_0(0)-z)^2}{2w(0)} + \frac{1}{2}\int_0^1 \sigma_V w(s) + \frac{\beta_2^2(s)\sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_V \sigma_Z^2(s)}{w(s)} - \frac{2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{w(s)} ds. \tag{3.54}$$

Proof. Note that for any admissible strategy θ, we can apply Itô's formula to obtain

$$\begin{aligned} dJ(t,X_t,X_t^M,Z_t) &= J_t dt + J_x dX_t + J_u dX_t^M + J_z dZ_t \\ &\quad + J_{xu} d[X,X^M]_t + J_{xz} d[X,Z]_t + J_{uz} d[X^M,Z]_t \\ &\quad + \frac{1}{2}J_{xx} d[X,X]_t + \frac{1}{2}J_{uu} d[X^M,X^M]_t + \frac{1}{2}J_{zz} d[Z,Z]_t \\ &= (\varphi_t - f'(t))dt \\ &\quad + J_x\left[w(t)(dB_t + d\theta_t) + r(t,X_t,X_t^M)dt\right] \\ &\quad + J_u(z-u)\frac{\sigma_M^2(t)}{1-\Sigma_M(t)}dt + J_u\sigma_M(t)dB_t^M \\ &\quad + J_{xu}\sigma_M(t)\gamma_1(t)w(t)dt + J_{xz}w(t)(\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t))dt \\ &\quad + \frac{1}{2}J_{xx}w^2(t)(1+\gamma_0^2(t)+\gamma_1^2(t))dt + \frac{1}{2}J_{uu}\sigma_M^2(t)dt \\ &\quad + \frac{1}{2}J_{zz}(\lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t))dt \end{aligned} \tag{3.55}$$

Note that $J_t(t,x,u,z) = \varphi_t(t,x,u,z) - f'(t)$ having φ being defined as in equation (3.36). Thus, $J_x = \varphi_x$, $J_u = \varphi_u$, and $\tilde{\varphi}(t) \equiv \varphi_t + J_x r(t,x,u) + J_u\sigma_M^2(t)\frac{z-u}{1-\Sigma_M(t)} = 0$ as a consequence of equation (3.24).

Furthermore, as

$$f'(t) = \varphi_{zu}\lambda_1(t)\sigma_M^2(t) + \frac{1}{2}\varphi_{xx}w^2(t) + \frac{1}{2}\varphi_{uu}\sigma_M^2(t) + \frac{1}{2}\varphi_{zz}\left(\lambda_0^2\sigma_I^2(t) + \lambda_1^2\sigma_M^2(t)\right), \tag{3.56}$$

equation (3.55) can be rewritten as

$$\begin{aligned} dJ(t,X_t,X_t^M,Z_t) &= J_x w(t)\,(dB_t + d\theta_t) + J_u\sigma_M(t)d\beta_t^M \\ &+ J_z(t,X_t,X_t^M,Z_t)dZ_t \\ &+ J_{xu}\sigma_M(t)\gamma_1(t)w(t)dt \\ &+ J_{xz}w(t)(\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t))dt \\ &+ \frac{1}{2}J_{xx}w^2(t)(\gamma_0^2(t) + \gamma_1^2(t))dt \end{aligned} \tag{3.57}$$

or, equivalently, in the integral form

$$\begin{aligned} J(t,X_t,X_t^M,Z_t) &= J(0,0,0,z) + \int_0^1 J_x w(t)dB_t + \int_0^1 J_x w(t)d\theta_t \\ &+ \int_0^1 J_u\sigma_M(t)dB_+^M \int_0^1 J_z(t,X_t,X_t^M,Z_t)dZ_t \\ &+ \int_0^1 \beta_2(t)\sigma_M(t)\gamma_1(t)dt \\ &- \int_0^1 (\lambda_0(t)\gamma_0(t)\sigma_I(t) + \lambda_1(t)\gamma_1(t)\sigma_M(t))dt \\ &+ \frac{1}{2}\int_0^1 \sigma_V w(t)(\gamma_0^2(t) + \gamma_1^2(t))dt \end{aligned} \tag{3.58}$$

Next, for $T < 1$, let

$$W_T := \int_0^T \theta_t dZ_t + \int_0^T (Z_t - S_t)d\theta_t + [\theta, Z]_T - [\theta, S]_T,$$

and recall from (3.1) that $\lim_{T\to 1} W_T = W_1$; which the terminal wealth of the insider.

Moreover, applying equations (3.4), (3.5), and (3.58) into the above equation, we get that

$$\begin{aligned} J(T,X_T,X_T^M,Z_T) = J(0,0,0,z) - W_T &+ \int_0^T J_x(t,X_t,X_t^M,Z_t)w(t)dB_t \\ &+ \int_0^T J_u(t,X_t,X_t^M,Z_t)\sigma_M(t)d\beta_t^M - \frac{1}{2}\sigma_V\int_0^T w(t)(\gamma_0^2(t) + \gamma_1^2(t))dt \\ &+ \int_0^T J_z(t,X_t,X_t^M,Z_t)dZ_t. \end{aligned}$$

Observe that $\lim_{T\to 1}\phi(T, X_T, X_T^M, Z_T) \geq 0$, where the existence of limits follow from the continuity of $(W_t)_{t\in[0,1]}$ and the stochastic integrals. Thus,

$$
\begin{aligned}
W_1 \leq J(0,0,0,z) &+ \int_0^1 J_x(t, X_t, X_t^M, Z_t) w(t) dB_t + \int_0^1 J_u(t, X_t, X_t^M, Z_t)\sigma_M(t) d\beta_t^M \\
&+ \int_0^1 J_z(t, X_t, X_t^M, Z_t) dZ_t - \frac{1}{2}\sigma_V \int_0^T w(t)(\gamma_0^2(t) + \gamma_1^2(t)) dt.
\end{aligned}
$$

The admissibility conditions in equations (1.10) and (1.11) imply that the stochastic integrals above have zero mean. Therefore,

$$
\mathbb{E}^z[W_1] \leq J(0,0,0,z) - \frac{1}{2}\sigma_V \mathbb{E}^z\left[\int_0^T w(t)(\gamma_0^2(t) + \gamma_1^2(t)) dt\right],
$$

where if $\gamma_0 = \gamma_1 = 0$ then the inequality is an equality if and only if $\lim_{T\to 1} J(T, X_T, X_T^M, Z_T) = 0$. Therefore, an admissible strategy with the properties given in the statement is optimal.

□

Chapter 4

Rationality Condition

In this chapter, we address the issue of the rationality condition of the market maker. At first glance, it should not be so different than what we had in the previous case of the literature: the market maker projects V into her filtration to find her valuation of the risky asset. Therefore, we aim to find $\mathbb{E}(V|\mathscr{F}_t^M)$ for any given optimal strategy of the insider[1], but now the complexity of this task is much greater.

Indeed, the original condition Back (1992) has found for the price process to be a martingale in the market maker's filtration was that the pricing rule should follow a heat equation - since H was originally just a function of the demand, it would mean that in equilibrium the pricing rule should be such that $H_t + \frac{1}{2}\sigma^2 H_{yy} = 0$ where σ^2 was the variance of $\eta \sim N(0, \sigma^2)$. Proceeding further in the literature we can mention Campi et al. (2011) where the market maker also uses a statistic for the demand process such that $dX_t = w(t, X_t)dY_t$ where Y is the demand process. In this model, the rationality condition is satisfied in equilibrium if $H_t(t,x) + \frac{w(t,x)^2}{2} + H_{xx}(t,x) = 0$ and $w_t(t,x) + \frac{w(t,x)^2}{2} + w_{xx}(t,x) = 0$. As we shall see in this chapter, even with some simplifying hypothesis, such as the one given by Equation (1.8), we have a more complex task.

In Chapter 2 we mentioned that there was no counterpart in the literature to the insider's projection of η into her filtration because either she would know the outcome of this random variable from the start of the trading period or she would receive a private signal such that her valuation of the risky asset would trivially follow this process.

However, now the projection of V into the market maker is much more complex than

[1] Recall that any strategy that satisfies the conditions of Theorem 3.1 is optimal.

before. The market maker now has two sources of information. An endogenous one, resulting from her interaction with the insider by trading with her, as in the cited previous literature, and an exogenous one, X^I.

Furthermore, one can say that now there are two possible ways to project η into the filtration of the market maker. The first is η directly into the filtration of the market maker, but the second is the projection of η first into the insider's filtration, hence Z_t, and further into the filtration of the market maker. As we shall discuss shortly, not only do we have to deal with both projections now, but also they must coincide in some sense. In the previous cases in the literature, there would be no projection of the projection as, without X^M, X^I would determine the insider's valuation of the asset, and we would only worry about the projection of X^I into the market maker's projection.

Let us first note that the price process should follow the projection of V directly into the market maker's filtration as follows:

$$\mathbb{E}(V|\mathscr{F}_t^M) = \sigma_V \mathbb{E}(\eta|\mathscr{F}_t^M) + \mu.$$

How we find this projection is discussed in Section 4.2. However, as we mentioned before, since $\mathscr{F}_t^M \subset \mathscr{F}_t^I$ for all t in $[0,1]$, it is immediate from the tower property that

$$\mathbb{E}(V|\mathscr{F}_t^M) = \sigma_V \mathbb{E}(\eta|\mathscr{F}_t^M) + \mu = \sigma_V \mathbb{E}(\mathbb{E}(\eta|\mathscr{F}_t^I)|\mathscr{F}_t^M) + \mu = \sigma_V \mathbb{E}(Z_t|\mathscr{F}_t^M) + \mu \qquad (4.1)$$

The projection of Z into the market maker's filtration is developed in section 4.3. The fact that those projections must coincide is also discussed in that section. As presented in section 2.1, the stochastic filtering gives us an ODE for the variance of the projection of the signal process. As in our case, not only do we have to make sure that the projections coincide but also that we have made an important assertive about the form of the r_1 coefficient in Assumption 3.2 while solving the insider's maximization problem, we end up with a system of ODEs to be solved.

The solution of the ODEs is not trivial. In fact, we developed an innovative strategy to prove the existence of a modified version of the system in Section 4.4. The equivalence to the modified version with the original relies on a particularly suitable initial condition to the system that is defined in a circular manner itself. We developed a fixed-point algorithm to prove the existence of such an initial condition in Section 4.5 and ultimately the existence and uniqueness of the whole system. Hence, we use non-standard machinery in both sections 4.4 and 4.5 to prove the existence and uniqueness of this system of ODEs.

In Section 4.6 we investigate some further results on w that will be relevant to prove the existence of equilibrium in Chapter 6. In addition to that, those results are also required to show that $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s. in Chapter 5. Section 4.7 is a small optional section showing that indeed the price process converges to V. We call it optional as we use some more robust mathematical techniques to prove that $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s. in Chapter 5 which is what we actually need to prove the existence of an equilibrium.

However, before all that, we restrict ourselves to a smaller class of trading strategies than we have so far. From section 4.1 onwards we only consider linear strategies as will be defined in equation (4.4). Our aim in this model is to show the existence of an equilibrium such that it satisfies definition 1.3. We do not claim that the equilibrium we present in Theorem 6.1 is in any form unique, as is the case in the literature (see Section 7.2 of Çetin, Danilova (2018)). The only reason why we do not embed such restriction in equation (1.9) in Definition 1.2 is because we want to make sure that it is clear that such hypothesis is not necessary to prove Theorem 3.1.

4.1 Linear Strategy

Before we proceed presenting the linear strategies to which we restrict ourselves, let us recall two important facts that will be relevant to our discussion of the linear strategies.

The first fact is that the admissibility condition for the insider's strategy given by Definition 1.2, in particular Equation (1.9), and Theorem 3.1 ensure us that $\gamma_0 = \gamma_1 = 0$. As a consequence, we have that any optimal admissible trading strategy must be of the form $d\theta_t = \alpha_t dt$.

The second is from Theorem 2.1 we can find the innovation processes associated with the observation processes for the market maker. Indeed, applying the theorem mentioned above in equations (1.1) and (1.3) considering equation (1.9) with $\gamma_0 = \gamma_1 = 0$ we have that:

$$
\begin{aligned}
dN_t^{(1)} &= dB_t + (\alpha_t - \hat{\alpha}_t)dt, \text{ and} \\
dN_t^{(2)} &= dB_t^M + \sigma_M(t)\frac{X_t^M - \hat{\eta}_t}{1 - \Sigma_M(t)}dt.
\end{aligned}
\tag{4.2}
$$

where $N^{(1)}$ and $N^{(2)}$ are innovation processes associated with the filtration of the market maker. In line with the notation developed in Chapter 2, we denote by $\hat{\alpha}$ the $\mathscr{F}^M$-optional projection of α, which makes $\hat{\alpha}$ a version of $\mathbb{E}(\alpha|\mathscr{F}_t^M)$. Furthermore, we denote the optional projection of Z_t with respect to the insider's filtration by $\hat{Z}_t$.

Replace the above equations with (1.1) and (1.5), one gets the following:

$$
\begin{aligned}
dX_t^M &= \sigma_M(t)dN_t^{(2)} + \sigma_M^2(t)\frac{\hat{\eta}_t - X_t^M}{1 - \Sigma_M(t)}dt \\
dX_t &= w(t)\left(\hat{\alpha}_t dt + dN_t^{(1)}\right) + \left(r_0(t) + r_1(t)X_t + r_2(t)X_t^M\right)dt,
\end{aligned}
\tag{4.3}
$$

Recall from the conditions of Theorem 3.1 that if $\gamma_0 = \gamma_1 = 0$ a sufficient and necessary condition for an admissible strategy that is absolutely continuous with respect to the Lebesgue measure to be optimal is that $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.. Since our aim in this chapter is to show that there exists at least one optimal strategy, we shall do so by showing that there is an optimal linear strategy of the form:

$$
\alpha_t = \alpha_0(t) + \alpha_1(t)X_t + \alpha_2(t)X_t^M + \alpha_3(t)Z_t, \tag{4.4}
$$

for some measurable deterministic functions α_i, $i = 0, 1, 2$ and 3. Before we explain why we have chosen such functional form, let us take a moment to look at $\hat{\alpha}$, the projection α into the market maker's filtration under the above condition:

$$
\hat{\alpha}_t = \alpha_0(t) + \alpha_1(t)X_t + \alpha_2(t)X_t^M + \alpha_3(t)\hat{Z}_t. \tag{4.5}
$$

Therefore, the first interesting fact we get from this linear structure is that due to the linear statement of Theorem 2.1 and the fact that both X and X^M are observable to the market maker, we have that as long as we know the projection of Z into the market maker's filtration, we can easily get the value of $\hat{\alpha}$.

Furthermore, from the very definition of a rational pricing rule given in Definition 1.3, the price process must be a martingale in the filtration of the market maker. As a consequence, from Equation (3.33), we have that a necessary condition to have S_t as a martingale is that $\beta_1(t)w(t)\alpha_t$ is equal to zero for all $t \in [0,1]$. Therefore, in the next section, Section 4.2, we shall find the coefficients of $(\alpha_i)_{i=1}^3$ such that the price process is a martingale.

4.2 Optional Projection of V

As we mentioned at the end of the last section, by the very definition of a rational pricing rule given by Definition 1.3, the projection of V into the market maker's filtration, hence the price process, must be a martingale in the market maker's filtration. Furthermore, equation (3.33) shows that a necessary condition for it is that $\beta_1(t)w(t)\alpha_t$ is equal to zero for all $t \in [0,1]$.

Our strategy in this section to ensure that S is a martingale is to find the coefficients of $(\alpha_i)_{i=1}^3$ such that $\hat{\alpha}_t = 0$ for all $t \in [0,1]$. In fact, Lemma 4.1 below shows exactly that this is the case if we set $(\alpha_i)_{i=1}^3$ to be as in (4.6).

Lemma 4.1. *Suppose (H,w,r) is an admissible pricing rule and Assumption 3.2 holds. Assume further that the insider's trading strategy from (4.4) satisfies the following:*

$$\begin{aligned}
\alpha_0(t) &= (\mu - \beta_0(t))\frac{\alpha_3(t)}{\sigma_V}, \\
\alpha_1(t) &= -\alpha_3(t), \\
\alpha_2(t) &= -\beta_2(t)\frac{\alpha_3(t)}{\sigma_V}.
\end{aligned} \tag{4.6}$$

Then, the price process S admits the following $\mathscr{F}^M$-dynamics if $S_t = H(t, X_t, X_t^M)$ for each t and (H,w,r) is a rational pricing rule:

$$dS_t = \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)}, \tag{4.7}$$

where $N^{(i)}$s are the innovations processes given by (4.2).

Proof. First, we can apply the Ito formula to the price process as we did in the previous chapter and rewrite the equation (3.32) under the conditions of Assumptions 3.1 and 3.2:

$$
\begin{aligned}
dS_t &= \sigma_V w(t)\left(\alpha_0(t)+\alpha_1(t)X_t+\alpha_2(t)X_t^M+\alpha_3(t)\frac{S_t-\mu}{\sigma_V}\right)dt \\
&\quad +\sigma_V w(t)dN_t^{(1)}+\beta_2(t)\sigma_M(t)dN_t^{(2)}.
\end{aligned}
$$

Applying the coefficients (4.6): one gets:

$$
\begin{aligned}
dS_t &= w(t)\left(\alpha_3(t)(\mu-\beta_0(t))-\alpha_3(t)\sigma_V X_t-\alpha_3(t)\beta_2(t)X_t^M+\alpha_3(t)(S_t-\mu)\right)dt \\
&\quad +\sigma_V w(t)dN_t^{(1)}+\beta_2(t)\sigma_M(t)dN_t^{(2)} \\
&= w(t)\alpha_3(t)\left(-(\beta_0-\mu)-\sigma_V X_t-\beta_2(t)X_t^M+(S_t-\mu)\right)dt \\
&\quad +\sigma_V w(t)dN_t^{(1)}+\beta_2(t)\sigma_M(t)dN_t^{(2)} \\
&= w(t)\alpha_3(t)\left(-(S_t-\mu)+(S_t-\mu)\right)dt \\
&\quad +\sigma_V w(t)dN_t^{(1)}+\beta_2(t)\sigma_M(t)dN_t^{(2)} \\
&= \sigma_V w(t)dN_t^{(1)}+\beta_2(t)\sigma_M(t)dN_t^{(2)}.
\end{aligned}
$$

□

Therefore, the price process is a martingale in the filtration of the market maker. As a consequence, so must the projection of Z into the filtration of the market maker. We shall address such projection in the next section.

However, before that, as an immediate consequence of the fact that the price process is a martingale in the market maker's filtration, we have that so the demand is. In fact, we call the next proposition a corollary because they are the same parameters that make the price a martingale that also make the demand a martingale because $\mathbb{E}(\alpha_t|\mathscr{F}_t^M)=0$ for all $t\in[0,1]$.

Corollary 4.1. *Under the conditions of Lemma 4.1, the demand process, Y, is a $\mathscr{F}^M$-Brownian motion.*

Proof. From equation (4.3), we have $dY_t=\hat{\alpha}_t dt+dN_t^{(2)}$. Combining that with equation (4.5) and (4.6), we have that

$$
\begin{aligned}
dY_t &= \left(\alpha_0(t) + \alpha_1(t)X_t + \alpha_2(t)X_t^M + \alpha_3(t)\hat{Z}_t\right) dt + dN_t^{(2)} \\
&= \left((\mu - \beta_0(t))\frac{\alpha_3(t)}{\sigma_V} - \alpha_3(t)X_t - \beta_2(t)\frac{\alpha_3(t)}{\sigma_V}X_t^M + \alpha_3(t)\hat{Z}_t\right) dt + dN_t^{(2)} \\
&= \frac{\alpha_3(t)}{\sigma_V}\left(-\beta_0(t) - \sigma_V X_t - \beta_2(t)X_t^M + \sigma_V \hat{Z}_t + \mu\right) dt + dN_t^{(2)} \\
&= \frac{\alpha_3(t)}{\sigma_V}\left(-S_t + S_t\right) dt + dN_t^{(2)} = dN_t^{(2)} \qquad (4.8)
\end{aligned}
$$

4.3 Optional Projection of Z_t

As we mentioned in the beginning of the chapter, we can now proceed to find the projection of Z into the market maker's filtration. We can now apply the theory of stochastic filtering to find the distribution of $\hat{Z}_t = \mathbb{E}\left[Z_t|\mathscr{F}_t^M\right]$ as we did in Chapter 2 for the insider's filtration.

Before we proceed, it is important to realise that the distribution of $Z_t|\mathscr{F}_t^M$ is not the same as the distribution of $\eta|\mathscr{F}_t^M$ as they only share the same expectation. In this section, we will also discuss the distribution of $\eta|\mathscr{F}_t^M$. As they are both Gaussian random variables, the only thing that remains to be shown is the variance as we know, from the previous section, in particular Lemma 4.1, that

$$
\mathbb{E}\left[Z_t|\mathscr{F}_t^M\right] = \mathbb{E}\left[\mathbb{E}[\eta|\mathscr{F}_t^I]|\mathscr{F}_t^M\right] = \mathbb{E}\left[\eta|\mathscr{F}_t^M\right] = \frac{S_t - \mu}{\sigma_V}
$$

Let us now define $v(t) := \mathbb{E}[Z_t^2|\mathscr{F}_t^M] - \hat{Z}_t^2$. Now we can easily find the distribution of $\eta|\mathscr{F}_t^M$. Indeed, applying the law of total variance to η into the market maker's filtration one gets

$$
\begin{aligned}
Var(\eta|\mathscr{F}_t^M) &= \mathbb{E}[Var(\eta|\mathscr{F}_t^I)|\mathscr{F}_t^M] + Var(\mathbb{E}[\eta|\mathscr{F}_t^I)]|\mathscr{F}_t^M) \\
&= \mathbb{E}[1 - \Sigma_Z(t)|\mathscr{F}_t^M] + Var(Z_t|\mathscr{F}_t^M) \\
&= 1 - \Sigma_Z(t) + v(t) \qquad (4.9)
\end{aligned}
$$

An immediate consequence of it that will be relevant to us in the next section is that since $Var(\eta|\mathscr{F}_0^M)$ is the unconditional variance of η, we have that

$$1 = 1 - \Sigma_Z(0) + v(0). \tag{4.10}$$

Now we can proceed with the projection of Z into the market maker's filtration through Lemma 4.2 below. This lemma is just an application of Theorem 2.2. Usually, one would go further to find a process that would be a solution for the system given by (4.11) as we did for the insider. However, we cannot do it yet because, as we discussed previously, there are other requirements we need to fulfil as we need the projection of η directly into the market maker's filtration to coincide with the projection of η first into the insider's filtration than further into the market maker's filtration.

Lemma 4.2. *Suppose that (H, w, r) is an admissible pricing rule and α satisfies (4.4). Then, for each $t \in [0, 1]$, the conditional distribution of Z_t given $\mathscr{F}_t^M$ is Gaussian with mean $\hat{Z}_t$ and variance v_t, where*

$$\begin{aligned} d\hat{Z}_t &= v(t)\alpha_3(t)dN_t^{(1)} + \frac{\sigma_M(t)}{1 - \Sigma_M(t)}\left(v(t) + 1 - \Sigma_Z(t)\right)dN_t^{(2)}, \quad \hat{Z}_0 = 0; \\ \sigma_Z^2(t) - v'(t) &= v^2(t)\alpha_3^2(t) + \frac{\sigma_M^2(t)}{(1 - \Sigma_M(t))^2}\left(v(t) + 1 - \Sigma_Z(t)\right)^2, \quad v(0) = \Sigma_Z(0). \end{aligned} \tag{4.11}$$

Proof. We shall use the same setting as we did in Chapter 2 to find the optional projection of Z into the market maker's filtration. First, note that now we have Z_t as the signal process and X_t and X_t^M as the observational processes; we can rewrite it in terms of Z_t, X_t and X_t^M:

$$dZ_t = [A_0(t) + A_1(t)Z_t + A_2\bar{X}_t]dt + B(t)dW_t \tag{4.12}$$

$$dX_t = [C_0(t) + C_1(t)Z_t + C_2\bar{X}_t]dt + D(t)dW_t \tag{4.13}$$

where

$$\bar{X}_t = \begin{bmatrix} X_t^M \\ X_t \end{bmatrix} \tag{4.14}$$

and W_t is a $(m+n)$-dimensional Brownian motion given by

$$W_t = \begin{bmatrix} \beta_t^I \\ \beta_t^M \\ B_t \end{bmatrix}. \tag{4.15}$$

In our case, we can apply the filtering equations from the insider's filtration into the market maker's filtration. As we have seen, we can write Z_t as

$$dZ_t = \lambda_0(t)\sigma_I(t)d\beta_t^I + \lambda_1(t)\sigma_M(t)d\beta_t^M$$

where $d\beta^I$ and $d\beta^M$ are independent Brownian motions in the insider's filtration. Furthermore, according to Theorem 2.3, in particular (2.34), we have that the public signal can be written as

$$dX_t^M = \sigma_M(t)d\beta_t^M + \sigma_M^2(t)\frac{Z_t - X_t^M}{1-\Sigma_M(t)}dt$$

and the demand process as

$$\begin{aligned} dX_t &= w(t)\left[\left(\alpha_0(t) + \alpha_1(t)X_t + \alpha_2(t)X_t^M + \alpha_3(t)Z_t\right)dt + dB_t\right] \\ &\quad + \left(r_0(t) + r_1(t)X_t + r_2(t)X_t^M\right)dt \end{aligned} \tag{4.16}$$

Therefore, the coefficients of Theorem 2.2 are as follows:

$$
\begin{aligned}
A_0(t) &= A_1(t) = A_1(t) = A_2(t) = 0 && (4.17)\\
B(t) &= \begin{bmatrix} \lambda_0(t)\sigma_I(t) & \lambda_1(t)\sigma_M(t) & 0 \end{bmatrix} && (4.18)\\
C_0(t) &= \begin{bmatrix} w(t)\alpha_0(t) + r_0(t) \\ 0 \end{bmatrix} && (4.19)\\
C_1(t) &= \begin{bmatrix} w(t)\alpha_3(t) \\ \frac{\sigma_M^2(t)}{1-\Sigma_M(t)} \end{bmatrix} && (4.20)\\
C_2(t) &= \begin{bmatrix} w(t)\alpha_1(t) + r_1(t) & w(t)\alpha_2(t) + r_2(t) \\ 0 & \frac{-\sigma_M^2(t)}{1-\Sigma_M(t)} \end{bmatrix} && (4.21)\\
D(t) &= \begin{bmatrix} 0 & 0 & w(t) \\ 0 & \sigma_M(t) & 0 \end{bmatrix} && (4.22)
\end{aligned}
$$

Equation (2.4) of Theorem 2.2 gives the general form

$$
\begin{aligned}
d\hat{Z}_t &= [A_0(t) + A_1(t)\hat{Y}_t + A_2 X_t]dt \\
&\quad +[v(t)C_1^*(t) + B(t)D^*(t)][D(t)D^*(t)]^{-\frac{1}{2}} dN_t && (4.23)
\end{aligned}
$$

since the innovation process must have the same dimension as the observational process, N_t is a two-dimensional $\mathscr{F}_t^M$-Brownian motion with coordinates $N_t^{(1)}$ and $N_t^{(2)}$.

Moreover, * stands for the transpose matrix. Applying (2.4) to our particular case, we have the following.

$$
\begin{aligned}
d\hat{Z}_t &= \Bigg\{ v(t) \left[w(t)\alpha_3(t) \quad \tfrac{\sigma_M^2(t)}{1-\Sigma_M(t)} \right] \\
&\quad + \left[\lambda_0(t)\sigma_I(t) \quad \lambda_1(t)\sigma_M(t) \quad 0 \right] \begin{bmatrix} 0 & 0 \\ 0 & \sigma_M(t) \\ w(t) & 0 \end{bmatrix} \Bigg\} \begin{bmatrix} \frac{1}{w(t)} & 0 \\ 0 & \frac{1}{\sigma_M(t)} \end{bmatrix} dN_t \\
&= \left\{ \left[v(t)w(t)\alpha_3(t) \quad v(t)\tfrac{\sigma_M^2(t)}{1-\Sigma_M(t)} \right] + \left[0 \quad \lambda_1(t)\sigma_M^2(t) \right] \right\} \begin{bmatrix} \frac{1}{w(t)} & 0 \\ 0 & \frac{1}{\sigma_M(t)} \end{bmatrix} dN_t \\
&= \left\{ \left[v(t)w(t)\alpha_3(t) \quad v(t)\tfrac{\sigma_M^2(t)}{1-\Sigma_M(t)} + \lambda_1(t)\sigma_M^2(t) \right] \right\} \begin{bmatrix} \frac{1}{w(t)} & 0 \\ 0 & \frac{1}{\sigma_M(t)} \end{bmatrix} dN_t \\
&= \left\{ \left[v(t)\alpha_3(t) \quad v(t)\tfrac{\sigma_M(t)}{1-\Sigma_M(t)} + \lambda_1(t)\sigma_M(t) \right] \right\} dN_t \\
&= v(t)\alpha_3(t)dN_t^{(1)} + \left(v(t)\frac{\sigma_M(t)}{1-\Sigma_M(t)} + \lambda_1(t)\sigma_M(t) \right) dN_t^{(2)} \\
&= v(t)\alpha_3(t)dN_t^{(1)} + \sigma_M(t) \left(\frac{v(t)}{1-\Sigma_M(t)} + \lambda_1(t) \right) dN_t^{(2)} \\
&= v(t)\alpha_3(t)dN_t^{(1)} + \sigma_M(t) \left(\frac{v(t)}{1-\Sigma_M(t)} + \frac{1-\Sigma_Z(t)}{1-\Sigma_M(t)} \right) dN_t^{(2)} \\
&= v(t)\alpha_3(t)dN_t^{(1)} + \frac{\sigma_M(t)}{1-\Sigma_M(t)} \left(v(t) + 1 - \Sigma_Z(t) \right) dN_t^{(2)}. \qquad (4.24)
\end{aligned}
$$

Recall that (2.5) from Theorem 2.2 also provides a formula for the variance of the process $v(t)$ as follows:

$$
\begin{aligned}
\frac{dv(t)}{dt} &= A_1(t)v(t) + v(t)A_1^*(t) + B(t)B^*(t) - [v(t)C_1^*(t) \\
&\quad + B(t)D^*(t)][D(t)D^*(t)]^{-1}[C_1(t)v(t) + D(t)B^*(t)]
\end{aligned}
$$

Replacing equations (4.17) - (4.22) in them gives us the following.

$$
\begin{aligned}
\frac{dv(t)}{dt} &= \lambda_0^2(t)\sigma_I^2(t) + \lambda_1^2(t)\sigma_M^2(t) \\
&\quad - \begin{bmatrix} v(t)\alpha_3(t) & v(t)\frac{\sigma_M(t)}{1-\Sigma_M(t)} + \lambda_1(t)\sigma_M(t) \end{bmatrix} \begin{bmatrix} \frac{1}{w^2(t)} & 0 \\ 0 & \frac{1}{\sigma_M^2(t)} \end{bmatrix} \begin{bmatrix} v(t)\alpha_3(t) \\ v(t)\frac{\sigma_M(t)}{1-\Sigma_M(t)} + \lambda_1(t)\sigma_M(t) \end{bmatrix} \\
&= \sigma_Z^2(t) \\
&\quad - \begin{bmatrix} v(t)\alpha_3(t) & \frac{\sigma_M(t)}{1-\Sigma_M(t)}\left(v(t)+1-\Sigma_Z(t)\right) \end{bmatrix} \begin{bmatrix} \frac{1}{w^2(t)} & 0 \\ 0 & \frac{1}{\sigma_M^2(t)} \end{bmatrix} \begin{bmatrix} v(t)\alpha_3(t) \\ \frac{\sigma_M(t)}{1-\Sigma_M(t)}\left(v(t)+1-\Sigma_Z(t)\right) \end{bmatrix}.
\end{aligned}
$$

Hence,

$$
\sigma_Z^2(t) - v'(t) = v^2(t)\alpha_3^2(t) + \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}\left(v(t)+1-\Sigma_Z(t)\right)^2 \tag{4.25}
$$

Now note that $v(0) = Var(Z_0|\mathscr{F}_0^M)$ which is the unconditional variance of Z_0. Since $Var(Z_0) = \Sigma_Z(0)$, then $v(0) = \Sigma_Z(0)$. □

4.4 Existence and Uniqueness

We now have all the ingredients to set the system of ODEs that must be solved to find the solution of v, hence the distribution of $\eta|\mathscr{F}_t^M$.

The first thing to mention is that from equation (4.1), we have

$$
dS_t = \sigma_V d\hat{Z}_t
$$

Combining the above equation with equation (4.11) from Lemma 4.2, which gives us the projection of η into the market maker's filtration through the projection of Z into the market maker's filtration, with equation (4.7) from Lemma 4.1 obtained by projecting η directly into the market maker's filtration and the martingale requirement given by the rational pricing rule, gives us the following equality:

$$
\begin{aligned}
dS_t &= \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)}. \\
dS_t &= \sigma_V v(t)\alpha_3(t) dN_t^{(1)} + \sigma_V \frac{\sigma_M(t)}{1-\Sigma_M(t)}\left(v(t)+1-\Sigma_Z(t)\right) dN_t^{(2)}.
\end{aligned}
$$

Therefore, we must have, for all $t < 1$:

$$\begin{aligned} \sigma_V \frac{\sigma_M(t)}{1-\Sigma_M(t)}\left(v(t)+1-\Sigma_Z(t)\right) &= \beta_2(t)\sigma_M(t) & (4.26)\\ v(t)\alpha_3(t) &= \beta_1(t)w(t) & (4.27)\end{aligned}$$

However, before setting the system, one should recall that from Assumption 3.2 the following identity also holds for all $t < 1$:

$$-\frac{w'(t)}{w(t)} = 2\frac{\beta_2(t)}{\sigma_V}\frac{\sigma_M^2(t)}{1-\Sigma_M(t)}, \tag{4.28}$$

Therefore, considering the ODE for the variance given by equation (4.11), we have an ODE system that can be summarised by the following:

$$\begin{aligned} \frac{w'(t)}{w(t)} &= -2(v(t)+1-\Sigma_Z(t))\frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2},\\ \sigma_Z^2(t) - v'(t) &= w^2(t) + \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}(v(t)+1-\Sigma_Z(t))^2, \quad v(0)=c^2. \end{aligned} \tag{4.29}$$

for some initial condition $w(0)$ yet to be determined.

Indeed, we cannot look at this system purely from an ODE point of view. Recall from equation (4.9) that $v + 1 - \Sigma_Z$ is the variance of $\eta|\mathscr{F}_t^M$. Therefore, we do not want any solution to v. We need a solution for v such that $v+1-\Sigma_Z$ is such that $v(0)+1-\Sigma_Z(0) = 1$, $v(1) + 1 - \Sigma_Z(1) = 0$, and $v + 1 - \Sigma_Z$ are decreasing.

The main result of this section, Proposition 4.1, proves the existence and uniqueness of a solution for a modified version of the above system that only considers the positive part of $v+1-\Sigma_Z$ with all the above requirements satisfied. Furthermore, this proposition shows that if $w^{-\frac{1}{2}}(0) \geq \int_0^t w^{\frac{3}{2}}(s)ds$, $\forall t \in [0,1)$, then $v(t) + 1 - \Sigma_Z(t) > 0$ for all $t \in [0,1)$. Therefore, if the initial condition of w is such that the above inequality holds, the positive part restriction we set for $v + 1 - \Sigma_Z$ is not triggered.

As a consequence, in section 4.5 we develop a fixed point algorithm to find $w(0)$ such that $w(0) = \left(\int_0^t w^{\frac{3}{2}}(s)ds\right)^{-2}$. Therefore, we are able to show that under that particular initial condition the system has a unique solution such that $v + 1 - \Sigma_Z$ is indeed the variance of

$\eta | \mathscr{F}_t^M$.

In order to keep this thesis self-contained, we present two important results from Khalil (2002) that will be the key to proving Proposition 4.1. Indeed, the major result that allows us to show the existence and uniqueness of our case is the following:

Theorem 4.1 (Theorem 3.2 of Khalil (2002))**.** *Suppose that $f(t,x)$ is piecewise continuous in t and satisfies*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

for all $x, y \in \mathbb{R}^n$, $\forall\, t \in [t_0, t_1]$. Then, the state equation $\dot{x} = f(t,x)$, with $x(t_0) = x_0$, has a unique solution over $[t_0, t_1]$.

As a consequence of the above theorem, it becomes clear that a key element in the proof of the existence and uniqueness of the system in (4.30) is to show that the solutions are indeed Lipschitz. The way we use to prove that the pair of solutions is indeed Lipschitz is by showing that the equations in (4.30) are uniformly bounded and by evoking the following lemma:

Lemma 4.3 (Lemma 3.3 of Khalil (2002))**.** *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b] \times \mathbb{R}^n$, then f is globally Lipschitz in x on $[a,b] \times \mathbb{R}^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b] \times \mathbb{R}^n$.*

We can now state and prove the main proposition of this section:

Proposition 4.1. *Consider the system*

$$\begin{aligned} \frac{w'(t)}{w(t)} &= -2(v(t)+1-\Sigma_Z(t))^+ \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}, \\ \sigma_Z^2(t) - v'(t) &= w^2(t) + \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}(v(t)+1-\Sigma_Z(t))(v(t)+1-\Sigma_Z(t))^+, \quad v(0) = c^2, \end{aligned} \tag{4.30}$$

where $x^+ := \max\{x, 0\}$ for any $x \in \mathbb{R}$. For any initial condition $w(0) > 0$, there exists a unique continuous solution (w, v) to (4.30) on $[0,1]$. Moreover, the following statements are valid:

1. *w and $v + 1 - \Sigma_Z$ are decreasing and $v \leq \Sigma_Z$.*

2. *w and $v + 1 - \Sigma_Z$ do not depend on Σ_Z.*

3. For all $t \leq 1$,

$$v(t) + 1 - \Sigma_Z(t) = w^{\frac{1}{2}}(t) \left(w^{-\frac{1}{2}}(0) - \int_0^t w^{\frac{3}{2}}(s) ds \right), \text{ and} \tag{4.31}$$

$$w(t) = w(0) \exp \left(-2 \int_0^t (v(s) + 1 - \Sigma_Z(s))^+ \frac{\sigma_M^2(s)}{\sigma_V (1 - \Sigma_M(s))^2} ds \right) \tag{4.32}$$

4. If

$$w^{-\frac{1}{2}}(0) - \int_0^t w^{\frac{3}{2}}(s) ds \geq 0, \quad \forall t \in [0, 1), \tag{4.33}$$

then $0 < v(t) + 1 - \Sigma_Z(t) \leq 1 - \Sigma_M(t)$ *for all* $t \in [0, 1)$.

Proof. We shall begin the proof by the existence and uniqueness of the system given by (4.30).

Let us fix $T < 1$ and define

$$x = (w, u) \tag{4.34}$$

where w and u are the solutions of (4.30) defined on $[0, 1)$. In particular, we denote $v + 1 - \Sigma_Z$ by u. Hence, as a consequence of equation (4.10), we have that $u(0) = 1$. Therefore, the system (4.30) can be written as

$$\nabla x(t) = f(t, x) \tag{4.35}$$

where $\nabla x(t) := (w'(t), u'(t))$ and $f(t, x) = (f_1(t, x), f_2(t, x))$ with

$$f_1(t, x) = -2wu^+ \frac{\sigma_M^2(t)}{(1 - \Sigma_M(t))^2}, \quad f_2(t, x) = -w^2 - uu^+ \frac{\sigma_M^2(t)}{(1 - \Sigma_M(t))^2}.$$

Note that both derivatives of the system are negative, so w and u are nonincreasing. In particular, note that

$$w(t) \geq w(0) \exp \left(\frac{2}{\sigma_V} \frac{\Sigma_M(t)}{1 - \Sigma_M(t)} \right)$$

which is finite for all $t \leq T < 1$.

Now, suppose that there is an $t^o < T$ such that $u(t^o) = 0$. Then for all t such that $t^o < t < T$

$$u(t) = -\int_{t^o}^t w^2(s) ds$$

as the second term in the r.h.s. of the second equation of (4.30) has disappeared. Since we have just seen that w cannot be negative, u will remain negative. However, if that is the case, the rate of w becomes zero by the first equation of (4.30). Recall also that $w(0) \geq w(t)$ for all $t \in [0, t]$. Hence, as a limiting case, suppose that $u(0) = 0$, then the farthest $u(t)$ can go up to time T is $-w^2(0)T$.

Hence, the images of w and u are in compacts $[0, w(0)]$ and $[-w^2(0)T, 1]$, hence making the derivatives w' and u' uniformly bounded. Next, observe that, for each $t \leq T$, the function $f(t, \cdot) : [0, w(0)] \times [-w^2(0)T, 1] \to \mathbb{R}$ is Lipschitz since σ_M is continuous. Therefore, by Lemma 4.3 we find that f as defined above is globally Lipschitz.

As a consequence, it allows us to claim Theorem 4.1 to guarantee the existence and uniqueness of the system given by (4.30) in $[0, T]$.

At this point, we do not need to calculate the limits of W and u as they go to one. All that is relevant now is that since they are both monotonic functions, Theorem 4.29 of Rudin (1976) guarantees the existence of the limit.

Now, note that since u is decreasing, we have $u \leq u(0) = 1$, hence $v + 1 - \Sigma_Z \leq 1$, which implies that $v \leq \Sigma_Z$.

Note that u does not depend on Σ_Z even through its initial condition. Therefore, if u is the solution of (4.30) and Σ_Z is an exogenous function, we can define

$$v \equiv u - (1 - \Sigma_Z).$$

Since w is only a function of u, which is independent of Σ_Z, it also does not depend on Σ_Z.

Let us now prove (4.32), take w as given by it. Obviously $\frac{\partial}{\partial t} \ln w(t)$ is indeed the first equation of (4.35). Since (w, v) is a unique solution for the system, we have the desired result.

Let w be the unique solution of of (4.35) as defined above. Lets define $\bar{u}$ as the following:

$$\bar{u}(t) = \sqrt{w(t)} \left(k - \int_0^t w^{3/2}(s) ds \right) \tag{4.36}$$

for some $k \in \mathbb{R}$.

$$\begin{aligned}
\bar{u}'(t) &= \frac{1}{2}\frac{w'(t)}{\sqrt{w}(t)}\left(k - \int_0^t w^{3/2}(s)ds\right) - \sqrt{w(t)}w^{3/2}(t) \\
&= \frac{1}{2}\frac{w'(t)}{w(t)}\sqrt{w}(t)\left(k - \int_0^t w^{3/2}(s)ds\right) - w^2(t) \\
&= \frac{1}{2}\frac{w'(t)}{w(t)}\bar{u}(t) - w^2(t) \\
&= -\frac{\sigma_V \sigma_M^2(t)}{(1-\Sigma_M(t))^2}\bar{u}(t)\left(v(t)+1-\Sigma_Z(t)\right)^+ - w^2(t)
\end{aligned}$$

the last equation being true because of the first equation of (4.30). Set $k = \frac{1}{\sqrt{w(0)}}$, so we have $\bar{u}(0) = 1$ Since now $\bar{u}$ is one solution to u', by the uniqueness of u, we must have $\bar{u} = u$.

Now suppose (4.33) holds; then

$$w^{-\frac{1}{2}}(0) - \int_0^t w^{\frac{3}{2}}(s)ds > 0, \quad \forall t \in [0,1).$$

As we did before, suppose that there is $t^0 < 1$ such that the above equation is zero. Since w does not vanish in $[0,T]$, we have $\int_{t^0}^t w^{\frac{3}{2}}(s) > 0$ making the above integral negative. Hence, by Equation (4.31), we have $u > 0$ in $[0,T]$. In fact, if the above is 0 for some $t^0 < 1$, it will become strictly negative for all $t > t^o$ since w never is 0. Thus, $u > 0$. Therefore, if (4.33) holds, we have

$$\sigma_Z^2(t) - v'(t) = w^2(t) + \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}(v(t)+1-\Sigma_Z(t))(v(t)^2$$

Hence,

$$\frac{\partial}{\partial t}\left(\frac{1}{u(t)}\right) \geq \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}$$

Therefore,

$$\frac{1}{u(t)} - 1 \geq \int_0^t \frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2} = \frac{1}{(1-\Sigma_M(t)} - 1$$

Rearranging the terms and recalling that we have already concluded that, if (4.33) holds, $u > 0$ leads to our statement. □

We have proven the existence and uniqueness of a system that coincides with (4.29) if

$u > 0$ for all t in $[0, 1)$. As we have seen in the proof of the last proposition, such a condition is satisfied when equation (4.33) is satisfied. Therefore, our next task, that we shall develop in the next section, is to find an initial condition such that (4.33) is satisfied.

4.5 Initial Condition

From Proposition 4.1 we see that as long as $w(0) \leq \left(\int_0^1 w^{3/2}(t)ds\right)^{-2}$, there is a unique solution to (4.29) such that $v + 1 - \Sigma_Z$ behaves as a variance should.

Therefore, if we are able to set the initial condition such that

$$w(0) = \left(\frac{1}{\int_0^1 w^{3/2}(t)dt}\right)^2. \tag{4.37}$$

we would be able to finally guarantee the existence and uniqueness of the system given by (4.29).

However, one may note that there is a circular reasoning in the above equation: $w(0)$ is determined by the function w, which obviously depends on knowing the value of $w(0)$ to be calculated. In order to show that equation (4.37) has a solution, we present a fixed point algorithm in Theorem 4.3 to show that there is a fixed point function for the ODE system (4.30) with initial condition $v(0) + 1 - \Sigma_Z(0) = 1$ and $w(0)$ given by equation (4.37).

With the result provided by Theorem 4.3, we are able to prove the main theorem of this section, which is Theorem 4.4. It shows that under the conditions of Theorem 4.3 all the first three numbered statements of Proposition 4.1 are true.

Before explaining the steps of the fixed point algorithm, note that, in view of equation (4.30), we may rewrite equation (4.37) as follows:

$$w(0) = \Big(\int_0^1 \exp\left(-3\int_0^t (v(s) + 1 - \Sigma_Z(s))^+ \frac{\sigma_M^2(s)}{(1 - \Sigma_M(s))^2} ds\right) dt\Big)^{-\frac{1}{2}}. \tag{4.38}$$

We can easily check that the above equation is true by noticing that if $w(0)$ is defined as above, we have that

$$
\begin{aligned}
\sqrt{w(0)} &= \frac{w^2(0)}{w^{3/2}(0)} \\
&= \frac{1}{\int_0^1 w^{3/2}(0)\exp\left(-3\int_0^t (v(s)+1-\Sigma_Z(s))^+ \frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2}ds\right)dt} \\
&= \frac{1}{\int_0^1 w^{3/2}(t)dt} \qquad (4.39)
\end{aligned}
$$

which is equivalent to (4.37).

We can now move on to the description of the fixed-point algorithm we use in this section.

Let us begin the description by defining the operator T. Denote by $C([0,1)$ the space of continuous functions in $[0,1)$ and let the function $T:(0,\infty)\to C([0,1)\times C([0,1)$ defined by

$$T(r) = (T^1(r), T^2(r)) = (w,v),$$

where (w,v) is the unique solution of (4.30) with $w(0)=r$. Therefore, for any initial condition $w(0)=r$, the operator provides the solution to our system (4.30).

We can now consider the initial value for our algorithm. If we start with $r_0 = \frac{\Gamma}{2}$ where Γ is given by

$$\Gamma = \left(\int_0^1 \exp\Big(-3\frac{\Sigma_M(t)}{1-\Sigma_M(t)}\Big)dt\right)^{-\frac{1}{2}} \qquad (4.40)$$

so that $w_0 = T^1(r_0)$ and $v_0 = T^2(r_0)$. Note that $\Gamma < \infty$ since $\int_0^1 (1-\Sigma_M(s))^3 ds = 0$ if and only if $\Sigma_M(0) = 1$ which goes against the condition that $\Sigma_M(t) < 1$ for all $t < 1$ - recall that we impose this condition because if $\Sigma_M(t) = 1$ for some $t < 1$ we would have the public information about the true value of the asset being released before time 1.

Once we have the initial condition for our algorithm, we can move forward presenting it by showing its updating rule. Define for $n \geq 1$,

$$r_n = \Big(\int_0^1 \exp\left(-3\int_0^t (v_{n-1}(s)+1-\Sigma_Z(s))^+ \frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2}ds\right)dt\Big)^{-\frac{1}{2}}. \qquad (4.41)$$

Hence, applying the operator T defined above, we can define the sequence ϕ such that

$$T(r_n) = \phi_n(t) = (v_n + 1 - \Sigma_Z, w_n) \qquad (4.42)$$

on $[0,1]$ where $v_n(t)+1-\Sigma_Z$ and $w_n(t)$ are the solution to the following system of ODEs:

$$\frac{w_n'(t)}{w_n(t)} = -2\left(v_n(t)+1-\Sigma_Z(t)\right)^+ \frac{\sigma_M^2}{(1-\Sigma_M)^2} \tag{4.43}$$

$$\sigma_Z^2 - v_n' = w_n^2(t) + \sigma_M^2\left(\frac{v_n(t)+1-\Sigma_Z(t)}{(1-\Sigma_M(t))^2}\right)\left(v_n(t)+1-\Sigma_Z(t)\right)^+ \tag{4.44}$$

with initial conditions given by $v_n(0)+1-\Sigma_Z(0)=1$ for all $n\in\mathbb{N}\cup\{0\}$ and $w_n(0)=r_{n-1}$ for all $n\in\mathbb{N}$ and $w_0(0)=\frac{\Gamma}{2}$.

Our goal is to show that one can extract a convergent subsequence $(r_n, w_n.v_n)$ so that the limit (w,v) solves (4.30) satisfying (4.33). That will be the case because r_n has a limit.

It should be clear now what the rationale is behind the fixed-point algorithm. Every iteration of the algorithm produces a new value for $w_n(0)$ and following produces new functions w_n and $v_n+1-\Sigma_Z$. If at some step we find that the value $w_n(0)=w_{n+1}(0)$ the functions w_n and $v_n+1-\Sigma_Z$ will not be updated any more and we will have reached a fixed point.

As the initial value of ϕ_n, $w_n(0)$ depends only on the functions $w_{n-1}(0)$ and $v_{n-1}(0)$, so that the operator T such that

$$\phi_{n+1}=T\phi_N.$$

Our strategy to show the existence of a fixed point is to show that there exists a sequence of $(v_n'(t)+\sigma_Z(t), w_n'(t))$ that converges uniformly on $[0,T]$ for any $T<1$ and that $(w_n(0))$ converges to $(w(0))$. By doing that, we can apply theorem 4.2 below to guarantee that there exists a limit function such that the limit function is indeed the derivative of the limit of ϕ_n.

Theorem 4.2 (Rudin (1976) 7.17)**.** *Suppose $\{f_n\}$ is a sequence of functions differentiable from $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a,b]$. If $\{f_n'\}$ converges uniformly in $[a,b]$, then $\{f_n\}$ converges uniformly in $[a,b]$, to a function f, and*

$$f'(x)=\lim_{n\to n} f_n'(x) \quad (a\le x\le b).$$

We have a lengthy lemmata until we are able to reach Theorem 4.3. The structure of it is such that each lemma is used to prove the following. Hence, the reader could first read the proof of Theorem 4.3 to check why every lemma is important and work it backwards until they come back to this point. Our task begins by showing that ϕ_n is bounded.

Lemma 4.4. *$(\phi_n)_{i=0}^{\infty}$ as defined by equation (4.42) is a bounded sequence of functions.*

Proof. Note from equation (4.42) that the sequence $(r_n)_{i=0}^{\infty}$ defines a sequence of functions $(\phi_n)_{i=0}^{\infty}$. By Proposition 4.1, we find that $(\phi_n)_{i=0}^{\infty}$ is bounded as long as $(r_n)_{i=0}^{\infty}$ is bounded. Therefore, all we need to show is that the sequence $(r_n)_{i=0}^{\infty}$ is bounded.

First, note that by the construction of (4.43)

$$\frac{w_n'(t)}{w_n(t)} = -2\left(v_n(t) + 1 - \Sigma_Z(t)\right)^+ \frac{\sigma_M^2}{(1-\Sigma_M)^2} \leq 0$$

therefore, we have $r_n^2 \geq 1$.

Moreover, equation (4.43) also shows that since $(v_n(t) + 1 - \Sigma_Z(t))^+ \leq 1$:

$$\begin{aligned} r_n^{-2} = \left(-3\int_0^t u^+(s)\frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2}ds\right) &\geq \exp\left(-3\int_0^t \frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2}ds\right) \\ &= \exp\Big(-3\frac{\Sigma_M(t)}{1-\Sigma_M(t)}\Big). \end{aligned}$$

Thus, for $n \geq 1$,

$$1 \leq r_n^2 \leq \frac{1}{\int_0^1 \exp\Big(-3\frac{\Sigma_M(t)}{1-\Sigma_M(t)}\Big)dt} = \Gamma^2 < \infty. \tag{4.45}$$

□

The previous lemma guarantees that ϕ_n with image in $[0,1]\times[0,\Gamma]$ is well-defined for all $n \in \mathbb{N}\cup\{0\}$. (i.e., $\phi_n : \mathbb{R} \to [0,1]\times[0,\Gamma]$).

We can now show that the sequence of functions defined by our algorithm,$(v_n+1-\Sigma_Z, w_n)$, is indeed equicontinuous.

Lemma 4.5. *$(v_n+1-\Sigma_Z, w_n)$ are equicontinuous sequences of functions $[0,T]$ for any $T<1$.*

Proof. One may start by noticing that by equation (4.44):

$$\begin{aligned} |v_n'(t) - \sigma_Z| &= w_n^2 + \sigma_M^2\left(\frac{(v_n+1-\Sigma_Z)^+}{1-\Sigma_M}\right)^2 \\ &\leq \Gamma^2 + \frac{\bar{\sigma}_M^2}{1-\Sigma_M(T)} \end{aligned} \tag{4.46}$$

where $\bar{\sigma_M^2}$ is an upper bound for σ_M^2.

The above shows that $(v_n+1-\Sigma_Z)$ is a bounded sequence of Lipschitz continuous functions with the same Lipschitz constant, so it is equicontinuous.

Analogously, by equation (4.43)

$$\begin{aligned} |w_n'| &= 2w_n\,(v_n+1-\Sigma_Z)^+\frac{\sigma_M^2}{(1-\Sigma_M)^2} \\ &\leq 2\bar{\Gamma}\frac{\bar{\sigma_M^2}}{(1-\Sigma_M(T))^2} \end{aligned} \tag{4.47}$$

for any $T \in [0,1)$. Like the previous case, (w_n) is a bounded sequence of Lipschitz continuous functions and hence it is equicontinuous. □

Once we have established that both (w_n) and $(v_n+1-\Sigma_Z)$ are equicontinuous sequences of functions on $[0,T]$ for any $T<1$ by Lemma 4.5, we can proceed to show that since their derivatives are linear combinations of them, they are also equicontinuous.

Lemma 4.6. $(v_n'-\sigma_Z, w_n')$ *are bounded equicontinuous sequences of functions* $[0,T]$ *for any* $T<1$.

Proof. As both w_n and $v_n+1-\Sigma_Z$ are nonincreasing functions, their derivatives are always bounded from above by 0. In addition to that, (4.46) shows that v_n' is also bounded from below and equation (4.47) does the same for w_n'.

Since (w_n) and $(v_n+1-\Sigma_Z)$ are equicontinuous sequences of functions on $[0,T]$ for any $T<1$, $\forall\,\epsilon>0\;\exists\,\delta>0$ s.t.

$$d(w_n^2(t_1), w_n^2(t_2)) < \epsilon/2$$

and

$$d(v_n(t_1)+1-\Sigma_Z(t_1), v_n(t_2)+1-\Sigma_Z(t_2)) < \frac{(1-\Sigma_M(T))^2}{2\bar{\sigma_M^2}}\epsilon$$

whenever $d(t_1,t_2)<\delta$.

As a consequence, for all n:

$$d(\sigma_Z(t_1)-v_n'(t_1), \sigma_Z(t_2)-v_n'(t_2) \leq \frac{\epsilon}{2}+\frac{\bar{\sigma_M^2}}{(1-\Sigma_M(T))^2}\frac{(1-\Sigma_M(T))^2}{2\bar{\sigma_M^2}}\epsilon=\epsilon$$

for $d(t_1, t_2) < \delta$ making $\sigma_Z - v'_n$ itself an equicontinuous sequence of functions.

Analogously, $\forall\, \epsilon > 0\, \exists\, \delta > 0$ s.t.

$$d(v_n(t_1) + 1 - \Sigma_Z(t_1), v_n(t_2) + 1 - \Sigma_Z(t_2)) < \frac{(1 - \Sigma_M(T))^2}{2\bar{\sigma_M^2}\bar{\Gamma}}\epsilon$$

whenever $d(t_1, t_2) < \delta$, so for all n

$$d(v_n(t_1) + 1 - \Sigma_Z(t_1), v_n(t_2) + 1 - \Sigma_Z(t_2)) < \frac{(1 - \Sigma_M(T))^2}{2\bar{\sigma_M^2}\bar{\Gamma}} \frac{2\bar{\sigma_M^2}\bar{\Gamma}}{(1 - \Sigma_M(T))^2}\epsilon = \epsilon$$

for $d(t_1, t_2) < \delta$ also making w'_n an equicontinuous sequence of functions. □

As a consequence of the equicontinuity of the functions and the fact that $(w_n(0))_{n=0}^{\infty}$ is bounded, we can prove the following lemma that guarantees the existence of a converging subsequence of our functions.

One should bear in mind that due to the fixed point nature of the algorithm, a convergence subsequence will be enough to make it work properly, as we shall see in Theorem 4.3.

Lemma 4.7. *There exists a converging subsequence* $(w_{n_{ijk}}(0), \phi_{n_{ijk}})$ *of* $(w_n(0), \phi_n)$ *that converges to* $(w(0), \phi)$ *such that the convergences of* $(w_{n_{ijk}}(0))$ *and* $(v'_{n_{ijk}} - \sigma_Z)$ *are uniform.*

Proof. The algorithm constructed in the beginning of this section defined by equations (4.43), (4.44), and (4.41) defines a bounded sequence $(w_n(0))_{n=0}^{\infty}$. Therefore, by the Bolzano-Weierstrass theorem (see Bartle, Sherbert (2011) Theorem 3.4.8) there exists a converging subsequence $(w_{n_i}(0))$.

Since (w'_n) is a bounded sequence of equicontinuous functions in $[0, T]$ for any $T < 1$ by Lemma 4.6, (w'_{n_i}) is also. Therefore, by the Arzelà-Ascoli theorem (see Rudin (1976) Theorem 7.17) there is a converging subsequence $(w'_{n_{ij}})$.

Analogously, $(v'_n - \sigma_Z)$ is a bounded sequence of equicontinuous functions in $[0, T]$ for any $T < 1$ by Lemma 4.6, $(v'_{n_{ij}} - \sigma_Z)$ is also. Therefore, by the Arzelà-Ascoli theorem, there is a converging subsequence $(v'_{n_{ijk}} - \sigma_Z)$.

□

Theorem 4.3. *There exists a fixed point* ϕ *such that the system of ODEs given by the equations* (4.30) *is satisfied with the initial conditions given by* $v(0) + 1 - \Sigma_Z(0) = 1$ *and* $w(0)$

given by equation (4.38)*.*

Proof. For a matter of notation, let's rearrange the terms of the subsequence given in Lemma 4.7 such that we have that

$$\begin{aligned} \lim_{n\to\infty} r_n &= w(0) \\ \lim_{n\to\infty} \sigma_Z^2 - v_n' &= \sigma_Z^2 - v' \quad \text{uniformly} \\ \lim_{n\to\infty} w_n' &= w' \quad \text{uniformly} \end{aligned}$$

Therefore, there is a limit to equations (4.43) and (4.44). Theorem 4.2 guarantees that (w_n) converges to a function such that the ODEs (4.38) are satisfied in $[0, 1)$ as they are the limits of equations (4.43) and (4.44). Furthermore, the theorem also guarantees that $\lim_{n\to\infty} w_n(0) = w(0)$ are their initial conditions with that $v(0) + 1 - \Sigma_Z(0) = 1$.

As a consequence, we have that if we apply the ODEs to the initial condition $w(0)$ given by equation (4.37) we would get the output $w(0) = \left(\frac{v(0)+1-\Sigma_Z(0)}{\int_0^1 w^{3/2}(t)dt}\right)^2$ again.

Therefore, the fixed point property is satisfied for $(v(t) + 1 - \Sigma_Z(t), w(t))$ with initial conditions (4.37) and $v(0) + 1 - \Sigma_Z(0) = 1$.

Or equivalently

$$\phi = T\phi.$$

□

Now that we have a suitable value of $w(0)$ that can guarantee that $v + 1 - \Sigma_Z > 0$ for all t in $[0, 1)$, we are ready to prove the main theorem of this section:

Theorem 4.4. *There exists a pair* (w, v) *that solves* (4.29) *and satisfies*

$$w^{-\frac{1}{2}}(0) = \int_0^1 w^{\frac{3}{2}}(s)ds < \infty.$$

Moreover, the following statements are valid:

1. w *and* $v + 1 - \Sigma_Z$ *are decreasing, and* $v \le \Sigma_Z - \Sigma_M$*.*

2. w *and* $v + 1 - \Sigma_Z$ *do not depend on* Σ_Z*.*

3. *For all* $t \leq 1$,

$$v(t) + 1 - \Sigma_Z(t) = w^{\frac{1}{2}}(t)\left(w^{-\frac{1}{2}}(0) - \int_0^t w^{\frac{3}{2}}(s)ds\right), \text{ and} \tag{4.48}$$

$$w(t) = w(0)\exp\left(-2\int_0^t (v(s) + 1 - \Sigma_Z(s))\frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2}ds\right) \tag{4.49}$$

Proof. The first claim is the same as the one in Theorem 4.3. Recall that finiteness is a consequence of the existence of the limit in the previous theorem and Lemma 4.4.

The existence of a fixed point in the previous theorem also guarantees that the limit functions inherit the second and third statements of Proposition 4.1.

Note that the equations in (4.30) would be the same as (4.48) and (4.49) if $u > 0$. However, that is the case for w as described in the statement of this theorem due to Proposition 4.1.

The fact that $u > 0$ also leads to the fact that

$$\frac{\partial}{\partial t}\left(\frac{1}{u(t)}\right) \geq \frac{\sigma_M^2(t)}{(1-\Sigma_M(t))^2}$$

As a consequence,

$$\frac{1}{u(t)} - 1 \geq \int_0^t \frac{\sigma_M^2(s)}{(1-\Sigma_M(s))^2} = \frac{1}{(1-\Sigma_M(t)} - 1$$

So, $v + 1 - \Sigma_Z \leq 1 - \Sigma_M$, which leads to the final claim. □

4.6 Further Results on w

In this section, we show two propositions related to the behaviour of the function w that will be relevant in further discussions on the equilibrium. We have already established that w does not vanish in $[0, T]$ for any $T < 1$. Perhaps the most interesting fact of this section is that $\lim_{t\to 1} w(t) = 0$.

Together, these two facts are quite relevant for our high-frequency motivation. First, the fact that $w(t) > 0$ for all t in $[0, 1)$ shows that the changes in the level of the cumulative demand are always relevant in the trading period, but the fact that the limit of w is zero when time approaches one also shows that such impact is very reduced close to the end of the trading period.

Proposition 4.2. *Let w be the function of Theorem 4.4. Then,* $\lim_{t\to 1} w(t) = 0$. *Moreover,*

$$\frac{w(t)}{w(0)} \geq (1 - \Sigma_M(t))^2. \tag{4.50}$$

Proof. Using integration by parts and the ODE (4.29), we have

$$\begin{aligned}\int_0^t (v(s) + 1 - \Sigma_Z(s)) \frac{\sigma_M^2(s)}{(1 - \Sigma_M(s))^2} ds &\geq -1 + \int_0^t \frac{w^2(s)}{1 - \Sigma_M(s)} ds \\ &\geq -1 + \frac{w^2(t)}{K} \int_0^t \frac{\sigma_M^2(s)}{1 - \Sigma_M(s)} ds \\ &= -1 - \frac{w^2(t)}{K} \ln(1 - \Sigma_M(t)),\end{aligned}$$

where K is an upper bound for σ_M^2 on $[0, 1]$, since w is decreasing. Now, suppose $\lim_{t\to 1} w(t) > 0$. Then, as $\Sigma_M(1) = 1$,

$$\lim_{t\to 1} w(t) \leq e^2 \lim_{t\to 1} \exp\left(\frac{2w^2(t)}{K} \ln(1 - \Sigma_M(t)),\right) = 0,$$

which is a contradiction.

Similarly, since $v + 1 - \Sigma_Z \leq 1 - \Sigma_M$ by Theorem 4.4, we have

$$\ln \frac{w(t)}{w(0)} \geq -2 \int_0^t \frac{\sigma_M^2(s)}{1 - \Sigma_m(s)} ds = 2 \ln(1 - \Sigma_M(t)).$$

This proves the remaining claim. □

We can now prove a proposition that, even though it seems rather ordinary, will be very important in showing the equilibrium condition of our model.

Proposition 4.3. *Let w and v be the functions from Theorem 4.4. Assume $\sigma_M(1) \neq 0$. Then,*

$$\lim_{t\to 1} \frac{v(t) + 1 - \Sigma_Z(t)}{w(t)} = 0. \tag{4.51}$$

Proof. Let $u = v + 1 - \Sigma_Z$. Note that

$$\frac{u(t)}{w(t)} = \frac{\frac{1}{\sqrt{w(0)}} - \int_0^t w^{3/2}(s) ds}{\sqrt{w(t)}} \leq \frac{\frac{1}{\sqrt{w(0)}} - \int_0^t w^{3/2}(s) ds}{\sqrt{w(0)}(1 - \Sigma_M(t))}.$$

Then, L'Hospital's rule yields

$$\lim_{t\to 1}\frac{u(t)}{w(t)} \le w^{-1/2}(0)\lim_{t\to 1}\frac{w^{3/2}(t)}{\sigma_M^2(t)},$$

which converges to 0 by Proposition 4.2.

4.7 Price Process for the Market Maker

This section should be considered optional. We are not addressing anything relevant to equilibrium. As we have mentioned before, what we actually need to have an admissible strategy to be optimal is that it drives the mispricing to zero almost surely, i.e. $\lim_{t\to 1}\phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.. This question will finally be addressed in the pages following this section in Chapter 5. As will be presented in the chapter, we use Doob's h-transform to show that the strategy given by equation (4.6) in Lemma 4.1 is indeed optimal to the insider.

However, as a consequence of equation (5.2) we find that the mispricing requires that the price converges to V $\mathbb{P}^z$-a.s. Combining that with the fact that the market maker now observes a signal that converges to a linear function of the final price, it would be expected that the price process converges to V also in the market maker's filtration. In this section, we show that this is indeed the case. As an exercise that is not required to prove the equilibrium of the model, one should not find a counterpart to this section in other Kyle-Back models.

Furthermore, the end of this section is dedicated to showing an interesting result on the coefficient β_2. In equilibrium, it either converges to zero or one. That means that by the end of the trading period, the market maker is either absorbing all the marginal information coming from the public signal or ignoring it completely. Each possibility has different consequences on the liquidity of the market during the final moments of the trading period and will be carefully investigated in Chapter 7.

Before we proceed to the next proposition, we need to collect the following result:

Lemma 4.8. *Under the conditions of Theorem 4.4 and β_2 as defined in equation (4.26), i.e.,*

$$\beta_2(t) = \sigma_V\frac{v(t) + 1 - \Sigma_Z(t)}{\Sigma_Z(t)}$$

Then,

$$\lim_{t\to 1}\frac{1-\Sigma_M(t)}{\beta_2(t)}=0 \tag{4.52}$$

Proof. Suppose that there is $k\in\mathbb{R}_{++}$ such that $\lim_{t\to 1}\frac{1-\Sigma_M(t)}{\beta_2(t)}=\frac{1}{k}$. As a consequence, $\lim_{t\to 1}\frac{\beta_2(t)}{1-\Sigma_M(t)}=k$. Therefore, $\forall\,\epsilon>0\;\exists\,t^*:\;\forall\,t>t^*$

$$\frac{\beta_2(t)}{1-\Sigma_M(t)}\le k+\epsilon. \tag{4.53}$$

Therefore, by equation (4.28), $\frac{w'(t)}{w(t)}\ge -2(k+\epsilon)\sigma_M^2(t)$. Hence,

$$\ln\frac{w(t)}{w(t^*)}\ge -2(k+\epsilon)(\Sigma_M(t)-\Sigma_M(t^*))\ge -2(k+\epsilon) \tag{4.54}$$

leading to

$$w(t)\ge w(t^*)\exp\left(-2(k+\epsilon\right)>0 \tag{4.55}$$

the above equation is in contradiction to Proposition 4.2. As a consequence, the limit of equation (4.52) is true. □

With the previous lemma proved, we can now investigate the limit of the process X_t in the market maker's filtration. This limit will be key in proving Proposition 4.4.

Lemma 4.9. *Under the conditions of Theorem 4.4, then*

$$\lim_{t\to 1}X_t=-\sigma_V^{-1}\beta_0(1)+\left(1-\frac{\beta_2(t)}{\sigma_V}\right)\eta\ a.s.$$

Proof. Firstly, note that under the assumption 3.2, we have

$$r_1(t)=\frac{1}{2}\frac{w'(t)}{w(t)} \tag{4.56}$$

As a consequence, r_0 and r_2 can be rewritten as

$$r_0(t)\;=\;\sigma_V^{-1}(\beta_0(t)-\mu)\frac{1}{2}\frac{w'(t)}{w(t)}-\sigma_V^{-1}\beta_0'(t) \tag{4.57}$$

$$r_2(t)\;=\;\frac{1}{2}\frac{w'(t)}{w(t)}\left(\frac{\beta_2(t)}{\sigma_V}-1\right)-\frac{\beta_2'(t)}{\sigma_V} \tag{4.58}$$

Now note that

$$
\begin{aligned}
d\exp\left(-\int_0^t r_1(s)ds\right)X_t &= -r_1(t)X_t\exp\left(-\int_0^t r_1(s)ds\right)dt + \exp\left(-\int_0^t r_1(s)ds\right)dX_t \\
&= \exp\left(-\int_0^t r_1(s)ds\right)(dX_t - r_1(t)X_t dt) \qquad (4.59)
\end{aligned}
$$

Due to equation (4.56),

$$
\begin{aligned}
\exp\left(-\int_0^t r_1(s)ds\right) &= \exp\left(-\int_0^t \frac{1}{2}\frac{w'(s)}{w(s)}ds\right) \\
&= \exp\left(-\int_0^t \frac{1}{2}\frac{w'(s)}{w(s)}ds\right) \\
&= \exp\left(-\frac{1}{2}\ln\frac{w(t)}{w(0)}\right) \\
&= \sqrt{\frac{w(0)}{w(t)}} \qquad (4.60)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
d\sqrt{\frac{w(0)}{w(t)}}X_t &= \sqrt{\frac{w(0)}{w(t)}}(dX_t - r_1(t)X_t dt) \\
&= \sqrt{\frac{w(0)}{w(t)}}\left(w(t)dB_t + r_0(t)dt + r_2(t)X_t^M dt\right)
\end{aligned}
$$

by the definition of the process X. Using equation (4.58), we have that

$$
\frac{r_2(t)}{\sqrt{w(t)}} = \frac{1}{2}\frac{w'(t)}{(w(t))^{3/2}}\left(\frac{\beta_2(t)}{\sigma_V} - 1\right) - \frac{\beta_2'(t)}{\sigma_V\sqrt{w(t)}}
$$

Furthermore, applying Itô formula,

$$
\begin{aligned}
d\frac{(1-\frac{\beta_2(t)}{\sigma_V})}{\sqrt{w(t)}}X_t^M &= \left(\frac{-\beta_2'(t)}{\sigma_V\sqrt{w(t)}} + \frac{1}{2}\frac{w'(t)}{(w(t))^{3/2}}\left(\frac{\beta_2(t)}{\sigma_V} - 1)\right)X_t^M dt + \frac{(1-\frac{\beta_2(t))}{\sigma_V})}{\sqrt{w(t)}}dX_t^M \\
&= \frac{r_2(t)}{\sqrt{w(t)}}X_t^M dt + \frac{(1-\frac{\beta_2(t))}{\sigma_V})}{\sqrt{w(t)}}dX_t^M \qquad (4.61)
\end{aligned}
$$

Combining the last equation with equation (4.61) and considering the fact that

$$\frac{\partial}{\partial t}\frac{-(\beta_0(t)-\mu)}{\sigma_V\sqrt{w(t)}} = \frac{-\beta_0'(t)}{\sigma_V\sqrt{w(t)}} + (\beta_0(t)-\mu)\frac{1}{2}\frac{w'(t)}{\sigma_V(w(t))^{(3/2)}},$$

we have that

$$d\sqrt{\frac{1}{w(t)}}X_t = \sqrt{w(t)}dB_t + d\left(\frac{-(\beta_0(t)-\mu)}{\sigma_V\sqrt{w(t)}}\right) + d\left(\frac{(1-\frac{\beta_2(t)}{\sigma_V})}{\sqrt{w(t)}}X_t^M\right) + \frac{(\frac{\beta_2(t)}{\sigma_V}-1)}{\sqrt{w(t)}}dX_t^M \quad (4.62)$$

Lets define $F_t = \int_0^t \sqrt{w(s)}dB_s$ and $G_t = \int_0^t \frac{(\beta_2(s)-1)}{\sqrt{w(s)}}dX_s^M$. Then, we know that $F_t \sim N(0, \int_0^t w(s)ds)$ which has a finite variance as w is bounded by $w(0)$. Regarding the integrability of G, it does not depend on the filtration we consider the process. In the filtration generated by X^M, we have that $G_t = \int_0^t \frac{(\frac{\beta_2(s)}{\sigma_V}-1)}{\sqrt{w(s)}}\sigma_M(s)dI_s^M$, where I is the innovation process of that filtration and hence a Brownian motion. We can now consider two scenarios, the first one is if $\int_0^1 \frac{(\frac{\beta_2(s)}{\sigma_V}-1)^2}{w(s)}\sigma_M^2(s)ds < \infty$. In this case, G_1 is integrable.

In the second case, if $\int_0^1 \frac{(\frac{\beta_2(s)}{\sigma_V}-1)^2}{w(s)}\sigma_M^2(s)ds = \infty$ the quadratic covariance of $\sqrt{w(1)}Z_1$ seems to be unspecified as

$$\lim_{t\to 1}\frac{\int_0^t \frac{(\frac{\beta_2(s)}{\sigma_V}-1)^2}{w(s)}\sigma_M^2(s)ds}{\frac{1}{w(t)}} = \frac{\infty}{\infty}.$$

However, if we apply L'Hopital rule to the above equation, one gets that

$$\begin{aligned}\lim_{t\to 1}\frac{\int_0^t \frac{(\frac{\beta_2(s)}{\sigma_V}-1)^2}{w(s)}\sigma_M^2(s)ds}{\frac{1}{w(t)}} &= \lim_{t\to 1}\frac{\frac{(\frac{\beta_2(t)}{\sigma_V}-1)^2\sigma_M^2(s)}{w(t)}}{\frac{-w'(t)}{w^2(t)}} \\ &= \lim_{t\to 1}\frac{(\frac{\beta_2(t)}{\sigma_V}-1)(1-\Sigma_M(t))}{2\beta_2(t)} \\ &= 0 \qquad (4.63)\end{aligned}$$

the second equality being true due to equation (4.28) considering Assumption 3.2 and the last one being true due to lemma 4.8 . Therefore, in both cases $\sqrt{w(t)G_t}$ converges to zero as $t \to 1$. We can now proceed to write equation (4.62) in its non-differential form:

$$\begin{aligned}\frac{X_t}{\sqrt{w(t)}} &= \frac{X_0 + \sigma_V^{-1}(\beta_0(t)-\mu) - (1-\frac{\beta_2(0)}{\sigma_V})X_0^M}{\sqrt{w(0)}} + F_t + G_t \\ &+ \frac{-\sigma_V^{-1}(\beta_0(t)-\mu)}{\sqrt{w(t)}} + \frac{(1-\frac{\beta_2(t)}{\sigma_V})}{\sqrt{w(t)}}X_t^M\end{aligned}$$

Hence,

$$\begin{aligned}X_t &= \sqrt{w(t)}\left(\frac{X_0 + \sigma_V^{-1}(\beta_0(t)-\mu) - (1-\frac{\beta_2(0)}{\sigma_V})X_0^M}{\sqrt{w(0)}} + F_t + G_t\right) \\ &-\sigma_V^{-1}\beta_0(t) + \left(1-\frac{\beta_2(t)}{\sigma_V}\right)X_t^M\end{aligned}$$

Since $\frac{X_0+\sigma_V^{-1}(\beta_0(t)-\mu)-(1-\frac{\beta_2(0)}{\sigma_V})X_0^M}{\sqrt{w(0)}} + F_t + G_t$ is just a real-valued random variable, by proposition 4.2 the limit of $\sqrt{w(t)}\left(\frac{X_0+\sigma_V^{-1}(\beta_0(t)-\mu)-(1-\frac{\beta_2(0)}{\sigma_V})X_0^M}{\sqrt{w(0)}} + F_t + G_t\right)$ when $t \to 1$ must be zero. As a consequence, the limit of X_t given by equation (4.64) when $t \to 1$ is $-\sigma_V^{-1}(\beta_0(t)-\mu) + (1-\frac{\beta_2(0)}{\sigma_V})X_1^M$. □

Proposition 4.4. *Under Assumption 3.2, then, in equilibrium, the price process is an $\mathscr{F}^M$-martingale converging to $\sigma_V\eta + \mu$.*

Proof. Considering Assumption 3.2, the price process becomes $S_t = \beta_0(t) + \sigma_V X_t^M + \beta_2(t)X_t^M$. Since, by Proposition 4.9, $X_1 = \sigma_V^{-1}(\beta_0(t)-\mu) + (1-\frac{\beta_2(1)}{\sigma_V}X_1^M)$, we have that

$$\lim_{t\to 1} S_t = \beta_0(1) - (\beta_0(1)-\mu) + (\sigma_V - \beta_2(t))X_1^M + \beta_2(1)X_1^M = \mu + \sigma_V X_1^M$$

hence, $S_1 = \sigma_V\eta + \mu = V$ since X^M is a Markov Bridge converging to η by construction. □

Even though we do not need to study the behaviour of β_2 as time goes to one as Theorem 4.4 does not require any particular value of β_2 in the limit, it is interesting to notice that under the very reasonable assumption that $\sigma_M^2(1) > 0$ it could be either 0 or 1.

One can start by noticing that

$$\lim_{t\to 1}\beta_2(t) = \lim_{t\to 1}\frac{v(t) + 1 - \Sigma_Z(t)}{1 - \Sigma_M(t)} = \frac{0}{0}.$$

Therefore, to find the limit of β_2 when t goes to 1 one must apply L'Hopital rule:

$$\begin{aligned}
\lim_{t\to 1}\beta_2(t) &= \lim_{t\to 1}\frac{v'(t)-\sigma_Z^2(t)}{-\sigma_M^2(t)}\\
&= \lim_{t\to 1}\frac{w^2(t)+\sigma_M^2\beta_2^2(t)}{\sigma_M^2(t)}\\
&= \lim_{t\to 1}\frac{w^2(t)}{\sigma_M^2(t)}+\lim_{t\to 1}\beta_2^2(t) \qquad (4.64)
\end{aligned}$$

By Proposition 4.2, that the limit of w is zero when time goes to one, as long as $\lim_{t\to 1}\sigma_M^2(t)\neq 0$ the limit of $\frac{w^2}{\sigma_M^2}$ goes to zero. Hence, equation (4.64) shows that the only two possible limits such that the limits for β_2 are either zero or one. In our numerical analysis, we only found the β_2 converging to one, but one cannot ensure that there is not a setting in which β_2 could go to zero.

Chapter 5

Mispricing

In this chapter, we finally address the issue of the mispricing of the market maker. Recall that from Theorem 3.1 we see that for any strategy to be optimal, we need that $\gamma_0 = \gamma_1 = 0$ and $\lim_{t\to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.. In the last chapter, we have confined ourselves to the set of linear strategies of the form of equation (4.4). In the previous chapter, we have also seen that such a strategy could only be compatible with a rational pricing rule if the coefficients of $(\alpha)_{i=0}^2$ follow equations (4.6). Therefore, the main aim of this chapter is to show that the strategy θ such that

$$d\theta_t = w(t)\frac{Z_t - X_t - \frac{\beta_2(t)}{\sigma_V}X_t^M}{v(t)}dt, \quad \theta_0 = 0$$

is indeed optimal for the insider for the functions v and w from Theorem 4.4.

We shall in a moment explain in words what the misprice means in terms of the model, but, in order to do so, note that by equation (1.8), we have that the function H is given by $H(t, x, x_1) = \beta_0(t) + \beta_1(t)x + \beta_2(t)x_1$. Therefore, we find that ϕ, as defined in Theorem 3.1, is:

$$\phi(t, x, u, z) = \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{\beta_0(t) + \beta_1(t)y + \beta_2(t)x_1 - \sigma_V z - \mu}{w(t)}dy$$

As a consequence,

$$
\begin{aligned}
\phi(t,x,u,z) &= (x - H^{-1}(t,\sigma_V z+\mu,u))\frac{\beta_0(t)+\beta_2(t)x_1-\sigma_V z-\mu}{w(t)} \\
&\quad +\frac{\beta_1(t)}{w(t)}\frac{1}{2}(x^2-(H^{-1}(t,\sigma_V z+\mu,u))^2) \\
&= (x - H^{-1}(t,\sigma_V z+\mu,u)) \times \\
&\quad \Bigg(\frac{\beta_0(t)+\beta_2(t)x_1-\sigma_V z-\mu}{w(t)} \\
&\quad + \frac{\beta_1(t)}{w(t)}\frac{1}{2}(x+(H^{-1}(t,\sigma_V z+\mu,u))\Bigg) \qquad (5.1)
\end{aligned}
$$

Now, one can note that

$$
\begin{aligned}
\big(x+H^{-1}(t,\sigma_V z+\mu,u)\big) &= \left(x+\frac{\sigma_V z+\mu-\beta_0(t)-\beta_2(t)u}{\beta_1(t)}\right) \\
&= \left(2x+\frac{\sigma_V z+\mu-\beta_0(t)-\beta_1(t)x-\beta_2(t)u}{\beta_1(t)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\big(x-H^{-1}(t,\sigma_V z+\mu,u)\big) &= \left(x-\frac{\sigma_V z+\mu-\beta_0(t)-\beta_2(t)u}{\beta_1(t)}\right) \\
&= \left(\frac{-\sigma_V z-\mu+\beta_0(t)+\beta_1(t)x+\beta_2(t)u}{\beta_1(t)}\right) \\
&= \frac{H(t,x,x_1)-\sigma_V z-\mu}{w(t)}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\phi(t,x,u,z) &= \frac{(H(t,x,x_1)-\sigma_V z-\mu)}{w(t)}\frac{(H(t,x,x_1)-\sigma_V z-\mu)}{2} \\
&= \frac{(H(t,x,x_1)-\sigma_V z-\mu)^2}{2w(t)}. \qquad (5.2)
\end{aligned}
$$

First, note that from a modelling point of view, we can only have $\lim_{t\to 1}\phi(t,X_t,X_t^M,Z_t)=0$, $\mathbb{P}^z$-a.s. if $\lim_{t\to 1}(S_t-\sigma_V Z_t-\mu)^2=0$, $\mathbb{P}^z$-a.s.. If we consider that $(S_t-\sigma_V Z_t-\mu)^2$ is the mispricing of the asset under a quadratic loss function, it is a necessary condition[1] that this mispricing goes to zero as the trading period ends.

The misprice condition is not uncommon in the literature (see, for example, condition *(ii)* in Theorem 6.1 of Çetin, Danilova (2018)). However, when there is no public signal, one can say that the insider drives the price process to the final value of the asset at the end of the trading period. In our case, on the other hand, one cannot say the same. Since the market maker observes a public signal that is itself a Markov bridge converging to a linear combination of the correct value of the asset, one cannot claim that the insider is the one bridging the price as the market maker no longer needs the insider to do so and still has the misprice condition satisfied.

Furthermore, if we are able to show that $\lim_{t\to 0} \frac{\mathbb{E}^z((Z_t - \frac{S_t-\mu}{\sigma_V})^2)}{w(s)} = 0$, then by Fatou's lemma, $\lim_{t\to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s..

The way we are going to deal with that is through the use of *Doob's h transform*. The role of it will be explained in Section 5.1. In Section 5.2 we describe the law of the process X in the filtration of the market maker, and in Section 5.3 we use the technology described in the previous section to find the law of the mispricing in the filtration of the insider.

5.1 Doob's h-transform

In this section, we develop the intuition for the role of *Doob's h-transform* for the change of measure necessary to find the law of a bridge related to a given Markov process. The reader is invited to read Chapter 4 of Çetin, Danilova (2018) for a much more formal and wholesome review of this tool. However, for the sake of formality, we also borrow from the aforementioned book the required theorem, Theorem 5.1, to properly prove Theorem 5.2 which is the main theorem of this chapter that will be proven in Section 5.3.

Suppose that we have a Markov process X defined in a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$with transition density $p(\cdot,\cdot,\cdot)$. If we want to find the density of this process given time t conditional on its position in s and 1 we could easily do it by conditioning

$$\frac{P^x(X_s \in dv, X_t \in dy, X_1 \in dz)}{P^x(X_s \in dv, X_1 \in dz)} = p(t-s, v, y)\frac{p(1-t, y, z)}{p(1-s, v, z)}dy \tag{5.3}$$

[1]Yet not sufficient since we know from Proposition 4.2 that $\lim_{t\to 1} w(t) = 0$, we need the mispricing to go to zero at least at a certain speed.

for $0 \leq s < t \leq 1$ where P^x is the law of the process starting at x.

The above conditioning defines a relationship between the stochastic process X and what we call its bridge. Indeed, 5.3 defines a new law, $P^{x \to z}_{0 \to 1}$, for every pair x, z that is the law of the Markov bridge starting at x and ending at z at time 1. Therefore, for every measurable function $F : C([0,t], \mathbb{R}) \mapsto \mathbb{R}$, we have

$$E^{x \to z}_{0 \to 1}\left[F(X_s; s \leq t)\right] = E^x\left[F(X_s; s \leq t)\frac{p(1-t, X_t, z)}{p(1, x, z)}\right].$$

One may have noticed that the role of the coefficient $\frac{p(1-t,X_t,z)}{p(1,x,z)}$ is to define a change in measure from P^x to $P^{x \to z}_{0 \to 1}$. Indeed, if we define $h(t,x) = p(1-t,x,z)$, one can rewrite equation (5.3) as:

$$p(t-s,x,y)\frac{h(t,y)}{h(0,x)}.$$

In fact, we can define a probability measure, Q^T, using the density above in $\mathcal{F}_T$ by

$$\frac{dQ^T}{dP^x}\bigg|_{\mathcal{F}_T} = \frac{h(T, X_T)}{h(0,x)} \quad \text{for} \quad T \in [0,T],$$

where $T < 1$. The process $(X_t)_{t \in [0,T]}$ under the new measure Q^T is called the *h-transformed* of X and the function h is called *h-transform*. One should note that in fact by doing that we have a martingale Z such that

$$Z_t \equiv \frac{dQ^T}{dP^x}\bigg|_{\mathcal{F}_t} = \frac{h(t, X_t)}{h(0,x)}$$

The remark about the martingale is relevant as in practice what we aim at in this chapter is to show that a candidate martingale, as properly defined in definition 5.1, defines a law that is indeed the law of a Markov bridge. That is exactly what we do in Theorem 5.2 which cuts many corners in the general theory presented in Çetin (2018).

Such a procedure is allowed us due to the Girsanov Theorem (see Theorem 38.4 of Rogers, Williams (2000)). Basically, if Z is a continuous density process as above. Then, for any continuous P^x-local martingale M the process

$$M' = M - \int_0^t \frac{1}{Z} d[M, Z]$$

is a Q^T-local martingale. Therefore, if a candidate h is such that

$$dM'_t = dM_t - \frac{h(0,x)}{h(t,X_t)} h_x(t,X_t) d[M,M]_t,$$

where h_x is the derivative of h with respect to the second component, it defines the SDE of a Markov bridge, then the measure it defines Q^T is indeed $P^{x\to z}_{0\to 1}$.

As a consequence, we may now properly define an *h-transform* as the following:

Definition 5.1. *(Definition 4.1 of Çetin, Danilova (2018)) We call a function* $h : [0,T^*] \times \mathbf{E} \mapsto [\mathbf{0}, \infty)$*, an h-function if it is strictly positive on* $[0,T^*) \times \mathbf{E}$*, belongs to* $C^{1,2}([0,T^*] \times \mathbf{E})$*, and*

$$(h(t,X_t)_{t\geq 0}, (\mathcal{B}_t)_{t\geq 0})$$

is a martingale under every $P^{0,x}$*.*

The main theorem we shall use to prove the main result of this chapter is the following:

Theorem 5.1. *(Theorem 4.1 of Çetin, Danilova (2018), modified) Let* X *be a strong Markov process under any* $P^{s,\mu}$ *for all* $s \geq 0$*, where* μ *is a probability measure on* $\mathbf{E}$ *and* $T^* < \infty$*. Let* h *be an h-function such that* $h(T^*, \cdot) > 0$ *and* $h \in C^{1,2}([0,T^*] \times \mathbf{E})$*. Define* $P^{h;s,x}$ *on* $(\Omega, \mathcal{B}_{T^*})$ *by*

$$\frac{dP^{h;s,x}}{dP^{s,x}} = \frac{h(T^*, X_{T^*})}{h(s,x)}.$$

Then, $P^{h;s,x}$ *is the unique solution of the local martingale problem for* A^h *starting from* x *at* s*, where*

$$A^h_t = A_t + \sum_{i,j=1}^{d} a_{ij}(t,x) \frac{\frac{\partial h}{\partial x_j}(t,x)}{h(t,x)} \frac{\partial}{\partial x_i}. \tag{5.4}$$

and

$$A_t = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(t,\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,\cdot) \frac{\partial}{\partial x_i},$$

is the generator associated with the transition function $(P_{s,t})$*.*

Consequently, X *is a strong Markov process under every* $P^{h;s,x}$ *for* s*,* T^**, and* $x \in \mathbb{E}$*, and the associated transition function* $(P^h_{s,t})$ *is related to* $(P_{s,t})$ *via*

$$P^h_{s,t}(x,A) = \frac{1}{h(s,x)} \int_A h(t,y) P_{s,t}(x,dy), \quad x \in \mathbf{E},\ \mathbf{A} \in \mathcal{E},\ \mathbf{t} \in \mathbf{T_s}. \tag{5.5}$$

A couple of remarks about the above theorem are needed, as we took it out of context. The first one is that as things are defined above we do not claim that $P^h_{s,t}$ is a Markov bridge. What it is saying is that every martingale $h(T, X_T)$ defines a change of measure and that the original process has density $P^h_{s,t}$ and operator A^h_t. The second one is that in both equations (5.4) and (5.5) should be clear the role of the change of measure as we very briefly discussed above. In particular, in Equation (5.4) the role of the Girsanov theorem should be clear.

5.2 Equilibrium demand process

In this section, we shall take a deeper look into market makers' signal X that is expected to appear in equilibrium. Given the hypothesis of the results in the previous section, we shall assume the following throughout.

Assumption 5.1. *w and v are the functions from Theorem 4.4, $\beta_0 \equiv \mu$, and equations (4.26) and (4.27) are satisfied. Moreover, Assumption 3.2 holds and α_is satisfy (4.6).*

All assumptions, except the one of $\beta_0 \equiv \mu$ have already been made at some point before. Since all r_0 does is adapt to changes in β_0, we can set such a restriction. Furthermore, one can note that as we reach the last fundamental result of the thesis and we have not required to impose any further restriction on β_0 what we are doing is to kill a free parameter that we do not need to use.

Moreover, setting $\beta_0 \equiv \mu$ also sets $r_0 \equiv 0$.

Given the above condition, let us consider the following SDE on $(\Omega, \mathscr{F}, (\mathscr{F}^I_t)_{t\in[0,1]}, \mathbb{P})$:

$$dX_t = w(t)dB_t + \{r_1(t)X_t + r_2(t)X^M_t\}dt + w(t)\alpha_3(t)\Big(Z_t - X_t - \frac{\beta_2(t)}{\sigma_V}X^M_t\Big)dt. \qquad (5.6)$$

One should note that all the coefficients of the above SDE depend only on time. Moreover, all coefficients are bounded for any $t \in [0, T]$ with $T < 1$. As a consequence, the linear structure implies the existence of a unique strong solution on $[0, 1)$ by Theorem 2.7 of Çetin, Danilova (2018).

Once we have the uniqueness of equation (5.6), we can be sure that the process X as defined above is indeed the same as the one in equation (4.3) under the conditions given by (4.6), the restrictions given by equations (4.26) and (4.27), and the processes X^M and the Brownian motion B. Therefore, we may apply Lemmas 4.1 and 4.2. Furthermore, recalling that $\hat{Z}$ is the $\mathscr{F}^M$-optional projection of Z, we may summarise both lemmas in Proposition 5.1:

Proposition 5.1. *Suppose Assumption 5.1 holds and consider X in (5.6). Then, $S := \beta_0 + \sigma_V X + \beta_2 X^M$ is a $(\mathscr{F}^M, \mathbb{P})$ martingale. Moreover,*

$$\begin{aligned} dX_t &= w(t)dN_t^{(1)} + \{r_0(t) + r_1(t)X_t + r_2(t)X_t^M\}dt \\ dX_t^M &= \sigma_M(t)dN_t^{(2)} + \sigma^2(t)\frac{\hat{Z}_t - X_t^M}{1 - \Sigma_M(t)}dt, \end{aligned} \tag{5.7}$$

where $(N^{(1)}, N^{(2)})$ is a two-dimensional $(\mathbb{P}, \mathscr{F}^M)$-Brownian motion. In particular, $S = \mu + \sigma_V \hat{Z}$, and the random variable $S_t \sim N(\mu, \Sigma_S(t))$ for every $t \in [0,1]$, where

$$\Sigma_S(t) := \int_0^t \sigma_V^2 w^2(s)ds + \int_0^t \beta_2^2(s)\sigma_M^2(s)ds. \tag{5.8}$$

Proof. First note that Lemma 4.2 shows that, given $\mathscr{F}_t^M$, Z_t is normally distributed with mean $\hat{Z}_t$ and variance v in view of Theorem 4.4 and the particular form of w given by equation (4.27).

Equation (4.27) with Assumption 3.2 sets the conditions for the application of Lemma 4.1 that shows the martingale property for S, which readily yields the identity $\frac{S-\mu}{\sigma_V} = \hat{Z}$.

Moreover, from Lemma 4.1 we have that

$$dS_t = \sigma_V w(t)dN_t^{(1)} + \beta_2(t)\sigma_M(t)N_t^{(2)}.$$

Therefore, $Var(\eta|\mathscr{F}_t^M) = [S,S]_t = \Sigma_S(t)$ as defined in (5.8). Furthermore, recall that $\mathbb{E}(\eta) = 0$, so $\mathbb{E}(S_t) = \mu$. □

We can now address the issue of the second moment of $S_t - \mu - \sigma_V Z_t$ that we have raised in the beginning of this section.

Recall from Theorem 3.1 that the optimality condition on insider strategy requires seemingly strong conditions on the second moment of mispricing, that is, $S_t - \mu - \sigma_V Z_t$, under the insider's probability measure. Our next goal is to show that this condition will be satisfied under the existing conditions for the candidate equilibrium strategy of the insider.

5.3 On the second moments of mispricing

In this section, we shall find the second moment of mispricing, $S_t - \mu - \sigma_V Z_t$, under the insider's probability measure. Our strategy for doing so is to define a process with the same law of $S_t - \mu - \sigma_V Z_t$ under the equilibrium condition by a *h-transform* of (X, X^M) under the probability measure $\mathbb{P}$.

We shall start with the case when the insider has the perfect knowledge of V; that is the private signal is such that $\Sigma_I \equiv 1$.

Recall that $\mathbb{P}^z$ is the law of (B, X^M, X^I) given that $z = X_0^I$. As a result of $\Sigma_I \equiv 1$, we replace Z by η in (5.6) as a consequence of 2.3. Now, we can find the dynamics of X^M under $\mathbb{P}$ using the theory of enlargement of filtrations theory (see, e.g., Theorem 4.2 in Çetin, Danilova (2018) that

$$dX_t^M = \sigma_M(t) d\beta_t^M + \sigma_M^2(t) \frac{\eta - X_t^M}{1 - \Sigma_M(t)} dt, \tag{5.9}$$

for some $(\mathbb{P}, \mathscr{F}^I)$-Brownian motion β_M. Therefore, writing the dynamics under the measure $\mathbb{P}^z$ and assuming that Assumption 5.1 is satisfied, leads to the following system:

$$\begin{aligned} dX_t &= w(t) dB_t + \{r_0(t) + r_1(t) X_t + r_2(t) X_t^M\} dt + \frac{w^2(t)}{v(t)} \Big(z - X_t - \frac{\beta_2(t)}{\sigma_V} X_t^M \Big) dt \\ dX_t^M &= \sigma_M(t) d\beta_t^M + \sigma_M^2(t) \frac{z - X_t^M}{1 - \Sigma_M(t)} dt, \end{aligned} \tag{5.10}$$

where (w, v) are the functions from Theorem 4.4 with $\Sigma_Z \equiv 1$, and β^M is (with an abuse of notation) a $(\mathbb{P}^z, \mathscr{F}^I)$-Brownian motion.

To get a description of the $\mathbb{P}^z$-moments of $S_t - V$, we shall apply a particular h-transformation to (X, X^M) from (5.7). The first step is to show that our candidate h-transform is indeed an h-transform. That will be done by Lemma 5.1 showing that our transformation is a martingale. Theorem 5.2 shows that the h-transformation to (X, X^M) has the same law of (X, X^M) under $\mathbb{P}^z$.

Lemma 5.1. *Let q be the transition density of a standard Brownian motion; that is,*

$$q(t, x, z) = (2\pi t)^{-1/2} \exp(-(x - z)^2/(2t)).$$

Let (X, X^M) be the solution of (5.7) on $(\Omega, \mathscr{F}, (\mathscr{F}_t^M), \mathbb{P})$ and fix $z \in \mathbb{R}$. Then, for any $T < 1$,

$(L_t(z))_{t\in[0,T]}$ *is* $(\mathbb{P}, \mathscr{F}^M)$*-martingale, where*

$$L_t(z) := q(v(t), X_t + \sigma_V^{-1}\beta_2(t)X^M, z) \tag{5.11}$$

and (w, v) *is the pair from Theorem 4.4 with* $\Sigma_Z \equiv 1$.

Proof. Note, under the hypothesis of the lemma, that for any $t < 1$, $\mathbb{P}(\eta \in dz|\mathscr{F}_t^M) = q(v(t), X_t + \sigma_V^{-1}\beta_2(t)X^M, z)dz$ by Proposition 5.1. Thus, for any test function,

$$\mathbb{E}[f(\eta)|\mathscr{F}_t^M] = \int_{\mathbb{R}} f(z)q(v(t), X_t + \sigma_V^{-1}\beta_2(t)X^M, z)dz.$$

Since the left-hand side is a martingale, so is the right-hand side. This yields the martingale property for almost all z due to the arbitrariness of f, and thus for all z due to the joint continuity of the transition density. □

Since L is a $\mathbb{P}$-martingale, one can use it to change the probability measure using Theorem 5.1.

Theorem 5.2. *Suppose Assumption 5.1 holds and consider the solution* (X, X^M) *of* (5.7) *on* $(\Omega, \mathscr{F}, (\mathscr{F}_t^M), \mathbb{P})$. *For any* $T < 1$, *define* Q^z *on* $\mathscr{F}_T$ *by* $\frac{dQ^Z}{d\mathbb{P}} = L_T(z)$, *where* $L(z)$ *is given by* (5.11). *Then, for any bounded and measurable* F

$$\mathbb{E}^{Q^z}[F((X_t)_{t\in[0,T]}, ((X_t^M)_{t\in[0,T]}))] = L_0^{-1}(z)\mathbb{E}[L_T(z)F((X_t)_{t\in[0,T]}, ((X_t^M)_{t\in[0,T]}))].$$

Moreover, under Q^z

$$\begin{aligned} dX_t &= w(t)dW_t^{(1)} + \{r_1(t)X_t + r_2(t)X_t^M\}dt + \frac{w^2(t)}{v(t)}\Big(z - X_t - \frac{\beta_2(t)}{\sigma_V}X_t^M\Big)dt \\ dX_t^M &= \sigma_M(t)dW_t^{(2)} + \sigma_M^2(t)\frac{z - X_t^M}{1 - \Sigma_M(t)}dt, \end{aligned} \tag{5.12}$$

where $(W^{(1)}, W^{(2)})$ *is a two-dimensional* $(Q^z, \mathscr{F}^M)$*-Brownian motion, and* (w, v) *is the pair from Theorem 4.4 with* $\Sigma_Z \equiv 1$.

Proof. First lets rewrite the dynamic of the process X given by equation (5.7) under the new generator given by (5.4) from Theorem 5.1 under Assumption 5.1:

$$dX_t \quad = w(t)dW_t^{(1)} + \{r_1(t)X_t + r_2(t)X_t^M\}dt + \frac{w^2(t)}{v(t)}\left(z - X_t - \frac{\beta_2(t)}{\sigma_V}X_t^M\right)dt$$

since

$$\frac{\partial L(z)(x_1,x_2)}{\partial x_1} = -L(z)(x_1,x_2)\frac{x_1 + \frac{B_2(t)x_2}{\sigma_V} - z}{v(t)}$$

and $a_{11} = w$ and, by the Independence of the original Brownian motions, $a_{12} = 0$ from the original generator.

Before we proceed it is worth recalling that $\hat{Z}_t = \frac{S_t - \mu}{\sigma_V}$, hence,

$$\begin{aligned}\hat{Z}_t &= \frac{S_t - \mu}{\sigma_V} \\ &= \frac{\beta_0(t) + \sigma_V X_t + \beta_2(t)X_t^M - \mu}{\sigma_V} \\ &= X_t + \frac{\beta_2(t)}{\sigma_V}X_t^M\end{aligned}$$

Furthermore, since $\Sigma_I \equiv 1$ implies that $\Sigma_Z \equiv 1$ by Corollary 2.2 leading to the fact that equation (4.26) becomes $\frac{\beta_2}{\sigma_v} = \frac{v}{1-\Sigma_M}$.

Now, as we did for the signal X, we can find the dynamic for X^M given by Equation (5.7) under the new generator given by (5.4) from Theorem 5.1 under Assumption 5.1.

Furthermore, also from the original generator we have $a_{21} = 0$ and $a_{22} = \sigma_M$. Likewise,

$$\frac{\partial L(z)(x_1,x_2)}{\partial x_2} = -L(z)(x_1,x_2)\frac{\beta_2(t)}{\sigma_V}\frac{x_1 + \frac{\beta_2(t)x_2}{\sigma_V} - z}{v(t)}$$

Therefore,

$$\begin{aligned}dX_t^M &= \sigma_M(t)dW_t^{(2)} + \sigma_M^2(t)\frac{\hat{Z}_t - X_t^M}{1 - \Sigma_M(t)}dt \\ &\quad + \frac{\beta_2(t)}{\sigma_V}\frac{\sigma_M^2(t)}{v(t)}\left(z - X_t - \frac{\beta_2(t)}{\sigma_V}X_t^M\right)dt \\ &= \sigma_M(t)dW_t^{(2)} + \sigma_M^2(t)\frac{\hat{Z}_t - X_t^M}{1 - \Sigma_M(t)}dt \\ &\quad + \frac{\sigma_M^2(t)}{1 - \Sigma_M(t)}\left(z - \hat{Z}_t\right)dt\end{aligned}$$

As a consequence, (X, X^M) follow the dynamic given by (5.12) for some $(W^{(1)}, W^{(2)})$ is a two-dimensional $(Q^z, \mathscr{F}^M)$-Brownian motion $\square$

Note that since (5.12) has a unique strong solution in $[0, T]$, the law of the solution coincides with that of (X, X^M) in $\mathbb{P}^z$, where (X, X^M) has the dynamics given by (5.10). This leads to the following representation of the second moment of $\hat{Z} - z$ under $\mathbb{P}^z$.

Corollary 5.1. *Suppose Assumption 5.1 holds and $\Sigma_I \equiv 1$. Let (X, X^M) be as in (5.10) and consider $\hat{Z} = X + \frac{\beta_2}{\sigma_V} X^M$. Then,*

$$\mathbb{E}^z[(\hat{Z}_t - z)^2] = \frac{q(\Sigma_S(t)\sigma_V^{-2} + v(t), 0, z)}{q(1, 0, z)} \Big(\frac{v(t)\Sigma_S(t)\sigma_V^{-2}}{v(t) + \Sigma_S(t)\sigma_V^{-2}} + \frac{v^2(t) z^2}{(v(t) + \Sigma_S(t)\sigma_V^{-2})^2} \Big),$$

where v is the function of Theorem 4.4 with $\Sigma_Z \equiv 1$.

Proof. In view of the absolute continuity relationship in Theorem 5.2 and the distribution of $\hat{Z}$ from Proposition 5.1,

$$\mathbb{E}^z[(\hat{Z}_t - z)^2] = \frac{1}{q(1, 0, z)} \int_{\mathbb{R}} q(v(t), x, z) q(\Sigma_S(t)\sigma_V^{-2}, 0, x)(x - z)^2 dx.$$

However, using the explicit structure of the Gaussian density, we can rewrite the above as

$$\begin{aligned}
\mathbb{E}^z[(\hat{Z}_t - z)^2] &= -2v(t) \frac{d}{dr} \frac{1}{q(1, 0, z)} \frac{1}{\sqrt{r}} \int_{\mathbb{R}} q(r^{-1}v(t), x, z) q(\Sigma_S(t)\sigma_V^{-2}, 0, x) dx \\
&= -2 \frac{v(t)}{q(1, 0, z)} \frac{d}{dr} \frac{1}{\sqrt{r}} q(r^{-1}v(t) + (\Sigma_S(t)\sigma_V^{-2}, 0, z) \\
&= 2 \frac{v(t)}{q(1, 0, z)} \Big(\frac{1}{2} r^{-3/2} q(r^{-1}v(t) + \Sigma_S(t)\sigma_V^{-2}, 0, z) + r^{-5/2} q_t(r^{-1}v(t) + \Sigma_S(t)\sigma_V^{-2}, 0, z) v(t) \Big).
\end{aligned}$$

Note that the above is independent of r. Thus, we may take $r = 1$ and arrive at the claim using the fact that

$$\frac{q_t(t, x, y)}{q(t, x, y)} = -\frac{1}{2t} + \frac{(x - y)^2}{2t^2}.$$

$\square$

Although the above corollary computes the second moment when the insider has full information about the value of η, it also yields the result for the general case. To see this, first observe that in general

$$\mathbb{P}^z = \int_{\mathbb{R}} \mathbb{Q}^z q(1 - c^2, 0, z) dz,$$

where $\mathbb{Q}^z$ is the law of (B, X^M, X^I) given that $X_0^I = z$ and $\Sigma_I \equiv 1$. Then, the following result is immediate in view of the independence of $v + 1 - \Sigma_Z$ from Σ_Z by Theorem 4.4.

Corollary 5.2. *Suppose Assumption 5.1 holds. Let (X, X^M) be as in (5.10) and consider $\hat{Z} = X + \frac{\beta_2}{\sigma_V} X^M$. Then,*

$$\mathbb{E}^z[(\hat{Z}_t - Z_1)^2] = \int_{\mathbb{R}} \frac{q(1-c^2, 0, z) q(\Sigma_S(t)\sigma_V^{-2} + u(t), 0, z)}{q(1,0,z)} \Big(\frac{u(t)\Sigma_S(t)\sigma_V^{-2}}{u(t) + \Sigma_S(t)\sigma_V^{-2}} + \frac{u^2(t) z^2}{u(t) + \Sigma_S(t)\sigma_V^{-2}} \Big)^2 dz, \tag{5.13}$$

where $u = v + 1 - \Sigma_Z$ with v being the function from Theorem 4.4.

Corollary 5.3. *Suppose Assumption 5.1 holds. Let (X, X^M) be as in (5.10) and consider $\hat{Z} = X + \frac{\beta_2}{\sigma_V} X^M$. Then, $\lim_{t \to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.*

Proof. First note that,

$$\begin{aligned}
\mathbb{E}^z((Z_s - \hat{Z}_s)^2) &= \mathbb{E}^z[Z_s^2 + (\hat{Z})_s^2 - 2Z_s\hat{Z}_s] = \mathbb{E}^z[Z_s^2 + (\hat{Z})_s^2 - 2Z_1\hat{Z}_s] \\
&= \mathbb{E}^z[Z_s^2 - Z_1^2 + (\hat{Z}_s - Z_1)^2] = \mathbb{E}^z[(\hat{Z}_s - Z_1)^2] - (1 - \Sigma_Z(s)) \\
&\leq \mathbb{E}^z[(\hat{Z}_s - Z_1)^2]
\end{aligned} \tag{5.14}$$

Observe that $\phi(t, x, u, z) = \frac{(H^*(t,x,u) - \sigma_v - \mu)^2}{2\sigma_V w(t)}$ by equation (5.2). Therefore, $\phi(t, X_t, X_t^M, Z_t) = \sigma_V \frac{(\hat{Z}_t - Z_t)^2}{w(t)}$.

Now, note that in equation (5.13) for every $t \in [0, 1)$, we calculate the second moment of a random variable with kernel $\frac{q(1-c^2,0,z)q(\Sigma_S(t)\sigma_V^{-2} + u(t),0,z)}{q(1,0,z)}$. Hence, as long as the second part of the integral goes to zero as time goes to one, we have $\lim_{t \to 0} \frac{\mathbb{E}^z((Z_s - \hat{Z}_s)^2)}{w(s)} = 0$, which is a consequence of Proposition (4.3) that states $\lim_{t \to 1} \frac{u(t)}{v(t)} = 0$.

Since we have the expectation of non-negative stochastic process converging to zero, the above implies $\lim_{t \to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s. by Fatou's lemma.

□

Chapter 6

Equilibrium

We are now ready to obtain an equilibrium for this economy. We have collected all the results we need to show that there is a pair of an admissible pricing rule, as given by Definition 1.1, and an admissible strategy, as given by Definition 1.2, that is an equilibrium under Definition 1.3.

First, let us recall that Theorem 3.1 guarantees that any trading strategy of the form given by equation (1.9) is optimal if $\gamma_0 = \gamma_1 \equiv 0$ and $\lim_{t\to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s. where $\phi(t,x,u,z) = \int_{H^{-1}(t,\sigma_V z+\mu,u)}^{x} \frac{H(t,y,u)-\sigma_V z-\mu}{w(t)} dy$. In Corollary 5.3 we prove that the above limit is achieved for a trading strategy given by equation (4.4) where Z is given by Theorem 2.3 and the coefficients of $(\alpha_i)_{i=0}^2$ are given by(4.6) and a particular pricing rule.

The particular pricing rule we show provides the equilibrium with the mentioned trading strategy, which must be a linear one as given in equation (1.8) of Definition 1.1. Furthermore, in order to be a pair for the given trading strategy, it is also required that the coefficients $(r_i)_{i=0}^2$ of the process X given by (1.5) follow Assumption 3.2 and the coefficients of $(\beta_i)_{i=0}^2$ are such that $\beta_0 \equiv \mu$, $\beta_1(t) = \sigma_V$, as of Assumption 3.1, and $\beta_2 = \sigma_V \frac{v+1-\Sigma_Z}{1-\Sigma_M}$ from equation (4.26).

Note that all the coefficients of the α's, r_1, r_2 and of β_2 depend on the pair of functions (w, v). The existence and uniqueness of these functions are given by Theorem 4.4.

Therefore, we have used all the major theorems of this thesis to show that there is a pair of an admissible pricing rule and an admissible trading strategy such that for this trading strategy the pricing rule is rational and given the pricing rule the trading strategy is optimal.

We can now combine everything we just mentioned in one single theorem that will prove the equilibrium for the particular pair $((H^*, w^*, r^*), (\theta^*, \gamma_0^*, \gamma_1^*)$:

Theorem 6.1. *Suppose that Assumptions 3.1 and 3.2 hold, w^* and v^* are the functions w and v, respectively, from Theorem 4.4, $\gamma_0 \equiv \gamma_1 \equiv 0$, $\beta_0 \equiv \mu$, and $\beta_2 = \sigma_V \frac{v^*+1-\Sigma_Z}{1-\Sigma_M}$. Then, $((H^*, w^*, r^*), \theta^*)$ is an equilibrium, where*

$$
\begin{aligned}
H^*(t, x, u) &= \mu + \sigma_V x + \beta_2(t) u \\
d\theta_t^* &= w^*(t) \frac{Z_t - X_t - \frac{\beta_2(t)}{\sigma_V} X_t^M}{v^*(t)} dt, \quad \theta_0^* = 0, \\
r_0^* &\equiv 0,\ r_1^* = \frac{\frac{dw^*}{dt}}{2w}, \ \text{ and } r_2^* = \frac{\beta_2}{\sigma_V}\left(\left(1 - \frac{\beta_2}{\sigma_V}\right)\frac{\sigma_M^2}{1-\Sigma_M} - \frac{\beta_2'}{\beta_2}\right).
\end{aligned}
$$

Moreover, the expected wealth of the insider in equilibrium is given by

$$
\sigma_V \left(\frac{z^2}{2w(0)} + \int_0^1 w(t)\, dt - \frac{c^2}{2w(0)} \right)
$$

Proof. Note that given θ^*, (H^*, w^*, r^*) is admissible as it satisfies all the condition of 1.1. Moreover, it is a rational pricing rule by Proposition 5.1.

Next, given H^*, θ^* is an admissible trading strategy for the insider as satisfying all the conditions of Definition 1.2. First note that θ^* satisfies equation (1.9). In the previous chapter we have seen that X as defined in (5.6) has indeed an strong solution. To guarantee that no doubling strategies are allowed just note that $(H^*)^{-1}(s, \sigma_V Z_s + \mu, X_s^M) - X_s = Z_s - \hat{Z}_s$. by equation (5.2).

Moreover, by equation (5.14), we have that $\mathbb{E}^z((Z_s - \hat{Z}_s)^2) \leq \mathbb{E}^z[(\hat{Z}_s - Z_1)^2]$. Proposition 4.3 says that $\lim_{t \to 1} \frac{u(t)}{v(t)} = 0$ which in turn implies that $\frac{\mathbb{E}^z((Z_s - \hat{Z}_s)^2)}{w(s)}$ is bounded on $[0, 1]$. This yields the desired admissibility since β_2 is bounded in view of Theorem 4.4.

Corollary 5.3 says that $\lim_{t \to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s., hence establishing the optimality of θ^* via Theorem 3.1.

Theorem 3.1 gives a sufficient condition for the optimality of the insider provided an additional integrability property is valid. In particular, we need to check that the integral in equation (3.54) is finite. The strict positivity and continuity of w implies we only need to check that

$$
\lim_{t \to 1} \int_0^t \left\{ \sigma_V w(s) + \frac{\beta_2^2(s) \sigma_M^2(s)}{\sigma_V w(s)} + \frac{\sigma_V \sigma_Z^2(s)}{w(s)} - \frac{2\sigma_M^2(s) \beta_2(s) \lambda_1(s)}{w(s)} \right\} ds < \infty.
$$

Indeed, letting $\beta_V := \frac{\beta_2}{\sigma_V}$, we have

$$\begin{aligned}
\bar{f}(t) &:= \frac{\sigma_V}{2}\int_0^t \frac{w^2(s) + \beta_V^2(s)\sigma_M^2(s) + \sigma_Z^2(s) - 2\sigma_M^2(s)\beta_V(s)\lambda_1(s)}{w(s)}ds \\
&= \frac{\sigma_V}{2}\int_0^t \frac{2w^2(s) + 2\beta_V^2(s)\sigma_M^2(s) + v'(s) - 2\sigma_M^2(s)\beta_2(s)\lambda_1(s)}{w(s)}ds \\
&= \frac{\sigma_V}{2}\int_0^t 2w(s) + \frac{v'(s) + 2\beta_V(s)\sigma_M^2(s)(\beta_V(s) - \lambda_1(s))}{w(s)}ds.
\end{aligned}$$

However,

$$\beta_V(t) - \lambda_1(t) = \frac{v(t) + 1 - \Sigma_Z(t)}{1 - \Sigma_M(t)} - \frac{1 - \Sigma_Z(t)}{1 - \Sigma_M(t)} = \frac{v(t)}{1 - \Sigma_M(t)}.$$

Consequently,

$$\begin{aligned}
\bar{f}(t) &= \frac{\sigma_V}{2}\int_0^t 2w(s) + \frac{v'(s)}{w(s)} + \frac{2\beta_V(s)\sigma_M^2(s)v(s)}{w(s)(1 - \Sigma_M(t))}ds \\
&= \frac{\sigma_V}{2}\int_0^t 2w(s) + \frac{v'(s)}{w(s)} - \frac{w'(s)v(s)}{w^2(s)}ds \\
&= \sigma_V\int_0^t w(s)ds + \frac{\sigma_V}{2}\left(\frac{v(t)}{w(t)} - \frac{v(0)}{w(0)}\right) \\
&= \sigma_V\int_0^t w(s)ds + \frac{\sigma_V}{2}\frac{v(t)}{w(t)} - \frac{\sigma_V}{2}\frac{v(0)}{w(0)}.
\end{aligned}$$

The above limit is finite due to Proposition 4.3 that shows that $\lim_{t\to 1} \frac{v(t)+1-\Sigma_Z(t)}{w(t)} = 0$. □

The next corollaries collect some properties of the equilibrium above. The first has two relevant statements. The first statement is that the equilibrium demand is a martingale in the filtration of the market maker, as is the case in the literature since Back (1992), which is relevant to the model, as it shows that the market maker cannot predict any movement of the demand. This is a direct consequence of the rational pricing rule, as we have shown in Corollary 4.1.

The second statement of the first corollary says that in equilibrium the price process converges almost surely to the final value of the asset in the insider's filtration. This has always been the case since the very first continuous-time version of Kyle-Back models. However, there is an important fact about the interpretation of it. In the previous models before this thesis, the only source of information would come from the insider - or insiders as in Holden, Subrahmanyam (1992), Foster, Viswanathan (1996), and Back et al. (2000) - hence it was clear that the insider was the one driving the price to V at the end of the trading period.

However, in our model, if the insider did not trade, the final price of the asset would also go to V at $t = 1$ because the market maker's signal would drive the price by its own. Therefore, we can no longer say that the insider drives the price, but rather that the price is driven - both by the insider and the market maker's public signal - to V at the end of the trading period.

Corollary 6.1. *Let $((H^*, w^*, r^*), \theta^*)$ be the equilibrium in Theorem 6.1. Then, the following statements are valid.*

1. *$Y^* := B + \theta^*$ is an $(\mathbb{P}, \mathscr{F}^M)$-Brownian motion.*
2. *$\lim_{t\to 1} H^*(t, X_t, X_t^M) = V$, $\mathbb{P}^z$-a.s., where X is the unique strong solution of (1.5) with Y replaced by Y^*.*

Proof. The first statement is a consequence of Corollary 4.1 under the optimality condition of the Theorem above. The second is a consequence of the fact that $\lim_{t\to 1} \phi(t, X_t, X_t^M, Z_t) = 0$, $\mathbb{P}^z$-a.s.. □

The second statement is about the ex-ante expected wealth of the insider. We have added a proof of the corollary just to remind the reader that $\mathbb{P}$ is the unconditional probability measure in our filtered probability space.

Both insider's expected wealth in equilibrium and her ex-ante expected wealth have a closed form in the literature, as is possible to see in Theorem 6.1 of Çetin, Danilova (2018) and in the discussion at the end of Chapter 6 of the cited book. The ex-ante expected wealth is also known as the value of information. Since the insider is risk neutral, that is the value that an agent would be willing to pay to become an insider, hence it is the price the agent is willing to pay to observe Z.

It is interesting to note that the value of the information does not depend on the insider's private signal, as is the case in the literature. In our case, it is even more remarkable to realize that the value of information depends only on the weighted value of the demand for the market maker when applying the rational pricing rule in equilibrium. Since we only know that there exists a function w such as presented in equilibrium, but we do not have an analytical solution to it, we estimate this value in the numerical analysis developed in Chapter 7.

Corollary 6.2. *Under the equilibrium conditions of Theorem 6.1, the ex-ante expected wealth of the insider, i.e. the expected wealth of the insider under* $\mathbb{P}$*, is*

$$\mathbb{E}(W_1) = \sigma_V \left(\int_0^1 w(t) \right). \tag{6.1}$$

Proof. Recall that from Theorem 2.3 that $Z_0 \sim N(0, c^2)$. Hence, combining it with the definition of $\mathbb{P}^z$ and applying the tower property, we get that

$$\begin{aligned} \mathbb{E}(W_1) &= \mathbb{E}(\mathbb{E}^{Z_0}(W_1) = \mathbb{E}(\mathbb{E}(W_1|Z_0)) \\ &= \mathbb{E}\left(\sigma_V \left(\frac{Z_0^2}{2w(0)} + \int_0^1 w(t)\, dt - \frac{c^2}{2w(0)} \right) \right) \\ &= \sigma_V \left(\int_0^1 w(t) \right). \end{aligned}$$

□

Chapter 7

Numerical Analysis

In Chapter 6, we have shown that a specific pair of an admissible trading strategy and an admissible pricing rule is an equilibrium for our model. However, both the strategy and the pricing rule depend on a pair of functions (w, v) for which we do not have a closed form. Therefore, one would like to know how those functions would look like for different values of public and private signals.

Moreover, it is not only curiosity that drives one to want to know the form of both w and v. They are also important with respect to the interpretability of the model we have at hand.

Let us begin recalling that in Corollary 6.2 states that the ex-ante value of information - i.e., the maximum amount a risk-neutral agent would be willing to pay to become an insider in our model - would be given by

$$\sigma_V \int_0^1 w(t)dt$$

according to equation (6.1) which depends on knowing the value of the function w. Therefore, if we would like to know the expected advantage that the insider has by receiving the signal X^I. Furthermore, we also would like to understand whether the speed with which the insider receives X^I would affect their expected wealth.

The second matter we would like to address is the limiting behaviour of β_2 when time goes to one. From equation (4.64), we see that in equilibrium the limit of β_2 must be either zero or one. This is particularly important to understand the liquidity of the model. Recall that equation (4.7) tells us that the price process is given by

$$dS_t = \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)},$$

From Proposition 4.2 we already know that $\lim_{t\to 1} w(t) = 0$, therefore if we want to know what the behaviour of the price is towards the end of the trading period, we must understand what happens with β_2. If it goes to zero, then the analysis is rather complicated. The speed in which w and β_2 could change a lot from what one could expect. However, if $\lim_{t\to 1} \beta_2(t) = 0$ we would have a price that was still going to V because it would be driven by the public signal, but, on the other hand, the liquidity of the market would increase to a perfect liquid one.

As a consequence, we would expect the two elements to behave in the same way as we have seen in Foucault et al. (2016). There would still be some room for profit, as there would be an opening in $(V - S_t)$, but very little feedback effect, as the marginal trading of the insider would affect the price very little. As a consequence, we would have the insider trading very aggressively towards the end of the trading period as predicted by the mentioned paper. Again, it is interesting to point out that unlike in Foucault et al. (2016) the insider does not trade aggressively because she has short-lived information, but because the life of the information she possesses is ending.

Therefore, we are very interested in knowing what the behaviour is like for β_2 towards the end of the trading period.

If we go back to Chapter 4, we will see that there are two things we would like to know to have the solution for the system in (4.29). The first is the functional form of v and the second would be the initial condition $w(0)$.

As a consequence, the aim of this section would be to analyse the behaviour of (w, v) and therefore of the ex-ante value of information and β_2 for different speeds of X^I and X^M. However, from Theorem 4.4, we know that Σ_Z is independent of $v + 1 - \Sigma_Z$ therefore, we do not need to conjugate different public and private signals to understand the behaviour of the aforementioned parameters.

The fact that the parameters w are β_2 are not influenced if the signal of the insider is static or dynamic allows us to compare $v + 1 - \Sigma_Z$ with the static case. Recall from Back (1992) that when the signal is static, then $v + 1 - \Sigma_Z = v = 1 - t$. We should not take that benchmark as an iron law once as long as the insider strategy drives the price process to V

that we have optimally[1]. We refer to Çetin (2018) for a specific study of the static case and to Çetin, Danilova (2018) for a more wholesome analysis of the dynamic case. The rule of $v = 1 - t$ is used as a reference for the static case, as we have a constant trading rate for the insider, making it the most smooth trading strategy among inconspicuous ones.

Therefore, the aim of this chapter becomes to find the initial value of w and to solve the ODE system (4.29) for different values of $1 - \Sigma_M$. In particular, we set $1 - \Sigma_M$ to be equal to $t(1 - \ln(t))$, $1 - \sqrt{t}$, $1 - t$,$1 - t^2$, $1 - t^6$, and $1 - t^{12}$.

The most complicated step in this task is to find $w(0)$. For each value of $1 - \Sigma_M$, we must implement the fixed-point algorithm we developed in Chapter 4. The details on how we implement the algorithm will be left to sections 7.1 and 7.2, however the basic ideas are the same in the cases developed in both sections.

The first step would be to find a suitable starting value for r_0 our updating scheme. In Chapter 4, as we were concerned with the computational efficiency of the algorithm, we have set r_0 to be given by $\frac{\Gamma}{2}$ where Γ is given by the equation (4.40). However, now that we are concerned with efficiency, we suggest a different method to begin the algorithm in sections 7.1 and 7.2.

Once we have an initial value $w_0(0)$, we can describe the update scheme for $w(0)$. Suppose that you have $w_n(0)$ as the initial condition of w for the n-th step of the algorithm. We use an ODE solver package to find the solutions of (w, u), where u is defined by equation (4.30) as $v + 1 - \Sigma_Z$, as given by (4.29) for the initial value $w_n(0)$. Now we can update the value of $w(0)$ as described by equation (4.39) as follows:

$$r_n = w_{n+1}(0) = \left(\int_0^1 w_n^{3/2}(t)dt \right)^{-2}. \tag{7.1}$$

As described in Lemma 4.7, there is a suitable converging subsequence for $(r_i)_{i=0}^{\infty}$ such that from Theorem 4.3 we know it converges to $w(0)$. Once we know the value $(r_i)_{i=0}^{\infty}$ converges to, we can use the ODE solver again to find (w, u) as the proper solutions to (4.30) and perform all the mentioned analysis.

As we can see in figure 7.1, all of the functions we considered for $1 - \Sigma_M$ start at one for $t = 0$ and go to zero as time approaches one. The difference between them is the speed at

[1]It is relevant to mentioned that there is also not uniqueness for the pricing rule as well. Considering the model presented in Chapter 6 of Çetin, Danilova (2018), which is a generalisation of Back (1992), any pricing rule that satisfies equations (6.24) and (6.25) is a rational pricing rule. However, it is important to note that these functions are generally affected by demand that is an output of a given trading strategy.

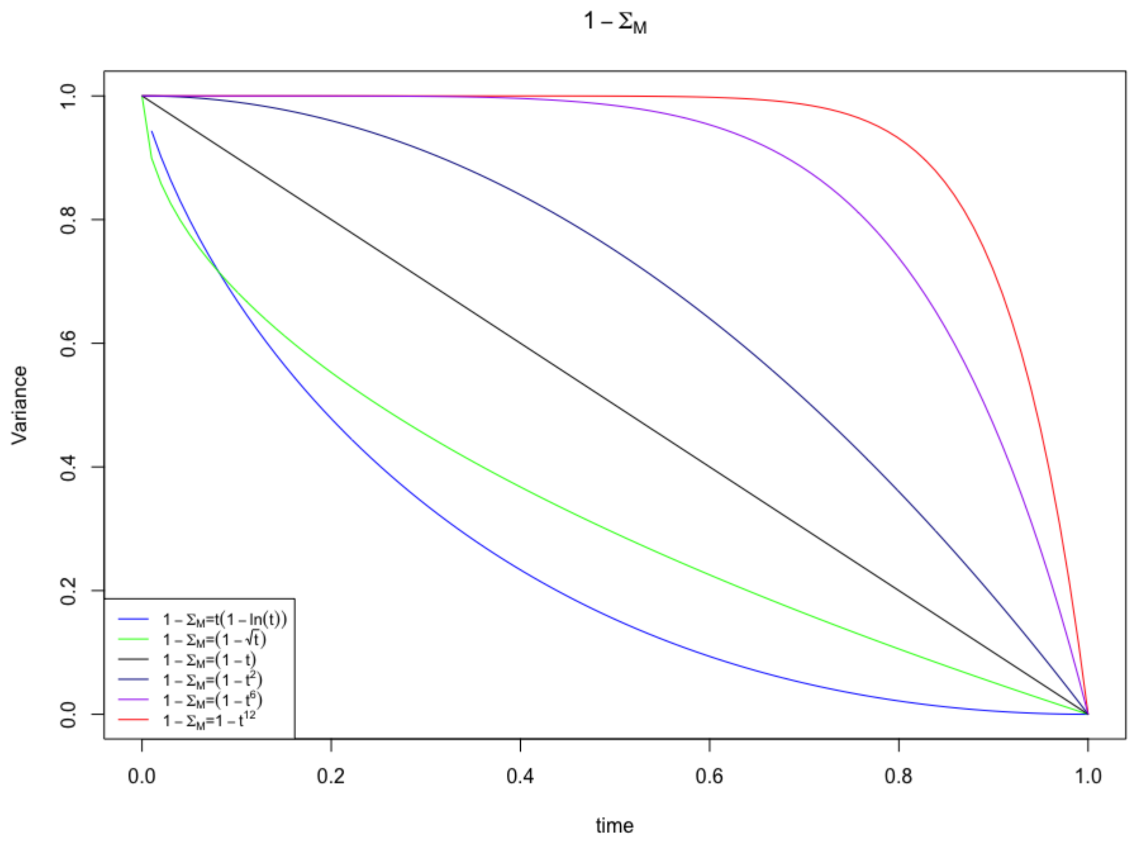

Figure 7.1: $1 - \Sigma_M$

which they go to zero. Compared with the linear case, both convex functions $t(1 - \ln(t))$, $1 - \sqrt{t}$ start to go much faster to zero than $1 - t$ and start to decelerate when approaching one. This means that a lot of information is released at the beginning of the trading period, but the speed with which this information is released diminishes as time evolves.

On the other hand, $1-\Sigma_M$ equals $1-t^2$, $1-t^6$, and $1-t^{12}$ have the opposite behaviour: they start delivering very little information and the closer they get to the end of the trading period, they give a lot of information. As we shall discuss later, we believe that $1 - \Sigma_M = 1 - t^{12}$ would work almost as a benchmark for the case where the insider trades without a public signal for most of the trading period.

In a modelling point of view the convex cases are not as interesting as the concave ones. By delivering a lot of information through the public signal at the beginning of the trading period, it only reduces the uncertainty about the final value of the asset V very early in the game. As an exercise, suppose what would happen if all the information was delivered at the very beginning of the trading period, that is, $\Sigma_M(0) = 1$. In this case, both the insider and the market maker would agree on the valuation of the risky asset. Furthermore, at $t = 1$, it would be made public that their valuation is indeed the price of the asset at the end of the trading period. It is also interesting to point out that this limiting case is exactly the one in which there is no insider. If $\Sigma_M(0) = 1$ there is no uncertainty for the insider, but there is also no uncertainty for the market maker throughout the trading period. Therefore, the insider has no informational advantage with respect to the market maker from $t = 0$ onwards, and hence no way to profit from this trading. As a consequence, we know that in this limiting case, the ex-ante profit of the insider is zero as no one would be interested in paying for information that brings no advantage for those who possess it. Obviously, that is a limiting case. It could be the case that the insider could explore their informational advantage in the beginning of the trading period so fiercely that her expected outcome profit-wise would be the same. As will be made clear by the end of this chapter, that is not the case and the insider makes less profit as the information is released very quickly close to $t = 0$.

On the other hand, as more information is delivered later in time, the greater the insider's informational advantage. Our numerical studies also show that the later in time the information is released, the higher the insider's profit.

We can now proceed to the specifics of our numerics. The convex functions impose some additional challenges when solving the ODEs. Therefore, we have developed a specific code for dealing with them, which is explained in Section 7.2. In that section, we also discuss the conclusions regarding both the estimates for (6.1) and the behaviour of β_2 when $1 - \Sigma_M$ is concave. Obviously, we also want to understand how $v + 1 - \Sigma_Z$ behaves for the different concave functional forms of $1 - \Sigma_M$ as it is proportional to the uncertainty the market maker has about the value of V. In Section 7.2 we also address that.

In the following section, Section 7.1, we explain the details of the algorithm we developed for convex functions. As we do for the concave functions in Section 7.2, we analyse how the behaviour of $1 - \Sigma_M$ affects the values of β_2, the value of information, and $v + 1 - \Sigma_Z$ for the convex functions in Section 7.1. In both concave and convex cases, simulations were performed on the software `R`, R Core Team (2021).

We collect all the takeaways from our numerical analysis so we can finally say in Section 7.3 how the speed at which $1 - \Sigma_M$ goes from one to zero affects, if it does, the shape of β_2, particularly the value of $\lim_{t\to 1} \beta_2(t)$, the behaviour of the function $v + 1 - \Sigma_Z$, and the value of the integral (6.1) that is the ex-ante value of information.

We once again mention the independence between Σ_Z and $u = v + 1 - \Sigma_Z$ given by Theorem 4.4 to explain that the numerical analysis developed in Section 7.4 is more an illustration of how the function v would behave for the approximation of u we have performed. Since Σ_Z and u are independent, we obtain the estimated value of $Var(Z_t|\mathscr{F}^M)$ simply by taking the value of $1 - \Sigma_Z(t)$ from the value of $u(t)$ we have found for all the values of t in our grid.

7.1 Convex case

In this section, we are going to describe the algorithm we develop to study the convex cases we have selected. As is also the case for the concave functions of $1 - \Sigma_M$, the most difficult task we have is to find the initial value of w. Therefore, as discussed in the previous section, we must implement the algorithm we developed in Chapter 4.

Let us begin by describing how we performed any given step of the algorithm. The method used to solve the ODEs was `ode45` from the package `deSolve`. We have made a time grid to input in the function of size 20 from 0.01 to 0.99 with lengths of the same size. Using this grid as input to the package with initial conditions $u(0) = 1$ and some $w_n(0)$, the package returned us approximate values for v and w. We do not use an index for neither the functions u nor w because we do not keep track of these functions before we find the limiting value of $w_n(0)$ to which the algorithm converges.

Therefore, the next step of the algorithm would be to find a new value of $w_{n+1}(0)$ according to the update scheme given by equation (7.1). Due to the fact that we have estimated the function w and the time lengths are also the same size, we could calculate the above equation in an ordinary way. With the values of w applied to the time grid, we could calculate $w_n^{3/2}$ applying the correct exponential. After that, we took the average value of $w_n^{3/2}(t)$ for every t in the time grid. By doing that, we would be therefore calculating $\int_0^1 w_n^{3/2}(t)dt$. Finally, once we have the integral value, all that was left to do to find the updated value $w_{n+1}(0)$ was apply the exponential -2.

It is important to note that any package developed to solve the ODEs system does not aim to produce an integral as described before, so we may have the error propagation from the estimate solutions to the integral, particularly because we are not only integrating, but also applying transformations before and after the integration.

As we mentioned in the previous section, while developing the fixed-point algorithm, we did not need to worry about computational efficiency, as all we needed for Lemma 4.7 was a convergent subsequence. Considering that and the fact that while producing the approximation for the functions u and w with the computational package, we may have a propagation of errors when finding the integral in equation (7.1), we have made an effort to start the algorithm in a suitable candidate for the value of $w(0)$. Otherwise, starting too far away from the value the algorithm should converge to, we could have it diverging due to the aforementioned possible propagation of error.

In addition to that, it is interesting to note that 1 is a lower bound for the sequence r_n for all t as given by equation (4.45) in the proof of the lemma 4.4. While performing the algorithm we have notice that if the sequence r_n would increase by a certain threshold that lower bound would be reached. Furthermore, often by applying 1 to a given r_n we would get below 1 again by r_{n+2}. Therefore, I think it is safe to say that the propagation of error could be big enough to prevent our algorithm to converge if we started r_0 very far away from the limit value $w(0)$.

On that account, we have decided to add an additional routine to our code. This routine sets the initial value r_0 to perform the algorithm mentioned above.

We developed a grid starting from the lower bound 1 up to 3 by steps of size 0.1. For each candidate value for r_0 we recorded if r_1 would be smaller or greater than r_0. We then select the smaller value of r_0 so that it generates an updated value r_1 smaller than r_0. Taking into account the function $1 - \Sigma_M = t(1 - \ln(t))$, such a value was 2.5. I.e., all r_0's from 1 to 2.4 generated values of r_1 greater than their respective r_0's. On the other hand, values from 2.5 and above would generate values smaller than themselves. Since we are interested in a value that is the result of an converging subsequence, we know that for $1 - \Sigma_M = t(1 - \ln(t))$ the converging value $w(0)$ should be in the interval from 2.5 to 2.6.

For that reason, the same procedure was repeated for the values between 2.5 and 2.6 with steps of size 0.01 for the case $1 - \Sigma_M = t(1 - \ln(t))$. The value we found was 2.57. The approach was repeated for the following decimal term such that we have found an initial value to start the proper algorithm of 2.570.

Accordingly, we could now start the algorithm described in Chapter 4 with an initial value $r_0 = 2.570$ for $1 - \Sigma_M = t(1 - \ln(t))$. However, to manage the instability of the algorithm, we decided to work with a moving average of size seven. Hence, the input to calculate r_n

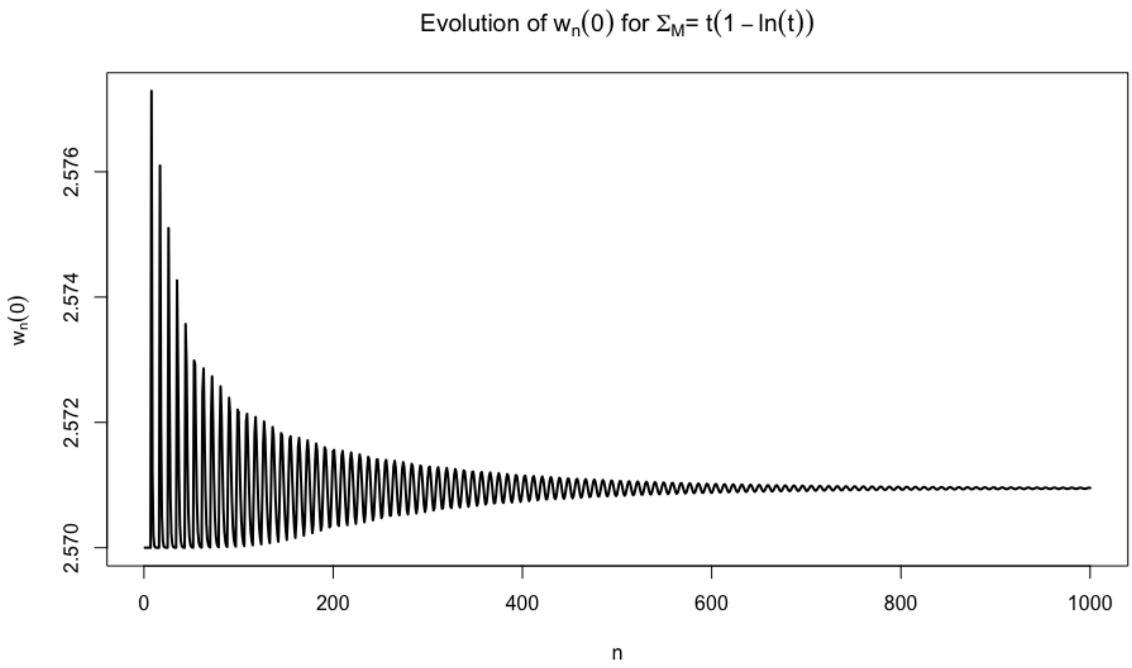

Figure 7.2: Evolution of w_n for $1 - \Sigma_M = t(1 - \ln(t))$

for any given n was the average of the previous seven results. All the values for $(r_n)_{n=0}^{6}$ in the actual code were just the initial condition developed in the previous term, that is, $r_i = 2.570 \; \forall \; i = 0, \ldots, 6$.

From figure 7.2 we can see that our attempts to avoid volatility paid off as we managed to get convergence for the case when $1 - \Sigma_M = t(1 - \ln(t))$. Note that, as expected due to our initial routine, we have found a value quite close to the initial condition we started with: 2.570948.

Now that we have found our estimate for the initial value $w(0)$, we can now apply the function `ode45` from the package `deSolve` to find the approximation for the functions $u = v + 1 - \Sigma_Z$ and w. In figure 7.3 we can see the estimated function w for $w(0) = 2.570948$. Such function led to an integral $\int_0^1 w(t)dt$ of 0.57. The estimated function $v + 1 - \Sigma_Z$ is represented in figure 7.4.

In figure 7.4 it is possible to see that the market maker takes a lot of information from the public signal as a lot of information is delivered in the beginning of the trade period. Throughout the entire trading period the uncertainty of the market maker is much smaller than she would have without a public signal. However, it is interesting to note that she also

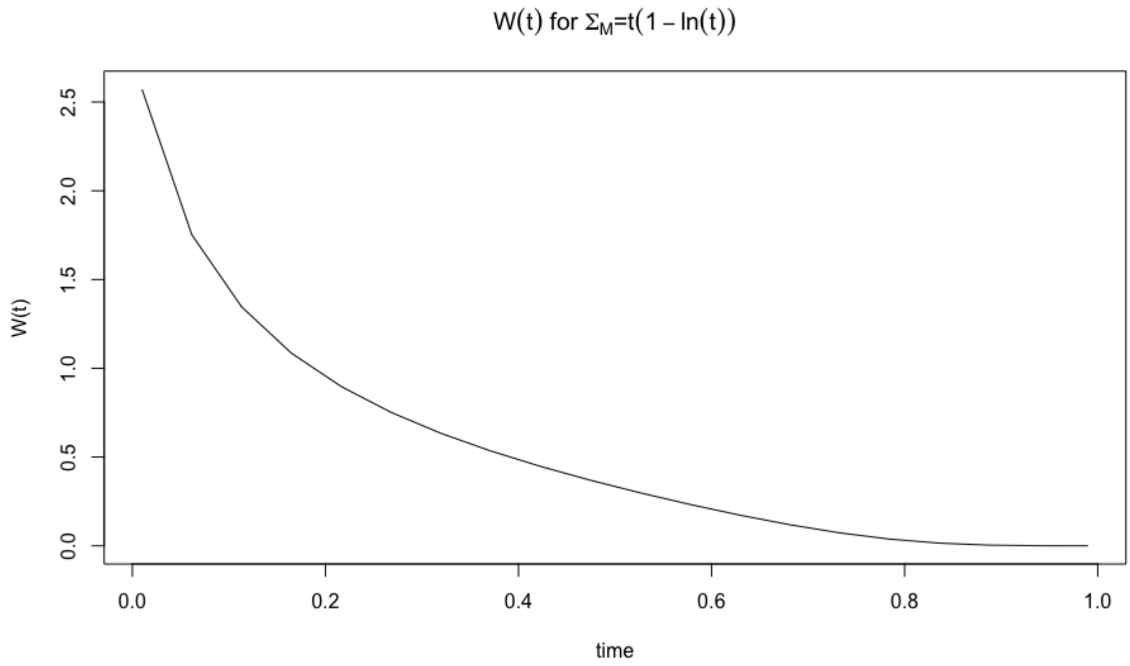

Figure 7.3: w for $1 - \Sigma_M = t(1 - \ln(t))$

uses quite some information coming from the trading as the estimated curve is substantially smaller than she would have without any information coming from the demand.

This behaviour is in line with the estimate function β_2. As will be the case in all numerics, we have a β_2 beginning at one. That is because in order to perform the numerical analysis we have set σ_V to be equal to one. As a consequence, since $u(0) = v(0) + 1 - \Sigma_Z(0) = 1$ and $1 - \Sigma_M(0) = 1$ we have from (4.26) that $\beta_2(0) = 1$.

In this particular case, as one can see in Figure 7.5, we have β_2 with relatively high values showing that during most of the trading period the market maker heavily relies on the information coming from the public signal to build their valuation about the risky asset.

We can now move forward to the case when $1 - \Sigma_M = 1 - \sqrt{t}$. Let us begin with our search method for the candidate r_0. The same package in `R` and the same time grid used for the previous case was used for this case. By performing an analogous code for the previous case, we have found a suitable value for r_0 of of 2.248. The main algorithm has started with a little bit less volatility than in the previous case, as it is possible to see in Figure 7.6, and with 1000 iterations we had already found convergence for $w(0) = 2.248458$. Again, our efforts in finding a candidate that would produce a stable candidate for the algorithm have

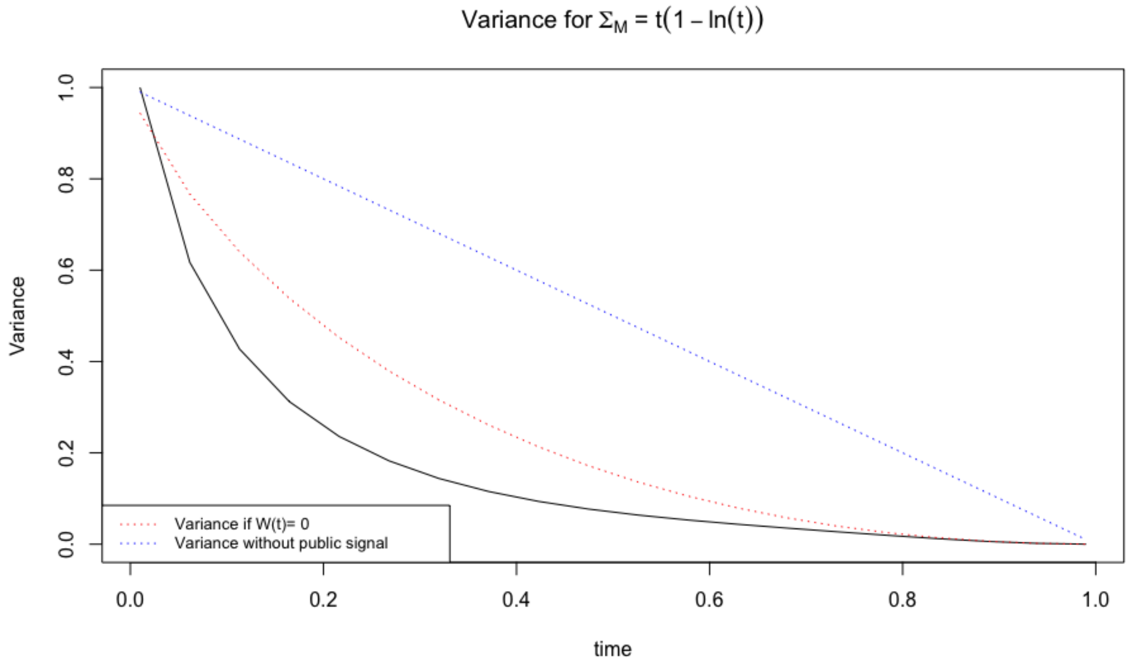

Figure 7.4: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = t(1 - \ln(t))$

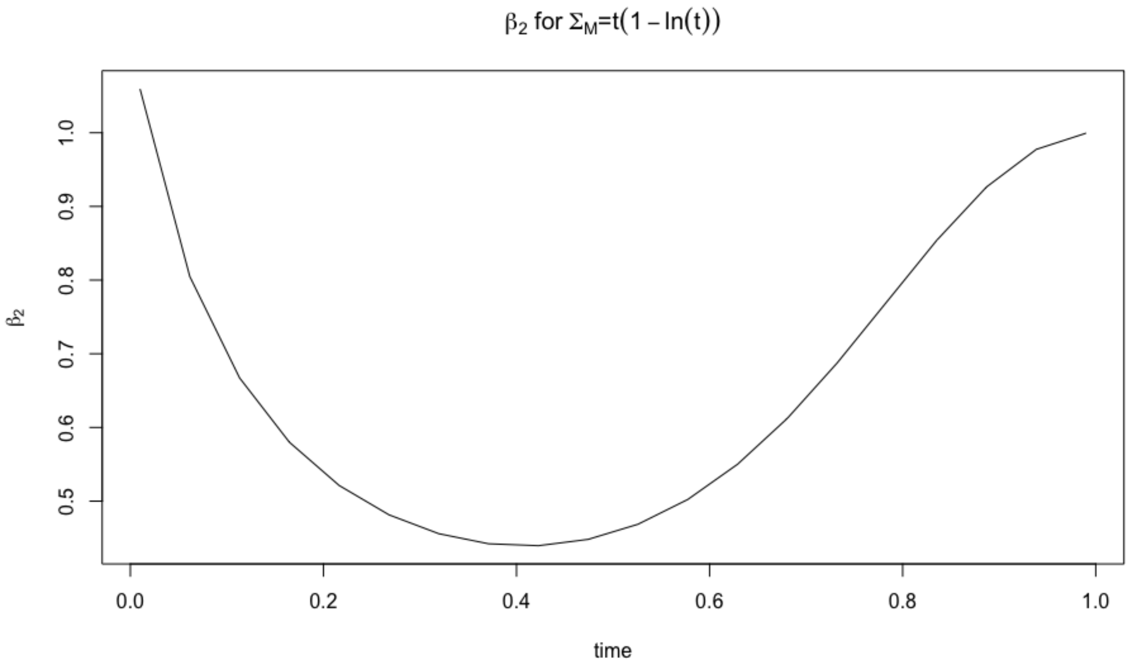

Figure 7.5: β_2 for $1 - \Sigma_M = t(1 - \ln(t))$

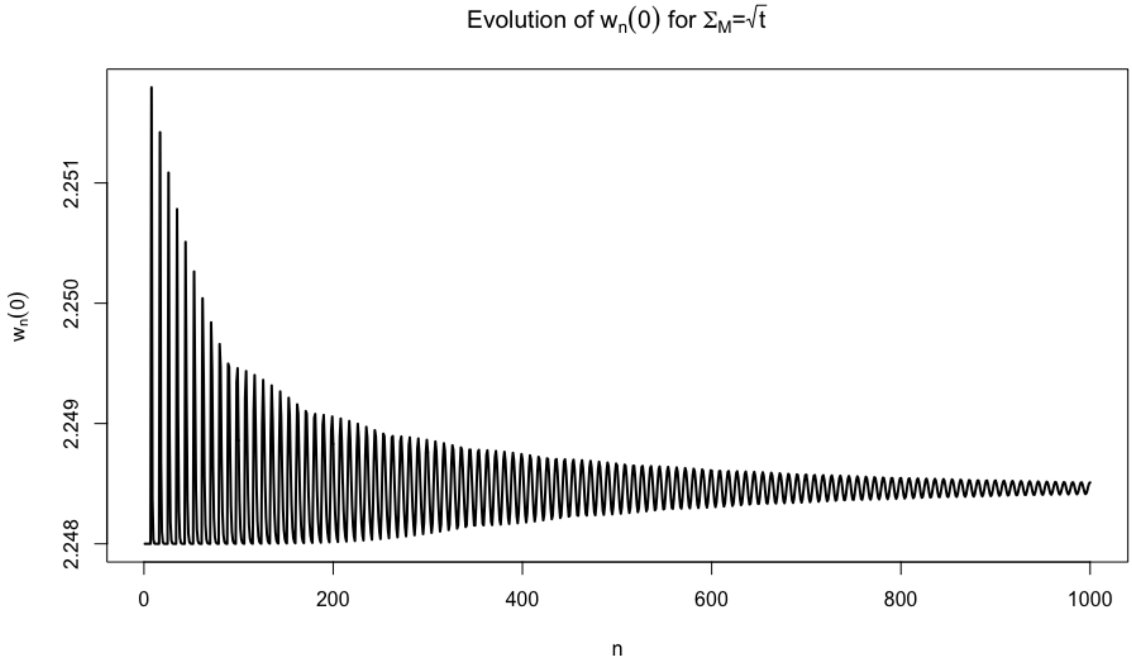

Figure 7.6: Evolution of w_n for $1 - \Sigma_M = 1 - \sqrt{t}$

been successful, as r_0 is very close to $w(0)$.

The estimated value of the integral $\int_0^1 w(t)dt$ was 0.6650335. This shows a trend that will be confirmed with the other cases: the slower the speed of Σ_M in the beginning of the trading period, the higher is the value of $\int_0^1 w(t)dt$ and the smaller is the value of $w(0)$. With the information provided by the other numerics in this chapter, the implications of this consideration will be addressed in Section 7.3.

Again, once we have the value of $w(0)$ we can produce the approximations of (u, w) using `ode45` now to evaluate the behaviour of u, w and β_2.

The behaviour of β_2 is quite similar as in the previous case as we can see by comparing the figure 7.5 with the figure 7.7.The main difference is that the minimum value of β_2 is a little longer in time, occurring about $t = 0.7$. From figure 7.1 it is possible to see that the square root case has a delivery quite slower than the previous case. We understand that the behaviour for the case when $1 - \Sigma_M = t(1 - \ln(t))$ has a minimum early because the uncertainty for the market maker becomes small very fast so she does not use a lot of incremental information.

On the other hand, figure 7.8 shows that the uncertainty for the market maker is quite

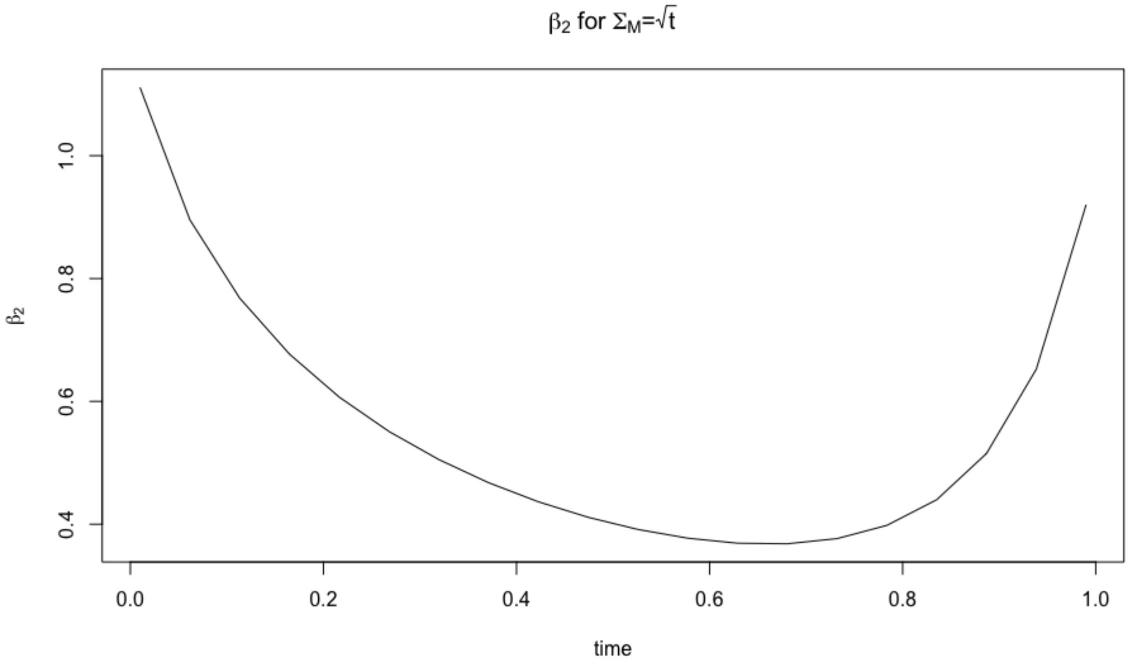

Figure 7.7: β_2 for $1 - \Sigma_M = 1 - \sqrt{t}$

similar in both convex cases. Again, the value of $v + 1 - \Sigma_Z$ is quite small from the beginning, but the gain from having access to the information coming from trading with the insider is far from negligible, as it is possible to see from the difference between the red dotted line and the black line in Figure 7.8.

The estimate function w when $1 - \Sigma_M = 1 - \sqrt{t}$ is also quite similar to the previous case as it is possible to see in figure 7.9.

We can proceed to the next section, where we analyse the remaining cases for $1 - \Sigma_M$, $1 - t$,$1 - t^2$, $1 - t^6$, and $1 - t^{12}$.

7.2 Concave Cases

We can now start describing the numerics we developed for the linear and concave cases. The main challenge of the ODEs solver package is the fact that we have two stiff equations, i.e. they have fast-varying parameters. As we can see in the images 7.3 and 7.9, in the convex cases, the estimated w decreases very fast in the beginning of the trading period, but towards the end of the period, it becomes more flat. Therefore, we had to use a specific method to deal with the convex cases, as we explained in Section 7.1.

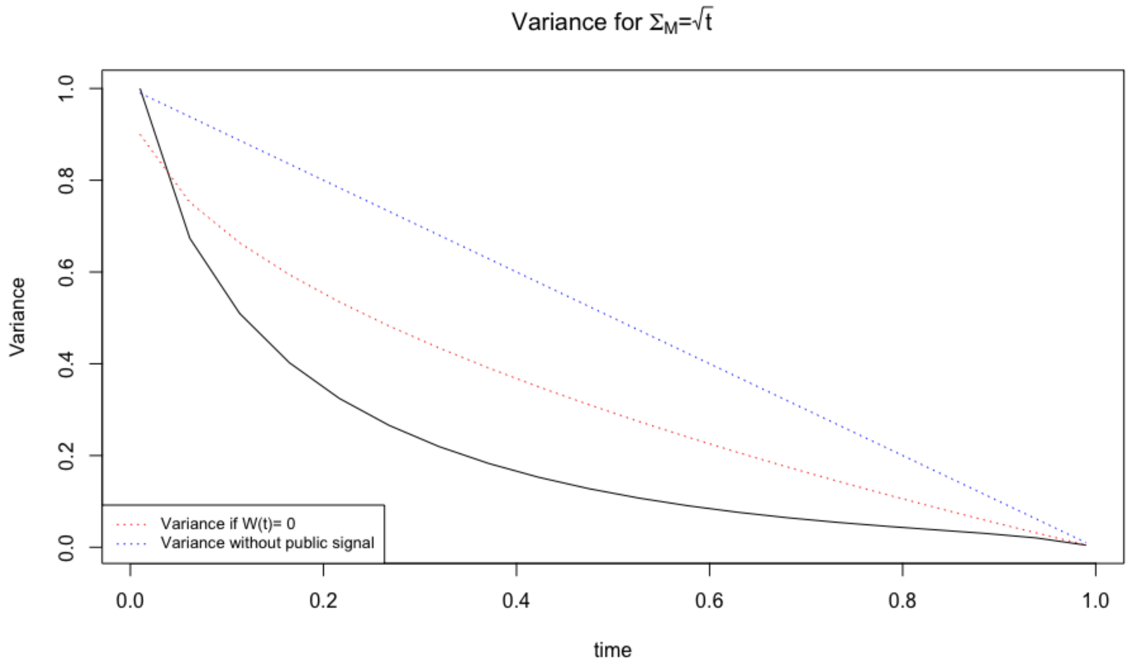

Figure 7.8: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = 1 - \sqrt{t}$

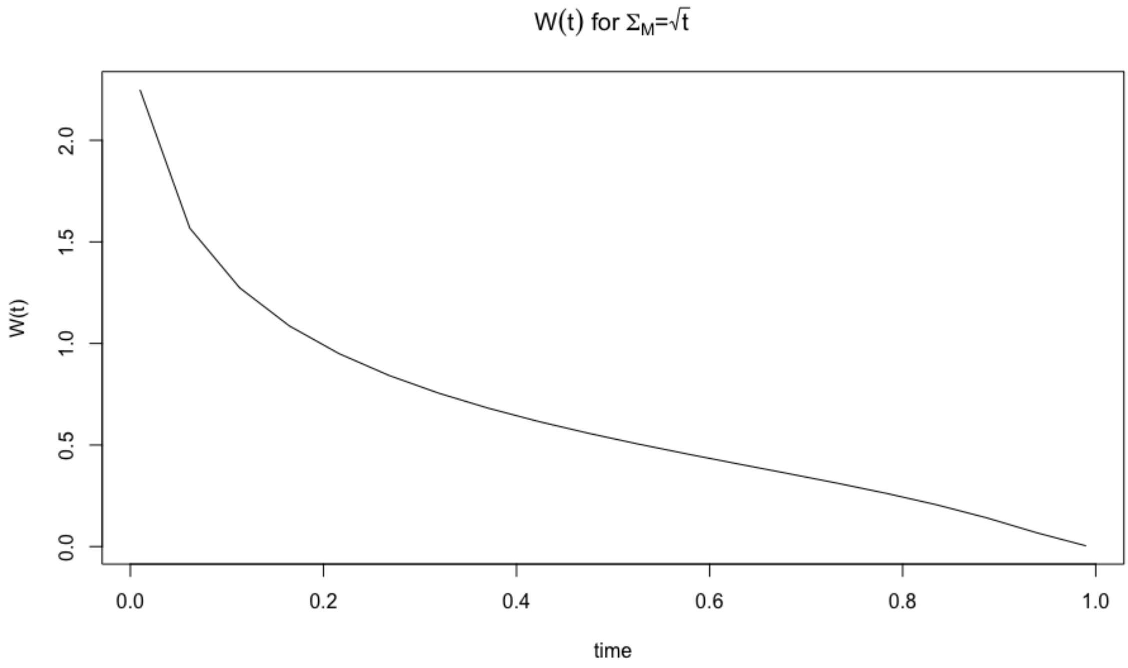

Figure 7.9: w for $1 - \Sigma_M = 1 - \sqrt{t}$

However, for the concave cases, as it is possible to see from figures 7.10, 7.11, 7.19, and 7.22, the behaviour of w is much more stable than in the previous cases, particularly in the beginning of the trading period.

To develop our numerical analysis for these cases we have also used the software `R`, but now we have used the function `ode78` of the package `pracma`. We have opted for this package because it produces more accurate results than the `DeSolve`. Hence, we took advantage of the stability of the concave functions to improve the accuracy. Recall that any estimation errors we have on the estimates of the solutions of the ODEs are propagated when calculating the integral of (7.1), hence it is important to seek accuracy whenever it is possible.

To further increase the precision of the estimates, we use the function `ode78` to estimate $v + 1 - \Sigma_Z$ and $ln(w)$ instead of w. It is interesting to mention that, unlike `DeSolve`, `pracma` has the time grid as an output of the function. In particular, for the linear case, we also had a time vector of size 20.

From the estimated function of w - i.e. the exponential of the estimated curve for $ln(w)$ - in figures 7.10 and 7.11 it is possible to see that the `R` functions produce trapeziums below the curve of the functions. Indeed, the trapezium shape comes from the fact that the estimated values of w are connected to each other in a straight line.

The trapezoidal shape will be key for us to estimate r_n for all the steps. At every step of the algorithm, we are given two functions $v + 1 - \Sigma_Z$ and $ln(w)$. Now, the first step to find r_n is applying the transformation $x \mapsto \frac{3}{2}\exp(x)$ to the function $ln(w)$, which gives us an estimate for $w^{3/2}(t)$ to every t belonging to the time grid the package provided as an output.

Let $(t_i)_{i=1}^k$ be the times provided by the time grid. Note that we have $k-1$ trapeziums with sides $w^{3/2}(t_i)$ and $w^{3/2}(t_{i+1})$ and height $t_{i+1} - t_i$. Therefore, we can easily calculate r_n by finding the area of each trapezium, then summing all the areas of the trapeziums and finally applying the exponential -2 as required by equation (7.1).

Since the output produced by the function is now $ln(w)$ we adapted our search mechanism for the initial value of w. Now we do the same procedure with a grid starting at zero up to one with steps of size 0.1 likewise the searching algorithm for the concave case.

The smallest value for $ln(w)$ that generated an updated value smaller than itself for $1 - \Sigma_M = 1 - t$ was now 3. Due to the greater variability of the logarithm, we extended our search procedure up to 10^{-4}. Therefore, we initialised our main algorithm with an initial condition $ln(w_0(0)) = 0.3261$, or equivalently, $w_0(0) = 1.385554$.

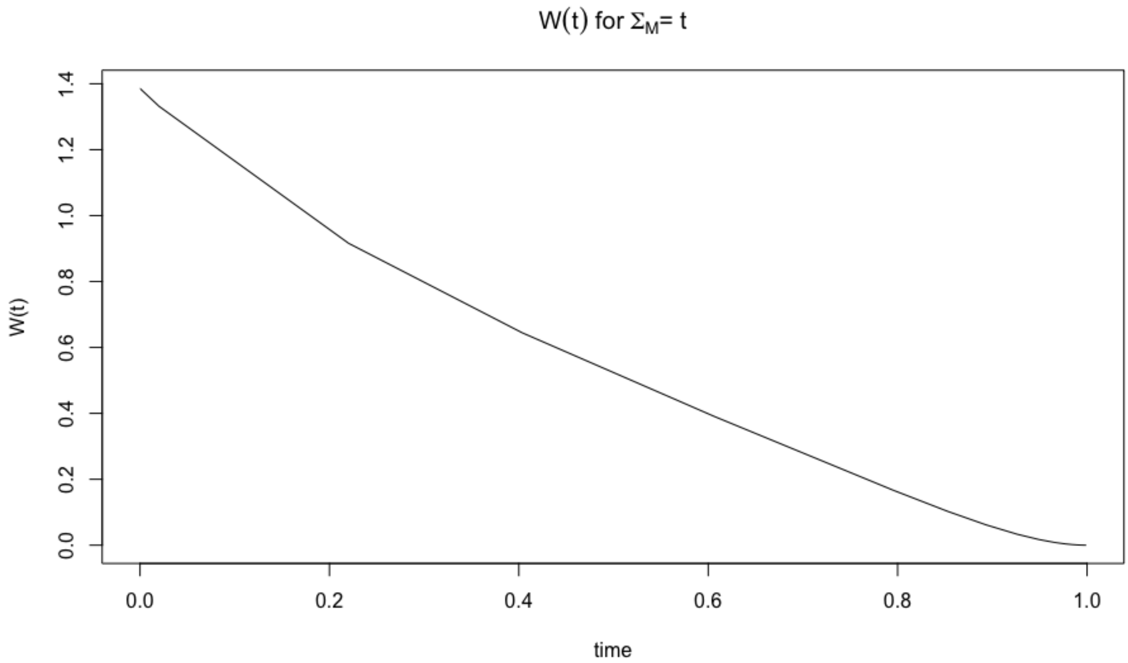

Figure 7.10: w for $1 - \Sigma_M = 1 - t$

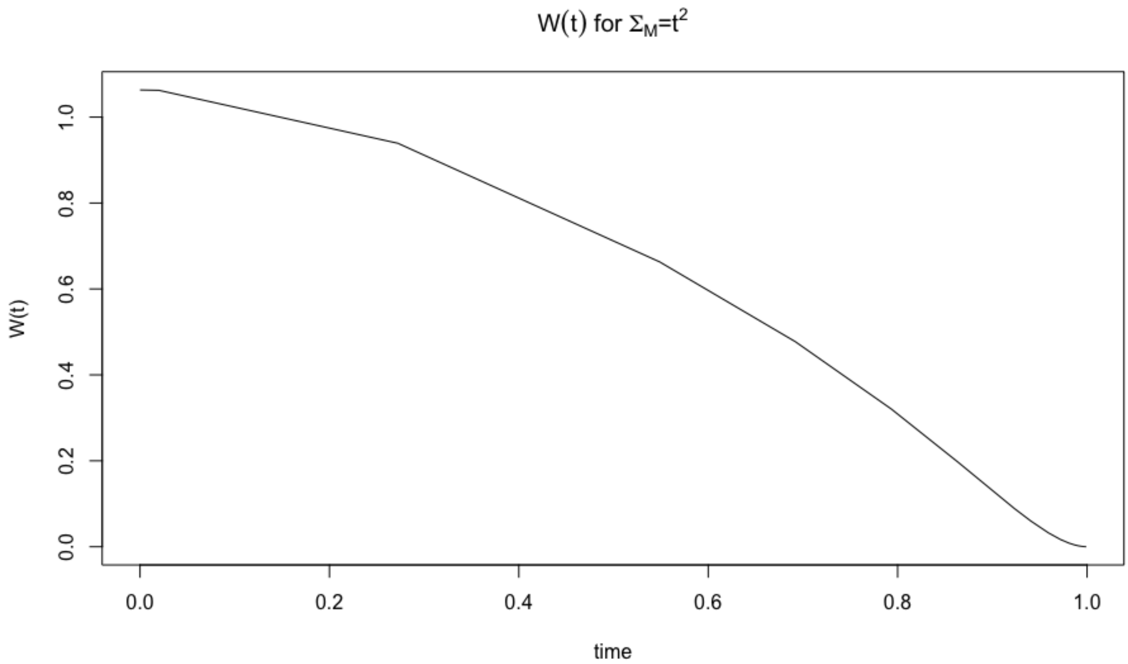

Figure 7.11: w for $1 - \Sigma_M = 1 - t^2$

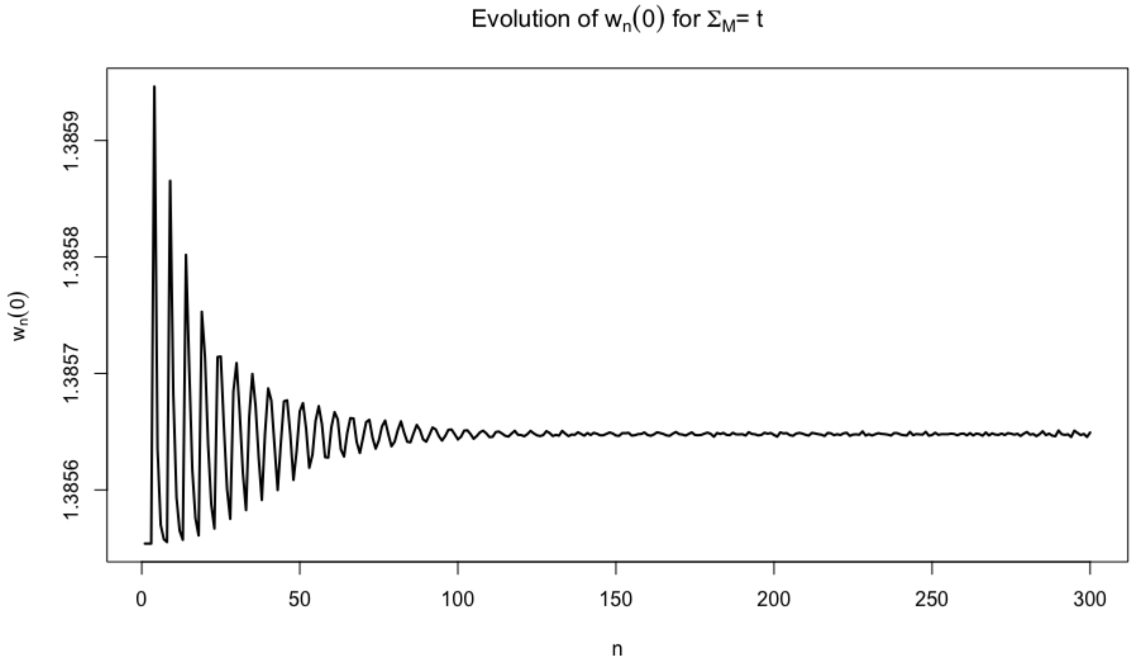

Figure 7.12: Evolution of w_n for $1 - \Sigma_M = 1 - t$

Our efforts to increase the accuracy have shown their role while performing the algorithm as well. We managed to reduce the moving average from the previous algorithm, for concave functions, from seven to three. Furthermore, note from figure 7.12 that even in the linear scale for w the convergence was much faster.

Even though we have performed 300 iterations of the algorithm, it is possible to see that by about 100 iterations we already had almost reached convergence. Again, the value found by the main algorithm for $w(0)$, 1.385647 was quite close to the initial condition. Indeed, one by now could say that the proper algorithm to find $w(0)$ is acting more as an inspection criterion for the search algorithm.

Again, once we have reached an initial condition for w, we can use package `pracma` once more to produce the approximation functions w and u. We do that by applying function `ode78` on `R`. In each iteration of the algorithm, we have calculated $\int_0^1 w^{3/2}(t)dt$ for a different function w; hence now we can use the same procedure we used to calculate $\int_0^1 w^{3/2}(t)dt$ to find $\int_0^1 w(t)dt$, but now instead of applying the transformation $x \mapsto \frac{3}{2}\exp(x)$, we apply the transformation $x \mapsto \exp(x)$.

From figure 7.10 we can see that w for the linear case decays much slower than the ones

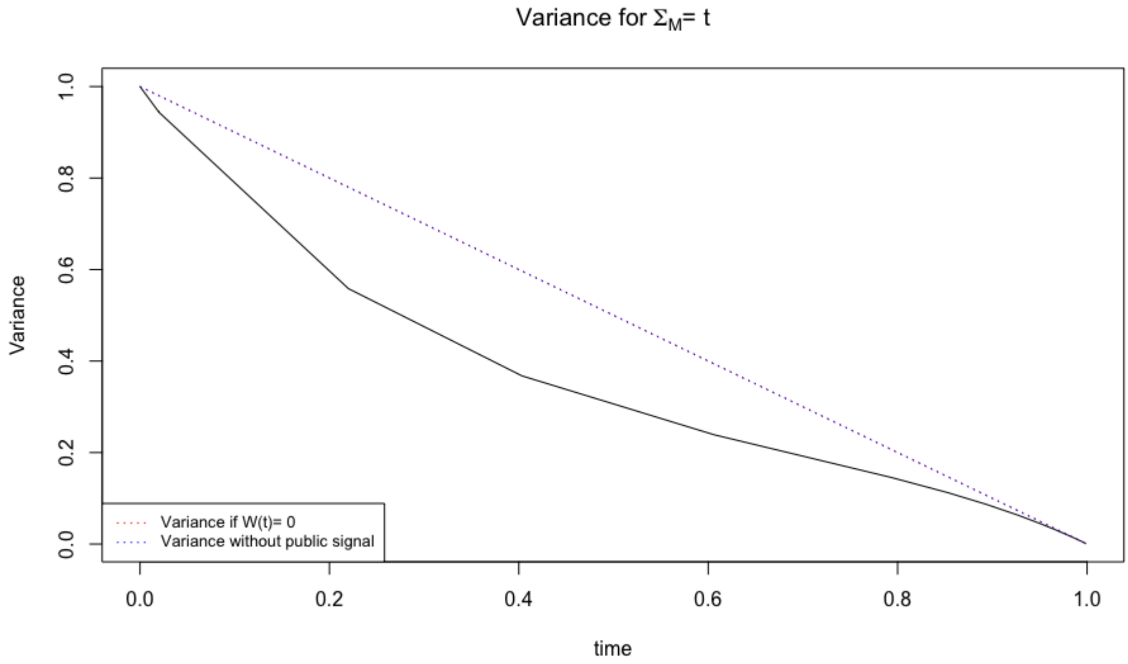

Figure 7.13: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = 1 - t$

represented in figures 7.3 and 7.9, almost in a linear fashion. It is interesting to note that concave values for the public information produce concave functions w and convex functions of $1 - \Sigma_M$ produce convex functions of w. The linear case is still convex, but it is very close to linear.

It is not possible to compare the estimate of the integral $\int_0^1 w(t)dt$ with the previous ones as it is a different method, but we shall compare the estimate of 0.5663463 that was calculated for the linear case with the convex cases.

The uncertainty of the market maker with respect to η is quite interesting in the linear case. This is so because both the benchmark when there is no public signal and the case when the market maker only receives the public signal coincide. One can see from figure 7.14 that the gap between both benchmarks opens, reaching the maximum around 0.4, which is a little bit before the minimum for β_2 in this case. As we will confirm later, indeed the more delayed the information is the further in time is the minimum of β_2.

Once we are done with the numerics for the linear case, we can proceed with our numerical analysis for the case when $1 - \Sigma_M = 1 - t^2$.

While performing the search algorithm for $1 - t^2$ we have reached an interesting case. The

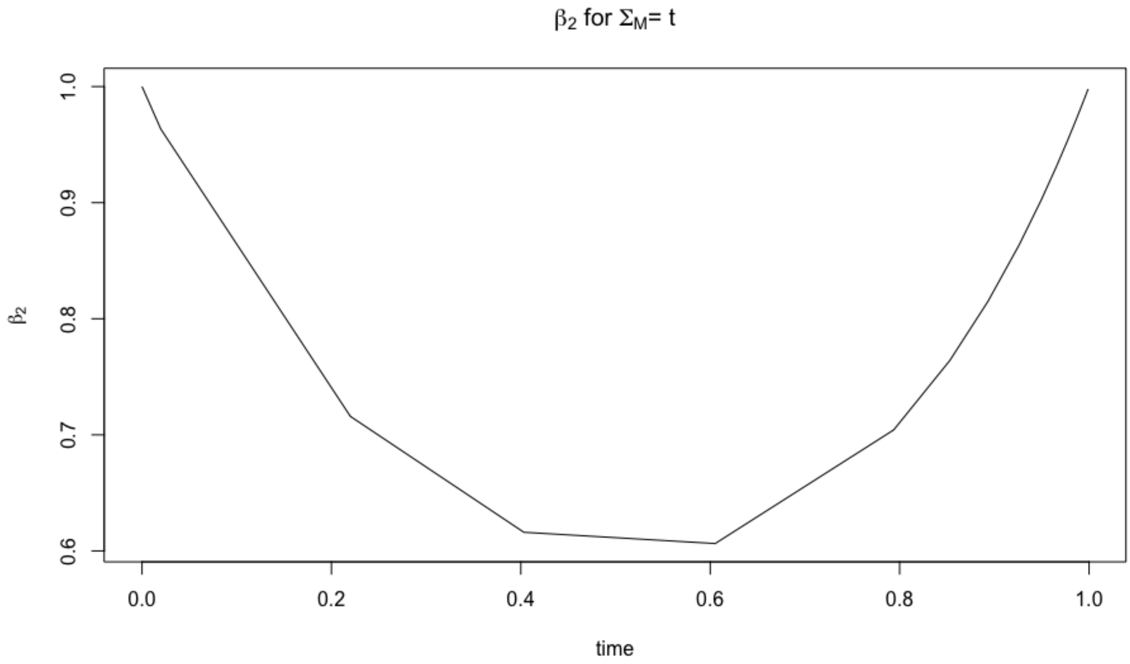

Figure 7.14: β_2 for $1 - \Sigma_M = 1 - t$

value found for r_0 was 1.061837, which is very close to the lower bound found by equation (4.45). Indeed, after performing the main algorithm with the same moving average of three we got an estimate of 1.063186 as we can see in figure 7.15. That would allow us to conjecture that for convex cases in which the delivery of information would happen quite late in the trading period we would have $w(0)$ going to one. Once again the searching mechanism for the initial value of the algorithm allowed us to start in a very close neighbourhood of the final solution.

However, the most interesting thing about the case when $1 - \Sigma_M = 1 - t^2$ was to consider the estimate of $v + 1 - \Sigma_Z$. Note in figure 7.16 that before time $t = 0.8$ we have the estimate for $v + 1 - \Sigma_Z$ crossing the curve of $1 - t$.

This behaviour could mean that the public information comes at a certain speed such that the uncertainty for the market maker is greater than it would be if there were no public signal. We need to be cautious about the previous assertive as it could be the case that our estimate for $v + 1 - \Sigma_Z$ is not accurate enough to say that for sure. However, we cannot dismiss this hypothesis now. As the market maker receives two different signals: the direct one coming from the public signal and an indirect one coming from their interaction with

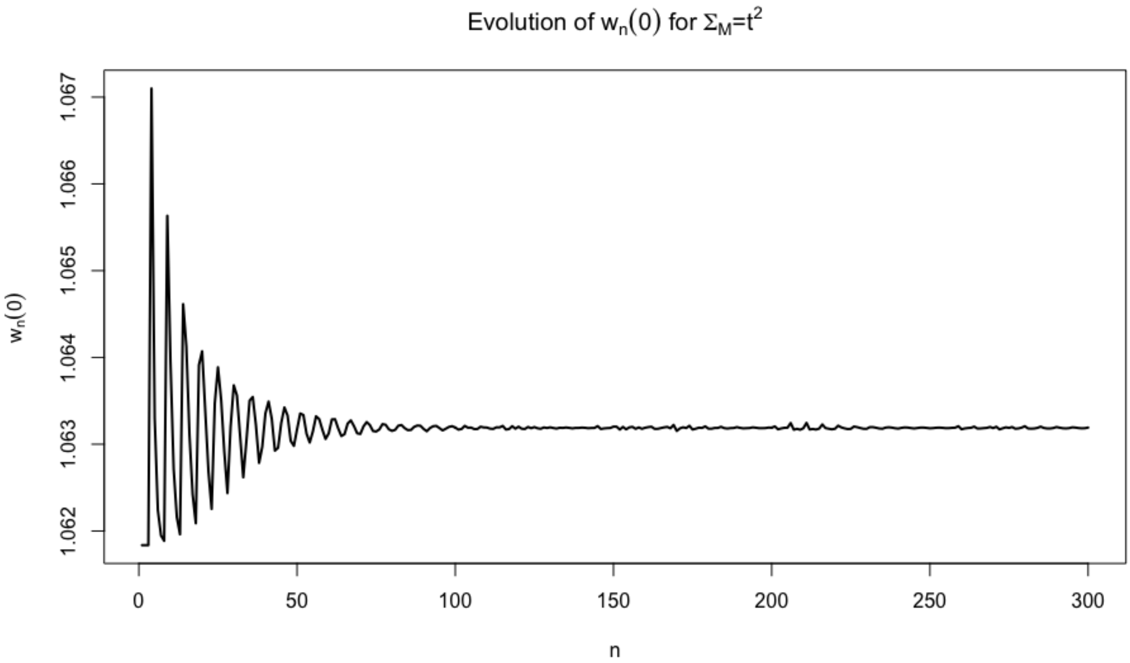

Figure 7.15: Evolution of w_n for $1 - \Sigma_M = 1 - t^2$

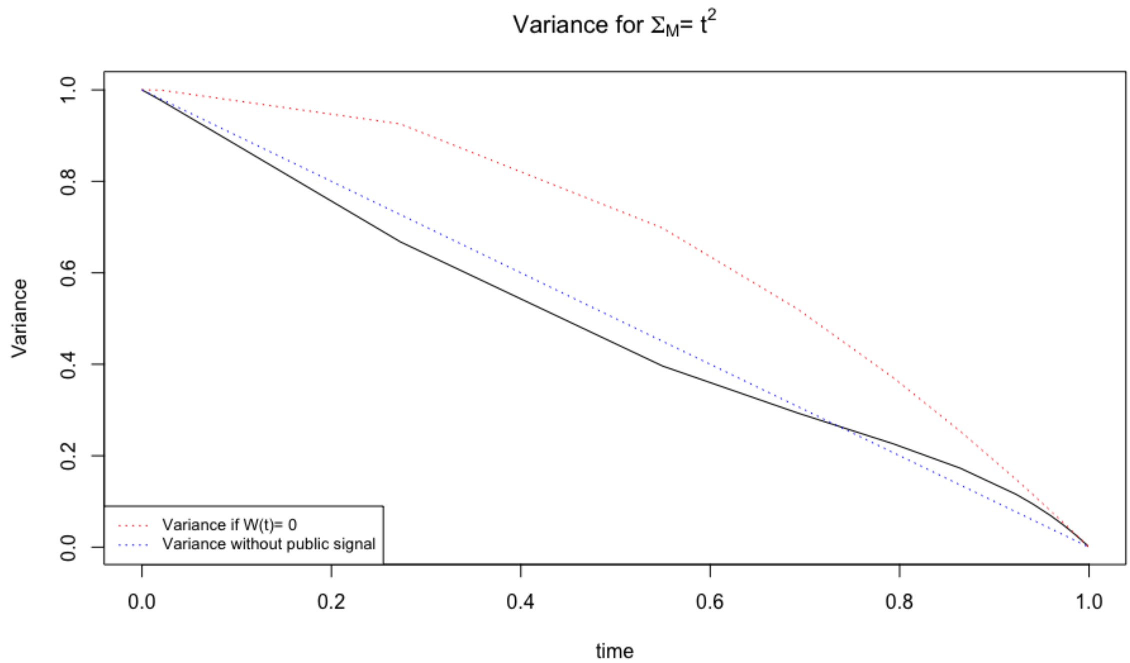

Figure 7.16: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = 1 - t^2$

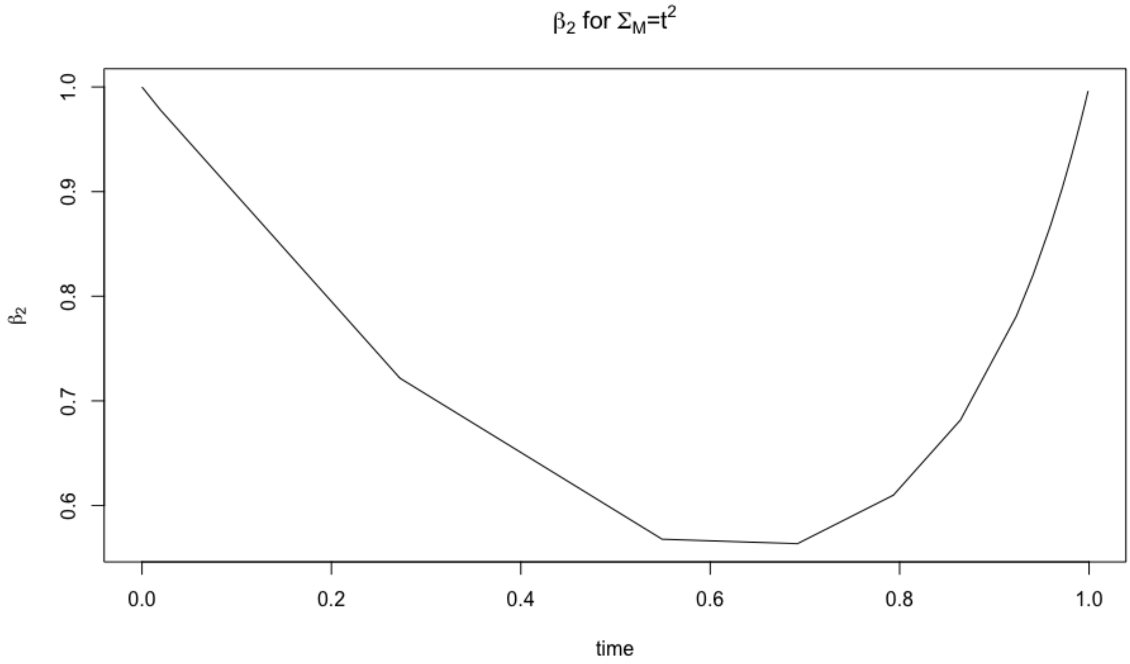

Figure 7.17: β_2 for $1 - \Sigma_M = 1 - t^2$

the insider by trading. The market maker knows she is learning about the price of V, but she does not know what she knows; hence it could be the case that she gets more puzzled by the different sources of information so her uncertainty increases by receiving a public signal. That is a matter that requires further investigation in future research in the theme of this thesis.

In figure 7.17 is possible to see that at the same time β_2 starts to increase and goes quite steeply to one. Going back to figure 7.16 we can see that after the crossing with $1 - t$ the variance attaches to the curve representing the public signal. Therefore, we can conclude that the market maker in the ending of the trading period relies most heavily on the public signal rather than trying to extract information from the demand.

The aforementioned conclusion is quite in line with the motivation of high-frequency trading. When approaching the end of the trading period, the market maker gives much more weight to the public signal while w is going to zero, hence making the feedback effect for the insider diminish. Therefore, the insider is able to trade very aggressively without affecting the price process, so she does that in order to maximise her profit.

The numerical value found for $\int_0^1 w(t)dt$ was 0.6465513. Again, while dealing with the

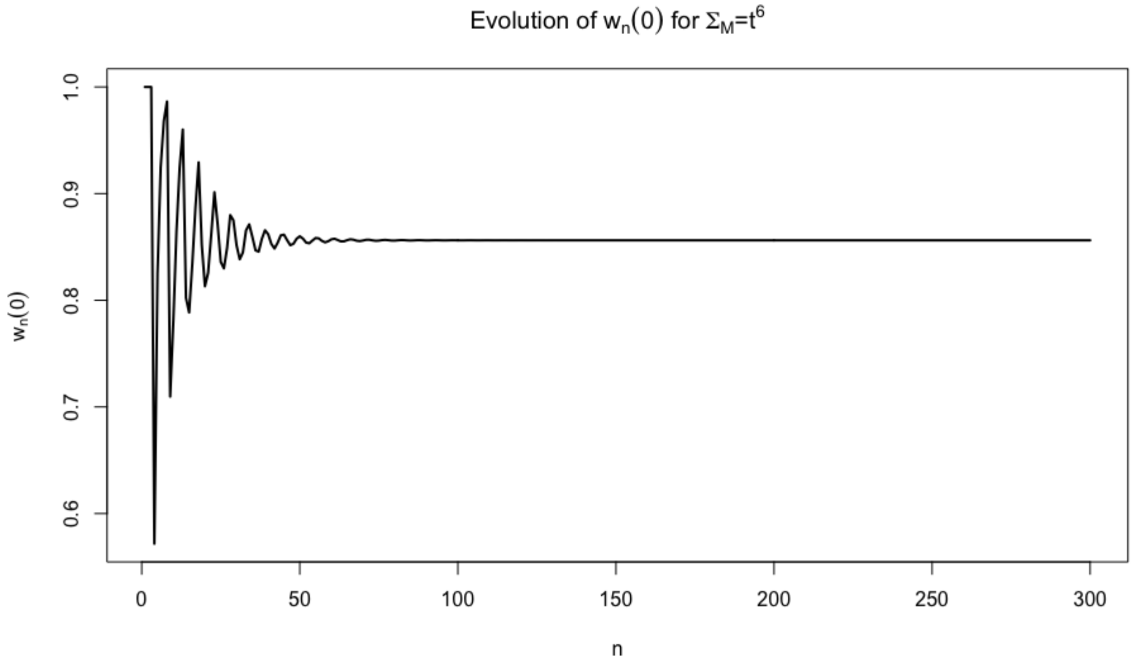

Figure 7.18: Evolution of w_n for $1 - \Sigma_M = 1 - t^6$

release of the information, we increased the ex-ante expected profit. We can also see from figure 7.11 that the function w became clearly concave.

As the reader may now expect, while performing the searching algorithm for the initial value r_0 for the public signal such that $1 - \Sigma_M = 1 - t^6$, we could not find a value that was greater than one that would produce an updated value also greater than one.

Therefore, the logical thing to do was to use 1 as the initial value r_0 and perform the algorithm to check what would be the limit of $(r_n)_{i=1}^{\infty}$. The results are shown in figure 7.18. As we can see from the figure, the algorithm has also found equilibrium below the lower bound of 1 given by equation (4.45). It is very interesting to note that the value found of 0.856 is substantially smaller than the lower bound. As a consequence, we have used 1 as the initial value of $w(0)$.

In figure 7.19 we can see the behaviour of the function w. Even though it has started in the lowest of all values studied in this chapter, it has stayed almost flat during the first half of the trading period, hence also very clearly concave.

The behaviour of $v + 1 - \Sigma_Z$ for the case when $1 - \Sigma_M = 1 - t^6$ also tells us an interesting story. Like was the case when $1 - \Sigma_M = 1 - t^2$, the uncertainty of the insider follows very

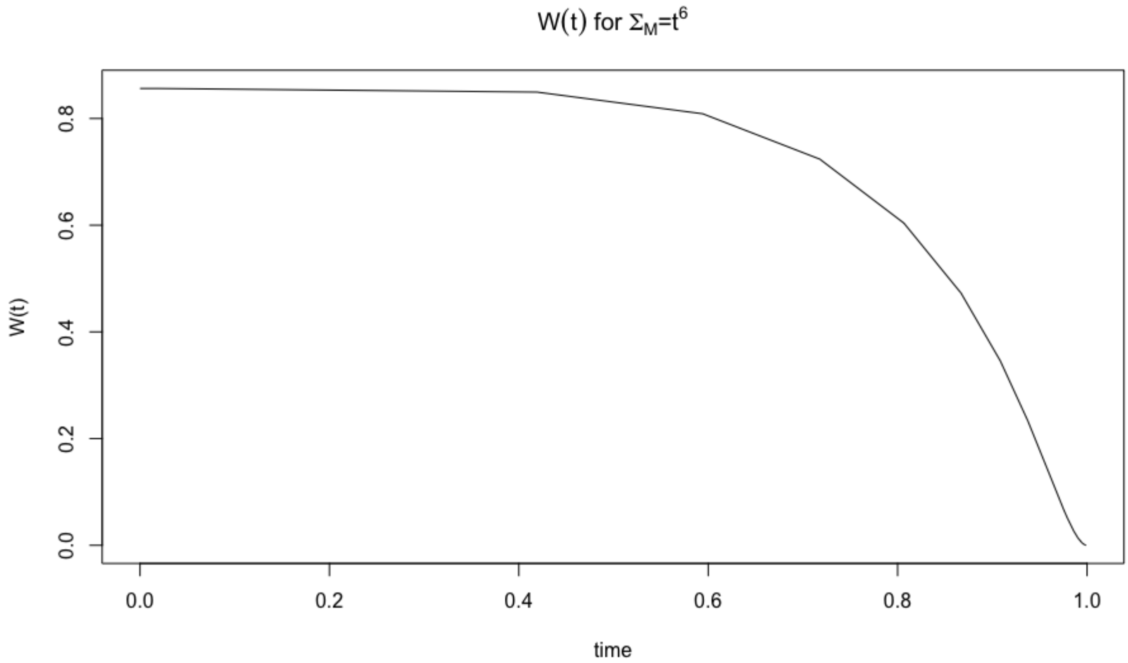

Figure 7.19: w for $1 - \Sigma_M = 1 - t^6$

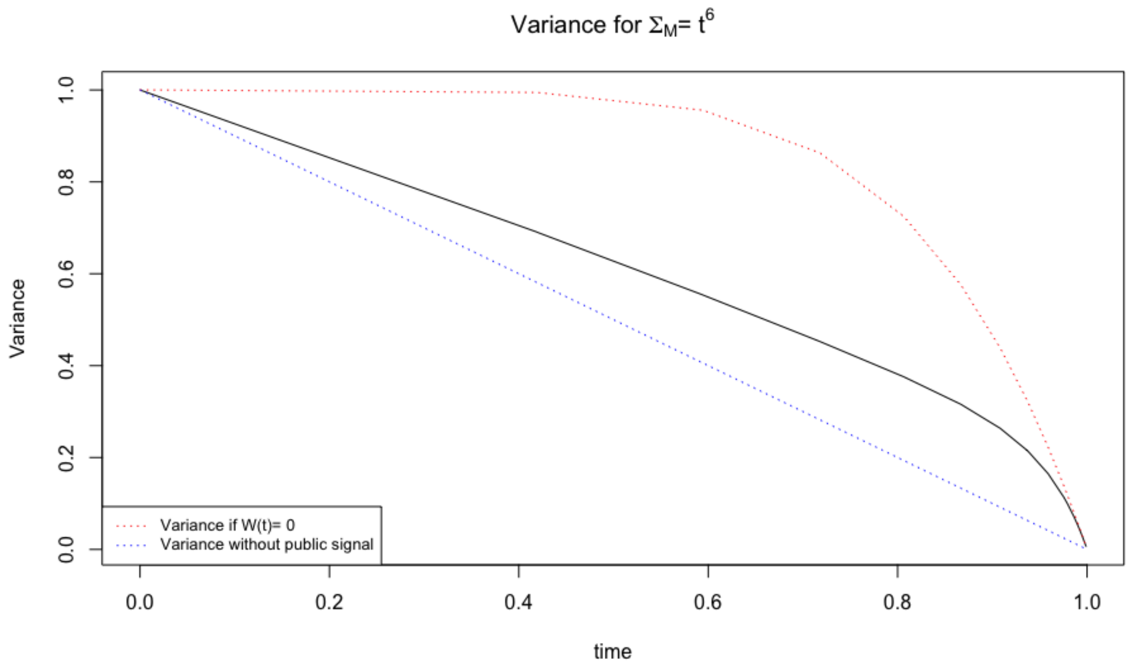

Figure 7.20: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = 1 - t^6$

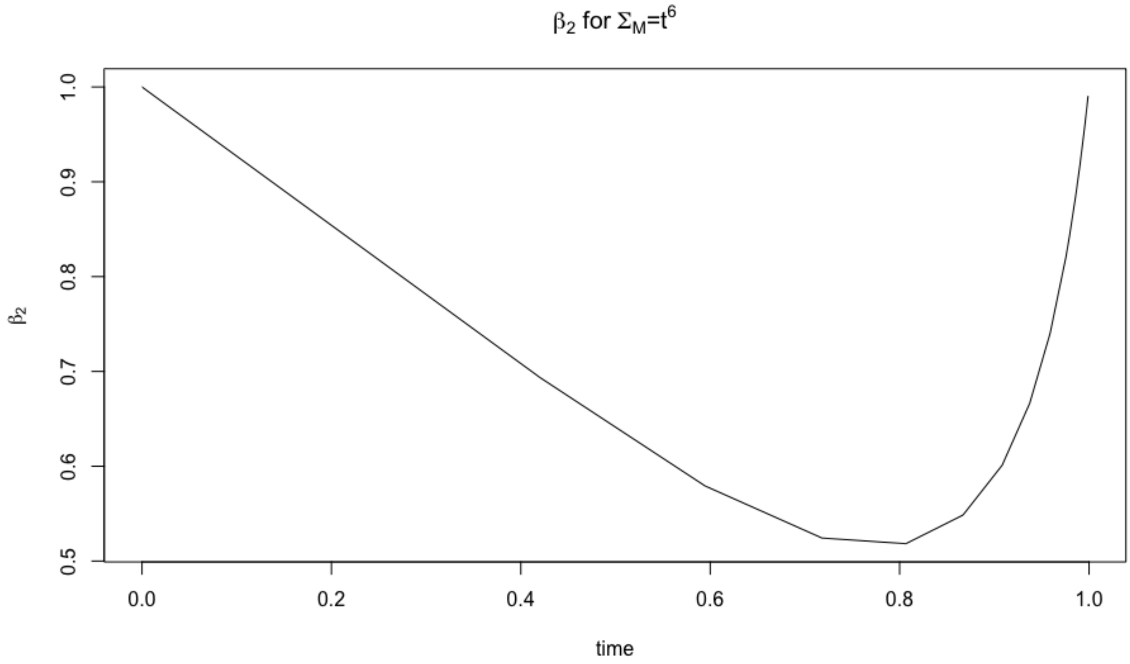

Figure 7.21: β_2 for $1 - \Sigma_M = 1 - t^6$

closely the constant rate line, but, by the end of the trading period, we have it attaching to the public signal line. Again, that movement of relying very closely on the public signal is represented by a sharp rise of the parameter β_2.

It is also very interesting to note that the presence of an uninformative signal actually increased the uncertainty of the market maker. Unlike the square case in which the function $v + 1 - \Sigma_Z$ only crosses the $1 - t$ curve at the end of the trading period to attach itself with the public signal, in this case, when $1 - \Sigma_M = 1 - t^6$, we have the curve $v + 1 - \Sigma_Z$ above the $1 - t$ line throughout the whole trading period. One could say that the role of an informative signal seems to be working more like noise.

There was a substantial increase in the value of the ex-ante profit of the insider in this case, going to 0.7204667.

If $1 - \Sigma_M = 1 - t^6$ was already quite elucidating of the limit behaviour when there is a big flow of information released by the end of the trading period. From figure 7.1, it is possible to observe that the line $1 - \Sigma_M = 1 - t^{12}$ is almost flat until at least 60% of the trading period.

The numerics for the case when $1 - \Sigma_M = 1 - t^{12}$ were much simpler than the previous

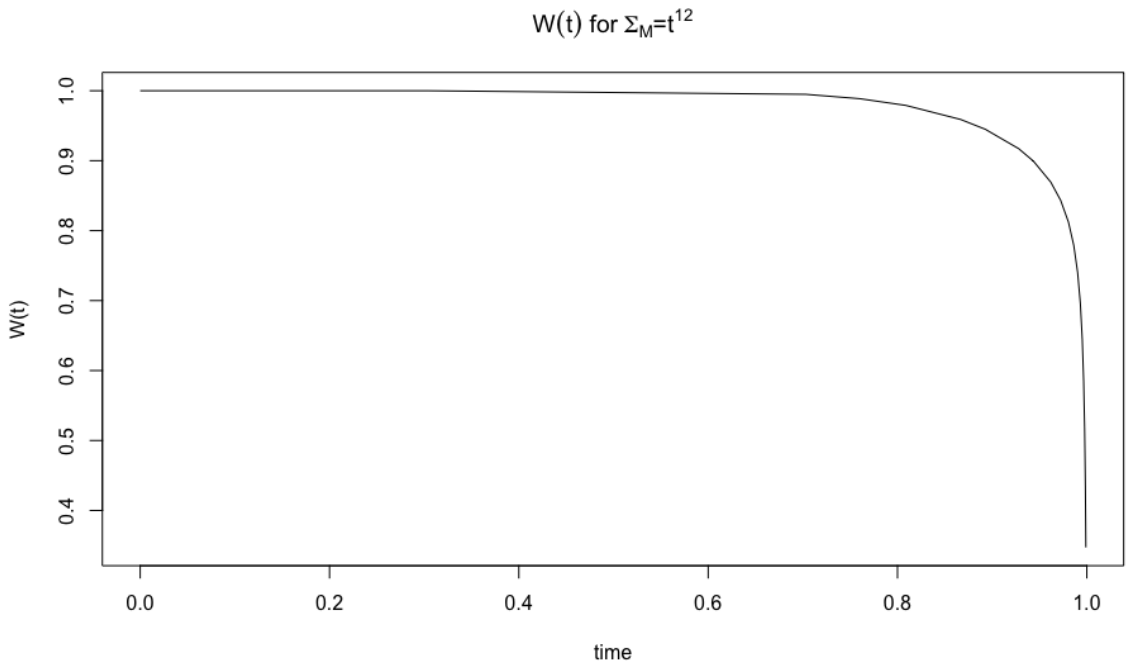

Figure 7.22: w for $1-\Sigma_M = 1-t^{12}$

cases. That is because if both the searching algorithm when started at 1 would lead to an r_0 below one and the proper algorithm for $(r_n)_{i=1}^{\infty}$ would lead to an initial value below one as well for $1-\Sigma_M = 1-t^6$, that would be aggravated in the case when $1-\Sigma_M = 1-t^{12}$. Therefore, all the numerics the reader sees for $1-\Sigma_M = 1-t^{12}$ were done considering $w(0) = 1$.

By comparing figure 7.22 with the other estimated values functions w it becomes clear that in the limit case when all public information is delayed at the latest possible moment, w would be a flat line over the value one. As a consequence, $\int_0^1 w(t)dt$ would be one. Indeed, the estimated value of the integral for $1-\Sigma_M = 1-t^{12}$ was 0.9787778.

Furthermore, the value of $beta_2$ decreased as the insider gathered information about the value of the asset through the demand process and increased as the public information became more relevant.

The behaviour of the uncertainty is also very interesting. As during most of the period there is almost no information coming from the public signal, $v+1-\Sigma_Z$ follows the benchmark of the $1-t$ line. Unlike the case in which $1-\Sigma_M = 1-t^6$, when the existence of an uninformative signal would increase the level of uncertainty, there is so little information during most of the trading period that the market maker's uncertainty just follows the $1-t$

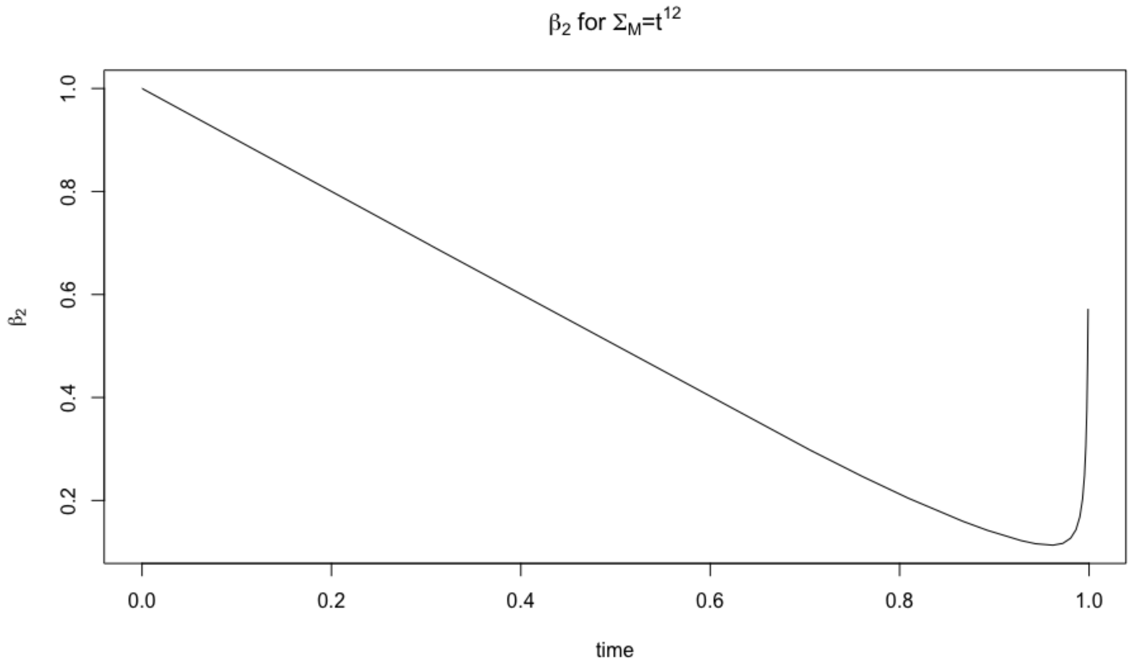

Figure 7.23: β_2 for $1 - \Sigma_M = 1 - t^{12}$

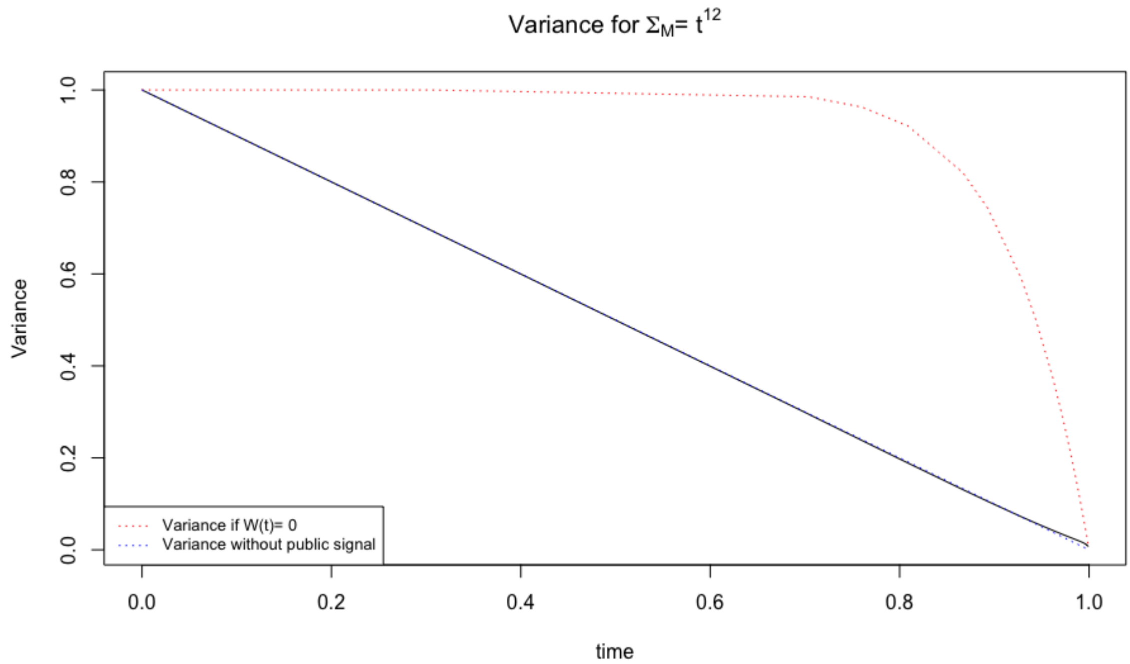

Figure 7.24: $v + 1 - \Sigma_Z$ for $1 - \Sigma_M = 1 - t^{12}$

line, which is smaller than the uncertainty of the case when $1 - \Sigma_M = 1 - t^6$. One could wonder whether the market maker does not take into account the value of the public signal, but not only do we know that $\lim_{t \to 1} w(t) = 0$, but also we can see in figure 7.22 that our numerics capture that feature of the model.

Analysing back figures 7.3 and 7.9 it becomes clear that the greater the speed of the flow of information in the beginning of the trading period, the faster we have w going to zero. As a consequence, we shall postulate that $\int_0^1 w(t)dt$ should be zero if all the information was made public at the beginning of the trading period. Such conjecture is very intuitive. In the limit, if all the information was made public at $t = 0$ the insider would have no relevant information to exploit; hence, her profit should be zero.

7.3 Conclusions of the Numerical Analysis

In this section we discuss and summarize our main findings from the previous two sections of this chapter. In those sections we were concerned with describing the algorithm we implemented and how all of the outputs for every functional form for $1 - \Sigma_M$ were. Now we are interested in describing the general trends for every parameter we were interested in studying across the different functional forms for $1 - \Sigma_M$.

Let us begin with the functional form of w. In figures 7.3, 7.9, 7.10 we have w being convex functions. The faster the signal X^M in the beginning of the trading period, the faster the function w decreases in that period. Figures 7.11, 7.19, and 7.22 show the same pattern. In those functions the speed at which w decreases is so slow that those functions became concave. Another interesting aspect of w is its initial value $w(0)$ for the different shapes of $1 - \Sigma_M$. The slower X^M in the beginning of the trading period, the higher the initial value $w(0)$. That is interesting because just by the form of the function w one would not be able to say if the ex-ante profit of the insider increases or decreases as $1 - \Sigma_M$ goes to zero faster. If on one hand the faster is the signal, the faster w goes to zero in the beginning of the trading period, on the other hand, the smaller is the initial value $w(0)$. Fortunately, we did estimate the value of $\int_0^1 w(t)dt$.

As we mentioned previously, we know that if $\Sigma_M(0) = 1$ the insider would not have any ex-ante profit as the information would also be public, hence she would not have any informational advantage. However, it was not clear considering only the analytical results of this thesis if the insider could profit greatly in the beginning of the trading period while the market maker is receiving a lot of information, but has not acquired a substantial amount of it. That case would be analogous to the dynamic case when Back, Pedersen (1998) and later Danilova (2010) in which the ex-ante value of information does not depend on how the information is revealed. Nevertheless, our numerical analysis shows that the slower the information is made public the greater the insider's expected profit. Indeed, since the later the information is released the smaller is the initial value $w(0)$ such that we are able to reach the lower bound given by equation (4.45) we can say that the maximum of the integral $\int_0^1 w(t)dt$ is one. Therefore, the maximum the insider's ex-ante expected profit is σ_V. Conversely, our numerics suggest that the minimum of such expectation is actually zero.

Perhaps the most unclear conclusion we can take is that depending on how fast Σ_M goes to one, the uncertainty about V could be greater than what it would be if there were no public signal. In the original paper Back (1992) considers a strategy for the insider such that $\eta|\mathscr{F}^M$ would variance of $1-t$. Indeed, the literature has shown that there are several optimal strategies such that v, the variance of $\mathbb{E}(\eta|\mathscr{F}^M)$, goes from $v(0) = 1$ to $v(1) = 0$. Nevertheless, $v(t) = 1-t$ is the one with a constant rate that represents an insider's strategy in which she does not increase or decrease the volume of stocks traded for no reason. As a consequence, one can use $1-t$ as the benchmark for the market maker's uncertainty about η if there was no public signal. Recall that $v + 1 + \Sigma_Z$ is independent of Σ_Z, hence there is no difference between a dynamic and a static signal when regarding this matter.

In two occasions, our numerics have given $v(t) + 1 + \Sigma_Z(t) > 1 - t$ for at least some t. In figures 7.16 and 7.20 it is possible that it happened for $1 - \Sigma_M = 1 - t^2$ and $1 - \Sigma_M = 1 - t^6$. The conjecture that this could be possible because the market maker has difficulties dealing with both the direct public signal and the information she learns by trading with someone with greater knowledge about the value of η is supported by figure 7.24. There we can see that when the information is delayed greatly we have $v(t) + 1 + \Sigma_Z(t)$ very close to $1 - t$ throughout most of the trading period. As a result, we can conjecture that depending on the shape of Σ_M, the uncertainty about V could be greater than what it would be if there were no public signal.

One of the reasons we have decided to implement numerical analysis for our model was due to the discussion about the limit of β_2. From equation (4.64), we know that the limit of β_2 must be either zero or one. In all the cases we considered, both packages `DeSolve` and `pracma` have given as output β_2 going to one as time goes to one, as we can see from figures 7.5, 7.7, 7.14, 7.17, 7.21, and 7.23.

This is particularly interesting because it is quite relevant regarding the price process. Recall that the price process given by (4.7) reads:

$$dS_t = \sigma_V w(t) dN_t^{(1)} + \beta_2(t)\sigma_M(t) dN_t^{(2)}.$$

Furthermore, from Proposition 4.2 we know that $\lim_{t\to 1} w(t) = 0$. Therefore, as time gets closer to one we have the price process start relying more on the public signal as β_2 gets closer to one and less on the information coming from the demand as w goes to zero. This behaviour is so powerful that in all the cases studied, we have $v+1-\Sigma_Z$ attaching to $1-\Sigma_M$ - as it is possible to see in figures 7.4, 7.8, 7.13, 7.16, 7.20, and 7.24 - suggesting that by the end of the trading period the market maker relies almost solely on the information coming from the public signal.

Since Kyle (1985) the reason the insider does not have infinite profits is because there is a feedback effect in the trading of the insider. That means that if the insider knows the price is below the value of the asset (recall that in Kyle (1985) the insider knows the value of the asset in advance) she will buy the cheap stock and increase the price by doing so. If the market were perfectly liquid the insider's profit would be infinite. In our model, the market is never perfectly liquid as, also by Proposition 4.2, $w(t) > 0$ for all $t \in (0, 1)$, but the effect of the insider trading declines over time. For that reason, the insider can trade more and more aggressively as the end of the trading period approaches.

Furthermore, by Corollary 5.3, as it is the case in the literature, the price is converging to the true price of the risky asset. Therefore, as time approaches one, the gap between the price in which the price is being traded and the final value is closing, making the marginal profit on each transaction relatively small. Hence, the insider must have plenty of volume to compensate for the reduced marginal profit.

Like Foucault et al. (2016) we have found that as the informational advantage of the insider is about to expire, she trades more aggressively, in a high-frequency fashion. However, in contrast with the authors, we show that such behaviour does not depend on the existence of short-term information being fed to the insider. It can also happen when the long-lived information is about to become void when competing with a public signal. Therefore, high-frequency trading does not mean high-frequency information.

7.4 Dynamic Comparison

In the very last section of this thesis, we present another optional topic. By Theorem 4.4, we know that $u = v + 1 - \Sigma_Z$ and w do not depend on Σ_Z. That is a very interesting result for us as it shows that the market maker's uncertainty about η does not depend on the signal X^I. In particular, the variance of $\eta|\mathscr{F}^M$ does not depend if the insider has a dynamic or a static signal. As a consequence, we developed all the numerics of this chapter without worrying about either the signal X^I nor the function v which is the variance of $Z_t|\mathscr{F}^M$, as given by Lemma 4.2.

However, one may be interested in analysing the behaviour of the function v. Once we have the estimates for u, we can find the function v by just doing $u - (1 - \Sigma_Z)$. The only thing that one must be cautious of while doing that is that given Σ_M, Σ_Z cannot be anything. Note that by the construction of the model we have as impute two different signals that determine Z. Indeed, we have from equation (2.36) that

$$\Sigma_Z(t) = c^2 + \int_0^t \sigma_Z^2(s)ds = c^2 + \int_0^t \left(\left(\frac{1 - \Sigma_Z(s)}{1 - \Sigma_I(s)} \right)^2 \sigma_I^2(s) + \left(\frac{1 - \Sigma_Z(s)}{1 - \Sigma_M(s)} \right)^2 \sigma_M^2(s) \right) ds.$$

As a consequence, we have from Corollary 2.1 that $\Sigma_Z(t) \leq \Sigma_M(t)$ $\forall t \in [0, 1]$. For this reason, while pairing two functions for Σ_Z and Σ_M one must make sure they are compatible. Indeed, Theorem 4.4 gives us a more restrictive condition, which is that $v + 1 - \Sigma_Z \leq 1 - \Sigma_M$.

Therefore, our task here was to find values of Σ_Z that were compatible with the above restriction. In figure 7.25 it is possible to see the several possible private signals,$1 - \Sigma_I$, for the public signal $1 - \Sigma_M = 1 - t^2$. Hence, what we did was to pick a private signal for each function $v + 1 - \Sigma_Z$ such that v would never be negative.

In figures 7.26 and 7.27, it is possible to see the value of v for the public signal $1 - \Sigma_M = 1 - t^2$ for $1 - \Sigma_Z = 0.7(1 - t)$ and $1 - \Sigma_Z = 0.7(1 - \sqrt{t})$ respectively. One interesting fact about those images is that, as it is the case in figure 7.27, we may have the variance of $Z_t|cF^M$ increasing over time as the variance of Z_t may increase more than it is revealed for the market maker. The rationale behind the fact that v is exactly the same as the restriction that $s(t)$,

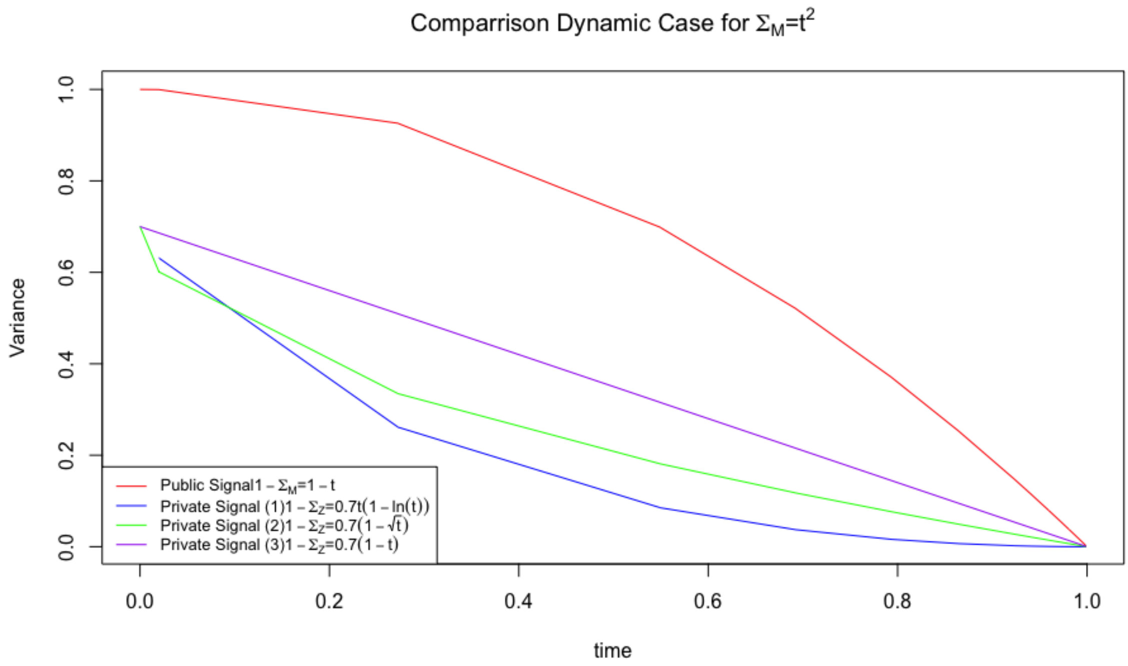

Figure 7.25: $1 - \Sigma_M = 1 - t^2$

the variance of η with respect to the insider's filtration when she receives a dynamic signal, in Danilova (2010) must be smaller than $1 - t$ as the uncertainty about the final value of the asset must be always greater for the market maker than it is for the insider.

While we performed several combinations of Σ_Z and Σ_M, we believe that the most interesting one is the one with $1 - \Sigma_M = 1 - t^2$. In figure 7.25 it is possible to check that the combinations did not lead to a clash with our analytical discoveries, so we assume there are private signals that would lead to those pair of signals.

Analysing v for $\Sigma_Z = 0.7(1-t)$ and $\Sigma_M - t^2$ in figure 7.26 we see what one would expect: as times goes by the uncertainty about Z reduces. However, one must be aware that variance of Z increases over time. Indeed, v for $\Sigma_M = t^2$ and $\Sigma_Z = 0.7t(1 - \ln(t))$ shows an increase in the variance before going to zero.

Indeed, if we keep $\Sigma_Z = 0.7t(1-\ln(t))$, but change Σ_M to t^6 we get a considerable increase before going to zero.

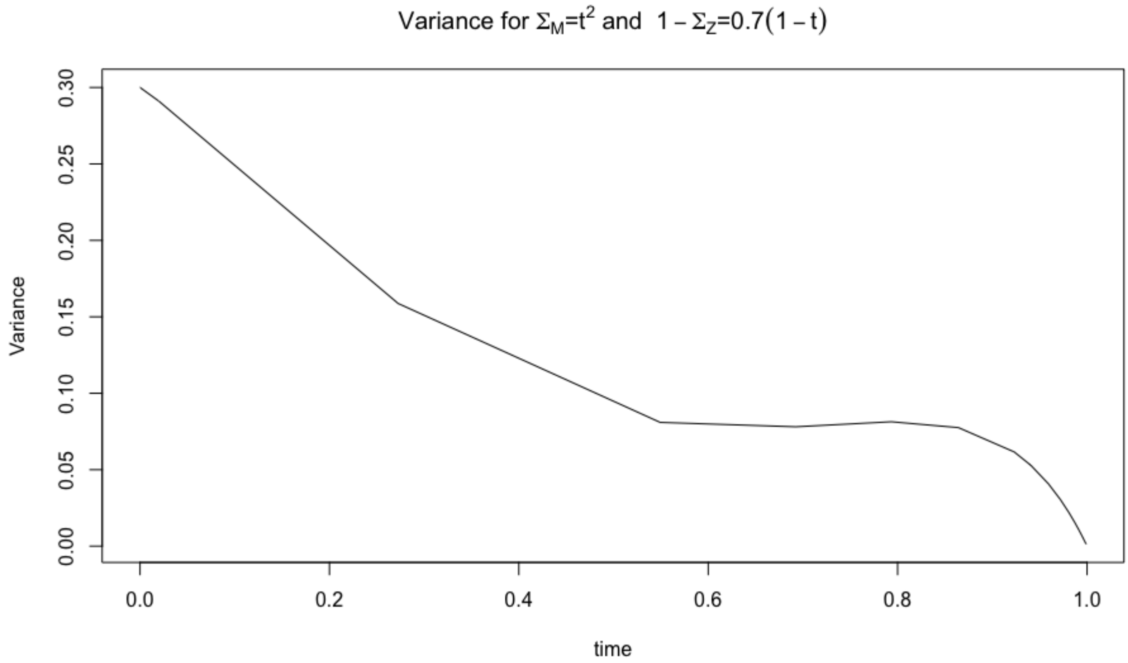

Figure 7.26: v for $\Sigma_M = t^2$ and $\Sigma_Z = 0.7(1-t)$

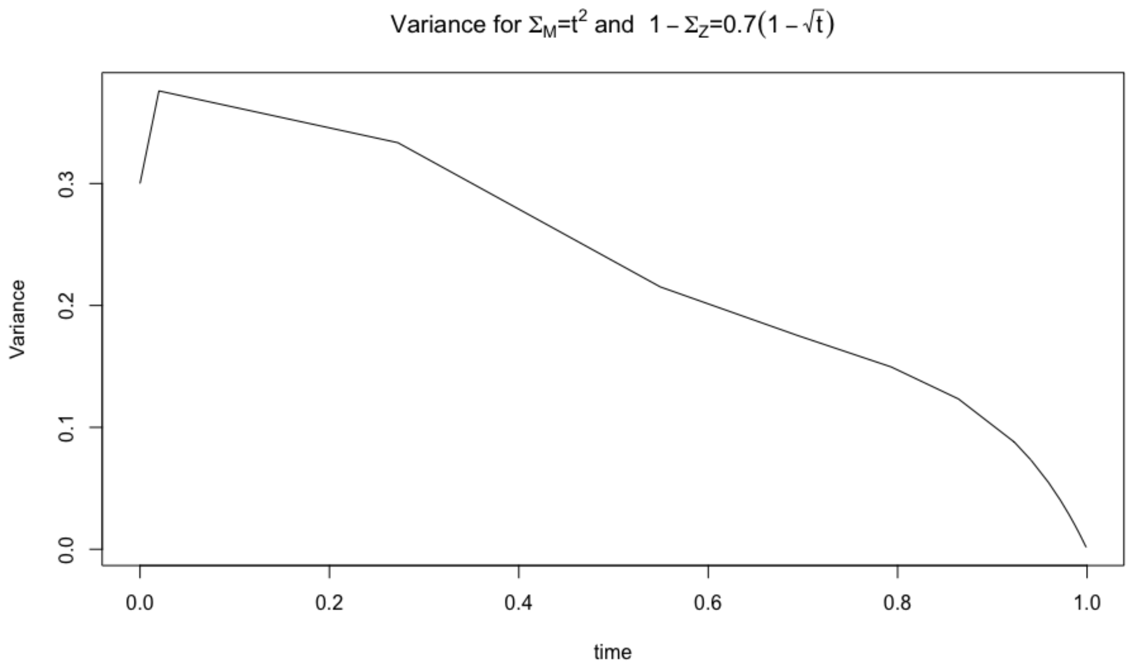

Figure 7.27: v for $\Sigma_M = t^2$ and $\Sigma_Z = 0.7t(1-\ln(t))$

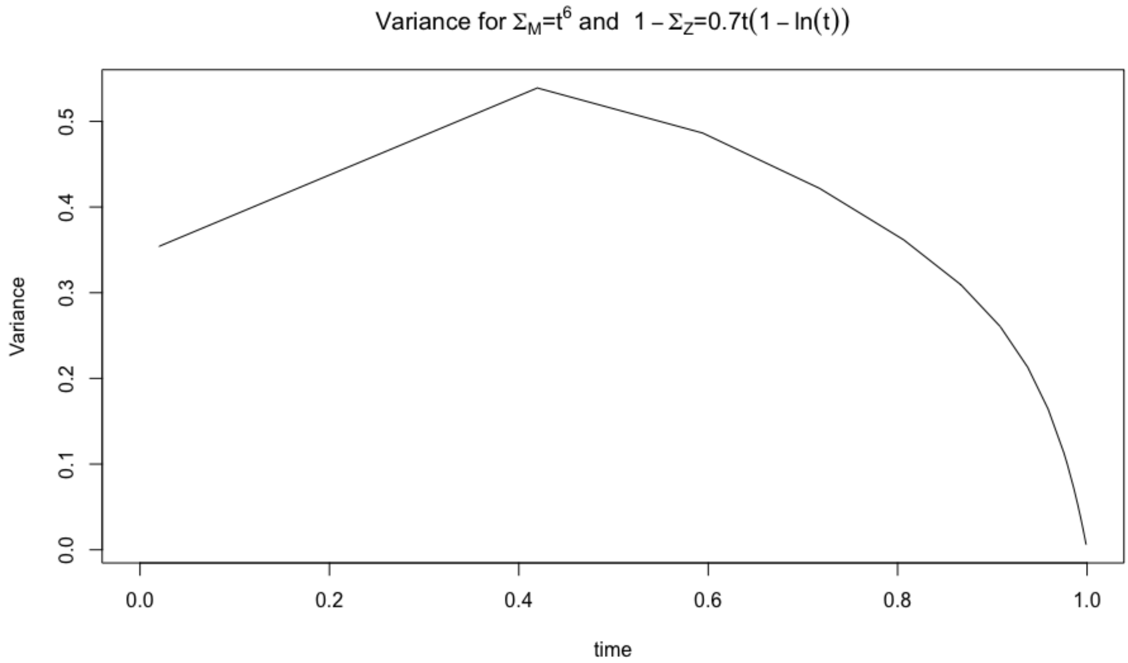

Figure 7.28: v for $\Sigma_M = t^6$ and $\Sigma_Z = 0.7t(1 - \ln(t))$

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