Singular and impulse stochastic control problems motivated by optimal harvesting and portfolio optimisation

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorisation does not, to the best of my belief, infringe the rights of any third party.

Statement of co-authored work

I confirm that Chapter 2 was jointly co-authored with Professor Mihail Zervos, based on Liu and Zervos [64].

I confirm that Chapter 3 was jointly co-authored with Professor Mihail Zervos and Dr. Gechun Liang, based on Liang, Liu and Zervos [58].

I confirm that Chapter 4 was jointly co-authored with Professor Mihail Zervos and Dr. Christoph Czichowsky.

Abstract

In this thesis, we investigate stochastic optimal control problems motivated by (a) the optimal sustainable exploitation of an ecosystem, and (b) trading in a financial market with proportional transaction costs.

In the context of optimal harvesting, we study two models. The first one is an impulse control problem with a discounted performance criterion. In this problem, the objective is to maximise a discounted performance criterion that rewards the effect of control action but involves a fixed cost at each time of a control intervention. The second problem is a singular control one, with an expected discounted criterion, an expected ergodic criterion and a pathwise ergodic criterion. We derive the explicit solutions to these stochastic control problems under general assumptions. We solve these problems by first constructing suitable solutions to their associated HJB equations. It turns out that the solution to the impulse control problem can take four qualitatively different forms, several of which have not been observed in the literature. We also show that the boundary classification of 0 may play a critical role in the solution of the problem. In the singular ergodic control problems, we develop a suitable new variational argument. Furthermore, we establish the convergence of the solution of the discounted control problem to the one of the ergodic control problems as the discounting rate function tends to zero in an Abelian sense.

In the portfolio optimisation problem, we determine the growth optimal portfolio under proportional transaction costs for an investor trading a risk-free asset and a risky asset with stochastic investment opportunities given by a linear diffusion. Despite extensive research, our results are the first that construct optimal trading strategies in continuous time beyond the restrictive setting of constant parameters. This allows us to investigate the tradeoff between active trading due to the random parameters and the proportional transaction costs. We solve this problem by explicitly constructing a shadow price process and provide the asymptotic expansions of the non-trade region, the stock-cash ratio and the proportion of wealth invested in the risky asset.

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Contents

Chapter 1

Introduction

In the first and second parts of the thesis, we consider two types of stochastic control problems motivated by the optimal sustainable exploitation of an ecosystem, such as a natural fishery. In particular, we consider a stochastic system whose uncontrolled state dynamics are modelled by a non-explosive positive linear diffusion. The control that can be applied to the first type of problems takes the form of one-sided impulsive action. In the second part of the thesis, we study the singular stochastic control problems of the monotone follower type.

We consider a stochastic dynamical system whose controlled state process satisfies the SDE

$$\mathrm{d}X_t^\zeta = b(X_t^\zeta)\,\mathrm{d}t - \mathrm{d}\zeta_t + \sigma(X_t^\zeta)\,\mathrm{d}W_t, \quad X_{0-}^\zeta = x > 0, \tag{1.1}$$

where W is a standard one-dimensional Brownian motion and ζ is a controlled càdlàg increasing process. Furthermore, ζ is piece-wise constant in the impulse control problem. The objective of the impulse optimisation problem is to maximise over all admissible processes ζ the performance criterion

$$J_x(\zeta) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\geq 0} e^{-\Lambda_t^\zeta}\left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}\right)\right], \tag{1.2}$$

where $\Delta\zeta_t = \zeta_t - \zeta_{t-}$, with the convention that $\zeta_{0-} = 0$, and

$$\Lambda_t^\zeta = \int_0^t r(X_u^\zeta)\,\mathrm{d}u. \tag{1.3}$$

Throughout the thesis, we write $\mathbb{E}_x$ to denote expectation so that we account for the dependence of X^ζ on its initial value x. In the singular control problems, we associate, with each controlled process ζ, the expected discounted performance index

$$I_x(\zeta) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \int_0^\infty e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t\right], \tag{1.4}$$

the expected long-term average performance index

$$J^{\mathrm{e}}_x(\zeta) = \limsup_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\biggl[\int_0^T h(X^\zeta_t)\, \mathrm{d}t + \int_0^T k(X^\zeta_t) \circ \mathrm{d}\zeta_t\biggr], \tag{1.5}$$

as well as the pathwise long-term average performance criterion

$$J^{\mathrm{p}}_x(\zeta) = \limsup_{T\uparrow\infty} \frac{1}{T}\biggl(\int_0^T h(X^\zeta_t)\, \mathrm{d}t + \int_0^T k(X^\zeta_t) \circ \mathrm{d}\zeta_t\biggr), \tag{1.6}$$

where

$$\int_0^T k(X^\zeta_t) \circ \mathrm{d}\zeta_t = \int_0^T k(X^\zeta_t)\, \mathrm{d}\zeta^{\mathrm{c}}_t + \sum_{0\le t\le T} \int_0^{\Delta\zeta_t} k(X^\zeta_{t-} - u)\, \mathrm{d}u. \tag{1.7}$$

In the last of these definitions, ζ^{c} is the continuous part of the càdlàg increasing process ζ. The objective of the resulting singular stochastic control problems is to maximise each of the objective criteria (1.4), (1.5) and (1.6) over all admissible controlled processes ζ.

In the context of optimal management of a natural resource, the state process X^ζ models the population density of a harvested species, while ζ_t is the cumulative amount of the species that has been harvested by time t. The constant $c > 0$ models a fixed cost associated with each harvesting cycle, while the function k models the marginal profit arising from each harvest. On the other hand, the function h models the utility arising from having a population level X_t of the harvested species at time t, which could reflect the role that the species plays in the stability of the overall ecosystem. Alternatively, the function h can be used to model running costs.

In the third part of the thesis, we study the optimal portfolio allocation under transaction costs. We consider a financial market consisting of a bond with a constant price equal to 1 and a stock, whose price is modelled by the strong solution to the SDE

$$\frac{\mathrm{d}S_t}{S_t} = \mu(S_t)\, \mathrm{d}t + \sigma(S_t)\, \mathrm{d}W_t, \quad S_0 = s_0 > 0, \tag{1.8}$$

under proportional transaction costs $\lambda \in \,]0,1[$, namely, the investor pays the ask price S when buying the stock, but receives the bid price $(1-\lambda)S$ when selling it. We denoted by ϑ^0 (resp., ϑ) the number of shares held in the bond (resp., stock) and

$$V_t(\vartheta^0,\vartheta) = \vartheta^0_t + (\vartheta_t \wedge 0)S_t + (1-\lambda)(\vartheta_t \vee 0)S_t = \vartheta^0_t + \bigl(1 - \lambda \mathbf{1}_{\{\vartheta_t>0\}}\bigr)\vartheta_t S_t \tag{1.9}$$

is the liquidation value. With each trading strategy (ϑ^0,ϑ), we associate the expected growth rate

$$J(\vartheta^0,\vartheta) = \limsup_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\bigl[\ln\bigl(V_T(\vartheta^0,\vartheta)\bigr)\bigr]. \tag{1.10}$$

The objective of the optimisation problem is to maximise (1.10) over all admissible self-financing strategies (see Definition 4.1 in Section 2.1).

1.1 Stochastic impulse control problems

Stochastic impulse control problems arise in various fields. In the context of mathematical finance, economics and operations research, notable contributions include Harrison, Sellke and Tayor [36], Harrison and Taksar [42], Mundaca and Øksendal [71], Korn [54, 55], Bielecki and Pliska [12], Cadenillas [16], Bar-Ilan, Sulem and Zanello [15], Bar-Ilan, Perry and Stadje [10], Ohnishi and Tsujimura [72], Cadenillas, Sarkar and Zapatero [22], LyVath, Mnif and Pham [62], and several references therein. Also, impulse control models motivated by the optimal management of a natural resource have been studied by Alvarez [1, 2], Alvarez and Koskela [4] and Alvarez and Lempa [5], and several references therein. In view of the wide range of applications, the general mathematical theory of stochastic impulse control is well-developed: apart from the contributions mentioned above, see also Richard [77], Stettner [81], Lepeltier and Marchal [59], Perthame [75], Egami [30], Davis, Guo and Wu [25], Djehiche, Hamadène and Hdhiri [26], Christensen [17], Helmes, Stockbridge and Zhu [43, 44], Menaldi and Robin [70], Palczewski and Stettner [74], Christensen and Strauch [23], as well as the books by Bensoussan and Lions [11], Davis [24], Øksendal and Sulem [73], Pham [76], and several references therein.

Relative to related references, such as the ones mentioned in the previous section, we generalise by considering (a) state-space discounting, (b) a state-dependent, rather than proportional, payoff associated with each harvest size, and (c) a running payoff such as the one modelled by the function h. On the other hand, the assumptions that we make are of a rather similar nature.

In light of standard impulse control theory, a "β-γ" strategy should be a prime candidate for an optimal one in the problem that we study here. Such a strategy is characterised by two points $\gamma < \beta$ in $]0, \infty[$, which are both chosen by the controller, and can be described informally as follows. If the state process takes any value $x \geq \beta$, then it is optimal for the controller to push it in an impulsive way down to level γ. On the other hand, the controller should wait and take no action at all for as long as the state process takes values in the interval $]0, \beta[$.

We show that a β-γ strategy is indeed optimal, provided that a critical parameter $\underline{x}$ is finite and the fixed cost c is sufficiently small (see Case I of Theorem 2.4.5 in Section 2.4). Otherwise, we show that only ε-optimal strategies may exist (see Case II or Case IV of Theorem 2.4.5) or that never making an intervention may be optimal (see Case III of Theorem 2.4.5). The absence of an optimal strategy in Case IV of Theorem 2.4.5 is due to the relatively rapid growth of the function k at infinity. It can therefore be eliminated if we make a suitable additional growth assumption. On the other hand, the absence of an optimal strategy in Case II of Theorem 2.4.5 is due to the nature of the problem that we solve.

The family of admissible controlled strategies that we consider do not allow for the state process to hit the boundary point 0 and be absorbed by it, which would amount to "switching off" the system. If we enlarged the set of admissible controls to allow for such a possibility and 0 were a natural boundary point, then we would face only the following difference: a β-0 strategy would be optimal in Case II of Theorem 2.4.5 and we would not need to consider ε-optimal strategies. On the other hand, the situation would be radically different if 0 were an entrance boundary point: in this case, β-0 strategies would become an indispensable part of the optimal tactics. We discuss these observations more precisely in Remark 2.1 at the end of Section 2.4. To the best of our knowledge, this is the first stochastic control problem in which the boundary classification of the problem's state space has such a fundamental influence on the problem's solution. We do not investigate this issue any further because this would require substantial extra analysis that would go beyond the scope of the present article.

The evolution of an impulse control problem's state process is quite intuitive, provided that the corresponding uncontrolled dynamics are well-posed. For this reason, several references simply assume the existence of such processes. In the context of SDEs in $\mathbb{R}^d$, the state process of an impulse control problem can be derived by pasting together suitable strong solutions to the underlying uncontrolled SDE with random initial conditions (e.g., see Bensoussan and Lions [11, Section 6.1.1]). In the context of general Markov processes, the classical construction of an impulse control strategy is substantially more technical and may involve countable products of canonical spaces (e.g., see Stettner [81] and Lepeltier and Marchal [59]). If the uncontrolled state space process is a general Markov process with continuous sample paths, then comprehensive constructions of impulse control models have been derived by Helmes, Stockbridge and Zhu [44].

Impulse control problems with SDEs in $\mathbb{R}^d$ can be formulated as in (1.1)–(1.3). In itself such a formulation is straightforward. Indeed, an SDE in $\mathbb{R}^d$ such as (1.1) has a unique strong solution under suitable Lipschitz assumptions on b and σ for a wide class of controlled processes ζ (e.g., see Krylov [56, Theorem 2.5.7]). On the other hand, a rigorous construction of an optimally controlled process ζ, such as a β-γ strategy, is rather non-trivial. In the context of this chapter, we construct a unique strong solution to the SDE (1.1) when the controlled process ζ is a β-γ strategy (see Theorem 2.3.1 in Section 2.3). Despite the central role that such strategies play in stochastic impulse control, we are not aware of any such rigorous SDE result. Furthermore, this construction allows for a probabilistic derivation of the optimal expected discounted running reward as well as the optimal expected discounted reward from control exdenditure functionals (see (2.48) and (2.49) in Theorem 2.3.1). The construction that we make can most easily be adapted to derive the existence of strong solutions to optimally controlled SDEs that arise in other stochastic impulse control problems, even in dimensions higher than one.

1.2 Singular stochastic control problems of the monotone follower type

Motivated by applications to the optimal harvesting of stochastically fluctuating populations, similar singular stochastic control problems with $h = 0$, constant k and with a discounted performance criterion with constant r have been studied by Alvarez [6, 7], Alvarez and

Shepp [9], and Lungu and Øksendal [60]. Extensions of these earlier works have been studied by Framstad [32], who considers a state process X with jumps, Song, Stockbridge and Zhu [80], who consider a state process X with regime switching, Morimoto [65], who considers the finite time horizon case, Alvarez, Lungu and Øksendal [8] and Lungu and Øksendal [61], who consider multidimensional state processes X, Hening, Tran, Phan and Yin [41], who consider multidimensional state processes X as well as allow for the modelling of both seeding and harvesting, and Gaïgi, Ly Vath and Scotti [33], who consider constraints of no-take areas. On the other hand, control problems with an expected ergodic performance criterion, similar to the one that we study here with $h = 0$ and constant k, have been solved by Hening, Nguyen, Ungureanu and Wong [45], Alvarez and Hening [3], as well as Cohen, Hening and Sun [18], who consider a performance criterion with model ambiguity. Several other closely related contributions can be found in the literature of all these papers.

We solve the control problems that we consider by deriving explicit solutions to their corresponding HJB equations. In generalising the special cases arising when $h = 0$ and k is constant, our main contributions include (a) the determination of sufficiently general assumptions on the functions h and k that give rise to threshold optimal strategies without making extra assumptions on the data b and σ of the underlying diffusion, and (b) the derivation of explicit solutions to the problems' HJB equations that are way more complicated than the ones associated with the special case arising when $h = 0$ and k is constant (e.g., we are faced with integral equations, such as (3.24), instead of algebraic equations, such as the one in Remark 3.3 from Alvarez and Hening [3]).

We derive the solution to the discounted singular stochastic control problem in Section 3.3. On the other hand, we solve the ergodic singular stochastic control problems in Sections 3.4 and 3.5. The analysis of these problems, which are in the so-called monotone follower singular stochastic control setting, has been influenced by Karatzas [49], Menaldi, Robin and Taksar [69], Weerasinghe [85], and Jack and Zervos [48], who consider different formulations. In the solution to the ergodic control problems that we solve, a notable difficulty arises from the fact that the solution $(w, \lambda^\star)$ to their corresponding HJB equation may involve functions w that are unbounded from below (see Remark 3.4), which makes the establishment of a suitable verification theorem intractable. We overcome this complication by means of a variational argument involving suitable pairs (w_λ, λ) with bounded from below functions w_λ that converge to $(w, \lambda^\star)$ as $\lambda \downarrow \lambda^\star$. The introduction of this technique is a further contribution of this chapter.

In Section 3.6, we establish the convergence of the solution to the discounted control problem to the one of the ergodic control problems as the discounting rate function r tends to 0 in an Abelian sense. In particular, we will prove that, if r depends on a parameter $\iota > 0$ and tends to zero as $\iota \downarrow 0$ in the sense of Assumption 3.7, then

$$\lim_{\iota \downarrow 0} \beta^\star(\iota) = \beta^\star, \quad \lim_{\iota \downarrow 0} r(y;\iota) w(x;\iota) = \lambda^\star \quad \text{and} \quad \lim_{\iota \downarrow 0} w'(x;\iota) = w'(x) \quad \text{for all } x, y > 0, \tag{1.11}$$

where $\beta^\star(\iota)$ (resp., $\beta^\star$) is the threshold point characterising the optimal strategy of the discounted problem (resp., the ergodic problems) and $w(\cdot;\iota)$ (resp., $(w, \lambda^\star)$) is the solution to the HJB equation of the discounted problem (resp., ergodic problems). In a singular stochastic control setting, Abelian limits, such as the first two ones in (1.11), have been obtained for constant r by Karatzas [49], Weerasinghe [86], Hynd [37], Alvarez and Hening [3], and Kunwai, Xi, Yin and Zhu [53] using different techniques. To the best of our knowledge, no results (a) with non-constant discounting rate $r(\iota)$, or (b) such as the third limit in (1.11), exist in the singular stochastic control literature, with the exception of Karatzas [49, Proposition 6], who establishes a limit such as the third one in (1.11) for a model with a standard Brownian motion and constant r.

1.3 Portfolio maximization under proportional transaction costs

In the frictionless market when $\lambda = 0$, Merton's seminal works [66, 67] have shown that it is optimal to invest fraction $\Theta(S)$ of wealth in the stock, i.e.

$$\frac{\vartheta S}{V(\vartheta^0, \vartheta)} = \Theta(S), \tag{1.12}$$

where

$$\Theta(s) = \frac{\mu(s)}{\sigma^2(s)}. \tag{1.13}$$

Magill and Constantinides [68] introduced the proportional transaction costs into Merton's problem. Maximization of the expected growth rate under proportional transaction costs in the Black-Scholes model has been studied by Taksar, Klass and Assaf [83]. They use the classical way of solving the Hamilton-Jacobi-Bellman (HJB) equation, and find that it is optimal to do minimal action to keep the fraction of wealth in the stock in an interval $[A, B]$

$$A \leq \frac{\vartheta S}{V(\vartheta^0, \vartheta)} \leq B,$$

for some $A < \Theta < B$. Other works that use the HJB equation in the Black-Scholes model include Davis and Norman [28], who consider the infinite time consumption problem, also extensions have been studied by Shreve and Soner [82], Janeček and Shreve [47] and Hobson, Tse and Zhu [40], Dumas and Luciano [27], who consider the power utility of the terminal wealth on long-term asymptotics, Liu and Loewenstein [57], who consider the finite time horizon case approximated by exponentially distributed horizon, Dai and Yi [29], who consider the finite time horizon problem.

An alternative approach to tackle this problem is to construct a shadow price process $\hat{S}$ (see Definition 4.3 in Section 4.3) in a fictitious frictionless market. The idea of the shadow price process goes back to Cvitanivic and Karatzas [19] and Jouini and Kallal [46]. The investor trade a bond with a constant price equal to 1 and a stock modelled by a price process $\hat{S}$ that takes value in the bid-ask spread $\big[(1-\lambda)S, S\big]$. The objective of the control problem is to maximise over all admissible self-financing strategy in this fictitious frictionless market the expected growth rate

$$\hat{J}(\hat{\vartheta}^0, \hat{\vartheta}) = \limsup_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\Big[\ln\big(\hat{V}_T(\hat{\vartheta}^0, \hat{\vartheta})\big)\Big], \tag{1.14}$$

where $\hat{\vartheta}^0$ (resp., $\hat{\vartheta}$) denotes the number of shares held in the bond (resp., stock) and

$$\hat{V}_T(\hat{\vartheta}^0, \hat{\vartheta}) = \hat{\vartheta}^0_T + \hat{\vartheta}_T\hat{S}_T \tag{1.15}$$

is the total wealth in the fictitious market. If $\hat{S}$ is a shadow price process in the bid-ask spread $\big[(1-\lambda)S, S\big]$, then the investor will have an identical optimal trading strategy between trading a price process $\hat{S}$ in the frictionless shadow market and trading the price process S in the original market, and the optimal growth rate will be the same. The optimal trading strategy is such that the investor only buys (resp., sells) when $\hat{S} = S$ (resp., $\hat{S} = (1-\lambda)S$).

We construct explicitly a shadow price process $\hat{S}$ of the form $g(S, A, B, \lambda)$. The process A (resp., B) is the buying (resp., selling) boundary, i.e. the investor only buys (resp., sells) when $S = A$ (resp., $S = B$). The shadow price in the Black-Scholes model has been studied by Gerhold, Muhle-Karbe and Schachermayer [34]. In such a case $B = cA$ for some $c > 1$, if $0 < \Theta < 1$, or $0 < c < 1$, if $\Theta > 1$, and they scale the boundaries and the price process S and construct a shadow price process in the domain $[1, c]$, if $0 < \Theta < 1$, or $[c, 1]$, if $\Theta > 1$. Such a process is a doubly reflected geometric Brownian motion. Other works that use the shadow price include Kallsen and Muhle-Karbe [50], who construct the shadow price process for the infinite time consumption problem with logarithmic utility, and extensions to the power utility have been studied by Choi, Sîrbu and Žitković [21], and Herczegh and Prokaj [38], Gerhold, Guasoni,Muhle-Karbe and Schachermayer [35], who consider the power utility of the terminal wealth, and Czichowsky, Peyre, Schachermayer and Yang [20],who consider a model based on fractional Brownian motion, etc.. Recently, an extension of earlier works has been studied by Herdegen, Hobson and Tse [39], who consider the Epstein-Zin stochastic differential utility.

The main contribution of this chapter is that it is the first work solve explicitly the Merton's problem under transaction beyond the Black-Scholes model. The main difficulty for the problem is that the optimal strategy is no longer static, since Θ is not constant. We consider general assumptions on the problem data, which includes a plenty of well-known processes, for instance exponential Ornstein-Uhlenbeck process, mean-reverting square-root process, Verhulst-Pearl logistic process, etc..

We will show in Section 4.3.2 that under our assumptions, A and B are continuous and $A \wedge B \le S \le A \vee B$ on $\{A \neq \underline{\rho}\} \cup \{B \neq \bar{\eta}\}$, for some turning points $\underline{\rho}$ and $\bar{\eta}$. When $B = \bar{\eta}$, A could have jumps, while $A = \underline{\rho}$, B could have jumps. If $A_{t+} \wedge A_{t-} < S_t < A_{t+} \vee < A_{t-}$, or $B_{t+} \wedge B_{t-} < S_t < B_{t+} \vee B_{t-}$, then the investor should take no action. When $S = A$ (resp., $S = B$), the investor should do minimum action to keep the stock-cash ratio

$$\mathcal{Q}_t := \frac{\vartheta_t}{\vartheta^0_t}\mathbf{1}_{\{\vartheta^0_t \neq 0\}} = \underline{Q}(A_t) = \overline{Q}(B_t) \tag{1.16}$$

for some continuous function $\underline{Q}$ (resp., $\overline{Q}$). Furthermore, $\underline{\rho}$ (resp., $\bar{\eta}$) is a turning point of $\underline{Q}$ (resp., $\overline{Q}$). In view of the Merton's proportion (1.12), we can see that in the frictionless market when $\lambda = 0$, the stock-cash ratio is

$$\mathcal{Q}_t = Q(S_t) := \frac{\Theta(S_t)}{(1 - \Theta(S_t))S_t}. \tag{1.17}$$

As a matter of fact, the turning points $\underline{\rho}$ and $\bar{\eta}$ arise from the turning point of Q.

Chapter 2

The solution to an impulse control problem motivated by optimal harvesting

The chapter is organised as follows. Section 2.1 presents the precise formulation of the control problem that we solve, including all of the assumptions that we make. In Section 2.2, we derive several results associated with a linear ODE that we need for the solution to the stochastic control problem we consider. In Section 2.3, we prove that the SDE (1.1) has a unique strong solution when the controlled process ζ is a β-γ strategy and we derive analytic expressions for certain associated functionals using probabilistic techniques. We derive the complete solution to the control problem that we consider in Section 2.4. Finally, we present several examples illustrating the assumptions that we make and the results that we establish in Section 2.5.

2.1 Formulation of the stochastic control problem

Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions and carrying a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a dynamical system, the uncontrolled stochastic dynamics of which are modelled by the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x > 0, \tag{2.1}$$

and we make the following assumption.

Assumption 2.1 The functions $b, \sigma : [0, \infty[\to \mathbb{R}$ are locally Lipschitz continuous and $\sigma(x) > 0$ for all $x > 0$.

This assumption implies that the scale function p and the speed measure m of the diffusion associated with the SDE (2.1), which are given by

$$p(1) = 0 \quad \text{and} \quad p'(x) = \exp\left(-2\int_1^x \frac{b(s)}{\sigma^2(s)}\,\mathrm{d}s\right) \tag{2.2}$$

$$\text{and} \quad m(\mathrm{d}x) = \frac{2}{\sigma^2(x)p'(x)}\,\mathrm{d}x, \tag{2.3}$$

are well-defined. Additionally, we make the following assumption on the boundary classification of the diffusion associated with (2.1).

Assumption 2.2 The boundary point 0 is inaccessible while the boundary point ∞ is natural.

The state space of the linear diffusion associated with the SDE (2.1) is the interval $\mathcal{I} = \,]0, \infty[$. Recall that the boundary point $p \in \{0, \infty\}$ of $\mathcal{I}$ is called *inaccessible* if $\mathbb{P}_x(T_p < \infty) = 0$ for all $x \in \mathcal{I}$ and *accessible* otherwise. Furthermore, if the boundary p is inaccessible, then it is *natural* if

$$\lim_{x\in\mathcal{I},\, x\to p} \mathbb{P}_x(T_y < t) = 0 \quad \text{for all } y \in \mathcal{I} \text{ and } t > 0$$

and *entrance* otherwise, namely, if

$$\lim_{x\in\mathcal{I},\, x\to p} \mathbb{P}_x(T_y < t) > 0 \quad \text{for some } y \in \mathcal{I} \text{ and } t > 0$$

(e.g., see Revuz and Yor [78, Definition VII.3.9]). In these expressions, T_y is the first hitting time of the set $\{y\}$, which is defined by

$$T_y = \inf\{t \geq 0 \mid \ X_t = y\}, \quad \text{for } y > 0. \tag{2.4}$$

In Borodin and Salminen [13, II.1.6], an inaccessible boundary point is called *not-exit*, while a natural (resp., entrance) boundary point is called *natural* (resp., *entrance-not-exit*). Integral conditions for the classification of a boundary point $p \in \{0, \infty\}$ of $\mathcal{I}$ in terms of the scale function p and the speed measure m can be found in this reference.

We next consider the stochastic control problem defined by (1.1)–(1.3).

Definition 2.1 The family of all admissible controlled strategies is the set of all $(\mathcal{F}_t)$-adapted càdlàg processes ζ with increasing and piece-wise constant sample paths such that the SDE (1.1) has a unique non-explosive strong solution and

$$\mathbb{E}_x\left[\sum_{t\geq 0} e^{-\Lambda_t^\zeta}\mathbf{1}_{\{\Delta\zeta_t>0\}}\right] < \infty. \tag{2.5}$$

Assumption 2.3 The discounting rate function r is bounded and continuous. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$ for all $x \geq 0$.

To complete the set of our assumptions, we consider the operator $\mathscr{L}$ acting on C^1 functions with absolutely continuous first-order derivatives that is defined by

$$\mathscr{L}w(x) = \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) - r(x)w(x). \tag{2.6}$$

In the presence of Assumptions 2.1, 2.2 and 2.3, the second-order linear ODE $\mathscr{L}w(x) = 0$ has two fundamental C^2 solutions φ and ψ such that

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x > 0, \tag{2.7}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x > 0 \tag{2.8}$$

$$\text{and} \quad \lim_{x \downarrow 0} \varphi(x) = \lim_{x \uparrow \infty} \psi(x) = \infty. \tag{2.9}$$

If 0 is a natural boundary point, then

$$\lim_{x \downarrow 0} \frac{\varphi'(x)}{p'(x)} = -\infty, \quad \lim_{x \downarrow 0} \psi(x) = 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{\psi'(x)}{p'(x)} = 0, \tag{2.10}$$

while, if 0 is an entrance boundary point, then

$$\lim_{x \downarrow 0} \frac{\varphi'(x)}{p'(x)} > -\infty, \quad \lim_{x \downarrow 0} \psi(x) > 0 \quad \text{and} \quad \lim_{x \downarrow 0} \frac{\psi'(x)}{p'(x)} = 0. \tag{2.11}$$

Symmetric results hold for the boundary point ∞ (e.g., see Borodin and Salminen [13, II.10]).

The functions φ and ψ admit the probabilistic representations

$$\varphi(y) = \varphi(x)\, \mathbb{E}_y\big[e^{-\Lambda_{T_x}}\big] \quad \text{and} \quad \psi(x) = \psi(y)\, \mathbb{E}_x\big[e^{-\Lambda_{T_y}}\big] \quad \text{for all } x < y, \tag{2.12}$$

where Λ is defined by (1.3) with X in place of X^ζ and T_y is defined by (2.4).

Furthermore, φ and ψ are such that

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp'(x) \quad \text{for all } x > 0, \tag{2.13}$$

where $C = \varphi(1)\psi'(1) - \varphi'(1)\psi(1)$ and p is the scale function defined by (2.2). To simplify the notation, we also define

$$\Phi(x) = \frac{2\varphi(x)}{C\sigma^2(x)p'(x)} = \frac{1}{C}\varphi(x)\frac{m(\mathrm{d}x)}{\mathrm{d}x} \quad \text{and} \quad \Psi(x) = \frac{2\psi(x)}{C\sigma^2(x)p'(x)} = \frac{1}{C}\psi(x)\frac{m(\mathrm{d}x)}{\mathrm{d}x}. \tag{2.14}$$

Beyond involving standard integrability and growth assumptions, the conditions in the following assumption may appear involved. However, they are standard in the relevant literature and are satisfied by a wide range of problem data choices (see Examples 2.1-2.4 in Section 2.5).

Assumption 2.4 The following conditions hold true:
(i) The function h is continuous as well as bounded from below. Also, the limit $\lim_{x\downarrow 0} h(x)/r(x)$ exists in $\mathbb{R}$ and

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|h(X_t)\big|\,\mathrm{d}t\right] < \infty.$$

(ii) The function k is absolutely continuous,

$$\int_0^1 \big|k(s)\big|\,\mathrm{d}s < \infty \quad \text{and the function } x \mapsto \int_0^x k(s)\,\mathrm{d}s \text{ is bounded from below.} \tag{2.15}$$

Furthermore,

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|\mathfrak{K}(X_t)\big|\,\mathrm{d}t\right] < \infty \quad \text{and} \quad \limsup_{x\uparrow\infty} \frac{1}{\psi(x)}\int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+ \tag{2.16}$$

hold true, where

$$\mathfrak{K}(x) = \mathscr{L}\left(\int_0^\cdot k(s)\,\mathrm{d}s\right)(x), \quad \text{for } x > 0. \tag{2.17}$$

(iii) If we define

$$\Theta(x) = h(x) + \mathfrak{K}(x), \tag{2.18}$$

then Θ is continuous and there exists a constant $\xi \in \,]0,\infty[$ such that the restriction of Θ/r in $]0,\xi[$ (resp., in $]\xi,\infty[$) is strictly increasing (resp., strictly decreasing).

2.2 Results associated with a linear ODE

Unless stated otherwise, the results in this section hold true if the coefficients of (2.1) satisfy the usual Engelbert and Schmidt conditions, rather than the stronger Assumption 2.1, and the boundary points 0, ∞ are inaccessible. We start by recalling some standard results that we will need and can be found in, e.g., Lamberton and Zervos [63, Section 4]. Consider a Borel measurable function $F : \,]0,\infty[\to \mathbb{R}$ such that

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\big|F(X_t)\big|\,\mathrm{d}t\right] < \infty \quad \text{for all } x > 0, \tag{2.19}$$

where Λ is defined by (1.3) for $X^\zeta = X$. This integrability condition is equivalent to

$$\int_0^x \big|F(s)\big|\Psi(s)\,\mathrm{d}s + \int_x^\infty \big|F(s)\big|\Phi(s)\,\mathrm{d}s < \infty \quad \text{for all } x > 0, \tag{2.20}$$

where Φ and Ψ are defined by (2.14). Given such a function F, we define

$$R_F(x) = \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}F(X_t)\,\mathrm{d}t\right], \quad \text{for } x > 0. \tag{2.21}$$

The function R_F admits the analytic presentation

$$R_F(x) = \varphi(x) \int_0^x F(s)\Psi(s)\,\mathrm{d}s + \psi(x) \int_x^\infty F(s)\Phi(s)\,\mathrm{d}s \tag{2.22}$$

and satisfies the ODE $\mathscr{L} R_F + F = 0$. Furthermore,

$$\lim_{x\downarrow 0} \frac{\left|R_F(x)\right|}{\varphi(x)} = 0 \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{\left|R_F(x)\right|}{\psi(x)} = 0. \tag{2.23}$$

Conversely, consider any function $f : \,]0,\infty[\to \mathbb{R}$ that is C^1 with absolutely continuous first-order derivative and such that

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t}\left|\mathscr{L} f(X_t)\right| \mathrm{d}t\right] < \infty, \quad \limsup_{z\downarrow 0} \frac{\left|f(z)\right|}{\varphi(z)} < \infty \quad \text{and} \quad \limsup_{z\uparrow\infty} \frac{\left|f(z)\right|}{\psi(z)} < \infty.$$

Such a function is such that

$$\text{both of the limits } \lim_{z\downarrow 0} \frac{f(z)}{\varphi(z)} \text{ and } \lim_{z\uparrow\infty} \frac{f(z)}{\psi(z)} \text{ exist} \tag{2.24}$$

$$\text{and} \quad f(x) = \lim_{z\downarrow 0} \frac{f(z)}{\varphi(z)}\varphi(x) - R_{\mathscr{L} f}(x) + \lim_{z\uparrow\infty} \frac{f(z)}{\psi(z)}\psi(x) \quad \text{for all } x > 0. \tag{2.25}$$

Part (ii) of the following result will be important in appreciating the role that the boundary classification of 0 has on whether switching off the system might be optimal (see Remark 2.1 at the end of Section 2.4). In general, (2.26) is not true if 0 is an entrance boundary point (see (2.97) in Example 2.8 in Section 2.5).

Lemma 2.2.1 *Suppose that Assumptions 2.1 and 2.3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (2.1) are both inaccessible. Let F be any Borel measurable function satisfying the equivalent integrability conditions (2.19) and (2.20), and consider the function R_F defined by (2.21) and (2.22). The following statements hold true:*

(i) *Suppose that F is bounded from below. If K is any constant such that $F(x)/r(x) \geq K$ for all $x > 0$, then $R_F(x) \geq K$ for all $x > 0$.*

(ii) *If 0 is a natural boundary point, then*

$$\liminf_{x\downarrow 0} \frac{F(x)}{r(x)} \leq \liminf_{x\downarrow 0} R_F(x) \leq \limsup_{x\downarrow 0} R_F(x) \leq \limsup_{x\downarrow 0} \frac{F(x)}{r(x)}. \tag{2.26}$$

Proof. Part (i) of the lemma follows immediately from the calculation

$$\inf_{x>0} R_F(x) = \inf_{x>0} \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t} F(X_t)\,\mathrm{d}t\right] \geq \inf_{x>0} \frac{F(x)}{r(x)}\, \mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t} r(X_t)\,\mathrm{d}t\right] = \inf_{x>0} \frac{F(x)}{r(x)},$$

where we have used the definition (1.3) of Λ.

To establish part (ii) of the lemma suppose in what follows that 0 is a natural boundary point. Assuming that $\limsup_{x\downarrow 0} F(x)/r(x) \in \mathbb{R}$, fix any $\varepsilon > 0$ and let $x_\varepsilon > 0$ be any point such that

$$\frac{F(x)}{r(x)} \leq \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} + \varepsilon \quad \text{for all } x \in \,]0, x_\varepsilon].$$

In view of (2.21), (2.22), the definition (1.3) of Λ and the second limit in (2.10), we can see that

$$\begin{aligned}
\limsup_{x\downarrow 0} R_F(x) - \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} - \varepsilon \\
&= \limsup_{x\downarrow 0} \mathbb{E}_x \left[\int_0^\infty e^{-\Lambda_t} \left(\frac{F(X_t)}{r(X_t)} - \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} - \varepsilon \right) r(X_t)\,\mathrm{d}t \right] \\
&= \limsup_{x\downarrow 0} \Bigg(\varphi(x) \int_0^x \left(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} - \varepsilon \right) r(s)\Psi(s)\,\mathrm{d}s \\
&\quad + \psi(x) \int_x^\infty \left(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} - \varepsilon \right) r(s)\Phi(s)\,\mathrm{d}s \Bigg) \\
&\leq \lim_{x\downarrow 0} \psi(x) \int_{x_\varepsilon}^\infty \left(\frac{F(s)}{r(s)} - \limsup_{x\downarrow 0} \frac{F(x)}{r(x)} - \varepsilon \right) r(s)\Phi(s)\,\mathrm{d}s \Bigg) \\
&= 0,
\end{aligned}$$

which implies that $\limsup_{x\downarrow 0} R_F(x) \leq \limsup_{x\downarrow 0} F(x)/r(x)$ because ε has been arbitrary. Similarly, we can show that $\lim_{x\downarrow 0} R_F(x) = -\infty$ if $\lim_{x\downarrow 0} F(x)/r(x) = -\infty$, and the third inequality in (2.26) follows. Using similar arguments, we can establish the first inequality in (2.26). □

Lemma 2.2.2 *Suppose that Assumptions 2.1 and 2.3 hold true, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (2.1) are both inaccessible and consider any Borel measurable function F satisfying the equivalent integrability conditions (2.19) and (2.20). The function $G_F :]0,\infty[\to \mathbb{R}$ defined by*

$$G_F(x) := R_F(x) - \frac{R_F'(x)}{\psi'(x)}\psi(x) = \frac{Cp'(x)}{\psi'(x)} \int_0^x F(s)\Psi(s)\,\mathrm{d}s \tag{2.27}$$

is such that

$$\liminf_{x\downarrow 0} \frac{F(x)}{r(x)} \leq \liminf_{x\downarrow 0} G_F(x) \leq \limsup_{x\downarrow 0} G_F(x) \leq \limsup_{x\downarrow 0} \frac{F(x)}{r(x)}. \tag{2.28}$$

Furthermore, if the boundary point ∞ is natural, then

$$\liminf_{x\uparrow\infty} \frac{F(x)}{r(x)} \leq \liminf_{x\uparrow\infty} G_F(x) \leq \limsup_{x\uparrow\infty} G_F(x) \leq \limsup_{x\uparrow\infty} \frac{F(x)}{r(x)}. \tag{2.29}$$

Proof. We first note that the equality in (2.27) follows immediately from the definition (2.22) of R_F and the identity (2.13). In view of (2.10) and (2.11), the assumption that the boundary point 0 is inaccessible implies that

$$\lim_{x\downarrow 0} \frac{\psi'(x)}{p'(x)} = 0. \tag{2.30}$$

This limit and the calculation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\psi'(x)}{p'(x)} = \frac{2}{\sigma^2(x)p'(x)}\left(\frac{1}{2}\sigma^2(x)\psi''(x) + b(x)\psi'(x)\right) = \frac{2r(x)\psi(x)}{\sigma^2(x)p'(x)} = Cr(x)\Psi(x)$$

imply that

$$\int_0^x r(s)\Psi(s)\,\mathrm{d}s = \frac{\psi'(x)}{Cp'(x)}. \tag{2.31}$$

Similarly, the calculation

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\varphi'(x)}{p'(x)} = Cr(x)\Phi(x) \tag{2.32}$$

and the assumption that the boundary point ∞ is inaccessible imply that

$$\int_x^\infty r(s)\Phi(s)\,\mathrm{d}s = -\frac{\varphi'(x)}{Cp'(x)}. \tag{2.33}$$

In view of (2.31) and the expression of G_F on the right-hand side of (2.27), we can see that

$$\begin{aligned} & G_F(x) \geq \frac{Cp'(x)}{\psi'(x)} \inf_{y<x} \frac{F(y)}{r(y)} \int_0^x r(s)\Psi(s)\,\mathrm{d}s = \inf_{y<x} \frac{F(y)}{r(y)} \\ \text{and} \quad & G_F(x) \leq \frac{Cp'(x)}{\psi'(x)} \sup_{y<x} \frac{F(y)}{r(y)} \int_0^x r(s)\Psi(s)\,\mathrm{d}s = \sup_{y<x} \frac{F(y)}{r(y)}. \end{aligned}$$

These inequalities imply (2.28).

Next, we additionally assume that ∞ is a natural boundary point, which implies that $\lim_{x\uparrow\infty} \psi'(x)/p'(x) = \infty$ (e.g., see Borodin and Salminen [13, II.10]). The expression of G_F on the right-hand side of (2.27), the strict positivity of Ψ and the identity (2.31) imply that, given any $x > z > 0$,

$$\begin{aligned} & \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \inf_{y>z} \frac{F(y)}{r(y)} \left(1 - \frac{p'(x)}{\psi'(x)} \frac{\psi'(z)}{p'(z)}\right) \\ & \qquad = \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \inf_{y>z} \frac{F(y)}{r(y)} \frac{Cp'(x)}{\psi'(x)} \int_z^x r(s)\Psi(s)\,\mathrm{d}s \\ & \qquad \leq G_F(x) \leq \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \sup_{y>z} \frac{F(y)}{r(y)} \frac{Cp'(x)}{\psi'(x)} \int_z^x r(s)\Psi(s)\,\mathrm{d}s \\ & \qquad = \frac{Cp'(x)}{\psi'(x)} \int_0^z F(s)\Psi(s)\,\mathrm{d}s + \sup_{y>z} \frac{F(y)}{r(y)} \left(1 - \frac{p'(x)}{\psi'(x)} \frac{\psi'(z)}{p'(z)}\right). \end{aligned}$$

Combining these observations, we can see that

$$\inf_{y>z} \frac{F(y)}{r(y)} \leq \liminf_{x\uparrow\infty} G_F(x) \leq \limsup_{x\uparrow\infty} G_F(x) \leq \sup_{y>z} \frac{F(y)}{r(y)} \quad \text{for all } z > 0,$$

and (2.29) follows. □

Lemma 2.2.3 *Suppose that Assumption 2.1 and 2.3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the SDE (2.1) are both inaccessible. Given any Borel measurable function F satisfying the equivalent integrability conditions (2.19) and (2.20), if the boundary point 0 (resp., ∞) is inaccessible, then*

$$\liminf_{x\downarrow 0} \frac{R_F'(x)}{\varphi'(x)} \leq 0 \leq \limsup_{x\downarrow 0} \frac{R_F'(x)}{\varphi'(x)} \quad \left(\textit{resp.,} \ \liminf_{x\uparrow\infty} \frac{R_F'(x)}{\psi'(x)} \leq 0 \leq \limsup_{x\uparrow\infty} \frac{R_F'(x)}{\psi'(x)} \right). \quad (2.34)$$

Furthermore, if there exists $x_\dagger > 0$ (resp., $x^\dagger > 0$) such that the restriction of F/r in $]0, x_\dagger[$ (resp., $]x^\dagger, \infty[$) is a monotone function, then

$$\lim_{x\downarrow 0} \frac{R_F'(x)}{\varphi'(x)} = 0 \quad \left(\textit{resp.,} \ \lim_{x\uparrow\infty} \frac{R_F'(x)}{\psi'(x)} = 0 \right). \quad (2.35)$$

Proof. To establish the very first inequality in (2.34), we argue by contradiction. To this end, we assume that $\liminf_{x\downarrow 0} R_F'(x)/\varphi'(x) > 0$, which implies that there exist $\varepsilon > 0$ and $x_\varepsilon > 0$ such that

$$\frac{R_F'(x)}{\varphi'(x)} > \varepsilon \ \Leftrightarrow \ R_F'(x) < \varepsilon \varphi'(x) \quad \text{for all } x \in \,]0, x_\varepsilon[.$$

However, this observation and the fact that $\lim_{x\downarrow 0} \varphi(x) = \infty$ imply that $\lim_{x\downarrow 0} R_F(x)/\varphi(x) \geq \varepsilon$, which contradicts (2.23). The proof of the other inequalities in (2.34) is similar.

To proceed further, we first note that (2.13) and the fact that $\mathscr{L}\varphi = \mathscr{L}\psi = 0$, where $\mathscr{L}$ is the differential operator defined by (2.6), imply that

$$\psi'(x)\varphi''(x) - \varphi'(x)\psi''(x) = \frac{2Cr(x)}{\sigma^2(x)} p'(x). \quad (2.36)$$

In view of this observation and the definition (2.22) of R_F, we can see that the function R_F'/ψ' is absolutely continuous with derivative

$$\frac{(\sigma(x)\psi'(x))^2}{2Cr(x)p'(x)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{R_F'(x)}{\psi'(x)} = \left(\int_0^x F(s)\Psi(s)\,\mathrm{d}s - \frac{F(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)} \right) =: Q_F(x). \quad (2.37)$$

Now, suppose that there exists a point $x^\dagger > 0$ such that F/r is monotone in $[x^\dagger, \infty[$. Given any points $x_1 < x_2$ in $[x^\dagger, \infty[$, we use (2.31) to calculate

$$\begin{aligned} Q_F(x_2) - Q_F(x_1) &= \int_{x_1}^{x_2} F(s)\Psi(s)\,\mathrm{d}s - \frac{F(x_2)}{r(x_2)}\frac{\psi'(x_2)}{Cp'(x_2)} + \frac{F(x_1)}{r(x_1)}\frac{\psi'(x_1)}{Cp'(x_1)} \\ &= \int_{x_1}^{x_2} \left(\frac{F(s)}{r(s)} - \frac{F(x_2)}{r(x_2)}\right) r(s)\Psi(s)\,\mathrm{d}s + \frac{\psi'(x_1)}{Cp'(x_1)}\left(\frac{F(x_1)}{r(x_1)} - \frac{F(x_2)}{r(x_2)}\right) \\ &\begin{cases} \geq 0, & \text{if } F/r \text{ is decreasing in } [x^\dagger, \infty[, \\ \leq 0, & \text{if } F/r \text{ is increasing in } [x^\dagger, \infty[. \end{cases} \end{aligned} \tag{2.38}$$

Therefore, Q_F is monotone in $[x^\dagger, \infty[$ and the limit $\lim_{x\uparrow\infty} Q_F(x)$ exists in $[-\infty, \infty]$. However, this observation and (2.37) imply that there exists $\widetilde{x} \geq x^\dagger$ such that R_F'/ψ' is monotone in $[\widetilde{x}, \infty[$. Therefore, the limit $\lim_{x\uparrow\infty} R_F'(x)/\psi'(x)$ exists, which, combined with the last two inequalities in (2.34), implies the corresponding limit in (2.35).

Finally, we can establish the other limit in (2.35) using symmetric arguments and (2.32). □

The following result will play a critical role in our analysis. Example 2.9 in Section 2.5 shows that the point $\underline{x}$ introduced in part (i) of the lemma can be equal to ∞ if the sufficient conditions in (2.41) fail to be true. Also, in contrast to the limit in (2.39), Examples 2.5 and 2.6 in Section 2.5 show that the limit $\lim_{x\downarrow 0} R_\Theta'(x)/\psi'(x)$, which characterises part (iii) of the lemma, can take any value in $]-\infty, \infty]$.

Lemma 2.2.4 *Suppose that Assumption 2.1 and 2.3 hold true. Also, suppose that the boundary points 0 and ∞ are both inaccessible. Given a function Θ satisfying the conditions of Assumption 2.4.(iii), as well as the equivalent integrability conditions (2.19) and (2.20), the following statements are true:*

(i) *There exists a unique $\underline{x} \in]\xi, \infty]$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{R_\Theta'(x)}{\psi'(x)} \begin{cases} < 0 & \text{for all } x \in]0, \underline{x}[, \\ > 0 & \text{for all } x \in]\underline{x}, \infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{R_\Theta'(x)}{\psi'(x)} = 0, \tag{2.39}$$

where we adopt the convention $]\infty, \infty[= \emptyset$.

(ii) $\underline{x} < \infty$ *if and only if* $\lim_{x\uparrow\infty} Q_\Theta(x) > 0$, *where*

$$Q_\Theta(x) = \int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x)}{r(x)}\frac{\psi'(x)}{Cp'(x)}. \tag{2.40}$$

In particular, this is the case if

$$\lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} = -\infty \quad \text{or} \quad \lim_{x\downarrow 0} \frac{\Theta(x)}{r(x)} > \lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)}. \tag{2.41}$$

(iii) *If* $\underline{x} < \infty$ *and we define*

$$\bar{x} = \inf\left\{ s > 0 \;\Big|\; \frac{R'_\Theta(s)}{\psi'(s)} > \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} \right\}, \tag{2.42}$$

with the usual convention that $\inf \emptyset = \infty$*, then* $\bar{x} > \underline{x}$*,*

$$\bar{x} = \infty \quad \Leftrightarrow \quad \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} \geq 0 \tag{2.43}$$

$$\text{and} \quad \lim_{x \downarrow 0} \frac{\Theta(x)}{r(x)} = -\infty \quad \Rightarrow \quad \lim_{x \downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \infty \quad \Rightarrow \quad \bar{x} = \infty. \tag{2.44}$$

Proof. The limit in (2.39) follows from Lemma 2.2.3 and the assumption that Θ/r is strictly decreasing in $]\xi, \infty[$. Using (2.31) and (2.37), we can see that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} = \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} \int_0^x \left(\frac{\Theta(s)}{r(s)} - \frac{\Theta(x)}{r(x)} \right) r(s)\Psi(s)\,\mathrm{d}s = \frac{2Cr(x)p'(x)}{(\sigma(x)\psi'(x))^2} Q_\Theta(x).$$

These expressions imply that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \quad \text{and} \quad Q_\Theta(x) < 0 \quad \text{for all } x \leq \xi$$

because Θ/r is strictly increasing in $]0, \xi[$. On the other hand, (2.38) for $F = \Theta$ implies that Q_Θ is strictly increasing in $[\xi, \infty[$ because Θ/r is strictly decreasing in $[\xi, \infty[$. It follows that there exists a unique $\underline{x} \in]\xi, \infty]$ such that the inequalities in (2.39) hold true. Furthermore, $\underline{x} < \infty$ if and only if $\lim_{x \uparrow \infty} Q_\Theta(x) > 0$.

To establish the sufficient conditions in part (ii) of the lemma, we first use the integration by parts formula and (2.31) to observe that

$$\begin{aligned} Q_\Theta(x) &= \int_0^\xi \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(\xi)}{r(\xi)} \frac{\psi'(\xi)}{Cp'(\xi)} - \int_\xi^x \frac{\psi'(s)}{Cp'(s)}\,\mathrm{d}\frac{\Theta(s)}{r(s)} \\ &\geq \int_0^\xi \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x)}{r(x)} \frac{\psi'(\xi)}{Cp'(\xi)} \quad \text{for all } \xi < x. \end{aligned} \tag{2.45}$$

This inequality reveals that $\lim_{x \uparrow \infty} Q_\Theta(x) = \infty$ if $\lim_{x \uparrow \infty} \Theta(x)/r(x) = -\infty$.

The identity (2.31) implies that, given any constant K,

$$\int_0^x Kr(s)\Psi(s)\,\mathrm{d}s - \frac{Kr(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)} = 0$$

Combining this observation with the definition of Q_Θ, we can see that $Q_\Theta = Q_{\Theta + Kr}$. If Θ/r satisfies the inequality in (2.41), then, for all K such that

$$-\lim_{x \downarrow 0} \frac{\Theta(x)}{r(x)} < K < -\lim_{x \uparrow \infty} \frac{\Theta(x)}{r(x)},$$

there exists $\eta(K) \in]\xi, \infty[$ such that

$$\Theta\big(\eta(K)\big) + Kr\big(\eta(K)\big) = 0 \quad \text{and} \quad Q_{\Theta+Kr}\big(\eta(K)\big) = \int_0^{\eta(K)} \big(\Theta(s) + Kr(s)\big)\Psi(s)\,\mathrm{d}s > 0.$$

It follows that

$$\lim_{x\uparrow\infty} Q_\Theta(x) = \lim_{x\uparrow\infty} Q_{\Theta+Kr}(x) > 0,$$

thanks to the fact that Q_Θ is strictly increasing in $[\xi, \infty[$.

The equivalence (2.43) follows immediately from (2.39) and the definition (2.42) of $\bar{x}$. To establish the implications in (2.44), we first note that (2.32) implies that the function φ'/p' is strictly increasing, so the limit $\lim_{x\downarrow 0} \varphi'(x)/p'(x)$ exists in $[-\infty, 0[$. Therefore,

$$\lim_{x\downarrow 0} \frac{\psi'(x)}{\varphi'(x)} = \lim_{x\downarrow 0} \frac{\psi'(x)}{p'(x)} \lim_{x\downarrow 0} \frac{p'(x)}{\varphi'(x)} = 0, \tag{2.46}$$

where we have also used (2.30). Using the first of these two observations, the definition (2.22) of R_Θ, (2.33), (2.36) and integration by parts, we can see that, if $\lim_{x\downarrow 0} \Theta(x)/r(x) = -\infty$, then

$$\begin{aligned}
\lim_{x\downarrow 0} \frac{(\sigma(x)\varphi'(x))^2}{2Cr(x)p'(x)} \frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\varphi'(x)} &= -\lim_{x\downarrow 0} \left(\int_x^\infty \Theta(s)\Phi(s)\,\mathrm{d}s + \frac{\Theta(x)}{r(x)} \frac{\varphi'(x)}{Cp'(x)} \right) \\
&= -\int_1^\infty \Theta(s)\Phi(s)\,\mathrm{d}s - \frac{\Theta(1)}{r(1)} \frac{\varphi'(1)}{Cp'(1)} + \lim_{x\downarrow 0} \int_x^1 \frac{\varphi'(s)}{Cp'(s)}\,\mathrm{d}\frac{\Theta(s)}{r(s)} \\
&= -\infty.
\end{aligned}$$

On the other hand, we use (2.36) to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\psi'(x)}{\varphi'(x)} = -\frac{2Cr(x)p'(x)}{(\sigma(x)\varphi'(x))^2}.$$

In view of (2.35) and (2.46), these calculations and L' Hôpital's formula imply that

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \lim_{x\downarrow 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\varphi'(x)}}{\frac{\mathrm{d}}{\mathrm{d}x} \frac{\psi'(x)}{\varphi'(x)}} = \infty.$$

The implications in (2.44) follow from this analysis and the definition (2.42) of $\bar{x}$. □

2.3 The "β-γ" strategy

In this section, we consider the β-γ strategy that is characterised by two points $0 < \gamma < \beta < \infty$ and takes the following form. If the state process takes any value $x \geq \beta$, the controller pushes it in an impulsive way down to the level γ. For as long as the state process takes values inside the interval $]0, \beta[$, the controller waits and takes no action. Accordingly, such a strategy is characterised by a controlled process ζ such that

$$\Delta\zeta_t = \big(X^\zeta_{t-} - \gamma\big)\mathbf{1}_{\{X^\zeta_{t-} \geq \beta\}} \quad \text{for all } t \geq 0, \tag{2.47}$$

where X^ζ is the associated solution to the SDE (1.1).

Theorem 2.3.1 *Suppose that Assumptions 2.1 and 2.3 hold true. Also, suppose that the boundary points 0 and ∞ of the diffusion associated with the uncontrolled SDE (2.1) are both inaccessible. Given any points $\gamma < \beta$ in $]0, \infty[$, there exists a controlled process $\zeta = \zeta(\beta, \gamma)$ that is admissible in the sense of Definition 2.1 and is such that (2.47) holds true. Furthermore, given any $x \in]0, \beta[$,*

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda^\zeta_t} h(X^\zeta_t)\, \mathrm{d}t\right] = R_h(x) + \frac{\psi(x)}{\psi(\beta) - \psi(\gamma)}\big(R_h(\gamma) - R_h(\beta)\big) \tag{2.48}$$

and

$$\mathbb{E}_x\left[\sum_{t \geq 0} e^{-\Lambda^\zeta_t}\mathbf{1}_{\{\Delta\zeta_t > 0\}}\right] = \frac{\psi(x)}{\psi(\beta) - \psi(\gamma)}. \tag{2.49}$$

Proof. We start with a recursive construction of the required process ζ and its associated solution to the SDE (2.1). To this end, we first consider any initial state $x \in]0, \beta[$, we denote by X^1 the solution to the uncontrolled SDE (2.1) and we define

$$\tau_1 = \inf\big\{t \geq 0 \mid \ X^1_t \geq \beta\big\} \quad \text{and} \quad \zeta^1_t = (\beta - \gamma)\mathbf{1}_{\{\tau_1 \leq t\}}. \tag{2.50}$$

Given $\ell \geq 1$, suppose that we have determined X^j, τ_j and ζ^j, for $j = 1, \ldots, \ell$.

The process $\widetilde{W}^{\ell+1}$ defined by $\widetilde{W}^{\ell+1}_t = \big(W_{\tau_\ell + t} - W_{\tau_\ell}\big)\mathbf{1}_{\{\tau_\ell < \infty\}}$ is a standard $(\mathcal{F}_{\tau_\ell + t})$-Brownian motion that is independent of $\mathcal{F}_{\tau_\ell}$ under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ (see Revuz and Yor [78, Exercise IV.3.21]). We denote by $\widetilde{X}^{\ell+1}$ the unique solution to the uncontrolled SDE (2.1) with $\widetilde{X}^{\ell+1}_0 = \gamma$ that is driven by the Brownian motion $\widetilde{W}^{\ell+1}$ and is defined on the probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_{\tau_\ell + t}), \mathbb{P}(\cdot \mid \tau_\ell < \infty)\big)$. Since $(t - \tau_\ell)^+$ is an $(\mathcal{F}_{\tau_\ell + t})$-stopping time for all $t \geq 0$,

$$\tau_\ell + (t - \tau_\ell)^+ = t \vee \tau_\ell \quad \text{and} \quad \widetilde{W}^{\ell+1}_{(t-\tau_\ell)^+} = \big(W_{t \vee \tau_\ell} - W_{\tau_\ell}\big)\mathbf{1}_{\{\tau_\ell < \infty\}},$$

we can see that, on the event $\{\tau_\ell < \infty\}$,

$$\begin{aligned}
\widetilde{X}^{\ell+1}_{(t-\tau_\ell)^+} &= \gamma + \int_0^{(t-\tau_\ell)^+} b\big(\widetilde{X}^{\ell+1}_s\big)\, \mathrm{d}s + \int_0^{(t-\tau_\ell)^+} \sigma\big(\widetilde{X}^{\ell+1}_s\big)\, \mathrm{d}\widetilde{W}^{\ell+1}_s \\
&= \gamma + \int_0^t b\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\, \mathrm{d}(s - \tau_\ell)^+ + \int_0^t \sigma\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\, \mathrm{d}\widetilde{W}^{\ell+1}_{(s-\tau_\ell)^+} \\
&= \gamma + \int_{\tau_\ell}^{t \vee \tau_\ell} b\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\, \mathrm{d}s + \int_{\tau_\ell}^{t \vee \tau_\ell} \sigma\Big(\widetilde{X}^{\ell+1}_{(s-\tau_\ell)^+}\Big)\, \mathrm{d}W_s,
\end{aligned}$$

where we have time changed the Lebesgue as well as the Itô integral (see Revuz and Yor [78, Propositions V.1.4, V.1.5]). It follows that, if we define

$$\overline{X}_t^{\ell+1} = \widetilde{X}_{(t-\tau_\ell)^+}^{\ell+1} \mathbf{1}_{\{\tau_\ell<\infty\}}, \quad \text{for } t \geq 0, \tag{2.51}$$

then

$$\overline{X}_t^{\ell+1} = \gamma + \int_{\tau_\ell}^{t\vee\tau_\ell} b\big(\overline{X}_s^{\ell+1}\big)\, \mathrm{d}s + \int_{\tau_\ell}^{t\vee\tau_\ell} \sigma\big(\overline{X}_s^{\ell+1}\big)\, \mathrm{d}W_s. \tag{2.52}$$

Furthermore, we define

$$X_t^{\ell+1} = X_t^\ell \mathbf{1}_{\{t<\tau_\ell\}} + \overline{X}_t^{\ell+1} \mathbf{1}_{\{\tau_\ell\leq t\}}, \tag{2.53}$$

$$\tau_{\ell+1} = \inf\big\{t > \tau_\ell \mid \ X_t^{\ell+1} \geq \beta\big\} \quad \text{and} \quad \zeta_t^{\ell+1} = \zeta_t^\ell + (\beta-\gamma)\mathbf{1}_{\{\tau_{\ell+1}\leq t\}}. \tag{2.54}$$

Also, we note that

$$\tau_{\ell+1} - \tau_\ell = \widetilde{T}_\beta^{\ell+1} := \inf\big\{t \geq 0 \mid \ \widetilde{X}_t^{\ell+1} \geq \beta\big\}. \tag{2.55}$$

Given the recursive construction we have just considered, we define

$$X_t^\zeta = \sum_{\ell=0}^\infty X_t^{\ell+1} \mathbf{1}_{\{\tau_\ell\leq t<\tau_{\ell+1}\}} \quad \text{and} \quad \zeta_t = \sum_{\ell=0}^\infty \zeta_t^{\ell+1} \mathbf{1}_{\{\tau_\ell\leq t<\tau_{\ell+1}\}}. \tag{2.56}$$

In view of (2.52)–(2.54), the process X^ζ given by (2.56) provides the unique solution to the SDE (1.1) for ζ being as in (2.56). Furthermore, these processes are such that (2.47) holds true. In the case that arises when the initial state $x \geq \beta$, the only modification of the arguments above involves X^1 being the solution to the uncontrolled SDE (2.1) for $x = \gamma$ and ζ^1 being the same as in (2.50) translated by adding the constant $x - \gamma$ to it.

We next establish (2.49), which implies the admissibility condition (2.5). The process $\widetilde{X}^{\ell+1}$ introduced at the beginning of the proof is independent of $\mathcal{F}_{\tau_\ell}$ under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ and its distribution under $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ is the same as the distribution of the solution X to the uncontrolled SDE (2.1) with initial state $X_0 = \gamma$ under $\mathbb{P}$. In particular,

$$\mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\Big] = \mathbb{E}_\gamma\big[F(X)\big]$$

for every bounded measurable functional F mapping continuous functions on $\mathbb{R}_+$ to $\mathbb{R}_+$, where we denote by $\mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}$ expectations computed under the conditional probability measure $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$. In view of these observations and the definition of conditional expectation,

$$\mathbb{E}_\gamma\Big[F\big(\widetilde{X}^{\ell+1}\big) \mid \mathcal{F}_{\tau_\ell}\Big] \mathbf{1}_{\{\tau_\ell<\infty\}} = \mathbb{E}_\gamma\big[F(X)\big] \mathbf{1}_{\{\tau_\ell<\infty\}}. \tag{2.57}$$

To see this claim, we first note that the Radon-Nikodym derivative of $\mathbb{P}(\cdot \mid \tau_\ell < \infty)$ with respect to $\mathbb{P}$ is given by

$$\frac{\mathrm{d}\,\mathbb{P}(\cdot \mid \tau_\ell < \infty)}{\mathrm{d}\,\mathbb{P}} = \frac{1}{\mathbb{P}(\tau_\ell < \infty)} \mathbf{1}_{\{\tau_\ell<\infty\}}.$$

Given any event $\Gamma \in \mathcal{F}_{\tau_\ell}$,

$$\begin{aligned}\frac{1}{\mathbb{P}(\tau_\ell<\infty)}\,\mathbb{E}_\gamma\Big[\mathbb{E}_\gamma\big[F(X)\big]\mathbf{1}_{\{\tau_\ell<\infty\}}\mathbf{1}_\Gamma\Big] &= \mathbb{E}_\gamma\big[F(X)\big]\frac{1}{\mathbb{P}(\tau_\ell<\infty)}\,\mathbb{E}_\gamma\big[\mathbf{1}_{\{\tau_\ell<\infty\}\cap\Gamma}\big]\\ &= \mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\Big]\,\mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\big[\mathbf{1}_\Gamma\big]\\ &= \mathbb{E}^{\mathbb{P}(\cdot|\tau_\ell<\infty)}\Big[F\big(\widetilde{X}^{\ell+1}\big)\mathbf{1}_\Gamma\Big]\\ &= \frac{1}{\mathbb{P}(\tau_\ell<\infty)}\,\mathbb{E}_\gamma\Big[F\big(\widetilde{X}^{\ell+1}\big)\mathbf{1}_{\{\tau_\ell<\infty\}}\mathbf{1}_\Gamma\Big],\end{aligned}$$

and (2.57) follows.

In view of (2.51)–(2.57), we can see that

$$\begin{aligned}\mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_{\ell+1}}}\big] &= \mathbb{E}_x\bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\bigg[\exp\bigg(-\int_{\tau_\ell}^{\tau_{\ell+1}} r(X^\zeta_u)\,\mathrm{d}u\bigg)\,\bigg|\,\mathcal{F}_{\tau_\ell}\bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\bigg]\\ &= \mathbb{E}_x\bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\bigg[\exp\bigg(-\int_{\tau_\ell}^{\tau_{\ell+1}} r\big(\overline{X}^{\ell+1}_u\big)\,\mathrm{d}u\bigg)\,\bigg|\,\mathcal{F}_{\tau_\ell}\bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\bigg]\\ &= \mathbb{E}_x\bigg[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\bigg[\exp\bigg(-\int_0^{\widetilde{T}^{\ell+1}_\beta} r\big(\widetilde{X}^{\ell+1}_u\big)\,\mathrm{d}u\bigg)\,\bigg|\,\mathcal{F}_{\tau_\ell}\bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\bigg]\\ &= \mathbb{E}_x\bigg[e^{-\Lambda^\zeta_{T_\ell}}\,\mathbb{E}_\gamma\bigg[\exp\bigg(-\int_0^{T_\beta} r(X_u)\,\mathrm{d}u\bigg)\bigg]\mathbf{1}_{\{\tau_\ell<\infty\}}\bigg]\\ &= \mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_\ell}}\big]\,\mathbb{E}_\gamma\big[e^{-\Lambda_{T_\beta}}\big],\end{aligned}$$

where Λ is defined by (1.3) with X in place of X^ζ and T_β is defined as in (2.4). Given any $x \in \,]0,\beta[$, we iterate this result and use (2.12) to obtain

$$\mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_{\ell+1}}}\big] = \mathbb{E}_x\big[e^{-\Lambda_{T_\beta}}\big]\Big(\mathbb{E}_\gamma\big[e^{-\Lambda_{T_\beta}}\big]\Big)^\ell = \frac{\psi(x)}{\psi(\beta)}\bigg(\frac{\psi(\gamma)}{\psi(\beta)}\bigg)^\ell. \tag{2.58}$$

It follows that

$$\begin{aligned}\mathbb{E}_x\bigg[\sum_{t\geq 0} e^{-\Lambda^\zeta_t}\mathbf{1}_{\{\Delta\zeta_t>0\}}\bigg] &= \mathbb{E}_x\bigg[\sum_{\ell=1}^\infty e^{-\Lambda^\zeta_{\tau_\ell}}\bigg]\\ &= \sum_{\ell=1}^\infty \mathbb{E}_x\big[e^{-\Lambda^\zeta_{\tau_\ell}}\big] = \frac{\psi(x)}{\psi(\beta)}\sum_{\ell=0}^\infty\bigg(\frac{\psi(\gamma)}{\psi(\beta)}\bigg)^\ell = \frac{\psi(x)}{\psi(\beta)-\psi(\gamma)},\end{aligned}$$

which establishes (2.49).

To show (2.48), we consider any $x \in \,]0,\beta[$ and we use (2.51)–(2.57) as well as (2.58) to derive the expression

$$
\begin{aligned}
&\mathbb{E}_x\biggl[\int_{\tau_\ell}^{\tau_{\ell+1}} e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\biggr] \\
&= \mathbb{E}_x\biggl[e^{-\Lambda^\zeta_{\tau_\ell}}\,\mathbb{E}_\gamma\biggl[\int_{\tau_\ell}^{\tau_{\ell+1}} \exp\biggl(-\int_{\tau_\ell}^t r\bigl(\overline{X}^{\ell+1}_u\bigr)\,\mathrm{d}u\biggr) h\bigl(\overline{X}^{\ell+1}_t\bigr)\,\mathrm{d}t \,\bigg|\, \mathcal{F}_{\tau_\ell}\biggr]\biggr] \\
&= \mathbb{E}_x\bigl[e^{-\Lambda^\zeta_{\tau_\ell}}\bigr]\,\mathbb{E}_\gamma\biggl[\int_0^{T_\beta} e^{-\Lambda_t} h(X_t)\,\mathrm{d}t\biggr] = \mathbb{E}_x\bigl[e^{-\Lambda^\zeta_{\tau_\ell}}\bigr]\Bigl(R_h(\gamma) - \mathbb{E}_\gamma\bigl[e^{-\Lambda_{T_\beta}}\bigr]R_h(\beta)\Bigr) \\
&= \frac{\psi(x)}{\psi(\beta)}\biggl(\frac{\psi(\gamma)}{\psi(\beta)}\biggr)^{\ell-1}\biggl(R_h(\gamma) - \frac{\psi(\gamma)}{\psi(\beta)}R_h(\beta)\biggr).
\end{aligned}
$$

Similarly, we can show that

$$
\mathbb{E}_x\biggl[\int_0^{\tau_1} e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\biggr] = R_h(x) - \frac{\psi(x)}{\psi(\beta)}R_h(\beta).
$$

Recalling the assumption that h is bounded from below, we can use the monotone convergence theorem and these results to obtain

$$
\begin{aligned}
&\mathbb{E}_x\biggl[\int_0^{\infty} e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\biggr] \\
&= \mathbb{E}_x\biggl[\int_0^{\tau_1} e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\biggr] + \sum_{\ell=1}^{\infty}\mathbb{E}_x\biggl[\int_{\tau_\ell}^{\tau_{\ell+1}} e^{-\Lambda^\zeta_t} h(X^\zeta_t)\,\mathrm{d}t\biggr] \\
&= R_h(x) - \frac{\psi(x)}{\psi(\beta)}R_h(\beta) + \frac{\psi(x)}{\psi(\beta)}\biggl(R_h(\gamma) - \frac{\psi(\gamma)}{\psi(\beta)}R_h(\beta)\biggr)\sum_{\ell=1}^{\infty}\biggl(\frac{\psi(\gamma)}{\psi(\beta)}\biggr)^{\ell-1},
\end{aligned}
$$

which proves (2.48). □

2.4 The solution to the control problem

We will solve the control problem we have considered by deriving a C^1 with absolutely continuous first-order derivative function $w :]0,\infty[\to \mathbb{R}$ that satisfies the HJB equation

$$
\max\biggl\{\mathscr{L}w(x) + h(x),\ -c + \sup_{z\in[0,x[}\int_{x-z}^{x}\bigl(k(s) - w'(s)\bigr)\,\mathrm{d}s\biggr\} = 0, \tag{2.59}
$$

Lebesgue-a.e. in $]0,\infty[$. Given such a solution, the optimal strategy can be characterised as follows. The controller should wait and take no action for as long as the state process X takes values in the interior of the set in which the ODE

$$
\mathscr{L}w(x) + h(x) = 0
$$

is satisfied and should take immediate action with an impulse in the negative direction if the state process takes values in the set of all points $x > 0$ such that

$$-c + \sup_{z \in [0,x[} \int_{x-z}^{x} \bigl(k(s) - w'(s)\bigr)\, \mathrm{d}s = 0.$$

We first consider the possibility for a β-γ strategy with $\gamma < \beta$ in $]0,\infty[$ to be optimal. The optimality of such a strategy is associated with a solution w to the HJB equation (2.59) such that

$$\mathscr{L}w(x) + h(x) = 0, \quad \text{for } x \in]0,\beta[, \tag{2.60}$$

$$\text{and} \quad w(x) = w(\gamma) + \int_{\gamma}^{x} k(s)\, \mathrm{d}s - c, \quad \text{for } x \in [\beta,\infty[. \tag{2.61}$$

To determine such a solution w, we first consider the so-called "principle of smooth fit", which requires that w' should be continuous, in particular, at the free-boundary point β. This condition suggests the free-boundary equation

$$\lim_{x \uparrow \beta} w'(x) = k(\beta). \tag{2.62}$$

Next we consider the inequality

$$-c + \sup_{z \in [0,x[} \int_{x-z}^{x} \bigl(k(s) - w'(s)\bigr)\, \mathrm{d}s \leq 0,$$

which is associated with impulsive action. For $x = \beta$ and $z = \beta - u$, we can see that this implies that

$$-c + \int_{u}^{\beta} \bigl(k(s) - w'(s)\bigr)\, \mathrm{d}s \leq 0 \quad \text{for all } u \in]0,\beta].$$

This inequality and the identity

$$-c + \int_{\gamma}^{\beta} \bigl(k(s) - w'(s)\bigr)\, \mathrm{d}s = 0, \tag{2.63}$$

which follows from (2.61), can both be true if and only if the function $u \mapsto \int_u^\beta \bigl(k(s)-w'(s)\bigr)\, \mathrm{d}s$ has a local maximum at γ. This observation gives rise to the free-boundary condition

$$w'(\gamma) = k(\gamma). \tag{2.64}$$

Every solution to (2.60) that can satisfy the so-called "transversality condition", which is required for a solution w to the HJB equation to identify with the control problem's value function, is given by

$$w(x) = R_h(x) + A\psi(x), \tag{2.65}$$

for some constant A, where R_h is given by (2.21) and (2.22) for $F = h$. In view of the definition (2.18) of Θ in Assumption 2.4, the expression of R_Θ as in (2.22) and the representation (2.25), we can see that

$$R_h(x) = R_\Theta(x) + \int_0^x k(s)\,\mathrm{d}s - \mathfrak{K}_\infty \psi(x), \tag{2.66}$$

where

$$\mathfrak{K}_\infty = \lim_{x\uparrow\infty} \frac{1}{\psi(x)} \int_0^x k(s)\,\mathrm{d}s \in \mathbb{R}_+. \tag{2.67}$$

Note that the limit $\mathfrak{K}_\infty$ indeed exists in $\mathbb{R}_+$, thanks to the last condition in Assumption 2.4.(ii) and (2.24). The identity (2.66) implies that (2.65) is equivalent to

$$w(x) = R_\Theta(x) + \int_0^x k(s)\,\mathrm{d}s + (A - \mathfrak{K}_\infty)\psi(x).$$

Therefore, the solution to (2.60) that satisfies the boundary condition (2.62) is given by

$$\begin{aligned} w(x) &= \int_0^x k(s)\,\mathrm{d}s + R_\Theta(x) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi(x) \\ &= R_h(x) + \left(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right)\psi(x), \quad \text{for } x \in \,]0, \beta[. \end{aligned} \tag{2.68}$$

Furthermore, the boundary conditions (2.64) and (2.63) are equivalent to

$$\frac{R'_\Theta(\gamma)}{\psi'(\gamma)} = \frac{R'_\Theta(\beta)}{\psi'(\beta)} \quad \text{and} \quad F(\gamma, \beta) = -c, \tag{2.69}$$

respectively, where

$$F(\gamma, \beta) := G_\Theta(\beta) - G_\Theta(\gamma) = \int_\gamma^\beta \left(\frac{R'_\Theta(s)}{\psi'(s)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right)\psi'(s)\,\mathrm{d}s, \tag{2.70}$$

and G_Θ is defined by (2.27) in Lemma 2.2.2.

The following result is about the solvability of the system of equations given by (2.69) for the unknowns γ and β. Note that Lemma 2.2.4.(i) implies that a pair $0 \le \gamma < \beta < \infty$ satisfying the first equation in (2.69) might exist only if $\underline{x} < \infty$.

Lemma 2.4.1 *Consider the stochastic control problem formulated in Section 2.1 and suppose that the point $\underline{x}$ introduced in Lemma 2.2.4.(i) is finite. There exist a unique strictly decreasing function $\gamma^\star :]0, c^\star[\to]0, \underline{x}[$ and a unique strictly increasing function $\beta^\star :]0, c^\star[\to]\underline{x}, \bar{x}[$, where $c^\star > 0$ is defined by (2.80) in the proof below and $\underline{x}$, $\bar{x}$ are as in Lemma 2.2.4, such that*

$$\frac{R'_\Theta(x)}{\psi'(x)} - \frac{R'_\Theta\big(\beta^\star(c)\big)}{\psi'\big(\beta^\star(c)\big)} \begin{cases} > 0, & \text{if } x \in \big]0, \gamma^\star(c)\big[, \\ = 0, & \text{if } x = \gamma^\star(c), \\ < 0, & \text{if } x \in \big]\gamma^\star(c), \beta^\star(c)\big[, \end{cases} \quad \text{and} \quad F\big(\gamma^\star(c), \beta^\star(c)\big) = -c \tag{2.71}$$

for all $c \in \left]0, c^\star\right[$. There exist no other points $0 < \gamma < \beta < \infty$ satisfying the system of equations (2.69). The functions $\beta^\star$ and $\gamma^\star$ are such that

$$\lim_{c\downarrow 0} \beta^\star(c) = \lim_{c\downarrow 0} \gamma^\star(c) = \underline{x}, \tag{2.72}$$

$$\lim_{c\uparrow c^\star} \beta^\star(c) = \bar{x} \quad \textit{and} \quad \lim_{c\uparrow c^\star} \gamma^\star(c) \begin{cases} > 0, & \textit{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} > 0 \ (\bar{x} = \infty), \\ = 0, & \textit{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = 0 \ (\bar{x} = \infty), \\ = 0, & \textit{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \ (\bar{x} < \infty). \end{cases} \tag{2.73}$$

Furthermore, $c^\star < \infty$ if and only if

$$\textit{either} \quad \text{(I) } \bar{x} < \infty \quad \textit{or} \quad \text{(II) } \bar{x} = \infty \textit{ and } \lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} > -\infty. \tag{2.74}$$

Proof. In view of (2.39) and (2.42) in Lemma 2.2.4, we can see that there exists a point $\gamma \in \left]0, \beta\right[$ such that the first equation in (2.69) holds true if and only if $\beta \in \left]\underline{x}, \bar{x}\right[$, in which case, $\gamma \in \left]0, \underline{x}\right[$. In particular, there exists a unique strictly decreasing function $\Gamma : \left]\underline{x}, \bar{x}\right[\to \left]0, \underline{x}\right[$ such that

$$\frac{R'_\Theta(x)}{\psi'(x)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)} \begin{cases} > 0, & \text{if } x \in \left]0, \Gamma(\beta)\right[, \\ = 0, & \text{if } x = \Gamma(\beta), \\ < 0, & \text{if } x \in \left]\Gamma(\beta), \beta\right[, \end{cases} \tag{2.75}$$

$$\left(\frac{R'_\Theta}{\psi'}\right)'\big(\Gamma(\beta)\big) < 0, \qquad \left(\frac{R'_\Theta}{\psi'}\right)'(\beta) > 0, \tag{2.76}$$

$$\lim_{\beta\downarrow\underline{x}} \Gamma(\beta) = \underline{x} \quad \text{and} \quad \lim_{\beta\uparrow\bar{x}} \Gamma(\beta) \begin{cases} > 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} > 0 \ (\bar{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = 0 \ (\bar{x} = \infty), \\ = 0, & \text{if } \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} < 0 \ (\bar{x} < \infty). \end{cases} \tag{2.77}$$

It follows that the system of equations (2.69) has a unique solution $\gamma < \beta$ if and only if the equation

$$F\big(\Gamma(\beta), \beta\big) = -c \tag{2.78}$$

has a unique solution $\beta^\star(c) \in \left]\underline{x}, \bar{x}\right[$. Using the first expression in (2.70), the identity in (2.75), the second of the inequalities in (2.76) and the fact that ψ is strictly increasing, we calculate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\beta} F\big(\Gamma(\beta), \beta\big) &= -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\psi(\beta) + \left(\frac{R'_\Theta}{\psi'}\right)'\big(\Gamma(\beta)\big)\psi\big(\Gamma(\beta)\big)\Gamma'(\beta) \\ &= -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\Big(\psi(\beta) - \psi\big(\Gamma(\beta)\big)\Big) < 0. \end{aligned}$$

Combining this result with the fact that

$$\lim_{\beta\downarrow\underline{x}} F\bigl(\Gamma(\beta),\beta\bigr) = 0, \tag{2.79}$$

which follows from the first limit in (2.77), we can see that the equation $F\bigl(\Gamma(\beta),\beta\bigr) = -c$ has a unique solution $\beta^\star(c) \in \,]\underline{x},\bar{x}[$ if and only if

$$c < -\lim_{\beta\uparrow\bar{x}} F\bigl(\Gamma(\beta),\beta\bigr) =: c^\star. \tag{2.80}$$

We conclude this part of the analysis by noting that the points $\beta^\star(c) \in \,]\underline{x},\bar{x}[$ and $\gamma^\star(c) := \Gamma\bigl(\beta^\star(c)\bigr) \in \,]0,\underline{x}[$ provide the unique solution to the system of equations (2.69) if $c \in \,]0,c^\star[$, while the system of equations (2.69) has no solution such that $0<\gamma<\beta<\infty$ if $c \geq c^\star$. In particular, the inequalities in (2.71) follow from the corresponding ones in (2.75).

The fact that the function $\beta \mapsto F\bigl(\Gamma(\beta),\beta\bigr)$ is strictly decreasing, which we have established above, implies that the function $c \mapsto \beta^\star(c)$ is strictly increasing because $\beta^\star(c)$ is the unique solution to equation (2.78) for each $c \in \,]0,c^\star[$. In turn, this result and the fact that Γ is strictly decreasing imply that the function $\gamma^\star = \Gamma\circ\beta^\star$ is strictly decreasing. The first limit in (2.73) follows immediately from (2.80). On the other hand, the second limit in (2.73) follows immediately from the first limit in (2.73) and the second limit in (2.77). Furthermore, the identities in (2.72) follow from the first limit in (2.77) and (2.79).

To establish the equivalence of the inequality $c^\star < \infty$ with the condition in (2.74), we first use the first expression of F in (2.70) and the definition (2.80) of $c^\star$ to observe that

$$c^\star = -\lim_{\beta\uparrow\bar{x}}\Bigl(G_\Theta(\beta) - G_\Theta\bigl(\Gamma(\beta)\bigr)\Bigr).$$

We next use the second limit in (2.77) as well as Lemmas 2.2.2 and 2.2.4. If $\bar{x}<\infty$, then

$$c^\star = -G_\Theta(\bar{x}) + \lim_{x\downarrow 0} G_\Theta(x) \overset{(2.28)}{=} -G(\bar{x}) + \lim_{x\downarrow 0}\frac{\Theta(x)}{r(x)} < \infty,$$

the inequality following because Θ/r is strictly increasing in $]0,\xi[$. If $\bar{x}=\infty$ and $\lim_{x\downarrow 0} R'_\Theta(x)/\psi'(x) = 0$, then $\lim_{x\downarrow 0}\Theta(x)/r(x) > -\infty$ thanks to the first implication in (2.44). In this case,

$$c^\star \overset{(2.29)}{=} -\lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} + \lim_{x\downarrow 0}\frac{\Theta(x)}{r(x)} \begin{cases} <\infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} > -\infty, \\ =\infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} = -\infty, \end{cases}$$

where we have also used the assumption that Θ/r is strictly decreasing in $]\xi,\infty[$. Finally, if $\bar{x}=\infty$ and $\lim_{x\downarrow 0}\frac{R'_\Theta(x)}{\psi'(x)} > 0$, then $\lim_{x\uparrow\infty}\Gamma(x) > 0$ (see (2.77)),

$$c^\star \overset{(2.29)}{=} -\lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} + \lim_{x\uparrow\infty} G\bigl(\Gamma(x)\bigr) \begin{cases} <\infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} > -\infty, \\ =\infty, & \text{if } \lim_{x\uparrow\infty}\frac{\Theta(x)}{r(x)} = -\infty, \end{cases}$$

and the proof is complete. □

In light of (2.61), (2.68) and the previous lemma, we now establish the following result, which provides the solution to the HJB equation (2.59) identifying with the control problem's value function when a β-γ strategy with $\gamma < \beta$ in $]0, \infty[$ is indeed optimal.

Lemma 2.4.2 *Consider the stochastic control problem formulated in Section 2.1 and suppose that the point $\underline{x}$ introduced in Lemma 2.2.4.(i) is finite. Also, fix any $c \in]0, c^\star[$, where $c^\star > 0$ is as in Lemma 2.4.1. The function w defined by*

$$w(x) = \begin{cases} R_h(x) + \left(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)}\right)\psi(x), & \text{for } x \in]0, \beta^\star[, \\ w(\gamma^\star) + \int_{\gamma^\star}^x k(s)\,\mathrm{d}s - c, & \text{for } x \in [\beta^\star, \infty[, \end{cases} \tag{2.81}$$

where we write $\gamma^\star$ and $\beta^\star$ in place of the points $\gamma^\star(c)$ and $\beta^\star(c)$ given by Lemma 2.4.1, is C^1 in $]0, \infty[$ and C^2 in $]0, \infty[\setminus \{\beta^\star\}$. Furthermore, this function is a solution to the HJB equation (2.59) that is bounded from below.

Proof. The boundedness from below of w follows immediately from Assumption 2.3, the conditions in (i) and (ii) of Assumption 2.4 and Lemma 2.2.1.(i).

By construction, we will establish all of the lemma's other claims if we prove that

$$-c + \int_u^x \big(k(s) - w'(s)\big)\,\mathrm{d}s \le 0 \quad \text{for all } 0 < u < x < \beta^\star \tag{2.82}$$

$$\text{and} \quad \mathscr{L}w(x) + h(x) \le 0 \quad \text{for all } x > \beta^\star. \tag{2.83}$$

To this end, we use the first expression of w in (2.68) and (2.71) to note that

$$k(s) - w'(s) = \psi'(s)\left(\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} - \frac{R'_\Theta(s)}{\psi'(s)}\right) \begin{cases} < 0 & \text{for all } s \in]0, \gamma^\star[, \\ > 0 & \text{for all } s \in]\gamma^\star, \beta^\star[. \end{cases}$$

The inequality (2.82) follows from this observation and the fact that

$$-c + \int_{\gamma^\star}^{\beta^\star} \big(k(s) - w'(s)\big)\,\mathrm{d}s = 0.$$

To show (2.83), we first use the expression

$$w(x) = w(\beta^\star) + \int_{\beta^\star}^x k(s)\,\mathrm{d}s, \quad \text{for } x > \beta^\star,$$

the definition (2.18) of Θ in Assumption 2.4 and the first expression in (2.68) to calculate

$$\begin{aligned} \mathscr{L}w(x) + h(x) &= -r(x)w(\beta^\star) + \mathscr{L}\left(\int_0^\cdot k(s)\,\mathrm{d}s\right)(x) + r(x)\int_0^{\beta^\star} k(s)\,\mathrm{d}s + h(x) \\ &= \Theta(x) - r(x)\left(R_\Theta(\beta^\star) - \frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)}\psi(\beta^\star)\right) \\ &= \Theta(x) - r(x)G_\Theta(\beta^\star), \end{aligned} \tag{2.84}$$

where G_Θ is given by (2.27) in Lemma 2.2.2 for $F = \Theta$. In view of the calculations

$$G'_\Theta(x) = -\psi(x) \frac{\mathrm{d}}{\mathrm{d}x} \frac{R'_\Theta(x)}{\psi'(x)} = -\frac{2Cr(x)p'(x)\psi(x)}{(\sigma(x)\psi'(x))^2} \biggl(\int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s - \frac{\Theta(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)} \biggr),$$

the inequalities (2.39) in Lemma 2.2.4 and the fact that $\beta^\star > \underline{x}$, we can see that

$$G_\Theta(x) < G_\Theta(\beta^\star) \quad \text{and} \quad \int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x)}{r(x)} \frac{\psi'(x)}{Cp'(x)} \quad \text{for all } x > \beta^\star. \tag{2.85}$$

The second of these inequalities and the second expression of G_Θ in (2.27) imply that

$$G_\Theta(x) = \frac{Cp'(x)}{\psi'(x)} \int_0^x \Theta(s)\Psi(s)\,\mathrm{d}s > \frac{\Theta(x)}{r(x)} \quad \text{for all } x > \beta^\star.$$

However, this result, (2.84) and the first inequality in (2.85) yield

$$\mathscr{L}w(x) + h(x) < r(x) \biggl(\frac{\Theta(x)}{r(x)} - G_\Theta(x) \biggr) < 0 \quad \text{for all } x > \beta^\star,$$

and (2.83) follows. □

To proceed further, we assume that the problem data is such that $c^\star < \infty$, which is the case if and only if one of the two conditions of (2.74) in Lemma 2.4.1 holds true. In the first case, when $\bar{x} < \infty$, the limits in (2.73) suggest the possibility for the function w defined by (2.61) and (2.68) for $\gamma = 0$ and some $\beta > \bar{x}$ to provide a solution to the HJB equation (2.59) that identifies with the control problem's value function. In this case, a free-boundary condition such as (2.64) is not relevant anymore and we are faced with only the free-boundary condition (2.63) with $\gamma = 0$, which is equivalent to the equation $F(0, \beta) = -c$, where F is defined by (2.70).

Lemma 2.4.3 *Consider the stochastic control problem formulated in Section 2.1 and suppose that the point $\underline{x}$ introduced in Lemma 2.2.4.(i) is finite. Also, suppose that the problem data is such that $\bar{x} < \infty$, where $\bar{x}$ is defined by (2.42) in Lemma 2.2.4. The following statements hold true:*

(I) *There exists $c^\circ \in \,]c^\star, \infty]$ and a strictly increasing function $\beta^\circ : [c^\star, c^\circ[\, \to [\bar{x}, \infty[$ such that*

$$F\bigl(0, \beta^\circ(c)\bigr) = -c \text{ for all } c \in [c^\star, c^\circ[\quad \text{and} \quad \lim_{c \uparrow c^\circ} \beta^\circ(c) = \infty, \tag{2.86}$$

where $c^\star \in \,]0, \infty[$ is as in Lemma 2.4.1.

(II) *$c^\circ = \infty$ if and only if $\lim_{x \uparrow \infty} \Theta(x)/r(x) = -\infty$.*

(III) *Given any $c \in [c^\star, c^\circ[$, the function w defined by*

$$w(x) = \begin{cases} R_h(x) + \Bigl(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)} \Bigr) \psi(x), & \text{for } x \in \,]0, \beta^\circ[, \\ R_h(0) + \Bigl(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)} \Bigr) \psi(0) + \int_0^x k(s)\,\mathrm{d}s - c, & \text{for } x \in [\beta^\circ, \infty[, \end{cases} \tag{2.87}$$

where we write β° in place of $\beta^\circ(c)$, is C^1 in $]0, \infty[$ and C^2 in $]0, \infty[\setminus \{\beta^\circ\}$. Furthermore, this function is a solution to the HJB equation (2.59) that is bounded from below.

Proof. The definition of G_Θ as in (2.27), the limits (2.28) in Lemma 2.2.2 and the implications (2.44) in Lemma 2.2.4 imply that the limit $\lim_{x\downarrow 0} G_\Theta(x)$ exists in $\mathbb{R}$ thanks to Assumption 2.4.(iii). On the other hand, (2.39) and (2.43) in Lemma 2.2.4 imply that the limit $\lim_{x\downarrow 0} R'_\Theta(x)/\psi'(x)$ exists in $]-\infty, 0[$. In view of these observations and the definition of G_Θ as in (2.27), we can see that the limit $\lim_{x\downarrow 0} R_\Theta(x)$ exists in $\mathbb{R}$. Therefore, the limit $R_h(0) := \lim_{x\downarrow 0} R_h(x)$ exists in $\mathbb{R}$ thanks to (2.66). It follows that the function w is well-defined.

The second expression in (2.71) and the limits in (2.73) imply that

$$F(0, \bar{x}) \equiv G_\Theta(\bar{x}) - \lim_{x\downarrow 0} G_\Theta(x) = -c^\star \in \,]-\infty, 0[.$$

Part (I) of the lemma follows from this observation and the calculation

$$\frac{\mathrm{d}}{\mathrm{d}\beta} F(0, \beta) = -\left(\frac{R'_\Theta}{\psi'}\right)'(\beta)\big(\psi(\beta) - \psi(0)\big) < 0 \quad \text{for all } \beta \geq \bar{x},$$

where the inequality follows from (2.39) in Lemma 2.2.4 and the fact that the strictly positive function ψ is strictly increasing, for $c^\circ = -\lim_{\beta\uparrow\infty} F(0, \beta)$. Furthermore, this definition of c°, Assumption 2.4.(iii) and the limits (2.29) in Lemma 2.2.2 imply immediately part (II) of the lemma.

Finally, we can show the rest of the claims on w by using exactly the same arguments as in the proof of Lemma 2.4.2 (see (2.82) and (2.83) in particular). □

To close the "gap" in the parameter space, we still need to derive a solution to the HJB equation (2.59) if

$$\underline{x} < \bar{x} = \infty,\ c^\star < \infty \text{ and } c \geq c^\star, \quad \text{or} \quad \bar{x} < \infty,\ c^\circ < \infty \text{ and } c \geq c^\circ, \quad \text{or} \quad \underline{x} = \infty \text{ and } c > 0.$$

In the first case, the first limit in (2.73) implies that $\lim_{c\uparrow c^\star} \beta^\star(c) = \infty$. In the second case, the limit in (2.86) implies that $\lim_{c\uparrow c^\circ} \beta^\circ(c) = \infty$. In all cases, we are faced with the possibility for the problem's value function to identify with a solution to the ODE $\mathscr{L}w(x) + h(x) = 0$ for all $x > 0$.

Lemma 2.4.4 *Consider the stochastic control problem formulated in Section 2.1 and suppose that the problem data is such that one of the following cases holds true:*
(a) *The point $\underline{x}$ introduced in Lemma 2.2.4.(i) is finite,*

$$\lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} > -\infty \quad \Leftrightarrow \quad \text{either } (\bar{x} = \infty \text{ and } c^\star < \infty) \text{ or } (\bar{x} < \infty \text{ and } c^\circ < \infty) \tag{2.88}$$

and $c \geq c^\star$ or $c \geq c^\circ$, depending on the case in (2.88).
(b) *The point $\underline{x}$ introduced in Lemma 2.2.4.(i) is equal to infinity.*
In either of these two cases, the function w defined by

$$w(x) = R_h(x) + \mathfrak{K}_\infty \psi(x), \quad \text{for } x > 0, \tag{2.89}$$

is a C^2 solution to the HJB equation (2.59) that is bounded from below.

Proof. The equivalence (2.88) follows immediately from the statement related to (2.74) in Lemma 2.4.1 and part (II) of Lemma 2.4.3. On the other hand, the boundedness from below of w follows immediately from Assumption 2.3, the conditions in (i) and (ii) of Assumption 2.4 and Lemma 2.2.1.(i).

To establish the fact that w satisfies the HJB equation (2.59), we have to show that

$$\int_u^x \big(k(s) - w'(s)\big)\,\mathrm{d}s \le c \quad \Leftrightarrow \quad R_\Theta(u) - R_\Theta(x) \le c \quad \text{for all } 0 < u < x < \infty, \tag{2.90}$$

where the equivalence follows from the identity (2.66) and the definition (2.89) of w. To this end, fix any $u < x$ in $]0, \infty[$. First, suppose that $\bar{x} = \infty$ and $c^\star < \infty$. In this case, the limits in (2.73) imply that $x < \beta^\star(c)$ for all $c < c^\star$ sufficiently close to $c^\star$. For such a c, the identity (2.66) and the fact that the function w defined by (2.81) in Lemma 2.4.2 satisfies the HJB equation (2.59) imply that

$$R_\Theta(u) - R_\Theta(x) \le c + \frac{R'_\Theta\big(\beta^\star(c)\big)}{\psi'\big(\beta^\star(c)\big)}\big(\psi(u) - \psi(x)\big).$$

Passing to the limit as $c \uparrow c^\star$ and using the fact that $\lim_{c\uparrow c^\star} \beta^\star(c) = \infty$ together with the limit in (2.39), we can see that $R_\Theta(u) - R_\Theta(x) \le c^\star$. It follows that (2.90) holds true for all $c \ge c^\star$.

If $\bar{x} < \infty$, $c^\circ < \infty$ and $c \ge c^\circ$, then we can show that the function w given by (2.89) satisfies the HJB equation (2.59) in exactly the same way using the results of Lemma 2.4.3.

Finally, suppose that the point $\underline{x}$ introduced in Lemma 2.2.4.(i) is equal to infinity and consider any points $u < x < \beta$ in $]0, \infty[$. In this case, the inequalities in (2.39) imply that

$$R'_\Theta(s) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi'(s) = \psi'(s)\left(\frac{R'_\Theta(s)}{\psi'(s)} - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right) > 0 \quad \text{for all } s < \beta.$$

In view of this observation, we can see that

$$R_\Theta(u) - R_\Theta(x) \le -\frac{R'_\Theta(\beta)}{\psi'(\beta)}\big(\psi(x) - \psi(u)\big).$$

Passing to the limit as $\beta \to \infty$, we can see that $R_\Theta(u) - R_\Theta(x) \le 0$, thanks to the limit in (2.39). It follows that (2.90) holds true for all $c > 0$. □

We conclude the section with the main result of the chapter.

Theorem 2.4.5 *Consider the stochastic control problem formulated in Section 2.1. Depending on the problem data, the function w defined by (2.81), (2.87) or (2.89) in Lemmas 2.4.2, 2.4.3 or 2.4.4, respectively, identifies with the control problem's value function, namely,*

$$w(x) = \sup_{\zeta \in \mathcal{A}} J_x(\zeta). \tag{2.91}$$

Furthermore, the following cases hold true:
(I) *If the problem data is as in Lemma 2.4.2, then the β-γ strategy characterised by the points $\beta^\star$ and $\gamma^\star$ in Lemma 2.4.2 is optimal.*
(II) *If the problem data is as in Lemma 2.4.3, then there exists no optimal strategy. In this case, if (ε_n) is any sequence such that $\varepsilon_1 < \beta^\circ$ and $\lim_{n\uparrow\infty} \varepsilon_n = 0$, then the β-γ strategies characterised by the points $\beta = \beta^\circ$ and $\gamma = \varepsilon_n$, where β° is as in Lemma 2.4.3, provide a sequence of ε-optimal strategies.*
(III) *If the problem data is as in Lemma 2.4.4 and $\mathfrak{K}_\infty = 0$, then $\zeta^\star = 0$ is an optimal strategy.*
(IV) *If the problem data is as in Lemma 2.4.4 and $\mathfrak{K}_\infty > 0$, then there exists no optimal strategy. In this case, if γ is an arbitrary point in $]0,\infty[$ and (ε_n) is any sequence such that $\varepsilon_1^{-1} > \gamma$ and $\lim_{n\uparrow\infty} \varepsilon_n^{-1} = \infty$, then the β-γ strategies characterised by the points $\beta = \varepsilon_n^{-1}$ and γ provide a sequence of ε-optimal strategies.*

Proof. Fix any initial value $x > 0$, consider any admissible controlled process $\zeta \in \mathcal{A}$ and denote by X^ζ the associated solution to the SDE (1.1). Using Itô's formula, we obtain

$$e^{-\Lambda_T^\zeta} w(X_T^\zeta) = w(x) + \int_0^T e^{-\Lambda_t^\zeta} \mathscr{L} w(X_t^\zeta)\,\mathrm{d}t + \sum_{0\le t\le T} e^{-\Lambda_t^\zeta} \big(w(X_t^\zeta) - w(X_{t-}^\zeta)\big) \mathbf{1}_{\{\Delta\zeta_t > 0\}} + M_T^\zeta,$$

where

$$M_T^\zeta = \int_0^T e^{-\Lambda_t^\zeta} \sigma(X_t^\zeta) w'(X_t^\zeta)\,\mathrm{d}W_t.$$

Since $\Delta X_t^\zeta \equiv X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \le 0$, we can see that

$$\begin{aligned} w(X_t^\zeta) - w(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}} \\ = \left(\int_{X_{t-}^\zeta - \Delta\zeta_t}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u - c \right) \mathbf{1}_{\{\Delta\zeta_t>0\}}. \end{aligned}$$

In view of these observations and the fact that w satisfies the HJB equation (2.59), we derive

$$\begin{aligned} \int_0^T e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\in[0,T]} e^{-\Lambda_t^\zeta} \left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}} \right) \\ = w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + \int_0^T e^{-\Lambda_t^\zeta} \Big(\mathscr{L} w(X_t^\zeta) + h(X_t^\zeta) \Big)\,\mathrm{d}t \\ + \sum_{0\le t\le T} \left(e^{-\Lambda_t^\zeta} \int_{X_{t-}^\zeta - \Delta\zeta_t}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big)\,\mathrm{d}u - c \right) \mathbf{1}_{\{\Delta\zeta_t>0\}} + M_T^\zeta \\ \le w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + M_T^\zeta. \end{aligned} \tag{2.92}$$

We next consider any sequence (τ_n) of bounded localising times for the local martingale M^ζ. Recalling Assumption 2.4.(ii) as well as the fact that h and w are both bounded from below, we use Fatou's lemma, the monotone convergence theorem and the admissibility condition (2.5) to observe that (2.92) implies that

$$\begin{aligned} J_x(\zeta) &\le \liminf_{n\uparrow\infty} \mathbb{E}_x\left[\int_0^{\tau_n} e^{-\Lambda_t^\zeta} h(X_t^\zeta)\,\mathrm{d}t + \sum_{t\in[0,\tau_n]} e^{-\Lambda_t^\zeta}\left(\int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u)\,\mathrm{d}u - c\mathbf{1}_{\{\Delta\zeta_t>0\}}\right)\right] \\ &\le \lim_{n\uparrow\infty} \mathbb{E}_x\left[w(x) + e^{-\Lambda_{\tau_n}^\zeta} w^-(X_{\tau_n}^\zeta)\right] = w(x), \end{aligned} \tag{2.93}$$

where $w^-(x) = -\min\{0, w(x)\}$.

Proof of (I). First, consider any $x \in\,]0,\beta[$. In view of the results in Theorem 2.3.1, the β-γ strategy $\zeta^\star$ characterised by the points $\beta^\star$ and $\gamma^\star$ is such that

$$J_x(\zeta^\star) = R_h(x) + \frac{\psi(x)}{\psi(\beta^\star) - \psi(\gamma^\star)}\left(R_h(\gamma^\star) - R_h(\beta^\star) + \int_{\gamma^\star}^{\beta^\star} k(s)\,\mathrm{d}s - c\right). \tag{2.94}$$

On the other hand, the identity $F(\gamma^\star, \beta^\star) = -c$ and the definition (2.70) of F imply that

$$\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} = \frac{R_\Theta(\beta^\star) - R_\Theta(\gamma^\star) + c}{\psi(\beta^\star) - \psi(\gamma^\star)}.$$

In view of the identity (2.66), this expression is equivalent to

$$\frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)} = \frac{1}{\psi(\beta^\star) - \psi(\gamma^\star)}\left(R_h(\beta^\star) - R_h(\gamma^\star) - \int_{\gamma^\star}^{\beta^\star} k(s)\,\mathrm{d}s + c + \mathfrak{K}_\infty\big(\psi(\beta^\star) - \psi(\gamma^\star)\big)\right).$$

However, this result, the definition (2.81) of w and (2.94) imply that $J_x(\zeta^\star) = w(x)$, which, combined with (2.93), establishes (2.91) as well as the optimality of $\zeta^\star$. The corresponding claims for $x \ge \beta$ are immediate.

Proof of (II). In this case, the identity $F(0, \beta^\circ) = -c$ implies that the sequence (c_n) defined by $c_n = -F(\varepsilon_n, \beta^\circ)$ is such that $\lim_{n\uparrow\infty} c_n = c$. By following reasoning similar to the one in the previous part of the proof, we can see that, given any $x \in\,]0,\beta[$, the β-γ strategy ζ^{ε_n} characterised by the points $\beta = \beta^\circ$ and $\gamma = \varepsilon_n$ is such that

$$J_x(\zeta^{\varepsilon_n}) = w(x) - \frac{(c - c_n)\psi(x)}{\psi(\beta^\circ) - \psi(\varepsilon_n)},$$

and the required results follow.

Proof of (III). This case follows immediately from (2.93) and the probabilistic expression of R_h as in (2.21).

Proof of (IV). In view of the results in Theorem 2.3.1, the β-γ strategy ζ^{ε_n} characterised by the points $\beta = \varepsilon_n^{-1}$ and γ is such that

$$J_x(\zeta^{\varepsilon_n}) = R_h(x) + \frac{\psi(x)}{\psi(\varepsilon_n^{-1}) - \psi(\gamma)}\left(R_h(\gamma) - R_h(\varepsilon_n^{-1}) + \int_\gamma^{\varepsilon_n^{-1}} k(s)\,\mathrm{d}s - c\right).$$

Combining this observation with the second limit in (2.23) and the definition (2.67) of $\mathfrak{K}_\infty$, we can see that $\lim_{n\uparrow\infty} J_x(\zeta^{\varepsilon_n}) = R_h(x) + \mathfrak{K}_\infty\psi(x)$. However, this limit and (2.93) imply the required results. □

Remark 2.1 Suppose that we enlarged the family of admissible strategies to allow for switching the system off. In particular, suppose that we allowed for the controlled process X^ζ to hit 0 at some time and be absorbed by 0 after that time. In this context, we would face the HJB equation

$$\max\Bigg\{ \mathscr{L}w(x) + h(x),\ -c + \sup_{z\in[0,x[} \int_{x-z}^{x} \big(k(s) - w'(s)\big)\,\mathrm{d}s,\\ -w(0) - c + \frac{h(0)}{r(0)} + \int_0^x \big(k(s) - w'(s)\big)\,\mathrm{d}s \Bigg\} = 0, \tag{2.95}$$

where we assume that both of the limits $h(0) := \lim_{x\downarrow 0} h(x)$ and $r(0) := \lim_{x\downarrow 0} r(x)$ exist in $\mathbb{R}$, instead of just the limit $\lim_{x\downarrow 0} h(x)/r(x)$. The third term of this HJB equation incorporates the inequality

$$w(x) \geq -c + \int_0^x k(s)\,\mathrm{d}s + \int_0^\infty e^{-r(0)s}h(0)\,\mathrm{d}s$$

that should hold with equality for those values x of the state space at which it is optimal to switch off the system.

In view of the second limit in (2.10) and Lemma 2.2.1.(ii), if 0 is a natural boundary point, then, in all of the cases appearing in Lemmas 2.4.2-2.4.4,

$$w(0) = R_h(0) = \frac{h(0)}{r(0)}$$

and the inequality associated with the third term of (2.95) follows from the one associated with the second term of (2.95). In view of this observation, we can see that the results of Theorem 2.4.5 hold true with the following modification: in Case II, the β-0 strategy that switches off the system as soon as the uncontrolled process X takes any value greater than or equal to $\beta = \beta^\circ$ is optimal. In Case IV of the theorem, an optimal strategy still does not exist.

The situation is entirely different if 0 is an entrance boundary point. In this case, Theorem 2.4.5 with a modification such as the one in the previous paragraph still provides the solution to the control problem if the problem data is such that the solution w to the HJB equation (2.59) satisfies the inequality $w(0) \geq h(0)/r(0)$. In Example 2.8 in the next section, we can see that this inequality may or may not be true. In particular, a β-0 strategy that may switch the system off can indeed be optimal and be associated with a payoff that is strictly greater than the value function derived in Theorem 2.4.5. Investigating the solution to the control problem if we allowed for the system to be switched off would require substantial extra analysis that goes beyond the scope of the present article. □

2.5 Examples

The first four examples that we consider in this section present choices for the problem data that satisfy our assumptions. In these examples, the functions r and k are strictly positive constants, so the function Θ introduced in Assumption 2.4 takes the form

$$\Theta(x) = h(x) + kb(x) - rkx.$$

Furthermore, $\lim_{x\uparrow\infty} \Theta(x) = -\infty$ in each of the Examples 2.1-2.4, which implies that $\underline{x} < \infty$ thanks to Lemma 2.2.4.(ii), where $\underline{x} \in]\xi, \infty]$ is as in Lemma 2.2.4.(i).

Example 2.1 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = bX_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad X_0 = x > 0, \tag{2.96}$$

for some constants b and $\sigma > 0$. Furthermore, if $r > b$ and h is any bounded from below strictly concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r-b) \quad \text{and} \quad \lim_{x\uparrow\infty} h'(x) = 0,$$

then Θ is strictly concave and satisfies the requirements of Assumption 2.4.

Example 2.2 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)X_t\,\mathrm{d}t + \sigma X_t^{\ell}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [1, \frac{3}{2}]$. Note that the celebrated stochastic Verhulst-Pearl logistic model of population growth arises in the special case $\ell = 1$. Assumptions 2.1-2.3 hold true if $\ell \in]1, \frac{3}{2}]$ or if $\ell = 1$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if h is any bounded from below concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r - \kappa\gamma),$$

then Θ is strictly concave and satisfies the requirements of Assumption 2.4. □

Example 2.3 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \left(\kappa\gamma + \frac{1}{2}\sigma^2 - \kappa\ln(X_t)\right)X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constants $\kappa, \gamma, \sigma > 0$, namely, the logarithm of the uncontrolled state process is the Ornstein-Uhlenbeck process given by

$$\mathrm{d}\ln(X_t) = \kappa\big(\gamma - \ln(X_t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad \ln(X_0) = \ln(x) \in \mathbb{R}.$$

Furthermore, if h is any bounded from below concave function, then Θ is strictly concave and satisfies the requirements of Assumption 2.4. □

Example 2.4 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)\,\mathrm{d}t + \sigma X_t^{\ell}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [\frac{1}{2}, 1]$. Note that, in the special case that arises for $\ell = \frac{1}{2}$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$, the process X identifies with the short rate process in the Cox-Ingersoll-Ross interest rate model. Assumptions 2.1-2.3 hold true if $\ell \in]\frac{1}{2}, 1]$ or if $\ell = \frac{1}{2}$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if h is any bounded from below strictly concave function such that

$$\lim_{x\downarrow 0} h'(x) > k(r+\kappa) \quad \text{and} \quad \lim_{x\uparrow\infty} h'(x) = 0,$$

then Θ is strictly concave and satisfies the requirements of Assumption 2.4. □

The next three examples illustrate the four different cases that appear in Theorem 2.4.5, our main result. In the next three ones, X is the geometric Brownian motion that is given by (2.96). In this context, it is well-known that

$$\varphi(x) = x^m, \quad \psi(x) = x^n \quad \text{and} \quad p'(x) = x^{m+n-1},$$

where the constants $m < 0 < n$ are given by

$$m, n = \frac{1}{2} - \frac{b}{\sigma^2} \mp \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}},$$

while the constant C defined by (2.13) is equal to $n - m$. Furthermore, the identities

$$mn = -\frac{2r}{\sigma^2} \quad \text{and} \quad m + n = 1 - \frac{2b}{\sigma^2}$$

hold true, while

$$r < b \;\Leftrightarrow\; 0 < n < 1 \quad \text{and} \quad b = r \;\Leftrightarrow\; 1 = n.$$

Example 2.5 Suppose that $r > b$ and consider the functions

$$h(x) = x^{\alpha} \quad \text{and} \quad k(x) = 1, \quad \text{for } x > 0,$$

where $\alpha \in]0, 1[$ is a constant. In this case, the function Θ defined by (2.18) is given by

$$\Theta(x) = x^{\alpha} - (r - b)x,$$

and all of the conditions in Assumption 2.4 hold true. Furthermore,

$$R_{\Theta}(x) = \frac{2}{\sigma^2(\alpha - m)(n - \alpha)} x^{\alpha} - x,$$

which implies that

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \lim_{x\downarrow 0} \left(\frac{2\alpha}{\sigma^2 n(\alpha - m)(n-\alpha)} x^{\alpha-n} - \frac{1}{n} x^{1-n} \right) = \infty$$

because $m < 0 < \alpha < 1 < n$. In view of Lemmas 2.2.4 and 2.4.1, we can see that

$$\underline{x} < \bar{x} = c^\star = \infty.$$

Therefore, a β-γ strategy is optimal (Case I of Theorem 2.4.5) for all $c > 0$. □

Example 2.6 Suppose that $r + b - \sigma^2 > 0 \Leftrightarrow m < -1$ and $b > r$. Also, consider the functions

$$h(x) = \begin{cases} -\alpha x, & \text{if } x \in \,]0,1[, \\ -\alpha, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} 3 - 2x, & \text{if } x \in \,]0,1[, \\ x^{-2}, & \text{if } x \geq 1, \end{cases}$$

for some constant $\alpha \in \big]-\infty, 3(b-r)\big[$. In this case,

$$\Theta(x) = \begin{cases} (r - 2b - \sigma^2)x^2 + (3b - 3r - \alpha)x, & \text{if } x \in \,]0,1[, \\ (r + b - \sigma^2)x^{-1} - 3r - \alpha, & \text{if } x \geq 1, \end{cases}$$

and all of the conditions in Assumption 2.4 hold true. In view of the assumption that $m < -1$ and the identity in (2.45), we can see that $\lim_{x\uparrow\infty} Q_\Theta(x) = \infty$, which implies that $\underline{x} < \infty$ thanks to Lemma 2.2.4.(ii). Furthermore,

$$R_\Theta(x) = R_h(x) - \int_0^x k(s)\,\mathrm{d}s = \begin{cases} x^2 + \left(\frac{\alpha}{b-r} - 3\right)x - \frac{2\alpha}{\sigma^2(n-m)n(1-n)} x^n, & \text{if } x \in \,]0,1[, \\ -\frac{\alpha}{r} - 3 + x^{-1} - \frac{2\alpha}{\sigma^2(n-m)m(1-m)} x^m, & \text{if } x \geq 1, \end{cases}$$

which implies that

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = -\frac{2\alpha}{\sigma^2(n-m)n(1-n)} \in \left]-\frac{6(b-r)}{\sigma^2(n-m)n(1-n)}, \infty\right[.$$

In view of Lemmas 2.2.4, 2.4.1, 2.4.3 and 2.4.4, we can see that, if $\alpha \leq 0$, then

$$\bar{x} = \infty, \quad \text{and} \quad c^\star \in \,]0, \infty[,$$

while, if $\alpha \in \big]0, 3(b-r)\big[$, then

$$\bar{x} < \infty \quad \text{and} \quad 0 < c^\star < c^\circ = 3 + \frac{\alpha}{r}.$$

If $\alpha \leq 0$ and $c \in \,]0, c^\star[$, then a β-γ strategy is optimal (Case I of Theorem 2.4.5), while, if $\alpha \leq 0$ and $c \geq c^\star$, then no intervention at all is optimal (Case III of Theorem 2.4.5). On the other hand, if $\alpha \in \big]0, 3(b-r)\big[$, then any of the Cases I, II or III of Theorem 2.4.5 arises depending on whether $0 < c < c^\star$, $c^\star \leq c < 3 + \frac{\alpha}{r}$ or $c \geq 3 + \frac{\alpha}{r}$ is the case, respectively. □

Example 2.7 Suppose that $b = r > \frac{1}{2}\sigma^2$, which implies that $m < -1$ and $n = 1$. Also, consider the functions

$$h(x) = \begin{cases} ax^\alpha, & \text{if } x \in \,]0,1[, \\ a, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \begin{cases} 4 - 2x, & \text{if } x \in \,]0,1[, \\ x^{-2} + 1, & \text{if } x \geq 1, \end{cases}$$

for some constants $\alpha \in \,]1,2]$ and $a \in \big]0, \frac{3}{2}(2b+\sigma^2)(1-\frac{1}{\alpha}) \vee 3r\big[$. In this case, all of the conditions in Assumption 2.4 hold true,

$$\lim_{x\uparrow\infty} \psi^{-1}(x) \int_0^\infty k(s)\,\mathrm{d}s = 1, \quad \Theta(x) = \begin{cases} (r - 2b - \sigma^2)x^2 + ax^\alpha, & \text{if } x \in \,]0,1[, \\ (r + b - \sigma^2)x^{-1} - 3r + a, & \text{if } x \geq 1, \end{cases}$$

$$\text{and} \quad R_\Theta(x) = \begin{cases} x^2 + \frac{2a}{\sigma^2(n-\alpha)(\alpha-m)}x^\alpha - \big(3 + \frac{2a\alpha}{\sigma^2(n-m)n(n-\alpha)}\big)x, & \text{if } x \in \,]0,1[, \\ \frac{a}{r} - 3 + x^{-1} + \frac{2a\alpha}{\sigma^2(n-m)m(\alpha-m)}x^m, & \text{if } x \geq 1. \end{cases}$$

Furthermore,

$$\lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = -3 + \frac{2a\alpha}{(2b+\sigma^2)(\alpha-1)} < 0.$$

In view of Lemmas 2.2.4, 2.4.1, 2.4.3 and 2.4.4, we can see that

$$\underline{x} < \bar{x} < \infty \quad \text{and} \quad 0 < c^\star < c^\circ = 3 - \frac{a}{r}.$$

Any of the Cases I, II or IV of Theorem 2.4.5 may arise, depending on whether $0 < c < c^\star$, $c^\star \leq c < 3 - \frac{a}{r}$ or $c \geq 3 - \frac{a}{r}$ is the case, respectively. □

The next example shows that (2.26) in Example 2.2.1 is not necessarily true if 0 is an entrance boundary point. Furthermore, it shows that β-0 strategies would be an indispensable part of the optimal tactics if we allowed for switching off the system and 0 were an entrance boundary point (see Remark 2.1 at the end of the previous section).

Example 2.8 Suppose that X is the mean-reverting square-root process that is given by

$$\mathrm{d}X_t = \alpha(2 - X_t)\,\mathrm{d}t + \sqrt{2\alpha X_t}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constant $\alpha > 0$. Also, suppose that

$$r(x) = \alpha, \quad h(x) = \begin{cases} e^x - 1, & \text{if } x \in \,]0,1[, \\ e - e^{\gamma+3} - 1 + e^{\gamma x + 3}, & \text{if } x \geq 1, \end{cases} \quad \text{and} \quad k(x) = \kappa, \quad \text{for } x > 0,$$

for some constants $\gamma < 0$ and $\kappa \in \big]0, \frac{1}{2\alpha}\big[$. In this case,

$$\varphi(x) = \frac{1}{x}, \quad \psi(x) = \frac{e^x - 1}{x} \quad \text{and} \quad p'(x) = \frac{1}{x^2}e^{x-1}.$$

In particular, 0 is an entrance boundary point. The function Θ defined by (2.18) is given by

$$\Theta(x) = \begin{cases} 2\alpha\kappa - 1 - 2\alpha\kappa x + e^x, & \text{if } x \in \,]0,1[, \\ 2\alpha\kappa + e - e^{\gamma+3} - 1 - 2\alpha\kappa x + e^{\gamma x+3}, & \text{if } x \geq 1, \end{cases}$$

all of the conditions in Assumption 2.4 hold true,

$$\lim_{x\downarrow 0} R_h(x) = \frac{1}{\alpha}\left(1 + \frac{\gamma}{1-\gamma}e^{\gamma+2}\right) =: \frac{1}{\alpha}f(\gamma) \quad \text{and} \quad \lim_{x\downarrow 0} \frac{R'_\Theta(x)}{\psi'(x)} = \frac{1}{\alpha}f(\gamma) - 2\kappa.$$

The function f is strictly decreasing in the interval $\big]-\infty, (1-\sqrt{5})/2\big[$, strictly increasing in the interval $\big](1-\sqrt{5})/2, 0\big[$,

$$\lim_{\gamma\downarrow -\infty} f(\gamma) = 1, \quad f\left(\frac{1-\sqrt{5}}{2}\right) = 1 - \frac{1}{2}\big(3-\sqrt{5}\big)e^{(5-\sqrt{5})/2} < 0 \quad \text{and} \quad f(0) = 1.$$

Therefore, there exist constants $(1-\sqrt{5})/2 < \gamma_1 < \gamma_2 < 0$ such that $f(\gamma) < 0$ for all $\gamma \in \big[(1-\sqrt{5})/2, \gamma_1\big[$ and $f(\gamma) \in \,]0, 2\alpha\kappa[$ for all $\gamma \in \,]\gamma_1, \gamma_2[$. In view of these observations, we can see that

$$\lim_{x\downarrow 0} R_h(x) \neq 0 = \lim_{x\downarrow 0} \frac{h(x)}{r(x)} \quad \text{for all } \gamma \in \big[(1-\sqrt{5})/2, \gamma_1\big[\setminus \{\gamma_2\}, \tag{2.97}$$

which shows that (2.26) in Lemma 2.2.1 is not in general true if 0 is an entrance boundary point. On the other hand, Lemma 2.2.4 implies that, if $\gamma \in \big[(1-\sqrt{5})/2, \gamma_1\big[$, then $0 < \underline{x} < \bar{x} < \infty$ and we are in the context of Lemma 2.4.3 with $c^\circ = \infty$. In this context, (2.87) yields the expression

$$w(0) = \lim_{x\downarrow 0} w(x) = \frac{1}{\alpha}f(\gamma) - \frac{R'_\Theta(\beta^\circ)}{\psi'(\beta^\circ)}.$$

In view of (2.39) in Lemma 2.2.4, (2.86) in Lemma 2.4.3, Remark 2.1 and the analysis thus far, we can see the following:
(a) If $\gamma \in \,]\gamma_1, 0[$, then $w(0) > 0 = h(0)/r(0)$ and a β-0 strategy would be strictly sub-optimal.
(b) If $\gamma \in \big](1-\sqrt{5})/2, \gamma_1\big[$, then $w(0) < 0 = h(0)/r(0)$ for all c sufficiently large, in which case, a β-0 strategy would be optimal. □

Our final example shows that the conditions in (2.41) are only sufficient for the point $\underline{x}$ introduced in part (i) of Lemma 2.2.4 to be finite.

Example 2.9 Suppose that X is the geometric Brownian motion that is given by (2.96) with $b = \frac{1}{4}$ and $\sigma = \frac{1}{\sqrt{2}}$. Also, suppose that $r = 1$, so that $m = -2$, $n = 2$ and $C \equiv n - m = 4$. The functions defined by

$$k(x) = \begin{cases} 6 - 5x, & \text{if } x \leq 1, \\ x^{-5}, & \text{if } x > 1, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 7x, & \text{if } x \leq 1, \\ 6 + x^{-4}, & \text{if } x > 1, \end{cases}$$

are such that

$$\Theta(x) = \begin{cases} \frac{5}{2}x, & \text{if } x \le 1, \\ \frac{9}{4} + \frac{1}{4}x^{-4}, & \text{if } x > 1, \end{cases}$$

and the function Q_Θ defined by (2.40) satisfies $\lim_{x\uparrow\infty} Q_\Theta(x) = -\frac{1}{6}$. In this case, the necessary and sufficient condition of Lemma 2.2.4.(ii) implies that $\underline{x} = \infty$. On the other hand, the functions defined by

$$k(x) = \begin{cases} 6 - 5x, & \text{if } x \le 1, \\ x^{-5}, & \text{if } x > 1, \end{cases} \quad \text{and} \quad h(x) = \begin{cases} 5x, & \text{if } x \le 1, \\ 4 + x^{-4}, & \text{if } x > 1, \end{cases}$$

are such that

$$\Theta(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \le 1, \\ \frac{1}{4} + \frac{1}{4}x^{-4}, & \text{if } x > 1, \end{cases}$$

and the function Q_Θ defined by (2.40) satisfies $\lim_{x\uparrow\infty} Q_\Theta(x) = \frac{1}{6}$. In this case, the necessary and sufficient condition of Lemma 2.2.4.(ii) implies that $\underline{x} < \infty$. $\square$

Chapter 3

Singular stochastic control problems motivated by the optimal sustainable exploitation of an ecosystem

The chapter is organised as follows. In Section 3.1, we introduce the singular stochastic control problems that we solve and present three examples satisfying the assumptions that we make. In Sections 3.2–3.4, we derive the solutions to the problems' HJB equations that characterises the optimal strategy by solving suitable free-boundary problems. We fully characterise the solution to all the control problems in Section 3.5. Finally, we establish the Abelian limits in Section 3.6.

3.1 Problem formulation

In this chapter, we consider a biological system, the uncontrolled stochastic dynamics of which are also modelled by the SDE (2.1), for some deterministic functions $b, \sigma :]0, \infty[\to \mathbb{R}$. We make some similar assumptions as in Section 2.1, but with additional smooth requirements.

Assumption 3.1 The function b is C^1 and the limit $b(0) := \lim_{x \downarrow 0} b(x)$ exists in $\mathbb{R}$. On the other hand, the function σ is C^1, the limit $\sigma(0) := \lim_{x \downarrow 0} \sigma(x)$ exists in $\mathbb{R}$ and

$$0 < \sigma^2(x) \leq C_1(1 + x^\eta) \quad \text{for all } x > 0, \tag{3.1}$$

for some constant $C_1, \eta > 0$.

This assumption implies that the scale function p given by (2.2) and the speed measure m given by (2.3) of the diffusion associated with the SDE (2.1) are well-defined. We also make the following assumption, which, together with Assumption 3.1, implies that the SDE (2.1) has a unique non-explosive strong solution (e.g., see Karatzas and Shreve [52, Proposition 5.5.22]).

Assumption 3.2 The scale function p and the speed measure m defined by (2.2) and (2.3) satisfy

$$\lim_{x\downarrow 0} p(x) = -\infty, \quad \lim_{x\uparrow\infty} p(x) = \infty \quad \text{and} \quad m\big(]0,1[\big) < \infty.$$

For the solution to the ergodic control problems, we need the following additional assumption, which implies that the diffusion associated with the SDE (2.1) is ergodic.

Assumption 3.3 The integrability condition

$$\int_0^\infty \big(s^\eta + 1\big)\, m(\mathrm{d}s) < \infty$$

holds true, where $\eta > 0$ is as in (3.1).

If the system is subject to harvesting, then its state process X satisfies the controlled one-dimensional SDE (1.1).

Definition 3.1 An *admissible harvesting strategy* is any $(\mathcal{F}_t)$-adapted process ζ with càdlàg increasing sample paths such that $\zeta_{0-} = 0$ and the SDE (1.1) has a unique non-explosive strong solution. We denote by $\mathcal{A}$ the family of all admissible strategies.

With each admissible harvesting strategy $\zeta \in \mathcal{A}$, we associate the expected discounted performance index $I_x(\zeta)$ given by (1.4), the expected ergodic performance index J_x^{e} given by (1.5) and the pathwise performance criterion J_x^{p} given by (1.6). The objective of the control problem that we consider is to maximise each of $I_x(\zeta)$, $J_x^{\mathrm{e}}(\zeta)$ and $J_x^{\mathrm{p}}(\zeta)$ over all $\zeta \in \mathcal{A}$.

Assumption 3.4 (i) The function h is C^1 as well as bounded from below and the limit $h(0) := \lim_{x\downarrow 0} h(x)$ exists in $\mathbb{R}$.
(ii) The function k is positive, bounded and C^2. Also, the limits $k(0) := \lim_{x\downarrow 0} k(x)$ and $k'(0) := \lim_{x\downarrow 0} k'(x)$ both exists in $\mathbb{R}_+$.

For the discounted control problem, we make the following additional assumption.

Assumption 3.5 (i) The discounting rate function r is bounded and C^1. Also, there exists $r_0 > 0$ such that $r(x) \geq r_0$ for all $x \geq 0$.
(ii) The limit $\lim_{x\downarrow 0} h(x)/r(x)$ exists in $\mathbb{R}$ and

$$\mathbb{E}_x\left[\int_0^\infty e^{-\Lambda_t} \big|h(X_t)\big|\, \mathrm{d}t\right] < \infty.$$

(iii) The conditions in (2.16) hold true.
(iv) The limit $\lim_{x\downarrow 0} \mathfrak{K}(x)/r(x)$ exists in $\mathbb{R}$, where $\mathfrak{K}$ is given by (2.17), and there exists a constant $\xi \in\,]0,\infty[$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\Theta(x)}{r(x)} \begin{cases} > 0, & \text{for } x \in\,]0,\xi[, \\ < 0, & \text{for } x \in\,]\xi,\infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{\Theta(x)}{r(x)} < \frac{\Theta(0)}{r(0)},$$

where Θ is defined by (2.18).

For the ergodic control problems, we make the following additional assumption.

Assumption 3.6 (i) The following integrability condition is satisfied:

$$\int_0^\infty |h(s)|\, m(\mathrm{d}s) < \infty.$$

(ii) If we define

$$K(x) = \frac{1}{2}\sigma^2(x)k'(x) + b(x)k(x), \quad \text{for } x > 0, \tag{3.2}$$

then there exists a constant $\xi \in \,]0, \infty[$ such that

$$K'(x) + h'(x) \begin{cases} > 0, & \text{for } x \in \,]0, \xi[, \\ < 0, & \text{for } x \in \,]\xi, \infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty} \bigl(K(x) + h(x)\bigr) < K(0) + h(0).$$

Remark 3.1 In the presence of Assumptions 3.2 and 3.4.(ii), the definitions of the scale function p and the speed measure m imply that

$$\int_0^x b(s)k(s)\, m(\mathrm{d}s) = \int_0^x k(s)\left(\frac{1}{p'}\right)'(s)\, \mathrm{d}s = \frac{k(x)}{p'(x)} - \frac{1}{2}\int_0^x \sigma^2(s)k'(s)\, m(\mathrm{d}s).$$

In turn, these identities and the definition (3.2) of K imply that

$$\int_0^x K(s)\, m(\mathrm{d}s) = \frac{k(x)}{p'(x)}. \tag{3.3}$$

Remark 3.2 In view of Assumption 3.6.(ii), we define

$$\underline{\lambda} = \lim_{x\uparrow\infty} K(x) + h(x) \quad \text{and} \quad \overline{\lambda} = K(\xi) + h(\xi), \tag{3.4}$$

and we note that the equation $K(x) + h(x) - \lambda = 0$ has

- no strictly positive solutions if $\lambda > \overline{\lambda}$,
- two strictly positive solutions if $\lambda \in \bigl]K(0) + h(0),\, \overline{\lambda}\bigr[$, and
- one strictly positive solution if $\lambda \in \bigl]\underline{\lambda},\, K(0) + h(0)\bigr]$ or $\lambda = \overline{\lambda}$

(see also Figure 3.1). In particular, there exists a unique function ϱ such that

$$\xi < \varrho(\lambda) \quad \text{and} \quad K\bigl(\varrho(\lambda)\bigr) + h\bigl(\varrho(\lambda)\bigr) - \lambda = 0 \quad \text{for all } \lambda \in \,]\underline{\lambda}, \overline{\lambda}[. \tag{3.5}$$

Furthermore, this function is such that

$$K(x) + h(x) - \lambda \begin{cases} > 0, & \text{for all } x \in \bigl[\xi, \varrho(\lambda)\bigr[, \\ < 0, & \text{for all } x > \varrho(\lambda), \end{cases} \tag{3.6}$$

and

$$K'\bigl(\varrho(\lambda)\bigr) + h'\bigl(\varrho(\lambda)\bigr) < 0 \quad \text{for all } \lambda \in \,]\underline{\lambda}, \overline{\lambda}[. \tag{3.7}$$

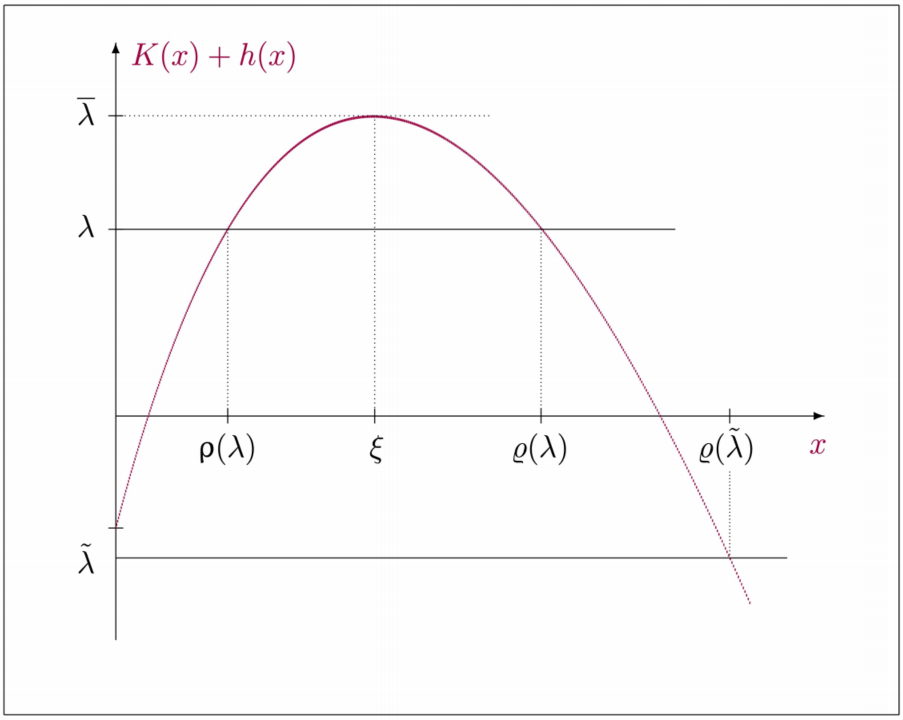

Figure 3.1: Notation associated with the graph of the function $K(\cdot) + h(\cdot)$.

On the other hand, there is a unique function ρ such that

$$0 < \rho(\lambda) < \xi \quad \text{and} \quad K\big(\rho(\lambda)\big) + h\big(\rho(\lambda)\big) - \lambda = 0 \quad \text{for all } \lambda \in \big]K(0) + h(0),\, \overline{\lambda}\big[. \tag{3.8}$$

Given any $\lambda \in \big]K(0) + h(0),\, \overline{\lambda}\big[$, this function is such that

$$K(x) + h(x) - \lambda < 0 \quad \text{for all } x \in \big]0, \rho(\lambda)\big[. \tag{3.9}$$

We conclude this section with the following three examples.

Example 3.1 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t)X_t\,\mathrm{d}t + \sigma X_t^{\ell}\,\mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [1, \frac{3}{2}]$. Note that the celebrated stochastic Verhulst-Pearl logistic model of population growth arises in the special case $\ell = 1$. The derivative of the scale function admits the expression

$$p'(x) = x^{-2\kappa\gamma/\sigma^2} \exp\left(\frac{2\kappa}{\sigma^2}(x-1)\right), \quad \text{if } \ell = 1,$$

$$p'(x) = \exp\left(\frac{\kappa\gamma}{(\ell-1)\sigma^2}\left[x^{-2(\ell-1)} - 1\right] + \frac{2\kappa}{(3-2\ell)\sigma^2}\left[x^{3-2\ell} - 1\right]\right), \quad \text{if } \ell \in \,]1, 1.5[,$$

$$\text{and} \quad p'(x) = x^{2\kappa/\sigma^2} \exp\left(\frac{2\kappa\gamma}{\sigma^2}(x^{-1} - 1)\right), \quad \text{if } \ell = 1.5.$$

Assumptions 3.1–3.3 hold true if $\ell \in \,]1, \frac{3}{2}]$ or if $\ell = 1$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if r, k are constant and either (a) $h = 0$ and $r < \kappa\gamma$, or (b) h is a strictly concave function satisfying the Inada conditions

$$\lim_{x \downarrow 0} h'(x) = \infty \quad \text{and} \quad \lim_{x \uparrow \infty} h'(x) = 0, \tag{3.10}$$

as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 3.4–3.6 are satisfied.

Example 3.2 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \left(\kappa\gamma + \frac{1}{2}\sigma^2 - \kappa \ln(X_t)\right) X_t \, \mathrm{d}t + \sigma X_t \, \mathrm{d}W_t, \quad X_0 = x > 0,$$

for some constants $\kappa, \gamma, \sigma > 0$, namely, the logarithm of the uncontrolled state process is the Ornstein-Uhlenbeck process given by

$$\mathrm{d}\ln(X_t) = \kappa\big(\gamma - \ln(X_t)\big) \, \mathrm{d}t + \sigma \, \mathrm{d}W_t, \quad \ln(X_0) = \ln(x) \in \mathbb{R}.$$

In this case, the derivative of scale function admits the expression

$$p'(x) = x^{\frac{\kappa}{\sigma^2}\ln(x) - \frac{2\kappa\gamma}{\sigma^2} - 1}$$

and all of Assumptions 3.1–3.3 hold true. Furthermore, if r, k are constant and either $h = 0$ or h is a strictly concave function satisfying the Inada conditions (3.10), as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 3.4–3.6 are satisfied.

Example 3.3 Suppose that the uncontrolled dynamics of the state process are modelled by the SDE

$$\mathrm{d}X_t = \kappa(\gamma - X_t) \, \mathrm{d}t + \sigma X_t^{\ell} \, \mathrm{d}W_t, \quad X_0 = x > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [\frac{1}{2}, 1]$. Note that, in the special case that arises for $\ell = \frac{1}{2}$ and $\kappa\gamma - \frac{1}{2}\sigma^2 > 0$, the process X identifies with the short rate process in the Cox-Ingersoll-Ross interest rate model. The derivative of the scale function admits the expression

$$p'(x) = x^{-2\kappa\gamma/\sigma^2} \exp\left(\frac{2\kappa}{\sigma^2}(x-1)\right), \quad \text{if } \ell = 0.5,$$

$$p'(x) = \exp\left(\frac{2\kappa\gamma}{(2\ell-1)\sigma^2}\left(x^{-(2\ell-1)} - 1\right) + \frac{\kappa}{(1-\ell)\sigma^2}\left(x^{2(1-\ell)} - 1\right)\right), \quad \text{if } \ell \in \,]0.5, 1[,$$

$$\text{and} \quad p'(x) = x^{2\kappa/\sigma^2} \exp\left(\frac{2\kappa\gamma}{\sigma^2}(x^{-1} - 1)\right), \quad \text{if } \ell = 1.$$

Assumptions 3.1–3.3 hold true if $\ell \in \,]\frac{1}{2}, 1]$ or if $\ell = \frac{1}{2}$ and $k\gamma - \frac{1}{2}\sigma^2 > 0$. Furthermore, if r, k are constant and h is any strictly concave function satisfying the Inada conditions (3.10), as well as the inequality $h(0) > -\infty$, then all of the conditions in Assumptions 3.4–3.6 are satisfied.

3.2 The HJB equations of the control problems

We will solve the discounted control problem by deriving a suitable C^2 solution w to the HJB equation

$$\max\left\{\mathscr{L}w(x) + h(x),\ k(x) - w'(x)\right\} = 0, \tag{3.11}$$

where $\mathscr{L}$ is the differential operator defined by (2.6), that identifies with the control problem's value function. In particular, we will construct a solution to this HJB equation such that

$$\sup_{\zeta \in \mathcal{A}} I_x(\zeta) = w(x) \quad \text{for all } x > 0.$$

On the other hand, we will solve both of the ergodic control problems by constructing a C^2 function w and finding a constant λ such that the HJB equation

$$\max\left\{\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda,\ k(x) - w'(x)\right\} = 0 \tag{3.12}$$

holds true for all $x > 0$. Given such a solution, we will prove that

$$\sup_{\zeta \in \mathcal{A}} J_x^{\mathrm{e}}(\zeta) = \lambda \quad \text{and} \quad \sup_{\zeta \in \mathcal{A}} J_x^{\mathrm{p}}(\zeta) = \lambda \quad \text{for all } x > 0.$$

Given suitable solutions to the HJB equations, the optimal strategies can be characterised as follows. In the discounted control problem, the controller should wait and take no action for as long as the state process X takes values in the interior of the set in which the ODE

$$\mathscr{L}w(x) + h(x) = 0 \tag{3.13}$$

is satisfied. On the other hand, the "no-action" region of the ergodic control problems is the interior of the set in which the ODE

$$\frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda = 0 \tag{3.14}$$

is satisfied. In all problems, the controller should take the minimal action required so that the state process is kept outside the interior of the set defined by $w'(x) = k(x)$ at all times.

We are going to establish that, in all of the problems that we consider, the optimal strategy takes the following qualitative form. There exists a point β in the state space $]0, \infty[$ such that it is optimal to push in an impulsive way the state process down to level β if the initial state x is strictly greater than β and otherwise take minimal action so that the state process X is kept inside the set $]0, \beta]$ at all times, which amounts at reflecting X in β in the negative direction. In view of the discussion in the previous paragraph, the optimality of such a strategy is associated with a solution w to the HJB equation (3.11) such that

$$\begin{aligned} \mathscr{L}w(x) + h(x) &= 0, \quad \text{for } x \in\,]0, \beta[, \\ w'(x) &= k(x), \quad \text{for } x \in [\beta, \infty[, \end{aligned}$$

and a solution (w, λ) to the HJB equation (3.12) such that

$$\begin{aligned} \frac{1}{2}\sigma^2(x)w''(x) + b(x)w'(x) + h(x) - \lambda &= 0, \quad \text{for } x \in\,]0, \beta[, \\ w'(x) &= k(x), \quad \text{for } x \in [\beta, \infty[. \end{aligned}$$

In both cases, we will determine the free-boundary point β using the so-called "smooth pasting condition" of singular stochastic control, which requires that w should be C^2, in particular, at the free-boundary point β. This condition suggests the free-boundary equations

$$\lim_{x \uparrow \beta} w'(x) = k(\beta) \quad \text{and} \quad \lim_{x \uparrow \beta} w''(x) = k'(\beta). \tag{3.15}$$

3.3 The solution to the discounted harvesting problem

In view of the arguments in Section 2.4, we consider the solution to the ODE (3.13) that is given by

$$\begin{aligned} w(x) &= \int_0^x k(s)\,\mathrm{d}s + R_\Theta(x) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi(x) \\ &= R_h(x) + \left(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\right)\psi(x), \quad \text{for } x \in\,]0, \beta]. \end{aligned} \tag{3.16}$$

Furthermore, this function satisfies the second boundary condition in (3.15) if and only if $\beta > 0$ is a solution to the algebraic equation

$$R''_\Theta(\beta) - \frac{R'_\Theta(\beta)}{\psi'(\beta)}\psi''(\beta) = 0. \tag{3.17}$$

Lemma 3.3.1 *Consider the discounted control problem formulated in Section 3.1. There exists a unique point $\beta^\star \in]\xi, \infty[$ satisfying equation (3.17), which is equivalent to*

$$\int_0^\beta \Theta(s)\Psi(s)\,\mathrm{d}s = \frac{\Theta(\beta)}{r(\beta)} \int_0^\beta r(s)\Psi(s)\,\mathrm{d}s. \tag{3.18}$$

Furthermore, the function w defined by

$$w(x) = \begin{cases} R_h(x) + \left(\mathfrak{K}_\infty - \frac{R'_\Theta(\beta^\star)}{\psi'(\beta^\star)}\right)\psi(x), & \text{for } x \in]0, \beta^\star], \\ w(\beta^\star) + \int_{\beta^\star}^x k(s)\,\mathrm{d}s, & \text{for } x \in]\beta^\star, \infty[, \end{cases} \tag{3.19}$$

is a C^2 solution to the HJB equation (3.11) that is bounded from below.

Proof. In the presence of the assumptions that we have made, Lemma 2.2.4.(i) shows that that there exists a unique point $\beta^\star = \underline{x} \in]\xi, \infty[$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{R'_\Theta(x)}{\psi'(x)} \begin{cases} < 0 & \text{for all } x \in]0, \beta^\star[, \\ > 0 & \text{for all } x \in]\beta^\star, \infty[, \end{cases} \quad \text{and} \quad \lim_{x\uparrow\infty} \frac{R'_\Theta(x)}{\psi'(x)} = 0.$$

The first set of these inequalities and the continuity of the derivative $(R'_\Theta/\psi')'$ imply that $\beta^\star$ is the unique solution to equation (3.17). To establish the equivalence of (3.17) and (3.18), we use (2.37) to see that that (3.17) is equivalent to

$$\int_0^\beta \Theta(s)\Psi(s)\,\mathrm{d}s = \frac{\Theta(\beta)}{r(\beta)} \frac{\psi'(\beta)}{Cp'(\beta)}.$$

Combining this observation with (2.31), we obtain the equivalence of (3.17) and (3.18).

The rest results are straightforward by using arguments similar to the ones in Lemma 2.4.2. □

Theorem 3.3.2 *Consider the discounted control problem formulated in Section 3.1. If the point $\beta^\star \in]\xi, \infty[$ and the function w are as in Lemma 3.3.1, then*

$$w(x) = \sup_{\zeta\in\mathcal{A}} J_x(\zeta) \quad \text{for all } x > 0, \tag{3.20}$$

while the harvesting strategy $\zeta^\star \in \mathcal{A}$ that has a jump of size $\Delta\zeta_0^\star = (x - \beta^\star)^+$ at time 0 and then reflects the state process $X^\star$ at the level $\beta^\star$ in the negative direction is optimal.

Proof. Fix any initial value $x > 0$, consider any admissible controlled process $\zeta \in \mathcal{A}$ and denote by X^ζ the associated solution to the SDE (1.1). Using Itô's formula, we obtain

$$\begin{aligned} e^{-\Lambda_T^\zeta} w(X_T^\zeta) = w(x) &+ \int_0^T e^{-\Lambda_t^\zeta}\left(\frac{1}{2}\sigma^2(X_t^\zeta)w''(X_t^\zeta) + b(X_t^\zeta)w'(X_t^\zeta) - r(X_t^\zeta)w(X_t^\zeta)\right)\mathrm{d}t \\ &- \int_{[0,T]} e^{-\Lambda_t^\zeta} w'(X_{t-}^\zeta)\,\mathrm{d}\zeta_t + \sum_{0\le t\le T} e^{-\Lambda_t^\zeta}\left(w(X_t^\zeta) - w(X_{t-}^\zeta) - w'(X_{t-}^\zeta)\,\Delta X_t^\zeta\right) \\ &+ M_T^\zeta, \end{aligned}$$

where

$$M_T^\zeta = \int_0^T e^{-\Lambda_t^\zeta} \sigma(X_t^\zeta) w'(X_t^\zeta) \, \mathrm{d}W_t.$$

Since $\Delta X_t^\zeta \equiv X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \leq 0$, we can see that

$$w(X_t^\zeta) - w(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u) \, \mathrm{d}u = \int_{X_t^\zeta}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big) \, \mathrm{d}u.$$

In view of the facts that ζ^c is an increasing process, $X_t^\zeta < X_{t-}^\zeta$ and w satisfies the HJB equation (3.11), we can see that these observations imply that

$$\begin{aligned}
&\int_0^T e^{-\Lambda_t^\zeta} h(X_t^\zeta) \, \mathrm{d}t + \int_0^T e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t \\
&\quad = w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) \\
&\qquad + \int_0^T e^{-\Lambda_t^\zeta} \left(\frac{1}{2}\sigma^2(X_t^\zeta) w''(X_t^\zeta) + b(X_t^\zeta) w'(X_t^\zeta) - r(X_t^\zeta) w(X_t^\zeta) + h(X_t^\zeta) \right) \mathrm{d}t \\
&\qquad + \int_0^T e^{-\Lambda_t^\zeta} \big(k(X_t^\zeta) - w'(X_t^\zeta)\big) \, \mathrm{d}\zeta_t^{\mathrm{c}} + \sum_{0 \leq t \leq T} e^{-\Lambda_t^\zeta} \int_{X_t^\zeta}^{X_{t-}^\zeta} \big(k(u) - w'(u)\big) \, \mathrm{d}u + M_T^\zeta \\
&\quad \leq w(x) - e^{-\Lambda_T^\zeta} w(X_T^\zeta) + M_T^\zeta.
\end{aligned} \tag{3.21}$$

We next consider any sequence (τ_n) of bounded localising stopping times for the local martingale M^ζ. Recalling that h and w are both bounded from below, k is positive and ζ is an increasing process, we use the dominated and the monotone convergence theorems to observe that (3.21) implies that

$$\begin{aligned}
I_x(\zeta) &= \lim_{n\uparrow\infty} \mathbb{E}_x \left[\int_0^{\tau_n} e^{-\Lambda_t^\zeta} h(X_t^\zeta) \, \mathrm{d}t + \int_0^{\tau_n} e^{-\Lambda_t^\zeta} k(X_t^\zeta) \circ \mathrm{d}\zeta_t \right] \\
&\leq \lim_{n\uparrow\infty} \mathbb{E}_x \Big[w(x) + e^{-\Lambda_{\tau_n}^\zeta} w^-(X_{\tau_n}^\zeta) \Big] = w(x),
\end{aligned} \tag{3.22}$$

where $w^-(x) = -\min\big\{0, w(x)\big\}$.

Consider the harvesting strategy $\zeta^\star \in \mathcal{A}$ that is as in the statement of the theorem: such a strategy indeed exists (see Tanaka [84, Theorem 4.1]). This strategy is such that (3.21) holds true with equality, namely,

$$\int_0^T e^{-\Lambda_t^\star} h(X_t^\star) \, \mathrm{d}t + \int_0^T e^{-\Lambda_t^\star} k(X_t^\star) \circ \mathrm{d}\zeta_t^\star = w(x) - e^{-\Lambda_T^\star} w(X_T^\star) + M_T^\star.$$

Furthermore, the processes $h(X^\star)$, $k(X^\star)$ and $w(X^\star)$ are all bounded because $X_t^\star \in \,]0, \beta^\star]$ for all $t > 0$. In view of these observations, we can use the dominated convergence theorem to obtain

$$
\begin{aligned}
I_x(\zeta^\star) &= \lim_{n\uparrow\infty} \mathbb{E}_x\left[\int_0^{\tau_n^\star} e^{-\Lambda_t^\star} h(X_t^\star)\,\mathrm{d}t + \int_0^{\tau_n^\star} e^{-\Lambda_t^\star} k(X_t^\star)\circ \mathrm{d}\zeta_t^\star\right] \\
&= \lim_{n\uparrow\infty} \mathbb{E}_x\left[w(x) - e^{-\Lambda_{\tau_n^\star}^\star} w\big(X_{\tau_n^\star}^\star\big)\right] = w(x),
\end{aligned}
$$

where $(\tau_n^\star)$ is a sequence of bounded localising stopping times for the local martingale $M^\star$. However, these identities and (3.22) imply (3.20) as well as the optimality of $\zeta^\star$. □

3.4 The solution to the ergodic problem's HJB equation

In view of (3.3) in Remark 3.1, we can verify that a solution to the ODE (3.14) is given by

$$
w'(x) \equiv w'(x;\lambda) = p'(x)\int_0^x \big(\lambda - h(s)\big)\,m(\mathrm{d}s) = k(x) - p'(x)\Xi(x,\lambda), \tag{3.23}
$$

where

$$
\Xi(x,\lambda) = \int_0^x \big(K(s) + h(s) - \lambda\big)\,m(\mathrm{d}s). \tag{3.24}
$$

This function satisfies the boundary conditions (3.15) if and only if

$$
\Xi(\beta,\lambda) = 0 \quad \text{and} \quad K(\beta) + h(\beta) - \lambda = 0. \tag{3.25}
$$

In the next result, we derive a unique solution to the system of equations in (3.25) as well as a solution to the HJB equation (3.12). It turns out that the solution to the HJB equation (3.12) may be unbounded from below (see Remark 3.4 at the end of the section), which gives rise to a non-trivial complication in the verification arguments we use for the proof of Theorem 3.5.1. Part (III) of the next result provides a way to overcome this complication.

Proposition 3.4.1 *In the presence of Assumptions 3.1, 3.2, 3.4 and 3.6, the following statements hold true:*

(I) *There exists a unique pair $(\beta^\star, \lambda^\star)$ with $\beta^\star > 0$ satisfying the system of equations (3.25). This pair is such that*

$$
K(0) + h(0) < \lambda^\star = \frac{1}{m\big(]0,\beta^\star[\big)}\int_0^{\beta^\star} \big(K(s)+h(s)\big)\,m(\mathrm{d}s) < \overline{\lambda} \quad \textit{and} \quad \beta^\star = \varrho(\lambda^\star), \tag{3.26}
$$

where $\overline{\lambda}$ and ϱ are given by (3.4) and (3.5).

(II) *The unique, modulo an additive constant, function w that is defined by*

$$
w'(x) = \begin{cases} k(x) - p'(x)\Xi(x,\lambda^\star), & \textit{for } x \in\,]0,\beta^\star[, \\ k(x), & \textit{for } x \geq \beta^\star, \end{cases} \tag{3.27}
$$

is a C^2 solution to the HJB equation (3.12).
(III) *Given any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$, there exists a point $\alpha(\lambda) \in \, \big]0, \rho(\lambda)\big[$, where ρ is as in (3.8), such that the unique, modulo an additive constant, function w_λ that is defined by*

$$w'_\lambda(x) = \begin{cases} k(x) - p'(x) \int_{\alpha(\lambda)}^x \big(K(s) + h(s) - \lambda\big) \, m(\mathrm{d}s), & \text{for } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \\ k(x), & \text{for } x \in \big]0, \alpha(\lambda)\big] \cup \big[\varrho(\lambda), \infty\big[, \end{cases} \tag{3.28}$$

is C^1 in $\mathbb{R}_+$ and C^2 in $\mathbb{R}_+ \setminus \big\{\alpha(\lambda), \varrho(\lambda)\big\}$,

$$\lim_{\lambda \downarrow \lambda^\star} \alpha(\lambda) = 0, \quad \lim_{\lambda \downarrow \lambda^\star} \varrho(\lambda) = \beta^\star \quad \text{and} \quad \lim_{\lambda \downarrow \lambda^\star} w'_\lambda(x) = w'(x) \text{ for all } x > 0. \tag{3.29}$$

Furthermore, this function is such that

$$w'_\lambda(x) \geq k(x), \quad \text{if } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \tag{3.30}$$

$$\frac{1}{2}\sigma^2(x) w''_\lambda(x) + b(x) w'_\lambda(x) + h(x) - \lambda \begin{cases} = 0, & \text{if } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[, \\ < 0, & \text{if } x \in \big]0, \alpha(\lambda)\big[\cup \big]\varrho(\lambda), \infty\big[, \end{cases} \tag{3.31}$$

$$\text{and} \quad \big|w'_\lambda(x)\big| \leq C_2 \quad \text{for all } x > 0, \tag{3.32}$$

for some constant $C_2 = C_2(\lambda) > 0$.

Proof. We develop the proof in four steps.

<u>*Preliminary results*</u>. Given any $\beta > 0$, the definition (3.24) of Ξ implies that

$$\lambda(\beta) = \frac{1}{m\big(]0, \beta[\big)} \int_0^\beta \big(K(s) + h(s)\big) \, m(\mathrm{d}s)$$

is the unique solution to the equation $\Xi(\beta, \lambda) = 0$. In light of Assumption 3.6.(ii) (see also Figure 3.1), a straightforward inspection of the definition of Ξ reveals that this solution is such that one of the following two cases holds true:

$$\text{(i) } K(0) + h(0) < \lambda(\beta) < \overline{\lambda} \quad \text{or} \quad \text{(ii) } \underline{\lambda} < \lambda(\beta) \leq K(0) + h(0) \text{ and } \varrho\big(\lambda(\beta)\big) < \beta, \tag{3.33}$$

where $\underline{\lambda} < \overline{\lambda}$ are defined by (3.4) and ϱ is introduced by (3.5). In particular, we note that $\lambda(\beta) \in \,]\underline{\lambda}, \overline{\lambda}[$, which is the domain of the function ϱ.

Differentiating the identity

$$\Xi\big(\beta, \lambda(\beta)\big) = 0, \quad \text{for } \beta > 0, \tag{3.34}$$

which defines λ, with respect to β, we calculate

$$\lambda'(\beta) = \frac{2}{\sigma^2(\beta) p'(\beta) m\big(]0, \beta[\big)} \big(K(\beta) + h(\beta) - \lambda(\beta)\big).$$

On the other hand, differentiating the identity

$$K\big(\varrho(\lambda(\beta))\big) + h\big(\varrho(\lambda(\beta))\big) - \lambda(\beta) = 0,$$

which follows from (3.5), with respect to β, we derive the expression

$$\lambda'(\beta) = \Big(K'\big(\varrho(\lambda(\beta))\big) + h'\big(\varrho(\lambda(\beta))\big)\Big) \frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\lambda(\beta)\big).$$

Combining these calculations, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\lambda(\beta)\big) = \frac{2\big(K(\beta) + h(\beta) - \lambda(\beta)\big)}{\sigma^2(\beta)p'(\beta)m\big(]0,\beta[\big)\big(K'\big(\varrho(\lambda(\beta))\big) + h'\big(\varrho(\lambda(\beta))\big)\big)}.$$

In view of this result and the inequality

$$K'\big(\varrho(\lambda(\beta))\big) + h'\big(\varrho(\lambda(\beta))\big) < 0 \quad \text{for all } \beta > 0,$$

which follows from (3.7), we can see that

$$\operatorname{sgn}\left(\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\lambda(\beta)\big)\right) = -\operatorname{sgn}\Big(K(\beta) + h(\beta) - \lambda(\beta)\Big) \quad \text{for all } \beta > 0, \tag{3.35}$$

where sgn is the sign function defined by

$$\operatorname{sgn}(x) = \begin{cases} \frac{x}{|x|}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

Proof of (I). In view of (3.34), we can see that there exists a pair $(\beta^\star, \lambda^\star)$ with $\beta^\star > 0$ satisfying the system of equations (3.25) if and only if

$$K(\beta^\star) + h(\beta^\star) - \lambda^\star = 0 \quad \text{and} \quad \lambda^\star = \lambda(\beta^\star). \tag{3.36}$$

The structure of the function $K + h$, which we have discussed in Remark 3.2 (see also Figure 3.1), implies that there exists no $\beta^\star$ satisfying (3.36) if $\lambda(\beta^\star)$ is as in case (ii) of (3.33). We therefore need to show that there exists $\beta^\star > 0$ such that, if we define $\lambda^\star = \lambda(\beta^\star)$, then $\lambda^\star$ satisfies the inequalities (3.26), and

$$\text{either} \quad \beta^\star = \rho\big(\lambda(\beta^\star)\big) = \rho(\lambda^\star) \quad \text{or} \quad \beta^\star = \varrho\big(\lambda(\beta^\star)\big) = \varrho(\lambda^\star), \tag{3.37}$$

where ϱ, ρ are as in (3.5), (3.8). Furthermore, the resulting solution $(\beta^\star, \lambda^\star)$ to the system of equations (3.25) is unique if and only if only one of the two equations in (3.37) has a unique solution and the other one has no solution.

If the equation $\beta = \rho\big(\lambda(\beta)\big)$ had a solution $\beta^\star > 0$, then (3.9) would imply that

$$K(s) + h(s) - \lambda(\beta^\star) < 0 \quad \text{for all } s < \rho\big(\lambda(\beta^\star)\big) = \beta^\star,$$

which would contradict the identity

$$\Xi\big(\beta^\star, \lambda(\beta^\star)\big) \equiv \int_0^{\beta^\star} \big(K(s) + h(s) - \lambda(\beta^\star)\big)\, m(\mathrm{d}s) = 0.$$

To establish part (I) of the theorem, we therefore have to prove that there exists a unique point $\beta^\star > 0$ such that

$$\lambda(\beta^\star) \in \big]K(0) + h(0), \overline{\lambda}\big[\quad \text{and} \quad \beta^\star = \varrho\big(\lambda(\beta^\star)\big). \tag{3.38}$$

To prove that there exists a unique $\beta^\star > 0$ satisfying (3.38), we first observe that the inequality in (3.5) implies that

$$\beta < \varrho\big(\lambda(\beta)\big) \quad \text{for all } \beta \leq \xi. \tag{3.39}$$

We next argue by contradiction and we assume that there is no $\beta^\star > 0$ satisfying the equation in (3.38). In view of (3.39) and the continuity of the functions ϱ and λ, we can see that such an assumption implies that

$$\beta < \varrho\big(\lambda(\beta)\big) \quad \text{for all } \beta > \xi. \tag{3.40}$$

In turn, this inequality and (3.6) imply that

$$K(\beta) + h(\beta) - \lambda(\beta) > 0 \quad \text{for all } \beta > \xi.$$

Combining this observation with (3.35), we obtain $\frac{\mathrm{d}}{\mathrm{d}\beta}\varrho\big(\lambda(\beta)\big) < 0$ for all $\beta > \xi$. Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\Big(\beta - \varrho\big(\lambda(\beta)\big)\Big) > 1 \quad \text{for all } \beta > \xi,$$

which contradicts (3.40). It follows that there exists $\beta^\star > 0$ satisfying the equation in (3.38).

To see that the solution $\beta^\star > \xi$ to the equation in (3.38) is indeed unique, we note that (3.5) implies that $K(\beta) + h(\beta) - \lambda(\beta) = 0$ for all $\beta > \xi$ such that $\beta = \varrho\big(\lambda(\beta)\big)$. This observation and (3.35) imply

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\big(\beta - \varrho(\beta)\big) = 1 \quad \text{for all } \beta > \xi \text{ such that } \beta = \varrho\big(\lambda(\beta)\big).$$

Based on this result, we can develop a simple contradiction argument to show that the equation in (3.38) has at most one solution $\beta^\star > \xi$.

We conclude this part of the proof by noting that the first statement in (3.38) can be seen by a straightforward inspection of the equation (3.24) that $\big(\beta^\star, \lambda(\beta^\star)\big)$ satisfies in light of the identity in (3.38) and Figure 3.1.

Proof of (II). By construction, we will show that the function w given by (3.27) is a C^2 solution to the HJB equation (3.12) if we prove that

$$w'(x) \geq k(x) \quad \text{for all } x \in \,]0, \beta^\star[$$

$$\text{and} \quad \frac{1}{2}\sigma^2(x) w''(x) + b(x) w'(x) + h(x) - \lambda^\star \leq 0 \quad \text{for all } x \in \,]\beta^\star, \infty[.$$

In view of the identities $\lambda^\star = \uplambda(\beta^\star)$ and $\beta^\star = \varrho\big(\uplambda(\beta^\star)\big)$, the second of these inequalities is equivalent to

$$K(x) + h(x) - \uplambda(\beta^\star) \le 0 \quad \text{for all } x > \beta^\star = \varrho\big(\uplambda(\beta^\star)\big),$$

which is true thanks to (3.6). On the other hand, the first of these inequalities follows immediately from the expression of w' in (3.27) and the inequalities

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_0^x \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s) \begin{cases} < 0 & \text{for all } x \in \big]0, \uprho(\lambda^\star)\big[, \\ > 0 & \text{for all } x \in \big]\uprho(\lambda^\star), \beta^\star\big[, \end{cases}$$

which hold true thanks to the identities $\lambda^\star = \uplambda(\beta^\star)$ and $\beta^\star = \varrho\big(\uplambda(\beta^\star)\big)$, the inequalities in (3.26) and Assumption 3.6.(ii) (see also Figure 3.1).

Proof of (III). Fix any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$. In view of Assumption 3.6.(ii) and the properties of the functions ϱ, $\uprho$ in (3.5), (3.8), we can see that

$$\begin{aligned} \int_0^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) &< \int_0^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s) \\ &< \int_0^{\beta^\star} \big(K(s) + h(s) - \lambda^\star\big)\, m(\mathrm{d}s) = 0 \\ \text{and} \quad \int_{\uprho(\lambda)}^{\varrho(\lambda)} &\big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) > 0, \end{aligned}$$

which imply that there exists a unique point $\alpha(\lambda) \in \big]0, \uprho(\lambda)\big[$ such that

$$\int_{\alpha(\lambda)}^{\varrho(\lambda)} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) = 0.$$

For this choice of $\alpha(\lambda)$, we can see that the function w'_λ defined by (3.28) is indeed C^1. In particular, the limits in (3.29) all hold true. Furthermore,

$$\int_{\alpha(\lambda)}^{x} \big(K(s) + h(s) - \lambda\big)\, m(\mathrm{d}s) < 0 \quad \text{for all } x \in \big]\alpha(\lambda), \varrho(\lambda)\big[. \tag{3.41}$$

The inequality (3.30) follows immediately from (3.41). On the other hand, it is straightforward to check that the function w'_λ defined by (3.28) satisfies the equality in (3.31), while the inequality in (3.31) is equivalent to

$$K(x) + h(x) - \lambda < 0, \quad \text{for } x \in \big]0, \alpha(\lambda)\big[\cup \big]\varrho(\lambda), \infty\big[,$$

which is true thanks to the inequalities (3.6), (3.9) and the fact that $\alpha(\lambda) \in \big]0, \uprho(\lambda)\big[$.

Finally, (3.32) follows immediately from the continuity of w'_λ and the boundedness of k.

□

Remark 3.3 The model studied by Alvarez and Hening [3] is the special case that arises when $h = 0$ and $k = 1$. In this case, the identity (3.3) in Remark 3.1 impies that

$$\int_0^x K(s)\, m(\mathrm{d}s) = \int_0^x b(s)\, m(\mathrm{d}s) = \frac{1}{p'(x)}. \tag{3.42}$$

In view of this identity, we can see that the system of equations in (3.25), which determines $(\beta^\star, \lambda^\star)$, and (3.27) reduce to

$$\lambda = b(\beta), \quad \lambda = \frac{1}{p'(\beta) m\big(]0,\beta[\big)} \quad \text{and} \quad w'(x) = \begin{cases} \lambda^\star p'(x) m\big(]0,x[\big), & \text{for } x \in\,]0,\beta^\star[, \\ 1, & \text{for } x \geq \beta^\star, \end{cases}$$

which are precisely the expressions (8) and (9) in Alvarez and Hening [3].

Remark 3.4 Consider the function w' defined by (3.27) and suppose that X is as in Example 3.1. Using L'Hôpital's formula, we calculate

$$\begin{aligned} \lim_{x\downarrow 0}\big(xw'(x)\big) &= -\lim_{x\downarrow 0} \frac{\frac{\mathrm{d}}{\mathrm{d}x}\Big(x\int_0^x \big(K(s)+h(s)-\lambda^\star\big)\, m(\mathrm{d}s)\Big)}{\frac{\mathrm{d}}{\mathrm{d}x}\big(1/p'(x)\big)} \\ &\geq -\lim_{x\downarrow 0} \frac{K(x)+h(x)-\lambda^\star}{\kappa(\gamma - x)} = \frac{\lambda^\star - K(0) - h(0)}{\kappa\gamma} > 0. \end{aligned}$$

It follows that, in the context of Example 3.1, $\lim_{x\downarrow 0} w(x) = -\infty$.

3.5 The solution to the ergodic harvesting problem

Theorem 3.5.1 *Consider the ergodic control problems formulated in Section 3.1, and let $(\beta^\star, \lambda^\star)$ be as in Proposition 3.4.1. Given any $x > 0$, the following statements hold true:*
(I) $J_x^{\mathrm{e}}(\zeta) \leq \lambda^\star$ *and* $J_x^{\mathrm{p}}(\zeta) \leq \lambda^\star$ *for all admissible harvesting strategies* $\zeta \in \mathcal{A}$.
(II) *If $\zeta^\star \in \mathcal{A}$ is the harvesting strategy that has a jump of size $\Delta\zeta_0^\star = (x-\beta^\star)^+$ at time 0 and then reflects the state process $X^\star$ at the level $\beta^\star$ in the negative direction, then*

$$\begin{aligned} J_x^{\mathrm{e}}(\zeta^\star) &\equiv \lim_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\left[\int_0^T h(X_t^\star)\,\mathrm{d}t + \int_0^T k(X_t^\star)\circ \mathrm{d}\zeta_t^\star\right] = \lambda^\star \\ \textit{and} \quad J_x^{\mathrm{p}}(\zeta^\star) &\equiv \lim_{T\uparrow\infty} \frac{1}{T}\left(\int_0^T h(X_t^\star)\,\mathrm{d}t + \int_0^T k(X_t^\star)\circ \mathrm{d}\zeta_t^\star\right) = \lambda^\star. \end{aligned}$$

Proof. Fix any initial state $x > 0$, let $\zeta \in \mathcal{A}$ be any admissible harvesting strategy and let X be the associated solution to the SDE (1.1). Also, consider the function w_λ defined by (3.28) for $\lambda \in\,]\lambda^\star, \overline{\lambda}[$. Using Itô's formula, we calculate

$$\begin{aligned} w_\lambda(X_T^\zeta) = w_\lambda(x) &+ \int_0^T \left(\frac{1}{2}\sigma^2(X_t^\zeta) w_\lambda''(X_t^\zeta) + b(X_t^\zeta) w_\lambda'(X_t^\zeta)\right)\mathrm{d}t - \int_{[0,T]} w_\lambda'(X_{t-}^\zeta)\,\mathrm{d}\zeta_t \\ &+ \sum_{0\leq t\leq T}\big(w_\lambda(X_t^\zeta) - w_\lambda(X_{t-}^\zeta) - w_\lambda'(X_{t-}^\zeta)\,\Delta X_t^\zeta\big) + M_T^\zeta, \end{aligned}$$

where

$$M_T^{\lambda,\zeta} = \int_0^T \sigma(X_t^\zeta) w_\lambda'(X_t^\zeta) \, \mathrm{d}W_t.$$

Since $\Delta X_t^\zeta \equiv X_t^\zeta - X_{t-}^\zeta = -\Delta\zeta_t \le 0$ and

$$w_\lambda(X_t^\zeta) - w_\lambda(X_{t-}^\zeta) + \int_0^{\Delta\zeta_t} k(X_{t-}^\zeta - u) \, \mathrm{d}u = \int_{X_t^\zeta}^{X_{t-}^\zeta} \bigl(k(u) - w_\lambda'(u)\bigr) \, \mathrm{d}u,$$

it follows that

$$\begin{aligned}
&\int_0^T h(X_t^\zeta) \, \mathrm{d}t + \int_0^T k(X_t^\zeta) \circ \mathrm{d}\zeta_t \\
&\qquad = \lambda T + w_\lambda(x) - w_\lambda(X_T^\zeta) + \int_0^T \left(\frac{1}{2}\sigma^2(X_t^\zeta) w_\lambda''(X_t^\zeta) + b(X_t^\zeta) w_\lambda'(X_t^\zeta) + h(X_t^\zeta) - \lambda \right) \mathrm{d}t \\
&\qquad\quad + \int_0^T \bigl(k(X_t^\zeta) - w_\lambda'(X_t^\zeta)\bigr) \, \mathrm{d}\zeta_t^{\mathrm{c}} + \sum_{0 \le t \le T} \int_{X_t^\zeta}^{X_{t-}^\zeta} \bigl(k(u) - w_\lambda'(u)\bigr) \, \mathrm{d}u + M_T^{\lambda,\zeta}.
\end{aligned}$$

Since ζ^c is an increasing process, $X_t^\zeta < X_{t-}^\zeta$ and the pair $(w_{\lambda^\star}, \lambda^\star)$ (resp., (w_λ, λ)) satisfies the HJB equation (3.12) (resp., the inequalities (3.30) and (3.31)), we can see that

$$\int_0^T h(X_t^\zeta) \, \mathrm{d}t + \int_0^T k(X_t^\zeta) \circ \mathrm{d}\zeta_t \le \lambda T + w_\lambda(x) - w_\lambda(X_T^\zeta) + M_T^{\lambda,\zeta}. \tag{3.43}$$

Proof of the inequality $J_x^{\mathrm{e}}(\zeta) \le \lambda^\star$. Fix any $\lambda \in \,]\lambda^\star, \overline{\lambda}[$ and let (τ_n) be a sequence of localising times for the corresponding local martingale $M^{\lambda,\zeta}$. Recalling the assumptions that h is bounded from below and k is positive, as well as the facts that ζ is an increasing process and w_λ is bounded from below (see (3.32) in Proposition 3.4.1.(III)), we take expectations in (3.43) and we use the monotone and the dominated convergence theorems to calculate

$$\begin{aligned}
\frac{1}{T} \mathsf{E}\left[\int_0^T h(X_t^\zeta) \, \mathrm{d}t + \int_0^T k(X_t^\zeta) \circ \mathrm{d}\zeta_t \right] &= \frac{1}{T} \lim_{n \uparrow \infty} \mathsf{E}\left[\int_0^{\tau_n \wedge T} h(X_t^\zeta) \, \mathrm{d}t + \int_0^{\tau_n \wedge T} k(X_t^\zeta) \circ \mathrm{d}\zeta_t \right] \\
&\le \frac{1}{T} \lim_{n \uparrow \infty} \mathsf{E}\left[\lambda(\tau_n \wedge T) + w_\lambda(x) + w_\lambda^-(X_{\tau_n \wedge T}^\zeta) \right] \\
&= \lambda + \frac{w_\lambda(x)}{T} + \frac{1}{T} \mathsf{E}\left[w_\lambda^-(X_T^\zeta) \right],
\end{aligned}$$

where $w_\lambda^-(x) = -\min\bigl\{w_\lambda(x), 0\bigr\}$. Using the fact that w_λ^- is bounded once again, we can pass to the limit as $T \uparrow \infty$ to obtain the inequality $J_x^{\mathrm{e}}(\zeta) \le \lambda$, which implies the required inequality $J_x^{\mathrm{e}}(\zeta) \le \lambda^\star$ by passing to the limit as $\lambda \downarrow \lambda^\star$.

Proof of the inequality $J_x^{\mathrm{p}}(\zeta) \le \lambda^\star$. Making a slight modification of the proof of the comparison Theorem V.43 in Rogers and Williams [79], we can show that $X_t^\zeta \le X_t$ for all $t \ge 0$,

P-a.s., where X is the solution to the SDE (2.1). In view of this observation, we can see that, given any $\lambda \in]\lambda^\star, \overline{\lambda}[$,

$$\langle M^{\lambda,\zeta}\rangle_T = \int_0^T \Bigl(\sigma(X_t^\zeta) w_\lambda'(X_t^\zeta)\Bigr)^2 \,\mathrm{d}t \leq C_1 C_2^2 \int_0^T \Bigl(1 + \bigl(X_t^\zeta\bigr)^\eta\Bigr) \,\mathrm{d}t \leq C_1 C_2^2 \biggl(T + \int_0^T X_t^\eta \,\mathrm{d}t\biggr),$$

where C_1, η and $C_2 = C_2(\lambda)$ are the constants in (3.1) and (3.32). Furthermore, the ergodic Theorem V.53 in Rogers and Williams [79] implies that

$$\begin{aligned}\limsup_{T\uparrow\infty} \frac{\langle M^{\lambda,\zeta}\rangle_T}{T} &\leq C_1 C_2^2 \biggl(1 + \lim_{T\uparrow\infty} \frac{1}{T} \int_0^T X_t^\eta \,\mathrm{d}t\biggr) \\ &= C_1 C_2^2 \biggl(1 + \frac{1}{m\bigl(]0,\infty[\bigr)} \int_0^\infty s^\eta \, m(\mathrm{d}s)\biggr) =: C_3 < \infty, \end{aligned} \tag{3.44}$$

with the second inequality following thanks to Assumption 3.3.

The Dambis, Dubins and Schwarz theorem (e.g., see Revuz and Yor [78, Theorem V.1.7]) asserts that there exists a standard Brownian motion B, which may be defined on a possible enlargement of the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, such that $M^{\lambda,\zeta} = B_{\langle M^{\lambda,\zeta}\rangle}$. Using this representation, (3.44) and the fact that $\lim_{T\uparrow\infty} B_T/T = 0$, we can see that

$$\lim_{T\uparrow\infty} \frac{\bigl|M_T^{\lambda,\zeta}\bigr|}{T} \mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty = \infty\}} \leq C_3 \lim_{T\uparrow\infty} \frac{|B_{\langle M^{\lambda,\zeta}\rangle_T}|}{\langle M^{\lambda,\zeta}\rangle_T} \mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty = \infty\}} = 0.$$

On the other hand,

$$\lim_{T\uparrow\infty} \frac{\bigl|M_T^{\lambda,\zeta}\bigr|}{T} \mathbf{1}_{\{\langle M^{\lambda,\zeta}\rangle_\infty < \infty\}} = 0$$

because $M^{\lambda,\zeta}$ converges in $\mathbb{R}$ on the event $\bigl\{\langle M^{\lambda,\zeta}\rangle_\infty < \infty\bigr\}$. In view of these results, we can pass to the limit as $T \uparrow \infty$ in (3.43) to obtain

$$J_x^{\mathrm{p}}(\zeta) \leq \lim_{T\uparrow\infty} \biggl(\lambda + \frac{w_\lambda(x)}{T} + \frac{w_\lambda^-(X_T^\zeta)}{T} + \frac{M_T^{\lambda,\zeta}}{T}\biggr) = \lambda.$$

The inequality $J_x^{\mathrm{p}}(\zeta) \leq \lambda^\star$ now follows by passing to the limit as $\lambda \downarrow \lambda^\star$.

Proof of (II). Let the harvesting strategy $\zeta^\star \in \mathcal{A}$ be as in the statement of the theorem: such a strategy indeed exists (see Tanaka [84, Theorem 4.1]). If we define

$$N_T = \int_0^T \sigma(X_t^\star) \,\mathrm{d}W_t,$$

then $\langle N\rangle_T/T \leq \max_{s\in[0,\beta^\star]} \sigma(s) < \infty$. Therefore, N is a square integrable martingale and $\mathsf{E}\bigl[N_T\bigr] = 0$ for all $T > 0$. Furthermore, reasoning as above, we can see that $\lim_{T\uparrow\infty} N_T/T = 0$. In view of these observations, the expression

$$\frac{\zeta_T^\star}{T} = \frac{x}{T} - \frac{X_t^\star}{T} + \frac{1}{T}\int_0^T b(X_t^\star) \,\mathrm{d}t + \frac{N_T}{T}, \tag{3.45}$$

and the fact that, beyond its possible initial jump, $\zeta^\star$ increases on the set $\{X^\star_t = \beta^\star\}$, we can see that

$$J^{\mathrm{e}}_x(\zeta^\star) = \lim_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\biggl[\int_0^T h(X^\star_t)\,\mathrm{d}t + k(\beta^\star)\zeta^\star_T\biggr] = \lim_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\biggl[\int_0^T \Bigl(h(X^\star_t) + k(\beta^\star)b(X^\star_t)\Bigr)\,\mathrm{d}t\biggr]$$

and

$$J^{\mathrm{p}}_x(\zeta^\star) = \lim_{T\uparrow\infty} \frac{1}{T}\biggl(\int_0^T h(X^\star_t)\,\mathrm{d}t + k(\beta^\star)\zeta^\star_T\biggr) = \lim_{T\uparrow\infty} \frac{1}{T}\int_0^T \Bigl(h(X^\star_t) + k(\beta^\star)b(X^\star_t)\Bigr)\,\mathrm{d}t.$$

These expressions and standard ergodic theorems (e.g., see Borodin and Salminen [14, Section II.6] and Rogers and Williams [79, Theorem V.53]) imply that

$$J^{\mathrm{e}}_x(\zeta^\star) = J^{\mathrm{p}}_x(\zeta^\star) = \frac{1}{m\bigl(]0,\beta^\star[\bigr)}\int_0^{\beta^\star} \bigl(h(s) + k(\beta^\star)b(s)\bigr)\,m(\mathrm{d}s).$$

Combining these observations with the identities

$$k(\beta^\star)\int_0^{\beta^\star} b(s)\,m(\mathrm{d}s) \overset{(3.42)}{=} \frac{k(\beta^\star)}{p'(\beta^\star)} \overset{(3.3)}{=} \int_0^{\beta^\star} K(s)\,m(\mathrm{d}s),$$

we obtain

$$J^{\mathrm{e}}_x(\zeta^\star) = J^{\mathrm{p}}_x(\zeta^\star) = \frac{1}{m\bigl(]0,\beta^\star[\bigr)}\int_0^{\beta^\star} \bigl(K(s) + h(s)\bigr)\,m(\mathrm{d}s) \overset{(3.26)}{=} \lambda^\star.$$

□

3.6 Abelian limits

In this section, we allow for the discounting rate function r to depend on a parameter $\iota > 0$ and we establish the convergence of the solution to the discounted control problem to the one of the ergodic control problems in an Abelian sense. In particular, we make the following assumption, which is the same as Assumption 3.5.(i) for each individual $\iota > 0$.

Assumption 3.7 The discounting rate function $(x,\iota) \mapsto r(x;\iota)$ is continuous. Also, given any $\iota > 0$, the function $r(\cdot;\iota)$ is C^1 and such that

$$\underline{r}(\iota) \le r(x;\iota) \le \overline{r}(\iota) \quad \text{for all } x \ge 0, \tag{3.46}$$

for some $\underline{r}(\iota)$ and $\overline{r}(\iota)$ such that

$$0 < \underline{r}(\iota) < \overline{r}(\iota) < \infty \text{ for all } \iota > 0, \quad \lim_{\iota\downarrow 0} \frac{\overline{r}(\iota)}{\underline{r}(\iota)} = 1 \quad \text{and} \quad \lim_{\iota\downarrow 0} \overline{r}(\iota) = 0. \tag{3.47}$$

The dependence of r on the parameter ι implies that the functions $\mathfrak{K}$, Θ, φ, ψ and R_h that we have considered in our analysis also depend on ι. Throughout this section, we will make such dependences explicit for clarity of the arguments.

The functions φ and ψ introduced at the beginning of Section 3.3 are unique up to a multiplicative constant. In this section, we assume that they have been scaled so that

$$\varphi(1;\iota) = 1 \quad \text{and} \quad \psi(1;\iota) = 1 \quad \text{for all } \iota > 0, \tag{3.48}$$

without loss of generality.

Lemma 3.6.1 *In the presence of Assumptions 3.1 and 3.2, the scaled as in (3.48) functions* $(x,\iota) \mapsto \varphi(x;\iota)$ *and* $(x,\iota) \mapsto \psi(x;\iota)$ *are continuous,*

$$\lim_{\iota\downarrow 0} \frac{\varphi(x;\iota)}{\varphi(y;\iota)} = \lim_{\iota\downarrow 0} \frac{\psi(x;\iota)}{\psi(y;\iota)} = 1 \quad \textit{for all } x, y > 0 \tag{3.49}$$

$$\textit{and} \quad \lim_{\iota\downarrow 0} \frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)} = p'(x) m\big(]0,x[\big) \quad \textit{for all } x, y > 0. \tag{3.50}$$

In the presence of the assumptions we have made in Section 3.1 and Assumption 3.7,

$$\lim_{\iota\downarrow 0} r(y;\iota)\left(R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)\right) = \frac{1}{m\big(]0,\beta[\big)} \int_0^\beta h(s)\, m(\mathrm{d}s) \tag{3.51}$$

for all $\beta > 0$, $x \in \,]0,\beta]$ *and* $y > 0$.

Proof. The continuity of the functions φ and ψ, as well as (3.49), follow immediately from (2.12) and the dominated convergence theorem. In turn, (3.50) follows from the definition (2.14), the identity (2.31), the limit (3.49) and Assumption 3.7, which imply that

$$\lim_{\iota\downarrow 0} \frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)} = p'(x) \lim_{\iota\downarrow 0} \int_0^x \frac{r(s;\iota)}{r(y;\iota)} \frac{\varphi(s;\iota)}{\varphi(x;\iota)}\, m(\mathrm{d}s) = p'(x) m\big(]0,x[\big).$$

Using the definitions (2.22) and (2.14), we can see that

$$\begin{aligned} & R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota) \\ & \quad = \frac{1}{C}\widetilde{\varphi}_\beta(x;\iota) \int_0^x h(s)\psi(s;\iota)\, m(\mathrm{d}s) + \frac{1}{C}\psi(x;\iota) \int_x^\beta h(s)\widetilde{\varphi}_\beta(s;\iota)\, m(\mathrm{d}s), \end{aligned}$$

where

$$\widetilde{\varphi}_\beta(x;\iota) = \varphi(x;\iota) - \frac{\varphi'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota).$$

In view of the observation that

$$\begin{aligned} \frac{\widetilde{\varphi}_\beta(x;\iota)}{\widetilde{\varphi}_\beta(\beta;\iota)} = 1 & + \left(\frac{\varphi(x;\iota)}{\varphi(\beta;\iota)} - 1\right) \frac{\varphi(\beta;\iota)\psi'(\beta;\iota)}{\varphi(\beta;\iota)\psi'(\beta;\iota) - \varphi'(\beta;\iota)\psi(\beta;\iota)} \\ & + \left(\frac{\psi(x;\iota)}{\psi(\beta;\iota)} - 1\right) \frac{-\varphi'(\beta;\iota)\psi(\beta;\iota)}{\varphi(\beta;\iota)\psi'(\beta;\iota) - \varphi'(\beta;\iota)\psi(\beta;\iota)} \end{aligned}$$

and the fact that the two long fractions on the right-hand side of this expression take values in $]0,1[$, we can see that $\lim_{\iota\downarrow 0} \widetilde{\varphi}_\beta(x;\iota)/\widetilde{\varphi}_\beta(\beta;\iota) = 1$, thanks to (3.49). On the other hand, we can use the probabilistic expression (2.21) to obtain

$$R_{r(\cdot;\iota)}(x) = \mathbb{E}_x\left[\int_0^\infty \exp\left(-\int_0^t r(X_u;\iota)\,\mathrm{d}u\right) r(X_t;\iota)\,\mathrm{d}t\right] = 1 \quad \text{for all } x, \iota > 0.$$

Combining these observations with (3.49) and the fact that

$$\lim_{\iota\downarrow 0} \frac{r(x;\iota)}{r(y;\iota)} = 1 \quad \text{for all } x, y > 0, \tag{3.52}$$

which follows from Assumption 3.7, and using the dominated convergence theorem, we obtain

$$\begin{aligned} \lim_{\iota\downarrow 0} r(y;\iota)&\left(R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)\right) \\ &= \lim_{\iota\downarrow 0} r(y;\iota) \frac{R_h(x;\iota) - \frac{R_h'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)}{R_{r(\cdot;\iota)}(x;p) - \frac{R_{r(\cdot;\iota)}'(\beta;\iota)}{\psi'(\beta;\iota)}\psi(x;\iota)} \\ &= \lim_{\iota\downarrow 0} \frac{\int_0^x h(s)\frac{\psi(s;\iota)}{\psi(x;\iota)}\, m(\mathrm{d}s) + \int_x^\beta h(s)\frac{\widetilde{\varphi}_\beta(s;\iota)}{\widetilde{\varphi}_\beta(x;\iota)}\, m(\mathrm{d}s)}{\int_0^x \frac{r(s;\iota)}{r(y;\iota)}\frac{\psi(s;\iota)}{\psi(x;\iota)}\, m(\mathrm{d}s) + \int_x^\beta \frac{r(s;\iota)}{r(y;\iota)}\frac{\widetilde{\varphi}_\beta(s;\iota)}{\widetilde{\varphi}_\beta(x;\iota)}\, m(\mathrm{d}s)} = \frac{\int_0^\beta h(s)\, m(\mathrm{d}s)}{m\big(]0,\beta[\big)}, \end{aligned}$$

namely, (3.51). □

Theorem 3.6.2 *Consider the control problems formulated in Section 3.1 and suppose that Assumption 3.7 also holds. If $\beta^\star(\iota)$, $w(\cdot;\iota)$ are as in Lemma 3.3.1 and $(\beta^\star,\lambda^\star)$, w are as in Proposition 3.4.1, then*

$$\lim_{\iota\downarrow 0} \beta^\star(\iota) = \beta^\star, \quad \lim_{\iota\downarrow 0} r(y;\iota)w(x;\iota) = \lambda^\star \quad \textit{and} \quad \lim_{\iota\downarrow 0} w'(x;\iota) = w'(x) \quad \textit{for all } x, y > 0. \tag{3.53}$$

Proof. In view of the definitions in (2.14), the equation (3.18) that $\beta^\star(\iota) > 0$ satisfies takes the form

$$\begin{aligned} \int_0^{\beta^\star(\iota)} &\big(\mathfrak{K}(s;\iota) + h(s)\big) \frac{\psi(s;\iota)}{\psi\big(\beta^\star(\iota);\iota\big)}\, m(\mathrm{d}s) \\ &= \Big(\mathfrak{K}\big(\beta^\star(\iota);\iota\big) + h\big(\beta^\star(\iota)\big)\Big) \int_0^{\beta^\star(\iota)} \frac{r(s;\iota)}{r\big(\beta^\star(\iota);\iota\big)} \frac{\psi(s;\iota)}{\psi\big(\beta^\star(\iota);\iota\big)}\, m(\mathrm{d}s). \end{aligned}$$

The functions r, $\mathfrak{K}$, h and ψ are all continuous, while $\lim_{\iota\downarrow 0} \mathfrak{K}(x;\iota) = K(x)$ (see the definitions (2.17) and (3.2) of $\mathfrak{K}$ and K). Therefore, we can use (3.49), (3.52) and the dominated convergence theorem to come to the conclusion that the limit $\beta^\star(0) = \lim_{\iota\downarrow 0} \beta^\star(\iota)$ exists and satisfies the equation

$$\int_0^{\beta^\star(0)} \big(K(s) + h(s)\big)\, m(\mathrm{d}s) = \Big(K\big(\beta^\star(0)\big) + h\big(\beta^\star(0)\big)\Big) m\big(\big]0, \beta^\star(0)\big[\big).$$

It follows that the first limit in (3.53) holds true because this is the equation that $\beta^\star$ satisfies (see (3.26)).

The second limit in (3.53) follows immediately from the first expression for $w(\cdot;\iota)$ in (3.16), Assumption 3.7 and (3.51) with $\mathfrak{K}(\cdot;\iota) + h$ in place of h. Finally, we use (2.13) and (3.49) to note that

$$\begin{aligned}\lim_{\iota\downarrow 0}\left(R'_{\mathfrak{K}+h} - \frac{\psi'}{\psi} R_{\mathfrak{K}+h}\right)(x;\iota) &= -p'(x) \lim_{\iota\downarrow 0} \int_0^x \big(\mathfrak{K}(s;\iota) + h(s)\big) \frac{\psi(s;\iota)}{\psi(x;\iota)}\, m(\mathrm{d}s) \\ &= -p'(x) \int_0^x \big(K(s) + h(s)\big)\, m(\mathrm{d}s).\end{aligned}$$

In light of this limit, the fact that $\lim_{\iota\downarrow 0} \beta^\star(\iota) = \beta^\star$, (3.50) and (3.51) with $\mathfrak{K}(\cdot;\iota) + h$ in place of h, we can see that

$$\begin{aligned}&\lim_{\iota\downarrow 0}\left(R'_{\mathfrak{K}+h}(x;\iota) - \frac{R'_{\mathfrak{K}+h}\big(\beta^\star(\iota);\iota\big)}{\psi'\big(\beta^\star(\iota);\iota\big)} \psi'(x;\iota)\right) \\ &\quad = \lim_{\iota\downarrow 0}\left(R'_{\mathfrak{K}+h} - \frac{\psi'}{\psi} R_{\mathfrak{K}+h}\right)(x;\iota) \\ &\qquad + \lim_{\iota\downarrow 0} r(y;\iota)\left(R_{\mathfrak{K}+h}(x;\iota) - \frac{R'_{\mathfrak{K}+h}\big(\beta^\star(\iota);\iota\big)}{\psi'\big(\beta^\star(\iota);\iota\big)} \psi(x;\iota)\right) \lim_{\iota\downarrow 0} \frac{\psi'(x;\iota)}{r(y;\iota)\psi(x;\iota)} \\ &\quad = p'(x)\left(\frac{m\big(\big]0, x\big[\big)}{m\big(\big]0, \beta^\star\big[\big)} \int_0^{\beta^\star} \big(K(s) + h(s)\big)\, m(\mathrm{d}s) - \int_0^x \big(K(s) + h(s)\big)\, m(\mathrm{d}s)\right).\end{aligned}$$

The third limit in (3.53) follows from this result, the first expression for $w(\cdot;\iota)$ in (3.16) and the second expression for w' in (3.23). $\square$

Chapter 4

Portfolio optimisation with proportional transaction costs and stochastic investment opportunities

The chapter is organised as follows. In Section 4.1, we introduce the formulation of the optimal portfolio selection problem we solve, including all the assumptions we make, and present three examples that satisfy our assumptions. In Section 4.2, we discuss the properties of the stock-cash ratio and the related functions, and present the optimal strategy without proof. In Section 4.3, we establish the stochastic control problem in the shadow market and introduce the shadow price process. We construct a shadow price process, find the buying and selling boundaries and derive the optimal trading strategy. Section 4.4 presents some asymptotic results. In Section 4.5, we discuss more cases of sufficient conditions for our results, and examples of each case are presented.

4.1 Formulation of the stochastic control problem

We first consider the assumptions on the dynamics of the price process S modelled by the SDE (1.8). The following conditions are satisfied by a wide variety of problem data choices (see Examples 4.1–4.3 at the end of this section).

Assumption 4.1 The following statements hold true:
(i) The functions $\mu, \sigma : \,]0, \infty[\, \to \mathbb{R}$ are locally Lipschitz continuous and $\sigma(x) > 0$ for all $x > 0$.
(ii) The function Θ given by (1.13) is C^1 and decreasing, and there exists $\xi > 0$ such that $\Theta(\xi) = 0$.

(iii) There exist $0 < \rho < \eta$ such that

$$Q'(s) = \frac{\Gamma(s)}{(1-\Theta(s))^2} \begin{cases} > 0, & \text{for } s < \rho, \\ < 0, & \text{for } \rho < s < \eta, \\ > 0, & \text{for } s > \eta, \end{cases}$$

where Q is as in (1.17) and

$$\Gamma(s) = \frac{-\Theta(s) + \Theta^2(s) + s\Theta'(s)}{s^2}. \tag{4.1}$$

In the frictionless market when $\lambda = 0$, it is optimal to keep the stock-cash ratio $\mathcal{Q} = Q(S)$ (see (1.17)). Furthermore, we will show in the next section that if the stock price S crosses the turning points ρ and η, then the investor would change the trading behaviour (see Remark 4.5). How the turning points ρ and η influence the optimal trading strategy for the market with $\lambda > 0$ will also be illustrated in Section 4.2. In Section 4.5, we will modify Assumption 4.1 and consider the problem data such that Q has less turning points, i.e. ρ or η are either 0 or infinity.

Remark 4.1 In light of the definition (1.17) of Q, if Assumption 4.1 holds true, then

$$\lim_{s\downarrow 0} \Theta(s) > 1 \quad \text{and} \quad \rho < \zeta := \Theta^{-1}(1) < \xi < \eta. \tag{4.2}$$

Given any $a > 0$, Assumption 4.1 implies that the scale function p_a and the speed measure m_a of the diffusion associated with the SDE (1.8), which are given by

$$p_a(a) = 0 \quad \text{and} \quad p_a'(s) = \exp\left(-2\int_a^s \frac{\Theta(u)}{u}\,\mathrm{d}u\right), \quad \text{for } s > 0, \tag{4.3}$$

and

$$m_a(\mathrm{d}s) = \frac{2}{s^2\sigma^2(s)p_a'(s)}\,\mathrm{d}s, \tag{4.4}$$

are well-defined. Furthermore, $\lim_{s\downarrow 0} p_a(s) = -\infty$ and $\lim_{s\uparrow\infty} p_a(s) = \infty$. These limits and (i) in Assumption (4.1) imply that the SDE (1.8) has a unique non-explosive strong solution with the state space $]0,\infty[$ (see e.g., Karatzas and Shreve [51, Proposition 5.22, Ch 5]).

Remark 4.2 Given any $a, b, c > 0$, the definitions (4.3) and (4.4) of p and m imply that

$$p_a(b) = -\frac{p_b(a)}{p_b'(a)} \quad \text{and} \quad p_a(c) = \frac{p_b(c)}{p_b'(a)} + p_a(b)$$

To simplify the notations, we define

$$p(s) := p_1(s) = p_a(s)p'(a) + p(a) \quad \text{and} \quad m(\mathrm{d}s) := \frac{2}{s^2\sigma^2(s)p'(s)}\,\mathrm{d}s = \frac{m_a(\mathrm{d}s)}{p'(a)}. \tag{4.5}$$

Additionally, we make the following assumption so that the optimal growth rate for the frictionless market is finite (see also Lemma 4.3.2 in Section 4.3).

Assumption 4.2 The problem data is such that

$$\limsup_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\left[\int_0^T \frac{\mu^2(S_t)}{\sigma^2(S_t)}\,\mathrm{d}t\right] < \infty.$$

Remark 4.3 If the problem data is such that

$$\begin{aligned} &\text{either} \quad \text{(i)}\ \frac{\mu^2}{\sigma^2} \le C, \quad \text{for some } C > 0,\\ &\text{or} \quad \text{(ii)}\ m\big(]0,\infty[\big) < \infty \quad \text{and} \quad \int_0^\infty \frac{\mu^2(s)}{\sigma^2(s)}\,m(\mathrm{d}s) < \infty, \end{aligned}$$

then Assumption 4.2 holds true. The geometric Brownian motion (gBm) is an example such that the condition in (i) holds true but the conditions in (ii) fails. In such a case, $m(]0,\infty[) = \infty$. The gBm does not satisfy Assumption 4.1, and will be discussed in Section 4.5. However, Examples 4.1–4.3 in this section satisfy the conditions in (ii) in Remark 4.3, as well as Assumption 4.1.

In addition, we make the following assumption

Assumption 4.3 If $m(]0,\infty[) = \infty$, then $|\Theta| < C_1$ for some $C_1 > 0$.

Definition 4.1 An admissible self-financing trading strategy under transaction cost $\lambda \in]0,1[$ belonging to $\mathcal{A}_\lambda(x,y)$ is a predictable finite variation process (ϑ^0,ϑ) with initial position $(\vartheta^0_{0-},\vartheta_{0-}) = (x,y)$ such that
(i) (self-financing condition)

$$\mathrm{d}\vartheta^{0,+}_t = (1-\lambda)S_t\,\mathrm{d}\vartheta^-_t \quad \text{and} \quad \mathrm{d}\vartheta^{0,-}_t = S_t\,\mathrm{d}\vartheta^+_t.$$

(ii) (admissibility) The liquidation value $V_t(\vartheta^0,\vartheta)$ as in (1.9) is positive P-a.s. for all $t \ge 0$.

If we denote by $|\vartheta^0|$ (resp., $|\vartheta|$) the total variance of ϑ^0 (resp., ϑ), then $\vartheta^{0,\pm}$ (resp., $\vartheta^\pm$) are the unique processes such that $\vartheta^0 = \vartheta^{0,+} - \vartheta^{0,-}$ and $|\vartheta^0| = \vartheta^{0,+} + \vartheta^{0,-}$ (resp., $\vartheta = \vartheta^+ - \vartheta^-$ and $|\vartheta| = \vartheta^+ + \vartheta^-$). With each admissible self-financing trading strategy $(\vartheta^0,\vartheta) \in \mathcal{A}_\lambda(x,y)$, we associate the expected growth rate J given by (1.10). The objective of the control problem that we consider is to maximise J over all $(\vartheta^0,\vartheta) \in \mathcal{A}_\lambda(x,y)$. The following result shows that different initial values do not influence the optimal growth rate. Therefore, we use $\mathcal{A}_\lambda$ in place of $\mathcal{A}_\lambda(x,y)$ in the rest of the chapter.

Lemma 4.1.1 *Given any (x, y) such that $x + (1 - \lambda \mathbf{1}_{\{y>0\}}) y s_0 > 0$, the following identity holds true:*

$$\sup_{\mathcal{A}_\lambda(x,y)} J(\vartheta^0, \vartheta) = \sup_{\mathcal{A}_\lambda(1,0)} J(\vartheta^0, \vartheta).$$

Proof. Given any $(\vartheta^0, \vartheta) \in \mathcal{A}_\lambda(x, y)$ and $c > 0$, we can see that

$$J(\vartheta^0, \vartheta) = \limsup_{T\uparrow\infty} \frac{1}{T} \mathsf{E}\big[\ln\big(V_T(\vartheta^0, \vartheta)\big)\big] = \limsup_{T\uparrow\infty} \frac{1}{T} \mathsf{E}\big[\ln\big(cV_T(\vartheta^0, \vartheta)\big)\big] = J(c\vartheta^0, c\vartheta). \quad (4.6)$$

It follows that

$$\sup_{\mathcal{A}_\lambda(x,y)} J(\vartheta^0, \vartheta) = \sup_{\mathcal{A}_\lambda(cx,cy)} J(\vartheta^0, \vartheta).$$

Let $c_1 = x + (1 - \lambda \mathbf{1}_{\{y>0\}}) y s_0$ and $c_2 = x + y s_0$. Note that a trading strategy in $\mathcal{A}_\lambda(x, y)$ (resp., in $\mathcal{A}_\lambda(c_2, 0)$) can be liquidating (resp., buying y shares of stock) at time 0 and follows a strategy in $\mathcal{A}_\lambda(c_1, 0)$ (resp., in $\mathcal{A}_\lambda(x, y)$). It follows that

$$\sup_{(c_1,0)} J(\vartheta^0, \vartheta) \le \sup_{\mathcal{A}_\lambda(x,y)} J(\vartheta^0, \vartheta) \le \sup_{\mathcal{A}_\lambda(c_2,0)} J(\vartheta^0, \vartheta).$$

Combining this observation with (4.6), we obtain

$$\sup_{\mathcal{A}_\lambda(x,y)} J(\vartheta^0, \vartheta) = \sup_{\mathcal{A}_\lambda(1,0)} J(\vartheta^0, \vartheta).$$

□

We conclude this section with the following three examples that satisfy Assumptions 4.1 and 4.2.

Example 4.1 Suppose that the price process S is modelled by the SDE

$$\mathrm{d}S_t = \left(\kappa\gamma + \frac{1}{2}\sigma^2 - \kappa \ln(S_t)\right) S_t \,\mathrm{d}t + \sigma S_t \,\mathrm{d}W_t, \quad S_0 = s_0 > 0,$$

for some constants $\kappa, \gamma, \sigma > 0$, namely, the logarithm of the uncontrolled state process is the Ornstein-Uhlenbeck process given by

$$\mathrm{d}\ln(S_t) = \kappa\big(\gamma - \ln(S_t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \quad \ln(S_0) = \ln(s_0) \in \mathbb{R}.$$

In such a case

$$\Theta(s) = \frac{\kappa\gamma}{\sigma^2} + \frac{1}{2} - \frac{\kappa}{\sigma^2} \ln(s)$$

is decreasing and $\xi = e^{\gamma + \frac{\sigma^2}{2\kappa}}$. Furthermore,

$$\Gamma(s) = \frac{4\kappa^2(\gamma^2 - 2\gamma \ln(s) + \ln(s)^2) - \sigma^4 - 4\kappa\sigma^2}{4\sigma^4 s^2}$$

$$\begin{cases} > 0, & \text{for } 0 < s < e^{\gamma - \frac{\sigma}{\kappa}\sqrt{4\sigma^2+\kappa}}, \\ < 0, & \text{for } e^{\gamma - \frac{\sigma}{\kappa}\sqrt{4\sigma^2+\kappa}} < s < e^{\gamma + \frac{\sigma}{\kappa}\sqrt{4\sigma^2+\kappa}}, \\ > 0, & \text{for } s > e^{\gamma + \frac{\sigma}{\kappa}\sqrt{4\sigma^2+\kappa}}, \end{cases}$$

and Assumption 4.1 holds true. The derivative of scale function admits the expression

$$p'(s) = s^{\frac{\kappa}{\sigma^2}\ln(s) - \frac{2\kappa\gamma}{\sigma^2} - 1}.$$

Example 4.2 Suppose that the price process S is modelled by the SDE

$$\mathrm{d}S_t = \kappa(\gamma - S_t)S_t\,\mathrm{d}t + \sigma S_t^{\ell+1}\,\mathrm{d}W_t, \quad S_0 = s_0 > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [0, \frac{1}{2}]$. Furthermore, if $\ell = 0$, then $\kappa\gamma > \frac{1}{2}\sigma^2$. In such a case,

$$\Theta(s) = \frac{\kappa}{\sigma^2}\left(\gamma s^{-2\ell} - s^{1-2\ell}\right)$$

is strictly decreasing and $\xi = \gamma$. Furthermore, we calculate

$$\Gamma(s) = \frac{\kappa s^{-4\ell-2}}{\sigma^4}\Big(\kappa(s-\gamma)^2 + \sigma^2(2\ell s^{2\ell+1} - (1+2\ell)\gamma s^{2\ell})\Big) =: \frac{\kappa s^{-4\ell-2}}{\sigma^4}u(s),$$

$$u(\gamma) = -\sigma^2\gamma^{2\ell+1} < 0, \qquad \lim_{s\uparrow\infty} u(s) = \infty$$

$$u(0) = \begin{cases} (\kappa\gamma - \sigma^2)\gamma, & \text{if } \ell = 0, \\ \kappa\gamma^2 > 0, & \text{if } 0 < \ell \le \frac{1}{2} \end{cases} \quad \text{and} \quad u'(s) = (s-\gamma)\Big(2\kappa + 2\ell(2\ell+1)\sigma^2 s^{2\ell-1}\Big).$$

It follows that there exist $\gamma < \eta < \infty$ such that $\Gamma(\eta) = 0$, and if $\ell = 0$ and $\kappa\gamma - \sigma^2 > 0$, or $0 < \ell$, then Assumption 4.1 holds true. The derivative of the scale function admits the expression

$$p'(x) = \exp\left(\frac{\kappa\gamma}{\ell\sigma^2}\left(x^{-2\ell} - 1\right) + \frac{2\kappa}{(1-2\ell)\sigma^2}\left(x^{1-2\ell} - 1\right)\right),$$

if $\ell \in \,]0, \frac{1}{2}[$,

$$p'(s) = \left(\frac{1}{s}\right)^{\frac{2\kappa\gamma}{\sigma^2}} \exp\left(\frac{2\kappa}{\sigma^2}(s-1)\right),$$

if $\ell = 0$, and

$$p'(s) = s^{\frac{2\kappa}{\sigma^2}} \exp\left(\frac{2\kappa\gamma}{\sigma^2}(s^{-1} - 1)\right),$$

if $\ell = \frac{1}{2}$.

Example 4.3 Suppose that the price process S is modelled by the SDE

$$\mathrm{d}S_t = \kappa(\gamma - S_t)\,\mathrm{d}t + \sigma S_t^{\ell+1}\,\mathrm{d}W_t, \quad S_0 = s_0 > 0,$$

for some strictly positive constants κ, γ, σ and $\ell \in [-\frac{1}{2}, 0]$. Furthermore, if $\ell = -\frac{1}{2}$, then $\kappa\gamma > \frac{1}{2}\sigma^2$. In such case, the function Θ is the same as Example 4.2.

4.2 The stock-cash ratio

In this section, we will discuss the stock-cash ratio process $\mathcal{Q}$ defined by (1.16). An advantage of the stock-cash process, compared to the proportional wealth process investing in stocks, is that it keeps constant if the investor does not trade. If the investor trades, then we differentiate $\mathcal{Q}$ and obtain

$$\mathrm{d}\mathcal{Q}_t = \mathbf{1}_{\{\vartheta_t^0 \neq 0\}} \frac{1}{\vartheta_t^0}\left(\frac{\vartheta_t^0 + \vartheta_t S_t}{\vartheta_t^0}\mathrm{d}\vartheta_t^+ - \frac{\vartheta_t^0 + (1-\lambda)\vartheta_t S_t}{\vartheta_t^0}\mathrm{d}\vartheta_t^-\right) \tag{4.7}$$

$$= \mathbf{1}_{\{\vartheta_t^0 \neq 0\}} \frac{1}{\vartheta_t^0}\Big(\big(1 + S_t\mathcal{Q}_t\big)\mathrm{d}\vartheta_t^+ - \big(1 + (1-\lambda)S_t\mathcal{Q}_t\big)\mathrm{d}\vartheta_t^-\Big). \tag{4.8}$$

The first identity and the admissible condition Definition 4.1.(ii) imply that if the investor buys (resp., sells), then $\mathcal{Q}$ increases (resp., decreases). Furthermore, a self-financing trading strategy is admissible if and only if (ϑ^0, ϑ) is in one of the following trading regions (denoted by TR)

$$\begin{cases} \mathrm{TR}_0 = \{(\vartheta^0, \vartheta) \mid \vartheta > 0 \quad \text{and} \quad \vartheta^0 = 0\}, \\ \mathrm{TR}_+ = \{(\vartheta^0, \vartheta) \mid 1 + SQ > 0 \quad \text{and} \quad \vartheta^0 > 0\}, \\ \mathrm{TR}_- = \{(\vartheta^0, \vartheta) \mid 1 + (1-\lambda)SQ < 0 \quad \text{and} \quad \vartheta^0 < 0\}. \end{cases} \tag{4.9}$$

In the frictionless market when $\lambda = 0$, the optimal strategy is such that $\mathcal{Q}_t = Q(S_t)$ (see (1.17)), if $S \neq \zeta$, and $\vartheta^0 = 0$, if $S = \zeta$, where ζ is given by (4.2). We present the graph of Q and the optimal strategy in Figure 4.1 (see also Remarks 4.4 and 4.5).

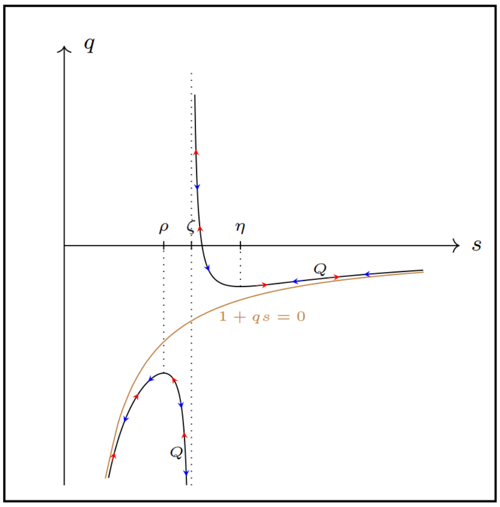

Figure 4.1: The function $Q(s) = \frac{7-50\ln s}{(-3+50\ln s)s}$ in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$ and $\sigma = 0.2$. In this context, $(\rho, Q(\rho)) \approx (0.83056, -1.5961)$, $(\xi, Q(\xi)) = (e^{0.06}, 0)$ and $(\eta, Q(\eta)) \approx (1.47058, -0.51296)$. In the frictionless market, where $\lambda = 0$, the investor keeps the stock-cash ratio $\mathcal{Q}$ identical to $Q(S)$. The optimal trading behaviour is as in Remark 4.5.

Remark 4.4 If Assumption 4.1 holds true (see also Remark 4.1), then $1+sQ(s)>0$ for all $s>\zeta$ and $1+Q(s)s<0$ for all $s<\zeta$. Furthermore, $\lim_{s\uparrow\zeta} Q(s)=-\infty$ and $\lim_{s\downarrow\zeta} Q(s)=\infty$, where ζ is defined by (4.2).

Remark 4.5 When the stock price S increases (resp., decreases), it is optimal to (i) buy (resp., sell) stocks if $S<\rho$, (ii) sell (resp., buy) stocks if $\rho<S<\eta$, (iii) buy (resp., sell) stocks if $S>\eta$, and keep $\mathcal{Q}=Q(S)$.

These remarks (see also Figure 4.1) imply that the optimal portfolio is in TR_- when $S<\zeta$, in TR_0 when $S=\zeta$ and in TR_+ when $S>\zeta$. Furthermore, the turning point ρ and η of Q are the signals for the investor to change their trading behaviours.

In the following subsections, we will provide a brief introduction of the optimal strategies characterized by the stock-cash ratio for $\lambda>0$. The proofs of the optimality will be given in Section 4.3.

In Lemma 4.3.6 and Theorem 4.3.7, we will construct the buying boundary $\underline{Q}$ and the selling boundary $\overline{Q}$ associated with the stock-cash ratio $\mathcal{Q}$. The optimal strategy for the optimal growth rate problem is characterized by $\underline{Q}$ an $\overline{Q}$ as in Figure 4.2. We next provide more details of $\underline{Q}$, $\overline{Q}$ and the associated notations in Figure 4.2. The functions $\underline{Q}$ and $\overline{Q}$ converge to Q when λ goes to 0 and are characterized as follows. There exist $\bar{\rho}_\ell(\lambda)<\underline{\rho}(\lambda)<\bar{\rho}_r(\lambda)<\bar{\zeta}(\lambda)$ and $\underline{\zeta}(\lambda)<\underline{\eta}_\ell(\lambda)<\bar{\eta}(\lambda)<\underline{\eta}_r(\lambda)$ such that

$$\underline{\rho}(\lambda)<\underline{\zeta}(\lambda)<\zeta<\bar{\zeta}(\lambda)<\bar{\eta}(\lambda),\quad \lim_{\lambda\downarrow 0}\underline{\zeta}(\lambda)=\lim_{\lambda\downarrow 0}\bar{\zeta}(\lambda)=\zeta \tag{4.10}$$

$$\lim_{\lambda\downarrow 0}\bar{\rho}_\ell(\lambda)=\lim_{\lambda\downarrow 0}\underline{\rho}(\lambda)=\lim_{\lambda\downarrow 0}\bar{\rho}_r(\lambda)=\rho \quad\text{and}\quad \lim_{\lambda\downarrow 0}\underline{\eta}_\ell(\lambda)=\lim_{\lambda\downarrow 0}\bar{\eta}(\lambda)=\lim_{\lambda\downarrow 0}\underline{\eta}_r(\lambda)=\eta. \tag{4.11}$$

The functions $\underline{Q}$ and $\overline{Q}$ are such that

$$\lim_{s\downarrow\underline{\zeta}}\underline{Q}(s)=\lim_{s\downarrow\bar{\zeta}}\overline{Q}(s)=\infty,\quad \lim_{s\uparrow\underline{\zeta}}\underline{Q}(s)=\lim_{s\uparrow\bar{\zeta}}\overline{Q}(s)=-\infty, \tag{4.12}$$

$$\underline{Q}(s)<Q(s),\quad \text{for } s\in\,]0,\underline{\zeta}[\,\cup\,]\zeta,\underline{\eta}_\ell[\,\cup\,]\underline{\eta}_r,\infty[\tag{4.13}$$

$$(1-\lambda)\overline{Q}(b)>Q(b),\quad \text{for } b\in\,]0,\bar{\rho}_\ell[\,\cup\,]\bar{\rho}_r,\zeta[\,\cup\,]\bar{\zeta},\infty[. \tag{4.14}$$

$$\underline{Q}(s)<\overline{Q}(s),\quad \text{for } s\in\,]0,\underline{\zeta}[\,\setminus\{\underline{\rho}\},\quad \overline{Q}(s)>\underline{Q}(s),\quad \text{for } s\in\,]\bar{\zeta},\infty[\,\setminus\{\bar{\eta}\}, \tag{4.15}$$

$$1+(1-\lambda)\overline{Q}(s)s<0,\quad \text{for } s\in\,]0,\bar{\zeta}[,\quad\text{and}\quad 1+s\underline{Q}(s)>0\quad \text{for } s\in\,]\underline{\zeta},\infty[, \tag{4.16}$$

$$\underline{Q}'(s)\begin{cases}=0, & \text{for } s\in\,]\underline{\eta}_\ell,\underline{\eta}_r[,\\ <0, & \text{for } s\in\,]\underline{\rho},\underline{\eta}_\ell[\setminus\{\underline{\zeta}\},\\ >0, & \text{for } s\in\,]0,\underline{\rho}[\,\cup\,]\underline{\eta}_r,\infty[,\end{cases}\quad\text{and}\quad \overline{Q}'(s)\begin{cases}=0, & \text{for } s\in\,]\bar{\rho}_\ell,\bar{\rho}_r[,\\ <0, & \text{for } s\in\,]\bar{\rho}_r,\bar{\eta}[\setminus\{\bar{\zeta}\},\\ >0, & \text{for } s\in\,]0,\underline{\rho}_\ell[\,\cup\,]\bar{\eta},\infty[,\end{cases} \tag{4.17}$$

All these observations (see also Figure 4.2) are consistent with Remarks 4.4 and 4.5 (see also Figure 4.1) as λ goes to 0. In a market with proportional transaction costs, the signals for the investor to change their trading behaviours are $\underline{\rho}$ and $\bar{\eta}$, as well as $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\underline{\eta}_\ell$ and $\underline{\eta}_r$.

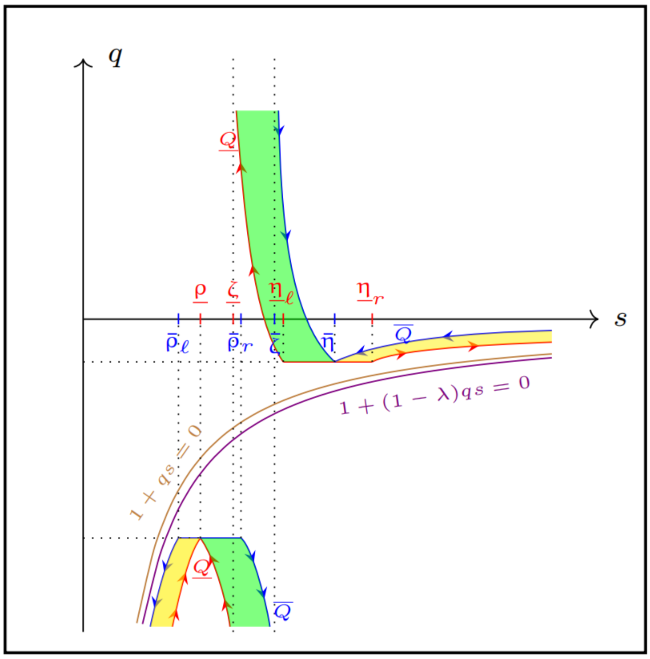

Figure 4.2: The optimal strategy is such that $(S, \mathcal{Q})$ is kept inside the blue and yellow areas or on their boundaries. The investor trades continuously, and only buys on the red curve, i.e. $(S, \mathcal{Q}) = (S, \underline{Q}(S))$, and sells on the blue curve i.e. $(S, \mathcal{Q}) = (S, \overline{Q}(S))$, and takes no actions inside the yellow and green areas. When $(\mathcal{Q}, S)$ hits the red (ress., blue) boundary from the yellow region, the investor buys (resp., sells) when the stock price increases (resp., decreases) and keeps $(S, \mathcal{Q}) = (S, \underline{Q}(S))$ (resp., $(S, \mathcal{Q}) = (S, \overline{Q}(S))$). While $(\mathcal{Q}, S)$ hits the red (ress., blue) boundary from the green region, the investor buys (resp., sells) when the stock price decreases (resp., increases).

The optimal trading strategy $(\vartheta^{0,\star}, \vartheta^\star)$ for the optimal growth rate problem is such that

$$\vartheta_T^{\star,+} = \int_0^T \frac{\vartheta_t^{0,\star}}{1 + S_t \underline{Q}(S_t)} \mathbf{1}_{\{\mathcal{Q}_t = \underline{Q}(S_t)\} \cap \{S_t \neq \underline{\zeta}\}} \, \mathrm{d}\underline{Q}(S_t) \tag{4.18}$$

$$\text{and} \quad \vartheta_T^{\star,-} = \int_0^T \frac{\vartheta_t^{0,\star}}{1 + (1-\lambda) S_t \overline{Q}(S_t)} \mathbf{1}_{\{\mathcal{Q}_t = \overline{Q}(S_t)\} \cap \{S_t \neq \bar{\zeta}\}} \, \mathrm{d}\overline{Q}(S_t), \tag{4.19}$$

for $T > 0$. Such a strategy also satisfies (4.8).

4.3 The optimal problem in the shadow market

In this section, we will consider the optimal control problem in a fictitious frictionless market with objective (1.14). We will show that there exists an optimal strategy in the fictitious frictionless market which is also optimal for the original problem formulated in the Section 4.1.

Definition 4.2 An admissible self-financing trading strategy $(\hat{\vartheta}^0, \hat{\vartheta})$, with initial condition $(\hat{\vartheta}^0_{0-}, \hat{\vartheta}_{0-}) = (x, y)$ belonging to $\hat{\mathcal{A}}_\lambda(x, y)$ is associated with a price process $\hat{S}$ such that
(i) (self-financing condition)

$$\hat{\vartheta}^0_T = \hat{\vartheta}^0_0 + \int_0^T \hat{S}_t \, \mathrm{d}\hat{\vartheta}_t.$$

(ii) (admissibility) The liquidation value $\hat{V}$ given by (1.15) is positive P-a.s. for all $t \geq 0$.

With each admissible self-financing trading strategy $(\hat{\vartheta}^0, \hat{\vartheta}) \in \hat{\mathcal{A}}$, we associate the expected growth rate $\hat{J}$ given by (1.14). The objective of the optimal problem is to maximise $\hat{J}$ over all $(\hat{\vartheta}^0, \hat{\vartheta}) \in \hat{\mathcal{A}}_\lambda(x, y)$. In light of Lemma 4.1.1, we write $\hat{\mathcal{A}}_\lambda$ in place of $\hat{\mathcal{A}}_\lambda(x, y)$ unless otherwise stated. We will show in Lemma 4.3.1 that if the optimal strategy is such that the controller only buys (resp., sells) when $\hat{S} = S$ (resp., $\hat{S} = (1-\lambda)S$), then this strategy is also the optimal strategy for the original growth rate problem.

Definition 4.3 A price process $\hat{S}$ is a shadow price for the bid-ask spread $\big[(1-\lambda)S, S\big]$ with associate optimal strategy $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) \in \hat{\mathcal{A}}$, if the following statements hold true.
(i) The price process $\hat{S}$ takes values in $\big[(1-\lambda)S, S\big]$.
(ii) The optimal problem in the fictitious frictionless market formulated in this section has an optimal trading strategy $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$ such that $\hat{\vartheta}^\star$ only increases (resp., decreases) on the set $\{\hat{S} = S\}$ (resp., $\{\hat{S} = (1-\lambda)S\}$).
(iii) The process $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$ is of finite variation.

Lemma 4.3.1 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. Suppose that $\hat{S}$ is a shadow price process for the bid-ask spread $\big[(1-\lambda)S, S\big]$ with associate optimal strategy $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$. Suppose that one of the following statements hold true:*
(i) *If $m(]0, \infty[) < \infty$, then*

$$\mathsf{E}[\tau_k] < \infty \quad \textit{for all } k = 0, 1, 2, \ldots, \quad \textit{and} \quad \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) = \lim_{T\uparrow\infty} \frac{1}{T} \, \mathsf{E}\left[\ln\bigl(\hat{V}_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)\bigr)\right],$$

where $\tau_k = \inf\{t > k \mid \hat{\vartheta}^{0,\star}_t, \hat{\vartheta}^\star_t > 0\}$.
(ii) *If $m(]0, \infty[) = \infty$, then*

$$\left| \frac{\hat{\vartheta}^\star S_T}{V_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} \right| \leq C_2, \tag{4.20}$$

for some $C_2 > 0$.
For sufficiently small λ, the equalities

$$\textit{and} \quad \sup_{\mathcal{A}_\lambda} J(\vartheta^0, \vartheta) = \sup_{\hat{\mathcal{A}}_\lambda} \hat{J}(\hat{\vartheta}^0, \hat{\vartheta}) = \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star). \tag{4.21}$$

hold true.

Proof. Given any $(\vartheta^0, \vartheta) \in \mathcal{A}_\lambda(x, y)$, let $\hat{\vartheta} = \vartheta$, which implies $\hat{\vartheta}\hat{S} \geq \vartheta(1 - \lambda \mathbf{1}_{\{\vartheta>0\}})S$, and a self-financing trading strategy $(\hat{\vartheta}^0, \hat{\vartheta})$ is such that

$$\hat{\vartheta}^{0,+}_T = \int_0^T \hat{S}_t \, \mathrm{d}\hat{\vartheta}^-_t \geq \int_0^T (1-\lambda) S_t \, \mathrm{d}\vartheta^-_t = \vartheta^{0,+}_T \quad \text{and} \quad \hat{\vartheta}^{0,-}_T = \int_0^T \hat{S}_t \, \mathrm{d}\hat{\vartheta}^+_t \leq \int_0^T S_t \, \mathrm{d}\vartheta^+_t = \vartheta^{0,-}_T.$$

It follows that $\hat{\vartheta}^0_T \geq \vartheta^0_T$ and $\hat{V}_T(\hat{\vartheta}^0, \hat{\vartheta}) \geq V_T(\vartheta^0, \vartheta)$ for all $T \geq 0$, and

$$\sup_{\mathcal{A}_\lambda} J(\vartheta^0, \vartheta) \leq \sup_{\hat{\mathcal{A}}_\lambda} \hat{J}(\hat{\vartheta}^0, \hat{\vartheta}) = \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star). \tag{4.22}$$

Other other hand, if $\hat{\vartheta}^{0,\star}_T, \hat{\vartheta}^\star_T \geq 0$, then

$$V_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) \geq (1 - \lambda)\hat{V}_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star).$$

This observation and (i) in this lemma imply that

$$\sup_{\mathcal{A}_\lambda} J(\vartheta^0, \vartheta) \geq \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) + \lim_{T \uparrow \infty} \frac{\ln(1 - \lambda)}{T} = \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star).$$

If (ii) in this lemma holds true and $\lambda < \frac{1}{C_2}$, then

$$1 \geq \frac{V_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)}{\hat{V}_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} = 1 - \frac{\hat{\vartheta}^\star(\hat{S}_T - (1 - \lambda \mathbf{1}_{\{\hat{\vartheta}^\star>0\}})S_T)}{\hat{V}_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} \geq 1 - \lambda \frac{|\hat{\vartheta}^\star| S_T}{V_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} \geq 1 - C_2\lambda$$

and

$$\sup_{\mathcal{A}_\lambda} J(\vartheta^0, \vartheta) \geq \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) + \lim_{T \uparrow \infty} \frac{\ln(1 - C_2\lambda)}{T} = \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star).$$

□

Remark 4.6 If the optimal strategy is as described in Section 4.2 (see also Figure 4.2) and Assumption 4.1 holds true, then the first condition in (i) in this lemma holds. We will show in Theorem 4.3.8 that the limit in (i) is indeed the case. We will show in the proof of Theorem 4.3.7 that $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$ constructed in Section 4.3.2 satisfies (ii) thanks to Assumption 4.3.

4.3.1 Heuristic derivation of the shadow price function

We will construct a shadow price process $\hat{S}$ of the form $\hat{S} = g(S, A, B, \lambda)$ within the bid-ask spread $[(1 - \lambda)S, S]$ with associated optimal strategy $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$, where A and B are càdlàg $(\mathcal{F}_t)$-adapted processes of finite variation such that

$$A_t = \int_0^t \mathbf{1}_{\{S_u = A_u\} \cup \{S_u = B_u\}} \, \mathrm{d}A_u \quad \text{and} \quad B_u = \int_0^t \mathbf{1}_{\{S_u = A_u\} \cup \{S_u = B_u\}} \, \mathrm{d}B_u, \quad \text{for } t > 0. \tag{4.23}$$

The process A (resp., B) denotes the buying (resp., selling) boundary, i.e.

$$\hat{\vartheta}_t^{\star,+} = \int_0^t \mathbf{1}_{\{S_u=A_u\}}\,\mathrm{d}\hat{\vartheta}_u^{\star,+} \quad \left(\text{resp., } \hat{\vartheta}_t^{\star,-} = \int_0^t \mathbf{1}_{\{S_u=B_u\}}\,\mathrm{d}\hat{\vartheta}_u^{\star,-}\right) \quad \text{for all } t>0. \tag{4.24}$$

The investor only buys (resp., sells) stocks when $S = A$ (resp., $S = B$). The function g is C^2 with respect to s and satisfies the so-called "principle of smooth fit"

$$g(A,A,B,\lambda) = A, \quad g_s(A,A,B,\lambda) = 1, \tag{4.25}$$

$$g(B,A,B,\lambda) = (1-\lambda)B \quad \text{and} \quad g_s(B,A,B,\lambda) = 1-\lambda. \tag{4.26}$$

Furthermore, the function g and the processes A and B are such that

$$\begin{aligned} &g_s(S,A,B,\lambda) > 0 \\ \text{and} \quad &g_a(A,A,B,\lambda) = g_a(B,A,B,\lambda) = g_b(A,A,B,\lambda) = g_b(B,A,B,\lambda) = 0. \end{aligned} \tag{4.27}$$

In such a case, we use Itô's formula to calculate

$$\frac{\mathrm{d}\hat{S}_t}{\hat{S}_t} = \hat{\mu}_t\,\mathrm{d}t + \hat{\sigma}_t\,\mathrm{d}W_t, \tag{4.28}$$

where

$$\hat{\mu}_t = \frac{g_s(S_t,A_t,B_t,\lambda)\mu(S_t)S_t + \frac{1}{2}g_{ss}(S_t,A_t,B_t,\lambda)\sigma^2(S_t)S_t^2}{g(S_t,A_t,B_t,\lambda)} \tag{4.29}$$

$$\text{and} \quad \hat{\sigma}_t = \frac{g_s(S_t,A_t,B_t,\lambda)\sigma(S_t)S_t}{g(S_t,A_t,B_t,\lambda)} > 0. \tag{4.30}$$

For examples of the shadow price function see Figure 4.3 and Figure 4.4.

We will show in Section 4.3.2 that $A \wedge B \le S \le A \vee B$, and A and B are continuous on $\{A \ne \underline{\rho}\} \cup \{B \ne \bar{\eta}\}$, where $\underline{\rho}$ and $\bar{\eta}$ are the turning points of $\overline{Q}$ and $\underline{Q}$ as in Section 4.2. When $B = \bar{\eta}$, A could have jumps, while $A = \underline{\rho}$, B could have jumps. We will show that for an appropriate turning point, the controller should not trade when the jumps happen. The optimal trading strategy is as illustrated in Table 4.1. The turning points $\underline{\rho}$, $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\bar{\eta}$, $\underline{\eta}_\ell$ and $\underline{\eta}_r$ of $\underline{Q}$ and $\overline{Q}$, which come from the turning points ρ and η of Q, are signals where the controller changes their trading behaviours. See also Figures 4.2 and 4.4 for these points.

The following lemma gives a sufficient condition for $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$ to be optimal.

Lemma 4.3.2 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. Suppose that $\hat{S}$ of the form $g(S,A,B,\lambda)$ is a shadow price process for the bid-ask spread $\big[(1-\lambda)S, S\big]$ such that A and B are of finite variation and satisfy (4.23), and the boundary conditions (4.25)–(4.27) hold true. If $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) \in \hat{\mathcal{A}}_\lambda$ satisfies (4.24) and is such that*

$$\frac{\hat{\vartheta}_t^\star \hat{S}_t}{\hat{V}_t(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} = \frac{\hat{\mu}_t}{\hat{\sigma}_t^2}, \tag{4.31}$$

Table 4.1: Optimal Trading Strategy for $\lambda > 0$.

Domain	Trading signal	Trading action
$S < \underline{\rho}$ or $S > \underline{\eta}_r$	$S = A$ and S increases	Buy and keep $\mathcal{Q} = \underline{Q}(A)$ and $A = S$
$S < \bar{\rho}_\ell$ or $S > \bar{\eta}$	$S = B$ and S decreases	Sell and keep $\mathcal{Q} = \overline{Q}(B)$ and $B = S$
$\underline{\rho} < S < \underline{\eta}_\ell$	$S = A$ and S decreases	Buy and keep $\mathcal{Q} = \underline{Q}(A)$ and $A = S$
$\bar{\rho}_r < S < \bar{\eta}$	$S = B$ and S increases	Sell and keep $\mathcal{Q} = \overline{Q}(B)$ and $B = S$

where $\hat{\mu}$ and $\hat{\sigma}$ are given by (4.29) and (4.30), then

$$\delta_\lambda := \sup_{\hat{\mathcal{A}}_\lambda} \hat{J}(\hat{\vartheta}^0, \hat{\vartheta}) = \hat{J}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star) = \limsup_{T\uparrow\infty} \frac{1}{T}\, \mathsf{E}\left[\int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\, \mathrm{d}t\right] \le \delta_0, \tag{4.32}$$

where

$$\delta_0 = \sup_{\mathcal{A}_0} J(\vartheta^0, \vartheta) \begin{cases} \le C, & \text{if (i) in Remark 4.3 is true,} \\ = \frac{1}{m(]0,\infty[)} \int_0^\infty \frac{\mu^2(s)}{\sigma^2(s)}\, m(\mathrm{d}s), & \text{if (ii) in Remark 4.3 is true,} \end{cases}$$

is the optimal growth rate for the original frictionless market with $\lambda = 0$.

Proof. Let $(\hat{\vartheta}^0, \hat{\vartheta})$ be any strategy in $\hat{\mathcal{A}}_\lambda$. With loss of generality, we assume $\hat{V}_0(\hat{\vartheta}^0, \hat{\vartheta}) = 1$. Using Itô's formula, we calculate

$$\ln \hat{V}_T(\hat{\vartheta}^0, \hat{\vartheta}) = \int_0^T \left(\frac{\hat{\vartheta}_t \hat{\mu}_t \hat{S}_t}{\hat{V}_t(\hat{\vartheta}^0, \hat{\vartheta})} - \frac{1}{2} \frac{\hat{\vartheta}_t^2 \hat{\sigma}_t^2 \hat{S}_t^2}{\hat{V}_t^2(\hat{\vartheta}^0, \hat{\vartheta})} \right) \mathrm{d}t + \hat{M}_T \le \int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\, \mathrm{d}t + \hat{M}_T, \tag{4.33}$$

where

$$\hat{M}_T = \int_0^T \frac{\hat{\vartheta}_t \hat{\sigma}_t \hat{S}_t}{\hat{V}_t(\hat{\vartheta}^0, \hat{\vartheta})}\, \mathrm{d}W_t,$$

and the equality holds true if and only if

$$\frac{\hat{\vartheta}_t \hat{S}_t}{\hat{V}_t(\hat{\vartheta}^0, \hat{\vartheta})} = \frac{\hat{\mu}_t}{\hat{\sigma}_t^2}. \tag{4.34}$$

Consider any sequence (τ_n) of localizing times of local martingale $\hat{M}$ and let

$$\mathbf{t}_m := \inf\left\{ t > 0 \,\middle|\, \hat{V}_t(\hat{\vartheta}^0, \hat{\vartheta}) < \frac{1}{m} \right\}.$$

Using the monotone convergence theorem and Fatou's lemma, (4.33) implies that

$$\mathsf{E}\big[\ln \hat{V}_T(\hat{\vartheta}^0, \hat{\vartheta})\big] \le \lim_{m\uparrow\infty} \liminf_{n\uparrow\infty} \mathsf{E}\big[\ln \hat{V}_{T\wedge \mathbf{t}_m \wedge \tau_n}(\hat{\vartheta}^0, \hat{\vartheta})\big] \le \mathsf{E}\left[\int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\, \mathrm{d}t\right].$$

On the other hand, we use the monotone convergence theorem and Fatou's lemma again to obtain

$$\mathsf{E}\big[\ln \hat{V}_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)\big] \geq \lim_{m\uparrow\infty} \limsup_{n\uparrow\infty} \mathsf{E}\big[\ln \hat{V}_{T\wedge \tilde{\mathbf{t}}_m \wedge \tau_n}(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)\big] = \mathsf{E}\bigg[\int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\,\mathrm{d}t\bigg],$$

where

$$\tilde{\mathbf{t}}_m := \inf\bigg\{t > 0 \bigg| \hat{V}_t(\hat{\vartheta}^0, \hat{\vartheta}) > m\bigg\}.$$

These two inequalities imply the equality

$$\mathsf{E}\big[\ln \hat{V}_T(\hat{\vartheta}^{\star,0}, \hat{\vartheta}^\star)\big] = \mathsf{E}\bigg[\int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\,\mathrm{d}t\bigg]$$

and the equality in (4.32) holds true. Furthermore, if we consider a frictionless market with price process S and let $\vartheta^f = \hat{\vartheta}^\star$, then we have $\vartheta_T^{0,f} \geq \hat{\vartheta}_T^{0,\star}$ and

$$\mathsf{E}\big[\ln V_T(\hat{\vartheta}^{\star,0}, \hat{\vartheta}^\star)\big] \leq \mathsf{E}\big[\ln V_T(\hat{\vartheta}^{f,0}, \hat{\vartheta}^f)\big] \leq \mathsf{E}\bigg[\int_0^T \frac{1}{2}\frac{\mu^2(S_t)}{\sigma^2(S_t)}\,\mathrm{d}t\bigg] \tag{4.35}$$

for all $T > 0$. Furthermore, if (i) in Remark 4.3 holds true, then

$$\limsup_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\bigg[\int_0^T \frac{1}{2}\frac{\mu^2(S_t)}{\sigma^2(S_t)}\,\mathrm{d}t\bigg] \leq C.$$

If (ii) in Remark 4.3 holds true, then we use the ergodic results (see e.g., Borodin and Salminen [14, Ch II.6]) to obtain.

$$\lim_{T\uparrow\infty} \frac{1}{T}\,\mathsf{E}\bigg[\int_0^T \frac{1}{2}\frac{\mu^2(S_t)}{\sigma^2(S_t)}\,\mathrm{d}t\bigg] = \frac{1}{m(]0,\infty[)}\int_0^\infty \frac{\mu^2(s)}{\sigma^2(s)}\,m(\mathrm{d}s).$$

□

The Merton proportion condition (4.31) shows that

$$\begin{aligned}\frac{\hat{\mu}_t}{\hat{\sigma}_t^2} = g(S_t)\frac{g_s(S_t)\mu(S_t)S_t + \frac{1}{2}g_{ss}(S_t)\sigma^2(S_t)S_t^2}{g_s^2(S_t)\sigma^2(S_t)S_t^2} &= \frac{\hat{\vartheta}_t\hat{S}_t}{\hat{\vartheta}_t^0 + \hat{\vartheta}_t\hat{S}_t} \\ &= \frac{\underline{h}_t g(S_t)}{1 - A_t\underline{h}_t + \underline{h}_t g(S_t)} \qquad (4.36)\\ &= \frac{\bar{h}_t g(S_t)}{1 - (1-\lambda)B_t\bar{h}_t + \bar{h}_t g(S_t)}, \qquad (4.37)\end{aligned}$$

where we write $g(\cdot)$ in place of $g(\cdot, A, B, \lambda)$ to simplify the notation, and

$$\underline{h}_t = \frac{\hat{\vartheta}_t}{\hat{\vartheta}_t^0 + \hat{\vartheta}_t A_t} \quad \text{and} \quad \bar{h}_t = \frac{\hat{\vartheta}_t}{\hat{\vartheta}_t^0 + \hat{\vartheta}_t(1-\lambda)B_t} = \frac{\underline{h}_t}{1 + ((1-\lambda)B_t - A_t)\underline{h}_t}. \tag{4.38}$$

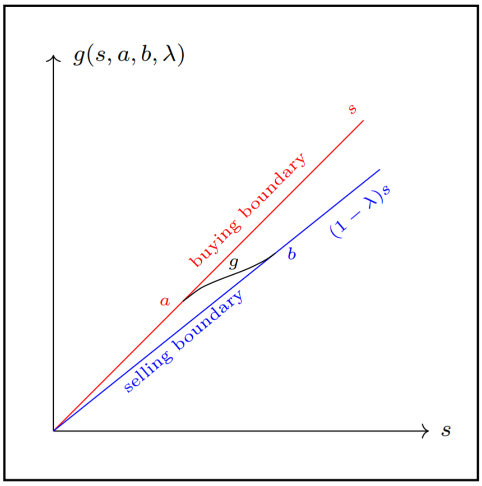

Figure 4.3: The shadow price function in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$, $\sigma = 0.2$, $\lambda = 0.2$, $a = 0.7930$ and $b = 1.3595$.

If $A_t \wedge B_t < S_t < A_t \vee B_t$, for some $\tau_1 < t < \tau_2$, where τ_1 and τ_2 are $(\mathcal{F}_t)$-stopping times, then A, B, $\underline{h}$ and $\bar{h}$ are constant, and the Merton proportion condition yields the ODE

$$g_{ss}(s) = \frac{2g_s^2(s)\underline{h}}{1 - a\underline{h} + \underline{h}g(s)} - 2g_s(s)\frac{\Theta(s)}{s} = \frac{2g_s^2(s)\bar{h}}{1-(1-\lambda)b\bar{h} + \bar{h}g(s)} - 2g_s(s)\frac{\Theta(s)}{s}, \tag{4.39}$$

for $a \wedge b < s < a \vee b$, where

$$a = A_t, \quad b = B_t, \quad \underline{h} = \underline{h}_t, \quad \bar{h} = \bar{h}_t \quad \text{and} \quad g(s) = g(s,a,b,\lambda),$$

and Θ is as in (1.13). In the following lemma, we show the existence of the solution to the ODE (4.39). See Figure 4.3 for an example of the shadow price function with some given $a, b > 0$.

Lemma 4.3.3 *Define*

$$g(s,a,b,\lambda) = a + \frac{p_a(s)}{1 - \underline{h}(a,b,\lambda)p_a(s)}, \tag{4.40}$$

for $a, b, s > 0$, $a \neq b$, $a \neq (1-\lambda)b$ *and* $1 - \underline{h}(a,b,\lambda)p_a(s) \neq 0$, *where* p_a *is as in (4.3), and*

$$\underline{h} = \underline{h}(a,b,\lambda) = \frac{1}{p_a(b)} - \frac{1}{(1-\lambda)b - a}. \tag{4.41}$$

If

$$G(a,b,\lambda) := \sqrt{1-\lambda}p_a(b) - ((1-\lambda)b - a)\sqrt{p_a'(b)} = 0, \tag{4.42}$$

then

$$\begin{cases} g_{ss}(s) = \frac{2g_s^2(s)\underline{h}(a,b,\lambda)}{1-a\underline{h}(a,b,\lambda)+\underline{h}(a,b,\lambda)g(s)} - 2g_s(s)\frac{\Theta(s)}{s}, & \text{for } s \in]a\wedge b, a \vee b[, \\ g(a) = a, \quad g(b) = (1-\lambda)b, \quad g_s(a) = 1, \quad g_s(b) = 1-\lambda, \\ g_a(a) = g_a(b) = g_b(a) = g_b(b) = 0, \end{cases} \tag{4.43}$$

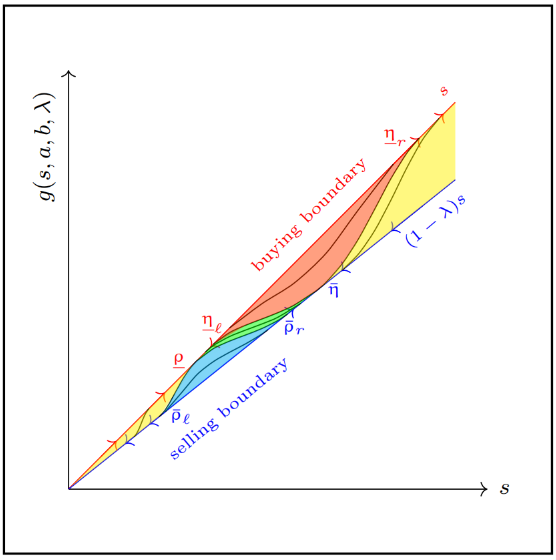

Figure 4.4: Shadow price curves in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$, $\sigma = 0.2$ and $\lambda = 0.2$, for different pair (a, b). In such a case, $\underline{\rho} \approx 0.7930$, $\bar{\rho}_\ell \approx 0.5608$, $\bar{\rho}_r \approx 1.3595$, $\bar{\eta} \approx 1.5403$, $\underline{\eta}_\ell \approx 0.8984$ and $\underline{\eta}_r \approx 2.1778$. The investor only buys on the red curve $(g(a) = a)$ and sells on the blue curve i.e. $g(b) = (1-\lambda)b)$. For the shadow price curves in the yellow region, the investor buys (resp., sells) when $(S, g(S))$ hits the red (resp., blue) curve and the stock price increases (resp., decreases). Furthermore, A (resp., B) increases (resp., decreases) and $A = S$ (resp., $B = S$). For the shadow price curves in the green region, the investor buys (resp., sells) when $(S, g(S))$ hits the red (resp., blue) curve and the stock price decreases (resp., increases). Furthermore, A (resp., B) decreases (resp., increases) and $A = S$ (resp., $B = S$). The shadow price curves do not exist in the orange and cyan regions for the optimal growth rate problem, but we could construct them in a similar way for the finite time scenario. Furthermore, the investor only buys (resp., sells) when $(S, g(S))$ hits the red (resp., blue) curve from the orange (resp., cyan) region. This would require substantial extra analysis that goes beyond the scope of the present chapter.

where we write $g(\cdot)$ in place of $g(\cdot, a, b, \lambda)$. Furthermore, the following statements hold true.

(i) *Either $b < a$ or $a < (1-\lambda)b$ holds true, and*

$$1 - \underline{h}(a, b, \lambda)p_a(s) \geq \min\left\{1 \wedge \frac{p_a(b)}{(1-\lambda)b - a}\right\} > 0 \quad \text{for all } s \in \,]a \wedge b, a \vee b[. \tag{4.44}$$

(ii) *$g_s(s, a, b, \lambda) > 0$ for all $s \in \,]a \wedge b, a \vee b[$.*

(iii) *The function g can be rewritten as*

$$g(s, a, b, \lambda) = (1-\lambda)b + \frac{(1-\lambda)p_b(s)}{1 - (1-\lambda)\bar{h}p_b(s)}, \tag{4.45}$$

where

$$\bar{h}(a, b, \lambda) := \frac{1}{(1-\lambda)p_b(a)} + \frac{1}{(1-\lambda)b - a} = \frac{\underline{h}}{1 + ((1-\lambda)b - a)\underline{h}}. \tag{4.46}$$

The function G can be rewritten as

$$G(a,b,\lambda) = p_a'(b)\mathbf{G}(a,b,\lambda) := p_a'(b)\big(-\sqrt{1-\lambda}p_b(a) - \big((1-\lambda)b - a\big)\sqrt{p_b'(a)}\big), \tag{4.47}$$

Also, the inequities

$$1 - \bar{h}(a,b,\lambda)p_b(s) \geq \min\left\{1 \wedge \frac{p_b(a)}{a-(1-\lambda)b}\right\} > 0 \quad \textit{for all } s \in \,]a\wedge b, a\vee b[\tag{4.48}$$

hold true.

Proof. If we define

$$f(s) = \frac{1}{1 - a\underline{h} + \underline{h}g(s)}, \tag{4.49}$$

for some constant $\underline{h}$, then we can calculate

$$f'(s) = -\frac{\underline{h}}{(1-a\underline{h}+\underline{h}g(s))^2}g_s(s)$$
$$\text{and} \quad f''(s) = \frac{\underline{h}}{(1-a\underline{h}+\underline{h}g(s))^2}\left(-g_{ss}(s) + \frac{2g_s^2(s)\underline{h}}{1-a\underline{h}+\underline{h}g(s)}\right),$$

and the ODE and the first set of boundary conditions in (4.43) becomes

$$\begin{cases} f''(s) = -\frac{2\Theta(s)}{s}f'(s), \\ f(a) = 1, \quad f(b) = \frac{1}{1-a\underline{h}+(1-\lambda)b\underline{h}}, \quad f'(a) = -\underline{h} \quad \text{and} \quad f'(b) = -\frac{(1-\lambda)\underline{h}}{(1-a\underline{h}+(1-\lambda)b\underline{h})^2}. \end{cases} \tag{4.50}$$

If

$$\underline{h} = \frac{1}{p_a(b)} - \frac{1}{(1-\lambda)b - a} \quad \text{and} \quad G(a,b,\lambda) = 0,$$

then the solution to the ODE (4.50) is given by

$$f(s) = 1 - \underline{h}p_a(s) \quad \Longleftrightarrow \quad g(s) = a + \frac{p_a(s)}{1-\underline{h}p_a(s)}.$$

To show the second set of boundary conditions in (4.43), we first use Remark 4.2 to show that

$$\frac{\partial}{\partial a}p_a(s) = -1 + \frac{2\Theta(a)}{a}p_a(s). \tag{4.51}$$

Using this calculation and the definitions of g and $\underline{h}$, we calculate

$$\underline{h}_a(a,b,\lambda) = \frac{1}{p_a^2(b)}\left(1 - \frac{2\Theta(a)}{a}p_a(b)\right) - \frac{1}{((1-\lambda)b-a)^2} \tag{4.52}$$

$$\text{and} \quad g_a(s) = 1 - \frac{1 - \frac{p_a^2(s)}{p_a^2(b)} + \frac{p_a^2(s)}{((1-\lambda)b-a)^2} - \frac{2\Theta(a)}{a}(1 - \frac{p_a(s)}{p_a(b)})p_a(s)}{(1 - \frac{p_a(s)}{p_a(b)} + \frac{p_a(s)}{((1-\lambda)b-a)})^2}. \tag{4.53}$$

It follows that $g_a(a) = g_a(b) = 0$. Also, we calculate

$$\begin{aligned} g_b(s) &= \frac{p_a^2(s)\underline{h}_b}{(1-\underline{h}p_a(s))^2} \\ &= G(a,b,\lambda)\frac{\sqrt{1-\lambda}p_a(b) + ((1-\lambda)b-a)\sqrt{p_a'(b)}}{p_a^2(b)((1-\lambda)b-a)^2}\frac{p_a^2(s)}{(1-\underline{h}p_a(s))^2} = 0 \end{aligned} \tag{4.54}$$

for all s.

The first statement in (i) follows from the fact that $G(a,s,\lambda) > 0$ for any $0 < a < s < \frac{a}{1-\lambda}$. To show (4.44), we use the definitions of (4.3), (4.41) of p and $\underline{h}$, and the equation (4.42) to calculate

$$1 - \underline{h}(a,b,\lambda)p_a(a) = 1 \quad \text{and} \quad 1 - \underline{h}(a,b,\lambda)p_a(b) = \frac{p_a(b)}{(1-\lambda)b-a} = \frac{\sqrt{p_a'(b)}}{\sqrt{1-\lambda}} > 0,$$

and notice that $p_a'(s) > 0$ for all $s > 0$. The proof of (4.48) is similar. The result in (ii) follows from

$$g_s(s) = \frac{p_a'(s)}{(1-\underline{h}(a,b,\lambda)p_a(s))^2} > 0.$$

The proof of (4.45)–(4.47) is straightforward by using Remark 4.2 and (4.42). □

Remark 4.7 An alternative solution to the ODE (4.43) is g given by the lemma, with (a,b) satisfies

$$\sqrt{1-\lambda}p_a(b) + ((1-\lambda)b-a)\sqrt{p_a'(b)} = 0.$$

However, this equation implies that $a < b < (1-\lambda)a$, and there exists $a < s < b$ such that $1 - \underline{h}p_a(s) = 0$.

Lemma 4.3.3, together with (4.36), (4.37) and Lemma 4.3.2, suggest that the buying and selling boundaries A and B should satisfy $G(A,B,\lambda) = 0$, and the trading strategy $(\hat{\vartheta}^0, \hat{\vartheta})$ is optimal if

$$\underline{h}(A_t, B_t, \lambda) = \frac{\hat{\vartheta}_t}{\hat{\vartheta}_t^0 + \hat{\vartheta}A_t} \quad \Leftrightarrow \quad \bar{h}(A_t, B_t, \lambda) = \frac{\hat{\vartheta}_t}{\hat{\vartheta}_t^0 + (1-\lambda)\hat{\vartheta}B_t}. \tag{4.55}$$

The process $Ah(A,b,\lambda)$ (resp., $(1-\lambda)B\bar{h}(A,b,\lambda)$) is the proportion of wealth invested into the stock when the price process S is on the buying (resp., selling) boundary A (resp., B). In the presence of $G(a,b,\lambda) = \mathbf{G}(a,b,\lambda) = 0$, we can rewrite $\underline{h}$ and $\bar{h}$ as follow.

$$\underline{h}(a,b,\lambda) = \frac{1}{(1-\lambda)b-a}\left(\frac{\sqrt{1-\lambda}}{\sqrt{p_a'(b)}} - 1\right) \tag{4.56}$$

$$\text{and} \quad \bar{h}(a,b,\lambda) = \frac{1}{(1-\lambda)b-a}\left(1 - \frac{1}{\sqrt{p_b'(a)}\sqrt{1-\lambda}}\right) = \frac{\sqrt{p_a'(b)}}{\sqrt{1-\lambda}}\underline{h}(a,b,\lambda). \tag{4.57}$$

Furthermore, if we define

$$L(a,b,\lambda) = \sqrt{1-\lambda}b\sqrt{p_a'(b)} - a \tag{4.58}$$

$$\text{and}\quad \hat{Q}(a,b,\lambda) = \frac{1-\frac{\sqrt{p_a'(b)}}{\sqrt{1-\lambda}}}{\sqrt{1-\lambda}b\sqrt{p_a'(b)}-a} = \frac{\underline{h}(a,b,\lambda)}{1-a\underline{h}(a,b,\lambda)} = \frac{\bar{h}(a,b,\lambda)}{1-(1-\lambda)b\bar{h}(a,b,\lambda)}, \tag{4.59}$$

then

$$1-a\underline{h}(a,b,\lambda) = \frac{\sqrt{1-\lambda}}{\sqrt{p_a'(b)}}\frac{1}{(1-\lambda)b-a}L(a,b,\lambda) \tag{4.60}$$

$$\text{and}\quad 1-(1-\lambda)b\bar{h}(a,b,\lambda) = \frac{1}{(1-\lambda)b-a}L(a,b,\lambda), \tag{4.61}$$

and the optimality condition can be equivalently characterized by the stock-cash ratio as follows:

$$\vartheta_t^0 = 0, \quad \text{if } L(A_t,B_t,\lambda)=0 \tag{4.62}$$

$$\text{and}\quad \mathcal{Q}_t = \frac{\vartheta_t}{\vartheta_t^0}\mathbf{1}_{\{\vartheta_t^0\neq 0\}} = \hat{Q}(A_t,B_t,\lambda), \quad \text{if } L(A_t,B_t,\lambda)\neq 0. \tag{4.63}$$

When $\vartheta^0 \neq 0$, (4.63) suggests us to construct the buying boundary $\underline{Q}$ and the selling boundary $\overline{Q}$ associated with the stock-cash ratio, that satisfy the equalities

$$\underline{Q}(A) = \hat{Q}(A,B,\lambda) = \overline{Q}(B). \tag{4.64}$$

The investor only buys when $\underline{Q}(S) = \mathcal{Q}$ ($\Leftrightarrow S = A$) and only sells when $\overline{Q}(S) = \mathcal{Q}$ ($\Leftrightarrow S = B$).

In the following subsections, we will construct the shadow price by (I) finding the solutions to the algebraic equation $G(a,b,\lambda)=0$, (II) deciding suitable solutions such that the shadow price process is within the bid-ask spread, (III) constructing the buying (resp., selling) boundary $\underline{Q}$ (resp., $\overline{Q}$) associated with the stock-cash ratio $\mathcal{Q}$, (IV) constructing A and B. To facilitate the exposition of our analysis, we collect all the results and proofs in Appendix 4.6.

4.3.2 The existence of the shadow price process

We will first consider the solutions to the algebraic equation $G(a,b,\lambda)=0$, where G is given by (4.42) in Lemma 4.3.3. To this end, we consider the families of functions $G(a,\cdot,\lambda)$ and $G(\cdot,b,\lambda)$. We will show in Proposition 4.3.4 that, fix $\lambda \in]0,1[$, there exists $\underline{\eta}_\ell(\lambda) < \eta < \underline{\eta}_r(\lambda)$ (resp., $0 < \bar{\rho}_\ell(\lambda) < \rho < \bar{\rho}_r(\lambda)$) such that

$$\lim_{\lambda\downarrow 0}\underline{\eta}_\ell(\lambda) = \lim_{\lambda\downarrow 0}\underline{\eta}_r(\lambda) = \eta \quad \big(\text{resp.,} \quad \lim_{\lambda\downarrow 0}\bar{\rho}_\ell(\lambda) = \lim_{\lambda\downarrow 0}\bar{\rho}_r(\lambda) = \rho\big)$$

and

$$G(a,.,\lambda)\begin{cases}\text{has 3 zeroes } \beta_1(a,\lambda)<\beta_2(a,\lambda)<\beta_3(a;\lambda), & \text{if } a\in]0,\underline{\eta}_\ell(\lambda)[\,\cup\,]\underline{\eta}_r(\lambda),\infty[,\\ \text{has 2 zeroes } \beta_1(a,\lambda)<\beta_2(a,\lambda)=\beta_3(a,\lambda), & \text{if } a\in \{\underline{\eta}_\ell(\lambda),\underline{\eta}_r(\lambda)\},\\ \text{has 1 zero } \beta_1(a,\lambda), & \text{if } a\in]\underline{\eta}_\ell(\lambda),\underline{\eta}_r(\lambda)[,\end{cases}$$

(resp.,

$$G(.,b,\lambda)\begin{cases}\text{has 3 zeroes } \alpha_1(b,\lambda)<\alpha_2(b,\lambda)<\alpha_3(b,\lambda), & \text{if } b\in]0,\bar{\rho}_\ell(\lambda)[\,\cup\,]\bar{\rho}_r(\lambda),\infty[,\\ \text{has 2 zeroes } \alpha_1(b,\lambda)=\alpha_2(b,\lambda)<\alpha_3(b,\lambda), & \text{if } b\in \{\bar{\rho}_\ell(\lambda),\bar{\rho}_r(\lambda)\},\\ \text{has 1 zero } \alpha_3(b,\lambda), & \text{if } b\in]\bar{\rho}_\ell(\lambda),\bar{\rho}_r(\lambda)[).\end{cases}$$

See Figures 4.5, 4.6 and 4.7 for examples. An outline of our proof of finding zeros (Proposition 4.3.4) is in Figure 4.7.

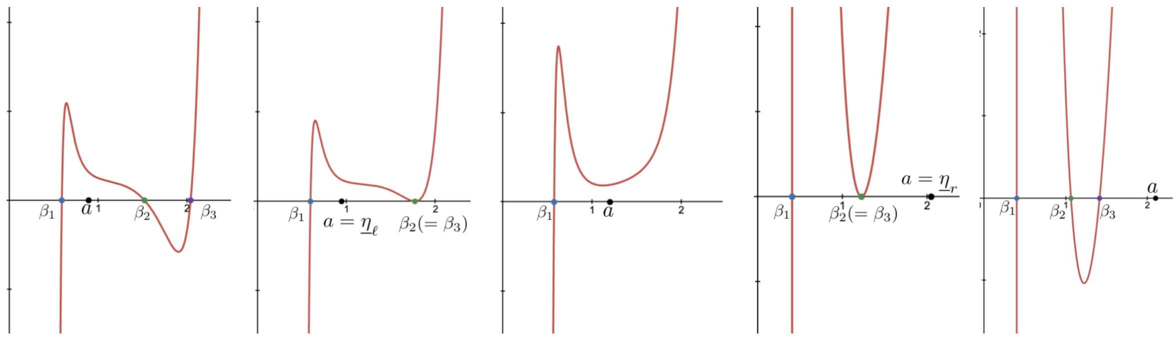

Figure 4.5: Function $G(a,\cdot,\lambda)$ in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$, $\sigma = 0.2$ and $\lambda = 0.2$. Graphs from left to right are such that $a = 0.89$, $a = \underline{\eta}_\ell \approx 0.944414$, $a = 1.2$, $a = \underline{\eta}_r \approx 2.0473255$ and $a = 2.1$.

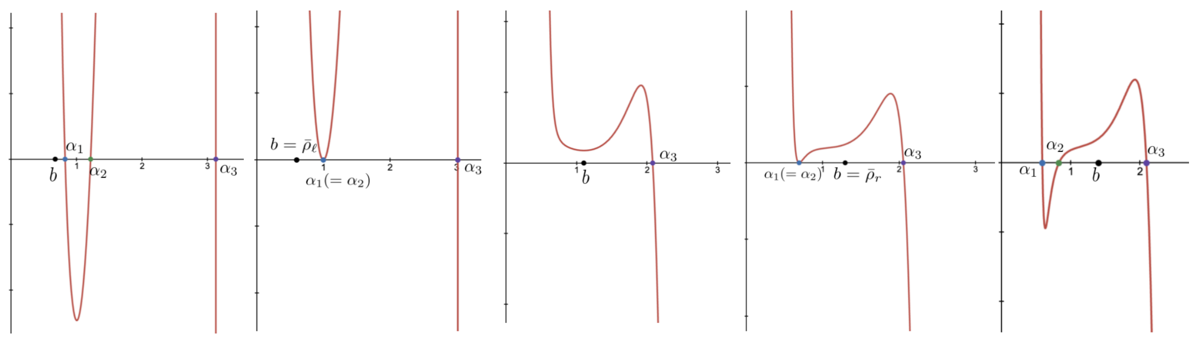

Figure 4.6: Function $\mathbf{G}(\cdot,b,\lambda)$ in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$, $\sigma = 0.2$ and $\lambda = 0.2$.. Graphs from left to right are such that $b = 0.57$, $b = \bar{\rho}_\ell \approx 0.596585$, $b = 1.1$, $b = \bar{\rho}_r \approx 1.29329$ and $b = 1.4$.

We include more properties related to points $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$, $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$ and the functions $G(a,\cdot,\lambda)$ and $G(\cdot,b,\lambda)$ in Proposition 4.3.4. In addition, we will show that

$$\bar{\rho}_r(\lambda) < \beta_3\big(\underline{\eta}_r(\lambda),\lambda\big) < \eta < \beta_2\big(\underline{\eta}_\ell(\lambda),\lambda\big) \quad \text{and} \quad \alpha_2\big(\bar{\rho}_r(\lambda),\lambda\big) < \rho < \alpha_1\big(\bar{\rho}_\ell(\lambda),\lambda\big) < \underline{\eta}_\ell(\lambda)$$

and the restriction and inverse of β_i (resp., α_i), for $i = 1, 2, 3$, are as in Table 4.2 (resp., Table 4.3) below.

Proposition 4.3.4 *Suppose that Assumptions 4.1 holds true. There exist* $\underline{\eta}_\ell, \underline{\eta}_r, \bar{\rho}_\ell, \bar{\rho}_r$: $]0,1[\, \to \,]0,\infty[$ *such that the following statements hold true:*

(i) $\bar{\rho}_\ell(0) = \bar{\rho}_r(0) = \rho$ *and* $\underline{\eta}_\ell(0) = \underline{\eta}_r(0) = \eta$.

(ii) *The restriction of* $\underline{\eta}_\ell$ *and* $\bar{\rho}_\ell$ *(resp.,* $\underline{\eta}_r$ *and* $\bar{\rho}_r$*) on* $[0,1[$ *are strictly decreasing (resp., increasing).*

In the following statements, we fixed $\lambda \in [0,1[$ *and write* $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\underline{\eta}_\ell$, $\underline{\eta}_r$, $\beta_i(\cdot)$ *and* $\alpha_i(\cdot)$ *in place of* $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$, $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$, $\beta_i(\cdot,\lambda)$ *and* $\alpha_i(\cdot,\lambda)$, *for* $i = 1, 2, 3$.

(iii) *For a fixed* $\lambda \in [0,1[$, *there exist* $\beta_1 :]0,\infty[\, \to \,]0,\infty[$ *and* $\beta_2, \beta_3 :]0,\underline{\eta}_\ell] \cup [\underline{\eta}_r,\infty[\to]0,\infty[$ *(resp.,* $\alpha_3 :]0,\infty[\to]0,\infty[$ *and* $\alpha_1, \alpha_2 :]0,\bar{\rho}_\ell] \cup [\bar{\rho}_r,\infty[\, \to \,]0,\infty[$*) such that*

$$G\big(a,\beta_i(a),\lambda\big) = 0 \quad (\textit{resp., } G\big(\alpha_i(b),b,\lambda\big) = 0), \tag{4.65}$$

for $i = 1, 2, 3$, *if* $a \in \,]0,\underline{\eta}_\ell] \cup [\underline{\eta}_r,\infty[$ *(resp.,* $b \in \,]0,\bar{\rho}_\ell] \cup [\bar{\rho}_r,\infty[$*) and for* $i = 1$ *(resp.,* $i = 3$*) , if* $a \in \,]\underline{\eta}_\ell,\underline{\eta}_r[$ *(resp.,* $b \in \,]\bar{\rho}_\ell,\bar{\rho}_r[$*). Furthermore,*

$$\beta_1(a) < \beta_2(a) < \beta_3(a), \quad \textit{for } a \in \,]0,\underline{\eta}_\ell[\, \cup \,]\underline{\eta}_r,\infty[,$$

$$\beta_2(\underline{\eta}_\ell) = \beta_3(\underline{\eta}_\ell), \quad \beta_2(\underline{\eta}_r) = \beta_3(\underline{\eta}_r) \quad \textit{and} \quad G_b(\underline{\eta}_\ell,\beta_2(\underline{\eta}_\ell),\lambda) = G_b(\underline{\eta}_r,\beta_3(\underline{\eta}_r),\lambda) = 0$$

$$(\textit{resp., } \alpha_1(b) < \alpha_2(b) < \alpha_3(b), \quad \textit{for } b \in \,]0,\bar{\rho}_\ell[\, \cup \,]\bar{\rho}_r,\infty[,$$

$$\alpha_1(\bar{\rho}_\ell) = \alpha_2(\bar{\rho}_\ell), \quad \alpha_1(\bar{\rho}_r) = \alpha_2(\bar{\rho}_r) \quad \textit{and} \quad G_a\big(\bar{\rho}_\ell,\alpha_1(\bar{\rho}_\ell),\lambda\big) = G_a\big(\bar{\rho}_r,\alpha_2(\bar{\rho}_r),\lambda\big) = 0\big).$$

(iv) *The following inequities hold true.*

$$G(a,b,\lambda)\begin{cases} < 0, & \textit{for } b \in \,]0,\beta_1(a)[\, \cup \,]\beta_2(a),\beta_3(a)[, \\ > 0, & \textit{for } b \in \,]\beta_1(a),\beta_2(a)[\, \cup \,]\beta_3(a),\infty[, \end{cases} \tag{4.66}$$

where we denote $]\beta_2(a),\beta_3(a)[\, = \emptyset$ *and* $]\beta_1(a),\beta_2(a)[\, \cup \,]\beta_3(a),\infty[\, = \,]\beta_1(a),\infty[$ *if* $a \in \,]\underline{\eta}_\ell,\underline{\eta}_r[$, *and*

$$G(a,b,\lambda)\begin{cases} > 0, & \textit{for } a \in \,]0,\alpha_1(b)[\, \cup \,]\alpha_2(b),\alpha_3(b)[, \\ < 0, & \textit{for } a \in \,]\alpha_1(b),\alpha_2(b)[\, \cup \,]\alpha_3(b),\infty[, \end{cases} \tag{4.67}$$

where we denote $]\alpha_1(b),\alpha_2(b)[\, = \emptyset$ *and* $]0,\alpha_1(b)[\, \cup \,]\alpha_2(b),\alpha_3(b)[\, = \,]0,\alpha_3(a)[$, *if* $b \in \,]\bar{\rho}_\ell,\bar{\rho}_r[$.

(v)

$$\alpha_1(\rho,0) = \alpha_2(\rho,0) = \rho \quad \textit{and} \quad \beta_2(\eta,0) = \beta_3(\eta,0) = \eta,$$

$$\textit{and} \quad \alpha_2(\bar{\rho}_r) < \rho < \alpha_1(\bar{\rho}_\ell) < \underline{\eta}_\ell \quad \textit{and} \quad \bar{\rho}_r < \beta_3(\underline{\eta}_r) < \eta < \beta_2(\underline{\eta}_\ell).$$

(vi) *The range, restriction and inverse of* β_i *(resp.,* α_i*), for* $i = 1, 2, 3$, *are as in Table 4.2 (resp., Table 4.3).*

Unless otherwise stated, we write $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\underline{\eta}_\ell$, $\underline{\eta}_r$, $\beta_i(\cdot)$ and $\alpha_i(\cdot)$ in place of $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$, $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$, $\beta_i(\cdot,\lambda)$ and $\alpha_i(\cdot,\lambda)$, for $i = 1, 2, 3$, for $\lambda > 0$. In the following lemma, we choose suitable candidates β_i and α_i such that the shadow price is in the bid-ask spread.

Table 4.2: Restriction and inverse of β_i.

Zeroes \ Domain	$]0, \alpha_2(\bar{\rho}_r)[$	$]\alpha_2(\bar{\rho}_r), \alpha_1(\bar{\rho}_\ell)[$	$]\alpha_1(\bar{\rho}_\ell), \underline{\eta}_\ell[$	$]\underline{\eta}_\ell, \underline{\eta}_r[$	$]\underline{\eta}_r, \infty[$
β_1	incr. $(\alpha_1\|_{]0,\bar{\rho}_\ell[})^{-1}$		decr. $(\alpha_2\|_{]0,\bar{\rho}_\ell[})^{-1}$		
β_2	decr. $(\alpha_1\|_{]0,\bar{\rho}_r[})^{-1}$	incr. $(\alpha_2\|_{]\bar{\rho}_r,\beta_2(\underline{\eta}_\ell)})^{-1}$		undefined	decr. $(\alpha_3\|_{]0,\beta_3(\underline{\eta}_r)[})^{-1}$
β_3	decr. $(\alpha_2\|_{]\beta_2(\underline{\eta}_\ell),\infty[})^{-1}$			undefined	incr. $(\alpha_3\|_{]\beta_3(\underline{\eta}_r),\infty[})^{-1}$

We denote by decr.(resp., incr.) the restriction of the function on the corresponding interval is strictly decreasing (resp., increasing).
We write $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\underline{\eta}_\ell$, $\underline{\eta}_r$, $\beta_i(\cdot)$ and $\alpha_i(\cdot)$ in place of $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$, $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$, $\beta_i(\cdot,\lambda)$ and $\alpha_i(\cdot,\lambda)$, for $i = 1, 2, 3$.

Table 4.3: Range, restriction and inverse of α_i.

Zeroes \ Domain	$]0, \bar{\rho}_\ell[$	$]\bar{\rho}_\ell, \bar{\rho}_r[$	$]\bar{\rho}_r, \beta_3(\underline{\eta}_r)[$	$]\beta_3(\underline{\eta}_r), \beta_2(\underline{\eta}_\ell)[$	$]\beta_2(\underline{\eta}_\ell), \infty[$
α_1	incr. $(\beta_1\|_{]0,\alpha_1(\bar{\rho}_\ell)[})^{-1}$	undefined	decr. $(\beta_2\|_{]0,\alpha_2(\bar{\rho}_r)[})^{-1}$		
α_2	decr. $(\beta_1\|_{]\alpha_1(\bar{\rho}_\ell),\infty[})^{-1}$	undefined	incr. $(\beta_2\|_{]\alpha_2(\bar{\rho}_r),\underline{\eta}_\ell[})^{-1}$		decr. $(\beta_3\|_{]0,\underline{\eta}_\ell[})^{-1}$
α_3	decr. $(\beta_2\|_{]\underline{\eta}_r,\infty[})^{-1}$			incr. $(\beta_3\|_{]\underline{\eta}_r,\infty[})^{-1}$	

We denote by decr.(resp., incr.) the restriction of the function on the corresponding interval is strictly decreasing (resp., increasing).
We write $\bar{\rho}_\ell$, $\bar{\rho}_r$, $\underline{\eta}_\ell$, $\underline{\eta}_r$, $\beta_i(\cdot)$ and $\alpha_i(\cdot)$ in place of $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$, $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$, $\beta_i(\cdot,\lambda)$ and $\alpha_i(\cdot,\lambda)$, for $i = 1, 2, 3$.

Lemma 4.3.5 *Suppose that Assumption 4.1 holds true and fix $a, b > 0$ and $\lambda \in]0, 1[$ such that $G(a, b, \lambda) = 0$, we have*

$$(1-\lambda)s \le g(s,a,b,\lambda) \le s \quad \textit{for all } s \in]a \wedge b, a \vee b[\quad \textit{if and only if either}$$
$$\text{(i.a) } a \in [\alpha_2(\bar{\rho}_r), \underline{\eta}_\ell] \quad \textit{and} \quad b = \beta_2(a) > a,$$
$$\textit{or} \quad \text{(ii.a) } a \in]0, \alpha_1(\bar{\rho}_\ell)] \quad \textit{and} \quad b = \beta_1(a) < a,$$
$$\textit{or} \quad \text{(iii.a) } a \in [\underline{\eta}_r, \infty[\quad \textit{and} \quad b = \beta_3(a) < a,$$

or equivalently either

$$\text{(i.b) } b \in [\bar{\rho}_r(\lambda), \beta_2(\underline{\eta}_\ell)] \quad \textit{and} \quad a = \alpha_2(b) < b,$$
$$\textit{or} \quad \text{(ii.b) } b \in]0, \bar{\rho}_\ell] \quad \textit{and} \quad a = \alpha_1(b) > b,$$
$$\textit{or} \quad \text{(iii.b) } b \in [\beta_3(\underline{\eta}_r), \infty[\quad \textit{and} \quad a = \alpha_3(b) > b,$$

holds true.

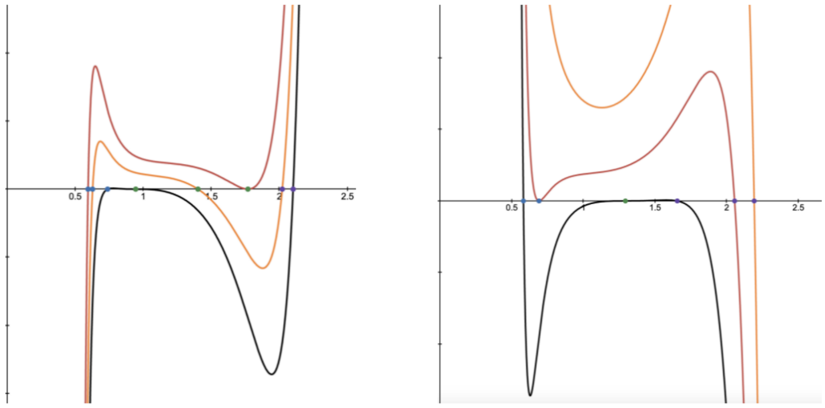

Figure 4.7: Function $G(0.944413, \cdot, \lambda)$ (left) and $\mathbf{G}(\cdot, 1.29329, \lambda)$ (right) in the context of Example 4.1 for $\kappa = 0.5$, $\gamma = 0.1$, $\sigma = 0.2$ and as well as $\lambda = 0$ (black curve), $\lambda = 0.1$ (orange curve) and $\lambda = 0.2$ (red curve). In the proof of Proposition 4.3.4, we show that $G(a, \cdot, 0)$ (resp., $G(\cdot, a, 0)$) has three zeroes except a (resp.,b) equals ρ or η, where two of the three zeroes coincide. Furthermore, $G(s, s, 0) =$ for all $s > 0$ implies that a (resp., b) itself is a zero. When λ become bigger, the graph of $G(a, \cdot, \lambda)$ (resp., $G(\cdot, a, \lambda)$) goes up, which indicate the possible existence of the zeros of $G(a, \cdot, \lambda)$ (resp., $G(\cdot, a, \lambda)$).

This lemma, together with Table 4.2 and 4.3, reveals that if $a \in [\alpha_2(\bar{\rho}_r), \alpha_1(\bar{\rho}_\ell)]$ (resp., $b \in [\beta_3(\underline{\eta}_r), \beta_2(\underline{\eta}_\ell)]$), then the corresponding b (resp., a), such that $G(a, b, \lambda) = 0$, has two possibilities, i.e. β_1 and β_2 (resp., α_2 and α_3). However, if we construct

$$\beta(a, \lambda) = \begin{cases} \beta_1(a), & \text{for } a \in \,]0, \alpha_2(\bar{\rho}_r)[, \\ \beta_1(a) \text{ or } \beta_2(a), & \text{for } a \in \,]\alpha_2(\bar{\rho}_r), \alpha_1(\bar{\rho}_\ell)[, \\ \beta_2(a), & \text{for } a \in \,]\alpha_1(\bar{\rho}_\ell), \underline{\eta}_\ell[, \end{cases}$$

where we write $\beta(\cdot)$ in place of $\beta(\cdot, \lambda)$, then β has at least one jump point $\alpha_2(\bar{\rho}_r) \leq \underline{\rho} \leq \alpha_1(\bar{\rho}_\ell)$ since $\bar{\rho}_\ell = \beta_1(\alpha_1(\bar{\rho}_\ell)) < \rho < \bar{\rho}_r = \beta_2(\alpha_2(\bar{\rho}_r))$, and β_1 and β_2 are increasing. Also, note that an admissible trading should be of finite variation and the optimal trading strategy is characterized $\hat{Q}$ as in (4.59). These observations suggest us to find a (unique) jump point $\alpha_2(\bar{\rho}_r) < \underline{\rho} < \alpha_1(\bar{\rho}_\ell)$ (resp., $\beta_3(\underline{\eta}_r) < \bar{\eta} < \beta_2(\underline{\eta}_\ell)$) such that

$$\hat{Q}(\underline{\rho}, \bar{\rho}_\ell, \lambda) = \hat{Q}(\underline{\rho}, \bar{\rho}_r, \lambda) \quad \big(\text{resp., } \hat{Q}(\underline{\eta}_\ell, \bar{\eta}, \lambda) = \hat{Q}(\underline{\eta}_r, \bar{\eta}, \lambda)\big), \tag{4.68}$$

where

$$\bar{\rho}_\ell = \beta_1(\underline{\rho}) < \underline{\rho} < \bar{\rho}_r = \beta_2(\underline{\rho}) \quad \text{and} \quad \underline{\eta}_\ell = \alpha_2(\bar{\eta}) < \bar{\eta} < \underline{\eta}_r = \alpha_3(\bar{\eta}). \tag{4.69}$$

See also Figures 4.2 and 4.4. We provide a sketch proof of determining $\underline{\rho}$ and $\bar{\eta}$ in Figure 4.8. Furthermore, the continuity of the shadow prices process suggests us to show that

$$g(s, \underline{\eta}_\ell, \bar{\eta}, \lambda) = g(s, \underline{\eta}_r, \bar{\eta}, \lambda) \quad \text{for all } s \in [\underline{\eta}_\ell, \underline{\eta}_r] \tag{4.70}$$

$$\text{and} \quad g(s, \bar{\rho}_\ell, \underline{\eta}, \lambda) = g(s, \bar{\rho}_r, \underline{\eta}, \lambda) \quad \text{for all } s \in [\bar{\rho}_\ell, \bar{\rho}_r]. \tag{4.71}$$

We construct β and α as follows.

$$\beta(a) = \begin{cases} \beta_1(a) < a, & \text{for } a \in]0, \underline{\rho}[, \\ \beta_2(a) > a, & \text{for } a \in [\underline{\rho}, \underline{\eta}_\ell], \\ \beta_3(a) < a, & \text{for } a \in [\underline{\eta}_r, \infty[, \end{cases} \quad \text{and} \quad \alpha(b) = \begin{cases} \alpha_1(b) > b, & \text{for } b \in]0, \bar{\rho}_\ell], \\ \alpha_2(b) < b, & \text{for } b \in [\bar{\rho}_r, \bar{\eta}], \\ \alpha_3(b) > b, & \text{for } b \in]\bar{\eta}, \infty[. \end{cases} \tag{4.72}$$

These construction and Tables 4.2 and 4.3 imply that

$$\beta\big(\alpha(b)\big) = b, \quad \text{for } b \in]0, \bar{\rho}_\ell[\cup [\bar{\rho}_r, \infty[\tag{4.73}$$

$$\text{and} \quad \alpha\big(\beta(a)\big) = a, \quad \text{for } a \in]0, \underline{\eta}_\ell] \cup]\underline{\eta}_r, \infty[. \tag{4.74}$$

In light of (4.64), we define $\underline{Q}$ and $\overline{Q}$ by

$$\underline{Q}(a) = \begin{cases} \hat{Q}\big(a, \beta(a), \lambda\big), & \text{for } a \in]0, \underline{\eta}_\ell[\cup]\underline{\eta}_r, \infty[\setminus \{\underline{\zeta}\}, \\ \hat{Q}\big(\underline{\eta}_\ell, \bar{\eta}, \lambda\big), & \text{for } a \in [\underline{\eta}_\ell, \underline{\eta}_r], \end{cases} \tag{4.75}$$

$$\text{and} \quad \overline{Q}(b) = \begin{cases} \hat{Q}\big(\alpha(b), b, \lambda\big), & \text{for } b \in]0, \bar{\rho}_\ell[\cup]\bar{\rho}_r, \infty[\setminus \{\bar{\zeta}\}, \\ \hat{Q}\big(\underline{\rho}, \underline{\rho}_r, \lambda\big), & \text{for } b \in [\underline{\rho}_\ell, \underline{\rho}_r], \end{cases}, \tag{4.76}$$

where

$$\underline{\zeta} = \big\{a \mid L\big(a, \beta(a, \lambda), \lambda\big) = 0\big\} \in]\bar{\rho}_r, \zeta[\quad \text{and} \quad \bar{\zeta} = \beta(\underline{\zeta}) \in]\zeta, \underline{\eta}_\ell[, \tag{4.77}$$

with L given by (4.58), will be given in the next lemma. The point $\underline{\zeta}$ (resp., $\bar{\zeta}$) is the place where $\underline{Q}$ (resp., $\overline{Q}$) crosses infinity. In view of these definitions and Section 4.2, we will show that

$$L\big(a, \beta(a), \lambda\big) \begin{cases} > 0, & \text{for } a \in]0, \underline{\rho}[\cup]\underline{\zeta}, \underline{\eta}_\ell[, \\ < 0, & \text{for } a \in]\underline{\rho}, \underline{\zeta}[\cup]\underline{\eta}_r, \infty[, \\ = 0, & \text{for } a = \underline{\zeta}, \end{cases} \tag{4.78}$$

and (4.12)–(4.17) in Section 4.2 hold true.

Lemma 4.3.6 *Suppose that Assumption 4.1 holds true. There exist unique* $\underline{\rho}, \underline{\eta}_\ell, \underline{\eta}_r, \bar{\eta}, \bar{\rho}_\ell, \bar{\rho}_r$: $]0,1[\to]0, \infty[$ *such that* $\alpha_2(\bar{\rho}_r) < \underline{\rho} < \alpha_1(\bar{\rho}_\ell)$ *and* $\beta_3(\underline{\eta}_r) < \bar{\eta} < \beta_2(\underline{\eta}_\ell)$, *and (4.68)–(4.71) hold true, where* α_i *and* β_i, *for* $i = 1, 2, 3$, *is as in Proposition 4.3.4. If we define* β *and* α *as in (4.72), as well as* $\underline{Q}$ *and* $\overline{Q}$ *as in (4.75) and (4.76), then (4.77), (4.78) and (4.12)–(4.16) hold true. Furthermore,*

$$\lim_{\lambda \downarrow 0} Q\big(a, \beta(a, \lambda), \lambda\big) = Q(a), \quad \textit{for } a \in \mathbb{R}_+ \setminus \{\zeta\}. \tag{4.79}$$

The final step to establish the shadow price process is to construct A and B mentioned at the beginning of Section 4.3.2. We define sequences of stopping times (τ_n) and (e_n) as well as A and B recursively as follows (see also Table 4.1 and Figure 4.4), as follows

$$\tau_0 = 0, \quad A_0 = \begin{cases} s_0, & \text{if } s_0, \notin]\underline{\eta}_\ell, \underline{\eta}_r], \\ \underline{\eta}_\ell & \text{if } s_0 \in]\underline{\eta}_\ell, \underline{\eta}_r], \end{cases} \quad B_0 = \beta(A_0), \tag{4.80}$$

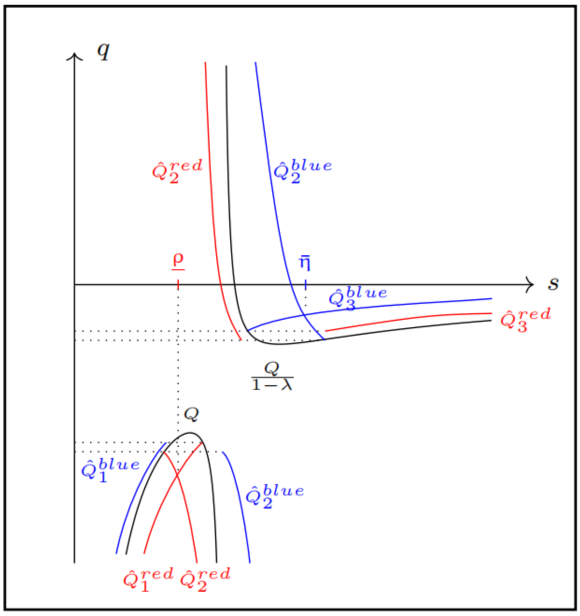

Figure 4.8: Sketch of determining $\underline{\varrho}$, $\bar{\eta}$, $\underline{\zeta}$ and $\bar{\zeta}$. We denote by $\hat{Q}_i^{red}$, $\hat{Q}_i^{red}$, the functions $\hat{Q}(\cdot, \beta_i(\cdot), \lambda)$ and $\hat{Q}(\alpha_i(\cdot), \cdot, \lambda)$, for $i = 1, 2, 3$. We will show that $\hat{Q}_1^{red}(\alpha_{1,2}(\bar{\rho}_\ell)) = Q(\alpha_{1,2}(\bar{\rho}_\ell))$ (resp., $\hat{Q}_2^{red}(\alpha_{1,2}(\bar{\rho}_r)) = Q(\alpha_{1,2}(\bar{\rho}_r)))$ and the restriction of $\hat{Q}_1^{red}$ (resp., $\hat{Q}_2^{red}$) on $]0, \alpha_{1,2}(\bar{\rho}_\ell)[$ (resp., $]\alpha_{1,2}(\bar{\rho}_r), \underline{\zeta}[\cup]\underline{\zeta}, \underline{\eta}_\ell[$) is strictly increasing (resp., decreasing). Similarly, $(1-\lambda)\hat{Q}_2^{blue}(\beta_{2,3}(\underline{\eta}_\ell)) = Q(\beta_{2,3}(\underline{\eta}_\ell))$ (resp., $(1-\lambda)\hat{Q}_2^{blue}(\beta_{2,3}(\underline{\eta}_r))) = Q(\beta_{2,3}(\underline{\eta}_r)))$ and the restriction of $\hat{Q}_2^{blue}$ on $]\bar{\rho}_r, \bar{\zeta}[\cup]\bar{\zeta}, \beta_{2,3}(\underline{\eta}_\ell)[$ (resp., $]\beta_{2,3}(\underline{\eta}_r), \infty[$) is strictly decreasing (resp., increasing). These observations imply the existence of $\underline{\varrho}$, $\bar{\eta}$, (see the intersections) $\underline{\zeta}$ and $\bar{\zeta}$ (see the crossing of $\pm\infty$).

$$\begin{cases} A_t = (\mathfrak{M}_t \wedge \underline{\varrho})\mathbf{1}_{\{A_{\tau_k} \le \underline{\varrho}\}} + (\mathfrak{m}_t \vee \bar{\varrho})\mathbf{1}_{\{\underline{\varrho} < A_{\tau_k} < \underline{\eta}_\ell\}} \\ \qquad\qquad + \big((\underline{\eta}_\ell \wedge \mathfrak{m}_t \vee \underline{\varrho})\mathbf{1}_{\{\mathfrak{M}_t \le \underline{\eta}_r\}} + \mathfrak{M}_t \mathbf{1}_{\{\mathfrak{M}_t > \underline{\eta}_r\}}\big)\mathbf{1}_{\{A_{\tau_k} = \underline{\eta}_\ell\}} + \mathfrak{M}_t \mathbf{1}_{\{A_{\tau_k} > \underline{\eta}_r\}}, \\ B_t = \beta(A_t), \\ e_k = \inf\{t > \tau_k \mid S_t = B_t \text{ and } B_t \neq \bar{\eta}, \quad \text{or} \quad A_t = \underline{\varrho} \text{ and } S_t < \bar{\varrho}_\ell\}, \end{cases} \tag{4.81}$$

for $t \in [\tau_k, e_k[$, and

$$\begin{cases} B_t = (\mathfrak{m}_t \vee \bar{\eta})\mathbf{1}_{\{\bar{\eta} \le B_{e_k}\}} + (\mathfrak{M}_t \wedge \bar{\eta})\mathbf{1}_{\{\bar{\varrho}_r < B_{e_k} < \bar{\eta}\}} \\ \qquad\qquad + \big((\bar{\varrho}_r \vee \mathfrak{M}_t \wedge \bar{\eta})\mathbf{1}_{\{\mathfrak{m}_t \ge \bar{\varrho}_\ell\}} + \mathfrak{m}_t \mathbf{1}_{\{\mathfrak{m}_t < \bar{\varrho}_\ell\}}\big)\mathbf{1}_{\{B_{e_k} = \bar{\rho}_r\}} + \mathfrak{m}_t \mathbf{1}_{\{B_{e_k} \le \bar{\varrho}_\ell\}}, \\ A_t = \alpha(B_t), \\ \tau_{k+1} = \inf\{t > e_k \mid S_t = A_t \text{ and } A_t \neq \underline{\varrho}, \quad \text{or} \quad B_t = \bar{\eta} \text{ and } S_t > \underline{\eta}_r\}, \end{cases} \tag{4.82}$$

for $t \in [e_k, \tau_{k+1}[$, where

$$\mathfrak{m}_t = \sum_{k=0}^{\infty}\Big(\min_{\tau_k \le u \le t} S_u \mathbf{1}_{\{\tau_k \le t < e_k\}} + \min_{e_k \le u \le t} S_u \mathbf{1}_{\{e_k \le t < \tau_{k+1}\}}\Big) \tag{4.83}$$

$$\text{and} \quad \mathfrak{M}_t = \sum_{k=0}^{\infty}\Big(\max_{\tau_k \le u \le t} S_u \mathbf{1}_{\{\tau_k \le t < e_k\}} + \max_{e_k \le u \le t} S_u \mathbf{1}_{\{e_k \le t < \tau_{k+1}\}}\Big). \tag{4.84}$$

For such A and B, we present the optimal strategy as follows. Without loss of generality, we let

$$\vartheta_0^\star = 1 \quad \text{and} \quad \vartheta_0^{0,\star} = \frac{1}{\mathcal{Q}(A_0)} \mathbf{1}_{\{A_0 \neq \underline{\zeta}\}}, \tag{4.85}$$

thanks to Lemma 4.1.1. The strategy $(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)$ as in Figures 4.2 and 4.4 is given by

$$\mathcal{Q}_T^\star = \frac{\vartheta_T^\star}{\vartheta_T^{0,\star}} \mathbf{1}_{\{\vartheta_T^{0,\star} \neq 0\}} = \overline{Q}(B_T) \mathbf{1}_{\{B_T \neq \bar{\zeta}\}} = \underline{Q}(A_T) \mathbf{1}_{\{A_T \neq \underline{\zeta}\}}, \tag{4.86}$$

$$\vartheta_T^{\star,+} = \int_0^T \frac{\vartheta_t^{0,\star}}{1 + S_t \underline{Q}(S_t)} \mathbf{1}_{\{S_t = A_t\}} \, \mathrm{d}\underline{Q}(S_t), \tag{4.87}$$

$$\vartheta_T^{\star,-} = \int_0^T \frac{\vartheta_t^{0,\star}}{1 + (1-\lambda) S_t \overline{Q}(S_t)} \mathbf{1}_{\{S_t = B_t\}} \, \mathrm{d}\overline{Q}(S_t), \tag{4.88}$$

$$\hat{\vartheta}_T^{0,\star,+} = \int_0^T (1-\lambda) S_t \, \mathrm{d}\hat{\vartheta}_t^{\star,-} \quad \text{and} \quad \hat{\vartheta}_T^{0,\star,-} = \int_0^T S_t \, \mathrm{d}\hat{\vartheta}_t^{\star,+}, \tag{4.89}$$

for $T > 0$. Such a strategy also satisfies (4.8).

We conclude the section with the main result of the chapter.

Theorem 4.3.7 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. The processes A and B, given by (4.80)–(4.82) are of finite variation, and the process $\hat{S} = g(S, A, B, \lambda)$, where g is given by (4.40) in Lemma 4.3.3, is a shadow price for the bid-ask spread $\big[(1-\lambda)S, S\big]$ with associated optimal strategy $(\vartheta^{0,\star}, \vartheta^\star)$ given by (4.85)–(4.89). Furthermore, either condition (i) or (ii) in Lemma 4.3.1 holds true, and $(\vartheta^{0,\star}, \vartheta^\star)$ is optimal for the original market.*

4.3.3 A lower bound of the optimal growth rate

In this section, we give a lower bound of the optimal growth rate. This bound goes to the optimal growth rate of the frictionless market as λ goes to 0.

Theorem 4.3.8 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. The optimal growth rate δ_λ satisfies the inequalities*

$$\delta_0 \ge \delta_\lambda \ge \limsup_{T \uparrow \infty} \frac{1}{T} \int_0^T \frac{1}{2} \mathcal{H}^2(S_t, \lambda) \, \mathrm{d}t \tag{4.90}$$

where where δ_0 is the optimal growth rate of the frictionless market as in Lemma 4.3.2 and

$$\mathcal{H}(s,\lambda) = \Big(\underline{h}\big(s,\beta(s),\lambda\big)\mathbf{1}_{]0,\underline{\xi}[}(s) + (1-\lambda)\bar{h}\big(\alpha(s),s,\lambda\big)\mathbf{1}_{[\bar{\xi},\infty[}(s)\Big)\sigma(s)s,$$

with

$$\underline{\xi} = \sup\{s>0 \mid \underline{Q}(s)\geq 0\}, \quad \bar{\xi} = \sup\{s>0 \mid \overline{Q}(s)\geq 0\} = \beta(\underline{\xi}). \tag{4.91}$$

The function $\mathcal{H}$ is such that

$$\lim_{\lambda\downarrow 0}\mathcal{H}(s,\lambda) = \frac{\mu(s)}{\sigma(s)} \quad \textit{and} \quad \lim_{\lambda\downarrow 0}\limsup_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{1}{2}\mathcal{H}^2(S_t,\lambda)\,\mathrm{d}t = \delta_0. \tag{4.92}$$

If $m(]0,\infty[) < 0$, then

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{1}{2}\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2}\,\mathrm{d}t \tag{4.93}$$

exists and

$$\lim_{T\uparrow\infty}\frac{1}{T}\int_0^T \frac{1}{2}\mathcal{H}^2(S_t,\lambda)\,\mathrm{d}t = \frac{1}{m(]0,\infty[)}\int_0^\infty \frac{1}{2}\mathcal{H}^2(s,\lambda)\,m(\mathrm{d}s). \tag{4.94}$$

Proof. In view of (4.28)–(4.30), (4.39) and (4.40), we can see that

$$\frac{\hat{\mu}_t^2}{\hat{\sigma}_t^2} = \frac{\mu_t^2}{\sigma_t^2} + \frac{g_{ss}}{g_s}\mu(S_t)S_t + \frac{1}{4}\frac{g_{ss}^2}{g_s^2}\sigma^2(S_t)S_t^2 = U^2(S_t,A_t,B_t,\lambda)\sigma^2(S_t)S_t^2, \tag{4.95}$$

where

$$U(s,a,b,\lambda) = \frac{p_a'(s)\underline{h}(a,b,\lambda)}{1-\underline{h}(a,b,\lambda)p_a(s)}.$$

Using (4.41), (4.51), (4.52) and (4.107) in Appendix 4.6, we calculate

$$\frac{\mathrm{d}}{\mathrm{d}a}U\big(s,a,\beta(a),\lambda\big) = -2\mathbf{G}_a\big(a,\beta(a),\lambda\big)\frac{\sqrt{p_a'(\beta(a))}p_a'(s)}{(1-\underline{h}p_a(s))^2}\frac{1}{((1-\lambda)\beta(a)-a)^2}. \tag{4.96}$$

This result and Corollary 4.6.1 in Appendix 4.6 imply that

$$\frac{\mathrm{d}}{\mathrm{d}a}U\big(s,a,\beta(a),\lambda\big)\begin{cases} >0, & \text{if } a>\beta(a),\\ <0, & \text{if } a<\beta(a).\end{cases} \tag{4.97}$$

Note that (4.59) implies that

$$\underline{h}(a,b,\lambda) = \frac{\hat{Q}(a,b,\lambda)}{1+a\hat{Q}(a,b,\lambda)}. \tag{4.98}$$

This observation, (4.64), (4.12)–(4.17) in Section 4.2 (see also Figure 4.2) and Lemma 4.3.3 imply that

$$\underline{h}\big(s,a,\beta(a),\lambda\big) < 0 \quad\Longleftrightarrow\quad U\big(s,a,\beta(a),\lambda\big)<0 \quad\Longleftrightarrow\quad a>\underline{\xi}.$$

Combining this result with (4.57), the definitions (4.91) of $\underline{\xi}$ and $\bar{\xi}$, (4.97), (4.98) and the fact that $G(\alpha(s), s, \lambda) = 0$, we obtain

$$U^2\big(s, a, \beta(a), \lambda\big) \geq U^2\big(s, \underline{\xi}, \bar{\xi}, \lambda\big) = 0, \quad \text{if } \underline{\xi} < s < \bar{\xi}$$
$$U^2(s, a, \beta(a), \lambda) \geq U^2(s, s, \beta(s), \lambda) = \underline{h}^2(s, s, \beta(s), \lambda), \quad \text{if } s \leq \underline{\xi},$$
$$\text{and} \quad U^2(s, a, \beta(a), \lambda) \geq U^2(s, \alpha(s), s, \lambda) = \big((1-\lambda)^2 \bar{h}(s, \alpha(s), s, \lambda)\big)^2, \quad \text{if } s \geq \bar{\xi}.$$

The inequality (4.90) follows from these inequalities, (4.32) and (4.95).

In view of (4.57), (4.79) in Lemma 4.3.6 and (4.98), we obtain the first limit in (4.92). The second limit in (4.92) follows from the first limit in (4.92), (4.90) and Fatou's Lemma. We again use the ergodic results (see e.g., Borodin and Salminen [14, Ch II.6]) to obtain (4.94). Finally, we show (4.93). Given any $b \leq \bar{\rho}_\ell \wedge B_0$ and $a \geq \underline{\eta}_r \vee A_0$, we define

$$M_0 = \inf\{t > 0 \mid S_t = a\}, \quad N_k = \inf\{t > M_k \mid S_t = b\} \quad \text{and} \quad M_{k+1} = \inf\{t > N_k \mid S_t = a\},$$

for $k = 0, 1, 2, \cdots$. In view of the definitions of A and B in Theorem 4.3.7, we can see that

$$S_{M_k} = a, \quad A_{M_k} = a, \quad B_{M_k} = \beta(b), \quad S_{N_k} = b, \quad A_{N_k} = \alpha(b) \quad \text{and} \quad B_{N_k} = b.$$

Combining this observation with (4.32) and (4.95), we use similar arguments in the proof of the ergodic results in Rogers and Williams [79, Theorem V53] to obtain the required results. □

4.4 Asymptotics

In this section, we derive the asymptotic expansion of the buying and selling boundaries. Note that around ρ, the function β could be β_1 and β_2, and

$$\beta_1(a, 0) \begin{cases} = a, & \text{for } a \leq \rho, \\ < a, & \text{for } \rho < a, \end{cases} \quad \text{and} \quad \beta_2(a, 0) \begin{cases} > a, & \text{for } a < \rho, \\ = a, & \text{for } \rho \leq a \leq \eta. \end{cases} \tag{4.99}$$

It follows that

$$\lim_{\lambda \downarrow 0} \beta_1(a, \lambda) \begin{cases} = a, & \text{for } a \leq \rho, \\ < a, & \text{for } \rho < a, \end{cases} \quad \text{and} \quad \lim_{\lambda \downarrow 0} \beta_2(a, \lambda) \begin{cases} > a, & \text{for } a < \rho, \\ = a, & \text{for } \rho \leq a \leq \eta. \end{cases} \tag{4.100}$$

Also, similar phenomenon happens for α around η. These observations inspire us to determine the asymptotic expansions of the functions β_i and α_i, for $i = 1, 2, 3$ in Proposition 4.3.4, in three different domains. We organize them in the following three propositions.

Proposition 4.4.1 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. For sufficiently small $\lambda > 0$, the following statements hold true.*

(i)

$$\beta_i(a,\lambda) = a - a\left(\frac{6}{\mathbf{\Gamma}(a)}\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \frac{3a\mathbf{\Gamma}(a) - a^2\mathbf{\Gamma}'(a)}{6^{\frac{1}{3}}\mathbf{\Gamma}^{\frac{5}{3}}(a)}\lambda^{\frac{2}{3}} + O(\lambda),$$

$$a\underline{h}\big(a,\beta_i(a,\lambda),\lambda\big) = \Theta(a) - \left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + a\mathbf{\Gamma}'(a)\left(\frac{1}{48\mathbf{\Gamma}^2(a)}\right)^{\frac{1}{3}}\lambda^{\frac{2}{3}} + O(\lambda),$$

and $$\hat{Q}(a,\beta_i(a,\lambda),\lambda) = Q(a) + \frac{1}{a(1-\Theta(a))^2}\left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \left(\frac{1}{(1-\Theta(a))^2}\mathbf{\Gamma}'(a)\left(\frac{1}{48\mathbf{\Gamma}^2(a)}\right)^{\frac{1}{3}} + \frac{1}{a(1-\Theta(a))^3}\left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{2}{3}}\right)\lambda^{\frac{2}{3}} + O(\lambda),$$

where $\mathbf{\Gamma}(a) = a^2\Gamma(a)$, *for* $i=1$, *if* $a \in\,]0,\rho[$, $i=2$, *if* $a \in\,]\rho, \underline{\eta}_\ell(\lambda)]$, *and* $i=3$, *if* $a \in [\underline{\eta}_r(\lambda),\infty[$, *and we exclude the point* $a = \Theta^{-1}(1) = \zeta$ *in the last expansion.*

(ii)

$$\alpha_i(b,\lambda) = b + b\left(\frac{6}{\mathbf{\Gamma}(b)}\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \frac{3a\mathbf{\Gamma}(b) - b^2\mathbf{\Gamma}'(a)}{6^{\frac{1}{3}}\mathbf{\Gamma}^{\frac{5}{3}}(b)}\lambda^{\frac{2}{3}} + O(\lambda),$$

$$(1-\lambda)b\bar{h}(\alpha_i(b,\lambda),b,\lambda) = \Theta(b) + \left(\frac{3}{4}\mathbf{\Gamma}^2(b)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + b\mathbf{\Gamma}'(b)\left(\frac{1}{48\mathbf{\Gamma}^2(b)}\right)^{\frac{1}{3}}\lambda^{\frac{2}{3}} + O(\lambda),$$

and $$\hat{Q}(\alpha_i(b,\lambda),b,\lambda) = Q(b) - \frac{1}{b(1-\Theta(b))^2}\left(\frac{3}{4}\mathbf{\Gamma}^2(b)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + \left(\frac{1}{(1-\Theta(b))^2}\mathbf{\Gamma}'(b)\left(\frac{1}{48\mathbf{\Gamma}^2(b)}\right)^{\frac{1}{3}} + \frac{1}{b(1-\Theta(b))^3}\left(\frac{3}{4}\mathbf{\Gamma}^2(b)\right)^{\frac{2}{3}}\right)\lambda^{\frac{2}{3}} + O(\lambda),$$

for $i=1$, *if* $b \in\,]0,\bar{\rho}_\ell(\lambda)]$, $i=2$, *if* $b \in [\bar{\rho}_r(\lambda),\eta[$, *and* $i=3$ *if* $b \in\,]\eta,\infty[$, *and we exclude the point* $b = \Theta^{-1}(1) = \zeta$ *in the last expansion.*

Proof. We only prove the case for the expansion of β_1 and the proof of other cases are similar. In view of (4.99) and (4.100) and the facts that $G(a,\beta_1(a,\lambda),\lambda) = 0$ and $G(a,\beta_1(a,0),0) = 0$, we can see that $\lambda = \Lambda(a,\beta_1(a,\lambda))$, where

$$\Lambda(a,b) = 1 - \frac{a}{b} - \frac{p_a^2(b) + p_a(b)\sqrt{p_a^2(b) + 4abp_a'(b)}}{2b^2p_a'(b)} \quad \text{and} \quad \Lambda(a,\beta_1(a,0)) = 0.$$

Furthermore, $\Lambda_b(a,\beta_1(a,\lambda)) = \Lambda_b(a,a) = 0$. Given any $a,b>0$, $G(a,b,\Lambda(a,b)) = 0$ holds true. We differentiate $G(a,b,\Lambda(a,b)) = 0$ with respect to b and use Faá di Bruno's formula to obtain

$$\frac{\mathrm{d}^nG}{\mathrm{d}b^n}\big(a,b,\Lambda(a,b)\big) = \sum \frac{n!}{k_0!k_1!k_2!\cdots k_n!}\frac{\partial^{k_0+k_1+\cdots+k_n}}{\partial b^{k_0}\partial\lambda^{k_1+\cdots k_n}}G\big(a,b,\Lambda(a,b)\big)\prod_{j=1}^{n}\left(\frac{1}{j!}\frac{\partial^j\Lambda}{\partial b^j}(a,b)\right)^{k_j} = 0, \qquad (4.101)$$

where the sum is over all non-negative integers $k_0, k_1, \cdots, k_n$ such that

$$k_0 + \sum_{j=1}^{n} nk_n = n.$$

Using the identities in (4.111)–(4.113) in Appendix 4.6, we calculate

$$G(a,a,0) = \frac{\partial G}{\partial b}(a,a,0) = \mathfrak{B}(a,a,0) = \frac{\partial^2 G}{\mathrm{d}b^2}(a,a,0) = \frac{\partial \mathfrak{B}}{\partial b}(a,a,0) = 0 \tag{4.102}$$

$$\text{and} \quad \frac{\partial^n G}{\partial b^n}\big(a,b,\Lambda(a,b)\big) = \sum_{j=0}^{n-1} \frac{(n-1)!}{j!(n-1-j)!}\frac{\partial^j \mathfrak{B}}{\partial b^j}\big(a,b,\Lambda(a,b)\big)\frac{\partial^{n-1-j} p_a'}{\partial b^{n-1-j}}(b), \tag{4.103}$$

for $n \geq 3$. The identities in (4.101) and (4.102) imply that

$$\frac{\partial \Lambda}{\partial b}(a,a) = \frac{\partial^2 \Lambda}{\partial b^2}(a,a) = 0$$

$$\text{and} \quad \sum \frac{n!}{k_0!k_3!\cdots k_n!}\frac{\partial^{k_3+\cdots+k_n}\partial^{k_0} G}{\partial \lambda^{k_3+\cdots k_n}\partial^{k_0} b}(a,a,0)\prod_{j=3}^{n}\left(\frac{1}{j!}\frac{\partial^n \Lambda}{\partial b^n}(a,a)\right)^{k_j} = 0, \quad \text{for } n \geq 3,$$

where the sum is over all non-negative integers $k_0, k_3, \cdots, k_n$ such that $k_0 + \sum_{j=3}^{n} nk_n = n$. Combining these results with (4.102), (4.103), (4.112), (4.113) and (4.126) we obtain

$$\frac{\partial^3 \Lambda}{\partial b^3}(a,a) = -\frac{\frac{\partial^3 G}{\partial b^3}(a,a,0)}{G_\lambda(a,a,0)} = -\frac{\Gamma(a)}{a},$$

$$\text{and} \quad \frac{\partial^4 \Lambda}{\partial b^4}(a,a) = -\frac{\frac{\partial^4 G}{\partial b^4}(a,a,0) + 4\frac{\partial^2 G}{\partial b \partial \lambda}(a,a,0)\frac{\partial^3 \Lambda}{\partial b^3}(a,a)}{G_\lambda(a,a,0)} = \frac{2\Gamma(a)}{a^2} - \frac{2\Gamma'(a)}{a},$$

and the analytic expressions of $\frac{\partial^n \Lambda}{\partial b^n}(a,a)$ for $n \geq 5$ can also be derived recursively. Now, we can see that the function Λ has the expansion

$$\Lambda\big(a,\beta_1(a,\lambda)\big) = -\frac{\Gamma(a)}{6a}\big(\beta_1(a,\lambda)-a\big)^3 + \left(\frac{\Gamma(a)}{12a^2} - \frac{\Gamma'(a)}{12a}\right)\big(\beta_1(a,\lambda)-a\big)^4 + O\Big(\big(\beta_1(a,\lambda)-a\big)^5\Big).$$

It follows that $\beta_1(a,\lambda)$ has the expansion

$$\beta_1(a,\lambda) = a + \sum_{k=1}^{\infty} \Upsilon_k \lambda^{\frac{k}{3}},$$

where

$$\Upsilon_1 = -\left(\frac{6a}{\Gamma(a)}\right)^{\frac{1}{3}} = -a\left(\frac{6}{\mathbf{\Gamma}(a)}\right)^{\frac{1}{3}} \quad \text{and} \quad \Upsilon_2 = \frac{\Gamma(a) - a\Gamma'(a)}{6^{\frac{1}{3}}a^{\frac{1}{3}}\Gamma^{\frac{5}{3}}(a)} = \frac{3a\mathbf{\Gamma}(a) - a^2\mathbf{\Gamma}'(a)}{6^{\frac{1}{3}}\mathbf{\Gamma}^{\frac{5}{3}}(a)}$$

and

$$\frac{1}{(1-\lambda)\beta_1(a,\lambda)-a} = \frac{\lambda^{-\frac{1}{3}}}{\Upsilon_1}\left(1-\frac{\Upsilon_2}{\Upsilon_1}\lambda^{\frac{1}{3}}+\left(\frac{\Upsilon_2^2}{\Upsilon_1^2}-\frac{\Upsilon_3-a}{\Upsilon_1}\right)\lambda^{\frac{2}{3}}+O(\lambda)\right). \tag{4.104}$$

Also, we derive the expansion

$$\begin{aligned}\frac{1}{\sqrt{p_a'(\beta_1(a,\lambda))}} &= 1+\frac{\Theta(a)}{a}\big(\beta_1(a,\lambda)-a\big)+\frac{1}{2}\Gamma(a)\big(\beta_1(a,\lambda)-a\big)^2\\ &\qquad +\frac{1}{6}\left(\frac{\Theta(a)}{a}\Gamma(a)+\Gamma'(a)\right)\big(\beta_1(a,\lambda)-a\big)^3\\ &= 1+\sum_{k=1}^{\infty}\check{\Upsilon}_k,\lambda^{\frac{k}{3}},\end{aligned}$$

where

$$\check{\Upsilon}_1 = \frac{\Theta(a)}{a}\Upsilon_1, \quad \check{\Upsilon}_2 = \frac{\Theta(a)}{a}\Upsilon_2+\frac{1}{2}\Gamma(a)\Upsilon_1^2$$
$$\text{and} \quad \check{\Upsilon}_3 = \frac{\Theta(a)}{a}\Upsilon_3+\Gamma(a)\Upsilon_1\Upsilon_2+\frac{1}{6}\left(\frac{\Theta(a)}{a}\Gamma(a)+\Gamma'(a)\right)\Upsilon_1^3.$$

It follows that

$$\frac{\sqrt{1-\lambda}}{\sqrt{p_a'(\beta_1(a,\lambda))}}-1 = \lambda^{\frac{1}{3}}\left(\check{\Upsilon}_1+\check{\Upsilon}_2\lambda^{\frac{1}{3}}+\left(\check{\Upsilon}_3-\frac{1}{2}\right)\lambda^{\frac{2}{3}}+O(\lambda)\right).$$

Using the definitions (4.56) and (4.59) of $\underline{h}$ and $\hat{Q}$, we calculate

$$\begin{aligned}a\underline{h}\big(a,\beta_1(a,\lambda),\lambda\big) &= \Theta(a)-\left(\frac{3}{4}a^4\Gamma^2(a)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}}+\left(\frac{a^2}{48}\right)^{\frac{1}{3}}\frac{2\Gamma(a)+a\Gamma'(a)}{\Gamma^{\frac{2}{3}}(a)}\lambda^{\frac{2}{3}}+O(\lambda)\\ &= \Theta(a)-\left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}}+a\mathbf{\Gamma}'(a)\left(\frac{1}{48\mathbf{\Gamma}^2(a)}\right)^{\frac{1}{3}}\lambda^{\frac{2}{3}}+O(\lambda)\end{aligned}$$

and

$$\begin{aligned}&\hat{Q}(a,\beta_1(a,\lambda),\lambda)\\ =&\frac{\underline{h}(a,\beta_1(a,\lambda),\lambda)}{1-a\underline{h}(a,\beta_1(a,\lambda),\lambda)}\\ =&Q(a)+\frac{1}{a(1-\Theta(a))^2}\left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}}\\ &\quad+\left(\frac{1}{(1-\Theta(a))^2}\mathbf{\Gamma}'(a)\left(\frac{1}{48\mathbf{\Gamma}^2(a)}\right)^{\frac{1}{3}}+\frac{1}{a(1-\Theta(a))^3}\left(\frac{3}{4}\mathbf{\Gamma}^2(a)\right)^{\frac{2}{3}}\right)\lambda^{\frac{2}{3}}+O(\lambda).\end{aligned}$$

□

Use similar arguments as the ones in Proposition 4.4.1, we obtain the following two propositions.

Proposition 4.4.2 , *Consider the stochastic control problem formulated in Section 4.1 and 4.3. Suppose that* $\Gamma'(\rho) < 0$ *and* $\Gamma'(\eta) > 0$. *For sufficient small* $\lambda > 0$, *the following statements hold true.*

(i)

$$\beta_{1,2}(\rho,\lambda) = \rho \mp \rho^{\frac{3}{4}}\left(-\frac{12}{\mathbf{\Gamma}'(\rho)}\right)^{\frac{1}{4}}\lambda^{\frac{1}{4}} - \frac{\sqrt{3\rho}}{20}\frac{17\mathbf{\Gamma}'(\rho)-3\mathbf{\Gamma}''(\rho)}{(-\mathbf{\Gamma}'(\rho))^{\frac{3}{2}}}\lambda^{\frac{1}{2}} + O\left(\lambda^{\frac{3}{4}}\right),$$

$$\rho\underline{h}\big(\rho,\beta_{1,2}(\rho,\lambda),\lambda\big) = \Theta(\rho) + \frac{\sqrt{-3\rho\mathbf{\Gamma}'(\rho)}}{3}\lambda^{\frac{1}{2}} \pm \frac{1}{5}\left(\frac{\rho\mathbf{\Gamma}''(\rho)}{\mathbf{\Gamma}'(\rho)}+1\right)\left(-\frac{\rho\mathbf{\Gamma}'(\rho)}{12}\right)^{\frac{1}{4}}\lambda^{\frac{3}{4}} + O(\lambda)$$

and $$\hat{Q}\big(\rho,\beta_{1,2}(\rho,\lambda),\lambda\big) = Q(\rho) + \frac{1}{\rho(1-\Theta(\rho))^2}\frac{\sqrt{-3\rho\mathbf{\Gamma}'(\rho)}}{3}\lambda^{\frac{1}{2}}$$
$$-\left(\frac{1}{(1-\Theta(\rho))^3}\frac{\mathbf{\Gamma}'(\rho)}{3} \mp \frac{1}{5\rho(1-\Theta(\rho))^2}\left(\frac{\rho\mathbf{\Gamma}''(\rho)}{\mathbf{\Gamma}'(\rho)}+1\right)\left(-\frac{\rho\mathbf{\Gamma}'(\rho)}{12}\right)^{\frac{1}{4}}\right)\lambda^{\frac{3}{4}} + O(\lambda).$$

(ii)

$$\alpha_{2,3}(\eta,\lambda) = \eta \mp \eta^{\frac{3}{4}}\left(\frac{12}{\mathbf{\Gamma}'(\eta)}\right)^{\frac{1}{4}}\lambda^{\frac{1}{4}} + \frac{\sqrt{3\rho}}{20}\frac{17\mathbf{\Gamma}'(\eta)-3\mathbf{\Gamma}''(\eta)}{(\mathbf{\Gamma}'(\eta))^{\frac{3}{2}}}\lambda^{\frac{1}{2}} + O\left(\lambda^{\frac{3}{4}}\right),$$

$$\eta\underline{h}\big(\eta,\alpha_{2,3}(\eta,\lambda),\lambda\big) = \Theta(\eta) + \frac{\sqrt{\eta\mathbf{\Gamma}'(\eta)}}{3}\lambda^{\frac{1}{2}} \pm \frac{1}{5}\left(\frac{\eta\mathbf{\Gamma}''(\eta)}{\mathbf{\Gamma}'(\eta)}-4\right)\left(\frac{\eta\mathbf{\Gamma}'(\eta)}{12}\right)^{\frac{1}{4}}\lambda^{\frac{3}{4}} + O(\lambda),$$

and $$\hat{Q}\big(\eta,\alpha_{2,3}(\eta,\lambda),\lambda\big) = Q(\eta) + \frac{1}{\eta(1-\Theta(\eta))^2}\frac{\sqrt{3\eta\mathbf{\Gamma}'(\eta)}}{3}\lambda^{\frac{1}{2}}$$
$$+\left(\frac{1}{(1-\Theta(\eta))^3}\frac{3\mathbf{\Gamma}'(\eta)}{3} \pm \frac{1}{5\eta(1-\Theta(\eta))^2}\left(\frac{\eta\mathbf{\Gamma}''(\eta)}{\mathbf{\Gamma}'(\eta)}-4\right)\left(\frac{\eta\mathbf{\Gamma}'(\eta)}{12}\right)^{\frac{1}{4}}\right)\lambda^{\frac{3}{4}} + O(\lambda).$$

Proposition 4.4.3 *Consider the stochastic control problem formulated in Section 4.1 and 4.3. For sufficient small* $\lambda > 0$, *the following statements holds true.*

(i)

$$\beta_i(a,\lambda) = \beta_i(a,0) - \frac{G_\lambda(a,\beta_i(a,0),0)}{G_b(a,\beta_i(a,0),0)}\lambda + O(\lambda^2),$$
$$a\underline{h}\big(a,\beta_i(a,\lambda),\lambda\big) = a\underline{h}(a,\beta_i(a,0),0) + F(a)\lambda + O(\lambda^2)$$
and $$\hat{Q}\big(a,\beta_i(a,\lambda),\lambda\big) = \hat{Q}\big(a,\beta_i(a,\lambda),0\big) + \frac{F(a)}{a(1-a\underline{h}(a,\beta_i(a,0),0))^2}\lambda + O(\lambda^2),$$

for $i = 1$, *if* $a \in]\rho, \alpha_1(\bar\rho_\ell(\lambda))[$, *and* $i = 2$, *if* $a \in]\alpha_2(\bar\rho_r(\lambda)), \rho[$, *where*

$$F(a) = -\frac{a}{(\beta_i(a,0)-a)\sqrt{p_a'(\beta_i(a,0))}}\left(\frac{\Theta(\beta_i(a,0))}{\beta_i(a,0)}\frac{G_\lambda(a,\beta_i(a,0),0)}{G_b(a,\beta_i(a,0),0)}+\frac{1}{2}\right)$$
$$+\frac{a}{(\beta_i(a,0)-a)^2}\left(\frac{G_\lambda(a,\beta_i(a,0),0)}{G_b(a,\beta_i(a,0),0)}+\beta_i(a,0)\right)\left(\frac{1}{\sqrt{p_a'(\beta_i(a,0))}}-1\right).$$

(ii)

$$\alpha_i(b,\lambda) = \alpha_i(b,0) - \frac{G_\lambda(\alpha_i(b,0),b,0)}{G_a(\alpha_i(b,0),b,0)}\lambda + O(\lambda^2),$$
$$(1-\lambda)b\bar h\big(\alpha_i(b,\lambda),b,\lambda\big) = b\bar h\big(\alpha_i(b,0),b,0\big) + \mathbf{F}(b)\lambda + O(\lambda^2)$$
$$\text{and}\quad (1-\lambda)\hat Q\big(\alpha_i(b,\lambda),b,\lambda\big) = \hat Q\big(\alpha_i(b,0),b,0\big) + \frac{\mathbf{F}(b)}{b(1-b\bar h\big(\alpha_i(b,0),b,0\big))^2}\lambda + O(\lambda^2),$$

for $i = 2$, *if* $b \in]\eta, \beta_2(\underline\eta_\ell(\lambda))[$ *and* $i = 3$, *if* $b \in]\beta_3(\underline\eta_r(\lambda)), \eta[$, *where*

$$\mathbf{F}(b) = -b\bar h\big(\alpha_i(b,0),b,0\big) - \frac{b}{(\alpha_i(b,0)-b)\sqrt{p_b'(\alpha_i(b,0))}}\left(\frac{\Theta(\alpha_i(b,0))}{\alpha_i(b,0)}\frac{G_\lambda(\alpha_i(b,0),b,0)}{G_a(\alpha_i(b,0),b,0)}-\frac{1}{2}\right)$$
$$+\frac{b}{(\alpha_i(b,0)-b)^2}\left(\frac{G_\lambda(\alpha_i(b,0),b,0)}{G_a(\alpha_i(b,0),b,0)}-\alpha_i(b,0)\right)\left(\frac{1}{\sqrt{p_b'(\alpha_i(b,0))}}-1\right).$$

Remark 4.8 All the functions have expansions of the form $\sum_{k=0}^\infty \nu_k\lambda^{\frac{k}{3}}$ in Proposition 4.4.1, $\sum_{k=0}^\infty \nu_k\lambda^{\frac{k}{4}}$ in Proposition 4.4.2, and $\sum_{k=0}^\infty \nu_k\lambda^k$ in Proposition 4.4.3. However, all the coefficients can be calculated explicitly.

4.5 Other cases of the problem data

In this section, we will break some of the conditions in Assumption 4.1, and divide the problem data into 5 different cases. The derivation of the shadow price process, the buying boundary, the selling boundary and the optimal strategy will be similar to the procedure in Section 4.3. On this account, we just provide the results and do not dive into details.

Assumption 4.4 The following conditions hold true:
(i) The function Θ is C^1 and decreasing. Furthermore, $\xi > \zeta$, where

$$\xi = \sup\{s>0 \mid \Theta(s)>0\} \in]0,\infty] \quad\text{and}\quad \zeta = \sup\{s>0 \mid \Theta(s)>1\} \in [0,\infty],$$

with the usual convention that $\sup\emptyset = 0$.
(ii) There exist $0 \le \rho \le \zeta \le \xi \le \eta$ such that

$$\Gamma(s)\begin{cases} >0, & \text{for } s<\rho,\\ <0, & \text{for } \rho<s<\eta,\\ >0, & \text{for } s>\eta.\end{cases} \quad .$$

Note that Q is the form of (1.17). We denote by Case 1, the case when

$$0 = \rho = \zeta < \xi < \eta < \infty \quad \Rightarrow \quad 1 \geq \Theta(0) > 0 > \Theta(\infty),$$

Case 2 the case when

$$0 < \rho < \zeta < \xi = \eta = \infty \quad \Rightarrow \quad \Theta(0) > 1 > \Theta(\infty) \geq 0,$$

Case 3 the case when

$$0 < \rho < \zeta = \xi = \eta = \infty \quad \Rightarrow \quad \Theta(\infty) = 1,$$

Case 4 the case when

$$\rho = \zeta = \xi = \eta = \infty \quad \Rightarrow \quad \Theta(\infty) \geq 1,$$

and Case 5 the case when

$$0 = \rho = \zeta < \xi = \eta = \infty \quad \Rightarrow \quad 1 \geq \Theta(0) \geq \Theta(\infty) > 0.$$

See Figure 4.9 for the function Q for Cases 1–5.

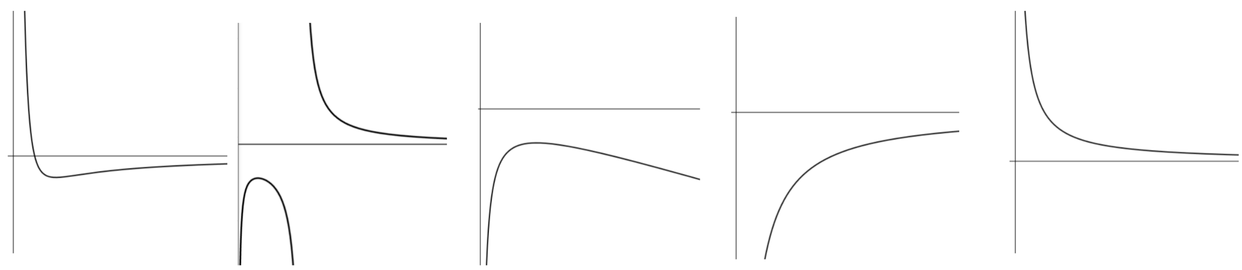

Figure 4.9: From left to right, the graphs are the function Q for Case 1–5

In Cases 1 and 5,

$$1 + Q(s)s > 0 \quad \text{for all } s > 0.$$

In Case 2,

$$1 + (1-\lambda)Q(s)s < 0 \quad \text{for all } s < \zeta \quad \text{and} \quad 1 + Q(s)s > 0 \quad \text{for all } s > \zeta.$$

In Cases 3 and 4,

$$1 + (1-\lambda)Q(s)s < 0 \quad \text{for all } s > 0.$$

The buying boundary, selling boundary, the shadow price process and the optimal strategy are similar to the ones we demonstrated in Section 4.2 and 4.3, but with the points $\bar{\rho}_\ell, \underline{\rho}, \bar{\rho}_r, \underline{\zeta}, \bar{\zeta}, \underline{\xi}, \bar{\xi}, \underline{\eta}_\ell, \bar{\eta}$, and $\underline{\eta}_r$ given by Lemma 4.3.6 and (4.91) as in Table 4.4. See also Figure 4.10 for the trading areas of the optimal strategy.

We conclude the section by providing some examples for Cases 1–5. Example 4.2 with $\ell = 0$ and $\kappa\gamma - \sigma^2 \leq 0$ is in Case 1.

Table 4.4: $\bar{\rho}_\ell, \underline{\rho}, \bar{\rho}_r, \underline{\zeta}, \bar{\zeta}, \underline{\xi}, \bar{\xi}, \underline{\eta}_\ell, \bar{\eta}$, and $\underline{\eta}_r$ for Cases 1–5

Case 1	$0 = \bar{\rho}_\ell = \underline{\rho} = \bar{\rho}_r = \underline{\zeta} = \bar{\zeta} < \underline{\xi} < \bar{\xi} < \underline{\eta}_\ell < \bar{\eta} < \underline{\eta}_r < \infty$
Case 2	$0 < \bar{\rho}_\ell < \underline{\rho} < \bar{\rho}_r < \underline{\zeta} < \bar{\zeta} < \underline{\xi} = \bar{\xi} = \underline{\eta}_\ell = \bar{\eta} = \underline{\eta}_r = \infty$
Case 3	$0 < \bar{\rho}_\ell < \underline{\rho} < \bar{\rho}_r < \underline{\zeta} = \bar{\zeta} = \underline{\xi} = \bar{\xi} = \underline{\eta}_\ell = \bar{\eta} = \underline{\eta}_r = \infty$
Case 4	$\bar{\rho}_\ell = \underline{\rho} = \bar{\rho}_r = \underline{\zeta} = \bar{\zeta} = \underline{\xi} = \bar{\xi} = \underline{\eta}_\ell = \bar{\eta} = \underline{\eta}_r = \infty$
Case 5	$0 = \bar{\rho}_\ell = \underline{\rho} = \bar{\rho}_r = \underline{\zeta} = \bar{\zeta} < \underline{\xi} = \bar{\xi} = \underline{\eta}_\ell = \bar{\eta} = \underline{\eta}_r = \infty$

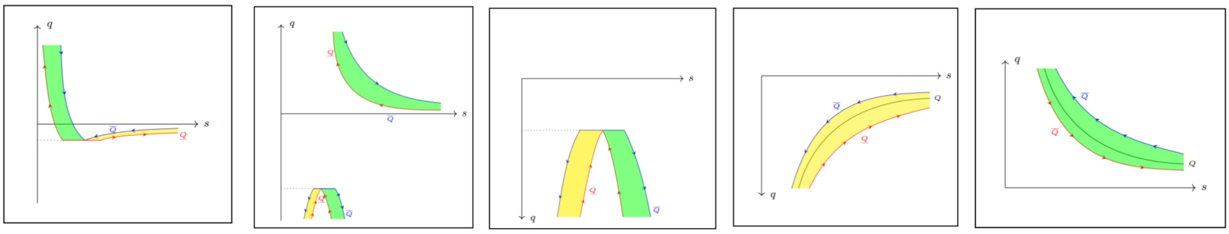

Figure 4.10: From left to right, the graphs are the buying boundary $\underline{Q}$ and selling boundary $\overline{Q}$ for Cases 1–5. The coloured areas reflect the optimal strategy as in Figure 4.2.

Example 4.4 (Cases 2 and 4) Suppose that the price process S is modelled by the SDE.

$$\mathrm{d}S_t = (\gamma + \mu S_t)\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$$

for some $\mu, \sigma, \gamma > 0$ such that $\mu < \frac{1}{2}\sigma^2$. In such a case,

$$\Theta(s) = \frac{\mu}{\sigma^2}(\gamma + s^{-1}), \quad p'(s) = s^{-\frac{2\mu}{\sigma^2}} \exp\left(\frac{2\mu\gamma}{\sigma^2}(s^{-1} - 1)\right)$$
$$\text{and} \quad \Gamma(s) = \frac{\mu}{\sigma^2}\left(\left(\frac{\mu\gamma}{\sigma^2} - 1\right)\gamma s^2 + 2\left(\frac{\mu\gamma}{\sigma^2} - 1\right)s + \frac{\mu}{\sigma^2}\right)s^{-4}.$$

If $\mu\gamma < \sigma^2$, then the problem data is in Case 2, and if $\mu\gamma \geq \sigma^2$, then the problem data is in Case 4.

Example 4.5 (Case 3) Suppose that the price process S is modelled by the SDE.

$$\mathrm{d}S_t = (\sigma^2 S_t^{2\alpha+1} + \mu S_t)\mathrm{d}t + \sigma S_t^{\alpha+1}\,\mathrm{d}t$$

for some $\mu, \sigma > 0$ and $\alpha > \frac{1}{2}$. In such a case,

$$\Theta(s) = 1 + \frac{\mu}{\sigma^2}s^{-2\alpha}, \quad p'(s) = \frac{1}{s^2}\exp\left(\frac{\mu\gamma}{\alpha\sigma^2}(s^{-2\alpha} - 1)\right)$$
$$\text{and} \quad Q'(s) = \left(\frac{(2\alpha - 1)\sigma^2}{\mu}s^{2\alpha} - 1\right)s^{-4}.$$

The geometric Brownian motion is an example such that the problem data could be in either Case 4 or 5. The model has been studied by Gerhold, Muhle-Karbe and Schachermayer [34]. In such a case, we can calculate explicitly that

$$\beta(a) = ca \quad \text{and} \quad \underline{Q}(s) = \overline{Q}(cs) = \frac{1 - \frac{c^{-\Theta}}{\sqrt{1-\lambda}}}{(c^{1-\Theta}-1)s},$$

for some $c > 0$. Furthermore,

$$c < 1, \quad \text{if } \Theta > 1 \text{ (Case 4)}, \quad \text{and} \quad c > 1, \quad \text{if } 0 < \Theta < 1 \text{ (Case 5)}.$$

4.6 Appendix

In this section, we give the proofs of the results in Sections 4.3.2.

Proof of Proposition 4.3.4 We develop the proof in 4 main steps

Step 1: Preliminary results. We will first provide some preliminary results and show the limits (see also Figures 4.5 and 4.6)

$$\lim_{b\downarrow 0} G(a,b,\lambda) = -\infty, \quad \text{and} \quad \lim_{b\uparrow\infty} G(a,b,\lambda) = \infty \quad \text{for all } a \in]0,\infty[\tag{4.105}$$

$$\text{and} \quad \lim_{a\downarrow 0} \mathbf{G}(a,b,\lambda) = \infty \quad \text{and} \quad \lim_{a\uparrow\infty} \mathbf{G}(a,b,\lambda) = -\infty \quad \text{for all } b \in]0,\infty[, \tag{4.106}$$

where G and $\mathbf{G}$ are given by (4.42) and (4.47). In view of the definition (4.3) of p and note that $\Theta(\infty) < 0$. We can see that

$$\lim_{a\uparrow\infty} p_b'(a) = \infty, \quad \lim_{a\uparrow\infty} \int_b^a \frac{1}{u} \frac{\Theta(u)}{\sqrt{p_b'(u)}}\, du = \lim_{a\uparrow\infty} \frac{1}{\sqrt{p_b'(a)}} - 1 = -1.$$

Using this result L'Hôpital's rule, we obtain

$$\lim_{a\uparrow\infty} \frac{a\sqrt{p_b'(a)}}{p_b(a)} = \lim_{a\uparrow\infty} \frac{1-\Theta(a)}{\sqrt{p_b'(a)}} = 0.$$

It follows that

$$\lim_{a\uparrow\infty} \mathbf{G}(a,b,\lambda) = \lim_{a\uparrow\infty} -p_b(a)\left(\sqrt{1-\lambda} + \frac{((1-\lambda)b - a)\sqrt{p_b'(a)}}{p_b(a)}\right) = -\infty \quad \text{for all } b > 0.$$

Also note that $\Theta(0) > 1$ (see Remark 4.1), we use L'Hôpital's rule to obtain

$$\lim_{a\downarrow 0} \frac{p_b(a)}{\sqrt{p_b'(a)}} = \lim_{a\downarrow 0} -\frac{\exp(\int_1^a \frac{1}{s}\, ds)\sqrt{p_b'(a)}}{\Theta(a)} = -\infty.$$

It follows that

$$\lim_{a\downarrow 0} \mathbf{G}(a,b,\lambda) = \lim_{a\downarrow 0} \sqrt{p_b'(a)}\left(-\sqrt{1-\lambda}\frac{p_b(a)}{\sqrt{p_b'(a)}} - (1-\lambda)b + a\right) = \infty \quad \text{for all } b>0.$$

Using similar arguments, we obtain (4.105).

We next derive the first and second order partial derivatives of G and $\mathbf{G}$. Differentiate $\mathbf{G}$ with respect to a, we obtain

$$\frac{\partial}{\partial a}\mathbf{G}(a,b,\lambda) = p_b'(a)\mathfrak{A}(a,b,\lambda) \tag{4.107}$$

where

$$\mathfrak{A}(a,b,\lambda) = -\sqrt{1-\lambda} + \frac{1}{\sqrt{p_b'(a)}} + ((1-\lambda)b - a)\frac{1}{\sqrt{p_b'(a)}}\frac{\Theta(a)}{a}. \tag{4.108}$$

It follows that

$$\frac{\partial}{\partial a}\mathfrak{A}(a,b,\lambda) = \frac{(1-\lambda)b-a}{\sqrt{p_b'(a)}}\Gamma(a), \tag{4.109}$$

where Γ is as in (4.1). In the presence of (4.47), the partial derivative $\mathbf{G}_a$ can be alternatively expressed by

$$\mathbf{G}_a(a,b,\lambda) = p_b'(a)G_a(a,b,\lambda) - \frac{2\Theta(a)}{a}p_b'(a)G(a,b,\lambda). \tag{4.110}$$

Similarly, we have

$$\frac{\partial}{\partial b}G(a,b,\lambda) = p_a'(b)\mathfrak{B}(a,b,\lambda), \tag{4.111}$$

where

$$\mathfrak{B}(a,b,\lambda) = \sqrt{1-\lambda} - \frac{1-\lambda}{\sqrt{p_a'(b)}} + ((1-\lambda)b - a)\frac{1}{\sqrt{p_a'(b)}}\frac{\Theta(b)}{b} \tag{4.112}$$

and

$$\frac{\partial}{\partial b}\mathfrak{B}(a,b,\lambda) = \frac{(1-\lambda)b-a}{\sqrt{p_a'(b)}}\Gamma(b). \tag{4.113}$$

Step 2: Results in the Proposition when $\lambda = 0$. In view of the definitions (4.42), (4.47), (4.108) and (4.112) of G, $\mathbf{G}$, $\mathfrak{A}$ and $\mathfrak{B}$, as well as (4.109), (4.113) and Assumption 4.1, we can see that

$$G(a,a,0) = \mathbf{G}(a,a,0) = \mathfrak{A}(a,a,0)\mathfrak{B}(b,b,0) = \mathfrak{A}_a(a,a,0) = \mathfrak{B}_b(b,b,0) = 0 \tag{4.114}$$

for all $a,b \in\,]0,\infty[$. Also, we can see that

$$G(a,b,0) = 0 \iff G(b,a,0) = 0. \tag{4.115}$$

We will prove all the results in this proposition for $\lambda = 0$ by showing that the function $G(a,\cdot,0) = 0$ (resp., $G(\cdot,b,0) = 0$) has 2 or 3 zeroes, and has 2 zeroes if and only if a (resp.,

b) is ρ or η. The zeroes satisfy the property in Table 4.6 and 4.7. Furthermore, there exists $0 < \rho^\natural(a) < \eta^\natural(a)$ and $\rho^\sharp(b) < \eta^\sharp(b)$ such that

$$\rho^\natural, \rho^\sharp(s) \begin{cases} = \rho, & \text{for } s = \rho, \\ < \rho, & \text{for } s > \rho \\ > \rho\ (< \eta), & \text{for } s < \rho \end{cases} \quad \text{and} \quad \eta^\natural, \eta^\sharp(s) \begin{cases} = \eta, & \text{for } s = \eta, \\ < \eta\ (> \rho), & \text{for } s > \eta, \\ > \eta, & \text{for } s < \eta, \end{cases} \tag{4.116}$$

and if $a \neq b$, then

$$G_b(a, b, 0) \begin{cases} > 0, & \text{for } b < \rho^\natural(a), \\ < 0, & \text{for } \rho^\natural(a) < b < \eta^\natural(a), \\ > 0, & \text{for } b > \eta^\natural(a) \end{cases} \tag{4.117}$$

$$\text{and} \quad \mathbf{G}_a(a, b, 0) \begin{cases} < 0, & \text{for } a < \rho^\sharp(b), \\ > 0, & \text{for } \rho^\sharp(b) < a < \eta^\sharp(b) \\ < 0, & \text{for } a > \eta^\sharp(b)). \end{cases} \tag{4.118}$$

We will show the results for $\lambda = 0$ in three sub-steps.

Step 2.1: On the interval $]\rho, \eta[$.

Fix $a \in]\rho, \eta[$ (resp., $b \in]\rho, \eta[$). In view of (4.105) (reps., (4.106)), (4.114), as well as (4.113), (resp., (4.109)), and Assumption 4.1, we can see that (4.117) (resp., 4.118)) holds true for some $\rho^\natural(a) < \rho < a < \eta < \eta^\natural(a)$ (resp., $\rho^\sharp(b) < \rho < b < \eta < \eta^\sharp(a)$), and there exists $\beta_1(a, 0) < \rho^\natural < \beta_2(a, 0) = a < \eta^\natural < \beta_3(a, 0)$ (resp., $\alpha_1(b, 0) < \rho^\sharp < \alpha_2(b, 0) = b < \eta^\sharp < \alpha_3(b, 0)$) such that the equation (4.65) holds true. Furthermore,

$$\begin{aligned} &\lim_{a \downarrow \rho} \beta_1(a, 0) = \lim_{a \downarrow \rho} \beta_2(a, 0) = \lim_{a \downarrow \rho} \rho^\natural(a) = \rho \\ \text{and} \quad &\lim_{a \uparrow \eta} \beta_1(a, 0) = \lim_{a \uparrow \eta} \beta_2(a, 0) = \lim_{a \uparrow \eta} \eta^\natural(a) = \eta \\ (\text{resp.,} \quad &\lim_{b \downarrow \rho} \alpha_1(b, 0) = \lim_{b \downarrow \rho} \alpha_2(b, 0) = \lim_{b \downarrow \rho} \rho^\sharp(b) = \rho \\ \text{and} \quad &\lim_{b \uparrow \eta} \alpha_1(b, 0) = \lim_{b \uparrow \eta} \alpha_2(b, 0) = \lim_{b \uparrow \eta} \eta^\sharp(b) = \eta). \end{aligned}$$

In addition, we can also show the restriction of $\beta_1(\cdot, 0)$ on $]\rho, \eta[$ is strictly decreasing by contradiction. If $\beta_1(\cdot, 0)$ is not strictly decreasing, then there exists $b < \rho = \beta_1(\rho, 0) < a_1 < a_2 < \eta$ such that $G(a_1, b, 0) = G(a_2, b, 0) = 0$. In view of (4.106) and (4.118), we can see that the exists at most one zero for $\mathbf{G}(., b, 0)$ on $[\rho, \eta]$ for any $b > 0$, which leads to contradiction. Similarly, we can show that the restriction of $\beta_3(\cdot, 0)$, $\alpha_1(\cdot, 0)$ and $\alpha_3(\cdot, 0)$ on $]\rho, \eta[$ are strictly decreasing. In view of (4.115), we can see that

$$\underline{x}_1 := \beta_1(\eta, 0) = \alpha_1(\eta, 0) < \rho =: \underline{x}_0 \quad \text{and} \quad \bar{x}_1 := \beta_3(\rho, 0) = \alpha_3(\rho, 0) > \eta =: \bar{x}_0.$$

A summary of the zeroes and their restrictions see the second column of Table 4.5.

Table 4.5: Range and restriction of $\beta_i(\cdot, 0)$ and $\alpha_i(\cdot, 0)$.

Zeroes \ Domain	$]\underline{x}_0, \bar{x}_0[$	$]\underline{x}_1, \underline{x}_0[$	$]\bar{x}_0, \bar{x}_1[$	$]\underline{x}_{n+1}, \underline{x}_n[$	$]\bar{x}_n, \bar{x}_{n+1}[$
$\beta_1(\cdot,0)$, $\alpha_1(\cdot,0)$	$]\underline{x}_1, \underline{x}_0[$ decr.	$]\underline{x}_1, \underline{x}_0[$ incr.	$]\underline{x}_1, \underline{x}_2[$ decr.	$]\underline{x}_{n+1}, \underline{x}_n[$ incr.	$]\underline{x}_{n+1}, \underline{x}_{n+2}[$ decr.
$\beta_2(\cdot,0)$, $\alpha_2(\cdot,0)$	$]\underline{x}_0, \bar{x}_0[$ incr.	$]\underline{x}_0, \bar{x}_0[$ decr.	$]\underline{x}_0, \bar{x}_0[$ decr.	$]\bar{x}_{n-1}, \bar{x}_n[$ decr.	$]\underline{x}_n, \underline{x}_{n-1}[$ decr.
$\beta_3(\cdot,0)$, $\alpha_3(\cdot,0)$	$]\bar{x}_0, \bar{x}_1[$ decr.	$]\bar{x}_1, \bar{x}_2[$ decr.	$]\bar{x}_0, \bar{x}_1[$ incr.	$]\bar{x}_{n+1}, \bar{x}_{n+2}[$ decr.	$]\bar{x}_n, \bar{x}_{n+1}[$ incr.

We denote by decr.(resp., incr.) the restriction of the function on the corresponding interval is strictly decreasing (resp., increasing).
$n = 1, 2, 3, \cdots$

Step 2.2: On the interval $]\underline{x}_1, \rho[\cup]\eta, \bar{x}_1[$. Combining (4.115) with the results and similar arguments in Step 2.1, we can see that for $a, b \in]\underline{x}_1, \rho[$ (resp., $a, b \in]\eta, \bar{x}_1[$) there exists

$$\begin{aligned} \beta_1(a,0) = a < \rho < \beta_2(a,0) = \big(\alpha_1(\cdot,0)\big)^{-1}(a) < \eta < \beta_3(b,0) \\ \text{and}\quad \alpha_1(b,0) = b < \rho < \alpha_2(b,0) = (\beta_1(\cdot,0))^{-1}(b) < \eta < \alpha_3(b,0) \\ (\text{resp.,}\quad \beta_1(a,0) < \rho < \beta_2(a,0) = (\alpha_3(\cdot,0))^{-1}(a) < \eta < \beta_3(a,0) = a \\ \text{and}\quad \alpha_1(b,0) < \rho < \alpha_2(b,0) = (\beta_3(\cdot,0))^{-1}(b) < \alpha_3(b,0) = b), \end{aligned}$$

where $(\alpha_1(\cdot,0))^{-1}$, $(\beta_1(\cdot,0))^{-1}$, $(\alpha_3(\cdot,0))^{-1}$ and $(\beta_3(\cdot,0))^{-1}$ are the inverse functions of $\alpha_1(\cdot,0)$, $\beta_1(\cdot,0)$, $\alpha_3(\cdot,0)$ and $\beta_3(\cdot,0)$ given in Step 2.1. Furthermore,

$$\beta_2(\underline{x}_1,0) = \alpha_2(\underline{x}_1,0) = \eta \quad \text{and} \quad \beta_2(\bar{x}_1,0) = \alpha_2(\bar{x}_1,0) = \rho, \tag{4.119}$$

(4.116)–(4.118) holds true on $]\underline{x}_1, \rho[\cup]\eta, \bar{x}_1[$, and the restriction of $\beta_2(\cdot,0)$(resp., $\alpha_2(\cdot,0)$) is strictly decreasing on $]\underline{x}_1, \rho[\cup]\eta, \bar{x}_1[$.

The restrictions of $\beta_3(\cdot,0)$ and $\alpha_3(\cdot,0)$ (resp., $\beta_1(\cdot,0)$ and $\alpha_1(\cdot,0)$) on $]\underline{x}_1, \rho[$ (resp., $]\eta, \bar{x}_1[$) are either strictly decreasing or strictly increasing. We will show that $\beta_3(\cdot,0)$ (resp., $\alpha_1(\cdot,0)$) is strictly decreasing on $]\underline{x}_1, \rho[$ (resp., $]\eta, \bar{x}_1[$) by contradiction. If $\beta_3(\cdot,0)$ is strictly increasing on $]\underline{x}_1, \rho[$, then for any $a \in]\underline{x}_1, \rho[$,we have $\beta_3(a,0) \in]\eta, \bar{x}_1[$. In view of this observation and (4.115)–(4.118), we can see that $\alpha_1(\cdot,0)$ should be strictly increasing on $]\eta, \bar{x}_1[$. Furthermore,

$$\beta_3(\underline{x}_1,0) = \eta \quad \text{and} \quad \alpha_1(\bar{x}_1,0) = \rho.$$

Combing this observation with (4.119), we have

$$\eta^\natural(\underline{x}_1) = \eta \quad \text{and} \quad \rho^\sharp(\bar{x}_1) = \rho,$$

which contradicts to (4.116). Similarly, we can show that the restriction of $\alpha_3(\cdot,0)$ (resp., $\beta_1(\cdot,0)$) on $]\underline{x}_1, \rho[$ (resp., $]\eta, \bar{x}_1[$) is strictly decreasing. We define

$$\underline{x}_2 = \beta_1(\bar{x}_1,0) = \alpha_1(\bar{x}_1,0) < \underline{x}_1 \quad \text{and} \quad \bar{x}_2 = \beta_3(\underline{x}_1,0) = \alpha_3(\underline{x}_1,0) > \bar{x}_1.$$

For a summary of the zeroes and their restrictions see the third and fourth columns of Table 4.5.

Step 2.3: On the interval $]0,\underline{x}_1[\cup]\bar{x}_1,\infty[$. Use arguments similar to the ones in Step 2.2, we can define recursively a strictly deceasing (resp., increasing) sequence $\underline{x}_n$ (resp., $\bar{x}_n$), given by

$$\underline{x}_n = \beta_1(\bar{x}_{n-1},0) = \alpha_1(\bar{x}_{n-1},0) < \underline{x}_{n-1} \quad (\text{resp., } \bar{x}_n = \beta_3(\underline{x}_{n-1},0) = \alpha_3(\underline{x}_{n-1},0) > \bar{x}_{n-1}),$$

such that the solutions $\beta_1(\cdot,0) < \beta_2(\cdot,0) < \beta_3(\cdot,0)$ (resp., $\alpha_1(\cdot,0) < \alpha_2(\cdot,0) < \alpha_3(\cdot,0)$) to the equalities (4.65) exist and satisfy the properties in Table 4.5 on $]\underline{x}_n,\underline{x}_{n-1}[\cup]\bar{x}_{n-1},\bar{x}_n[$. Furthermore, (4.116)–(4.118) hold true. The sequences $\{\underline{x}_n\}$ and $\{\bar{x}_n\}$ also satisfy

$$\underline{x}_n = \beta_2(\bar{x}_{n+1},0) = \alpha_2(\bar{x}_{n+1},0) \quad \text{and} \quad \bar{x}_n = \beta_2(\underline{x}_{n+1},0) = \alpha_2(\underline{x}_{n+1},0).$$

We only need to show that $\underline{x}_\infty := \lim_{n\uparrow\infty}\underline{x}_n = 0$ and $\bar{x}_\infty := \lim_{n\uparrow\infty}\bar{x}_n = \infty$ to complete the proof of this step. We prove the limits by contradiction. To this end, we first notice that

$$\mathbf{G}(\underline{x}_{n\pm1},\bar{x}_n,0) = 0, \quad \mathbf{G}_a(\underline{x}_{n+1},\bar{x}_n,0) < 0 \quad \text{and} \quad \mathbf{G}_a(\underline{x}_{n-1},\bar{x}_n,0) > 0, \tag{4.120}$$

$$\text{and} \quad G(\underline{x}_n,\bar{x}_{n\pm1},0) = 0, \quad G_b(\underline{x}_n,\bar{x}_{n-1},0) < 0 \quad \text{and} \quad G_b(\underline{x}_n,\bar{x}_{n+1},0) > 0. \tag{4.121}$$

In view of (4.108) and the facts that $\underline{x}_n < \rho < \xi = \Theta^{-1}(0) < \eta$, for $n = 1,2,3,\cdots$, and Θ is strictly decreasing, we can see that

$$\frac{\partial}{\partial b}\mathfrak{A}(\underline{x}_{n-1},b,0) = \sqrt{p'_{\underline{x}_{n-1}}(b)}\left(-\frac{\Theta(b)}{b} + \frac{\Theta(\underline{x}_{n-1})}{\underline{x}_{n-1}} - (b-\underline{x}_{n-1})\frac{\Theta(b)}{b}\frac{\Theta(\underline{x}_{n-1})}{\underline{x}_{n-1}}\right) > 0.$$

for all $\eta < b$. This observation and (4.120) imply that If $\underline{x}_\infty = 0$ and $\bar{x}_\infty < \infty$, we have

$$\mathbf{G}_a(\underline{x}_{n-1},\bar{x}_\infty,0) > 0 \quad \text{for all } n > 1,$$

which contradicts to the fact that $\lim_{a\downarrow0}\mathbf{G}(a,\bar{x}_\infty,0) = \infty$ (see (4.106)). The contradiction arguments are similar if $\underline{x}_\infty > 0$ and $\bar{x}_\infty = \infty$. If $\underline{x}_\infty > 0$ and $\bar{x}_\infty < \infty$, then we can use (4.120), (4.121) and arguments similar to the ones above to show that

$$\mathbf{G}_a(\underline{x}_\infty,\bar{x}_\infty,0) = G_b(\underline{x}_\infty,\bar{x}_\infty,0) = 0 \quad \Longleftrightarrow \quad \mathfrak{A}(\underline{x}_\infty,\bar{x}_\infty,0) = \mathfrak{B}(\underline{x}_\infty,\bar{x}_\infty,0) = 0. \tag{4.122}$$

Using the definitions (4.56) and (4.108) of $\underline{h}$ and $\mathfrak{A}$, we calculate

$$\frac{\Theta(a)}{a} - \underline{h}(a,b,\lambda) = \frac{1}{((1-\lambda)b-a)\sqrt{p'_a(b)}}\mathfrak{A}(a,b,\lambda), \tag{4.123}$$

Similarly, we use definitions (4.57) and (4.112) of $\bar{h}$ and $\mathfrak{B}$,

$$\begin{aligned}\frac{\Theta(b)}{b} - (1-\lambda)\bar{h}(a,b,\lambda) &= \frac{\Theta(b)}{b} - \sqrt{1-\lambda}\sqrt{p'_a(b)}\underline{h}(a,b,\lambda)\\ &= \frac{1}{((1-\lambda)b-a)\sqrt{p'_b(a)}}\mathfrak{B}(a,b,\lambda).\end{aligned} \tag{4.124}$$

Combining the identities in (4.122)–(4.124), we obtain

$$\underline{h}(\underline{x}_\infty, \bar{x}_\infty, \lambda) = \frac{\Theta(\underline{x}_\infty)}{\underline{x}_\infty} = \frac{1}{\sqrt{1-\lambda}\sqrt{p'_{\underline{x}}(\bar{x}_\infty)}} \frac{\Theta(\bar{x}_\infty)}{\bar{x}_\infty},$$

which contradicts to the fact that $\Theta(\underline{x}_\infty) < 0 < \Theta(\bar{x}_\infty)$. A summary of the zeroes and their restrictions see the fifth and sixth columns of Table 4.5. See also Table 4.6 for an overview. From the construction of zeroes, we can also derive the inverse functions in Table 4.7.

Table 4.6: Range and restriction of $\beta_i(\cdot, 0)$ and $\alpha(\cdot, 0)$.

Zeroes \ Domain	$]0, \rho[$	$]\rho, \eta[$	$]\eta, \infty[$
$\beta_1(\cdot, 0)$, $\alpha_1(\cdot, 0)$	$]0, \rho[$ incr.	$]0, \rho[$ decr.	
$\beta_2(\cdot, 0)$, $\alpha_2(\cdot, 0)$	$]\rho, \infty[$ decr.	$]\rho, \eta[$ incr.	$]0, \eta[$ decr.
$\beta_3(\cdot, 0)$, $\alpha_3(\cdot, 0)$	$]\eta, \infty[$ decr.		$]\eta, \infty[$ incr.

We denote by decr.(resp., incr.) the restriction of the function on the corresponding interval is strictly decreasing (resp., increasing).

Table 4.7: Inverse of $\beta_i(\cdot, 0)$ and $\alpha_i(\cdot, 0)$.

Zeroes \ Domain	$]0, \rho[$	$]\rho, \eta[$	$]\eta, \infty[$
$\beta_1(\cdot, 0)$, $\alpha_1(\cdot, 0)$	$(\alpha_1(\cdot,0)\vert_{]0,\rho[})^{-1}$, $(\beta_1(\cdot,0)\vert_{]0,\rho[})^{-1}$	$(\alpha_2(\cdot,0)\vert_{]0,\rho[})^{-1}$, $(\beta_2(\cdot,0)\vert_{]0,\rho[})^{-1}$	
$\beta_2(\cdot, 0)$, $\alpha_2(\cdot, 0)$	$(\alpha_1(\cdot,0)\vert_{]\rho,\infty[})^{-1}$, $(\beta_1(\cdot,0)\vert_{]\rho,\infty[})^{-1}$	$(\alpha_2(\cdot,0)\vert_{]\rho,\eta[})^{-1}$, $(\beta_2(\cdot,0)\vert_{]\rho,\eta[})^{-1}$	$(\alpha_3(\cdot,0)\vert_{]0,\eta[})^{-1}$, $(\beta_3(\cdot,0)\vert_{]0,\eta[})^{-1}$
$\beta_3(\cdot, 0)$, $\alpha_3(\cdot, 0)$	$(\alpha_2(\cdot,0)\vert_{]\eta,\infty[})^{-1}$, $(\beta_2(\cdot,0)\vert_{]\eta,\infty[})^{-1}$		$(\alpha_3(\cdot,0)\vert_{]\eta,\infty[})^{-1}$, $(\beta_3(\cdot,0)\vert_{]\eta,\infty[})^{-1}$

Step 3: The proof of (ii)–(v). In this step, we will first show that there exist a unique $\Lambda :]0,\infty[\to [0,1[$ and $\beta :]0,\infty[\to]0,\infty[$ such that

$$G\big(a, \beta(a), \Lambda(a)\big) = 0, \quad \mathfrak{B}\big(a, \beta(a), \Lambda(a)\big) = 0 \quad \text{and} \quad \beta(a) \begin{cases} > a & \text{for } a < \eta \\ < a, & \text{for } a > \eta \end{cases} \tag{4.125}$$

To this end, we first calculate

$$\frac{\partial}{\partial \lambda} G(a, b, \lambda) = -\frac{1}{2\sqrt{1-\lambda}} p_a(b) + b\sqrt{p'_a(b)} = \frac{(1-\lambda)b + a}{2(1-\lambda)} \sqrt{p'_a(b)} - \frac{G(a, b, \lambda)}{2(1-\lambda)} \tag{4.126}$$

and

$$\frac{\partial^2}{\partial \lambda^2} G(a, b, \lambda) = -\frac{1}{4}(1-\lambda)^{-\frac{3}{2}} p_a(b).$$

Combining these results and the definition of G, we obtain

$$\frac{\partial}{\partial\lambda}G(a,b,\lambda) > 0, \quad \text{if } b \le a \text{ or } G(a,b,\lambda) \le 0, \tag{4.127}$$

$$\frac{\partial^2}{\partial\lambda^2}G(a,b,\lambda) < 0 \quad \text{and} \quad \lim_{\lambda\uparrow 1}\frac{\partial}{\partial\lambda}G(a,b,\lambda) = -\infty, \quad \text{if } b > a,$$

$$\text{and} \quad \frac{\partial^2}{\partial\lambda^2}G(a,b,\lambda) \ge 0 \quad \text{and} \quad \lim_{\lambda\uparrow 1}\frac{\partial}{\partial\lambda}G(a,b,\lambda) = \infty, \quad \text{if } b \le a.$$

In view of these observations, we can see that for a fixed a and b, there exists $\lambda_{ab} \in [0,1]$ such that

$$\frac{\partial}{\partial\lambda}G(a,b,\lambda)\begin{cases} > 0, & \text{for } \lambda < \lambda_{ab}, \\ < 0. & \text{for } \lambda > \lambda_{ab}, \end{cases} \quad \text{and} \quad \lambda_{ab}\begin{cases} = 1, & \text{if } b \le a, \\ < 1, & \text{if } b > a. \end{cases} \tag{4.128}$$

Furthermore,

$$\lambda_{ab} > 0, \quad \text{if } G(a,b,0) \le 0. \tag{4.129}$$

For a fixed $a > 0$,

$$\lim_{\lambda\uparrow 1} G(a,b,\lambda) = a\sqrt{p_a'(b)} > 0 \quad \text{for all } b > 0. \tag{4.130}$$

Combining this observation with (4.128), (4.129) and (iv) for $\lambda = 0$ in this Proposition, as well as Lemma 4.3.3.(i), (4.112), (4.113) and Assumption 4.1.(iii), we can see that for a fixed $a > 0$, there exists Λ and β such that

$$G\big(a,\beta(a),\Lambda(a)\big) = 0, \quad \mathfrak{B}\big(a,\beta(a),\Lambda(a)\big) = 0, \quad \beta(\eta) = \eta, \quad \Lambda(\eta) = 0, \tag{4.131}$$

$$\text{and} \quad \begin{cases} a < \beta_2(a,0) < \beta(a) < \beta_3(a,0) & \text{and} \quad \eta < \beta(a), \quad \text{if } a < \rho, \\ \beta_2(a,0) = a < \beta(a) < \beta_3(a,0) & \text{and} \quad \eta < \beta(a), \quad \text{if } \rho < a < \eta, \\ \beta_2(a,0) < \beta(a) < a = \beta_3(a,0) & \text{and} \quad \rho < \beta(a) < \eta, \quad \text{if } \eta < a. \end{cases} \tag{4.132}$$

Using these results, (4.107)–(4.110) and Assumption 4.1.(iii), we have

$$G_a\big(a,\beta(a),\Lambda(a)\big)\begin{cases} < 0, & \text{for } a > \eta, \\ > 0, & \text{for } \rho < a < \eta. \end{cases} \tag{4.133}$$

Furthermore, we can show

$$G_a\big(a,\beta(a),\Lambda(a)\big) > 0, \quad \text{for } a \le \rho, \tag{4.134}$$

by contradiction. If this is not true, then there exists $a_1 \le \rho$ such that $G_a(a_1,\beta(a_1),\Lambda(a_1)) = 0$ thanks to (4.133). In view of the definitions (1.17) and (4.59) of Q and $\hat{Q}$, we can see that if $G(a,b,\lambda) = 0$, then

$$Q(a) - \hat{Q}(a,b,\lambda) = \frac{1}{\sqrt{1-\lambda}}\frac{\mathfrak{A}(a,b,\lambda)}{(\sqrt{1-\lambda}b\sqrt{p_a'(b)} - a)(1-\Theta(a))} \tag{4.135}$$

$$\text{and} \quad \frac{Q(b)}{1-\lambda} - \hat{Q}(a,b,\lambda) = \frac{\sqrt{p_a'(b)}}{1-\lambda}\frac{\mathfrak{B}(a,b,\lambda)}{(\sqrt{1-\lambda}b\sqrt{p_a'(b)} - a)(1-\Theta(b))}. \tag{4.136}$$

Combining these calculations with the fact that $G_a(a_1, \beta(a_1), \Lambda(a_1)) = G_b(a_1, \beta(a_1), \Lambda(a_1)) = 0$, we obtain $(1-\lambda)Q(a_1) = Q(\beta(a_1))$, which contradicts to Remark 4.4, since $a_1 \le \rho$ but $\beta(a_1) > \eta$.

We next show that $\Lambda(\infty) := \lim_{a\uparrow\infty} \Lambda(a) = 1$ by contradiction. Differentiating $G\big(a, \beta(a), \Lambda(a)\big) = 0$ with respect to a, we use (4.127), (4.110), (4.133) and (4.134) to obtain

$$\Lambda'(a) = -\frac{G_a(a, \beta(a), \Lambda(a))}{G_\lambda(a, \beta(a), \Lambda(a))} \begin{cases} > 0 & \text{if } a > \eta, \\ < 0 & \text{if } a < \eta. \end{cases} \tag{4.137}$$

Note that given any $\lambda \in]0,1[$ and $b > 0$, there exists $a > b$ such that $G(a, b, \lambda) = 0$ (see (iv) for $\lambda = 0$, the limits in (4.106), (4.126) and (4.128)). If $\lambda = \Lambda(\infty) < 1$, then we let $b > \eta$ and $G(a, b, \Lambda(\infty)) = 0$ for some $a > b$. Furthermore, $G(a, u, \lambda) < 0$ for all $u \in]b, a[$. Combining this observation with (4.128) and (4.129), we can see that $\Lambda(a) > \lambda$, which leads to contradiction. Similarly, we have $\lim_{a\downarrow 0} \Lambda(a) = 1$. Also, note the last two identities in (4.131). For any $\lambda \in]0,1[$, there exist $0 < \underline{\eta}_\ell(\lambda) < \eta < \underline{\eta}_r(\lambda)$ such that

$$\Lambda(\underline{\eta}_\ell(\lambda)) = \Lambda(\underline{\eta}_r(\lambda)) = \lambda.$$

Combining these observations with (4.128)–(4.130), (4.137) as well as this proposition for $\lambda = 0$ with $\underline{\eta}_\ell(0) = \underline{\eta}_r(0) = \eta$, we can see that for any $a \in]0, \underline{\eta}_\ell(\lambda)[\cup]\underline{\eta}_r(\lambda), \infty[$, there exist

$$\beta_1(a, \lambda) < \beta_1(a, 0) < \beta_2(a, 0) < \beta_2(a, \lambda) < \beta_3(a, \lambda) < \beta_3(a, 0) \tag{4.138}$$

such that (iii) and (iv) for β_i hold true, and for $a \in]\underline{\eta}_\ell(\lambda), \underline{\eta}_r(\lambda)[$ there exists $\beta_1(a, \lambda) < \beta_1(0)$ such that (iii) and (iv) hold true for β_1. Furthermore,

$$\beta_2\big(\underline{\eta}_\ell(\lambda)\big) = \beta_3\big(\underline{\eta}_\ell(\lambda)\big) > \eta > \beta_2\big(\underline{\eta}_r(\lambda)\big) = \beta_3\big(\underline{\eta}_r(\lambda)\big) > \rho. \tag{4.139}$$

We have the inequities thanks to (4.131) and (4.132). Similarly, we can show that there exist

$$\alpha_1(b, 0) < \alpha_1(b, \lambda) < \alpha_2(b, \lambda) < \alpha_2(b, 0) < \alpha_3(b, 0) < \alpha_3(b, \lambda) \tag{4.140}$$

for $]0, \bar{\rho}_\ell(\lambda)[\cup]\bar{\rho}_\ell(\lambda), \infty[$ such that (iii) and (iv) hold true for α_i, and there exist $\alpha_3(a, \lambda) > \alpha_3(a, 0)$ such that (iii) and (iv) hold true for α_3. Furthermore, we have the equalities $0 < \bar{\rho}_\ell(\lambda) < \rho < \bar{\rho}_r(\lambda)$ as well as

$$\eta > \alpha_1\big(\bar{\rho}_\ell(\lambda)\big) = \alpha_2\big(\bar{\rho}_\ell(\lambda)\big) > \rho > \alpha_1\big(\bar{\rho}_r(\lambda)\big) = \alpha_2\big(\bar{\rho}_r(\lambda)\big). \tag{4.141}$$

The results in (v) follow from (4.138)–(4.140). We conclude this step by showing that the restriction of $\underline{\eta}_\ell$ is strictly decreasing and proof of the rest of (ii) is similar. For any $1 > \tilde{\lambda} > \lambda > 0$, we have

$$G(\underline{\eta}_\ell(\lambda), s, \tilde{\lambda}) \begin{cases} > 0, & \text{for } s > \beta_1\big(\underline{\eta}_\ell(\lambda), \tilde{\lambda}\big), \\ < 0, & \text{for } s < \beta_1\big(\underline{\eta}_\ell(\lambda), \tilde{\lambda}\big), \end{cases}$$

thanks to (4.127)–(4.130) and (iv). Using this observation, $\underline{\eta}_\ell(\lambda) > \underline{\eta}_\ell(\tilde{\lambda})$ follows from (iii) and (iv).

Step 4: The proof of (vi). We only prove the restriction, range and inverse of $\beta_1(\cdot,\lambda)$ and proof of other cases are similar. In view of (4.138), (4.140) as well as Table 4.6, we can see that $\beta_1(\cdot,0)$ increases on $]0,\rho[$, decreases on $]\rho,\infty[$, $\beta_1(\rho,0)=\rho$ and $\beta_1(\cdot,\lambda)<\beta_1(\cdot,0)\le\rho$. Furthermore, for sufficient small a,

$$a=\alpha_1\big(\beta_1(a,\lambda)\big),\quad G_a(a,\beta_1(a,\lambda),\lambda)<0,\quad \text{and}\quad G_b(a,\beta_1(a,\lambda),\lambda)>0, \tag{4.142}$$

thanks to (4.109), (4.113), (iv) and Assumption 4.1, and

$$(\beta_1)'(a)=-\frac{G_a(a,\beta_1(a,\lambda),\lambda)}{G_b(a,\beta_1(a,\lambda),\lambda)}>0. \tag{4.143}$$

The sign of $(\beta_1)'$ changes only around $\{a>0 \mid G_a(a,\beta_1(a,\lambda),\lambda)=0\}$, for which is only possible when $\beta_1(a,\lambda)=\bar{\rho}_\ell$ or $\beta_1(a,\lambda)=\bar{\rho}_r$. In view of (4.138), (4.139), (4.141) and Table 4.6, we can see that

$$G\big(\alpha_1(\bar{\rho}_\ell,\lambda),\bar{\rho}_\ell,\lambda\big)=0,\quad \bar{\rho}_\ell<\rho<\alpha_1(\bar{\rho}_\ell,\lambda)<\eta\quad\text{and}\quad \beta_2(a,\lambda)>\beta_2(a,0)\ge a,\quad \text{for } a\le\eta.$$

It follows that $\beta_1(\alpha_1(\bar{\rho}_\ell,\lambda),\lambda)=\bar{\rho}_\ell$. Furthermore, β_1 is strictly increasing on $]0,\alpha_1(\bar{\rho}_\ell,\lambda)[$, since $G(a,\bar{\rho}_\ell(\lambda),\lambda)>0$ for all $a\in\,]0,\beta_3(\bar{\rho}_\ell)[\,\setminus\{\alpha_1(\bar{\rho}_\ell)\}$, and $\beta_1(\alpha_1(a,\lambda),\lambda)=a$ for all $a<\alpha_1(\bar{\rho}_\ell,\lambda)$, thanks to (4.142) and the fact that $\beta_1(a,\lambda)<\beta_2(a,\lambda)$. The

Note that for a fixed $b\in\,]\bar{\rho}_\ell,\bar{\rho}_r[$, the function $G(.,b,\lambda)$ only has one zero $\alpha_3(b,\lambda)>\alpha_3(b,0)\ge\eta>\alpha_1(\bar{\rho}_\ell,\lambda)$. Using similar arguments above, we obtain that the restriction of β_1 on $]\alpha_1(\bar{\rho}_\ell),\infty[$ is strictly decreasing,

$$G_b(a,\beta_1(a,\lambda),\lambda)>0,\quad G_a(a,\beta_1(a,\lambda),\lambda)>0\quad\text{and}\quad a=\alpha_2\big(\beta_1(a,\lambda)\big) \tag{4.144}$$

for all $a>\alpha_1(\bar{\rho}_\ell,\lambda)=\alpha_2(\bar{\rho}_\ell,\lambda)$.

□

The following corollary is straightforward by using (4.107)–(4.113) and Assumption 4.1, as well as (iii)–(vi) in Proposition 4.3.4.

Corollary 4.6.1 *Suppose that Assumption 4.1 holds true. Fix $\lambda\in[0,1[$, we write $\underline{\eta}_\ell$, $\underline{\eta}_r$ and $\bar{\rho}_\ell$, $\bar{\rho}_r$, in place of $\bar{\rho}_\ell(\lambda)$, $\bar{\rho}_r(\lambda)$, $\underline{\eta}_\ell(\lambda)$, $\underline{\eta}_r(\lambda)$. The following statements hold true:*
(I)

$$\beta_1(a)\le a,\quad \text{for } a>0,\quad \text{and}\quad \beta_{2,3}(a)\begin{cases}\ge a, & \text{for } a\le\underline{\eta}_\ell,\\ \le a, & \text{for } a\ge\underline{\eta}_r.\end{cases}$$

Furthermore, the equalities hold true if and only if $\lambda=0$ and (i) $a\le\rho$ for the β_1, (ii) $\rho\le a\le\eta$ for β_2 and (iii) $a\ge\eta$ for β_3.
(II)

$$\alpha_{1,2}(b)\begin{cases}\ge b, & \text{for } b\le\bar{\rho}_\ell,\\ \le b, & \text{for } b\ge\bar{\rho}_r,\end{cases}\quad \text{and}\quad \alpha_3(b)\ge b,\quad \text{for } b>0.$$

Furthermore, the equalities hold true if and only if $\lambda=0$ and (i) $b\le\rho$ for the first equality, (ii) $\rho\le b\le\eta$ for the second equality and (iii) $b\ge\eta$ for the third equality.

(III)

$$G_b\big(a,\beta_1(a),\lambda\big) > 0 \quad \text{for all } a>0$$
$$\text{and} \quad G_b\big(a,\beta_2(a),\lambda\big) < 0 \quad \text{and} \quad G_b\big(a,\beta_2(a),\lambda\big) > 0, \quad \text{for } a \in]0,\underline{\eta}_\ell[\cup]\underline{\eta}_r,\infty[.$$

(IV)

$$\mathbf{G}_a\big(\alpha_3(b),b,\lambda\big) < 0, \quad \text{for all } b>0$$
$$\text{and} \quad \mathbf{G}_a\big(\alpha_2(b),b,\lambda\big) > 0 \quad \text{and} \quad \mathbf{G}_a\big(\alpha_1(b),b,\lambda\big) < 0, \quad \text{for } b \in]0,\bar{\rho}_\ell[\cup]\bar{\rho}_r,\infty[.$$

Proof of Lemma 4.3.5. We first show that

$$(1-\lambda)s \le g(s) \quad \text{for all } s\in]a\wedge b, a\vee b[\iff \begin{cases} \text{(i) } a\in]0,\underline{\eta}_\ell] \text{ and } b=\beta_2(a)>a, & \text{or} \\ \text{(ii) } a\in]0,\underline{\eta}_r[\text{ and } b=\beta_1(a)<a, & \text{or} \\ \text{(iii) } a\in [\underline{\eta}_r,\infty[\text{ and } b=\beta_3(a)<a. \end{cases} \tag{4.145}$$

In light of (i) in Lemma 4.3.3, we first consider the case that $(1-\lambda)b > a$. In such a case, $a \le \underline{\eta}_\ell$ and $b=\beta_2(a)$ or $\beta_3(a)$, thanks to Corollary 4.6.1.(I). Note that Lemma 4.3.3 implies that

$$g(s) > a > (1-\lambda)s, \quad \text{for } s\in\left[a, \frac{a}{1-\lambda}\right].$$

For $s > \frac{a}{1-\lambda}$, we calculate

$$g(s)-(1-\lambda)s = \frac{((1-\lambda)s-a)p_a(s)}{1-\underline{h}(a,b,\lambda)p_a(s)}\big(\underline{h}(a,b,\lambda)-\underline{h}(a,s,\lambda)\big), \tag{4.146}$$

where

$$\underline{h}(a,s,\lambda) = \frac{1}{p_a(s)} - \frac{1}{(1-\lambda)s-a},$$

and

$$\underline{h}_s(a,s,\lambda) = G(a,s,\lambda)\frac{\sqrt{1-\lambda}p_a(s) + ((1-\lambda)s-a)\sqrt{p_a'(s)}}{p_a^2(s)((1-\lambda)s-a)^2}. \tag{4.147}$$

Combining these results with (i) in Lemma 4.3.3 and (iv) in Proposition 4.3.4, we can see that $g(s)\ge(1-\lambda)s$ if and only if $b=\beta_2(a)$.

When $a>b$, it is either (I) $b=\beta_1(a)$ or (II) $a\ge\underline{\eta}_r(\lambda)$ and $b=\beta_i(a)$, for $i=2,3$. In view of (4.146) and (4.147), we can see that if (a,b) is in (II) for $b=\beta_2(a)$, then the inequity $g(s)\le(1-\lambda)s$ holds true. While if (a,b) is either in (I) for $a\le\underline{\eta}_r$, or (II) for $b=\beta_3(a)$, then $g(s)\ge(1-\lambda)s$. We next exclude the possibility when (a,b) is (I) for $a\ge\underline{\eta}_r$. In light of (4.146), we only need to show that

$$\underline{h}(a,\beta_3(a),\lambda) < \underline{h}(a,\beta_1(a),\lambda). \tag{4.148}$$

Note that $\alpha_2(\beta_1(a)) = a$ and $\alpha_3(\beta_3(a)) = a$ (see Table 4.2). This observation and Corollary 4.6.1.(IV) imply that

$$\mathbf{G}_a\big(a,\beta_1(a),\lambda\big) = \mathbf{G}_a\big(\alpha_2(\beta_1(a)),\beta_1(a),\lambda\big) < 0$$
$$\text{and} \quad \mathbf{G}_a\big(a,\beta_3(a),\lambda\big) = \mathbf{G}_a\big(\alpha_3(\beta_3(a)),\beta_3(a),\lambda\big) > 0.$$

Combining these inequities with (4.123), we obtain

$$\underline{h}(a,\beta_3(a),\lambda) < \frac{\Theta(a)}{a} < \underline{h}(a,\beta_1(a),\lambda)$$

We next show that

$$g(s) \le s \quad \text{for all } s \in \,]a\wedge b, a\vee b[\iff \begin{cases} \text{(i) } b \in [\bar{\rho}_r,\infty[\text{ and } a=\alpha_2(b)<b, & \text{or} \\ \text{(ii) } b \in \,]0,\bar{\rho}_\ell] \text{ and } a=\alpha_1(b)>b, & \text{or} \\ \text{(iii) } b \in [\bar{\rho}_\ell,\infty[\text{ and } a=\alpha_3(b)>b. \end{cases} \tag{4.149}$$

To this end, we use the form of g in Lemma 4.3.3.(iii) to calculate

$$g(s)-s = -\frac{(1-\lambda)((1-\lambda)b-s)p_b(s)}{1-\bar{h}(a,b,\lambda)p_b(s)}\big(\bar{h}(a,b,\lambda)-\bar{h}(a,s,\lambda)\big),$$

where

$$\bar{h}(s,b,\lambda) = \frac{1}{(1-\lambda)p_b(s)} + \frac{1}{(1-\lambda)b-s},$$

and

$$\bar{h}_s(s,b,\lambda) = \mathbf{G}(s,b,\lambda)\frac{((1-\lambda)s-a)\sqrt{p_b'(s)}-\sqrt{(1-\lambda}p_b(s)}{p_a^2(s)((1-\lambda)s-a)^2}. \tag{4.150}$$

The required results as well as the equivalence are straightforward by using (4.145), (4.149) and Tables 4.2 and 4.3.

□

Proof of Lemma 4.3.6. We prove the theorem in the following 6 main steps.

Step 1: Properties of $\hat{Q}(.,\beta_1(.),\lambda)$ (resp., $\hat{Q}(\alpha_1(.),.,\lambda)$) on $]0,\alpha_1(\bar{\rho}_\ell)[$ (resp., $]0,\bar{\rho}_\ell[$). In view of the deification (4.59) of $\hat{Q}$, (iv) in Proposition 4.3.4, and (4.123), (4.124), (I) and (IV) in Corollary 4.6.1, Tables 4.2 and 4.3 and the fact that $G_a(a,\beta_1(a),\lambda)+G_b(a,\beta_1(a),\lambda)\beta_1'(a)=0$, we can see that

$$\frac{\Theta(\alpha_1(\bar{\rho}_\ell))}{\alpha_1(\bar{\rho}_\ell)} = \underline{h}(\alpha_1(\bar{\rho}_\ell),\beta_1(\alpha_1(\bar{\rho}_\ell)),\lambda) = \underline{h}(\alpha_1(\bar{\rho}_\ell),\bar{\rho}_\ell,\lambda), \tag{4.151}$$

$$G_a\big(a,\beta_1(a),\lambda\big)<0 \quad\text{and}\quad G_b\big(a,\beta_1(a),\lambda\big)>0 \quad \text{for all } a\in\,]0,\alpha_1(\bar{\rho}_\ell)[, \tag{4.152}$$

$$\frac{\Theta(a)}{a} > \underline{h}\big(a,\beta_1(a),\lambda\big) > \frac{1}{\sqrt{1-\lambda}\sqrt{p_a'(\beta_1(a))}}\frac{\Theta(\beta_1(a))}{\beta_1(a)} \quad \text{for all } a\in\,]0,\alpha_1(\bar{\rho}_\ell)[, \tag{4.153}$$

$$\text{and}\quad \frac{\mathrm{d}\hat{Q}(a,\beta_1(a),\lambda)}{\mathrm{d}a} = -\frac{2G_a(a,\beta_1(a),\lambda)}{\sqrt{1-\lambda}(\sqrt{1-\lambda}\beta_1(a)\sqrt{p_a'(\beta_1(a))}-a)^2} > 0, \tag{4.154}$$

for $a\in\,]0,\alpha_1(\bar{\rho}_\ell)[\backslash\Xi^0$, where

$$\Xi^0 := \{a\in\,]0,\alpha_1(\bar{\rho}_\ell)[|L\big(a,\beta_1(a),\lambda\big)=0\},$$

with L as in (4.58). We next show that

$$\underline{h}(a,\beta_1(a),\lambda) > \frac{1}{a} \quad\Leftrightarrow\quad L\big(a,\beta_1(a),\lambda\big)>0 \quad \text{for all } a\in\,]0,\alpha_1(\bar{\rho}_\ell)[\tag{4.155}$$

by contradiction. In view of (4.153) and the fact that $\bar\rho_\ell < \rho$, if

$$\underline{h}\big(a,\beta_1(a),\lambda\big) \le \frac{1}{a} \quad \Leftrightarrow \quad \sqrt{1-\lambda}\beta_1(a)\sqrt{p_a'(\beta_1(a))} - a \le 0, \quad \text{for some } a \in\,]0,\alpha_1(\bar\rho_\ell)[,$$

then

$$1 \ge a\underline{h}\big(a,\beta_1(a),\lambda\big) > \frac{a}{\sqrt{1-\lambda}\beta_1(a)\sqrt{p_a'(\beta_1(a))}}\Theta\big(\beta_1(a)\big) \ge \Theta\big(\beta_1(a)\big) > \Theta(\rho) > 1, \tag{4.156}$$

which leads to contradiction.

Using (4.135), (4.136), (4.151)–(4.155) and Remarks 4.1 and 4.4, we can see that

$$\alpha_1(\bar\rho_\ell) < \rho < \zeta, \quad \hat{Q}\big(\alpha_1(\bar\rho_\ell),\beta_1\big(\alpha_1(\bar\rho_\ell)\big),\lambda\big) = \hat{Q}\big(\alpha_1(\bar\rho_\ell),\bar\rho_\ell,\lambda\big) = Q(\alpha_1(\bar\rho_\ell)\big), \tag{4.157}$$

$$\text{and} \quad \frac{Q(\beta_1(a))}{1-\lambda} < \hat{Q}\big(a,\beta_1(a),\lambda\big) < Q(a), \quad \text{for } a \in\,]0,\alpha_1(\bar\rho_\ell)[. \tag{4.158}$$

Furthermore,

$$\hat{Q}\big(\alpha_1(b),b,\lambda\big) = \hat{Q}\big(\alpha_1(b),\beta_1\big(\alpha_1(b)\big),\lambda\big) \quad \text{for all } b \in]0,\bar\rho_\ell[,$$

and

$$\frac{Q(b)}{1-\lambda} < \hat{Q}\big(\alpha_1(b),b,\lambda\big) < Q\big(\alpha_1(b)\big), \quad L\big(\alpha_1(b),b,\lambda\big) > 0 \quad \text{and} \quad \frac{\mathrm{d}\hat{Q}(\alpha_1(b),b,\lambda)}{\mathrm{d}b} > 0, \tag{4.159}$$

for $b \in]0,\bar\rho_\ell[$, thanks to Table 4.2.

Step 2: Properties of $\hat{Q}(\cdot,\beta_2(\cdot),\lambda)$ (resp., $\hat{Q}(\alpha_2(\cdot),\cdot,\lambda)$) on $]\alpha_2(\bar\rho_r),\underline{\eta}_\ell[$ (resp., on $]\bar\rho_r,\beta_2(\underline{\eta}_\ell)[$) .
Using arguments similar to the ones in Step 1, we obtain

$$\hat{Q}\big(\alpha_2(\bar\rho_r),\beta_2(\alpha_2(\bar\rho_r)),\lambda\big) = Q\big(\alpha_2(\bar\rho_r,)\big), \quad (1-\lambda)\hat{Q}\big(\underline{\eta}_\ell,\beta_2(\underline{\eta}_\ell),\lambda\big) = Q\big(\beta_2(\underline{\eta}_\ell)\big), \tag{4.160}$$

$$\frac{\Theta(a)}{a} > \underline{h}(a,\beta_2(a),\lambda) > \frac{1}{\sqrt{1-\lambda}\sqrt{p_a'(\beta_1(a))}}\frac{\Theta(\beta_1(a))}{\beta_1(a)} \quad \text{for } a \in\,]\alpha_2(\bar\rho_r),\underline{\eta}_\ell[, \tag{4.161}$$

$$\text{and} \quad \frac{\mathrm{d}\hat{Q}(a,\beta_2(a),\lambda)}{\mathrm{d}a} < 0, \quad \text{for } a \in\,]\alpha_2(\bar\rho_r),\underline{\eta}_\ell[\backslash\Xi^\dagger, \tag{4.162}$$

where

$$\Xi^\dagger := \{a \in\,]\alpha_2(\bar\rho_r),\underline{\eta}_\ell[\,|L(a,\beta_2(a),\lambda)=0\}.$$

Note that if $\zeta > a \in \Xi^\dagger$, then $\beta_2(a) > \zeta$ by using arguments similar to the ones in (4.156). Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}a}L\big(a,\beta_2(a),\lambda\big) = \Theta(a) - 1 + \sqrt{1-\lambda}\sqrt{p_a'\big(\beta_2(a)\big)}\big(1-\Theta(\beta_2(a)\big)(\beta_2)'(a) > 0.$$

Combining all these results above in this step with Assumption 4.1 and Remarks 4.1 and 4.4 and the definition (4.59) of $\hat{Q}$, we can show that there exists $\alpha_2(\bar\rho_r) < \underline{\zeta} < \underline{\eta}_\ell$ such that

$$\Xi^\dagger = \{\underline{\zeta}\}, \quad \lim_{a\uparrow\underline{\zeta}} \hat{Q}(a,\beta_2(a),\lambda) = -\infty, \quad \lim_{a\downarrow\underline{\zeta}} \hat{Q}(a,\beta_2(a),\lambda) = \infty, \tag{4.163}$$

$$\underline{h}(a,\beta_2(a),\lambda)\begin{cases} > \frac{1}{a}, & \text{for } a \in\,]\alpha_2(\bar\rho_r),\underline{\zeta}[, \\ < \frac{1}{a}, & \text{for } a \in\,]\underline{\zeta},\underline{\eta}_\ell[, \end{cases} \quad L(a,\beta_2(a),\lambda)\begin{cases} < 0, & \text{for } a \in\,]\alpha_2(\bar\rho_r),\underline{\zeta}[, \\ > 0, & \text{for } a \in\,]\underline{\zeta},\underline{\eta}_\ell[, \end{cases} \tag{4.164}$$

$$\text{and} \quad \hat{Q}(a,\beta_2(a),\lambda)\begin{cases} < Q(a), & \text{for } a \in\,]\alpha_2(\bar\rho_r),\underline{\zeta}[\,\cup\,]\Theta^{-1}(1),\underline{\eta}_\ell[, \\ > Q(a), & \text{for } a \in\,]\underline{\zeta},\Theta^{-1}(1)[. \end{cases} \tag{4.165}$$

On the other hand, if we define

$$\bar{\zeta} = \beta_2(\underline{\zeta}),$$

then we have

$$\hat{Q}(\alpha_2(\bar{\rho}_r), \bar{\rho}_r, \lambda) = Q(\alpha_2(\bar{\rho}_r)) = Q(\alpha_2(\bar{\rho}_r)), \quad (1-\lambda)\hat{Q}(\alpha_2(\beta_2(\underline{\eta}_\ell)), \beta_2(\underline{\eta}_\ell), \lambda) = Q(\beta_2(\underline{\eta}_\ell)), \tag{4.166}$$

$$\frac{\mathrm{d}\hat{Q}(\alpha_2(b), b, \lambda)}{\mathrm{d}b} < 0, \quad \text{for } a \in]\bar{\rho}_r, \bar{\zeta}[\cup]\bar{\zeta}, \beta_2(\underline{\eta}_\ell)[, \tag{4.167}$$

$$\lim_{b \uparrow \bar{\zeta}} \hat{Q}\big(\alpha_2(b), b, \lambda\big) = -\infty, \quad \lim_{b \downarrow \bar{\zeta}} \hat{Q}(\alpha_2(b), b, \lambda) = \infty, \tag{4.168}$$

$$L\big(\alpha_2(b), b, \lambda\big) \begin{cases} < 0, & \text{for } b \in]\bar{\rho}_r, \bar{\zeta}[, \\ > 0, & \text{for } b \in]\bar{\zeta}, \beta_2(\underline{\eta}_\ell)[, \end{cases} \tag{4.169}$$

$$\text{and} \quad (1-\lambda)\hat{Q}(\alpha_2(b), b, \lambda) \begin{cases} > Q(b), & \text{for } a \in]\bar{\rho}_r, \Theta^{-1}(1)[\cup]\bar{\zeta}, \bar{\rho}_r[, \\ < Q(b), & \text{for } a \in]\Theta^{-1}(1), \bar{\zeta}[. \end{cases} \tag{4.170}$$

Step 3: Properties of $\hat{Q}(\cdot, \beta_3(\cdot), \lambda)$ (resp., $\hat{Q}(\alpha_3(\cdot), \cdot, \lambda)$) on $]\underline{\eta}_r, \infty[$ (resp., on $]\beta_3(\underline{\eta}_r), \infty[$) . Use arguments similar to the ones in Step 1, we obtain

$$\beta_3(\underline{\eta}_r) > \zeta, \quad (1-\lambda)\hat{Q}\big(\alpha_3(b), b, \lambda\big) \begin{cases} = Q(b), & \text{for } b = \beta_3(\underline{\eta}_r), \\ > Q(b), & \text{for } b \in]\beta_3(\underline{\eta}_r), \infty[, \end{cases} \tag{4.171}$$

$$\text{and} \quad L\big(\alpha_3(b), b, \lambda\big) < 0 \quad \text{and} \quad \frac{\mathrm{d}\hat{Q}(\alpha_3(b), b, \lambda)}{\mathrm{d}b} > 0, \quad \text{for } b \in [\beta_3(\underline{\eta}_r), \infty[, \tag{4.172}$$

as well as

$$L\big(a, \beta_3(a), \lambda\big) < 0, \quad \hat{Q}\big(a, \beta_3(a), \lambda\big) < Q(a) \quad \text{and} \quad \frac{\mathrm{d}\hat{Q}(a, \beta_3(a), \lambda)}{\mathrm{d}b} > 0, \quad \text{for } a \in [\underline{\eta}_r, \infty[. \tag{4.173}$$

Step 4: $\underline{\uprho}$, $\underline{\upeta}_\ell$, $\underline{\upeta}_r$, $\bar{\uprho}_\ell$,$\bar{\uprho}_r$ and $\bar{\upeta}$. In view of (4.154), (4.157) (4.158), (4.160), (4.162), (4.163), (4.165) and the fact that $\alpha_2(\bar{\rho}_r) < \rho < \alpha_1(\bar{\rho}_\ell)$, we can see that

$$\hat{Q}(\alpha_2(\bar{\rho}_r), \beta_1(\alpha_2(\bar{\rho}_r)), \lambda) - \hat{Q}(\alpha_2(\bar{\rho}_r), \beta_2(\alpha_2(\bar{\rho}_r)), \lambda) < 0,$$

$$\lim_{a \uparrow \alpha_1(\bar{\rho}_\ell) \wedge \underline{\zeta}} \Big(\hat{Q}(a, \beta_1(a), \lambda) - \hat{Q}(a, \beta_2(a), \lambda)\Big) > 0,$$

$$\text{and} \quad \frac{\mathrm{d}}{\mathrm{d}a}\big(\hat{Q}(a, \beta_1(a), \lambda) - \hat{Q}(a, \beta_2(a), \lambda)\big) > 0, \quad \text{for } a \in]\alpha_2(\bar{\rho}_r), \alpha_1(\bar{\rho}_\ell) \wedge \underline{\zeta}[.$$

It follows that there exists $\alpha_2(\bar{\rho}_r) < \underline{\uprho} < \alpha_1(\bar{\rho}_\ell) \wedge \underline{\zeta}$ such that

$$\hat{Q}(\underline{\uprho}, \bar{\uprho}_\ell, \lambda) = \hat{Q}(\underline{\uprho}, \bar{\uprho}_r, \lambda) \quad \text{and} \quad \bar{\uprho}_\ell < \bar{\rho}_\ell < \bar{\rho}_r < \bar{\uprho}_r,$$

where

$$\bar{\uprho}_\ell = \beta_1(\underline{\uprho}) \quad \text{and} \quad \bar{\uprho}_\ell = \beta_2(\underline{\uprho}). \tag{4.174}$$

Similarly, that there exists $\beta_3(\underline{\eta}_r) \vee \bar{\zeta} < \bar{\upeta} < \beta_2(\underline{\eta}_\ell)$ such that

$$\hat{Q}(\bar{\upeta}, \underline{\upeta}_\ell, \lambda) = \hat{Q}(\bar{\upeta}, \underline{\upeta}_r, \lambda) \quad \text{and} \quad \underline{\upeta}_\ell < \underline{\eta}_\ell < \underline{\upeta}_r < \underline{\eta}_r, \tag{4.175}$$

where

$$\underline{\eta}_\ell = \alpha_2(\bar{\eta}) \quad \text{and} \quad \underline{\eta}_r = \alpha_3(\bar{\eta}).$$

Combining all the arguments in Step 1–4, and in addition the calculations

$$1 + a\hat{Q}\big(a, \beta(a), \lambda\big) = \frac{1}{\sqrt{1-\lambda}} \frac{(1-\lambda)\beta(a) - a}{\sqrt{1-\lambda}\beta(a)\sqrt{p'_a(\beta(a))} - a}$$
$$\text{and} \quad 1 + (1-\lambda)b\hat{Q}(\alpha(b), b, \lambda) = \frac{(1-\lambda)b - \alpha(b)}{\sqrt{1-\lambda}b\sqrt{p'_{\alpha(b)}(b)} - \alpha(b)},$$

we obtain all the required results expect (i), (4.70) and (4.71).

Step 5: Proof of (4.70) and (4.71).

In view of the definitions (4.40) and (4.59) of g and $\hat{Q}$ and Lemma 4.3.6, we can see that

$$\underline{h}\big(\underline{\rho}, \bar{\rho}_\ell, \lambda\big) = \underline{h}\big(\underline{\rho}, \bar{\rho}_r, \lambda\big) \quad \text{and} \quad g\big(s, \underline{\rho}, \bar{\rho}_\ell, \lambda\big) = g\big(s, \underline{\rho}, \bar{\rho}_r, \lambda\big) \quad \text{for all } s \in [\bar{\rho}_\ell, \bar{\rho}_r].$$

Similarly, we use the third identity in (4.59) and the alternative expression (4.45) of g in Lemma 4.3.3.(iii) to obtain

$$\bar{h}\big(\underline{\eta}_\ell, \bar{\eta}, \lambda\big) = \bar{h}\big(\underline{\eta}_r, \bar{\eta}, \lambda\big) \quad \text{and} \quad g\big(s, \underline{\eta}_\ell, \bar{\eta}, \lambda\big) = g\big(s, \underline{\eta}_r, \bar{\eta}, \lambda\big) \quad \text{for all } s \in [\underline{\eta}_\ell, \underline{\eta}_r].$$

Step 6: Proof of (i) In view of the definition (4.72) of β and Step 1 in the proof of Proposition 4.3.4, we can see that

$$\lim_{\lambda \downarrow 0} \beta(a, \lambda) = a \quad \text{for all } a > 0.$$

Combining this result with the definition (4.56) of $\underline{h}$, we use L'Hôpital's rule to compute

$$\lim_{\lambda \downarrow 0} \underline{h}\big(a, \beta(a,\lambda), \lambda\big) = \lim_{\lambda \downarrow 0} \frac{\sqrt{p_{\beta(a,\lambda)}(a)}\big(-\frac{\Theta(a)}{a} + \frac{\Theta(\beta(a,\lambda))}{\beta(a)}\beta'(a,\lambda)\big)}{\beta'(a,\lambda) - 1} = \frac{\Theta(a)}{a}, \tag{4.176}$$

and (i) follows. □

Proof of Theorem 4.3.7. The only results left to be proven is A and B are of finite variation and the condition (i) in Lemma 4.3.1 is true. We first notice that the restriction of A on $[\tau_k, e_k[$ is increasing (resp., decreasing) if $S_{\tau_k} \in]0, \underline{\rho}[\cup]\underline{\eta}_r, \infty[$ (resp., $S_{\tau_k} \in [\underline{\rho}, \underline{\eta}_\ell[$). If $S_{\tau_k} \in [\underline{\eta}_\ell, \underline{\eta}_r[$, then

$$\mathsf{P}(\{\mathfrak{m}_{e_k} < \underline{\eta}_\ell\} \cup \{\mathfrak{M}_{e_k} > \underline{\eta}_r\}) = 1.$$

Furthermore, if $\mathfrak{M}_t > \underline{\eta}_r$ (resp., $\mathfrak{m}_t < \underline{\eta}_\ell$) for some $\tau_k < t < e_k$, then

$$\max_{t \le u \le e_k} S_u > \underline{\eta}_r \quad \text{and} \quad \min_{t \le u \le e_k} S_u \ge \beta(\mathfrak{M}_t) > \beta(\underline{\eta}_r) = \bar{\eta} > \underline{\eta}_\ell$$
$$\text{(resp.,} \min_{t \le u \le e_k} S_u < \underline{\eta}_\ell \quad \text{and} \quad \max_{t \le u \le e_k} S_u \le \beta(\mathfrak{M}_t) < \beta(\underline{\eta}_\ell) = \bar{\eta} < \underline{\eta}_r).$$

It follows that if $S_{\tau_k} \in [\underline{\eta}_\ell, \underline{\eta}_r[$, then $\{\mathfrak{m}_{e_k} < \underline{\eta}_\ell\} \cap \{\mathfrak{M}_{e_k} > \underline{\eta}_r\} = \emptyset$ and A is increasing (resp., decreasing) if $\mathfrak{M}_{e_k} > \underline{\eta}_r$ (resp., $\mathfrak{m}_{e_k} < \underline{\eta}_r$). In particular, A has a positive jump from $\underline{\eta}_\ell$ to $\underline{\eta}_r$ at $\inf\{t > \tau_k \mid \mathfrak{M}_t > \underline{\eta}_r\}$. Combining all these observations above and the fact the restriction of β on $]0, \underline{\eta}_\ell] \cup [\underline{\eta}_r, \infty[$ is increasing, we obtain the corresponding results in Table 4.8. Similarly, we can

Table 4.8: Restriction of A and B on time interval $[\tau_k, e_k]$ and $[e_k, \tau_{k+1}]$

S_{τ_k}	A and B on $[\tau_k, e_k[$	S_{e_k}	A and B on $[e_k, \tau_{k+1}[$
$]0, \underline{\rho}[$	incr.	$]0, \bar{\rho}_\ell[$	decr.
$[\underline{\rho}, \underline{\eta}_\ell[$	decr.	$[\bar{\rho}_\ell, \bar{\rho}_r]$	1) decr.($m_{\tau_{k+1}} < \bar{\rho}_\ell$) or 2) incr.($M_{\tau_{k+1}} > \bar{\rho}_r$)
$[\underline{\eta}_\ell, \underline{\eta}_r]$	1) incr.($M_{e_k} > \underline{\eta}_r$) or 2) decr.($m_{e_k} < \underline{\eta}_\ell$)	$]\bar{\rho}_r, \bar{\eta}]$	incr.
$]\underline{\eta}_r, \infty]$	incr.	$]\bar{\eta}, \infty[$	decr.

We denote by decr.(resp., incr.) the restriction of the process A and B on the time interval $[\tau_k, e_k]$ and $[e_k, \tau_{k+1}]$ is decreasing (resp., increasing).

obtain the restriction of B on $[e_k, \tau_{k+1}[$ in Table 4.8. These two tables imply that the processes A and B are of finite variation.

We will prove the condition (ii) in Lemma 4.3.1, if we show that (4.20) for sufficiently small and sufficiently big S_T hold true. If S_T is sufficiently small, then Lemma 4.3.6 and (4.9) imply that $\mathcal{Q}_T < 0$, $\vartheta_T^{0,\star} < 0$, $\vartheta_T^\star > 0$, $B_T < S_T < A_T$ and $\underline{Q}(A_T) < Q(A_T)$. Combining these observations with Assumption 4.3, we obtain

$$0 < \frac{\hat{\vartheta}^\star S_T}{V_T(\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} \leq \frac{\underline{Q}(A_T)A_T}{1 + \underline{Q}(A_T)A_T} \leq \frac{Q(A_T)A_T}{1 + Q(A_T)A_T} = \Theta(A_T) \leq C_1.$$

If S_T is sufficiently large, then we use similar arguments to obtain

$$0 > \frac{(1-\lambda)\hat{\vartheta}^\star S_T}{V_T((1-\lambda)\hat{\vartheta}^{0,\star}, \hat{\vartheta}^\star)} \geq \frac{(1-\lambda)\underline{Q}(B_T)B_T}{1 + (1-\lambda)\underline{Q}(B_T)A_T} \geq \frac{Q(B_T)B_T}{1 + Q(B_T)B_T} = \Theta(B_T) \geq -C_1.$$

□

Chapter 5

Conclusion and future work

In Chapter 2, we consider a classical one-sided impulse control problem for a linear diffusion on the positive half axis. The problem is motivated by the optimal exploitation of a renewable resource with a fixed cost at each time of a control action. The controlled process models the population density of a harvested species, where the decision-maker determines the timing and magnitude of harvesting interventions while accounting for fixed costs and state-dependent profits. The main contribution of the chapter is a complete characterization of the solution under general assumptions on the problem data. We find that a β-γ strategy, where the system is controlled impulsively when the state process reaches an upper threshold β and reset to a lower level γ, can be optimal under certain conditions. In cases where an optimal strategy does not exist, we identify a sequence of ε-optimal strategies that approximate the optimal payoff. Furthermore, we find the problem data with which it is optimal to take no intervention at all. Another novel contribution of this study is the analysis the boundary classification of the problem's state space, which may influence the optimal strategy. In addition, we give an SDE construction of the optimally controlled processes.

In Chapter 3, we study singular stochastic control problems in the context of optimal harvesting, extending previous works by incorporating sufficiently general problem data. The model is similar to Chapter 2 but does not involve a fixed cost. Three performance criteria are considered: an expected discounted performance criterion, an expected ergodic performance criterion, and a pathwise ergodic performance criterion. We deriving explicit solutions to the HJB equations associated with the three variants, and characterize the optimal payoff. The optimal strategy is to take the minimal action required to keep the state process below a certain threshold β. We develop a novel argument to address the potential unboundedness below of the solution to the HJB equation, which gives rise to a non-trivial complication in the verification arguments. Additionally, the discounted and ergodic versions of the problem are connected between the Abelian limits, with non-constant discounting rate functions.

A natural extension of Chapter 2 is the impulse control problem with an expected ergodic performance criterion and a pathwise ergodic performance criterion. In addition, we could extend impulse control problems to a mean field game (MFG), which studies the strategic interactions of a large number of harvesting companies. A representative player could follow with the dynamics and controls of the form of the controller in Chapter 2.

Future research of Chapter 3 could explore the extension of the results to game settings, incorporating interactions between multiple decision-makers. One player in such a game could be the harvesting company, while the other player could be a government agency, cooperative, international organization or private enterprise that leases or provides access to a harvesting area and manages harvesting with sustainability standards. An other possible extension to this game is to add a fixed cost associated each intervention as in Chapter 2.

In Chapter 4, we explored the Merton's problem of maximizing the long-term growth rate under transaction costs. The risky asset we considered is modelled by a linear diffusion, beyond the Black-Scholes model. By constructing a shadow price process, we could reformulate the portfolio optimization problem in a fictitious frictionless market while maintaining the optimal strategy and the expected growth rate. We construct explicitly a shadow price process, as well as the buying boundary and the selling boundary. The key insight from our analysis is that transaction costs create a dynamic no-trade region where investors refrain from adjusting their portfolios unless stock prices reach the boundaries. The derivation of an explicit shadow price process and the trading boundaries are way more complicated than the Black-Scholes setting, where the no-trade region is static. Furthermore, the asymptotic expansions of arbitrary order for the non-trade region, the stock-cash ratio and the proportion of wealth invested in the risky asset are also provided.

Future research may extend this framework by considering different optimization objectives, such as power utility of terminal wealth and/or consumption. Additionally, the price impact on the risky asset could be incorporated. Another extension is to consider a multi-asset model, where, for instance, a geometric Brownian motion asset without transaction costs is introduced into the market.

Chapter 6

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