Singular Stochastic Control in Risk-Sensitive Optimisation and Equilibrium Asset Pricing with Proportional Transaction Costs

Justin Gwee

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London School of Economics and Political Science
Department of Mathematics

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Declaration

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I confirm that Chapter 2 was jointly co-authored with Professor Mihail Zervos.

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Abstract

In this thesis, we investigate the applications of singular stochastic control in two optimization problems. In the first part, we consider a two-sided singular stochastic control problem with a risk-sensitive ergodic criterion. In particular, we consider a stochastic system whose uncontrolled dynamics are modelled by a linear diffusion. The control that can be applied to this system is modelled by an additive finite variation process. The objective of the control problem is to minimise a risk-sensitive long-term average criterion that penalises deviations of the controlled process from a given interval as well as the expenditure of control effort. We derive the complete solution to the problem under general assumptions by deriving a C^2 solution to its HJB equation. To this end, we use the solutions to a suitable family of Sturm-Liouville eigenvalue problems.

In the second part of this thesis, we study a risk-sharing equilibrium with proportional transaction costs. We consider an economy with two agents, each of whom receive a cumulative endowment flow which is modelled as a stochastic integral of a deterministic continuous function of the economy's state, which is modelled by means of a general Itô diffusion. Each of the two heterogeneous agents have mean-variance preferences and can also trade a risky asset to hedge against the fluctuations of their endowment streams. We determine the agents' optimal (Radner) equilibrium trading strategies in the presence of proportional transaction costs. In particular, we derive a new free-boundary problem that provides the solution to the agents' optimisation problem in equilibrium. Furthermore, we derive the explicit solution to this free-boundary problem when the problem data is such that the frictionless optimiser is a strictly increasing or a strictly increasing and then strictly decreasing function of the economy's state. Finally, we derive small transaction cost asymptotics.

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Contents

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Introduction

A wide variety of theoretical aspects and applications of singular stochastic control to optimisation problems in mathematical finance and economics have been studied. This thesis aims to contribute to the study of the optimality of singular stochastic controls in two types of optimisation problems: risk-sensitive control problems and equilibrium asset pricing with proportional transaction costs. In this chapter, we give an overview of singular stochastic control problems and some of their applications, followed by an overview of control problems with a risk-sensitive criterion. Finally, we give an overview of equilibrium asset pricing with transaction costs.

1.1 Singular stochastic control problems

Singular stochastic control problems are a class of stochastic control problems in which the displacement of the state due to control effort is not absolutely continuous in time. Such a problem was first formulated by Bather and Chernoff [5] and [6] (known as the finite fuel problem), in which a spaceship is controlled such that its path is as close to a target as possible, while expending a minimal amount of fuel. In this thesis, we focus on infinite horizon singular stochastic control problems involving one-dimensional diffusion processes. Therefore, we give a brief overview of one-dimensional infinite horizon singular stochastic control problems, based on Chapter VIII of Fleming and Soner [44] (the cases of $\mathbb{R}^n$-valued diffusions and finite horizon problems can be found in references such as Fleming and Soner [44] and Pham [100]). We will first outline what it means for a stochastic control problem to be *singular*, as opposed to a regular control problem, through a specific simple example. We then introduce *singular controls*, a type of control that is optimal for such singular stochastic control problems, which are essentially solutions to a Skorokhod problem.

Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. Suppose that X is a real-valued state process whose dynamics satisfy the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \big(b(X_t) + u_t\big)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R},$$

where the process u is the control variable. Suppose we have an infinite horizon with discounting rate $r > 0$, and an objective functional

$$J_x(u) = \mathsf{E}_x\left[\int_0^\infty \mathrm{e}^{-rt}\Big(h(X_t) + K|u_t|\Big)\,\mathrm{d}t\right]$$

that we wish to minimise, where h is the running cost, and K represents the costs incurred from expenditure of effort. We define the value function by

$$V(x) = \inf_{u \in \mathcal{A}} J_x(u),$$

where $\mathcal{A}$ is the set of $(\mathcal{F}_t)$-progressively measurable $\mathbb{R}$-valued processes. By the dynamic programming principle, V should satisfy the HJB equation

$$rV(x) + \mathcal{H}\big(x, V'(x), V''(x)\big) = 0,$$

where $\mathcal{H}$ is the Hamiltonian defined by

$$\begin{aligned}
\mathcal{H}(x,p,M) &= \sup_{v\in\mathbb{R}}\left[-p\big(b(x)+v\big) - \frac{1}{2}M\sigma^2(x) - h(x) - K|v|\right] \\
&= -pb(x) - \frac{1}{2}M\sigma^2(x) - h(x) + \sup_{v\in\mathbb{R}}\big[-(p+K)v^+ + (p-K)v^-\big],
\end{aligned}$$

where $v^{\pm} > 0$ are the positive and negative parts of v. We observe that

$$\mathcal{H}(x, p, m) \begin{cases} = \infty, & \text{if } |p| > K, \\ < \infty, & \text{if } |p| \leq K. \end{cases}$$

Hence, the control problem is called *singular* due to the singularity of the Hamiltonian. The value function V should then satisfy

$$|V'(x)| \leq K \quad \text{and} \quad rV(x) - b(x)V'(x) - \frac{1}{2}\sigma^2(x)V''(x) - h(x) \leq 0$$

for all $x \in \mathbb{R}$. If $|V'(x)| < K$ for some $x \in \mathbb{R}$, then, in a neighbourhood of x, the unique maximiser in the Hamiltonian is zero. Hence, the optimal control should also be equal to zero in a neighbourhood of x, in which case, we expect that

$$rV(x) - b(x)V'(x) - \frac{1}{2}\sigma^2(x)V''(x) - h(x) = 0, \quad \text{whenever } |V'(x)| < K.$$

In other words, the HJB equation takes the form of the following variational inequality:

$$\max\left\{ rV(x) - b(x)V'(x) - \frac{1}{2}\sigma^2(x)V''(x) - h(x),\ |V'(x)| - K \right\} = 0$$

for all $x \in \mathbb{R}$.

The optimal strategy can be characterised as follows. The controller should wait and take no action for as long as the state process X takes values in the set in which $|V'(x)| < K$ (the "no-action" region). Otherwise, the controller should take minimal action to keep the state process X outside the interior of the set in which $|V'(x)| = K$ (the "push" region) at all times. Starting from the push region, the optimal state process X is moved impulsively into the no-action region, and reflections at the boundaries of the no-action region in the appropriate directions prevent X from exiting the no-action region. In order to obtain such an optimal control process, we first reformulate the original problem. To this end, we define the increasing (finite-variation) processes

$$\xi_t^+ = \int_0^t (u_s)^+ \, \mathrm{d}s, \quad \xi_t^- = \int_0^t (u_s)^- \, \mathrm{d}s, \quad \text{and} \quad \xi_t = \int_0^t u_s \, \mathrm{d}s = \xi_t^+ - \xi_t^-,$$

where $\xi^\pm$ are the unique increasing processes such that $\xi = \xi^+ - \xi^-$. Denoting by $|\xi| = \xi^+ + \xi^-$ the total variation process of ξ, we observe that

$$|\xi|_t = \xi_t^+ + \xi_t^- = \int_0^t |u_s| \, \mathrm{d}s.$$

The dynamics of X can now be written as

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}\xi_t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R},$$

where ξ is now the control variable. The objective functional and corresponding value function can be written as

$$J_x(\xi) = \mathsf{E}_x\left[\int_0^\infty \mathrm{e}^{-rt} h(X_t)\,\mathrm{d}t + K \int_{[0,\infty[} \mathrm{e}^{-rt} \mathrm{d}|\xi|_t\right] \quad \text{and} \quad V(x) = \inf_{\xi \in \mathcal{A}} J_x(\xi),$$

where $\mathcal{A}$ is the set of admissible controls. In order to obtain optimal controls, the class of admissible controls has to be enlarged to consider controlled finite-variation processes ξ which are not absolutely continuous functions of t (in other words, singular with respect to the Lebesgue measure).

In the context of the finite fuel problem, X represents the deviation of the spaceship from the target, where the target is modelled by an uncontrolled diffusion with drift b and volatility σ, and h is the running cost of these deviations. The controlled process ξ represents the cumulative expenditure of fuel to minimise these deviations, and costs proportional to the expenditures are incurred, where $K > 0$ is the proportionality constant. In the finite fuel problem, there is an additional constraint that $|\xi|_t \leq y$ for all $t \geq 0$, where $y > 0$ is a constant.

A common approach to solving this problem is to construct a (classical) C^2 solution w satisfying the HJB equation

$$\max\left\{rw(x) - b(x)w'(x) - \frac{1}{2}\sigma^2(x)w''(x) - h(x),\ \ |w'(x)| - K\right\} = 0$$

for all $x \in \mathbb{R}$, and through a verification theorem, prove that $w = V$. A general verification theorem for singular control problems for classical solutions w can be found in Fleming and Soner [44, Chapter VIII Theorem 4.1]. In some cases, it is possible to solve for w explicitly. For example, Benes, Shepp and Witsenhaussen [7] obtain explicit solutions to a one-dimensional problem, and establish a *principle of smooth fit* property that implies that w is C^2. However, in more general settings such as the multi-dimensional case, w may not be smooth, and it is instead shown (in Fleming and Soner [44, Chapter VIII Theorem 5.1] for example) that V is a viscosity solution of the HJB equation. In both problems considered in this thesis, the underlying diffusion is one-dimensional, and we obtain explicit classical solutions.

When w is a classical solution of the HJB equation, the optimal control $\xi^\star$ can be constructed explicitly. In the above example, it is the solution to the following Skorokhod problem, satisfying

$$\mathrm{d}X_t^\star = b(X_t^\star)\,\mathrm{d}t + \mathrm{d}\xi_t^\star + \sigma(X_t^\star)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}, \tag{1.1}$$

$$X_t^\star \in \mathrm{cl}\big\{x \in \mathbb{R} : |w'(x)| < K\big\}$$

$$\text{and} \quad \int_{[0,t)} \big(|w'(X_s^\star)| - K\big)\,\mathrm{d}\xi_s^\star = 0 \quad \text{for all } t \geq 0. \tag{1.2}$$

If $\mathcal{D} = \big\{x \in \mathbb{R} : |w'(x)| \geq K\big\}$, then (1.2) is equivalent to

$$\int_{[0,t)} \mathbf{1}_{\{X_s^\star \in \mathcal{D}\}}\,\mathrm{d}\xi_s^\star = \xi_t^\star, \quad \text{for all } t \geq 0.$$

SDEs with reflecting boundary conditions such as (1.1) were first considered by Skorokhod [105], where solutions on the half-line $\mathbb{R}_+$ with a reflecting boundary condition at 0 are constructed via a deterministic mapping (called the *Skorokhod map*) on the space $\mathcal{C}[0,\infty[$ of continuous functions on $[0,\infty[$. This was extended to a multi-dimensional setting with reflecting boundary conditions in convex regions by Tanaka [108]. An explicit formula for the Skorokhod map on $[0, a]$ for any $a > 0$ is derived by Kruk et al. [76]. In the case $b \equiv 0$ and $\sigma(x) = \sigma > 0$ in (1.1) and $w'(x) = K$ whenever $x \in]a, b[$, the solution to (1.1) is a Brownian motion reflected at the boundary of $]a, b[$ and $\xi^\star$ is the sum of the local times (which are not absolutely continuous in t) of the Brownian motion $X^\star$ at a and b (see (3.8) in Chapter 6 of [71]).

Singular stochastic control problems have been studied extensively, both from a theoretical point of view and as an applied problem. Such problems have been motivated by applications in several areas in mathematical finance and economics such as target tracking, optimal investment in the presence of proportional costs, optimal harvesting and management of public debt and exchange rates. We mention a few of these papers; the list of papers is not exhaustive for the sake of brevity.

The earliest papers on singular stochastic control were by Benes et al. [7] and Karatzas [68], who study the optimal tracking of a Brownian motion by a finite-variation process and derive explicit solutions. Benes et al. [7] considers a discounted infinite horizon criterion and Karatzas [68] extends the study to ergodic and finite horizon criterion, and establishes Abelian and Cesaro limits. Menaldi and Robin [89] extend the problem considered by [68] to diffusion processes and establish existence results, as well as analyse asymptotically the discounted problem. Menaldi, Robin and Taksar [91] extend the study by [68] to multidimensional Gaussian processes, which is followed up on by Menaldi and Robin [90] to consider multidimensional Gaussian-Poisson processes. Weerasinghe [112] considers an ergodic singular control problem where the controller additionally chooses the drift and volatility of the diffusion process. Weerasinghe [113] considers a discounted problem for diffusion processes with general (not necessarily convex) running cost functions, and finds that for unbounded cost functions, the value function is C^2 and the optimal control is of Skorokhod reflection (local time) type, while for bounded cost functions, the value function is only C^1 and the optimal control is a mixture of jumps and local time processes. This is followed up by Weerasinghe [114] that considers the ergodic (via Abelian limits) and finite time horizon problems. Kunwai et al. [77] develop a new approach to establish Abelian and Cesaro limits, without the assumption of symmetry properties satisfied by the drift and volatility made in [68] and [114]. In Jack and Zervos [61], an ergodic singular stochastic control problem is explicitly solved, where the proportional costs are state-dependent, and considers a pathwise performance criterion in addition to the usual expected performance criterion.

Hynd [59] studies an n-dimensional elliptic PDE which is the HJB equation corresponding to the (risk-neutral) ergodic singular control problem for an n-dimensional Brownian motion, calling it an eigenvalue problem. He shows that there is a unique eigenvalue corresponding to a viscosity solution of the PDE, and this eigenvalue is the optimal (minimal) long-term average cost for the singular control problem. In [59] it is assumed that the running cost function is convex and superlinear. Wu and Chen [118] consider weaker assumptions on the running cost function and use a shooting method to solve the problem. We note that in our risk-sensitive ergodic singular control problem where we have a one-dimensional diffusion, we have an ODE which has a classical solution and the eigenvalue is a Sturm-Liouville eigenvalue.

Connections between singular control problems and optimal stopping problems and Dynkin games have been studied, where the stopping times are given by the times that the controlled process hits the reflection boundaries. It is shown that the value function of the optimal stopping problem is the derivative (with respect to initial condition of the controlled diffusion) of the value function of optimal stopping problems and Dynkin games. Karatzas and Shreve [69] and [70] are the first to explore this connection with optimal stopping problems for Brownian motion, and this is extended to diffusions by Boetius and Kohlmann [10]. The connections with Dynkin games are studied by Karatzas and Wang [72] and Boetius [9].

One of the earliest applications of singular control (after the finite fuel problem) was considered by Taksar, Klass and Assaf [107], which is a portfolio selection problem with proportional transaction costs, in a market consisting of a riskless and risky asset whose dynamics are governed by a geometric Brownian motion. The optimal policy to maximise the expected growth rate of funds is a reflection type policy that keeps the ratio of funds in the risky to the riskless asset within a certain interval with minimal effort. This was followed soon after by Davis and Norman [29], who study the Merton problem of optimal investment and consumption with proportional transaction costs, formalising the model introduced by Magill and Constantinides [85]. It is optimal to trade in such a way that the fraction of wealth invested in the risky asset is within an interval around the constant fraction that is optimal in the frictionless case (without transaction costs). This is followed up on by Shreve and Soner [104], who study the regularity of the value function using the theory of viscosity solutions. Irle and Sass [60] transform the problem of maximising asymptotic growth rate under fixed and proportional transaction costs to an impulse control problem of a $(0,1)$-valued diffusion, which represents the fraction of wealth invested in the risky asset. The corresponding singular control problem of this $(0,1)$-valued diffusion was then studied by Christensen et al. [24], by letting the fixed costs vanish. Martin [87] considers this problem where the drift and volatility of the traded risky asset are driven by stochastic factors. More recently, an extension of the Merton problem with proportional transaction costs from power and logarithmic utility to Epstein-Zin stochastic differential utility was first studied by Melnyk, Muhle-Karbe and Seifried [88] for small transaction costs, and then followed up on by Herdegen, Hobson and Tse [55].

Apart from the above-mentioned portfolio selection problems, other optimal investment problems in the presence of proportional costs have also been studied. For example, in Løkka and Zervos [82], the liquid reserves of a company are decreased and increased in a "singular" manner by dividend payments and issuance of a new equity respectively, and the objective is to maximise the expected discounted dividend payments minus the expected discounted costs of issuing new equity. Merhi and Zervos [92], Løkka and Zervos ([83] and [84]) and Federico and Pham [33] also consider optimal adjustments of capacity levels in investment projects to maximise certain payoffs, subjected to proportional costs of adjustment. Koch and Vargiolu [74] consider a company which aims to optimally increase its electricity generation through the installation of solar panels, which in turn has a permanent impact on the spot electricity price. Federico, Ferrari and Rodosthenous [31] study a problem of optimal inventory management with unknown demand trend, which gives rise to a singular stochastic control problem with partial observation.

Singular control has also been applied in optimal harvesting problems in papers such as Alvarez and Hening [1], where the controlled finite variation process represents the cumulative harvested quantity and the controlled diffusion process is the population, and the objective is to maximise the expected and almost sure long-term harvested yield. The optimal harvesting strategy is to harvest in such a way that the population density is maintained below a certain optimal threshold. This generalises the result by Hening et al. [54], where the harvesting rate is bounded, which results in a bang-bang type optimal harvesting strategy. Hu, Øksendal and Sulem [58] also study a similar optimal harvesting problem in a mean-field setting, but in a finite time horizon setting and using a BSDE approach to establish a maximum principle.

Singular control problems have also been applied in macroeconomic problems. Ferrari [34] investigates optimal debt reduction policy by a government to minimise total expected cost of having a debt as well as the cost of interventions on the debt-to-GDP ratio. This is extended to an N-state regime switching economy by Ferrari and Rodosthenous [35], and to a setting with partial observation of economic growth by Callegaro, Ceci and Ferrari [18]. Ferrari and Vargiolu [36] study the problem of a central bank that buys and sells foreign currency in order to manage the exchange rate between the domestic and foreign currency.

Singular control problems have also been studied in mean field settings, such as Hu, Øksendal and Sulem [58], Fu and Horst [46] and Cao, Dianetti and Ferrari [19]. Fu and Horst [46] establishes existence of solutions to mean field games with singular controls (in a finite time horizon setting) through an approximation by solutions to mean field games with regular controls. Cao, Dianetti and Ferrari [19] study the infinite horizon problem, using the characterisations of the stationary distributions of reflected diffusions by their speed measure in determining mean field equilibria.

1.2 Risk-sensitive control problems

In this section, we define the notion of risk-sensitivity and risk-sensitive control problems, based on Chapter VI of Fleming and Soner [44], as well as give a brief overview of risk-sensitive control problems in the existing literature. Suppose that $\mathcal{J}$ is a random variable that represents costs that depend on the sample paths of a Markov process X (in the case of a control problem, the controlled Markov process X). However, as not all values of $\mathcal{J}$ may be equally significant, a nonlinear function F may be applied to $\mathcal{J}$ to account for this by, for example, giving greater weight to larger values of $\mathcal{J}$. The function F satisfies $F'(x), F''(x) \neq 0$, and we define the risk-sensitivity parameter r_F by

$$r_F(x) = \frac{|F''(x)|}{|F'(x)|},$$

where larger values of r_F indicate greater risk-sensitivity. F can also be viewed as a disutility function, with r_F being the coefficient of absolute risk aversion. Moreover, minimising $\mathbb{E}[F(\mathcal{J})]$ is equivalent to minimising the certainty equivalent expectation defined by

$$\mathcal{E}(\mathcal{J}) = F^{-1}\left(\mathbb{E}[F(\mathcal{J})]\right),$$

which is the value that gives the same disutility as the expected disutility of costs. For most of Chapter VI of Fleming and Soner [44] as well as the risk-sensitive ergodic singular control problem that we study, F is assumed to be of exponential form $F(\mathcal{J}) = \exp(\theta\mathcal{J})$, for some $\theta > 0$. For such a function F, the risk-sensitivity is constant and $r_F(x) = \theta$, and F is also known as a *constant absolute risk aversion* (CARA) disutility function. In this case,

$$\mathcal{E}(\mathcal{J}) = \frac{1}{\theta}\ln\Big(\mathbb{E}[\exp(\theta\mathcal{J})]\Big),$$

and a Taylor expansion about $\theta = 0$ of this certainty equivalent expectation reveals that

$$\mathcal{E}(\mathcal{J}) = \mathbb{E}(\mathcal{J}) + \frac{\theta}{2}\operatorname{Var}(\mathcal{J}) + O\left(\theta^2\right).$$

This indicates that for small values of θ, the certainty equivalent expectation is approximately the (risk-neutral) expectation with a penalty on the variance of costs proportionate to the risk-sensitivity. It can also be shown that this certainty equivalent expectation can be rewritten as an ordinary expectation via a change in probability measure.

Define the conditional expectations

$$\mathbb{E}_{tx}\big(\mathcal{J}\big) = \mathbb{E}\big(\mathcal{J}\big|X_t = x\big) \quad \text{and} \quad \mathcal{E}_{tx}\big(\mathcal{J}\big) = \mathcal{E}\big(\mathcal{J}\big|X_t = x\big),$$

where $\mathcal{J}$ is a random variable. Suppose that

$$\begin{aligned}
V(t,x) &:= \mathcal{E}_{tx}\left(\int_t^T h(s,X_s)\,\mathrm{d}s + \psi(X_T)\right) \\
&= \mathcal{E}\left(\int_t^T h(s,X_s)\,\mathrm{d}s + \psi(X_T)\middle| X_t = x\right) \\
&= \frac{1}{\theta}\ln\left(\mathbb{E}\left[\exp\left(\theta\left(\int_t^T h(s,X_s)\,\mathrm{d}s + \psi(X_T)\right)\right)\middle| X_t = x\right]\right) \\
&= \frac{1}{\theta}\ln\left(\mathbb{E}_{tx}\left[\exp\left(\theta\left(\int_t^T h(s,X_s)\,\mathrm{d}s + \psi(X_T)\right)\right)\right]\right) =: \frac{1}{\theta}\ln\big(\phi(t,x)\big),
\end{aligned}$$

where X is a Markov diffusion process. It can be shown that ϕ is the solution to a linear PDE using the Feynman-Kac formula. Moreover, the certainty equivalent expectation V is a logarithmic transformation of ϕ and satisfies a nonlinear PDE. When X is a regularly controlled Markov process, the HJB equation is characterised by this nonlinear PDE. In the risk-sensitive ergodic singular control problem that we study, we minimise the long-term certainty equivalent expectation

$$\begin{aligned}
J_x(\xi) &= \limsup_{T\uparrow\infty} \frac{1}{T}\,\mathcal{E}\left(\int_0^T h(X_t)\,\mathrm{d}t + K|\xi|_T\right) \\
&= \limsup_{T\uparrow\infty} \frac{1}{\theta T}\ln\mathbb{E}\left[\exp\left(\theta\left(\int_0^T h(X_t)\,\mathrm{d}t + K|\xi|_T\right)\right)\right],
\end{aligned}$$

where $|\xi|$ is the total variation of the process ξ and $K > 0$ is a proportionality constant associated with effort expenditure costs. This is an extension of the risk-neutral problem studied by Jack and Zervos [61]. To solve this problem, we use the same logarithmic transformation to convert a nonlinear first order ODE satisfied in the continuation region to a second order linear ODE.

Risk-sensitive problems have been studied in a variety of settings. In [41], [42] and [43], Fleming and Sheu consider optimal investment models in which long-term growth of expected HARA utility of wealth is maximised, and reformulate the problem as a risk-sensitive control problem with regular controls, the control variable being the fraction of wealth invested in the risky asset(s). Pham [99] shows that ergodic risk-sensitive regular stochastic control problems are dual problems to optimal investment problems in which the probability of beating a given index is maximised (a large deviation probability control problem), and this is used by Hata and Iida [52] to solve an optimal investment problem with partial information. Lepeltier [79] develops a martingale optimality principle to establish the existence of optimal (regular) controls in risk-sensitive stochastic control problems. Arapostathis and Biswas [4] study infinite horizon risk-sensitive problems (with regular controls) under minimal assumptions, by relating it with the eigenfunctions and eigenvalues of a multiplicative Poisson equation. Optimal stopping problems with risk-sensitive criteria have been studied by Jelito et al. [64] and [65], and risk-sensitive Dynkin games with heterogeneous Poisson random intervention times in by Liang and Sun [80]. Fleming and McEneaney ([40] and [37]) consider a totally risk-sensitive limit (with regular controls), in which the noise becomes smaller as the risk-sensitivity increases. In the limit, a deterministic differential game is obtained, where there is a convergence in the Hamiltonians.

Most of the risk-sensitive control problems in the literature are restricted to regular controls. Moreover, in most singular control problems, only the risk-neutral criterion is considered (such as those mentioned in Section 1.1). To the best of our knowledge, there are only two papers that have studied risk-sensitive singular stochastic control problems. The first is by Park [97] in his PhD thesis, where he establishes general existence and uniqueness results for singular control problems with risk-sensitive ergodic criteria in a multi-dimensional setting, but it is assumed that the volatility matrix is constant. In our paper, we restrict to the one-dimensional setting, but we consider state-dependent volatility and obtain explicit results. The second paper that studies risk-sensitive singular stochastic control problems that we know of is by Chala [20], who establishes a singular risk-sensitive stochastic maximum principle in a finite time horizon setting through a BSDE approach.

1.3 Equilibrium asset pricing with transaction costs

We give a brief introduction to the concept of equilibria of financial markets in continuous time in the case of complete markets, which is adapted from Chapter 7 of Dana and Jeanblanc [28]. Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ satisfying the usual conditions and supporting a standard d-dimensional $(\mathcal{F}_t)$-Brownian motion W. Suppose the financial market consist of a riskless asset and d risky assets. The dynamics of the price process of the riskless asset S^0 is given by

$$\mathrm{d}S^0(t) = r(t)S^0(t)\,\mathrm{d}t, \quad S^0(0) = 1,$$

where r is the interest rate and we denote the discount factor by $R(t) = \exp\big(-\int_0^t r(s)\mathrm{d}s\big)$. The price process of the d risky assets is denoted by $S = (S^1, \dots, S^d)$ and these assets pay cumulative dividends $D = (D^1, \dots, D^d)$. The price and dividends process are assumed to follow an Itô process. We define the discounted cumulative dividend process D^{d} by $\mathrm{d}D^{\mathrm{d}}(t) = R(t)\mathrm{d}D(t)$, $D^{\mathrm{d}}(0) = 0$ and the discounted gains process by $G^{\mathrm{d}} = RS + D^{\mathrm{d}}$. The market is arbitrage-free and there exists an equivalent martingale measure. The economy consist of a single consumption good (associated with consumption process c) and m agents, and each Agent i receives an endowment stream e^i and aims to maximise her utility given by

$$U_i(c) = \mathsf{E}_{\mathsf{P}}\left[\int_0^T u_i(t, c(t))\,\mathrm{d}t + \tilde{u}_i(T, c(T))\right],$$

for given functions u_i and $\tilde{u}_i$. Denote the portfolio of Agent i by $\overline{\vartheta}_i = (\vartheta_i^0, \vartheta_i)$ and $\overline{S} = (S^0, S)$. Admissible trading strategies and consumption policies are defined as follows:

Definition 1.1. *Given the discounted gains process of the assets, the pair $(\overline{\vartheta}_i, c_i)$ is admissible for Agent i if it satisfies*

$$R(t)\big(\overline{\vartheta}_i(t)\cdot\overline{S}_t\big) = \int_0^t \vartheta_i(s)\,\mathrm{d}G^d(s) - \int_0^t R(s)\big(c_i(s)-e_i(s)\big)\mathrm{d}s$$
$$\mathsf{Q}-a.s.\ \text{for all } t\in[0,T],$$

where Q is an equivalent martingale measure, and

$$\overline{\vartheta}_i(T)\cdot\overline{S}_T = \vartheta_i^0(T)S^0(T) + \vartheta_i(T)\cdot S_T \geq 0 \quad a.s..$$

In other words, the discounted wealth is the the sum of discounted gains or losses from trades of the risky asset and consumption, and there is no debt remaining at the end of the period. The pair $(\overline{\vartheta}, c)$ is optimal for Agent i if it is admissible and maximises the utility function U_i over the set of admissible strategies. We can now define a Radner equilibrium.

Definition 1.2. *For a given dividend process D, $(\overline{\vartheta}_i, c_i, \overline{S})$ is a Radner equilibrium if:*

1. *the pair $(\overline{\vartheta}_i, c_i)$ is optimal for all $i = 1,\dots,m$,*
2. *the stock and consumption goods markets clear:*
   (a) $\sum_{i=1}^m \overline{\vartheta}_i = 0 \quad \mathsf{P}\otimes\mathrm{d}t\text{–}a.s..$
   (b) $\sum_{i=1}^m c_i = \sum_{i=1}^m e_i + \sum_{j=1}^d D^j \quad \mathsf{P}\otimes\mathrm{d}t\text{–}a.s..$

The pair $(\overline{\vartheta}_i, c_i)$ that is optimal and satisfies the market clearing conditions determines the equilibrium price $\overline{S}$. Further technical details can be found in Chapter 7 of Dana and Jeanblanc [28]. In an equilibrium model with transaction costs, the discounted payments due to transaction costs are deducted from the discounted wealth.

In the model that we consider, we make some simplifying assumptions for tractability. Firstly, the agents do not have a consumption policy, and the utility function is a function of the wealth of the agents instead. The market clearing condition then reduces to the clearing of the stock market only. Secondly, we assume that there is only one risky asset, which does not pay dividends. Finally, as there is no market clearing condition for consumption goods, we can assume without loss of generality (and for notational simplicity) that the interest rate $r = 0$ and the price of the riskless asset is constant and equal to 1.

The dependence of liquidity on asset prices has been a topic of interest in finance and economics, and has been empirically studied in papers such as Amihud and Mendelson [2], Brennan and Subrahmanyam [13] and Pastor and Stambaugh [98]. The effects of illiquidity due to the presence of transaction costs on optimal trading strategies, asset returns, volatility and interest rates have been studied extensively. Different types of transaction costs have different impacts on trading behaviour. Costs on the trading rate leads to sluggishness of trading, fixed costs lead to infrequency of trades and proportional costs cause a reduction in trading volume and frequency, and models aim to quantify such effects through deriving theoretical results or conducting numerical studies.

The impact of transaction costs on asset returns, volatility and interest rates are determined by endogenising these variables in equilibrium models. Equilibrium prices are determined through the stock market clearing condition under which the total demand of each traded asset is equal to its supply and equilibrium interest rates are determined by the consumption goods market clearing condition. We highlight a few papers that have adopted a more mathematical approach. Bouchard et al. [12] develop an equilibrium model where mean-variance investors subjected to quadratic costs on their trading rates, assuming constant exogenous volatility, and obtain a unique equilibrium that is the unique solution to a system of coupled but linear forward-backward stochastic differential equations (FBSDEs) for the optimal trading strategies. The sluggishness of the frictional portfolios induced by the quadratic costs results in mean-reverting equilibrium returns. Herdegen et al. [57] extend this model to one that endogenizes volatility by matching an exogenous terminal condition for the risky asset. This results in a system of nonlinear fully coupled FBSDEs, to which a unique solution exists if the risk aversion of the two agents are sufficiently similar. Gonon et al. [51] determine equilibrium return rates under general convex costs on trading rates as well as the limiting case of proportional costs. In the case of exogenous volatility, equilibrium return rates are given explicitly by solutions to an ODE. The BSDE approach is also used by Weston in [116], where equilibria in which some agents have more access to the financial markets than others is studied. Equilibria with proportional costs where agents are incentivized to trade towards a targeted number of shares have been studied in Noh and Weston [96] and Choi, Duraj and Weston [23]. Other equilibrium models without transaction costs have also been studied in papers such as [119], [73], [22], [117] and [116].

Equilibrium models are notoriously intractable, and simplifying assumptions are often made to improve tractability. For example, Lo et al. [81] assume that the equilibrium price is a constant (with zero market volatility) in their study of an equilibrium with fixed transaction costs. Vayanos and Vila [110] also assume zero market volatility, whereby two riskless assets (one being liquid and the other carrying proportional transaction costs) are traded in the economy, and determine interest rates endogenously. Similarly, Weston [115] studies an equilibrium with proportional transaction costs where interest rates are determined endogenously, assuming that there is zero market volatility, where the asset being traded is a deterministic annuity. The case of a stochastic annuity is studied by Weston and Zitkovic [117], but the model does not include transaction costs. Intractable models are also studied numerically, for example, in the case of endogenous volatility in [51], the system of nonlinear FBSDEs is solved numerically using a simulation based deep-learning approach. Other papers where numerical methods are adopted include Heaton and Lucas [53], Buss and Dumas [14] and Buss et al. [15]. In this paper, we assume for tractability that the volatility and interest rate are exogenous (as is done in Bouchard et al. [12], Gonon et al. [51], Vayanos [109], Sannikov and Skrzypacz [102] and Zitkovic [119]), in order to determine expected returns endogenously. For tractability, we assume that there are two agents in the economy having tractable mean-variance preferences who trade a safe and a risky asset.

It has been established that for small proportional transaction costs, the no-trade region has a width proportional to $\lambda^{1/3}$, where λ is the proportional transaction cost parameter. These small cost asymptotics were first observed by Constantinides [27] and then calculated by Rogers [101] for the classical Davis-Norman problem for optimal investment and consumption, and was further developed by Schachermayer [103] in the context of shadow price processes. Such asymptotics have been studied in both equilibrium and non-equilibrium settings with different types of transaction costs, examples of papers include Shreve and Soner [104], Kallsen and Muhle-Karbe ([66] and [67]), Herdegen and Muhle-Karbe [56], Choi and Larsen [21], Martin [87], Gerhold et al. ([49] and [50]) and Muhle-Karbe, Shi and Yang [94].

We develop an equilibrium model with proportional transaction costs that generalises the setting of Gonon et al. [51]. We adopt a similar but more general singular control approach, and solve a novel free-boundary problem. We also derive small transaction cost asymptotics.

2

The explicit solution to a risk-sensitive ergodic singular stochastic control problem

We consider a stochastic dynamical system whose state process satisfies the SDE

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \mathrm{d}\xi_t + \sigma(X_t)\,\mathrm{d}W_t, \quad X_0 = x \in \mathbb{R}, \tag{2.1}$$

where W is a standard one-dimensional Brownian motion and ξ is a controlled finite-variation process. Given a positive function h, $K > 0$ a proportionality constant associated with effort expenditure costs, and $\theta > 0$ the risk-sensitivity parameter, we associate with each controlled process ξ the risk-sensitive long-term average performance index

$$J_x(\xi) = \limsup_{T\uparrow\infty} \frac{1}{\theta T} \ln \mathsf{E}\left[\exp\left(\theta\left(\int_0^T h(X_t)\,\mathrm{d}t + K|\xi|_T\right)\right)\right], \tag{2.2}$$

where $|\xi|$ denotes the total variation process of ξ. The objective of the resulting ergodic risk-sensitive singular stochastic control problem is to minimise (2.2) over all admissible controlled processes ξ.

This stochastic control problem has been partly motivated by the problem faced by a central bank that wishes to control the exchange rate between its domestic currency and a foreign currency so that this fluctuates within a suitable target zone. In this context, the state process X models the log exchange rate's stochastic dynamics, while the controlled process ξ models the cumulative effect of the bank's interventions in the FX market to buy or sell the foreign currency. Furthermore, the running cost function h penalises deviations of the log exchange rate from a desired nominal value, while K models proportional transaction costs resulting from the bank's interventions.

Similar models, which endogenise an exchange rate's target zone by formulating its management as a stochastic control problem, have been studied by Jeanblanc- Picqué [62], Mundaca and Øksendal [95], Cadenillas and Zapatero [16], [17], Ferrari and Vargiolu [36], and references therein. The stochastic control problems solved in these references involve expected discounted performance criteria. Discounting is commonly used to estimate the present value of an asset or to model an economic agent's impatience. Since an exchange rate is not an asset and a central bank can be viewed as an institution as well as a regulator, a long-term average criterion may be more appropriate for this kind of applications.

Singular stochastic control problems have been motivated by several applications in areas including target tracking, optimal harvesting, optimal investment in the presence of proportional transaction costs and others. Singular stochastic control problems with risk-neutral ergodic criteria have been studied by Karatzas [68], Menaldi and Robin [89, 90], Taksar, Klass and Assaf [107], Menaldi, Robin and Taksar [91], Weerasinghe [112, 114], Jack and Zervos [61], Løkka and Zervos [83, 84], Hynd [59], Wu and Chen [118], Hening, Nguyen, Ungureanu and Wong [54], Alvarez and Hening [1], Kunwai, Xi, Yin and Zhu [77], listed in rough chronological order, and several references therein. On the other hand, Park [97, Chapter I] and Chala [20] study singular stochastic control problems with finite time horizon risk-sensitive criteria. In the context of this paper, Park [97, Chapter II] studies a risk-sensitive singular stochastic control problem with an ergodic criterion in $\mathbb{R}^n$, but with constant σ. In this reference, the existence of a suitable solution to the problem's HJB equation is established and a limiting connection with the solution to a certain deterministic ergodic differential game is established. This chapter extends the PhD thesis of Park (1996) [97, Chapter II] in the one-dimensional case by obtaining explicit solutions for more general σ. For other ergodic risk-sensitive control problems, see the recent review paper by Biswas and Borkar [8].

In this chapter, we derive the complete solution to the problem that we consider. In particular, we derive a C^2 solution w to the problem's HJB equation that determines the optimal strategy, which reflects the state process in the endpoints of an interval $[\alpha_\star, \beta_\star]$. To this end, we first use a suitable logarithmic transformation that gives rise to a family of Sturm-Liouville eigenvalue problems parametrised by their boundary points $\alpha < \beta$. We then use the optimality conditions suggested by the so-called smooth-fit of singular stochastic control, namely, the C^2 continuity of w, to derive the optimal free-boundary points $\alpha_\star < \beta_\star$. Furthermore, we show that the control problem's optimal growth rate identifies with the maximal eigenvalue of the corresponding Sturm-Liouville problem.

2.1 Problem formulation

Fix a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathsf{P}\big)$ satisfying the usual conditions and supporting a standard one-dimensional $(\mathcal{F}_t)$-Brownian motion W. We consider a dynamical system, the uncontrolled stochastic dynamics of which are modelled by the SDE

$$\mathrm{d}\overline{X}_t = b(\overline{X}_t)\,\mathrm{d}t + \sigma(\overline{X}_t)\,\mathrm{d}W_t, \quad \overline{X}_0 = x \in \mathbb{R}. \tag{2.3}$$

We make the following assumption, which also ensures that (2.3) has a unique strong solution up to a possible explosion time.

Assumption 2.1. The functions $b, \sigma : \mathbb{R} \to \mathbb{R}$ are C^1 and there exists constants $C_1 > 0$ and $\zeta \geq 1$ such that

$$0 < \sigma^2(x) \leq C_1\big(1 + |x|^\zeta\big) \quad \text{for all } x \in \mathbb{R}. \tag{2.4}$$

The growth condition (2.4) ensures that certain stochastic integrals and stochastic exponentials are true martingales, which we require to perform a measure change in the verification theorem. We require that b and σ are C^1 because solving the free-boundary problem will involve differentiating these functions. We next consider the stochastic control problem defined by (2.1)–(2.2). Given a finite variation $(\mathcal{F}_t)$-adapted process ξ with càglàd sample paths such that the SDE (2.1) has a unique non-explosive strong solution, we denote by $\mathcal{M}^\xi$ the family of all local martingales $M^{\xi,g}$ defined by

$$M_T^{\xi,g} = \int_0^T \sigma(X_t) g(X_t) \, \mathrm{d}W_t, \tag{2.5}$$

which is parametrised by all continuous functions $g : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\downarrow-\infty} g(x) = -K \quad \text{and} \quad \lim_{x\uparrow\infty} g(x) = K. \tag{2.6}$$

Furthermore, we define the stochastic exponential

$$\mathcal{E}_T\big(\theta M^{\xi,g}\big) = \exp\bigg(-\frac{1}{2}\theta^2 \big\langle M^{\xi,g}\big\rangle_T + \theta M_T^{\xi,g}\bigg). \tag{2.7}$$

The requirement for $\mathcal{E}\big(\theta M^{\xi,g}\big)$ to be a martingale motivates an integrability condition (so that Novikov's condition is satisfied) in the following definition of admissible controls.

Definition 2.2. The family of all admissible control strategies $\mathcal{A}$ is the set of all finite variation $(\mathcal{F}_t)$-adapted process ξ with càglàd sample paths such that $\xi_0 = 0$ and the SDE (2.1) has a unique non-explosive strong solution satisfying

$$\mathsf{E}\left[\exp\bigg(C \int_0^T |X_t|^\zeta \, \mathrm{d}t\bigg)\right] < \infty \quad \text{for all } C, T > 0, \tag{2.8}$$

where $\zeta \geq 1$ is as in Assumption 2.1, as well as

$$\limsup_{T\uparrow\infty} \frac{1}{\theta T} \ln \mathsf{E}^{\widetilde{\mathsf{P}}_T}\Big[\exp\big(-\theta K |X_T|\big)\Big] = 0 \quad \text{for all } \widetilde{\mathsf{P}}_T \in \mathcal{P}_T^\xi, \tag{2.9}$$

where $\mathcal{P}_T^\xi$ is the family of all probability measures $\widetilde{\mathsf{P}}_T$ on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym derivative with respect to P given by $\mathrm{d}\widetilde{\mathsf{P}}_T/\mathrm{d}\mathsf{P} = \mathcal{E}_T\big(\theta M^{\xi,g}\big)$, for $M^{\xi,g} \in \mathcal{M}^\xi$.

Remark 2.3. In view of (2.4), (2.6), (2.8) and Jensen's inequality, we can see that

$$\begin{aligned}\mathsf{E}\Big[\big\langle M^{\xi,g}\big\rangle_T\Big] &= \mathsf{E}\bigg[\int_0^T \sigma^2(X_t)g^2(X_t)\,\mathrm{d}t\bigg] \\ &\le C_1 \sup_{x\in\mathbb{R}} g^2(x)\left(1+\mathsf{E}\bigg[\int_0^T |X_t|^\zeta\,\mathrm{d}t\bigg]\right) < \infty.\end{aligned}$$

Therefore, the process $M^{\xi,g}$ is a square-integrable martingale for all $\xi \in \mathcal{A}$ and all continuous g satisfying (2.6). Furthermore, the observation that

$$\mathsf{E}\left[\exp\left(\frac{1}{2}\big\langle M^{\xi,g}\big\rangle_T\right)\right] \le \mathsf{E}\left[\exp\left(\frac{1}{2}C_1\sup_{x\in\mathbb{R}} g^2(x)\left(1+\int_0^T |X_t|^\zeta\,\mathrm{d}t\right)\right)\right] < \infty$$

implies that $\mathcal{E}\big(\theta M^{\xi,g}\big)$ is a martingale because it satisfies Novikov's condition.

Remark 2.4. Given points $\alpha < \beta$ in $\mathbb{R}$, let $\xi^{\alpha,\beta}$ be the controlled process that, beyond an initial jump at time 0 of size $\Delta\xi_0^{\alpha,\beta} = (x-\beta)^+ \vee (\alpha - x)^+$, reflects the corresponding state process $X^{\alpha,\beta}$ at α in the positive direction and at β in the negative direction. Such a controlled process indeed exists (e.g., see Tanaka [108, Theorem 4.1]). The process $X^{\alpha,\beta}$ satisfies the integrability condition (2.8) as well as the transversality condition (2.9) because $X_t^{\alpha,\beta} \in [\alpha,\beta]$ for all $t > 0$. Therefore, $\xi^{\alpha,\beta} \in \mathcal{A}$. Moreover, due to the fact that $X^{\alpha,\beta}$ takes values in a compact set, we observe that the processes $M^{\xi^{\alpha,\beta},g}$ and $\mathcal{E}\big(\theta M^{\xi^{\alpha,\beta},g}\big)$ are martingales for any b and σ such that (2.3) has a unique strong solution up to a possible explosion time. Therefore, if we restrict ourselves to controlled processes $\xi^{\alpha,\beta}$, the growth condition (2.4) on σ may be relaxed, and we may assume instead that the C^1 functions b and σ satisfy the conditions of the Yamada-Watanabe pathwise uniqueness theorem.

Assumption 2.5. $K > 0$. The function h is C^1 and positive. Furthermore, if we define

$$H_\pm(x) = \frac{1}{2}\theta K^2\sigma^2(x) \pm Kb(x) + h(x), \quad \text{for } x \in \mathbb{R}, \tag{2.10}$$

then

$$\lim_{x \downarrow -\infty} H_-(x) = \lim_{x \uparrow \infty} H_+(x) = \infty \tag{2.11}$$

and there exist constants $\alpha_- \leq \alpha_+$ such that

$$\text{the function } H_- \text{ is strictly } \begin{cases} \text{decreasing and positive in }]-\infty, \alpha_-[, \\ \text{negative in }]\alpha_-, \alpha_+[, \text{ if } \alpha_- < \alpha_+, \\ \text{increasing and positive in }]\alpha_+, \infty[, \end{cases} \tag{2.12}$$

as well as constants $\beta_- \leq \beta_+$ such that

$$\text{the function } H_+ \text{ is strictly } \begin{cases} \text{decreasing and positive in }]-\infty, \beta_-[, \\ \text{negative in }]\beta_-, \beta_+[, \text{ if } \beta_- < \beta_+, \\ \text{increasing and positive in }]\beta_+, \infty[. \end{cases} \tag{2.13}$$

Remark 2.6. If $h = 0$, then $\xi = 0$ clearly minimises the objective functional (2.2), and we therefore only consider non-trivial functions h.

Example 2.7. Suppose that $\overline{X} = x + \sigma W$ for some constant $\sigma > 0$. Also, let $h(x) = cx^2$ for some constant $c > 0$. In this context, the functions $H_\pm$ defined by (2.10) are given by

$$H_+(x) = H_-(x) = cx^2 + \frac{1}{2}\theta\sigma^2 K^2$$

and the conditions required by Assumptions 2.1 and 2.5 are satisfied with $\alpha_- = \alpha_+ = \beta_- = \beta_+ = 0$.

Example 2.8. Suppose that $\overline{X}$ is an Ornstein-Uhlenbeck process with the dynamics

$$d\overline{X}_t = \gamma(\mu - \overline{X}_t)\,dt + \sigma\,dW_t, \quad \overline{X}_0 = x \in \mathbb{R},$$

for some constants $\gamma, \sigma > 0$ and $\mu \in \mathbb{R}$. Also, let $h(x) = cx^2$ for some constant

$c > 0$. In this context, the functions $H_\pm$ defined by (2.10) admit the expressions

$$H_+(x) = c\left(x - \frac{\gamma K}{2c}\right)^2 + \frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 + \gamma\mu K$$
$$\text{and} \quad H_-(x) = c\left(x + \frac{\gamma K}{2c}\right)^2 + \frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 - \gamma\mu K.$$

If $\frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 - \gamma\mu K \geq 0$, then $\alpha_- = \alpha_+ = -\frac{1}{2c}\gamma K$, otherwise

$$\alpha_\pm = -\frac{\gamma K}{2c} \pm \sqrt{-\frac{1}{c}\left(\frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 - \gamma\mu K\right)}.$$

Similarly, if $\frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 + \gamma\mu K \geq 0$, then $\beta_- = \beta_+ = \frac{1}{2c}\gamma K$, otherwise,

$$\beta_\pm = -\frac{\gamma K}{2c} \pm \sqrt{-\frac{1}{c}\left(\frac{1}{2}\theta\sigma^2 K^2 - \frac{1}{4c}\gamma^2 K^2 + \gamma\mu K\right)}.$$

In all cases, the conditions required by Assumptions 2.1 and 2.5 are all satisfied.

2.2 The control problem's HJB equation and its associated Sturm-Liouville eigenvalue problem

We will solve the control problem that we consider by constructing a C^2 function w and finding a constant λ such that the HJB equation

$$\min\left\{\frac{1}{2}\sigma^2(x)w''(x) + \frac{1}{2}\theta\big(\sigma(x)w'(x)\big)^2 + b(x)w'(x) + h(x) - \lambda,\ K - \big|w'(x)\big|\right\} = 0 \tag{2.14}$$

holds true for all $x \in \mathbb{R}$. Given such a solution (w, λ) to this HJB equation,

$$\inf_{\xi \in \mathcal{A}} J_x(\xi) = \lambda \quad \text{for all } x \in \mathbb{R},$$

where J_x is defined by (2.2). Furthermore, an optimal strategy can be characterised as follows. The controller should wait and take no action for as long as the state process X takes value in the set in which $\left|w'(x)\right| < K$. Otherwise, the controller should take minimal action to keep the state process X outside the interior of the set in which $\left|w'(x)\right| = K$ at all times.

We will prove that the optimal control strategy is characterised by two points $\alpha < \beta$ and takes the following form. If the initial state x is strictly greater than β (resp., strictly less than α), then it is optimal to push the state process in an impulsive way down to level β (resp., up to level α). Beyond such a possible initial jump, it is optimal to take minimal action to keep the state process X inside the set $[\alpha, \beta]$ at all times, which amounts to reflecting X in β in the negative direction and in α in the positive direction. In view of the discussion in the previous paragraph, the optimality of such a strategy is associated with a solution (w, λ) to the HJB equation (2.14) such that

$$w'(x) = -K, \quad \text{for } x \in]-\infty, \alpha], \tag{2.15}$$

$$\frac{1}{2}\sigma^2(x)w''(x) + \frac{1}{2}\theta\big(\sigma(x)w'(x)\big)^2 + b(x)w'(x) + h(x) - \lambda = 0, \quad \text{for } x \in]\alpha, \beta[, \tag{2.16}$$

$$\text{and} \quad w'(x) = K, \quad \text{for } x \in [\beta, \infty[, \tag{2.17}$$

To determine the points $\alpha < \beta$, we consider the so-called "smooth pasting" condition of singular stochastic control, which suggests that w should be C^2, in particular, at the free-boundary points α and β. This condition gives rise to the equations

$$\lim_{x \downarrow \alpha} w'(x) = -K, \quad \lim_{x \downarrow \alpha} w''(x) = 0, \tag{2.18}$$

$$\lim_{x \uparrow \beta} w'(x) = K \quad \text{and} \quad \lim_{x \uparrow \beta} w''(x) = 0. \tag{2.19}$$

In view of (2.16), these free-boundary equations can be satisfied if and only if

$$H_-(\alpha) \equiv \frac{1}{2}\theta K^2\sigma^2(\alpha) - Kb(\alpha) + h(\alpha) = \lambda \tag{2.20}$$

$$\text{and} \quad H_+(\beta) \equiv \frac{1}{2}\theta K^2\sigma^2(\beta) + Kb(\beta) + h(\beta) = \lambda. \tag{2.21}$$

The ODE (2.16) is a Riccati equation. If we write

$$w'(x) = \frac{u'(x)}{\theta u(x)}, \quad \text{for } x \in \,]\alpha, \beta[, \tag{2.22}$$

then w is a solution to the ODE (2.16) only if u is a solution to the second order linear ODE

$$\frac{1}{2}\sigma^2(x)u''(x) + b(x)u'(x) + \theta\big(h(x) - \lambda\big)u(x) = 0,$$

which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(q(x)u'(x)\big) + \frac{2\theta}{\sigma^2(x)}\big(h(x) - \lambda\big)q(x)u(x) = 0, \tag{2.23}$$

where

$$q(x) = \exp\left(\int_0^x \frac{2b(y)}{\sigma^2(y)}\,\mathrm{d}y\right). \tag{2.24}$$

In view of this transformation and the boundary conditions (2.18) and (2.19), we are faced with the regular Sturm-Liouville eigenvalue problem defined by the ODE (2.23) with boundary conditions

$$\theta K u(\alpha) + u'(\alpha) = 0 \quad \text{and} \quad \theta K u(\beta) - u'(\beta) = 0. \tag{2.25}$$

This problem has infinitely many simple real eigenvalues

$$\lambda_0 > \lambda_1 > \cdots > \lambda_n > \cdots \quad \text{such that } \lim_{n\uparrow\infty} \lambda_n = -\infty$$

and no other eigenvalues, while the eigenfunction $\mathfrak{u}_n$ corresponding to λ_n has exactly n zeros in the interval $]\alpha, \beta[$ (e.g., see Walter [111, Theorem VI.27.II]). Furthermore, the eigenvalues are related to their corresponding eigenfunctions by means of the Rayleigh quotient

$$\begin{aligned}\lambda_n = \Bigg(& q(\beta)\mathfrak{u}_n(\beta)\mathfrak{u}_n'(\beta) - q(\alpha)\mathfrak{u}_n(\alpha)\mathfrak{u}_n'(\alpha) \\ & + \int_\alpha^\beta q(y)\left(\frac{2\theta h(y)}{\sigma^2(y)}\mathfrak{u}_n^2(y) - \big(\mathfrak{u}_n'(y)\big)^2\right) \mathrm{d}y\Bigg) \\ & \times \left(\int_\alpha^\beta \frac{2\theta}{\sigma^2(y)} q(y)\mathfrak{u}_n^2(y)\,\mathrm{d}y\right)^{-1}. \end{aligned} \tag{2.26}$$

The eigenfunction $\mathfrak{u}_0$ is the only one that has no zeros in $]\alpha, \beta[$. The function w' given by (2.22) is therefore clearly well-defined only for $u = \mathfrak{u}_0$. In other words, if we write

$$w'(x) = \frac{\mathfrak{u}_0'(x)}{\theta \mathfrak{u}_0(x)}, \quad \text{for } x \in \,]\alpha, \beta[,$$

then w is a solution to the ODE (2.16) if and only if $\mathfrak{u}_0$ is the solution to the ODE (2.23) corresponding to the maximal eigenvalue λ_0. In view of this observation, we consider the maximal eigenvalue λ_0 and its corresponding eigenfunction $\mathfrak{u}_0$ in what follows. We also write $\lambda(\alpha, \beta)$ and $\phi_{\alpha,\beta}$ instead of λ_0 and $\mathfrak{u}_0$ to stress their dependence on the free-boundary points α and β. Furthermore, we assume that $\phi_{\alpha,\beta}$ has been normalised by a multiplicative constant, so that

$$\int_\alpha^\beta \frac{2\theta}{\sigma^2(y)} q(y)\phi_{\alpha,\beta}^2(y)\,\mathrm{d}y = 1, \tag{2.27}$$

and we note that the boundary conditions (2.25) and the expression (2.26) imply that

$$\begin{aligned}\lambda(\alpha, \beta) = \theta K & \Big(q(\alpha)\phi_{\alpha,\beta}^2(\alpha) + q(\beta)\phi_{\alpha,\beta}^2(\beta)\Big) \\ & + \int_\alpha^\beta q(y)\left(\frac{2\theta h(y)}{\sigma^2(y)}\phi_{\alpha,\beta}^2(y) - \big(\phi_{\alpha,\beta}'(y)\big)^2\right)\mathrm{d}y. \end{aligned} \tag{2.28}$$

Lemma 2.9. *The function λ defined by (2.28) for $\alpha < \beta$ is $C^{1,1}$ and such that*

$$\lambda_\alpha(\alpha, \beta) = \frac{2\theta q(\alpha)\phi_{\alpha,\beta}^2(\alpha)}{\sigma^2(\alpha)}\big(\lambda(\alpha, \beta) - H_-(\alpha)\big) \tag{2.29}$$

$$\textit{and} \quad \lambda_\beta(\alpha, \beta) = -\frac{2\theta q(\beta)\phi_{\alpha,\beta}^2(\beta)}{\sigma^2(\beta)}\big(\lambda(\alpha, \beta) - H_+(\beta)\big), \tag{2.30}$$

where the functions $H_\pm$ are defined by (2.10). Furthermore, given any points $\alpha < \beta$ in $\mathbb{R}$,

$$\lambda(\alpha, \beta) = J_x\big(\xi^{\alpha,\beta}\big) > 0 \quad \text{for all } x \in \mathbb{R}, \tag{2.31}$$

where J_x is defined by (2.2) and $\xi^{\alpha,\beta} \in \mathcal{A}$ is the controlled process discussed in Remark 2.4.

Proof. We first prove (2.29) and (2.30) using a technique inspired by Kong and Zettl [75]. Given any $\varepsilon > 0$, we use integration by parts and the ODE (2.23) to calculate

$$\begin{aligned}
q(\beta)&\Big(\phi_{\alpha,\beta}(\beta)\phi'_{\alpha+\varepsilon,\beta}(\beta) - \phi'_{\alpha,\beta}(\beta)\phi_{\alpha+\varepsilon,\beta}(\beta)\Big) \\
&\quad - q(\alpha+\varepsilon)\Big(\phi_{\alpha,\beta}(\alpha+\varepsilon)\phi'_{\alpha+\varepsilon,\beta}(\alpha+\varepsilon) - \phi'_{\alpha,\beta}(\alpha+\varepsilon)\phi_{\alpha+\varepsilon,\beta}(\alpha+\varepsilon)\Big) \\
&= \int_{\alpha+\varepsilon}^{\beta} \Big(\phi_{\alpha,\beta}(y)\big(q\phi'_{\alpha+\varepsilon,\beta}\big)'(y) - \phi_{\alpha+\varepsilon,\beta}(y)\big(q\phi'_{\alpha,\beta}\big)'(y)\Big)\,\mathrm{d}y \\
&= \big(\lambda(\alpha+\varepsilon,\beta) - \lambda(\alpha,\beta)\big)\int_{\alpha+\varepsilon}^{\beta} \frac{2\theta}{\sigma^2(y)} q(y)\phi_{\alpha,\beta}(y)\phi_{\alpha+\varepsilon,\beta}(y)\,\mathrm{d}y.
\end{aligned}$$

In view of the boundary conditions (2.25), these identities imply that

$$\begin{aligned}
\big(\lambda(\alpha+\varepsilon,\beta) - \lambda(\alpha,\beta)\big)&\int_{\alpha+\varepsilon}^{\beta} \frac{2\theta}{\sigma^2(y)} q(y)\phi_{\alpha,\beta}(y)\phi_{\alpha+\varepsilon,\beta}(y)\,\mathrm{d}y \\
&= q(\alpha+\varepsilon)\Big(\theta K\phi_{\alpha,\beta}(\alpha+\varepsilon) + \phi'_{\alpha,\beta}(\alpha+\varepsilon)\Big)\phi_{\alpha+\varepsilon,\beta}(\alpha+\varepsilon).
\end{aligned}$$

Using integration by parts, the ODE (2.23) and the boundary conditions (2.25) once more, we obtain

$$\begin{aligned}
q(\alpha+\varepsilon)&\Big(\theta K\phi_{\alpha,\beta}(\alpha+\varepsilon) + \phi'_{\alpha,\beta}(\alpha+\varepsilon)\Big) \\
&= \int_{\alpha}^{\alpha+\varepsilon} \Big(\theta K\big(q\phi_{\alpha,\beta}\big)'(y) + \big(q\phi'_{\alpha,\beta}\big)'(y)\Big)\,\mathrm{d}y \\
&= \int_{\alpha}^{\alpha+\varepsilon} \frac{2\theta q(y)\phi_{\alpha,\beta}(y)}{\sigma^2(y)}\left(\lambda(\alpha,\beta) + \frac{1}{2}K\sigma^2(y)\frac{\phi'_{\alpha,\beta}(y)}{\phi_{\alpha,\beta}(y)} + Kb(y) - h(y)\right)\mathrm{d}y.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\big(\lambda(\alpha+\varepsilon,\beta)-\lambda(\alpha,\beta)\big)\int_{\alpha+\varepsilon}^{\beta}\frac{2\theta}{\sigma^2(y)}q(y)\phi_{\alpha,\beta}(y)\phi_{\alpha+\varepsilon,\beta}(y)\,\mathrm{d}y\\
&=\phi_{\alpha+\varepsilon,\beta}(\alpha+\varepsilon)\\
&\quad\cdot\int_{\alpha}^{\alpha+\varepsilon}\frac{2\theta q(y)\phi_{\alpha,\beta}(y)}{\sigma^2(y)}\left(\lambda(\alpha,\beta)+\frac{1}{2}K\sigma^2(y)\frac{\phi'_{\alpha,\beta}(y)}{\phi_{\alpha,\beta}(y)}+Kb(y)-h(y)\right)\mathrm{d}y.
\end{aligned}$$

Dividing by ε and passing to the limit as $\varepsilon\downarrow 0$ using (2.25) as well as (2.27), we can see that the right-hand derivative $\lambda_{\alpha+}(\alpha,\beta)$ exists and is equal to the expression on the right-hand side of (2.29).

Replacing α and $\alpha+\varepsilon$ by $\alpha-\varepsilon$ and α, respectively, in the analysis above, we can see that the left-hand derivative $\lambda_{\alpha-}(\alpha,\beta)$ also exists and is equal to $\lambda_{\alpha+}(\alpha,\beta)$.

The proof of (2.30) follows the same arguments.

To establish (2.31), we first consider a C^1 function $w:\mathbb{R}\to\mathbb{R}$ that is piece-wise C^2 and an admissible control strategy $\xi\in\mathcal{A}$. Using Itô-Tanaka's formula for general semimartingales and the identity $\Delta X_t\equiv X_{t+}-X_t=\Delta\xi_t$, we obtain

$$\begin{aligned}
w(X_{T+})&=w(x)+\int_0^T\left(\frac{1}{2}\sigma^2(X_t)w''(X_t)+b(X_t)w'(X_t)\right)\mathrm{d}t+\int_{[0,T]}w'(X_t)\,\mathrm{d}\xi_t\\
&\qquad+\sum_{0\le t\le T}\big(w(X_{t+})-w(X_t)-w'(X_t)\Delta X_t\big)+M_T\\
&=w(x)+\int_0^T\left(\frac{1}{2}\sigma^2(X_t)w''(X_t)+b(X_t)w'(X_t)\right)\mathrm{d}t+\int_0^T w'(X_t)\,\mathrm{d}\xi_t^{\mathrm{c}+}\\
&\qquad-\int_0^T w'(X_t)\,\mathrm{d}\xi_t^{\mathrm{c}-}+\sum_{0\le t\le T}\big(w(X_{t+})-w(X_t)\big)+M_T,
\end{aligned}$$

where M is defined by (2.5) for $g=w'$, while $\xi^{\mathrm{c}+}$, $\xi^{\mathrm{c}-}$ are the continuous parts of the increasing processes ξ^+, ξ^- providing the unique decomposition $\xi=\xi^+-\xi^-$ and $|\xi|=\xi^++\xi^-$. In view of the identity

$$w(X_{t+})-w(X_t)=\int_0^{\Delta\xi_t^+}w'(X_t+r)\,\mathrm{d}r-\int_0^{\Delta\xi_t^-}w'(X_t-r)\,\mathrm{d}r,$$

we can see that

$$\sum_{0\le t\le T}\big(w(X_{t+})-w(X_t)+K\Delta|\xi|_t\big)$$
$$=\sum_{0\le t\le T}\int_0^{\Delta\xi_t^+}\big(K+w'(X_t+r)\big)\,\mathrm{d}r+\sum_{0\le t\le T}\int_0^{\Delta\xi_t^-}\big(K-w'(X_t-r)\big)\,\mathrm{d}r.$$

It follows that

$$\begin{aligned}
&\int_0^T h(X_t)\,\mathrm{d}t+K|\xi|_T\\
&=\lambda(\alpha,\beta)T+w(x)-w(X_{T+})-\frac{1}{2}\theta\langle M\rangle_T+M_T\\
&\quad+\int_0^T\bigg(\frac{1}{2}\sigma^2(X_t)w''(X_t)+b(X_t)w'(X_t)+\frac{1}{2}\theta\big(\sigma(X_t)w'(X_t)\big)^2+h(X_t)\\
&\qquad\qquad-\lambda(\alpha,\beta)\bigg)\,\mathrm{d}t\\
&\quad+\int_0^T\big(K+w'(X_t)\big)\,\mathrm{d}\xi_t^{\mathrm{c}+}+\int_0^T\big(K-w'(X_t)\big)\,\mathrm{d}\xi_t^{\mathrm{c}-}\\
&\quad+\sum_{0\le t\le T}\int_0^{\Delta\xi_t^+}\big(K+w'(X_t+r)\big)\,\mathrm{d}r+\sum_{0\le t\le T}\int_0^{\Delta\xi_t^-}\big(K-w'(X_t-r)\big)\,\mathrm{d}r.
\end{aligned} \tag{2.32}$$

The controlled process $\xi^{\alpha,\beta}$ and the corresponding state process $X^{\alpha,\beta}$ discussed in Remark 2.4 are such that

$$\begin{aligned}
X_T^{\alpha,\beta}\in[\alpha,\beta],\quad \xi_T^{\alpha,\beta,+}&=\int_{[0,T]}\mathbf{1}_{\{X_t^{\alpha,\beta}=\alpha\}}\,\mathrm{d}\xi_t^{\alpha,\beta,+}\\
\text{and}\quad \xi_T^{\alpha,\beta,-}&=\int_{[0,T]}\mathbf{1}_{\{X_t^{\alpha,\beta}=\beta\}}\,\mathrm{d}\xi_t^{\alpha,\beta,-}
\end{aligned} \tag{2.33}$$

for all $T>0$. Let w be a function whose first derivative is given by

$$w'(x)=\begin{cases}-K, & \text{if } x\le\alpha,\\ \frac{1}{\theta}\frac{\mathrm{d}}{\mathrm{d}x}\ln\big(\phi_{\alpha,\beta}(x)\big), & \text{if } x\in\,]\alpha,\beta[,\\ K, & \text{if } x\ge\beta.\end{cases}$$

In view of (2.22), (2.25) and the in-between arguments, we can see that w is C^2 in $\mathbb{R}\setminus\{\alpha,\beta\}$ and C^1 at both of α and β. Furthermore, recalling that the eigenfunction $\mathfrak{u}_0 \equiv \phi_{\alpha,\beta}$ and the eigenvalue $\lambda_0 \equiv \lambda(\alpha,\beta)$ provide a solution to the Sturm-Liouville eigenvalue problem defined by the ODE (2.23) with boundary conditions (2.25), we can see that w satisfies the ODE (2.16). Combining these observations with (2.32) and (2.33), we obtain

$$\int_0^T h\big(X_t^{\alpha,\beta}\big)\,\mathrm{d}t + K|\xi^{\alpha,\beta}|_T = \lambda(\alpha,\beta)T + w(x) - w\big(X_{T+}^{\alpha,\beta}\big) - \frac{1}{2}\theta\big\langle M^{\alpha,\beta}\big\rangle_T + M_T^{\alpha,\beta},$$

where $M^{\alpha,\beta}$ is defined by (2.5) for $\xi = \xi^{\alpha,\beta}$ and $g = w'$. It follows that

$$\begin{aligned}
&\frac{1}{\theta T}\ln \mathsf{E}\left[\exp\left(\theta\left(\int_0^T h\big(X_t^{\alpha,\beta}\big)\,\mathrm{d}t + K|\xi^{\alpha,\beta}|_T\right)\right)\right] \\
&= \lambda(\alpha,\beta) + \frac{w(x)}{T} + \frac{1}{\theta T}\ln \mathsf{E}\left[\exp\left(\theta\left(-w\big(X_T^{\alpha,\beta}\big) - \frac{1}{2}\theta\big\langle M^{\alpha,\beta}\big\rangle_T + M_T^{\alpha,\beta}\right)\right)\right] \\
&= \lambda(\alpha,\beta) + \frac{w(x)}{T} + \frac{1}{\theta T}\ln \mathsf{E}^{\widetilde{\mathsf{P}}_T^{\alpha,\beta}}\Big[\exp\Big(-\theta w\big(X_T^{\alpha,\beta}\big)\Big)\Big].
\end{aligned}$$

Here, $\widetilde{\mathsf{P}}_T^{\alpha,\beta}$ is the probability measure on $(\Omega,\mathcal{F}_T)$ with Radon-Nikodym derivative with respect to P given by $\mathrm{d}\widetilde{\mathsf{P}}_T^{\alpha,\beta}/\mathrm{d}\mathsf{P} = \mathcal{E}_T\big(\theta M^{\alpha,\beta}\big)$ (see also (2.7)). Finally, using the fact that the process $w(X^{\alpha,\beta})$ is bounded, we can pass to the limit as $T\uparrow\infty$ to obtain $J_x(\xi^{\alpha,\beta}) = \lambda(\alpha,\beta)$. □

2.3 The solution to the control problem

Theorem 2.10. *In the presence of Assumptions 2.1 and 2.5, the following statements hold true:*

(I) *There exists a unique pair* $(\alpha_\star, \beta_\star)$ *such that*

$$\alpha_\star < \alpha_-, \quad \beta_+ < \beta_\star \quad \text{and} \quad \lambda_\star := \lambda(\alpha_\star, \beta_\star) = H_-(\alpha_\star) = H_+(\beta_\star), \tag{2.34}$$

where the function λ *is defined by (2.28) and the functions* $H_\pm$ *are defined by (2.10).*

(II) *The function* w *that is defined by*

$$w'(x) = \begin{cases} -K, & \text{if } x \leq \alpha_\star, \\ \frac{1}{\theta} \frac{\mathrm{d}}{\mathrm{d}x} \ln\bigl(\phi_{\alpha_\star,\beta_\star}(x)\bigr), & \text{if } x \in \,]\alpha_\star, \beta_\star[, \\ K, & \text{if } x \geq \beta_\star, \end{cases} \tag{2.35}$$

modulo an additive constant, is C^2*. Furthermore, this function and* $\lambda_\star$ *provide a solution to the HJB equation (2.14).*

Proof. The conditions (2.11)–(2.13) in Assumption 2.5 imply that there exists a unique function $\Gamma : [\beta_+, \infty[\,\to\,]-\infty, \alpha_-]$ such that $H_+(\beta) = H_-\bigl(\Gamma(\beta)\bigr)$ for all $\beta \geq \beta_+$. In particular, $\Gamma(\beta_+) = \alpha_-$. In view of this observation, we can see that, if the equation

$$\Lambda(\beta) := \lambda\bigl(\Gamma(\beta), \beta\bigr) = H_+(\beta) \tag{2.36}$$

has a unique solution $\beta_\star > \beta_+$, then part (I) of the theorem holds true with $\alpha_\star = \Gamma(\beta_\star)$.

To show that the equation (2.36) has a unique solution $\beta_\star > \beta_+$, we first use (2.29) and (2.30) in Lemma 2.9, as well as the identity $H_+(\beta) = H_-\bigl(\Gamma(\beta)\bigr)$, to calculate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\beta}\big(\Lambda(\beta) - H_+(\beta)\big) \\
&\qquad = \lambda_\alpha\big(\Gamma(\beta), \beta\big)\Gamma'(\beta) + \lambda_\beta\big(\Gamma(\beta), \beta\big) - H_+'(\beta) \\
&\qquad = 2\theta\left(\frac{q\big(\Gamma(\beta)\big)\phi^2_{\Gamma(\beta),\beta}\big(\Gamma(\beta)\big)}{\sigma^2\big(\Gamma(\beta)\big)}\Gamma'(\beta) - \frac{q(\beta)\phi^2_{\Gamma(\beta),\beta}(\beta)}{\sigma^2(\beta)}\right)\big(\Lambda(\beta) - H_+(\beta)\big) \\
&\qquad\quad - H_+'(\beta) \\
&\qquad =: \varrho(\beta)\big(\Lambda(\beta) - H_+(\beta)\big) - H_+'(\beta), \quad \text{for } \beta > \beta_+.
\end{aligned} \tag{2.37}$$

The solution to this first-order ODE is such that

$$\begin{aligned}
I(\beta)\big(\Lambda(\beta) - H_+(\beta)\big) &= \Lambda(\beta_+) - H_+(\beta_+) - \int_{\beta_+}^{\beta} I(u)H_+'(u)\,\mathrm{d}u \\
&= \lambda(\alpha_-, \beta_+) - \int_{\beta_+}^{\beta} I(u)H_+'(u)\,\mathrm{d}u =: F(\beta),
\end{aligned}$$

where $I(\beta) = \exp\big(-\int_{\beta_+}^{\beta}\varrho(u)\,\mathrm{d}u\big)$. The second equality here follows from the fact that $\Gamma(\beta_+) = \alpha_-$ and the assumption that $H_+(\beta_+) = 0$. It follows that equation (2.36) is equivalent to the equation

$$F(\beta) = 0. \tag{2.38}$$

In view of the inequalities

$$F'(\beta) = -I(\beta)H_+'(\beta) < 0 \text{ for all } \beta > \beta_+ \quad \text{and} \quad F(\beta_+) = \lambda(\alpha_-, \beta_+) \overset{(2.31)}{>} 0,$$

we can see that equation (2.38) has a unique solution $\beta_\star > \beta_+$ if and only if $\lim_{\beta\uparrow\infty} F(\beta) < 0$. To see that this inequality is indeed true, we argue by contradiction. To this end, we assume that $\lim_{\beta\uparrow\infty} F(\beta) \geq 0$, which can be true only if

$$\Lambda(\beta) - H_+(\beta) = \frac{F(\beta)}{I(\beta)} > 0 \quad \text{for all } \beta > \beta_+ \tag{2.39}$$

because $F'(\beta) < 0$ and $I(\beta) > 0$ for all $\beta > \beta_+$. In view of the inequalities $\Gamma' < 0$ and $q > 0$, we can see that the function ϱ introduced in (2.37) is such that $\varrho(\beta) < 0$ for all $\beta > \beta_+$. In view of this inequality and the contradiction hypothesis (2.39), we can see that (2.37) implies that $\Lambda'(\beta) < 0$ for all $\beta > \beta_+$. However, this conclusion and (2.11) imply that

$$\lim_{\beta\uparrow\infty}\big(\Lambda(\beta) - H_+(\beta)\big) \leq \Lambda(\beta_+) - \lim_{\beta\uparrow\infty} H_+(\beta) = -\infty,$$

which contradicts (2.39). Thus, we have proved that equation (2.38), which is equivalent to equation (2.36), has a unique solution $\beta_\star > \beta_+$ and we have established part (I) of the theorem.

By construction, we will prove that the function w given by (2.35) is a C^2 solution to the HJB equation (2.14) if we show that

$$\frac{1}{2}\theta K^2\sigma^2(x) + Kb(x) + h(x) - \lambda_\star \geq 0 \quad \text{for all } x \in \,]-\infty, \alpha_\star[\, \cup \,]\beta_\star, \infty[\tag{2.40}$$

$$\text{and} \quad \big|w'(x)\big| \leq K \quad \text{for all } x \in \,]\alpha_\star, \beta_\star[. \tag{2.41}$$

The inequality (2.40) follows immediately from (2.12), (2.13) in Assumption 2.5 and the observation that

$$\frac{1}{2}\theta K^2\sigma^2(x) + Kb(x) + h(x) - \lambda_\star = \begin{cases} H_-(x) - H_-(\alpha_\star), & \text{if } x < \alpha_\star, \\ H_+(x) - H_+(\beta_\star), & \text{if } x > \beta_\star, \end{cases}$$

where we have used the definition (2.10) of the functions $H_\pm$ and part (I) of the theorem.

To establish (2.41), we first note that the C^1 continuity of the functions b, σ and h implies that the restriction of w in $]\alpha_\star, \beta_\star[$ is C^3. In particular, we note that differentiation of the ODE (2.16) that w satisfies in $]\alpha_\star, \beta_\star[$ implies that

$$\frac{1}{2}\sigma^2(x)w'''(x) + \big(b(x) + \sigma(x)\sigma'(x) + \theta\sigma^2(x)w'(x)\big)w''(x) \\ + \theta\sigma(x)\sigma'(x)\big(w'(x)\big)^2 + b'(x)w'(x) + h'(x) = 0.$$

In view of this calculation, the inequalities in (2.34), the assumptions (2.12), (2.13) and the free-boundary equations (2.18), (2.19), we can see that

$$\lim_{x\downarrow\alpha_\star} w'''(x) = -\frac{2}{\sigma^2(\alpha_\star)} H'_-(\alpha_\star) > 0 \quad \text{and} \quad \lim_{x\uparrow\beta_\star} w'''(x) = -\frac{2}{\sigma^2(\beta_\star)} H'_+(\beta_\star) < 0.$$

It follows that there exists $\varepsilon > 0$ such that

$$w''(x) > 0 \text{ for all } x \in \,]\alpha_\star, \alpha_\star + \varepsilon[\, \cup \,]\beta_\star - \varepsilon, \beta_\star[. \tag{2.42}$$

We next argue by contradiction, we assume that there exist $x \in \,]\alpha_\star, \beta_\star[$ such that $w'(x) > K$ and we define

$$\begin{aligned} \alpha_\star < \underline{\gamma} &:= \min\bigl\{x \in \,]\alpha_\star, \beta_\star[\mid \ w'(x) = K\bigr\} \\ &< \max\bigl\{x \in \,]\alpha_\star, \beta_\star[\mid \ w'(x) = K\bigr\} =: \overline{\gamma} < \beta_\star, \end{aligned} \tag{2.43}$$

where the inequalities follow once we combine (2.42) with the boundary conditions $w'(\alpha_\star) = -K$ and $w'(\beta_\star) = K$. The points $\underline{\gamma}$ and $\overline{\gamma}$ are such that

$$\begin{aligned} w''(\underline{\gamma}) &= \frac{2}{\sigma^2(\underline{\gamma})}\bigl(H_+(\beta_\star) - H_+(\underline{\gamma})\bigr) \geq 0 \\ \text{and} \quad w''(\overline{\gamma}) &= \frac{2}{\sigma^2(\overline{\gamma})}\bigl(H_+(\beta_\star) - H_+(\overline{\gamma})\bigr) \leq 0. \end{aligned}$$

However, these inequalities and the ones in (2.43) contradict (2.13) in Assumption 2.5, and (2.41) follows. □

We can now prove the main result of the paper.

Theorem 2.11. *Suppose that Assumptions 2.1 and 2.5 hold true. If* $(\alpha_\star, \beta_\star)$ *and* $\lambda_\star$ *are as in Theorem 2.10, then, given any* $x \in \mathbb{R}$*,*

$$\begin{aligned} \inf_{\xi\in\mathcal{A}} J_x(\xi) &= J_x\bigl(\xi^{\alpha_\star,\beta_\star}\bigr) \\ &\equiv \lim_{T\uparrow\infty} \frac{1}{\theta T} \ln \mathsf{E}\left[\exp\left(\theta\left(\int_0^T h(X_t^\star)\,\mathrm{d}t + K|\xi^{\alpha_\star,\beta_\star}|_T\right)\right)\right] = \lambda_\star > 0, \end{aligned} \tag{2.44}$$

where $\xi^{\alpha_\star,\beta_\star} \in \mathcal{A}$ *is as in the statement of Lemma 2.9 and* $X^\star$ *is the corresponding solution to the SDE (2.1).*

Proof. Fix any $x \in \mathbb{R}$ and any $\xi \in \mathcal{A}$. Also, let w be a function defined by (2.35) in Theorem 2.10. Since w satisfies the HJB equation (2.14), the expression (2.32) in the proof of Lemma 2.9 implies that

$$\int_0^T h(X_t)\,\mathrm{d}t + K|\xi|_T \geq \lambda_\star T + w(x) - w(X_{T+}) - \frac{1}{2}\theta\big\langle M^{\xi,w'}\big\rangle_T + M_T^{\xi,w'},$$

where $M^{\xi,w'}$ is defined by (2.5) for $g = w'$. In view of this observation, we can see that

$$\begin{aligned}
&\frac{1}{\theta T}\ln \mathsf{E}\left[\exp\left(\theta\left(\int_0^T h(X_t)\,\mathrm{d}t + K|\xi|_T\right)\right)\right] \\
&\qquad \geq \lambda_\star + \frac{w(x)}{T} + \frac{1}{\theta T}\ln \mathsf{E}\left[\exp\left(-\theta w(X_{T+}) - \frac{1}{2}\theta^2\big\langle M^{\xi,w'}\big\rangle_T + \theta M_T^{\xi,w'}\right)\right] \\
&\qquad = \lambda_\star + \frac{w(x)}{T} + \frac{1}{\theta T}\ln \mathsf{E}^{\widetilde{\mathsf{P}}_T}\Big[\exp\big(-\theta w(X_T)\big)\Big],
\end{aligned} \tag{2.45}$$

where $\widetilde{\mathsf{P}}_T$ is the probability measure on $(\Omega, \mathcal{F}_T)$ that has Radon-Nikodym derivative with respect to P given by $\mathrm{d}\widetilde{\mathsf{P}}_T/\mathrm{d}\mathsf{P} = \mathcal{E}_T\big(\theta M^{\xi,w'}\big)$ (see also (2.7)). In view of the inequality $\big|w'(x)\big| \leq K$, we can see that

$$\mathsf{E}^{\widetilde{\mathsf{P}}_T}\Big[\exp\big(-\theta w(X_T)\big)\Big] \geq \mathsf{E}^{\widetilde{\mathsf{P}}_T}\Big[\exp\Big(-\theta\big|w(0)\big| - \theta K|X_T|\Big)\Big].$$

In view of this observation and the admissibility condition (2.9), we can pass to the limit as $T \uparrow \infty$ in (2.45) to obtain $J_x(\xi) \geq \lambda_\star$. Finally, the identity $J_x\big(\xi^{\alpha_\star,\beta_\star}\big) = \lambda_\star$ follows from Lemma 2.9. □

3

Equilibrium asset pricing with proportional transaction costs in a stochastic factor model

In this chapter, we study a risk-sharing equilibrium similar to Gonon et al. [51] where the agents trade to hedge against the fluctuations of their random endowment streams, while subjected to proportional transaction costs. Our aim is to generalise the setting in [51], where endowment rates are scalar multiples of a Brownian motion, to continuous functions of a one-dimensional diffusion. This allows us to investigate new qualitative behaviours of optimal trading strategies and incorporate new effects captured by the underlying diffusion such as seasonality.

In [51], a singular stochastic control approach specialised to their particular choice of endowment rate is adopted, and solutions to the associated free-boundary problem are available in closed form - the solution to the ODE is a cubic function and the free-boundaries are linear functions, and the deviations of the frictional from the frictionless optimiser (as well as the liquidity premium) is a doubly reflected Brownian motion with constant end points. We develop a more general dynamic programming approach where we obtain an explicit system of equations characterising the free-boundaries. The free-boundaries are no longer available in closed form, but existence and uniqueness results can still be proven. Moreover, the deviation of the frictional from the frictionless optimiser is a doubly reflected diffusion with non-constant end points that depend on the number of shares of risky asset held. To the best of our knowledge, the two-dimensional singular stochastic control problem and corresponding free-boundary problem providing the solution to the agents' optimisation problems are novel and non-standard, as we do not have the usual variational inequality in standard singular control problems. However, similar equations for the free-boundaries are still obtained. Examples of two-dimensional singular control problems in the literature include [84], [92], [30], [32], [31] and [74]. The problems considered by Løkka and Zervos [84] and Merhi and Zervos [92] are more closely related to our problem in the sense that the two variables in the two-dimensional problem are taken to be the controlled finite variation process and a one-dimensional diffusion whose dynamics are not affected by the controlled process. Merhi and Zervos [92] similarly considers a discounted infinite horizon problem, but specialises to a geometric Brownian motion as the state process, while we consider one-dimensional diffusions. In Løkka and Zervos [84], an ergodic criterion is considered and the underlying diffusion is assumed to be ergodic. In our setting, we consider a discounted infinite horizon criterion and do not assume ergodicity of the diffusion, and only require that the discount factor is sufficiently large. Moreover, in our setting, our "running cost" function does not satisfy the assumptions of the running payoff functions in [84] and [92].

The no-trade region in Gonon et al. [51] is exactly a constant multiplied by $\lambda^{1/3}$, due to the linearity of the free-boundary functions and the solution to the free-boundary problem being a cubic function. The free-boundaries in our setting are not available in closed form, but we can compute transaction cost asymptotics explicitly using the system of free-boundary equations. The calculation is similar to that in Schachermayer [103], through the inversion of the Taylor expansion of the transaction cost parameter, expressed as a function of the boundary.

The remainder of this chapter is organised as follows: Section 3.1 describes the model in the frictionless case as well as the case with proportional transaction costs. Section 3.2 establishes sufficient conditions for the existence of an equilibrium return rate with corresponding optimal trading strategies, as well as the associated free-boundary problem. In Sections 3.3 and 3.4, the free-boundary problem is solved when the problem data is such that the frictionless optimiser is a strictly increasing, and a strictly increasing and then strictly decreasing function of the economy's state respectively. In Section 3.5 the solution to the control problem is constructed. In Section 3.6, small transaction cost asymptotics are derived for the free-boundaries. Finally, Appendix A reviews results on ODEs and one-dimensional diffusions frequently used in our analysis.

3.1 Problem formulation

Fix a probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}\big)$ satisfying the usual conditions and supporting two independent standard one-dimensional $(\mathcal{F}_t)$-Brownian motions W_1 and W_2. We set $W = W_1$ and $B = \rho W_1 + \sqrt{1-\rho^2} W_2$, so that W and B are correlated Brownian motions with correlation coefficient $\rho \in [0,1]$. We note that the correlation coefficient ρ does not affect the remainder of our analysis. We consider two agents who start with 0 initial endowments at time 0 and have an infinite horizon as well as the same discounting rate $r > 0$. For $i = 1, 2$, Agent i receives a cumulative random endowment that is given by

$$\Xi_t^i = \int_0^t \xi_s^i \, \mathrm{d}W_s, \quad \text{for } t \geq 0. \tag{3.1}$$

Assumption 3.1. *For $i = 1, 2$, the processes ξ^i are $(\mathcal{F}_t)$-progressive and such that*

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} \big(\xi_t^i\big)^2 \mathrm{d}t\right] < \infty.$$

For the best part of our analysis, we are going to assume that

$$\xi_t^i = h_i(X_t), \quad \text{for } t \geq 0 \text{ and } i = 1, 2,$$

where $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are given functions and X is the solution to the SDE

$$\mathrm{d}X_t = \alpha(X_t) \, \mathrm{d}t + \beta(X_t) \, \mathrm{d}B_t, \quad X_0 = x \in \mathbb{R}. \tag{3.2}$$

In this context, the process X represents the state of the economy in which the two agents operate, while the functions h_i reflect the sensitivity of the agents' endowment rates to the state of the economy. We note that, in [51], $\rho = 1$,

$h_i(x) = \xi_i x$ where $\xi_i \in \mathbb{R}$ and X is a standard one-dimensional Brownian motion (that is, $\alpha = 0$ and $\beta = 1$).

Assumption 3.2. *The functions $\alpha : \mathbb{R} \to \mathbb{R}$ and $\beta : \mathbb{R} \to \mathbb{R}$ are continuous, the functions $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ are C^1 and there exist constants $C, K > 0$ and $p \geq 1$ such that*

$$\big|\alpha(x) - \alpha(y)\big| \leq K|x-y|, \quad \big|\beta(x) - \beta(y)\big| \leq K|x-y|, \tag{3.3}$$

$$\big|x\alpha(x)\big| + \frac{2p-1}{2}\beta^2(x) \leq C\big(1+x^2\big) \tag{3.4}$$

$$\text{and} \quad \big|h_1(x)\big| + \big|h_2(x)\big| \leq K\big(1+|x|^p\big) \tag{3.5}$$

for all $x, y \in \mathbb{R}$. Furthermore, these constants are such that

$$r > 2pC. \tag{3.6}$$

Remark 3.3. *The Lipschitz continuity assumption (3.3) ensures that the SDE (3.2) has a unique strong solution and the growth conditions (3.4) imply that*

$$\mathbb{E}\Big[|X_t|^{2p}\Big] \leq 2^{p-1}\big(1+|x|^{2p}\big)\mathrm{e}^{2pCt} \quad \textit{for all } t \geq 0 \tag{3.7}$$

(see Mao [86, Chapter 2.4, Theorem 4.1]). Furthermore, this estimate, the growth conditions (3.5) and (3.6) as well as Fubini's theorem imply that the endowment rates $\xi^i = h_i(X)$ satisfy Assumption 3.1, in other words, for $i = 1, 2$,

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} h_i^2(X_t)\mathrm{d}t\right] < \infty. \tag{3.8}$$

Remark 3.4. *If h_1 and h_2 are bounded, then we can relax the requirement* (3.6) *on the discounting rate (that is, taking $p = 0$ in* (3.5) *and* (3.6)*), since the integrability condition* (3.8) *is clearly satisfied for any $r > 0$.*

We assume that the two agents are risk-averse and have mean-variance preferences with risk-aversion parameters $\gamma_1 > 0$ and $\gamma_2 > 0$. Accordingly, the mean-variance payoff that Agent i expects from their individual income stream is given by

$$\lim_{T\uparrow\infty} \mathbb{E}\left[\int_0^T \mathrm{e}^{-rt}\Big(\mathrm{d}\Xi_t^i - \frac{\gamma_i}{2}\,\mathrm{d}\big\langle \Xi^i\big\rangle_t\Big)\right] = -\frac{\gamma_i}{2}\,\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\big(\xi_t^i\big)^2\,\mathrm{d}t\right].$$

To hedge against the random fluctuations of their individual endowments, the two agents may enter a risk-sharing agreement to trade a risky asset that is in zero-net supply. The stochastic dynamics of the risky asset's price process S are given by

$$\mathrm{d}S_t = \mu_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{3.9}$$

where the constant absolute volatility $\sigma > 0$ and the initial price $S_0 \in \mathbb{R}$ are determined as part of the risk-sharing agreement. On the other hand, the drift process μ will be determined in equilibrium. In this context, if we denote by ϑ_t^i the number of shares held by Agent i at time t, then the market clearing condition

$$\vartheta_t^1 + \vartheta_t^2 = 0 \tag{3.10}$$

should be satisfied at all times because the asset is in zero-net supply. Furthermore, if we denote by Y^i the total wealth process of Agent i, then

$$\begin{aligned} Y_t^i &= \theta_0^i S_0 + \int_0^t \vartheta_u^i\,\mathrm{d}S_u + \int_0^t \xi_u^i\,\mathrm{d}W_u \\ &= \theta_0^i S_0 + \int_0^t \vartheta_u^i \mu_u\,\mathrm{d}u + \int_0^t \big(\sigma\vartheta_u^i + \xi_u^i\big)\,\mathrm{d}W_u, \end{aligned} \tag{3.11}$$

where $\theta_0^i = \vartheta_{0-}^i$ are the number of the risky asset shares with which Agent i enters the agreement. If the two agents enter the risk-sharing agreement at time 0, then it is natural to assume that their starting holdings of the risky asset are null, namely, $\theta_0^1 = \theta_0^2 = 0$, particularly so, in the presence of transaction costs, which is the focus of the paper. However, assuming non-zero such values is essential for our analysis, which will be based on dynamic programming.

3.1.1 Frictionless equilibrium

In the frictionless setting, the agents face no transaction costs during their mutual trading. We note that this section is very closely based on the frictionless setting in [12]. In this context, the objective of Agent i is to maximise the performance index

$$\begin{aligned} I^{0,i}\big(\vartheta^i \mid \mu^0\big) &= \lim_{T\uparrow\infty} \mathbb{E}\bigg[\int_0^T \mathrm{e}^{-rt}\Big(\mathrm{d}Y_t^i - \frac{\gamma_i}{2}\,\mathrm{d}\big\langle Y^i\big\rangle_t\Big)\bigg] \\ &= \mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\Big(\vartheta_t^i\mu_t^0 - \frac{\gamma_i}{2}\big(\sigma\vartheta_t^i + \xi_t^i\big)^2\Big)\,\mathrm{d}t\bigg] \end{aligned} \tag{3.12}$$

over all trading strategies ϑ^i satisfying suitable integrability conditions. Here, we write μ^0 in place of μ to stress the absence of any transaction costs. Pointwise maximisation of the integrand on the right-hand side of (3.12) yields

$$\hat{\vartheta}_t^i = \frac{1}{\sigma^2\gamma_i}\mu_t^0 - \frac{1}{\sigma}\xi_t^i. \tag{3.13}$$

This result and the market clearing condition (3.10) imply that the frictionless equilibrium mean-rate of return is given by

$$\mu_t^0 = \sigma\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\big(\xi_t^1 + \xi_t^2\big) =: \sigma\widetilde{\gamma}\big(\xi_t^1 + \xi_t^2\big).$$

Substituting this expression for μ_t^0 into (3.13), we obtain

$$\hat{\vartheta}_t^{0,1} = \frac{\gamma_2\xi_t^2 - \gamma_1\xi_t^1}{\sigma(\gamma_1+\gamma_2)} \quad \text{and} \quad \hat{\vartheta}_t^{0,2} = \frac{\gamma_1\xi_t^1 - \gamma_2\xi_t^2}{\sigma(\gamma_1+\gamma_2)}.$$

Furthermore, the optimal mean-variance payoffs that the two agents receive in equilibrium are

$$\begin{aligned} & I^{0,1}\big(\hat{\vartheta}^{0,1} \mid \sigma\widetilde{\gamma}(\xi^1+\xi^2)\big) \\ & \quad = \frac{\widetilde{\gamma}}{\gamma_1+\gamma_2}\,\mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\big(\xi_t^1+\xi_t^2\big)\Big(\frac{\gamma_2}{2}\xi_t^2 - \Big(\gamma_1 + \frac{\gamma_2}{2}\Big)\xi_t^1\Big)\,\mathrm{d}t\bigg] \\ \text{and} \quad & I^{0,2}\big(\hat{\vartheta}^{0,2} \mid \sigma\widetilde{\gamma}(\xi^1+\xi^2)\big) \\ & \quad = \frac{\widetilde{\gamma}}{\gamma_1+\gamma_2}\,\mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\big(\xi_t^1+\xi_t^2\big)\Big(\frac{\gamma_1}{2}\xi_t^1 - \Big(\gamma_2 + \frac{\gamma_1}{2}\Big)\xi_t^2\Big)\,\mathrm{d}t,\bigg] \end{aligned}$$

which are well-defined and real-valued, thanks to Assumption 3.1.

It is straightforward to verify that

$$I^{0,i}\big(\hat{\vartheta}^{0,i} \mid \sigma\widetilde{\gamma}(\xi^1+\xi^2)\big) = \frac{\gamma_i}{2}\,\mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\Big(\big(\sigma\hat{\vartheta}_t^{0,i}\big)^2 - \big(\xi_t^i\big)^2\Big)\,\mathrm{d}t\bigg]$$
$$\geq -\frac{\gamma_i}{2}\,\mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\big(\xi_t^i\big)^2\,\mathrm{d}t\bigg].$$

This inequality holds with equality if and only if

$$\frac{\gamma_1}{\gamma_1+\gamma_2}\xi^1 = \frac{\gamma_2}{\gamma_1+\gamma_2}\xi^2,$$

namely, if and only if the agents' individual endowment rates are equal if weighted by the agents' relative risk aversions. In this case, $\hat{\vartheta}^i = 0$ and the risk-sharing agreement presents no benefit to either of the two parties. In the absence of this identity, entering the risk-sharing agreement strictly increases the expected mean-variance payoff of either of the two agents.

In the presence of the assumption that $\xi^i = h_i(X)$, the frictionless optimisers, as well as the frictionless equilibrium mean-rate of return, are also functions of X and are given by

$$\hat{\vartheta}_t^{0,1} = \frac{\gamma_2 h_2(X_t) - \gamma_1 h_1(X_t)}{\sigma(\gamma_1+\gamma_2)} =: \Theta(X_t), \hat{\vartheta}_t^{0,2} = \frac{\gamma_1 h_1(X_t) - \gamma_2 h_2(X_t)}{\sigma(\gamma_1+\gamma_2)} = -\Theta(X_t) \tag{3.14}$$

$$\text{and} \quad \mu_t^0 = \sigma\frac{\gamma_1\gamma_2}{\gamma_1+\gamma_2}\big(h_1(X_t)+h_2(X_t)\big) =: M(X_t). \tag{3.15}$$

In view of these expressions, the first agent's optimal strategy is to ensure that the joint process $(\hat{\vartheta}^{0,1}, X)$ takes values in the set $\big\{(\theta,x)\in\mathbb{R}^2 \mid \theta = \Theta(x)\big\}$ at all times. Unless Θ is identically equal to a constant, such a strategy is of infinite variation because the process X is. As a result, it cannot be optimal in the presence of proportional transaction costs because it would incur infinite losses. We note that (3.8) implies that $\Theta(X)$ satisfies the integrability condition

$$\mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\Theta^2(X_t)\,\mathrm{d}t\bigg]$$
$$\leq \mathbb{E}\bigg[\int_0^\infty \mathrm{e}^{-rt}\Big(\frac{\gamma_2^2+\gamma_1\gamma_2}{\sigma^2(\gamma_1+\gamma_2)^2}h_2^2(X_t) + \frac{\gamma_1^2+\gamma_1\gamma_2}{\sigma^2(\gamma_1+\gamma_2)^2}h_1^2(X_t)\Big)\,\mathrm{d}t\bigg] < \infty. \tag{3.16}$$

3.1.2 Proportional transaction costs

We now assume that the agents' transactions are subject to transaction costs that are proportional to the volume of their trades with a given proportionality constant $\lambda > 0$. In this context, trading strategies must be of finite variation because, otherwise, they would incur infinite costs. We assume that the admissible trading strategies of either of the two agents belong to the class introduced by the following definition.

Definition 3.5. *Given an initial condition $(\theta, x) \in \mathbb{R}^2$, a trading strategy ϑ is admissible if it is a càglàd $(\mathcal{F}_t)$-adapted process of finite variation such that $\vartheta_{0-} = \theta$ and*

$$\mathbb{E}\big[|\vartheta|_T\big] < \infty \quad \textit{for all } T \geq 0 \quad \textit{and} \quad \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} \vartheta_t^2 \, \mathrm{d}t\right] < \infty, \tag{3.17}$$

where, if we denote by $|\vartheta|$ the total variation process of ϑ, then $\vartheta^\pm$ are the unique $(\mathcal{F}_t)$-adapted càglàd increasing processes satisfying

$$\vartheta_t = \theta + \vartheta_t^+ - \vartheta_t^- \quad \textit{and} \quad |\vartheta|_t = \vartheta_t^+ + \vartheta_t^- \quad \textit{for all } t \geq 0. \tag{3.18}$$

We denote by $\mathcal{A}(\theta)$ the family of all admissible trading strategies that is parametrised by the convention $\vartheta_{0-} = \theta$, where $\theta \in \mathbb{R}$ is a constant.

Remark 3.6. *The integrability condition (3.17) implies the transversality condition*

$$\liminf_{T \uparrow \infty} \mathrm{e}^{-rT} \, \mathbb{E}\big[\vartheta_T^2\big] = 0. \tag{3.19}$$

To see this, we observe that since ϑ is a càglàd process, its sample paths can have at most countable discontinuities. This observation, Fubini's theorem and (3.17) imply that

$$\int_0^\infty \mathrm{e}^{-rt} \, \mathbb{E}\big[\vartheta_{t+}^2\big] \mathrm{d}t = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} \vartheta_{t+}^2 \, \mathrm{d}t\right] = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt} \vartheta_t^2 \, \mathrm{d}t\right] < \infty,$$

which implies (3.19).

The level of the proportionality constant $\lambda > 0$ plainly influences the optimal strategies in equilibrium. We therefore denote by μ^λ the frictional equilibrium mean-rate of return in what follows. Accordingly, we write the dynamics of the risky asset price process in the form

$$\mathrm{d}S_t = \mu_t^\lambda \,\mathrm{d}t + \sigma \,\mathrm{d}W_t$$

instead of (3.9). In the presence of proportional transaction costs, the objective of Agent i is to maximise the performance index

$$\mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}\Bigl(\vartheta_t^i \mu_t^\lambda - \frac{\gamma_i}{2}\bigl(\sigma\vartheta_t^i + \xi_t^i\bigr)^2\Bigr)\,\mathrm{d}t - \lambda \int_{[0,\infty[} \mathrm{e}^{-rt}\,\mathrm{d}|\vartheta^i|_t\biggr] \tag{3.20}$$

over all admissible trading strategies ϑ^i, where $|\vartheta^i|$ is the total variation process of ϑ^i.

Using completion of squares and substituting the formulae derived in the previous section for the frictionless optimal equilibrium strategies and mean-rate of return, we obtain

$$\vartheta_t^i \mu_t^\lambda - \frac{\gamma_i}{2}\bigl(\sigma\vartheta_t^i + \xi_t^i\bigr)^2 = \hat{\vartheta}_t^{0,i}\mu_t^0 - \frac{\gamma_i}{2}\bigl(\sigma\hat{\vartheta}_t^{0,i} + \xi_t^i\bigr)^2 + \vartheta_t^i\bigl(\mu_t^\lambda - \mu_t^0\bigr) - \frac{1}{2}\sigma^2\gamma_i\bigl(\vartheta_t^i - \hat{\vartheta}_t^{0,i}\bigr)^2.$$

In view of this identity, we can see that maximising the performance index in (3.20) is equivalent to maximising

$$\begin{aligned} &I^{0,i}\bigl(\hat{\vartheta}^{0,i} \mid \sigma\widetilde{\gamma}(\xi^1 + \xi^2)\bigr) \\ &\quad - \mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}\biggl(\frac{1}{2}\sigma^2\gamma_i\bigl(\vartheta_t^i - \hat{\vartheta}_t^{0,i}\bigr)^2 - \vartheta_t^i\bigl(\mu_t^\lambda - \mu_t^0\bigr)\biggr)\,\mathrm{d}t + \lambda\int_{[0,\infty[} \mathrm{e}^{-rt}\,\mathrm{d}|\vartheta^i|_t\biggr], \end{aligned}$$

where $I^{0,i}$ is defined by (3.12). Therefore, the objective of Agent i is to minimise the performance index

$$I_{\theta_i,x}^{\lambda,i}\bigl(\vartheta^i \mid \mu^\lambda\bigr) = \mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt} f^i\Bigl(\vartheta_t^i, X_t, \mu_t^\lambda - M(X_t)\Bigr)\,\mathrm{d}t + \lambda\int_{[0,\infty[} \mathrm{e}^{-rt}\,\mathrm{d}|\vartheta^i|_t\biggr]$$

over all admissible strategies $\vartheta^i \in \mathcal{A}(\theta_i)$, where

$$f^i(\theta_i, x, m) = \begin{cases} \frac{1}{2}\sigma^2\gamma_1\bigl(\theta_1 - \Theta(x)\bigr)^2 - \theta_1 m, & \text{if } i = 1, \\ \frac{1}{2}\sigma^2\gamma_2\bigl(\theta_2 + \Theta(x)\bigr)^2 - \theta_2 m, & \text{if } i = 2, \end{cases}$$

with Θ and M being defined by (3.14) and (3.15).

We are now faced with the Radner equilibrium problem that is introduced by the following definition, which incorporates the market clearing condition (3.10).

Definition 3.7. *Fix any $(\theta, x) \in \mathbb{R}^2$. A mean-rate of return process μ^λ and a strategy $\hat{\vartheta} \in \mathcal{A}(\theta)$ present an equilibrium if*

$$I^{\lambda,1}_{\theta,x}\big(\vartheta \mid \mu^\lambda\big) \geq I^{\lambda,1}_{\theta,x}\big(\hat{\vartheta} \mid \mu^\lambda\big) \quad \text{and} \quad I^{\lambda,2}_{-\theta,x}\big(-\vartheta \mid \mu^\lambda\big) \geq I^{\lambda,2}_{-\theta,x}\big(-\hat{\vartheta} \mid \mu^\lambda\big)$$

for all $\vartheta \in \mathcal{A}(\theta)$. In particular, the agents' optimal trading strategies in equilibrium are given by $\hat{\vartheta}^{\lambda,1} = \hat{\vartheta}$ and $\hat{\vartheta}^{\lambda,2} = -\hat{\vartheta}$.

Due to the symmetry arising from the market clearing condition, it is sufficient to solve for the optimal strategy for Agent 1. In a (θ, x)-graph depicting the optimal strategies of the agents, the optimal strategy of Agent 2 is obtained by reflecting the optimal strategy of Agent 1 about the x-axis. The next section establishes sufficient conditions for the existence of an equilibrium with corresponding optimal trading strategies, and formulates the relevant free-boundary problem.

3.2 Sufficient conditions for the existence of an equilibrium

Fix any initial state $(\theta, x) \in \mathbb{R}^2$. Given any trading strategies $\vartheta, \hat{\vartheta} \in \mathcal{A}(\theta)$, we first observe that

$$\begin{aligned} I^{\lambda,1}_{\theta,x}\big(\vartheta \mid \mu^\lambda\big) &= I^{\lambda,1}_{\theta,x}\big(\hat{\vartheta} \mid \mu^\lambda\big) + \gamma_1 Q_1(\vartheta, \hat{\vartheta}) + Q_2(\vartheta, \hat{\vartheta}, \mu^\lambda) \\ &\quad + J_{\theta,x}(\vartheta, \hat{\vartheta}) - J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta}) \\ \text{and} \quad I^{\lambda,2}_{-\theta,x}\big(-\vartheta \mid \mu^\lambda\big) &= I^{\lambda,2}_{-\theta,x}\big(-\hat{\vartheta} \mid \mu^\lambda\big) + \gamma_2 Q_1(\vartheta, \hat{\vartheta}) - Q_2(\vartheta, \hat{\vartheta}, \mu^\lambda) \\ &\quad + J_{\theta,x}(\vartheta, \hat{\vartheta}) - J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta}), \end{aligned}$$

where

$$
\begin{aligned}
&Q_1(\vartheta, \hat{\vartheta}) = \frac{1}{2}\sigma^2\,\mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}\bigl(\vartheta_t - \hat{\vartheta}_t\bigr)^2\,\mathrm{d}t\biggr],\\
&Q_2(\vartheta, \hat{\vartheta}, \mu^\lambda)\\
&\quad = \mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}\biggl(\frac{1}{2}\sigma^2(\gamma_1 - \gamma_2)\bigl(\hat{\vartheta}_t - \Theta(X_t)\bigr) + M(X_t) - \mu_t^\lambda\biggr)\bigl(\vartheta_t - \hat{\vartheta}_t\bigr)\,\mathrm{d}t\biggr]
\end{aligned}
$$

and $J_{\theta,x}(\vartheta, \hat{\vartheta})$

$$
= \mathbb{E}\biggl[\frac{1}{2}\sigma^2(\gamma_1 + \gamma_2)\int_0^\infty \mathrm{e}^{-rt}\bigl(\hat{\vartheta}_t - \Theta(X_t)\bigr)\vartheta_t\,\mathrm{d}t + \lambda\int_{[0,\infty[} \mathrm{e}^{-rt}\,\mathrm{d}|\vartheta|_t\biggr].
$$

If we can determine a process $\hat{\vartheta} \in \mathcal{A}(\theta)$ such that

$$
J_{\theta,x}(\vartheta, \hat{\vartheta}) \geq J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta}) \quad \text{for all } \vartheta \in \mathcal{A}(\theta), \tag{3.21}
$$

then these expressions imply immediately that the mean-rate of return process defined by

$$
\mu^\lambda = M(X) + \frac{1}{2}\sigma^2(\gamma_1 - \gamma_2)\bigl(\hat{\vartheta} - \Theta(X)\bigr)
$$

and the optimal strategy $\hat{\vartheta}$ present an equilibrium, and we note that μ^λ has the same form as in [51]. To this end, we consider the following free-boundary problem, where we determine a $C^{1,2}$ function v that satisfies $v(\theta, x)\theta = J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta})$. In the complement of the domain of this free-boundary problem, $v_\theta(\theta, x) = 0$ and $v(\theta, x)$ is the partial derivative of $J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta})$ with respect to the initial position θ of the process $\hat{\vartheta}$, having a magnitude of λ.

Problem 3.8. *Determine a continuous function $v : \mathbb{R}^2 \to \mathbb{R}$ such that $v(\cdot, x)$ is C^1 for all $x \in \mathbb{R}$,*

$$
\bigl|v(\theta, x)\bigr| \leq \lambda \quad \textit{for all } (\theta, x) \in \mathbb{R}^2 \tag{3.22}
$$

and

$$
\mathscr{L}v(\theta, x) + \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2)\bigl(\theta - \Theta(x)\bigr) = 0 \quad \textit{for all } (\theta, x) \in \mathcal{C}, \tag{3.23}
$$

where $\mathscr{L}$ is the infinitesimal generator of the diffusion associated with the SDE

(3.2) that is killed at a rate r, which is defined by

$$\mathscr{L}w(x) = \frac{1}{2}\beta^2(x)w_{xx}(x) + \alpha(x)w_x(x) - rw(x),$$

for C^2 functions w, and

$$\mathcal{C} = \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \big|v(\theta, x)\big| < \lambda\big\}.$$

Given a solution v to this free-boundary problem, we define

$$\mathcal{B} = \big\{(\theta, x) \in \mathbb{R}^2 \mid \ v(\theta, x) = -\lambda\big\} \quad \text{and} \quad \mathcal{S} = \big\{(\theta, x) \in \mathbb{R}^2 \mid \ v(\theta, x) = \lambda\big\}. \tag{3.24}$$

In what follows, we denote by $\mathrm{cl}\,\mathcal{C}$ the closure of a set $\mathcal{C} \subseteq \mathbb{R}^2$ in $\mathbb{R}^2$. Furthermore, given any $\vartheta \in \mathcal{A}(\theta)$, we recall the convention $\vartheta_{0-} = \theta$ that we have adopted as well as the unique processes $\vartheta^{\pm}$ satisfying (3.18). We can now prove the verification theorem.

Theorem 3.9. *Suppose that v is a solution to Problem 3.8. Also, suppose that, given any $(\theta, x) \in \mathbb{R}^2$, there exists a process $\hat{\vartheta} \in \mathcal{A}(\theta)$ such that*

$$(\hat{\vartheta}_t, X_t) \in \mathrm{cl}\,\mathcal{C}, \quad \hat{\vartheta}_t^+ = \int_{[0,t]} \mathbf{1}_{\{(\hat{\vartheta}_s, X_s) \in \mathcal{B}\}} \,\mathrm{d}\hat{\vartheta}_s^+ \quad \textit{and} \quad \hat{\vartheta}_t^- = \int_{[0,t]} \mathbf{1}_{\{(\hat{\vartheta}_s, X_s) \in \mathcal{S}\}} \,\mathrm{d}\hat{\vartheta}_s^- \tag{3.25}$$

for all $t \geq 0$. Then, for any process $\vartheta \in \mathcal{A}(\theta)$, (3.21) holds true. Moreover, $v(\theta, x)\theta = J_{\theta,x}(\hat{\vartheta}, \hat{\vartheta}) \in \mathbb{R}$.

Proof. Fix any initial state $(\theta, x) \in \mathbb{R}^2$ and consider any $\vartheta \in \mathcal{A}(\theta)$ as well the process $\hat{\vartheta}$ that is as in the statement of the theorem. The C^1 continuity of $v(\cdot, x)$ and the definitions of the regions $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{S}$ imply that

$$v_\theta(\theta, x) = 0 \quad \text{for all } (\theta, x) \in (\mathcal{B} \cap \mathrm{cl}\,\mathcal{C}) \cup (\mathcal{S} \cap \mathrm{cl}\,\mathcal{C}).$$

On the other hand, the ODE (3.23) that $v(\theta, \cdot)$ satisfies in $\mathcal{C}$ and Assumption 3.2 imply that the first two derivatives of $v(\theta, \cdot)$ exist in the limit as (θ, x) tends to the boundary of $\mathcal{C}$ and (3.23) is satisfied for all $(\theta, x) \in \mathrm{cl}\,\mathcal{C}$, acordingly. In view of these observations, we use Itô's and the integration by parts formulae to calculate

$$\begin{aligned}
&\mathrm{e}^{-rT} v(\hat{\vartheta}_T, X_T)\vartheta_T \\
&= v(\theta, x)\theta + \int_0^T \mathrm{e}^{-rt} \mathscr{L} v(\hat{\vartheta}_t, X_t)\vartheta_t \,\mathrm{d}t + \int_{[0,T]} \mathrm{e}^{-rt} v_\theta(\hat{\vartheta}_t, X_t)\vartheta_t \,\mathrm{d}\hat{\vartheta}_t \\
&\quad + \int_{[0,T]} \mathrm{e}^{-rt} v(\hat{\vartheta}_t, X_t) \,\mathrm{d}\vartheta_t + \int_0^T \mathrm{e}^{-rt} \beta(X_t) v_x(\hat{\vartheta}_t, X_t)\vartheta_t \,\mathrm{d}B_t \\
&= v(\theta, x)\theta - \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2) \int_0^T \mathrm{e}^{-rt} \big(\hat{\vartheta}_t - \Theta(X_t)\big)\vartheta_t \,\mathrm{d}t \\
&\quad + \int_{[0,T]} \mathrm{e}^{-rt} v(\hat{\vartheta}_t, X_t) \,\mathrm{d}\vartheta_t + \int_0^T \mathrm{e}^{-rt} \beta(X_t) v_x(\hat{\vartheta}_t, X_t)\vartheta_t \,\mathrm{d}B_t.
\end{aligned}$$

This expression implies that

$$\begin{aligned}
\mathrm{e}^{-rT} v(\hat{\vartheta}_T, X_T)\vartheta_T \geq v(\theta, x)\theta &- \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2) \int_0^T \mathrm{e}^{-rt} \big(\hat{\vartheta}_t - \Theta(X_t)\big)\vartheta_t \,\mathrm{d}t \\
&- \lambda \int_{[0,T]} \mathrm{e}^{-rt} \,\mathrm{d}|\vartheta|_t + \int_0^T \mathrm{e}^{-rt} \beta(X_t) v_x(\hat{\vartheta}_t, X_t)\vartheta_t \,\mathrm{d}B_t
\end{aligned}$$

as well as

$$\begin{aligned}
\mathrm{e}^{-rT} v(\hat{\vartheta}_T, X_T)\hat{\vartheta}_T = v(\theta, x)\theta &- \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2) \int_0^T \mathrm{e}^{-rt} \big(\hat{\vartheta}_t - \Theta(X_t)\big)\hat{\vartheta}_t \,\mathrm{d}t \\
&- \lambda \int_{[0,T]} \mathrm{e}^{-rt} \,\mathrm{d}|\hat{\vartheta}|_t + \int_0^T \mathrm{e}^{-rt} \beta(X_t) v_x(\hat{\vartheta}_t, X_t)\hat{\vartheta}_t \,\mathrm{d}B_t.
\end{aligned} \tag{3.26}$$

It follows that

$$\begin{aligned}
&\frac{1}{2}\sigma^2(\gamma_1 + \gamma_2) \int_0^T \mathrm{e}^{-rt} \big(\hat{\vartheta}_t - \Theta(X_t)\big)\hat{\vartheta}_t \,\mathrm{d}t + \lambda \int_{[0,T]} \mathrm{e}^{-rt} \,\mathrm{d}|\hat{\vartheta}|_t \\
&\qquad \leq \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2) \int_0^T \mathrm{e}^{-rt} \big(\hat{\vartheta}_t - \Theta(X_t)\big)\vartheta_t \,\mathrm{d}t + \lambda \int_{[0,T]} \mathrm{e}^{-rt} \,\mathrm{d}|\vartheta|_t \\
&\qquad\quad + \mathrm{e}^{-rT} v(\hat{\vartheta}_T, X_T)\big(\vartheta_T - \hat{\vartheta}_T\big) \\
&\qquad\quad + \int_0^T \mathrm{e}^{-rt} \beta(X_t) v_x(\hat{\vartheta}_t, X_t)\big(\hat{\vartheta}_t - \vartheta_t\big) \,\mathrm{d}B_t.
\end{aligned}$$

Taking expectations, we obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\frac{1}{2}\sigma^2(\gamma_1+\gamma_2)\int_0^{T\wedge\tau_n} \mathrm{e}^{-rt}\big(\hat{\vartheta}_t-\Theta(X_t)\big)\hat{\vartheta}_t\,\mathrm{d}t+\lambda\int_{[0,T\wedge\tau_n]}\mathrm{e}^{-rt}\,\mathrm{d}|\hat{\vartheta}|_t\bigg]\\
&\qquad\le \mathbb{E}\bigg[\frac{1}{2}\sigma^2(\gamma_1+\gamma_2)\int_0^{T\wedge\tau_n} \mathrm{e}^{-rt}\big(\hat{\vartheta}_t-\Theta(X_t)\big)\vartheta_t\,\mathrm{d}t+\lambda\int_{[0,T\wedge\tau_n]}\mathrm{e}^{-rt}\,\mathrm{d}|\vartheta|_t\bigg]\\
&\qquad\quad+\mathbb{E}\Big[\mathrm{e}^{-r(T\wedge\tau_n)}v(\hat{\vartheta}_{T\wedge\tau_n},X_{T\wedge\tau_n})\big(\vartheta_{T\wedge\tau_n}-\hat{\vartheta}_{T\wedge\tau_n}\big)\Big],
\end{aligned}\tag{3.27}$$

where (τ_n) is a localising sequence of $(\mathcal{F}_t)$-stopping times for the stochastic integral with respect to the Brownian motion B. For each n, (3.22) implies that

$$\Big|\mathrm{e}^{-r(T\wedge\tau_n)}v(\hat{\vartheta}_{T\wedge\tau_n},X_{T\wedge\tau_n})\big(\vartheta_{T\wedge\tau_n}-\hat{\vartheta}_{T\wedge\tau_n}\big)\Big|\le\lambda\big(|\hat{\vartheta}|_T+|\vartheta|_T\big).$$

Since $\vartheta,\hat{\vartheta}\in\mathcal{A}(\theta)$ satisfy (3.17), we can pass to the limit as $n\to\infty$ using the dominated convergence theorem to obtain

$$\begin{aligned}
&\lim_{n\to\infty}\mathbb{E}\Big[\mathrm{e}^{-r(T\wedge\tau_n)}v(\hat{\vartheta}_{T\wedge\tau_n},X_{T\wedge\tau_n})\big(\vartheta_{T\wedge\tau_n}-\hat{\vartheta}_{T\wedge\tau_n}\big)\Big]\\
&=\mathbb{E}\Big[\mathrm{e}^{-rT}v(\hat{\vartheta}_T,X_T)\big(\vartheta_T-\hat{\vartheta}_T\big)\Big].
\end{aligned}$$

Moreover, by (3.22),

$$\begin{aligned}
\Big|\mathbb{E}\Big[\mathrm{e}^{-rT}v(\hat{\vartheta}_T,X_T)\big(\vartheta_T-\hat{\vartheta}_T\big)\Big]\Big| &\le\lambda\mathrm{e}^{-rT}\,\mathbb{E}\Big[\big|\hat{\vartheta}_T-\vartheta_T\big|\Big]\\
&\le\lambda\mathrm{e}^{-rT}\Big(1+\mathbb{E}\Big[\big(\hat{\vartheta}_T-\vartheta_T\big)^2\Big]\Big)\\
&\le\lambda\mathrm{e}^{-rT}\Big(1+2\,\mathbb{E}\big[\hat{\vartheta}_T^2+\vartheta_T^2\big]\Big)\quad\text{for all } T>0.
\end{aligned}$$

By the transversality condition (3.19), there exists a sequence (T_n) such that $T_n\uparrow\infty$ as $n\to\infty$ and

$$\liminf_{n\to\infty}\mathrm{e}^{-rT_n}\,\mathbb{E}\big[\hat{\vartheta}_{T_n}^2+\vartheta_{T_n}^2\big]=\lim_{n\to\infty}\mathrm{e}^{-rT_n}\,\mathbb{E}\big[\hat{\vartheta}_{T_n}^2+\vartheta_{T_n}^2\big]=0.$$

Moreover, for each n,

$$\begin{aligned}\left|\mathbb{E}\Big[\mathrm{e}^{-rT_n}v(\hat{\vartheta}_{T_n},X_{T_n})\big(\vartheta_{T_n}-\hat{\vartheta}_{T_n}\big)\Big]\right| \leq& \lambda\mathrm{e}^{-rT_n}\Big(1+2\,\mathbb{E}\big[\hat{\vartheta}^2_{T_n}+\vartheta^2_{T_n}\big]\Big)\\ &\to 0 \quad \text{as } n\to\infty,\end{aligned}$$

which implies that

$$\liminf_{T\uparrow\infty}\mathbb{E}\Big[\mathrm{e}^{-rT}v(\hat{\vartheta}_T,X_T)\big(\vartheta_T-\hat{\vartheta}_T\big)\Big]=0.$$

Next, for any $\vartheta\in\mathcal{A}(\theta)$, by monotone convergence we may let $n\to\infty$ and then $T\uparrow\infty$ to obtain

$$\lim_{n\to\infty,T\uparrow\infty}\mathbb{E}\left[\int_{[0,T\wedge\tau_n]}\mathrm{e}^{-rt}\,\mathrm{d}|\vartheta|_t\right]=\mathbb{E}\left[\int_{[0,\infty[}\mathrm{e}^{-rt}\,\mathrm{d}|\vartheta|_t\right].$$

Finally, for any $\vartheta\in\mathcal{A}(\theta)$,

$$\left|\int_0^{T\wedge\tau_n}\mathrm{e}^{-rt}\big(\hat{\vartheta}_t-\Theta(X_t)\big)\vartheta_t\,\mathrm{d}t\right|\leq\frac{1}{2}\int_0^\infty\mathrm{e}^{-rt}\big(\hat{\vartheta}^2_t+\Theta^2(X_t)+2\vartheta^2_t\big)\mathrm{d}t,$$

and, by (3.16) and (3.17),

$$\mathbb{E}\left[\int_0^\infty\mathrm{e}^{-rt}\big(\hat{\vartheta}^2_t+\Theta^2(X_t)+2\vartheta^2_t\big)\mathrm{d}t\right]<\infty.$$

Therefore, by dominated convergence, we may let $n\to\infty$ and then $T\uparrow\infty$ to obtain

$$\lim_{n\to\infty,T\uparrow\infty}\mathbb{E}\left[\int_0^{T\wedge\tau_n}\mathrm{e}^{-rt}\big(\hat{\vartheta}_t-\Theta(X_t)\big)\vartheta_t\,\mathrm{d}t\right]=\mathbb{E}\left[\int_0^\infty\mathrm{e}^{-rt}\big(\hat{\vartheta}_t-\Theta(X_t)\big)\vartheta_t\,\mathrm{d}t\right].$$

Therefore, by passing to the limit as $n\to\infty$ and then to the limit inferior as $T\uparrow\infty$ in (3.27), we obtain (3.21).

Finally, we use (3.26) and perform similar calculations as above to obtain $v(\theta,x)\theta=J_{\theta,x}(\hat{\vartheta},\hat{\vartheta})$, which is clearly real-valued by the fact that v is a solution to Problem 3.8. □

3.3 Solving the free-boundary Problem 3.8 when Θ is strictly increasing

We first derive explicit solutions to the free-boundary Problem 3.8 in the special cases that arise if the problem data is such that the function Θ providing the agent's frictionless optimiser is strictly increasing. Economically, this means that as the value of the process X increases, the optimal number of shares held by Agent 1 (Agent 2) in the frictionless setting increases (decreases). This situation can arise for example when the gradients of h_1 and h_2 are both positive and h_2 is steeper than h_1 at all values of x - economically, Agent 2's endowment rate is more sensitive to changes to the state of the economy, in all states of the economy. Moreover, we also consider the case where Θ has horizontal asymptotes (economically, there is a maximal or minimal number of shares in the frictionless optimal trading strategy).

3.3.1 The structure of the solution

We first postulate the structure of the solution. To this end, we define

$$\underline{\theta} = \lim_{x \downarrow -\infty} \Theta(x) \quad \text{and} \quad \overline{\theta} = \sup_{x \in \mathbb{R}} \Theta(x) = \lim_{x \uparrow \infty} \Theta(x), \tag{3.28}$$

We postulate that the continuation region $\mathcal{C}$ is characterised by points $\utilde{\theta} \leq \widetilde{\theta}$ in $\left[\underline{\theta}, \overline{\theta}\right]$, $\hat{\theta} \leq \underline{\theta}$ and $\check{\theta} \geq \overline{\theta}$ such that

$$\hat{\theta} \begin{cases} < \underline{\theta}, & \text{if } \underline{\theta} > -\infty, \\ = -\infty, & \text{if } \underline{\theta} = -\infty, \end{cases} \qquad \utilde{\theta} \begin{cases} > \underline{\theta}, & \text{if } \underline{\theta} > -\infty, \\ = -\infty, & \text{if } \underline{\theta} = -\infty, \end{cases}$$

$$\widetilde{\theta} \begin{cases} < \overline{\theta}, & \text{if } \underline{\theta} < \infty, \\ = \infty, & \text{if } \underline{\theta} = \infty, \end{cases} \quad \text{and} \quad \check{\theta} \begin{cases} > \overline{\theta}, & \text{if } \underline{\theta} < \infty, \\ = \infty, & \text{if } \underline{\theta} = \infty, \end{cases}$$

as well as by strictly increasing continuous functions $\mathfrak{F} : \left]\utilde{\theta}, \check{\theta}\right[\to \mathbb{R}$ and $\mathfrak{G} : \left]\hat{\theta}, \widetilde{\theta}\right[\to \mathbb{R}$ such that

$$\mathfrak{F}(\theta) < \Theta^{-1}(\theta) < \mathfrak{G}(\theta) \text{ for all } \theta \in \left]\utilde{\theta}, \widetilde{\theta}\right[, \ \lim_{\theta \downarrow \utilde{\theta}} \mathfrak{F}(\theta) = -\infty \text{ and } \lim_{\theta \uparrow \widetilde{\theta}} \mathfrak{G}(\theta) = \infty.$$

In particular, we will show that $\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_m \cup \mathcal{C}_h$, where

$$\begin{aligned} \mathcal{C}_\ell &= \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \hat{\theta} < \theta \leq \utilde{\theta} \text{ and } x < \mathfrak{G}(\theta)\big\}, \\ \mathcal{C}_m &= \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \utilde{\theta} < \theta < \widetilde{\theta} \text{ and } \mathfrak{F}(\theta) < x < \mathfrak{G}(\theta)\big\} \\ \text{and} \quad \mathcal{C}_h &= \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \widetilde{\theta} \leq \theta < \check{\theta} \text{ and } \mathfrak{F}(\theta) < x\big\}, \end{aligned}$$

while the selling and buying regions are respectively given by

$$\begin{aligned} \mathcal{S} &= \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \check{\theta} \leq \theta\big\} \cup \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \utilde{\theta} < \theta < \check{\theta} \text{ and } x \leq \mathfrak{F}(\theta)\big\} \\ \text{and} \quad \mathcal{B} &= \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \theta \leq \hat{\theta}\big\} \cup \big\{(\theta, x) \in \mathbb{R}^2 \mid \ \hat{\theta} < \theta < \widetilde{\theta} \text{ and } \mathfrak{G}(\theta) \leq x\big\}. \end{aligned}$$

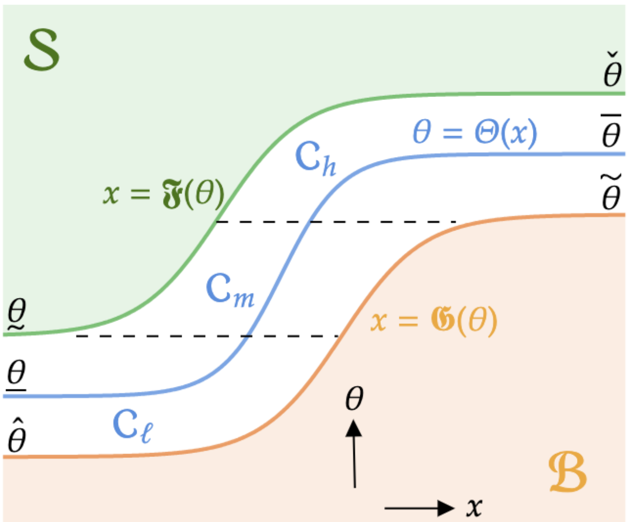

Figure 1: Regions for Θ strictly increasing

Figure 1 provides an illustration of the three regions arising in the context of this case. After an initial jump such that $(\vartheta_0, x) \in \operatorname{cl}\mathcal{C}$, the frictional optimiser ϑ reflects in the negative θ-direction whenever $X_t = \mathfrak{F}(\vartheta_t)$ for some $t \geq 0$, and reflects in the positive θ-direction whenever $X_t = \mathfrak{G}(\vartheta_t)$ for some $t \geq 0$, so that the joint process (ϑ, X) remains in $\operatorname{cl}\mathcal{C}$ at all times. We will show that when $\overline{\theta} < \infty$ and $\underline{\theta} > -\infty$ (when Θ is bounded), there will be one-sided boundaries (and hence one-sided optimal strategies) in $\mathcal{C}_\ell \cup \mathcal{C}_h$ and two-sided boundaries (with two-sided optimal strategies) in $\mathcal{C}_m$. Moreover, once the joint process (ϑ, X) reaches $\mathcal{C}_m$, it will remain there indefinitely. We will also show that when $\overline{\theta} = \infty$ and $\underline{\theta} = -\infty$, there will only be the region $\mathcal{C}_m$, in other words, only two-sided boundaries. If $\overline{\theta} = \infty$ and $\underline{\theta} > -\infty$ ($\overline{\theta} < \infty$ and $\underline{\theta} = -\infty$), then we will have the continuation region $\mathcal{C}_\ell \cup \mathcal{C}_m$ ($\mathcal{C}_h \cup \mathcal{C}_m$). Moreover, when $\overline{\theta} < \infty$ and $\underline{\theta} > -\infty$ and the transaction cost parameter λ is large enough, we will only have one-sided boundaries with continuation region as $\mathcal{C}_\ell \cup \mathcal{C}_h$. In Gonon et. al [51], where Θ is linear and X is a standard one-dimensional Brownian motion, there is only the region $\mathcal{C}_m$ with linear $\mathfrak{F}$ and $\mathfrak{G}$. In our more general setting, we consider non-linear Θ and a one-dimensional diffusion X. In particular, we will observe that even when Θ is linear, $\mathfrak{F}$ and $\mathfrak{G}$ are non-linear when X is a one-dimensional diffusion.

To proceed further, we note that the general solution to the ODE (3.23) that v satisfies inside the continuation region $\mathcal{C}$ is given by

$$v(\theta, x) = A(\theta)\varphi(x) + B(\theta)\psi(x) - \frac{r}{\zeta} R_\Theta(x) + \frac{\theta}{\zeta}, \tag{3.29}$$

for some functions A and B, where φ, ψ are as in Appendix A, R_Θ is defined by (A.10) for $h = \Theta$ and

$$\zeta = \frac{2r}{\sigma^2(\gamma_1 + \gamma_2)}. \tag{3.30}$$

In view of the structures of the continuation region $\mathcal{C}$ that we have postulated above (see Figure 1), we are faced with the following three "building blocks" associated with the functions that determine the boundary of $\mathcal{C}$.

Case I. Suppose that there exist points $\underset{\sim}{\theta} < \widetilde{\theta}$ in $[-\infty, \infty]$ and strictly increasing functions $\mathfrak{F}, \mathfrak{G} : \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[\to \mathbb{R}$ such that

$$\begin{gathered} \mathfrak{F}(\theta) < \mathfrak{G}(\theta) \quad \text{for all } \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[, \\ \left\{(\theta, x) \mid \ \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[\text{ and } x \leq \mathfrak{F}(\theta)\right\} \subseteq \mathcal{S}, \\ \left\{(\theta, x) \mid \ \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[\text{ and } \mathfrak{F}(\theta) < x < \mathfrak{G}(\theta)\right\} \subseteq \mathcal{C} \\ \text{and} \quad \left\{(\theta, x) \mid \ \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[\text{ and } \mathfrak{G}(\theta) \leq x\right\} \subseteq \mathcal{B}. \end{gathered}$$

In this context, the requirement that $v(\cdot, x)$ should be C^1 for all $x \in \mathbb{R}$ suggests the free-boundary conditions

$$v\big(\theta, \mathfrak{F}(\theta)\big) = \lambda, \quad v\big(\theta, \mathfrak{G}(\theta)\big) = -\lambda,$$
$$v_\theta\big(\theta, \mathfrak{F}(\theta)\big) = 0 \quad \text{and} \quad v_\theta\big(\theta, \mathfrak{G}(\theta)\big) = 0,$$

which, combined with (3.29), give rise to the system of equations

$$\begin{aligned}
A(\theta)\varphi\big(\mathfrak{F}(\theta)\big) + B(\theta)\psi\big(\mathfrak{F}(\theta)\big) - \frac{r}{\zeta}R_\Theta\big(\mathfrak{F}(\theta)\big) + \frac{\theta}{\zeta} &= \lambda,\\
A(\theta)\varphi\big(\mathfrak{G}(\theta)\big) + B(\theta)\psi\big(\mathfrak{G}(\theta)\big) - \frac{r}{\zeta}R_\Theta\big(\mathfrak{G}(\theta)\big) + \frac{\theta}{\zeta} &= -\lambda,\\
A'(\theta)\varphi\big(\mathfrak{F}(\theta)\big) + B'(\theta)\psi\big(\mathfrak{F}(\theta)\big) + \frac{1}{\zeta} &= 0\\
\text{and} \quad A'(\theta)\varphi\big(\mathfrak{G}(\theta)\big) + B'(\theta)\psi\big(\mathfrak{G}(\theta)\big) + \frac{1}{\zeta} &= 0.
\end{aligned}$$

Differentiating the first two of these equations with respect to θ and using the last two of these equations to eliminate $A'(\theta)$ and $B'(\theta)$, we obtain

$$\begin{aligned}
A(\theta)\varphi'\big(\mathfrak{F}(\theta)\big) + B(\theta)\psi'\big(\mathfrak{F}(\theta)\big) &= \frac{r}{\zeta}R'_\Theta\big(\mathfrak{F}(\theta)\big)\\
\text{and} \quad A(\theta)\varphi'\big(\mathfrak{G}(\theta)\big) + B(\theta)\psi'\big(\mathfrak{G}(\theta)\big) &= \frac{r}{\zeta}R'_\Theta\big(\mathfrak{G}(\theta)\big)
\end{aligned}$$

because we have made the ansatz that $\mathfrak{F}$ and $\mathfrak{G}$ are strictly increasing. In view of (A.10) in Appendix A, it follows that

$$\begin{aligned}
A(\theta) &= r\lambda \int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)\,\mathrm{d}y - \frac{r}{\zeta}\int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y\\
&= -r\lambda \int_{-\infty}^{\mathfrak{G}(\theta)} \Psi(y)\,\mathrm{d}y - \frac{r}{\zeta}\int_{-\infty}^{\mathfrak{G}(\theta)} \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y\\
\text{and} \quad B(\theta) &= r\lambda \int_{\mathfrak{F}(\theta)}^{\infty} \Phi(y)\,\mathrm{d}y - \frac{r}{\zeta}\int_{\mathfrak{F}(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y\\
&= -r\lambda \int_{\mathfrak{G}(\theta)}^{\infty} \Phi(y)\,\mathrm{d}y - \frac{r}{\zeta}\int_{\mathfrak{G}(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y,
\end{aligned}$$

where

$$\Phi(x) = \frac{2\varphi(x)}{C\beta^2(x)p'(x)} \quad \text{and} \quad \Psi(x) = \frac{2\psi(x)}{C\beta^2(x)p'(x)}.$$

These expressions imply the system of equations

$$\mathcal{H}_1\big(\theta, \mathfrak{F}(\theta), \mathfrak{G}(\theta), \lambda\big) = 0 \quad \text{and} \quad \mathcal{H}_2\big(\theta, \mathfrak{F}(\theta), \mathfrak{G}(\theta), \lambda\big) = 0, \tag{3.31}$$

for the free-boundary points $\mathfrak{F}(\theta)$ and $\mathfrak{G}(\theta)$, where

$$\begin{aligned} \mathcal{H}_1(\theta, F, G, \lambda) = {} & \int_F^G \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y \\ & + \zeta\lambda\bigg(\int_{-\infty}^F \Psi(y)\,\mathrm{d}y + \int_{-\infty}^G \Psi(y)\,\mathrm{d}y\bigg) \qquad (3.32) \\ \text{and} \quad \mathcal{H}_2(\theta, F, G, \lambda) = {} & \int_F^G \Phi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y \\ & - \zeta\lambda\bigg(\int_F^\infty \Phi(y)\,\mathrm{d}y + \int_G^\infty \Phi(y)\,\mathrm{d}y\bigg), \qquad (3.33) \end{aligned}$$

as well as the expressions

$$\begin{aligned} A(\theta) &= -\frac{r}{\zeta}\int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y \qquad (3.34) \\ \text{and} \quad B(\theta) &= -\frac{r}{\zeta}\int_{\mathfrak{F}(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y. \qquad (3.35) \end{aligned}$$

Case II. Suppose that there exist points $\hat{\theta} < \underset{\sim}{\theta}$ in $[-\infty, \infty[$ and a strictly increasing function $\mathfrak{G} : \big]\hat{\theta}, \underset{\sim}{\theta}\big[\to \mathbb{R}$ such that

$$\begin{aligned} & \big\{(\theta, x) \mid \ \theta \in \big]\hat{\theta}, \underset{\sim}{\theta}\big[\text{ and } x < \mathfrak{G}(\theta)\big\} \subseteq \mathcal{C} \\ \text{and} \quad & \big\{(\theta, x) \mid \ \theta \in \big]\hat{\theta}, \underset{\sim}{\theta}\big[\text{ and } \mathfrak{G}(\theta) \le x\big\} \subseteq \mathcal{B}. \end{aligned}$$

In this context, we must have $A(\theta) = 0$ in (3.29) because, otherwise, (3.22) cannot be satisfied. Furthermore, the requirement that $v(\cdot, x)$ should be C^1 for all $x \in \mathbb{R}$ suggests the free-boundary conditions

$$v\big(\theta, \mathfrak{G}(\theta)\big) = -\lambda \quad \text{and} \quad v_\theta\big(\theta, \mathfrak{G}(\theta)\big) = 0,$$

which give rise to the system of equations

$$B(\theta)\psi\big(\mathfrak{G}(\theta)\big) - \frac{r}{\zeta} R_\Theta\big(\mathfrak{G}(\theta)\big) + \frac{\theta}{\zeta} = -\lambda \quad \text{and} \quad B'(\theta)\psi\big(\mathfrak{G}(\theta)\big) + \frac{1}{\zeta} = 0.$$

Making calculations that are similar to the ones of the previous cases, we conclude this case with the algebraic equation

$$\underline{\mathcal{H}}\big(\theta, \mathfrak{G}(\theta), \lambda\big) = 0 \tag{3.36}$$

for the free-boundary point $\mathfrak{G}(\theta)$, where

$$\underline{\mathcal{H}}\big(\theta, G, \lambda\big) = \int_{-\infty}^{G} \Psi(y)\big(\theta - \Theta(y) + \zeta\lambda\big)\,\mathrm{d}y, \tag{3.37}$$

as well as the expressions

$$A(\theta) = 0 \quad \text{and} \quad B(\theta) = -\frac{r}{\zeta}\int_{\mathfrak{G}(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y) + \zeta\lambda\big)\,\mathrm{d}y. \tag{3.38}$$

Case III. Suppose that there exist points $\widetilde{\theta} < \check{\theta}$ in $]-\infty, \infty]$ and a strictly increasing function $\mathfrak{F} : \big]\widetilde{\theta}, \check{\theta}\big[\to \mathbb{R}$ such that

$$\begin{aligned} &\big\{(\theta, x) \mid \ \theta \in \big]\widetilde{\theta}, \check{\theta}\big[\text{ and } x \leq \mathfrak{F}(\theta)\big\} \subseteq \mathcal{S} \\ \text{and} \quad &\big\{(\theta, x) \mid \ \theta \in \big]\widetilde{\theta}, \check{\theta}\big[\text{ and } \mathfrak{F}(\theta) < x\big\} \subseteq \mathcal{C}. \end{aligned}$$

The same arguments and calculations as in the previous case give rise to the equation

$$\overline{\mathcal{H}}\big(\theta, \mathfrak{F}(\theta), \lambda\big) = 0, \tag{3.39}$$

where

$$\overline{\mathcal{H}}\big(\theta, F, \lambda\big) = \int_{F}^{\infty} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y, \tag{3.40}$$

and the expressions

$$A(\theta) = -\frac{r}{\zeta}\int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y \quad \text{and} \quad B(\theta) = 0. \tag{3.41}$$

3.3.2 The functions determining the boundary of the continuation region

We now derive the solution to the free-boundary equations arising in Cases I-III that we considered in Section 3.3.1. We first study the solvability of the system of equations (3.31) in the context of Case I in Section 3.3.1, as illustrated in Figure 2.

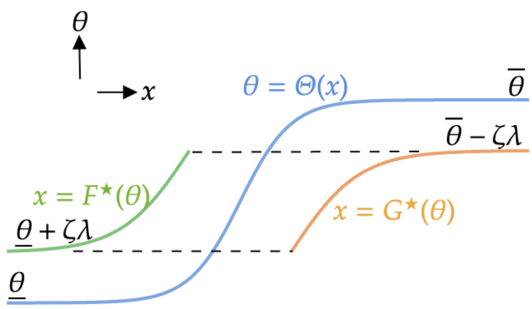

Figure 2: Case I (Θ strictly increasing)

Lemma 3.10. *Suppose that the function Θ is strictly increasing and recall the definitions of $\underline{\theta}$, $\overline{\theta}$ in (3.28). Consider the system of equations (3.31), where $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by (3.32) and (3.33). Then, there exist continuous functions $F^{\star}, G^{\star} : \mathscr{D} \to \mathbb{R}$, where*

$$\mathscr{D} = \left\{ (\theta, \lambda) \in \mathbb{R} \times \mathbb{R}_+ \mid \ \underline{\theta} + \zeta\lambda < \theta < \overline{\theta} - \zeta\lambda \quad \textit{and} \quad 0 < \lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta}) \right\},$$

such that

$$F^{\star}(\theta,\lambda) < \Theta^{-1}(\theta - \zeta\lambda) < \Theta^{-1}(\theta + \zeta\lambda) < G^{\star}(\theta,\lambda) \tag{3.42}$$

$$\textit{and} \quad \mathcal{H}_1\big(\theta, F^{\star}(\theta,\lambda), G^{\star}(\theta,\lambda), \lambda\big) = \mathcal{H}_2\big(\theta, F^{\star}(\theta,\lambda), G^{\star}(\theta,\lambda), \lambda\big) = 0. \tag{3.43}$$

Furthermore, given any $\lambda \in \left]0, \frac{\overline{\theta} - \underline{\theta}}{2\zeta}\right[$,

$$F^{\star}_{\theta}(\theta,\lambda) > 0, \quad G^{\star}_{\theta}(\theta,\lambda) > 0, \quad \lim_{\theta \downarrow \underline{\theta} + \zeta\lambda} F^{\star}(\theta,\lambda) = -\infty, \quad \textit{and} \quad \lim_{\theta \uparrow \overline{\theta} - \zeta\lambda} G^{\star}(\theta,\lambda) = \infty. \tag{3.44}$$

Moreover, if $\underline{\theta} = -\infty$*, then*

$$\lim_{\theta \downarrow -\infty} G^\star(\theta, \lambda) = -\infty, \tag{3.45}$$

and if $\overline{\theta} = \infty$*, then*

$$\lim_{\theta \uparrow \infty} F^\star(\theta, \lambda) = \infty, \tag{3.46}$$

Given any $\theta \in \left]\underline{\theta}, \overline{\theta}\right[$*,*

$$F^\star_\lambda(\theta, \lambda) < 0, \quad G^\star_\lambda(\theta, \lambda) > 0 \quad \textit{and} \quad \lim_{\lambda \downarrow 0} F^\star(\theta, \lambda) = \lim_{\lambda \downarrow 0} G^\star(\theta, \lambda) = \Theta^{-1}(\theta). \tag{3.47}$$

Proof. We organise the proof in five steps.

Step 1: Solvability of the equation $\mathcal{H}_1(\theta, F, G, \lambda) = 0$ for $G > \Theta^{-1}(\theta)$. Fix any $\theta \in \left]\underline{\theta}, \overline{\theta}\right[$, $0 < \lambda < \frac{1}{2\zeta}\big(\overline{\theta} - \underline{\theta}\big)$ and $F \in \left[-\infty, \Theta^{-1}(\theta)\right]$. Recalling that $\Psi > 0$, we calculate

$$\mathcal{H}_1(\theta, F, F, \lambda) = 2\zeta\lambda \int_{-\infty}^{F} \Psi(y)\, \mathrm{d}y \begin{cases} > 0, & \text{if } F > -\infty, \\ = 0, & \text{if } F = -\infty, \end{cases}$$

$$\text{and} \quad \frac{\partial \mathcal{H}_1}{\partial G}(\theta, F, G, \lambda) = \Psi(G)\big(\theta - \Theta(G) + \zeta\lambda\big) \begin{cases} > 0, & \text{if } \Theta(G) < \theta + \zeta\lambda, \\ < 0, & \text{if } \theta + \zeta\lambda < \Theta(G). \end{cases}$$

These calculations imply that the equation $\mathcal{H}_1(\theta, F, G, \lambda) = 0$ for $G > F$ has a unique solution if and only if

$$\lim_{G \uparrow \infty} \mathcal{H}_1(\theta, F, G, \lambda) < 0, \tag{3.48}$$

in which case, $G \in \left]\Theta^{-1}(\theta + \zeta\lambda), \infty\right[$.

We are now faced with the following two cases.

<u>*Case 1:* $\overline{\theta} = \infty$.</u> Lemma A.1 implies that

$$\lim_{G \uparrow \infty} \mathcal{H}_1(\theta, F, G, \lambda) = -\infty \quad \text{for all } \lambda > 0.$$

It follows that (3.48) is satisfied for all $F \leq \Theta^{-1}(\theta)$ and $\lambda > 0$. Furthermore, the equation $\mathcal{H}_1(\theta, F, G, \lambda) = 0$ defines uniquely a continuous function $\mathfrak{g} : \mathscr{D}_1 \to \mathbb{R}$ such that

$$F \leq \Theta^{-1}(\theta) < \Theta^{-1}(\theta + \zeta\lambda) < \mathfrak{g}(\theta, F, \lambda) \quad \text{and} \quad \mathcal{H}_1\big(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda\big) = 0 \tag{3.49}$$

for all $(\theta, F, \lambda) \in \mathscr{D}_1$, where

$$\mathscr{D}_1 = \big\{(\theta, F, \lambda) \in \mathbb{R}^3 \mid \ \theta \in \big]\underline{\theta}, \infty\big[, \ F \in \big[-\infty, \Theta^{-1}(\theta)\big] \text{ and } \lambda > 0\big\}. \tag{3.50}$$

For later reference, we stress that

$$\mathfrak{g}(\theta, -\infty, \lambda) < \infty \quad \text{for all } \theta \in \big]\underline{\theta}, \infty\big[\text{ and } \lambda > 0. \tag{3.51}$$

<u>*Case 2:* $\overline{\theta} < \infty$.</u> In this case, for $\theta < \overline{\theta} - \zeta\lambda$, Lemma A.1 implies that

$$\lim_{G \uparrow \infty} \mathcal{H}_1(\theta, F, G, \lambda) = -\infty \quad \text{for all } 0 < \lambda < \frac{\overline{\theta} - \theta}{\zeta}.$$

It follows that the equation $\mathcal{H}_1(\theta, F, G, \lambda) = 0$ defines uniquely a continuous function $\mathfrak{g} : \mathscr{D}_2 \to \mathbb{R}$ such that

$$F \leq \Theta^{-1}(\theta) < \Theta^{-1}(\theta + \zeta\lambda) < \mathfrak{g}(\theta, F, \lambda) \quad \text{and} \quad \mathcal{H}_1\big(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda\big) = 0 \tag{3.52}$$

for all $(\theta, F, \lambda) \in \mathscr{D}_2$, where

$$\begin{aligned}\mathscr{D}_2 = \Bigg\{&(\theta, F, \lambda) \in \mathbb{R}^3 \mid \ \theta \in \big]\underline{\theta}, \overline{\theta} - \zeta\lambda\big[, \ F \in \big[-\infty, \Theta^{-1}(\theta)\big] \\ &\text{and } 0 < \lambda < \frac{\overline{\theta} - \theta}{\zeta}\Bigg\}.\end{aligned} \tag{3.53}$$

For later reference, we stress that

$$\mathfrak{g}(\theta, -\infty, \lambda) < \infty \quad \text{for all } \theta \in \big]\underline{\theta}, \overline{\theta} - \zeta\lambda\big[\text{ and } 0 < \lambda < \frac{\overline{\theta} - \theta}{\zeta}. \tag{3.54}$$

Step 2: Common calculations for Cases 1 and 2.

Differentiation of the identity in (3.49) or (3.52) implies that

$$\mathfrak{g}_\theta(\theta,F,\lambda) = -\frac{\int_F^{\mathfrak{g}(\theta,F,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(\mathfrak{g}(\theta,F,\lambda)\big)\big(\theta-\Theta\big(\mathfrak{g}(\theta,F,\lambda)\big)+\zeta\lambda\big)}, \tag{3.55}$$

$$\mathfrak{g}_F(\theta,F,\lambda) = \frac{\Psi(F)\big(\theta-\Theta(F)-\zeta\lambda\big)}{\Psi\big(\mathfrak{g}(\theta,F,\lambda)\big)\big(\theta-\Theta\big(\mathfrak{g}(\theta,F,\lambda)\big)+\zeta\lambda\big)} \tag{3.56}$$

$$\text{and}\quad \mathfrak{g}_\lambda(\theta,F,\lambda) = -\zeta\frac{\int_{-\infty}^F \Psi(y)\,\mathrm{d}y + \int_{-\infty}^{\mathfrak{g}(\theta,F,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(\mathfrak{g}(\theta,F,\lambda)\big)\big(\theta-\Theta\big(\mathfrak{g}(\theta,F,\lambda)\big)+\zeta\lambda\big)}. \tag{3.57}$$

Furthermore, (3.56), the definition (3.33) of $\mathcal{H}_2$ and the definitions of Φ, Ψ in (A.8) imply that

$$\begin{aligned}\frac{\partial\mathcal{H}_2}{\partial F}\big(\theta,F,\mathfrak{g}(\theta,F,\lambda),\lambda\big) &= \Phi\big(\mathfrak{g}(\theta,F,\lambda)\big)\big(\theta-\Theta\big(\mathfrak{g}(\theta,F,\lambda)\big)+\zeta\lambda\big)\mathfrak{g}_F(\theta,F,\lambda)\\ &\quad -\Phi(F)\big(\theta-\Theta(F)-\zeta\lambda\big)\\ &= \Psi(F)\left(\frac{\varphi\big(\mathfrak{g}(\theta,F,\lambda)\big)}{\psi\big(\mathfrak{g}(\theta,F,\lambda)\big)}-\frac{\varphi(F)}{\psi(F)}\right)\big(\theta-\Theta(F)-\zeta\lambda\big).\end{aligned} \tag{3.58}$$

Step 3: Existence and uniqueness of $F^\star$ and $G^\star$ satisfying (3.42) and (3.43) in either Case 1 or Case 2 of Step 1.

Recalling the definition (3.50) of $\mathfrak{g}$'s domain $\mathscr{D}_1$ in Case 1 of Step 1 and the definition (3.53) of $\mathfrak{g}$'s domain $\mathscr{D}_2$ in Case 2 of Step 1, we fix any $\theta \in \left]\underline{\theta},\overline{\theta}-\zeta\lambda\right[$ (in Case 1 of Step 1, this is equivalent to fixing $\theta\in\left]\underline{\theta},\infty\right[$) and consider the equation $\mathcal{H}_2\big(\theta,F,\mathfrak{g}(\theta,F,\lambda),\lambda\big)=0$ for $F\le\Theta^{-1}(\theta)$. We first note that, since $\theta-\Theta(y)<0$ for all $y>\Theta^{-1}(\theta)$,

$$\begin{aligned}&\mathcal{H}_2\big(\theta,\Theta^{-1}(\theta),\mathfrak{g}\big(\theta,\Theta^{-1}(\theta),\lambda\big),\lambda\big)\\ &= \int_{\Theta^{-1}(\theta)}^{\mathfrak{g}(\theta,\Theta^{-1}(\theta),\lambda)} \Phi(y)\big(\theta-\Theta(y)\big)\,\mathrm{d}y\\ &\quad -\zeta\lambda\left(\int_{\Theta^{-1}(\theta)}^{\infty}\Phi(y)\,\mathrm{d}y+\int_{\mathfrak{g}(\theta,\Theta^{-1}(\theta),\lambda)}^{\infty}\Phi(y)\,\mathrm{d}y\right)<0.\end{aligned}$$

Furthermore, we use (3.58), the inequalities in (3.49) and the fact that φ/ψ is strictly decreasing to see that

$$\frac{\partial \mathcal{H}_2}{\partial F}\big(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda\big) \begin{cases} < 0, & \text{if } F < \Theta^{-1}(\theta - \zeta\lambda), \\ > 0, & \text{otherwise.} \end{cases} \tag{3.59}$$

These calculations imply that the equation $\mathcal{H}_2(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda) = 0$ for F has a unique solution if and only if

$$\lim_{F \downarrow -\infty} \mathcal{H}_2\big(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda\big) > 0, \tag{3.60}$$

in which case, $F \in \big]-\infty, \Theta^{-1}(\theta - \zeta\lambda)\big[$.

We are now faced with the following two cases.

<u>*Case 1:* $\underline{\theta} = -\infty$.</u> Lemma A.1 and (3.51) imply that

$$\lim_{F \downarrow \infty} \mathcal{H}_2(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda) = \infty \quad \text{for all } \lambda > 0.$$

It follows that (3.60) is satisfied for all $\theta \in \left]-\infty, \overline{\theta} - \zeta\lambda\right[$ and $\lambda > 0$.

<u>*Case 2:* $\underline{\theta} > -\infty$.</u> Lemma A.1 and (3.54) imply that, for all $\theta > \underline{\theta} + \zeta\lambda$,

$$\begin{aligned} &\lim_{F \downarrow -\infty} \mathcal{H}_2\big(\theta, F, \mathfrak{g}(\theta, F, \lambda), \lambda\big) \\ &= \lim_{F \downarrow -\infty} \int_F^{\mathfrak{g}(\theta, -\infty, \lambda)} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\, \mathrm{d}y - 2\zeta\lambda \int_{\mathfrak{g}(\theta, -\infty, \lambda)}^{\infty} \Phi(y)\, \mathrm{d}y = \infty. \end{aligned} \tag{3.61}$$

It follows that there exists a unique $F^\star(\theta, \lambda) \in \big]-\infty, \Theta^{-1}(\theta)\big[$ such that

$$\mathcal{H}_2\big(\theta, F^\star(\theta, \lambda), \mathfrak{g}\big(\theta, F^\star(\theta, \lambda), \lambda\big), \lambda\big) = 0 \tag{3.62}$$

for all

$$\theta \in \big]\underline{\theta} + \zeta\lambda, \overline{\theta} - \zeta\lambda\big[.$$

If we define $G^\star(\theta, \lambda) = \mathfrak{g}\big(\theta, F^\star(\theta, \lambda), \lambda\big)$, then (3.49), (3.52) and (3.62) imply that the functions $F^\star$ and $G^\star$ satisfy (3.42) and (3.43). Finally, we note that $\underline{\theta} + \zeta\lambda < \overline{\theta} - \zeta\lambda$ if and only if $\lambda < \frac{1}{2\zeta}\big(\overline{\theta} - \underline{\theta}\big)$.

Step 4: Proof of the monotonicity of the functions $F^\star$ and $G^\star$.
Differentiating the identity in (3.62) with respect to θ and λ, and using (3.55), (3.56) and (3.57), we obtain

$$\begin{aligned}
F^\star_\theta(\theta,\lambda) &= \frac{\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Phi(y)\,\mathrm{d}y - \frac{\varphi(G^\star(\theta,\lambda))}{\psi(G^\star(\theta,\lambda))}\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(F^\star(\theta,\lambda)\big)\Big(\frac{\varphi(F^\star(\theta,\lambda))}{\psi(F^\star(\theta,\lambda))} - \frac{\varphi(G^\star(\theta,\lambda))}{\psi(G^\star(\theta,\lambda))}\Big)\big(\theta - \Theta\big(F^\star(\theta,\lambda)\big) - \zeta\lambda\big)} > 0,\\
G^\star_\theta(\theta,\lambda) &= \mathfrak{g}_F\big(\theta, F^\star(\theta,\lambda),\lambda\big) F^\star_\theta(\theta,\lambda) + \mathfrak{g}_\theta\big(\theta, F^\star(\theta,\lambda),\lambda\big)\\
&= \frac{\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Phi(y)\,\mathrm{d}y - \frac{\varphi(F^\star(\theta,\lambda))}{\psi(F^\star(\theta,\lambda))}\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(G^\star(\theta,\lambda)\big)\Big(\frac{\varphi(F^\star(\theta,\lambda))}{\psi(F^\star(\theta,\lambda))} - \frac{\varphi(G^\star(\theta,\lambda))}{\psi(G^\star(\theta,\lambda))}\Big)\big(\theta - \Theta\big(G^\star(\theta,\lambda)\big) + \zeta\lambda\big)} > 0,
\end{aligned}$$

as well as (writing $F^\star$ and $G^\star$ in place of $F^\star(\theta,\lambda)$ and $G^\star(\theta,\lambda)$ respectively for notational simplicity)

$$\begin{aligned}
F^\star_\lambda(\theta,\lambda) &= \zeta\frac{\int_{F^\star}^\infty \Phi(y)\,\mathrm{d}y + \int_{G^\star}^\infty \Phi(y)\,\mathrm{d}y + \frac{\varphi(G^\star)}{\psi(G^\star)}\Big(\int_{-\infty}^{F^\star}\Psi(y)\,\mathrm{d}y + \int_{-\infty}^{G^\star}\Psi(y)\,\mathrm{d}y\Big)}{\Psi\big(F^\star\big)\Big(\frac{\varphi(G^\star)}{\psi(G^\star)} - \frac{\varphi(F^\star)}{\psi(F^\star)}\Big)\big(\theta - \Theta\big(F^\star\big) - \zeta\lambda\big)}\\
&< 0
\end{aligned}$$

and

$$\begin{aligned}
G^\star_\lambda(\theta,\lambda) &= \mathfrak{g}_F\big(\theta, F^\star(\theta,\lambda),\lambda\big) F^\star_\lambda(\theta,\lambda) + \mathfrak{g}_\lambda\big(\theta, F^\star(\theta,\lambda),\lambda\big)\\
&= \zeta\frac{\int_{F^\star}^\infty \Phi(y)\,\mathrm{d}y + \int_{G^\star}^\infty \Phi(y)\,\mathrm{d}y + \frac{\varphi(F^\star)}{\psi(F^\star)}\Big(\int_{-\infty}^{F^\star}\Psi(y)\,\mathrm{d}y + \int_{-\infty}^{G^\star}\Psi(y)\,\mathrm{d}y\Big)}{\Psi\big(G^\star\big)\Big(\frac{\varphi(G^\star)}{\psi(G^\star)} - \frac{\varphi(F^\star)}{\psi(F^\star)}\Big)\big(\theta - \Theta\big(G^\star\big) + \zeta\lambda\big)}\\
&> 0,
\end{aligned}$$

where the inequalities follow from the facts that φ/ψ is strictly decreasing, $F^\star(\theta,\lambda) < \Theta^{-1}(\theta - \zeta\lambda)$ and $G^\star(\theta,\lambda) > \Theta^{-1}(\theta - \zeta\lambda)$. We have proven the inequalities in (3.44) and (3.47).

Step 5: Proof of the limits in (3.44), (3.45), (3.46) and (3.47).
To establish the limits in (3.44), we argue by contradiction. By the inequalities in (3.44), $F^\star$ and $G^\star$ are strictly increasing in θ, and therefore their limits when θ goes to $\overline{\theta} - \zeta\lambda$ exist. Assume for a contradiction that $G^\star(\theta,\lambda) \uparrow \overline{G} < \infty$ as $\theta \uparrow \overline{\theta} - \zeta\lambda$. Since $F^\star(\theta,\lambda) < G^\star(\theta,\lambda)$ for all θ, $F^\star(\theta,\lambda) \uparrow \overline{F} < \infty$ as well. We

first suppose that $\overline{\theta} = \infty$. Then, by (3.43) and the limit in (3.44),

$$\zeta\lambda = \lim_{\theta \uparrow \infty} \frac{\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y}{\int_{-\infty}^{F^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y + \int_{-\infty}^{G^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y} = \infty,$$

which is a contradiction. If instead $\overline{\theta} < \infty$, we obtain

$$\begin{aligned} &\lim_{\theta \uparrow \overline{\theta} - \zeta\lambda} \mathcal{H}_1(\theta, F^\star(\theta,\lambda), G^\star(\theta,\lambda), \lambda) \\ &= \int_{\overline{F}}^{\overline{G}} \Psi(y)\big(\overline{\theta} - \Theta(y)\big)\,\mathrm{d}y + 2\zeta\lambda \int_{-\infty}^{\overline{F}} \Psi(y)\,\mathrm{d}y > 0, \end{aligned}$$

which contradicts (3.43). The other limit in (3.44) is proved similarly. To establish (3.45), we argue by contradiction. Recalling that $G^\star$ is strictly increasing in θ, we suppose that $\underline{\theta} = -\infty$ and $G^\star(\theta,\lambda) \downarrow \underline{G} > -\infty$ as $\theta \downarrow -\infty$. Then, by (3.43) and the limit in (3.44),

$$\zeta\lambda = \lim_{\theta \downarrow -\infty} \frac{\int_{F^\star(\theta,\lambda)}^{G^\star(\theta,\lambda)} \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y}{\int_{-\infty}^{F^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y + \int_{-\infty}^{G^\star(\theta,\lambda)} \Psi(y)\,\mathrm{d}y} = -\infty,$$

which is a contradiction. Therefore, when $\underline{\theta} = -\infty$, $G^\star(\theta,\lambda) \downarrow -\infty$ as $\theta \downarrow -\infty$, and similar arguments establish (3.46) when $\overline{\theta} = \infty$.

To establish the limits in (3.47), we first show that the system of equations

$$\begin{aligned} \mathcal{H}_1(\theta, F, G, 0) &\equiv \int_F^G \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y \\ &= \mathcal{H}_2(\theta, F, G, 0) \equiv \int_F^G \Phi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y = 0 \end{aligned} \tag{3.63}$$

can be satisfied only if $F = G$. To prove this claim, we argue by contradiction and we assume that there exist $F < G$ that satisfy (3.63). Using the fact that ψ/φ is strictly increasing, we can see that

$$0 = \int_F^G \Psi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y < \frac{\psi\big(\Theta^{-1}(\theta)\big)}{\varphi\big(\Theta^{-1}(\theta)\big)} \int_F^G \Phi(y)\big(\theta - \Theta(y)\big)\,\mathrm{d}y = 0,$$

which is a contradiction. Combining the fact that the system of equations

(3.63) is satisfied by any choice $F = G$, and only such a choice, with (3.42) and the continuity of the functions $F^\star$, $G^\star$, we obtain the limits in (3.47). $\square$

We next study the solvability of equations (3.36) and (3.39) that arise in the context of Cases II and III of Section 3.3.1. To this end, we first make the following observation: if $\underline{\theta} > -\infty$ and $\overline{\theta} < \infty$, we observe that

$$\lim_{\lambda \uparrow \frac{1}{2\zeta}\left(\overline{\theta}-\underline{\theta}\right)} \underline{\theta} + \zeta\lambda = \lim_{\lambda \uparrow \frac{1}{2\zeta}\left(\overline{\theta}-\underline{\theta}\right)} \overline{\theta} - \zeta\lambda = \frac{1}{2}\left(\underline{\theta} + \overline{\theta}\right).$$

This implies that when $\underline{\theta} > -\infty$, $\overline{\theta} < \infty$ and $\lambda \geq \frac{1}{2\zeta}\left(\overline{\theta} - \underline{\theta}\right)$, there are no solutions to the system of equations (3.31) in the context of Case I, and we only have solutions to equations (3.36) and (3.39) that arise in the context of Cases II and III. On the other hand, if $\underline{\theta} = -\infty$ and $\overline{\theta} = \infty$, there are no solutions to equations (3.36) and (3.39) that arise in the context of Cases II and III, and there are only solutions to the system of equations (3.31) in the context of Case I. This is illustrated in Figures 3 and 4.

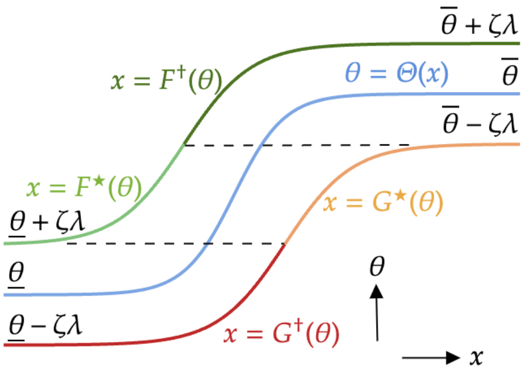

Figure 3: $0 < \lambda < \frac{\overline{\theta}-\underline{\theta}}{2\zeta}$

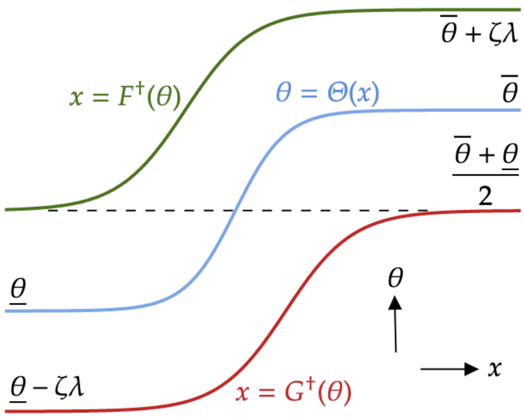

Figure 4: $\lambda \geq \frac{\overline{\theta}-\underline{\theta}}{2\zeta}$

In the next result, we study the solvability of equations (3.36) and (3.39).

Lemma 3.11. *The following statements hold true:*

(a) *Suppose that* $-\infty < \underline{\theta}$ *and let*

$$\underset{\sim}{\theta} = \begin{cases} \underline{\theta} + \zeta\lambda, & \textit{if } \lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta}), \\ \frac{1}{2}(\underline{\theta} + \overline{\theta}), & \textit{if } \lambda \geq \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta}), \end{cases}$$

Given any $\theta \in \left]\underline{\theta} - \zeta\lambda, \underset{\sim}{\theta}\right[$, there exists a unique point $G^\dagger(\theta, \lambda) \in \left]\Theta^{-1}(\theta), \infty\right[$ such that $\underline{\mathcal{H}}\big(\theta, G^\dagger(\theta, \lambda), \lambda\big) = 0$, where $\underline{\mathcal{H}}$ is defined by (3.37). Furthermore, if $\lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta})$, then $G^\dagger\big(\underset{\sim}{\theta}, \lambda\big) = G^\star\big(\underset{\sim}{\theta}, \lambda\big)$, where $G^\star$ is as defined in Lemma 3.10.

$$G^\dagger_\theta(\theta, \lambda) > 0, \quad G^\dagger_\lambda(\theta, \lambda) > 0 \quad \textit{and} \quad \lim_{\theta \downarrow \underline{\theta} - \zeta\lambda} G^\dagger(\theta, \lambda) = -\infty. \tag{3.64}$$

(b) *Suppose that $\overline{\theta} < \infty$ and let*

$$\widetilde{\theta} = \begin{cases} \overline{\theta} - \zeta\lambda, & \textit{if } \lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta}), \\ \frac{1}{2}(\underline{\theta} + \overline{\theta}), & \textit{if } \lambda \geq \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta}), \end{cases}$$

Given any $\theta \in \left]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right[$, there exists a unique point $F^\dagger(\theta, \lambda) \in \left]-\infty, \Theta^{-1}(\theta)\right[$ such that $\overline{\mathcal{H}}\big(\theta, F^\dagger(\theta, \lambda), \lambda\big) = 0$, where $\overline{\mathcal{H}}$ is defined by (3.40). Furthermore, if $\lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta})$, then $F^\dagger\big(\widetilde{\theta}, \lambda\big) = F^\star\big(\widetilde{\theta}, \lambda\big)$, where $F^\star$ is as defined in Lemma 3.10.

$$F^\dagger_\theta(\theta, \lambda) > 0, \quad F^\dagger_\lambda(\theta, \lambda) < 0 \quad \textit{and} \quad \lim_{\theta \uparrow \overline{\theta} + \zeta\lambda} F^\dagger(\theta, \lambda) = \infty. \tag{3.65}$$

Proof. In the context of (a), we can combine the observations that

$$\underline{\mathcal{H}}(\theta, \Theta^{-1}(\theta + \zeta\lambda), \lambda) = \int_{-\infty}^{\Theta^{-1}(\theta + \zeta\lambda)} \Psi(y)\big(\theta - \Theta(y) + \zeta\lambda\big)\,\mathrm{d}y > 0$$
$$\text{and} \quad \underline{\mathcal{H}}_G(\theta, G, \lambda) = \Psi(G)(\theta - \Theta(G) + \zeta\lambda) < 0 \quad \text{for all } G > \Theta^{-1}(\theta + \zeta\lambda)$$

with the limit

$$\lim_{G \uparrow \infty} \underline{\mathcal{H}}(\theta, G, \lambda) = -\infty \quad \text{for all } \lambda > 0 \quad \text{and} \quad \theta \in \left]\underline{\theta} - \zeta\lambda, \underset{\sim}{\theta}\right[,$$

which follows from Lemma A.1, to see that there exists a unique point $G^\dagger(\theta, \lambda) \in \left]\Theta^{-1}(\theta + \zeta\lambda), \infty\right[$ such that $\underline{\mathcal{H}}(\theta, G^\dagger(\theta, \lambda), \lambda) = 0$. The equality $G^\dagger\big(\underset{\sim}{\theta}, \lambda\big) = G^\star\big(\underset{\sim}{\theta}, \lambda\big)$ when $\lambda < \frac{1}{2\zeta}(\overline{\theta} - \underline{\theta})$ follows from the continuity of the functions $F^\star$ and $\mathcal{H}_1$ as well as the limits

$$\lim_{\theta \downarrow \underline{\theta} + \zeta\lambda} F^\star(\theta, \lambda) = -\infty \quad \text{and} \quad \lim_{F \downarrow -\infty} \mathcal{H}_1(\theta, F, G, \lambda) = \underline{\mathcal{H}}(\theta, G, \lambda).$$

Differentiating the equation $\underline{\mathcal{H}}(\theta, G^\dagger(\theta, \lambda), \lambda) = 0$ with respect to θ and λ, we obtain the inequalities in (3.64):

$$G^\dagger_\theta(\theta,\lambda) = -\frac{\int_{-\infty}^{G^\dagger(\theta,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(G^\dagger(\theta,\lambda)\big)\big(\theta - \Theta(G^\dagger(\theta,\lambda)) + \zeta\lambda\big)} > 0$$

$$\text{and} \quad G^\dagger_\lambda(\theta,\lambda) = -\frac{\zeta\int_{-\infty}^{G^\dagger(\theta,\lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(G^\dagger(\theta,\lambda)\big)\big(\theta - \Theta(G^\dagger(\theta,\lambda)) + \zeta\lambda\big)} > 0,$$

where the inequalities follows from the fact that $G^\dagger(\theta,\lambda) > \Theta^{-1}(\theta+\zeta\lambda)$. Moreover, taking the limit $\theta \downarrow \underline{\theta} - \zeta\lambda$, we obtain

$$\lim_{\theta\downarrow\underline{\theta}-\zeta\lambda} \underline{\mathcal{H}}(\theta, G^\dagger(\theta,\lambda),\lambda) = \lim_{\theta\downarrow\underline{\theta}-\zeta\lambda} \int_{-\infty}^{G^\dagger(\theta,\lambda)} \Psi(y)\big(\underline{\theta} - \Theta(y)\big)\,\mathrm{d}y = 0,$$

which implies that $\lim_{\theta\downarrow\bar{\theta}-\zeta\lambda} G^\dagger(\theta,\lambda) = -\infty$ by continuity of $\underline{\mathcal{H}}$. Finally, the proof of (b) follows symmetric arguments.

□

We conclude the section with growth estimates of the free-boundary functions in Lemma 3.10.

Lemma 3.12. *Fix $\lambda > 0$ and recall the free-boundary functions $F^\star$ and $G^\star$ in Lemma 3.10, as well as the constant $p \geq 1$ in Assumption 3.2.*

(a) *Suppose that $\overline{\theta} = \infty$. Then, there exists a constant $C_F > 0$ such that*

$$F^\star(\theta,\lambda) > C_F\theta^{\frac{1}{p}} \quad \textit{for all } \theta \gg 0. \tag{3.66}$$

(b) *Suppose that $\underline{\theta} = -\infty$. Then, there exists a constant $C_G > 0$ such that*

$$G^\star(\theta,\lambda) < -C_G(-\theta)^{\frac{1}{p}} \quad \textit{for all } \theta \ll 0. \tag{3.67}$$

Proof. In the context of (a), by the same arguments in the proof of Lemma 3.11, there exists a unique function $F^\sharp$ such that $\overline{\mathcal{H}}(\theta, F^\sharp(\theta,\lambda),\lambda) = 0$ for all $\theta \in \mathbb{R}$ and $F^\sharp(\theta,\lambda) < \Theta^{-1}(\theta - \zeta\lambda)$, where $\overline{\mathcal{H}}$ is defined by (3.40). By Lemma 3.10, $F^\star(\theta,\lambda) < \Theta^{-1}(\theta-\zeta\lambda)$ and

$$\overline{\mathcal{H}}(\theta, F^\star(\theta,\lambda),\lambda) = \int_{G^\star(\theta,\lambda)}^{\infty} \Phi(y)\big(\theta - \Theta(y) + \zeta\lambda\big)\,\mathrm{d}y < 0,$$

where the inequality follows from (3.42). This inequality combined with the fact that $\overline{\mathcal{H}}_F < 0$ for all $F < \Theta^{-1}(\theta - \zeta\lambda)$ imply that $F^\star(\theta,\lambda) > F^\sharp(\theta,\lambda)$. We therefore prove (3.66) by proving that $F^\sharp(\theta,\lambda) > C_F\theta^{\frac{1}{p}}$ for $\theta \gg 0$.

By (3.5), there exists a constant $\check{C} > 0$ such that $\Theta(x) \le \check{C}(1 + x^p)$ for $x \gg 0$. We first note that, by (3.6) and (3.7),

$$\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}|X_t|^p \,\mathrm{d}t\right] < \infty.$$

Therefore, by (A.10),

$$x^p \int_x^\infty \Phi(y)\,\mathrm{d}y \le \int_x^\infty \Phi(y)y^p\,\mathrm{d}y < \infty \quad \text{for all } \; x > 0,$$

which implies that

$$\lim_{x\uparrow\infty} x^p \int_x^\infty \Phi(y)\mathrm{d}y = 0.$$

By L'Hôpital's theorem, we calculate

$$\begin{aligned}\limsup_{x\uparrow\infty} \frac{\int_x^\infty \Phi(y)\Theta(y)\,\mathrm{d}y}{x^p\int_x^\infty \Phi(y)\,\mathrm{d}y} &\le \limsup_{x\uparrow\infty}\left(\frac{\check{C}}{x^p} + \frac{\check{C}\int_x^\infty \Phi(y)y^p\,\mathrm{d}y}{x^p\int_x^\infty \Phi(y)\,\mathrm{d}y}\right)\\ &= \check{C}\limsup_{x\uparrow\infty} \frac{\Phi(x)x^p}{\Phi(x)x^p + px^{p-1}\int_x^\infty \Phi(y)\,\mathrm{d}y} \le \check{C},\end{aligned}$$

where the final inequality follows from the strict positivity of Φ. This implies that

$$\begin{aligned}2\check{C}\left(F^\sharp(\theta,\lambda)\right)^p \int_{F^\sharp(\theta,\lambda)}^\infty \Phi(y)\,\mathrm{d}y &> \int_{F^\sharp(\theta,\lambda)}^\infty \Phi(y)\Theta(y)\,\mathrm{d}y\\ &= (\theta - \zeta\lambda)\int_{F^\sharp(\theta,\lambda)}^\infty \Phi(y)\,\mathrm{d}y,\end{aligned}$$

where the equality follows from the fact that $\overline{\mathcal{H}}(\theta, F^\sharp(\theta,\lambda),\lambda) = 0$. Therefore, for $\theta \gg 0$ sufficiently large, there exists a constant $\tilde{C} > 0$ such that

$$F^\sharp(\theta,\lambda) > \left(\frac{\theta - \zeta\lambda}{2\check{C}}\right)^{\frac{1}{p}} > \left(\frac{\tilde{C}\theta}{2\check{C}}\right)^{\frac{1}{p}},$$

and we obtain (3.66) with $C_F = \left(\frac{\tilde{C}}{2\check{C}}\right)^{\frac{1}{p}}$. The proof of (b) follows symmetric arguments. $\square$

3.3.3 The solution to the free-boundary problem

We now outline the solution to the free-boundary problem, having solved for the free-boundary functions. In view of Lemmas 3.10 and 3.11, the points $\underset{\sim}{\theta} < \widetilde{\theta} \in \left[\underline{\theta}, \overline{\theta}\right]$, $\hat{\theta} \leq \underline{\theta}$ and $\check{\theta} \geq \overline{\theta}$ considered in Section 3.3.1 are given by

$$\hat{\theta} = \begin{cases} -\infty, & \text{if } \underline{\theta} = -\infty, \\ \underline{\theta} - \zeta\lambda, & \text{if } \underline{\theta} > -\infty, \end{cases} \qquad \check{\theta} = \begin{cases} \infty, & \text{if } \underline{\theta} = \infty, \\ \overline{\theta} + \zeta\lambda, & \text{if } \underline{\theta} < \infty, \end{cases}$$

$$\widetilde{\theta} = \begin{cases} \infty, & \text{if } \underline{\theta} = \infty, \\ \overline{\theta} - \zeta\lambda, & \text{if } \underline{\theta} < \infty \text{ and } \lambda < \frac{\overline{\theta} - \underline{\theta}}{2\zeta}, \\ \frac{\overline{\theta} + \underline{\theta}}{2}, & \text{otherwise,} \end{cases}$$

$$\text{and} \quad \underset{\sim}{\theta} = \begin{cases} -\infty, & \text{if } \underline{\theta} = -\infty, \\ \underline{\theta} + \zeta\lambda, & \text{if } \underline{\theta} > -\infty \text{ and } \lambda < \frac{\overline{\theta} - \underline{\theta}}{2\zeta}, \\ \frac{\overline{\theta} + \underline{\theta}}{2}, & \text{otherwise.} \end{cases}$$

Furthermore, the functions $\mathfrak{F}$ and $\mathfrak{G}$ separating the continuation region $\mathcal{C}$ from the sell region $\mathcal{S}$ and the buy region $\mathcal{B}$, are given by

$$\mathfrak{F}(\theta) = \begin{cases} F^{\dagger}(\theta), & \text{if } \widetilde{\theta} < \check{\theta} \text{ and } \theta \in \left[\widetilde{\theta}, \check{\theta}\right[\\ F^{\star}(\theta), & \text{if } \underset{\sim}{\theta} < \widetilde{\theta} \text{ and } \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[, \end{cases}$$

$$\text{and} \quad \mathfrak{G}(\theta) = \begin{cases} G^{\dagger}(\theta), & \text{if } \underset{\sim}{\theta} > \hat{\theta} \text{ and } \theta \in \left]\hat{\theta}, \underset{\sim}{\theta}\right] \\ G^{\star}(\theta), & \text{if } \underset{\sim}{\theta} < \widetilde{\theta} \text{ and } \theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[, \end{cases}$$

as illustrated in Figures 5 and 6.

In these expressions, as well as the rest of this section, we suppress the dependence of $F^{\star}$, $G^{\star}$, $F^{\dagger}$ and $G^{\dagger}$ on λ, which we consider fixed. In this

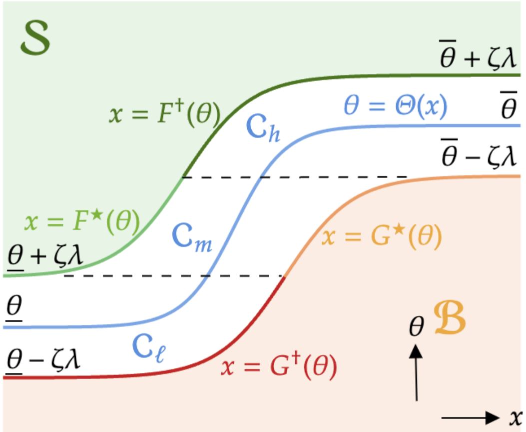

Figure 5: $0 < \lambda < \frac{\overline{\theta}-\underline{\theta}}{2\zeta}$

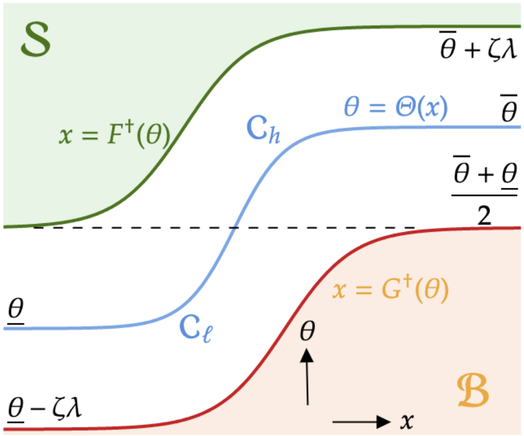

Figure 6: $\lambda \geq \frac{\overline{\theta}-\underline{\theta}}{2\zeta}$

context, we are faced with the solution v to the ODE (3.23) that is given by

$$v(\theta, x) = \begin{cases} A(\theta)\varphi(x) + B(\theta)\psi(x) - \frac{r}{\zeta}R_\Theta(x) + \frac{\theta}{\zeta}, & \text{if } (\theta, x) \in \operatorname{cl}\mathcal{C}, \\ v\big(\theta, \mathfrak{F}(\theta)\big) = \lambda, & \text{if } (\theta, x) \in \operatorname{int}\mathcal{S}, \\ v\big(\theta, \mathfrak{G}(\theta)\big) = -\lambda, & \text{if } (\theta, x) \in \operatorname{int}\mathcal{B}, \end{cases} \tag{3.68}$$

where

$$A(\theta) = 0, \quad \text{if } \hat{\theta} < \underset{\sim}{\theta} \text{ and } \theta \in \big]\hat{\theta}, \underset{\sim}{\theta}\big],$$
$$A(\theta) = -\frac{r}{\zeta}\int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y, \quad \text{if } \underset{\sim}{\theta} < \widetilde{\theta} \text{ and } \theta \in \big]\underset{\sim}{\theta}, \check{\theta}\big[, \tag{3.69}$$

$$B(\theta) = 0, \quad \text{if } \widetilde{\theta} < \check{\theta} \text{ and } \theta \in \big[\widetilde{\theta}, \check{\theta}\big[,$$
$$\text{and} \quad B(\theta) = -\frac{r}{\zeta}\int_{\mathfrak{F}(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\,\mathrm{d}y, \quad \text{if } \underset{\sim}{\theta} < \widetilde{\theta} \text{ and } \theta \in \big]\hat{\theta}, \widetilde{\theta}\big[. \tag{3.70}$$

Lemma 3.13. *The function v given by* (3.68) *is well-defined in the sense that the integrals in* (3.69) *and* (3.70) *are well-defined and real-valued. Furthermore, v is a $C^{1,2}$ solution to the ODE (3.23) that satisfies $|v(\theta, x)| < \lambda$ for all $(\theta, x) \in \mathcal{C}$.*

Proof. By the growth condition (3.5) and integrability condition (A.7),

$$\begin{aligned} \frac{\zeta}{r}|A(\theta)| &\leq \frac{|\theta-\zeta\lambda|\psi'(\mathfrak{F}(\theta))}{Cp'(\mathfrak{F}(\theta))} + K\int_{-\infty}^{\mathfrak{F}(\theta)} \Psi(y)(1+|y|^p)\,\mathrm{d}y < \infty \\ \text{and} \quad \frac{\zeta}{r}|B(\theta)| &\leq \frac{|\theta-\zeta\lambda|\varphi'(\mathfrak{F}(\theta))}{Cp'(\mathfrak{F}(\theta))} + K\int_{\mathfrak{F}(\theta)}^{\infty} \Phi(y)(1+|y|^p)\,\mathrm{d}y < \infty, \end{aligned}$$

which proves that v is well-defined and real-valued. To prove that $|v(\theta,x)| < \lambda$ in $\mathcal{C}$, we first observe that we can write, for all $(\theta,x) \in \mathcal{C}_m$,

$$\begin{aligned} v(\theta,x) &= \frac{\varphi(x)\psi'(\mathfrak{F}(\theta)) - \psi(x)\varphi'(\mathfrak{F}(\theta))}{Cp'(\mathfrak{F}(\theta))}\lambda \\ &\quad + \frac{r}{\zeta}\left(\varphi(x)\int_{\mathfrak{F}(\theta)}^{x} \Psi(y)(\theta-\Theta(y))\,\mathrm{d}y - \psi(x)\int_{\mathfrak{F}(\theta)}^{x} \Phi(y)(\theta-\Theta(y))\,\mathrm{d}y\right) \\ &= \frac{\psi(x)\varphi'(\mathfrak{G}(\theta)) - \varphi(x)\psi'(\mathfrak{G}(\theta))}{Cp'(\mathfrak{G}(\theta))}\lambda \\ &\quad - \frac{r}{\zeta}\left(\varphi(x)\int_{x}^{\mathfrak{G}(\theta)} \Psi(y)(\theta-\Theta(y))\,\mathrm{d}y - \psi(x)\int_{x}^{\mathfrak{G}(\theta)} \Phi(y)(\theta-\Theta(y))\,\mathrm{d}y\right). \end{aligned}$$

For $(\theta,x) \in \mathcal{C}_m$ such that $\theta > \Theta(x)$, the fact that φ is strictly decreasing, ψ is strictly increasing and ψ/φ is strictly increasing implies that

$$v(\theta,x) < \frac{\varphi(\mathfrak{F}(\theta))\psi'(\mathfrak{F}(\theta)) - \psi(\mathfrak{F}(\theta))\varphi'(\mathfrak{F}(\theta))}{Cp'(\mathfrak{F}(\theta))}\lambda = \lambda.$$

Similarly, for $(\theta,x) \in \mathcal{C}_m$ such that $\theta < \Theta(x)$,

$$v(\theta,x) > \frac{\psi(\mathfrak{G}(\theta))\varphi'(\mathfrak{G}(\theta)) - \varphi(\mathfrak{G}(\theta))\psi'(\mathfrak{G}(\theta))}{Cp'(\mathfrak{G}(\theta))}\lambda = -\lambda.$$

Therefore, $|v(\theta,x)| < \lambda$ for all $(\theta,x) \in \mathcal{C}_m$ by continuity of v. Assuming that

$\underline{\theta} > -\infty$ (otherwise $\mathcal{C}_\ell = \emptyset$), we can write, for all $(\theta, x) \in \mathcal{C}_\ell$,

$$\begin{aligned} v(\theta, x) &= \frac{\psi(x)\varphi'(\mathfrak{G}(\theta))}{Cp'(\mathfrak{G}(\theta))}\lambda \\ &\quad + \frac{r}{\zeta}\left(\varphi(x)\int_{-\infty}^{x} \Psi(y)(\theta - \Theta(y))\,\mathrm{d}y + \psi(x)\int_{x}^{\mathfrak{G}(\theta)} \Phi(y)(\theta - \Theta(y))\,\mathrm{d}y\right) \\ &= \frac{\psi(x)\varphi'(\mathfrak{G}(\theta)) - \varphi(x)\psi'(\mathfrak{G}(\theta))}{Cp'(\mathfrak{G}(\theta))}\lambda \\ &\quad - \frac{r}{\zeta}\left(\varphi(x)\int_{x}^{\mathfrak{G}(\theta)} \Psi(y)(\theta - \Theta(y))\,\mathrm{d}y - \psi(x)\int_{x}^{\mathfrak{G}(\theta)} \Phi(y)(\theta - \Theta(y))\,\mathrm{d}y\right), \end{aligned}$$

where the second equality follows from (3.36). Proving that $v(\theta, x) > -\lambda$ for $(\theta, x) \in \mathcal{C}_\ell$ such that $\theta > \Theta(x)$ is therefore analogous to the proof for $(\theta, x) \in \mathcal{C}_m$. To prove that $v(\theta, x) < \lambda$, using the fact that φ is strictly decreasing and the fact that $|\theta - \Theta(x)| \leq \underset{\sim}{\theta} - \underline{\theta} = \zeta\lambda$,

$$v(\theta, x) < \frac{r}{\zeta}\left(\varphi(x)\int_{-\infty}^{x} \Psi(y)(\underset{\sim}{\theta} - \underline{\theta})\,\mathrm{d}y + \psi(x)\int_{x}^{\infty} \Phi(y)(\underset{\sim}{\theta} - \underline{\theta})\,\mathrm{d}y\right) = \lambda.$$

The proof that $|v(\theta, x)| < \lambda$ for $(\theta, x) \in \mathcal{C}_h$ follows symmetric arguments to those for $\mathcal{C}_\ell$. □

3.4 Solving the free-boundary Problem 3.8 when $\boldsymbol{\Theta}$ is strictly increasing and then strictly decreasing

We now derive explicit solutions to the free-boundary Problem 3.8 if the problem data is such that the function Θ providing the agent's frictionless optimiser is strictly increasing and then strictly decreasing. Economically, this means that as the value of the process X increases up to a certain point, the optimal number of shares held by Agent 1 in the frictionless setting increases until it reaches a maximum number of shares at this point, and once the value of X increases beyond this point, the optimal number of shares decreases. This situation can arise for example when the gradients of h_1 and h_2 are both positive, with h_2 being steeper than h_1 up to a certain point, and h_1 being steeper than h_2 after this point. In this case, there are two regimes in the economy, one where Agent 2's endowment rate is more sensitive to changes to the state of the economy, and another where Agent 1's endowment rate is more sensitive to changes to the state of the economy.

3.4.1 The structure of the solution

We first postulate the structure of the solution. To this end, we first make the following assumption.

Assumption 3.14. *There exists a point $x^\dagger \in \left]-\infty, \infty\right[$ such that*

$$\Theta'(x) \begin{cases} > 0, & \text{if } x < x^\dagger, \\ < 0, & \text{if } x > x^\dagger. \end{cases}$$

This implies that

$$\overline{\theta} \equiv \sup_{x \in \mathbb{R}} \Theta(x) = \Theta(x^\dagger).$$

We also assume that

$$\underline{\theta} \equiv \lim_{x \downarrow -\infty} \Theta(x) = \lim_{x \uparrow \infty} \Theta(x) = -\infty. \tag{3.71}$$

Furthermore, given any $\theta \in \left]\underline{\theta}, \overline{\theta}\right[$, we define

$$\underline{\chi}(\theta) = \min\left\{x \in \mathbb{R} \mid \ \Theta(x) = \theta\right\} \quad \text{and} \quad \overline{\chi}(\theta) = \inf\left\{x > \underline{\chi}(\theta) \mid \ \Theta(x) = \theta\right\}, \tag{3.72}$$

with the usual convention that $\inf \emptyset = \infty$.

In this case, we postulate that the continuation region $\mathcal{C}$ is characterised by points $\widetilde{\theta} \in \left]-\infty, \overline{\theta}\right[$ and $\check{\theta} > \overline{\theta}$ as well as by strictly increasing continuous functions $\mathfrak{F}_\ell : \left]-\infty, \check{\theta}\right] \to \mathbb{R}$, $\mathfrak{G}_\ell : \left]-\infty, \widetilde{\theta}\right] \to \mathbb{R}$ and strictly decreasing continuous functions $\mathfrak{G}_r : \left]-\infty, \widetilde{\theta}\right] \to \mathbb{R}$, $\mathfrak{F}_r : \left]-\infty, \check{\theta}\right] \to \mathbb{R}$ such that

$$\begin{gathered}
\mathfrak{F}_\ell(\widetilde{\theta}) < \underline{\chi}(\widetilde{\theta}) < \mathfrak{G}_\ell(\widetilde{\theta}) = \mathfrak{G}_r(\widetilde{\theta}) < \overline{\chi}(\widetilde{\theta}) < \mathfrak{F}_r(\widetilde{\theta}), \\
\mathfrak{F}_\ell(\theta) < \underline{\chi}(\theta) < \mathfrak{G}_\ell(\theta) < \mathfrak{G}_r(\theta) < \overline{\chi}(\theta) < \mathfrak{F}_r(\theta) \quad \text{for all } \theta \in \left]-\infty, \widetilde{\theta}\right[, \\
\mathfrak{F}_\ell(\check{\theta}) = x^\dagger = \mathfrak{F}_r(\check{\theta}), \quad \mathfrak{F}_\ell(\theta) < \mathfrak{F}_r(\theta) \quad \text{for all } \theta \in \left]\widetilde{\theta}, \check{\theta}\right[\\
\lim_{\theta \downarrow -\infty} \mathfrak{F}_\ell(\theta) = \lim_{\theta \downarrow -\infty} \mathfrak{G}_\ell(\theta) = -\infty \quad \text{and} \quad \lim_{\theta \downarrow -\infty} \mathfrak{F}_r(\theta) = \lim_{\theta \downarrow -\infty} \mathfrak{G}_r(\theta) = \infty.
\end{gathered}$$

In particular, we will show that $\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_h \cup \mathcal{C}_r$, where

$$\begin{aligned}
\mathcal{C}_\ell &= \left\{(\theta, x) \in \mathbb{R}^2 \mid -\infty < \theta \le \widetilde{\theta} \text{ and } \mathfrak{F}_\ell(\theta) < x < \mathfrak{G}_\ell(\theta)\right\}, \\
\mathcal{C}_h &= \left\{(\theta, x) \in \mathbb{R}^2 \mid \widetilde{\theta} < \theta < \check{\theta} \text{ and } \mathfrak{F}_\ell(\theta) < x < \mathfrak{F}_r(\theta)\right\}, \\
\text{and} \quad \mathcal{C}_r &= \left\{(\theta, x) \in \mathbb{R}^2 \mid -\infty < \theta \le \widetilde{\theta} \text{ and } \mathfrak{G}_r(\theta) < x < \mathfrak{F}_r(\theta)\right\}
\end{aligned}$$

while the selling and buying regions are respectively given by

$$\begin{aligned}
\mathcal{S} &= \left\{(\theta, x) \in \mathbb{R}^2 \mid \check{\theta} < \theta\right\} \\
&\quad \cup \left\{(\theta, x) \in \mathbb{R}^2 \mid -\infty < \theta \le \check{\theta} \text{ and } x \le \mathfrak{F}_\ell(\theta) \text{ or } \mathfrak{F}_r(\theta) \le x\right\}, \\
\text{and} \quad \mathcal{B} &= \left\{(\theta, x) \in \mathbb{R}^2 \mid -\infty < \theta \le \widetilde{\theta} \text{ and } \mathfrak{G}_\ell(\theta) \le x \le \mathfrak{G}_r(\theta)\right\}.
\end{aligned}$$

Figure 7 provides an illustration of the three regions arising in the context of

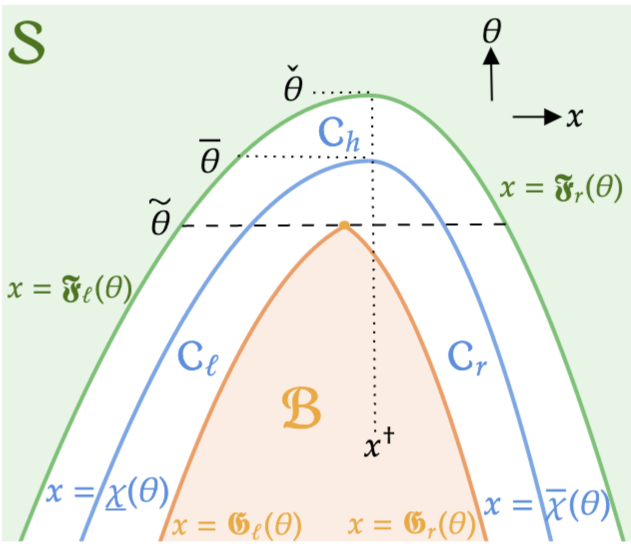

Figure 7: Regions for Θ strictly increasing and then strictly decreasing

this case. After an initial jump such that $(\vartheta_0, x) \in \operatorname{cl}\mathcal{C}$, the frictional optimiser ϑ reflects in the negative θ-direction whenever $X_t = \mathfrak{F}_\ell(\vartheta_t)$ or $X_t = \mathfrak{F}_r(\vartheta_t)$ for some $t \geq 0$, and reflects in the positive θ-direction whenever $X_t = \mathfrak{G}_\ell(\vartheta_t)$ or $X_t = \mathfrak{G}_r(\vartheta_t)$ for some $t \geq 0$, so that the joint process (ϑ, X) remains in $\operatorname{cl}\mathcal{C}$ at all times. In the region $\mathcal{C}_h$, we observe that there are "sell-sell" boundaries defined by the functions $\mathfrak{F}_\ell$ and $\mathfrak{F}_r$, and "buy-sell" boundaries in $\mathcal{C}_\ell \cup \mathcal{C}_r$. Moreover, we observe that, once the joint process (ϑ, X) reaches $\mathcal{C}_\ell \cup \mathcal{C}_r$, it will remain there indefinitely, and will never enter $\mathcal{C}_h$ again.

Similarly as before, the general solution to the ODE (3.23) that v satisfies inside the continuation region $\mathcal{C}$ is given by

$$v(\theta, x) = A(\theta)\varphi(x) + B(\theta)\psi(x) - \frac{r}{\zeta} R_\Theta(x) + \frac{\theta}{\zeta},$$

for some functions A and B, where φ, ψ are as in Appendix A, R_Θ is defined by (A.10) for $h = \Theta$ and

$$\zeta = \frac{2r}{\sigma^2(\gamma_1 + \gamma_2)}.$$

In view of the structures of the continuation region $\mathcal{C}$ that we have postulated, we are faced with the following three "building blocks" associated with the functions that determine the boundary of $\mathcal{C}$.

Case I. Suppose that there exists a point $\widetilde{\theta} \in \left[-\infty, \overline{\theta}\right[$ and strictly increasing functions $\mathfrak{F}_\ell, \mathfrak{G}_\ell : \left]-\infty, \widetilde{\theta}\right] \to \mathbb{R}$ such that

$$\begin{aligned} &\mathfrak{F}_\ell(\theta) < \mathfrak{G}_\ell(\theta) \quad \text{for all } \theta \in \left]-\infty, \widetilde{\theta}\right], \\ &\left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } x \leq \mathfrak{F}_\ell(\theta)\right\} \subseteq \mathcal{S}, \\ &\left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } \mathfrak{G}_\ell(\theta) \leq x\right\} \subseteq \mathcal{B} \\ \text{and} \quad &\left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } \mathfrak{F}_\ell(\theta) < x < \mathfrak{G}_\ell(\theta)\right\} \subseteq \mathcal{C}. \end{aligned}$$

Case II. This case is symmetric to the previous one: suppose that there exists a point $\widetilde{\theta} \in \left[-\infty, \overline{\theta}\right[$ and strictly decreasing functions $\mathfrak{F}_r, \mathfrak{G}_r : \left]-\infty, \widetilde{\theta}\right] \to$

$\mathbb{R}$ such that

$$
\begin{aligned}
\mathfrak{G}_r(\theta) < \mathfrak{F}_r(\theta) \quad \text{for all } \theta \in \left]-\infty, \widetilde{\theta}\right], \\
\left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } x \leq \mathfrak{G}_r(\theta)\right\} \subseteq \mathcal{B}, \\
\left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } \mathfrak{F}_r(\theta) \leq x\right\} \subseteq \mathcal{S} \\
\text{and} \quad \left\{(\theta, x) \mid \ \theta \in \left]-\infty, \widetilde{\theta}\right] \text{ and } \mathfrak{G}_r(\theta) < x < \mathfrak{F}_r(\theta)\right\} \subseteq \mathcal{C}.
\end{aligned}
$$

In Cases I and II, making calculations that are similar to the ones made in Case I of Section 3.3.1, we can see that the two pairs of free-boundary points $\mathfrak{F}_\ell(\theta)$ and $\mathfrak{G}_\ell(\theta)$, as well as $\mathfrak{G}_r(\theta)$ and $\mathfrak{F}_r(\theta)$, should satisfy the system of equations (3.31), while $A(\theta)$ and $B(\theta)$ should admit the expressions (3.34) and (3.35).

Case III. Suppose that there exists points $\widetilde{\theta} < \overline{\theta} < \check{\theta}$ in $]-\infty, \infty[$, a strictly increasing function $\mathfrak{F}_\ell : \left]\widetilde{\theta}, \check{\theta}\right] \to \mathbb{R}$ and a strictly decreasing function $\mathfrak{F}_r : \left]\widetilde{\theta}, \check{\theta}\right] \to \mathbb{R}$ such that

$$
\begin{aligned}
\mathfrak{F}_\ell(\theta) < \mathfrak{F}_r(\theta) \quad \text{for all } \theta \in \left]\widetilde{\theta}, \check{\theta}\right[, \quad \mathfrak{F}_\ell(\check{\theta}) = x^\dagger = \mathfrak{F}_r(\check{\theta}), \\
\left\{(\theta, x) \mid \ \theta \in \left]\widetilde{\theta}, \check{\theta}\right] \text{ and } x \leq \mathfrak{F}_\ell(\theta) \text{ or } \mathfrak{F}_r(\theta) \leq x\right\} \subseteq \mathcal{S} \\
\text{and} \quad \left\{(\theta, x) \mid \ \theta \in \left]\widetilde{\theta}, \check{\theta}\right[\text{ and } \mathfrak{F}_\ell(\theta) < x < \mathfrak{F}_r(\theta)\right\} \subseteq \mathcal{C}.
\end{aligned}
$$

In this context, the requirement that $v(\cdot, x)$ should be C^1 for all $x \in \mathbb{R}$ suggests the free-boundary conditions

$$
\begin{aligned}
v\big(\theta, \mathfrak{F}_\ell(\theta)\big) = \lambda, \quad v\big(\theta, \mathfrak{F}_r(\theta)\big) = \lambda, \\
v_\theta\big(\theta, \mathfrak{F}_\ell(\theta)\big) = 0 \quad \text{and} \quad v_\theta\big(\theta, \mathfrak{F}_r(\theta)\big) = 0.
\end{aligned}
$$

Making similar calculations to the ones made in Case I of the previous subsection, we obtain the system of equations

$$
\mathcal{H}_3\big(\theta, \mathfrak{F}_\ell(\theta), \mathfrak{F}_r(\theta), \lambda\big) = 0 \quad \text{and} \quad \mathcal{H}_4\big(\theta, \mathfrak{F}_\ell(\theta), \mathfrak{F}_r(\theta), \lambda\big) = 0 \tag{3.73}
$$

for the free-boundary points $\mathfrak{F}_\ell(\theta)$ and $\mathfrak{F}_r(\theta)$, where

$$\mathcal{H}_3(\theta, F_\ell, F_r, \lambda) = \int_{F_\ell}^{F_r} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \tag{3.74}$$

$$\text{and} \quad \mathcal{H}_4(\theta, F_\ell, F_r, \lambda) = \int_{F_\ell}^{F_r} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y, \tag{3.75}$$

as well as the expressions

$$A(\theta) = -\frac{r}{\zeta}\int_{-\infty}^{\mathfrak{F}_\ell(\theta)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \tag{3.76}$$

$$\text{and} \quad B(\theta) = -\frac{r}{\zeta}\int_{\mathfrak{F}_\ell(\theta)}^{\infty} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y. \tag{3.77}$$

3.4.2 The functions determining the boundary of the continuation region

We now derive the solution to the free-boundary equations arising in Cases I-III that we considered in the Section 3.4.1. This will be done in two major steps. In the first major step, we study the solvability of the system of equations (3.73) in the context of Case III of Section 3.4.1 as illustrated in Figure 8: given any $\widetilde{\theta} \in \big[-\infty, \overline{\theta} + \zeta\lambda\big[$, there exists free-boundary functions $F_\ell^\dagger$ and $F_r^\dagger$ such that $v\big(\theta, F_\ell^\dagger(\theta)\big) = v\big(\theta, F_r^\dagger(\theta)\big) = \lambda$ for all $\theta \in \big]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\big]$.

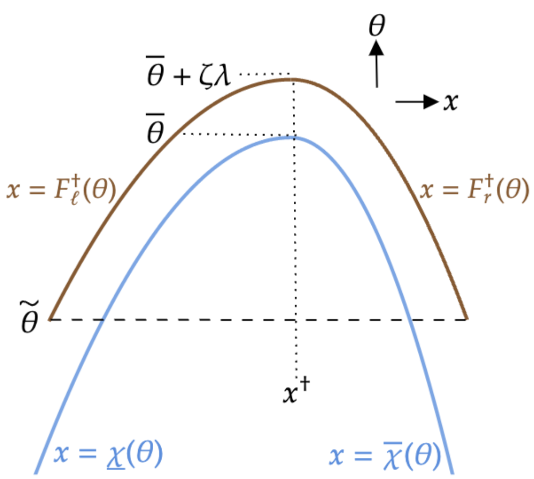

Figure 8: Case III

We will then show that for each $\theta \in \left]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right]$ and $x \in \left]F_\ell^\dagger(\theta), F_r^\dagger(\theta)\right[$ that $v(\theta, x) < \lambda$ and there exists a unique function H such that $v(\theta, H(\theta))$ is minimal for each θ, and that there exists $\widetilde{\theta}$ such that $v(\widetilde{\theta}, H(\widetilde{\theta})) = -\lambda$. Moreover, we will obtain that

$$\mathcal{H}_1\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0, \quad \mathcal{H}_2\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0,$$
$$\mathcal{H}_1\big(\widetilde{\theta}, F_r^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0, \quad \text{and} \quad \mathcal{H}_2\big(\widetilde{\theta}, F_r^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0,$$

which leads us to the second major step where we study the solvability of the system of equations (3.31) in the context of Cases I and II of Section 3.4.1 for functions $F_\ell^\star$, $G_\ell^\star$, $F_r^\star$ and $G_r^\star$. This is illustrated in Figure 9.

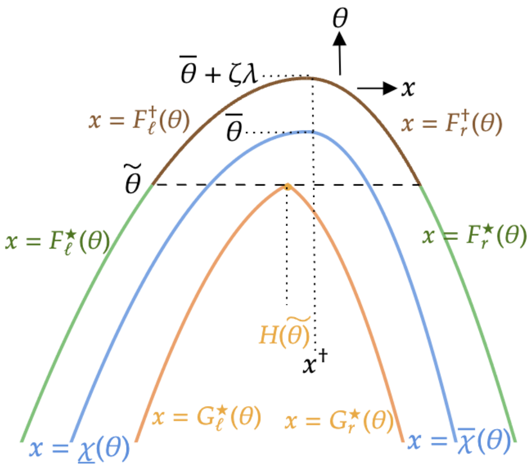

Figure 9: Cases I and II

In the following result, we study the solvability of the system of equations (3.73) in the context of Case III of Section 3.4.1.

Lemma 3.15. *Suppose that the function Θ satisfies Assumption 3.14. Consider the system of equations (3.73), where $\mathcal{H}_3$ and $\mathcal{H}_4$ are defined by (3.74) and (3.75). Then, given any $\widetilde{\theta} \in \left[-\infty, \overline{\theta}+\zeta\lambda\right[$, there exist continuous functions $F_\ell^\dagger, F_r^\dagger : \mathscr{D} \to \mathbb{R}$, where*

$$\mathscr{D} = \big\{(\theta, \lambda) \in \mathbb{R} \times \mathbb{R}_+ \mid \ \widetilde{\theta} < \theta < \overline{\theta} + \zeta\lambda \quad \textit{and} \quad \lambda > 0\big\},$$

such that

$$F_\ell^\dagger(\theta,\lambda) < \underline{\chi}(\theta-\zeta\lambda) < \overline{\chi}(\theta-\zeta\lambda) < F_r^\dagger(\theta,\lambda) \tag{3.78}$$

$$\textit{and} \quad \mathcal{H}_3\big(\theta, F_\ell^\dagger(\theta,\lambda), F_r^\dagger(\theta,\lambda),\lambda\big) = \mathcal{H}_4\big(\theta, F_\ell^\dagger(\theta,\lambda), F_r^\dagger(\theta,\lambda),\lambda\big) = 0. \tag{3.79}$$

Furthermore, given any $\lambda > 0$,

$$\frac{\partial F_\ell^\dagger}{\partial\theta}(\theta,\lambda) > 0, \quad \frac{\partial F_r^\dagger}{\partial\theta}(\theta,\lambda) < 0, \quad \lim_{\theta\uparrow\overline{\theta}+\zeta\lambda} F_\ell^\dagger(\theta,\lambda) = \lim_{\theta\uparrow\overline{\theta}+\zeta\lambda} F_r^\dagger(\theta,\lambda) = x^\dagger, \tag{3.80}$$

$$\lim_{\theta\downarrow-\infty} F_\ell^\dagger(\theta,\lambda) = -\infty \quad \textit{and} \quad \lim_{\theta\downarrow-\infty} F_r^\dagger(\theta,\lambda) = \infty. \tag{3.81}$$

Given any $\theta \in \big]\widetilde{\theta}, \overline{\theta}+\zeta\lambda\big[$,

$$\frac{\partial F_\ell^\dagger}{\partial\lambda}(\theta,\lambda) < 0 \quad \textit{and} \quad \frac{\partial F_r^\dagger}{\partial\lambda}(\theta,\lambda) > 0. \tag{3.82}$$

Proof. We organise the proof in four steps.

Step 1: Solvability of the equation $\mathcal{H}_3(\theta, F_\ell, F_r, \lambda) = 0$ for $F_r > \overline{\chi}(\theta)$. Fix any $\lambda > 0$ and $\theta \in \big]\widetilde{\theta}, \overline{\theta}+\zeta\lambda\big[$. We calculate

$$\mathcal{H}_3(\theta, \underline{\chi}(\theta-\zeta\lambda), \overline{\chi}(\theta-\zeta\lambda), \lambda) = \int_{\underline{\chi}(\theta-\zeta\lambda)}^{\overline{\chi}(\theta-\zeta\lambda)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y < 0$$

$$\text{and} \quad \frac{\partial\mathcal{H}_3}{\partial F_r}(\theta, F_\ell, F_r, \lambda) = \Psi(F_r)(\theta - \Theta(F_r) - \zeta\lambda) \begin{cases} > 0, & \text{if } \Theta(F_r) < \theta - \zeta\lambda, \\ < 0, & \text{otherwise.} \end{cases}$$

Assumption (3.14) and Lemma A.1 imply that

$$\lim_{F_r\uparrow\infty} \mathcal{H}_3(\theta, F_\ell, F_r, \lambda) = \infty \quad \text{for all } \lambda > 0 \quad \text{and } \theta \in \mathbb{R}.$$

Similarly,

$$\frac{\partial\mathcal{H}_3}{\partial F_\ell}(\theta, F_\ell, F_r, \lambda) = -\Psi(F_\ell)(\theta - \Theta(F_\ell) - \zeta\lambda) \begin{cases} < 0, & \text{if } \Theta(F_\ell) < \theta - \zeta\lambda, \\ > 0, & \text{otherwise,} \end{cases}$$

and

$$\lim_{F_\ell \downarrow -\infty} \mathcal{H}_3(\theta, F_\ell, F_r, \lambda) < \infty \quad \text{for all } \lambda > 0 \quad \text{and } \theta \in \mathbb{R}.$$

We are faced with two possible cases:

- If $\mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0$, then $\mathcal{H}_3(\theta, F_\ell, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0$ for all $F_\ell \in [-\infty, \underline{\chi}(\theta - \zeta\lambda)]$.

- If $\mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) \geq 0$, then there exists $\underline{F}(\theta, \lambda) \in [-\infty, \underline{\chi}(\theta - \zeta\lambda)[$ such that
$$\mathcal{H}_3(\theta, \underline{F}(\theta, \lambda), \overline{\chi}(\theta - \zeta\lambda), \lambda) = 0 \tag{3.83}$$
and $\mathcal{H}_3(\theta, F_\ell, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0$ if and only if $F_\ell \in]\underline{F}(\theta, \lambda), \underline{\chi}(\theta - \zeta\lambda)]$.

In either case, there exists a continuous function $\mathfrak{f} : \mathscr{D}_3 \to \mathbb{R}$ such that

$$F_\ell \leq \underline{\chi}(\theta - \zeta\lambda) < \overline{\chi}(\theta - \zeta\lambda) < \mathfrak{f}(\theta, F_\ell, \lambda) \quad \text{and} \quad \mathcal{H}_3\big(\theta, F_\ell, \mathfrak{f}(\theta, F_\ell, \lambda), \lambda\big) = 0 \tag{3.84}$$

for all $(\theta, F_\ell, \lambda) \in \mathscr{D}_3$, where

$$\begin{aligned} \mathscr{D}_3 = \big\{&(\theta, F_\ell, \lambda) \in \mathbb{R}^3 \mid \mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) \geq 0, \\ &\underline{F}(\theta, \lambda) < F_\ell \leq \underline{\chi}(\theta - \zeta\lambda) \text{ and } \lambda > 0\big\} \\ \cup \big\{&(\theta, F_\ell, \lambda) \in \mathbb{R}^3 \mid \mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0, \\ &F_\ell \leq \underline{\chi}(\theta - \zeta\lambda) \text{ and } \lambda > 0\big\}. \end{aligned} \tag{3.85}$$

We note that when $\mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0$, $\mathfrak{f}(\theta, -\infty, \lambda) < \infty$. Moreover, differentiating the identity in (3.84) implies that

$$\mathfrak{f}_\theta(\theta, F_\ell, \lambda) = -\frac{\int_{F_\ell}^{\mathfrak{f}(\theta, F_\ell, \lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(\mathfrak{f}(\theta, F_\ell, \lambda)\big)\big(\theta - \Theta(\mathfrak{f}(\theta, F_\ell, \lambda)) - \zeta\lambda\big)}, \tag{3.86}$$

$$\mathfrak{f}_{F_\ell}(\theta, F_\ell, \lambda) = \frac{\Psi(F_\ell)(\theta - \Theta(F_\ell) - \zeta\lambda)}{\Psi\big(\mathfrak{f}(\theta, F_\ell, \lambda)\big)\big(\theta - \Theta(\mathfrak{f}(\theta, F_\ell, \lambda)) - \zeta\lambda\big)} \tag{3.87}$$

$$\text{and} \quad \mathfrak{f}_\lambda(\theta, F_\ell, \lambda) = \zeta\frac{\int_{F_\ell}^{\mathfrak{f}(\theta, F_\ell, \lambda)} \Psi(y)\,\mathrm{d}y}{\Psi\big(\mathfrak{f}(\theta, F_\ell, \lambda)\big)\big(\theta - \Theta(\mathfrak{f}(\theta, F_\ell, \lambda)) - \zeta\lambda\big)}. \tag{3.88}$$

Step 2: Existence and uniqueness of $F_\ell^\dagger$ and $F_r^\dagger$ satisfying (3.78) and (3.79). Recalling the definition (3.85) of $\mathfrak{f}$'s domain $\mathscr{D}_3$, we fix any $\lambda > 0$

and $\theta \in \left]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right[$ and consider the equation $\mathcal{H}_4(\theta, F_\ell, \mathfrak{f}(\theta, F_\ell, \lambda), \lambda) = 0$ for $F_\ell \leq \underline{\chi}(\theta - \zeta\lambda)$. Recalling the definitions of Φ and Ψ in Appendix A, as well as the fact that φ (resp., ψ) is strictly decreasing (resp., strictly increasing), we can use the inequalities

$$\theta - \Theta(y) - \zeta\lambda \begin{cases} < 0, & \text{if } y \in]\underline{\chi}(\theta - \zeta\lambda), \overline{\chi}(\theta - \zeta\lambda)[, \\ > 0, & \text{if } y > \overline{\chi}(\theta - \zeta\lambda), \end{cases}$$

to obtain

$$\begin{aligned} &\mathcal{H}_4(\theta, \underline{\chi}(\theta - \zeta\lambda), \mathfrak{f}(\theta, \underline{\chi}(\theta - \zeta\lambda), \lambda), \lambda) \\ &= \int_{\underline{\chi}(\theta-\zeta\lambda)}^{\mathfrak{f}(\theta, \underline{\chi}(\theta-\zeta\lambda), \lambda)} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \\ &< \frac{\varphi(\overline{\chi}(\theta - \zeta\lambda))}{\psi(\overline{\chi}(\theta - \zeta\lambda))} \int_{\underline{\chi}(\theta-\zeta\lambda)}^{\mathfrak{f}(\theta, \underline{\chi}(\theta-\zeta\lambda), \lambda)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y = 0, \end{aligned}$$

where the final equality follows from the identity in (3.84). Using (3.87), the definition (3.75) of $\mathcal{H}_4$ and the definitions of Φ and Ψ in (A.8), we obtain

$$\begin{aligned} \frac{\partial \mathcal{H}_4}{\partial F_\ell}(\theta, F_\ell, \mathfrak{f}(\theta, F_\ell, \lambda), \lambda) &= \Psi(F_\ell)\left(\frac{\varphi(\mathfrak{f}(\theta, F_\ell, \lambda)}{\psi(\mathfrak{f}(\theta, F_\ell, \lambda))} - \frac{\varphi(F_\ell)}{\psi(F_\ell)}\right)(\theta - \Theta(F_\ell) - \zeta\lambda) \\ &\begin{cases} < 0, & \text{if } \Theta(F_\ell) < \theta - \zeta\lambda, \\ > 0, & \text{otherwise}, \end{cases} \end{aligned}$$

where the inequality follows from the fact that φ/ψ is strictly decreasing. Then, if $\mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) < 0$, we observe that by Lemma (A.1) and the fact that $\mathfrak{f}(\theta, -\infty, \lambda) < \infty$ that

$$\lim_{F_\ell \downarrow -\infty} \mathcal{H}_4(\theta, F_\ell, \mathfrak{f}(\theta, F_\ell, \lambda), \lambda) = \infty.$$

On the other hand, if $\mathcal{H}_3(\theta, -\infty, \overline{\chi}(\theta - \zeta\lambda), \lambda) \geq 0$,

$$\begin{aligned}
&\lim_{F_\ell \downarrow \underline{F}(\theta,\lambda)} \mathcal{H}_4(\theta, F_\ell, \mathfrak{f}(\theta, F_\ell, \lambda), \lambda) \\
&= \lim_{F_\ell \downarrow \underline{F}(\theta,\lambda)} \int_{F_\ell}^{\mathfrak{f}(\theta,F_\ell,\lambda)} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\, \mathrm{d}y \\
&> \lim_{F_\ell \downarrow \underline{F}(\theta,\lambda)} \int_{F_\ell}^{\overline{\chi}(\theta-\zeta\lambda)} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\, \mathrm{d}y \\
&= \int_{\underline{F}(\theta,\lambda)}^{\overline{\chi}(\theta-\zeta\lambda)} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\, \mathrm{d}y \\
&> \frac{\varphi(\underline{\chi}(\theta - \zeta\lambda))}{\psi(\underline{\chi}(\theta - \zeta\lambda))} \int_{\underline{F}(\theta,\lambda)}^{\overline{\chi}(\theta-\zeta\lambda)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\, \mathrm{d}y \overset{(3.83)}{=} 0.
\end{aligned}$$

In either case, we conclude that there exists a unique $F_\ell^\dagger(\theta, \lambda) < \underline{\chi}(\theta - \zeta\lambda)$ such that

$$\mathcal{H}_4\big(\theta, F_\ell^\dagger(\theta, \lambda), \mathfrak{f}(\theta, F_\ell^\dagger(\theta, \lambda), \lambda), \lambda\big) = 0. \tag{3.89}$$

If we define $F_r^\dagger(\theta, \lambda) = \mathfrak{f}\big(\theta, F_\ell^\dagger(\theta, \lambda), \lambda\big)$, then (3.84) and (3.89) imply that the functions satisfy (3.78) and (3.79).

Step 3: Proof of the monotonicity of the functions $F_\ell^\dagger$ and $F_r^\dagger$. Differentiating the identity in (3.89) with respect to θ and λ, and using (3.86), (3.87), and (3.88), we obtain

$$\frac{\partial F_\ell^\dagger}{\partial\theta}(\theta,\lambda)=\frac{\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Phi(y)\,\mathrm{d}y-\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Psi(y)\,\mathrm{d}y}{\Psi\big(F_\ell^\dagger(\theta)\big)\Big(\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}-\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\Big)\big(\theta-\Theta\big(F_\ell^\dagger(\theta)\big)-\zeta\lambda\big)}>0, \tag{3.90}$$

$$\begin{aligned}\frac{\partial F_r^\dagger}{\partial\theta}(\theta,\lambda)&=\mathfrak{f}_\theta(\theta,F_\ell^\dagger(\theta),\lambda)+\mathfrak{f}_{F_\ell}(\theta,F_\ell^\dagger(\theta),\lambda)\frac{\partial F_\ell^\dagger}{\partial\theta}(\theta,\lambda)\\&=\frac{\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Phi(y)\,\mathrm{d}y-\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Psi(y)\,\mathrm{d}y}{\Psi\big(F_r^\dagger(\theta)\big)\Big(\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}-\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\Big)\big(\theta-\Theta\big(F_r^\dagger(\theta)\big)-\zeta\lambda\big)}<0,\\\frac{\partial F_\ell^\dagger}{\partial\lambda}(\theta,\lambda)&=\zeta\frac{\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Psi(y)\,\mathrm{d}y-\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Phi(y)\,\mathrm{d}y}{\Psi\big(F_\ell^\dagger(\theta)\big)\Big(\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}-\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\Big)\big(\theta-\Theta\big(F_\ell^\dagger(\theta)\big)-\zeta\lambda\big)}<0\\\text{and}\quad\frac{\partial F_r^\dagger}{\partial\lambda}(\theta,\lambda)&=\mathfrak{f}_\lambda(\theta,F_\ell^\dagger(\theta),\lambda)+\mathfrak{f}_{F_\ell}(\theta,F_\ell^\dagger(\theta),\lambda)\frac{\partial F_\ell^\dagger}{\partial\lambda}(\theta,\lambda)\\&=\zeta\frac{\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Psi(y)\,\mathrm{d}y-\int_{F_\ell^\dagger(\theta)}^{F_r^\dagger(\theta)}\Phi(y)\,\mathrm{d}y}{\Psi\big(F_r^\dagger(\theta)\big)\Big(\frac{\varphi(F_\ell^\dagger(\theta))}{\psi(F_\ell^\dagger(\theta))}-\frac{\varphi(F_r^\dagger(\theta))}{\psi(F_r^\dagger(\theta))}\Big)\big(\theta-\Theta\big(F_r^\dagger(\theta)\big)-\zeta\lambda\big)}>0,\end{aligned}$$

where the inequalities follow from the facts that φ/ψ is strictly decreasing, $F_\ell^\dagger(\theta,\lambda)<\underline{\chi}(\theta-\zeta\lambda)$ and $F_r^\dagger(\theta,\lambda)>\overline{\chi}(\theta-\zeta\lambda)$. We have proven the inequalities in (3.80) and (3.82).

Step 4: Proof of the limits in (3.80) and (3.81). To establish the limits in (3.80), we first observe that the system of equations

$$\begin{aligned}\mathcal{H}_3(\overline{\theta}+\zeta\lambda, F_\ell, F_r, \lambda) &\equiv \int_{F_\ell}^{F_r} \Psi(y)\big(\overline{\theta}-\Theta(y)\big)\,\mathrm{d}y \\ &= \mathcal{H}_4(\overline{\theta}+\zeta\lambda, F_\ell, F_r, \lambda) \equiv \int_{F_\ell}^{F_r} \Phi(y)\big(\overline{\theta}-\Theta(y)\big)\,\mathrm{d}y = 0\end{aligned} \tag{3.91}$$

can be satisfied only if $F_\ell = F_r$. Combining the fact that the system of equations (3.91) is satisfied by any choice $F_\ell = F_r$, and only such a choice, with (3.78) and the continuity of the functions $F_\ell^\dagger$, $F_r^\dagger$, we obtain the limits in (3.80).

To establish the limits in (3.81), we argue by contradiction. By the inequality in (3.80), $F_\ell^\dagger$ is strictly increasing in θ and $F_r^\dagger$ is strictly decreasing in θ. Suppose that $F_\ell^\dagger(\theta,\lambda) \downarrow -\infty$ and $F_r^\dagger(\theta,\lambda) \uparrow \overline{F}_r < \infty$ as $\theta \downarrow -\infty$. Then, by (3.79),

$$\zeta\lambda = \lim_{\theta\downarrow-\infty} \frac{\int_{F_\ell^\dagger(\theta,\lambda)}^{F_r^\dagger(\theta,\lambda)} \Psi(y)\big(\theta-\Theta(y)\big)\,\mathrm{d}y}{\int_{F_\ell^\dagger(\theta,\lambda)}^{F_r^\dagger(\theta,\lambda)} \Psi(y)\,\mathrm{d}y} = -\infty,$$

which is a contradiction. Therefore, $F_r^\dagger(\theta,\lambda) \uparrow \infty$ as $\theta \downarrow -\infty$. Similarly, we can prove that $F_\ell^\dagger(\theta,\lambda) \downarrow -\infty$ as $\theta \downarrow -\infty$. $\square$

In what follows, we suppress the dependence of $F_\ell^\dagger$ and $F_r^\dagger$ on λ and determine $\widetilde{\theta} < \overline{\theta}$ such that $|v(\theta,x)| < \lambda$ for all

$$\big\{(\theta,x) \mid \ \theta \in \big]\widetilde{\theta}, \overline{\theta}+\zeta\lambda\big[\text{ and } F_\ell^\dagger(\theta) < x < F_r^\dagger(\theta)\big\}, \tag{3.92}$$

Moreover, the limits in (3.80) imply that $\lim_{\theta\uparrow\overline{\theta}+\zeta\lambda} v(\theta,\cdot)$ is only defined for $x = x^\dagger$, and $\lim_{\theta\uparrow\overline{\theta}+\zeta\lambda} v(\theta,x^\dagger) = \lambda$. We will first show that $v(\theta,x) < \lambda$ for all (θ,x) in the set (3.92) for any choice of $\widetilde{\theta}$. To proceed further, we observe that we can write, using (3.29) and the expressions (3.76) and (3.77),

$$
\begin{aligned}
&v(\theta,x)+\lambda\\
&=\frac{r}{\zeta}\bigg(\varphi(x)\Big(\int_{-\infty}^{x}\Psi(y)(\theta-\Theta(y)+\zeta\lambda)\,\mathrm{d}y-\int_{-\infty}^{F_\ell^\dagger(\theta)}\Psi(y)(\theta-\Theta(y)-\zeta\lambda)\,\mathrm{d}y\Big)\\
&\quad+\psi(x)\Big(\int_{x}^{\infty}\Phi(y)(\theta-\Theta(y)+\zeta\lambda)\,\mathrm{d}y-\int_{F_\ell^\dagger(\theta)}^{\infty}\Phi(y)(\theta-\Theta(y)-\zeta\lambda)\,\mathrm{d}y\Big)\bigg).
\end{aligned}
$$

Differentiating with respect to x, we obtain

$$
\begin{aligned}
&v_x(\theta,x)\\
&=\frac{r}{\zeta}\bigg(\varphi'(x)\Big(\int_{-\infty}^{x}\Psi(y)(\theta-\Theta(y)+\zeta\lambda)\,\mathrm{d}y-\int_{-\infty}^{F_\ell^\dagger(\theta)}\Psi(y)(\theta-\Theta(y)-\zeta\lambda)\,\mathrm{d}y\Big)\\
&\quad+\psi'(x)\Big(\int_{x}^{\infty}\Phi(y)(\theta-\Theta(y)+\zeta\lambda)\,\mathrm{d}y-\int_{F_\ell^\dagger(\theta)}^{\infty}\Phi(y)(\theta-\Theta(y)-\zeta\lambda)\,\mathrm{d}y\Big)\bigg).
\end{aligned}
$$

Using (A.12), we observe that $v_x(\theta,F_\ell^\dagger(\theta))=0$. Similar calculations show that $v_x(\theta,F_r^\dagger(\theta))=0$ as well. Moreover, by the ODE (3.23),

$$
v_{xx}(\theta,F_\ell^\dagger(\theta))=-\frac{\sigma^2(\gamma_1+\gamma_2)(\theta-\Theta(F_\ell^\dagger(\theta))-\zeta\lambda)}{\beta^2(F_\ell^\dagger(\theta))}<0,
$$

where the inequality follows from (3.78). Similarly, $v_{xx}(\theta,F_r^\dagger(\theta))<0$. We are now ready to prove the following lemma, which shows that, for each θ, there is exactly one minimum point of $v(\theta,\cdot)$ between $F_\ell^\dagger(\theta)$ and $F_r^\dagger(\theta)$ and no other stationary points, which in turn shows that $v(\theta,x)<\lambda$ for all $x\in\,]F_\ell^\dagger(\theta),F_r^\dagger(\theta)[$, because $v(\theta,F_\ell^\dagger(\theta))=v(\theta,F_r^\dagger(\theta))=\lambda$, $v_x(\theta,F_\ell^\dagger(\theta))=v_x(\theta,F_r^\dagger(\theta))=0$ and $v_{xx}(\theta,F_\ell^\dagger(\theta)),v_{xx}(\theta,F_r^\dagger(\theta))<0$.

Lemma 3.16. *Let $F_\ell^\dagger$ and $F_r^\dagger$ be the functions in Lemma 3.15. Then, for any $\widetilde{\theta}<\overline{\theta}$,*

$$
v(\theta,x)<\lambda \textit{ for all } (\theta,x)\in\big\{(\theta,x)\ \big|\ \theta\in\big]\widetilde{\theta},\overline{\theta}+\zeta\lambda\big[\textit{ and } F_\ell^\dagger(\theta)<x<F_r^\dagger(\theta)\big\}. \tag{3.93}
$$

Proof. Fix $\theta\in\big]\widetilde{\theta},\overline{\theta}+\zeta\lambda\big[$. We prove (3.93) by proving that there exists exactly one point $x^\star \in]F_\ell^\dagger(\theta), F_r^\dagger(\theta)[$ such that $v_x(\theta, x^\star) = 0$ and $v_{xx}(\theta, x^\star) > 0$ (that is, $x^\star$ is the unique minimum point of $v(\theta, \cdot)$). The existence of $x \in]F_\ell^\dagger(\theta), F_r^\dagger(\theta)[$ such that $v_x(\theta, x) = 0$ is implied by the fact that $v(\theta, F_\ell^\dagger(\theta)) = v(\theta, F_r^\dagger(\theta)) = \lambda$ and the mean value theorem. The fact that $v_x(\theta, F_\ell^\dagger(\theta)) = v_x(\theta, F_r^\dagger(\theta)) = 0$ and $v_{xx}(\theta, F_\ell^\dagger(\theta)), v_{xx}(\theta, F_r^\dagger(\theta)) < 0$ implies there exists at least one minimum point of $v(\theta, \cdot)$. We assume for a contradiction that there exists two local minimum points $\underline{x} < \overline{x}$, such that $v_x(\theta, \underline{x}) = v_x(\theta, \overline{x}) = 0$ and $v_{xx}(\theta, \underline{x}), v_{xx}(\theta, \overline{x}) > 0$. Then, there exists $\hat{x} \in]\underline{x}, \overline{x}[$ such that $v_x(\theta, \hat{x}) = 0$ and $v_{xx}(\theta, \hat{x}) < 0$ (in other words, $\hat{x}$ is a local maximum point), such that $v(\theta, \underline{x}) < v(\theta, \hat{x})$ and $v(\theta, \overline{x}) < v(\theta, \hat{x})$. By the ODE (3.23),

$$\begin{aligned}
0 &< \frac{1}{2}\beta^2(\underline{x}) v_{xx}(\theta, \underline{x}) - \frac{1}{2}\beta^2(\hat{x}) v_{xx}(\theta, \hat{x}) - r\big(v(\theta, \underline{x}) - v(\theta, \hat{x})\big) \\
&= \frac{1}{2}\sigma^2(\gamma_1 + \gamma_2)\big(\Theta(\underline{x}) - \Theta(\hat{x})\big),
\end{aligned}$$

which implies that $\Theta(\underline{x}) > \Theta(\hat{x})$. Similarly, we obtain that $\Theta(\hat{x}) < \Theta(\overline{x})$. However, this contradicts the assumption that Θ is strictly increasing and then strictly decreasing. Therefore, there is a unique minimum point $x^\star$ of $v(\theta, \cdot)$ and no other stationary points, which proves that $v(\theta, x) < \lambda$ for all $x \in]F_\ell^\dagger(\theta), F_r^\dagger(\theta)[$. □

We now determine $\widetilde{\theta}$ such that $v(\theta, x) > -\lambda$ for all (θ, x) in the set (3.92). Lemma 3.16 implies that there exists a unique function H such that $F_\ell^\dagger(\theta) < H(\theta) < F_r^\dagger(\theta)$ and $w_x(\theta, H(\theta)) = 0$ for each $\theta < \overline{\theta} + \zeta\lambda$. Having shown that for each θ, $v(\theta, \cdot)$ has a unique minimum, we will determine in the following lemma $\widetilde{\theta} < \overline{\theta}$ as the maximal θ satisfying

$$\min_{x \in \left] F_\ell^\dagger(\theta), F_r^\dagger(\theta) \right[} v(\theta, x) = -\lambda.$$

Lemma 3.17. *Define the function w by $w(\theta, x) = v(\theta, x) + \lambda$. Then, there exists $\widetilde{\theta} < \overline{\theta} - \zeta\lambda$ such that $w(\widetilde{\theta}, H(\widetilde{\theta})) = 0$ and $w(\theta, H(\theta)) > 0$ for all $\theta > \widetilde{\theta}$.*

Proof. We first calculate

$$\begin{aligned}
&w_x(\theta, x)\\
&= \frac{r}{\zeta}\Bigg(\varphi'(x)\bigg(\int_{-\infty}^{x} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y - \int_{-\infty}^{F_\ell^\dagger(\theta)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y\bigg)\\
&\quad + \psi'(x)\bigg(\int_{x}^{\infty} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y - \int_{F_\ell^\dagger(\theta)}^{\infty} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y\bigg)\Bigg)
\end{aligned}$$

and observe that $w_x(\theta, H(\theta)) = 0$ implies that

$$\begin{aligned}
&w(\theta, H(\theta))\\
&= \frac{r}{\zeta}\left(\varphi(H(\theta)) - \frac{\psi(H(\theta))\varphi'(H(\theta))}{\psi'(H(\theta))}\right)\\
&\quad \cdot \left(\int_{-\infty}^{H(\theta)} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y - \int_{-\infty}^{F_\ell^\dagger(\theta)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y\right)\\
&= \frac{rp'(H(\theta))}{\zeta\psi'(H(\theta))}\bigg(\int_{-\infty}^{H(\theta)} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y\\
&\qquad\qquad - \int_{-\infty}^{F_\ell^\dagger(\theta)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y\bigg) \qquad (3.94)\\
&= -\frac{rp'(H(\theta))}{\zeta\varphi'(H(\theta))}\bigg(\int_{H(\theta)}^{\infty} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y\\
&\qquad\qquad - \int_{F_\ell^\dagger(\theta)}^{\infty} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y\bigg). \qquad (3.95)
\end{aligned}$$

We observe that, for all $\theta \geq \overline{\theta} - \zeta\lambda$,

$$w(\theta, H(\theta)) \geq \frac{rp'(H(\theta))}{\zeta\psi'(H(\theta))}\left(\int_{F_\ell^\dagger(\theta)}^{H(\theta)} \Psi(y)(\overline{\theta} - \Theta(y))\,\mathrm{d}y + 2\zeta\lambda \int_{-\infty}^{F_\ell^\dagger(\theta)} \Psi(y)\,\mathrm{d}y\right) > 0. \qquad (3.96)$$

Next, (3.78) and (3.79) imply that

$$\int_{-\infty}^{F_\ell^\dagger(\theta)} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y > 0 \quad \text{and} \quad \int_{F_\ell^\dagger(\theta)}^{\infty} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y > 0$$

for all $\theta < \overline{\theta} + \zeta\lambda$, which implies that

$$w(\theta, x) < \frac{r}{\zeta}\Big(\varphi(x)\Big(\int_{-\infty}^{x} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y + \psi(x)\int_{x}^{\infty} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y\Big)$$

for each $\theta < \overline{\theta} + \zeta\lambda$ and all $x \in \,]F_\ell^\dagger(\theta), F_r^\dagger(\theta)[$. Since $H(\theta)$ is the minimum of $w(\theta, \cdot)$ for each θ,

$$\begin{aligned} w(\theta, H(\theta)) &\le w(\theta, x^\dagger) \\ &< \frac{r}{\zeta}\Big(\varphi(x^\dagger)\Big(\int_{-\infty}^{x^\dagger} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y \\ &\quad + \psi(x^\dagger)\int_{x^\dagger}^{\infty} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y\Big) \to -\infty \quad \text{as } \theta \downarrow -\infty. \end{aligned}$$

This limit, together with (3.96), imply that there exists $\widetilde{\theta} < \overline{\theta} - \zeta\lambda$ such that $w\big(\widetilde{\theta}, H(\widetilde{\theta})\big) = 0$ and $w\big(\theta, H(\theta)\big) > 0$ for all $\theta > \widetilde{\theta}$. □

By Lemmas 3.15 and 3.17, $v\big(\widetilde{\theta}, H(\widetilde{\theta})\big) = -\lambda$, and (3.94) and (3.95) imply that

$$\int_{-\infty}^{H(\widetilde{\theta})} \Psi(y)(\widetilde{\theta} - \Theta(y) + \zeta\lambda)\,\mathrm{d}y = \int_{-\infty}^{F_\ell^\dagger(\widetilde{\theta})} \Psi(y)(\widetilde{\theta} - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \tag{3.97}$$

$$= \int_{-\infty}^{F_r^\dagger(\widetilde{\theta})} \Psi(y)(\widetilde{\theta} - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \tag{3.98}$$

and
$$\int_{H(\widetilde{\theta})}^{\infty} \Phi(y)(\widetilde{\theta} - \Theta(y) + \zeta\lambda)\,\mathrm{d}y = \int_{F_\ell^\dagger(\widetilde{\theta})}^{\infty} \Phi(y)(\widetilde{\theta} - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \tag{3.99}$$

$$= \int_{F_r^\dagger(\widetilde{\theta})}^{\infty} \Phi(y)(\widetilde{\theta} - \Theta(y) - \zeta\lambda)\,\mathrm{d}y. \tag{3.100}$$

In other words, the following system of equations is satisfied:

$$\begin{gathered} \mathcal{H}_1\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0, \quad \mathcal{H}_2\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0, \\ \mathcal{H}_1\big(\widetilde{\theta}, F_r^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0 \quad \text{and} \quad \mathcal{H}_2\big(\widetilde{\theta}, F_r^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0. \end{gathered}$$

Moreover, by the ODE (3.23) and the fact that $H(\theta)$ is the minimum of $w(\theta,\cdot)$,

$$0 < \frac{1}{2}\beta^2\big(H(\widetilde{\theta})\big)w_{xx}\big(\widetilde{\theta}, H(\widetilde{\theta})\big) = \frac{r}{\zeta}\big(\Theta(H(\widetilde{\theta})) - \widetilde{\theta} - \zeta\lambda\big),$$

which implies that

$$\underline{\chi}\big(\widetilde{\theta}+\zeta\lambda\big) < H\big(\widetilde{\theta}\big) < \overline{\chi}\big(\widetilde{\theta}+\zeta\lambda\big).$$

We are now ready to prove the next result, where we study the solvability of the equations (3.31) in the context of Cases I and II of Section 3.4.1. We note that $\widetilde{\theta} = \widetilde{\theta}(\lambda)$ depends on λ, but we consider $\lambda > 0$ to be fixed in the following lemma, so we write $\widetilde{\theta}$ in place of $\widetilde{\theta}(\lambda)$ wherever possible for notational simplicity.

Lemma 3.18. *Suppose that the function Θ satisfies Assumption 3.14 and recall the definitions of $F_\ell^\dagger$ and $F_r^\dagger$ in Lemma 3.15 as well as the definitions of H and $\widetilde{\theta}$ in Lemma 3.17. Consider the system of equations (3.31), where $\mathcal{H}_1$ and $\mathcal{H}_2$ are defined by (3.32) and (3.33). Then, there exist continuous functions $F_\ell^\star, G_\ell^\star, F_r^\star, G_r^\star : \mathscr{D} \to \mathbb{R}$, where*

$$\mathscr{D} = \Big\{(\theta,\lambda) \in \mathbb{R}\times\mathbb{R}_+ \mid \; -\infty < \theta \le \widetilde{\theta}(\lambda) \quad \textit{and} \quad \lambda > 0\Big\},$$

such that, for each $\lambda > 0$,

$$F_\ell^\star(\widetilde{\theta},\lambda) = F_\ell^\dagger(\widetilde{\theta},\lambda), \quad G_\ell^\star(\widetilde{\theta},\lambda) = H(\widetilde{\theta},\lambda) = G_r^\star(\widetilde{\theta},\lambda), \quad F_r^\star(\widetilde{\theta},\lambda) = F_r^\dagger(\widetilde{\theta},\lambda), \tag{3.101}$$

and, for all $\theta < \widetilde{\theta}$,

$$F_\ell^\star(\theta,\lambda) < F_\ell^\dagger(\widetilde{\theta},\lambda) < \underline{\chi}(\theta-\zeta\lambda) < \overline{\chi}(\theta-\zeta\lambda) < F_r^\dagger(\widetilde{\theta},\lambda) < F_r^\star(\theta,\lambda), \tag{3.102}$$

$$\underline{\chi}(\theta+\zeta\lambda) < G_\ell^\star(\theta,\lambda) < H(\widetilde{\theta},\lambda) < G_r^\star(\theta,\lambda) < \overline{\chi}(\theta+\zeta\lambda), \tag{3.103}$$

$$\mathcal{H}_1\big(\theta, F_\ell^\star(\theta,\lambda), G_\ell^\star(\theta,\lambda),\lambda\big) = \mathcal{H}_2\big(\theta, F_\ell^\star(\theta,\lambda), G_\ell^\star(\theta,\lambda),\lambda\big) = 0 \tag{3.104}$$

$$\textit{and} \quad \mathcal{H}_1\big(\theta, F_r^\star(\theta,\lambda), G_r^\star(\theta,\lambda),\lambda\big) = \mathcal{H}_2\big(\theta, F_r^\star(\theta,\lambda), G_r^\star(\theta,\lambda),\lambda\big) = 0.$$

Furthermore, given any $\lambda > 0$,

$$\frac{\partial F_\ell^\star}{\partial \theta}(\theta, \lambda) > 0, \quad \frac{\partial G_\ell^\star}{\partial \theta}(\theta, \lambda) > 0, \quad \frac{\partial F_r^\star}{\partial \theta}(\theta, \lambda) < 0, \quad \frac{\partial G_r^\star}{\partial \theta}(\theta, \lambda) < 0, \tag{3.105}$$

$$\lim_{\theta \downarrow -\infty} F_\ell^\star(\theta, \lambda) = \lim_{\theta \downarrow -\infty} G_\ell^\star(\theta, \lambda) = -\infty \tag{3.106}$$

$$\text{and} \quad \lim_{\theta \downarrow -\infty} F_r^\star(\theta, \lambda) = \lim_{\theta \downarrow -\infty} G_r^\star(\theta, \lambda) = \infty.$$

Given any $\theta < \widetilde{\theta}$,

$$\frac{\partial F_\ell^\star}{\partial \lambda}(\theta, \lambda) < 0, \quad \frac{\partial G_\ell^\star}{\partial \lambda}(\theta, \lambda) > 0, \quad \frac{\partial F_r^\star}{\partial \lambda}(\theta, \lambda) > 0, \quad \frac{\partial G_r^\star}{\partial \lambda}(\theta, \lambda) < 0, \tag{3.107}$$

$$\lim_{\lambda \downarrow 0} F_\ell^\star(\theta, \lambda) = \lim_{\lambda \downarrow 0} G_\ell^\star(\theta, \lambda) = \underline{\chi}(\theta) \quad \text{and} \quad \lim_{\lambda \downarrow 0} F_r^\star(\theta, \lambda) = \lim_{\lambda \downarrow 0} G_r^\star(\theta, \lambda) = \overline{\chi}(\theta). \tag{3.108}$$

Proof. We prove the results for $F_\ell^\star$ and $G_\ell^\star$ only, as the proofs of the results for $F_r^\star$ and $G_r^\star$ follow symmetric arguments. We organise the proof in three steps.

Step 1: Solvability of $\mathcal{H}_1(\theta, F, G, \lambda) = 0$ for $G \in \left]\underline{\chi}(\theta + \zeta\lambda), H(\widetilde{\theta})\right[$. We fix any $\lambda > 0$ and $\theta < \widetilde{\theta}$. The calculation $\frac{\partial \mathcal{H}_1}{\partial \theta}(\theta, F, G, \lambda) = \int_F^G \Psi(y)\, \mathrm{d}y > 0$ implies that

$$\mathcal{H}_1\big(\theta, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) < \mathcal{H}_1\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0.$$

We calculate

$$\frac{\partial \mathcal{H}_1}{\partial F}(\theta, F, G, \lambda) = -\Psi(F)\big(\theta - \Theta(F) - \zeta\lambda\big) < 0 \text{ for all } F \le F_\ell^\dagger(\widetilde{\theta}),$$

because $F_\ell^\dagger(\widetilde{\theta}) < \underline{\chi}(\theta - \zeta\lambda)$, and

$$\frac{\partial \mathcal{H}_1}{\partial G}(\theta, F, G, \lambda) = \Psi(G)\big(\theta - \Theta(G) + \zeta\lambda\big) < 0 \quad \text{for all } G \in \left]\underline{\chi}(\theta + \zeta\lambda), H(\widetilde{\theta})\right[.$$

We then observe that for all $F \leq F_\ell^\dagger(\widetilde{\theta}) < \underline{\chi}(\theta - \zeta\lambda)$,

$$\begin{aligned}\mathcal{H}\big(\theta, F, \underline{\chi}(\theta+\zeta\lambda), \lambda\big) = & \int_F^{\underline{\chi}(\theta+\zeta\lambda)} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y \\ & + 2\zeta\lambda \int_{-\infty}^F \Psi(y)\,\mathrm{d}y > 0.\end{aligned}$$

We are now faced with two possible cases:

- If $\mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}), \lambda\big) \leq 0$, then $\mathcal{H}_1(\theta, F, H(\widetilde{\theta}), \lambda) < 0$ for all $F \leq F_\ell^\dagger(\widetilde{\theta})$.
- If $\mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}), \lambda\big) > 0$, then there exists $\underline{F}(\theta, \lambda) \in \,[-\infty, F_\ell^\dagger(\widetilde{\theta})[$ such that
$$\mathcal{H}_1\big(\theta, \underline{F}(\theta, \lambda), H(\widetilde{\theta}), \lambda\big) = 0 \tag{3.109}$$
and $\mathcal{H}_1\big(\theta, F, H(\widetilde{\theta}), \lambda\big) < 0$ if and only if $F \in \,]\underline{F}(\theta, \lambda), F_\ell^\dagger(\widetilde{\theta})]$.

In either case, there exists a continuous function $\hat{\mathfrak{g}} : \hat{\mathscr{D}} \to \mathbb{R}$ such that

$$\hat{\mathfrak{g}}(\theta, F, \lambda) \in \big]\underline{\chi}(\theta + \zeta\lambda), H(\widetilde{\theta})\big[\quad \text{and} \quad \mathcal{H}_1(\theta, F, \hat{\mathfrak{g}}(\theta, F, \lambda), \lambda) = 0 \tag{3.110}$$

for all $(\theta, F, \lambda) \in \hat{\mathscr{D}}$, where

$$\begin{aligned}\hat{\mathscr{D}} = & \big\{(\theta, F, \lambda) \in \mathbb{R}^3 \mid \ \theta \in \big]-\infty, \widetilde{\theta}(\lambda)\big[,\ \mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}(\lambda)), \lambda\big) > 0, \\ & \quad F \in \big[\underline{F}(\theta, \lambda), F_\ell^\dagger(\widetilde{\theta}(\lambda))\big] \text{ and } \lambda > 0\big\} \\ & \cup \big\{(\theta, F, \lambda) \in \mathbb{R}^3 \mid \ \theta \in \big]-\infty, \widetilde{\theta}(\lambda)\big[,\ \mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}(\lambda)), \lambda\big) \leq 0, \\ & \quad F \in \big[-\infty, F_\ell^\dagger(\widetilde{\theta}(\lambda))\big] \text{ and } \lambda > 0\big\}.\end{aligned} \tag{3.111}$$

Differentiation of the identity in (3.110) implies that

$$\hat{\mathfrak{g}}_F(\theta, F, \lambda) = \frac{\Psi(F)\big(\theta - \Theta(F) - \zeta\lambda\big)}{\Psi(\hat{\mathfrak{g}}(\theta, F, \lambda))\big(\theta - \Theta(\hat{\mathfrak{g}}(\theta, F, \lambda)) + \zeta\lambda\big)}. \tag{3.112}$$

Furthermore, we observe that

$$\begin{aligned}&\mathcal{H}_1\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) - \mathcal{H}_1\big(\theta, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) \\ &\quad = \mathcal{H}_1\big(\theta, F, \hat{\mathfrak{g}}(\theta, F, \lambda), \lambda\big) - \mathcal{H}_1\big(\theta, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big),\end{aligned}$$

in other words,

$$\bigl(\widetilde{\theta}-\theta\bigr)\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})}\Psi(y)\,\mathrm{d}y=\int_F^{F_\ell^\dagger(\widetilde{\theta})}\Psi(y)(\theta-\Theta(y)-\zeta\lambda)\,\mathrm{d}y\\
-\int_{\hat{\mathfrak{g}}(\theta,F,\lambda)}^{H(\widetilde{\theta})}\Psi(y)(\theta-\Theta(y)+\zeta\lambda)\,\mathrm{d}y.$$

Step 2: Existence and uniqueness of $F_\ell^\star$ and $G_\ell^\star$ satisfying (3.101)–(3.104). We first observe that (3.101) follows directly from equations (3.97)–(3.100). Recalling the definition (3.111) of $\hat{\mathfrak{g}}$'s domain $\hat{\mathscr{D}}$ in Step 1, we fix any $\theta\in\,]-\infty,\widetilde{\theta}[$ and consider the equation $\mathcal{H}_2\bigl(\theta,F_\ell,\hat{\mathfrak{g}}(\theta,F_\ell,\lambda),\lambda\bigr)=0$ for $F_\ell\le F_\ell^\dagger(\widetilde{\theta})$. We calculate, using (3.112), the definition (3.33) of $\mathcal{H}_2$ and the definitions of Φ, Ψ in (A.8),

$$\begin{aligned}\frac{\partial\mathcal{H}_2}{\partial F}\bigl(\theta,F,\hat{\mathfrak{g}}(\theta,F,\lambda),\lambda\bigr)&=\Phi\bigl(\hat{\mathfrak{g}}(\theta,F,\lambda)\bigr)\bigl(\theta-\Theta\bigl(\hat{\mathfrak{g}}(\theta,F,\lambda)\bigr)+\zeta\lambda\bigr)\hat{\mathfrak{g}}_F(\theta,F,\lambda)\\
&\quad-\Phi(F)\bigl(\theta-\Theta(F)-\zeta\lambda\bigr)\\
&=\Psi(F)\left(\frac{\varphi\bigl(\hat{\mathfrak{g}}(\theta,F,\lambda)\bigr)}{\psi\bigl(\hat{\mathfrak{g}}(\theta,F,\lambda)\bigr)}-\frac{\varphi(F)}{\psi(F)}\right)\bigl(\theta-\Theta(F)-\zeta\lambda\bigr)<0\end{aligned}$$

for all $F\le F_\ell^\dagger(\widetilde{\theta})<\underline{\chi}(\theta-\zeta\lambda)$. Next, the calculation $\frac{\partial\mathcal{H}_2}{\partial\theta}(\theta,F,G,\lambda)=\int_F^G\Phi(y)\,\mathrm{d}y>0$ implies that

$$\mathcal{H}_2\bigl(\theta,F_\ell^\dagger(\widetilde{\theta}),H(\widetilde{\theta}),\lambda\bigr)<\mathcal{H}_2\bigl(\widetilde{\theta},F_\ell^\dagger(\widetilde{\theta}),H(\widetilde{\theta}),\lambda\bigr)=0.$$

Then,

$$\begin{aligned}&\mathcal{H}_2\bigl(\widetilde{\theta},F_\ell^\dagger(\widetilde{\theta}),H(\widetilde{\theta}),\lambda\bigr)-\mathcal{H}_2\bigl(\theta,F_\ell^\dagger(\widetilde{\theta}),H(\widetilde{\theta}),\lambda\bigr)\\
&=\bigl(\widetilde{\theta}-\theta\bigr)\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})}\Phi(y)\,\mathrm{d}y\begin{cases}<\bigl(\widetilde{\theta}-\theta\bigr)\dfrac{\varphi(F_\ell^\dagger(\widetilde{\theta}))}{\psi(F_\ell^\dagger(\widetilde{\theta}))}\displaystyle\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})}\Psi(y)\,\mathrm{d}y,\\[2ex] >\bigl(\widetilde{\theta}-\theta\bigr)\dfrac{\varphi(H(\widetilde{\theta}))}{\psi(H(\widetilde{\theta}))}\displaystyle\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})}\Psi(y)\,\mathrm{d}y.\end{cases}\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{H}_2\big(\theta, F, \hat{\mathfrak{g}}(\theta, F, \lambda), \lambda\big) - \mathcal{H}_2\big(\theta, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) \\
&= \int_F^{F_\ell^\dagger(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y - \int_{\hat{\mathfrak{g}}(\theta,F,\lambda)}^{H(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y.
\end{aligned}$$

We first observe that, when $F = F_\ell^\dagger(\widetilde{\theta})$,

$$\begin{aligned}
&\big(\widetilde{\theta} - \theta\big)\frac{\varphi(H(\widetilde{\theta}))}{\psi(H(\widetilde{\theta}))}\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})} \Psi(y)\,\mathrm{d}y \\
&= -\frac{\varphi(H(\widetilde{\theta}))}{\psi(H(\widetilde{\theta}))}\int_{\hat{\mathfrak{g}}(\theta,F_\ell^\dagger(\widetilde{\theta}),\lambda)}^{H(\widetilde{\theta})} \Psi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y \\
&> -\int_{\hat{\mathfrak{g}}(\theta,F_\ell^\dagger(\widetilde{\theta}),\lambda)}^{H(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) + \zeta\lambda)\,\mathrm{d}y,
\end{aligned}$$

which implies that

$$\mathcal{H}_2\big(\theta, F_\ell^\dagger(\widetilde{\theta}), \hat{\mathfrak{g}}(\theta, F_\ell^\dagger(\widetilde{\theta}), \lambda), \lambda\big) < \mathcal{H}_2\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0.$$

Next, we consider the case $\mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}), \lambda\big) > 0$. We observe by (3.109) that

$$\hat{\mathfrak{g}}(\theta, \underline{F}(\theta, \lambda), \lambda) = H(\widetilde{\theta}),$$

which implies that

$$\begin{aligned}
\big(\widetilde{\theta} - \theta\big)\frac{\varphi(F_\ell^\dagger(\widetilde{\theta}))}{\psi(F_\ell^\dagger(\widetilde{\theta}))}\int_{F_\ell^\dagger(\widetilde{\theta})}^{H(\widetilde{\theta})} \Psi(y)\,\mathrm{d}y &= \frac{\varphi(F_\ell^\dagger(\widetilde{\theta}))}{\psi(F_\ell^\dagger(\widetilde{\theta}))}\int_{\underline{F}(\theta,\lambda)}^{F_\ell^\dagger(\widetilde{\theta})} \Psi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y \\
&< \int_{\underline{F}(\theta,\lambda)}^{F_\ell^\dagger(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) - \zeta\lambda)\,\mathrm{d}y,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathcal{H}_2\big(\theta, \underline{F}(\theta, \lambda), \hat{\mathfrak{g}}(\theta, \underline{F}(\theta, \lambda), \lambda), \lambda\big) \\
&= \mathcal{H}_2\big(\theta, \underline{F}(\theta, \lambda), H(\widetilde{\theta}), \lambda\big) > \mathcal{H}_2\big(\widetilde{\theta}, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) = 0.
\end{aligned}$$

On the other hand, if $\mathcal{H}_1\big(\theta, -\infty, H(\widetilde{\theta}), \lambda\big) \leq 0$, Lemma A.1 implies that

$$\begin{aligned}
&\lim_{F \downarrow -\infty} \Big(\mathcal{H}_2\big(\theta, F, \hat{\mathfrak{g}}(\theta, F, \lambda), \lambda\big) - \mathcal{H}_2\big(\theta, F_\ell^\dagger(\widetilde{\theta}), H(\widetilde{\theta}), \lambda\big) \Big) \\
&= \lim_{F \downarrow -\infty} \bigg(\int_F^{F_\ell^\dagger(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) - \zeta\lambda) \,\mathrm{d}y \\
&\qquad - \int_{\hat{\mathfrak{g}}(\theta, F, \lambda)}^{H(\widetilde{\theta})} \Phi(y)(\theta - \Theta(y) + \zeta\lambda) \,\mathrm{d}y \bigg) = \infty.
\end{aligned}$$

In either case, we conclude that there exists a unique $F_\ell^\star(\theta, \lambda) < F_\ell^\dagger(\widetilde{\theta})$ such that

$$\mathcal{H}_2\big(\theta, F_\ell^\star(\theta, \lambda), \hat{\mathfrak{g}}(\theta, F_\ell^\star(\theta, \lambda), \lambda), \lambda\big) = 0. \tag{3.113}$$

If we define $G_\ell^\star(\theta, \lambda) = \hat{\mathfrak{g}}(\theta, F_\ell^\star(\theta, \lambda), \lambda)$, then (3.110) and (3.113) imply that the functions $F_\ell^\star$ and $G_\ell^\star$ satisfy (3.102), (3.103) and (3.104).

Step 3: Proof of the monotonicity of the functions $F_\ell^\star$ and $G_\ell^\star$ and limits in (3.106) and (3.108). The inequalities in (3.105) and (3.107), as well as the limits in (3.106) and (3.108) follow the same calculations and arguments as those in Lemma 3.10. □

We conclude the section with growth estimates of the free-boundary functions in Lemma 3.18.

Lemma 3.19. *Fix $\lambda > 0$ and recall the free-boundary functions $F_\ell^\star$, $G_\ell^\star$, $F_r^\star$ and $G_r^\star$ in Lemma 3.18, as well as the constant $p \geq 1$ in Assumption 3.2. Then, there exists constants $C_\ell, C_r > 0$ such that*

$$G_\ell^\star(\theta, \lambda) < -C_\ell(-\theta)^{\frac{1}{p}} \quad \textit{and} \quad G_r^\star(\theta, \lambda) > C_r(-\theta)^{\frac{1}{p}} \quad \textit{for all } \theta \ll 0. \tag{3.114}$$

Proof. We first prove that for $\theta \ll 0$, there exists a unique function $G_\ell^\sharp$ such that $\underline{\mathcal{H}}\big(\theta, G^{\sharp}_{\ell}(\theta,\lambda),\lambda\big) = 0$, where $\underline{\mathcal{H}}$ is defined by (3.37). The calculations

$$\underline{\mathcal{H}}\big(\theta, \underline{\chi}(\theta+\zeta\lambda),\lambda\big) = \int_{-\infty}^{\underline{\chi}(\theta+\zeta\lambda)} \Psi(y)\big(\theta-\Theta(y)+\zeta\lambda\big)\,\mathrm{d}y > 0,$$

$$\underline{\mathcal{H}}_G(\theta,G,\lambda) = \Psi(G)\big(\theta-\Theta(G)+\zeta\lambda\big)\begin{cases} <0, & \text{if } G \in \big]\underline{\chi}(\theta+\zeta\lambda), \overline{\chi}(\theta+\zeta\lambda)\big[, \\ \geq 0, & \text{otherwise,}\end{cases}$$

$$\frac{\partial\underline{\mathcal{H}}}{\partial\theta}\big(\theta, \overline{\chi}(\theta+\zeta\lambda),\lambda\big) = \int_{-\infty}^{\overline{\chi}(\theta+\zeta\lambda)} \Psi(y)\,\mathrm{d}y > 0$$

$$\text{and} \quad \lim_{\theta\downarrow-\infty}\int_{-\infty}^{\overline{\chi}(\theta+\zeta\lambda)} \Psi(y)\,\mathrm{d}y \overset{(A.11)}{=} \infty$$

imply that for $\theta \ll 0$, there exists $G^{\sharp}_{\ell}(\theta,\lambda) \in \big]\underline{\chi}(\theta+\zeta\lambda), \overline{\chi}(\theta+\zeta\lambda)\big[$ such that $\underline{\mathcal{H}}\big(\theta, G^{\sharp}_{\ell}(\theta,\lambda),\lambda\big) = 0$. By Lemma 3.18, $G^{\star}_{\ell}(\theta,\lambda) \in \big]\underline{\chi}(\theta+\zeta\lambda), \overline{\chi}(\theta+\zeta\lambda)\big[$ and

$$\underline{\mathcal{H}}\big(\theta, G^{\star}_{\ell}(\theta,\lambda),\lambda\big) = \int_{-\infty}^{F^{\star}_{\ell}(\theta,\lambda)} \Psi(y)\big(\theta-\Theta(y)-\zeta\lambda\big)\,\mathrm{d}y > 0,$$

where the inequality follows from (3.102). This inequality combined with the fact that $\underline{\mathcal{H}}_G < 0$ for all $G \in \big]\underline{\chi}(\theta+\zeta\lambda), \overline{\chi}(\theta+\zeta\lambda)\big[$ imply that $G^{\star}_{\ell}(\theta,\lambda) < G^{\sharp}_{\ell}(\theta,\lambda)$. We therefore prove the first inequality in (3.114) by proving that $G^{\sharp}_{\ell}(\theta,\lambda) < -C_{\ell}(-\theta)^{\frac{1}{p}}$ for $\theta \ll 0$.

By (3.5), there exists a constant $\check{C} > 0$ such that $|\Theta(x)| \leq \check{C}(1+|x|^p)$ for $x \ll 0$. By similar arguments as in Lemma 3.12,

$$|x|^p \int_{-\infty}^{x} \Psi(y)\,\mathrm{d}y \leq \int_{-\infty}^{x} \Psi(y)|y|^p\,\mathrm{d}y < \infty \quad \text{for all } x < 0,$$

which implies that

$$\lim_{x\downarrow-\infty} |x|^p \int_{-\infty}^{x} \Psi(y)\,\mathrm{d}y = 0.$$

By L'Hôpital's theorem, we calculate

$$\begin{aligned}\liminf_{x\downarrow-\infty} \frac{\int_{-\infty}^{x}\Psi(y)\Theta(y)\,\mathrm{d}y}{|x|^p\int_{-\infty}^{x}\Psi(y)\,\mathrm{d}y} &\geq \liminf_{x\downarrow-\infty}\left(-\frac{\check{C}}{|x|^p} - \frac{\check{C}\int_{-\infty}^{x}\Psi(y)|y|^p\,\mathrm{d}y}{|x|^p\int_{-\infty}^{x}\Psi(y)\,\mathrm{d}y}\right) \\ &= -\check{C}\liminf_{x\downarrow-\infty}\frac{\Psi(x)|x|^p}{\Psi(x)|x|^p + p|x|^{p-1}\int_{-\infty}^{x}\Psi(y)\,\mathrm{d}y} \geq -\check{C},\end{aligned}$$

where the final inequality follows from the strict positivity of Ψ. This implies that

$$-2\check{C}\big(-G_\ell^\sharp(\theta,\lambda)\big)^p \int_{-\infty}^{G_\ell^\sharp(\theta,\lambda)} \Psi(y)\,\mathrm{d}y < \int_{-\infty}^{G_\ell^\sharp(\theta,\lambda)} \Psi(y)\Theta(y)\,\mathrm{d}y$$
$$= (\theta+\zeta\lambda)\int_{-\infty}^{G_\ell^\sharp(\theta,\lambda)} \Psi(y)\,\mathrm{d}y,$$

where the equality follows from the fact that $\underline{\mathcal{H}}(\theta, G_\ell^\sharp(\theta,\lambda),\lambda) = 0$. Therefore, for $\theta \ll 0$ sufficiently negative, there exists a constant $\tilde{C}_\ell > 0$ such that

$$-G_\ell^\sharp(\theta,\lambda) > \left(-\frac{\theta+\zeta\lambda}{2\check{C}}\right)^{\frac{1}{p}} > \left(-\frac{\tilde{C}_\ell\theta}{2\check{C}}\right)^{\frac{1}{p}},$$

and we obtain the first estimate in (3.114) with $C_\ell = \big(\frac{\tilde{C}_\ell}{2\check{C}}\big)^{\frac{1}{p}}$. For the second estimate in (3.114), we first prove in a similar way as before that for $\theta \ll 0$, there exists a unique function $G_r^\sharp$ such that $G_r^\sharp(\theta,\lambda) \in \big]\underline{\chi}(\theta+\zeta\lambda), \overline{\chi}(\theta+\zeta\lambda)\big[$ and

$$\int_{G_r^\sharp(\theta,\lambda)}^{\infty} \Phi(y)\big(\theta - \Theta(y) + \zeta\lambda\big)\,\mathrm{d}y = 0.$$

The remaining arguments are analogous to the previous ones, by considering $x \gg 0$. $\square$

3.4.3 The solution to the free-boundary problem

We now outline the solution to the free-boundary problem, having solved for the free-boundary functions. In view of Lemmas 3.15, 3.16 and 3.17, the points $\widetilde{\theta} < \check{\theta}$ considered in Section 3.4.1 are given by $\check{\theta} = \overline{\theta} + \zeta\lambda$, and $\widetilde{\theta}$ is as in Lemma 3.17. Furthermore, the functions $\mathfrak{F}_\ell$, $\mathfrak{F}_r$, $\mathfrak{G}_\ell$ and $\mathfrak{G}_r$ separating the continuation region $\mathcal{C}$ from the sell region $\mathcal{S}$ and the buy region $\mathcal{B}$, are given by

$$\mathfrak{F}_\ell(\theta) = \begin{cases} F_\ell^\dagger(\theta), & \text{if } \theta \in \left]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right], \\ F_\ell^\star(\theta), & \text{if } \theta \in \left]-\infty, \widetilde{\theta}\right], \end{cases}$$

$$\text{and} \quad \mathfrak{F}_r(\theta) = \begin{cases} F_r^\dagger(\theta), & \text{if } \theta \in \left]\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right], \\ F_r^\star(\theta), & \text{if } \theta \in \left]-\infty, \widetilde{\theta}\right]. \end{cases}$$

and the functions $\mathfrak{G}_\ell$ and $\mathfrak{G}_r$ are the functions $G_\ell^\star$ and $G_r^\star$ respectively in Lemma 3.18. This is illustrated in Figure 10.

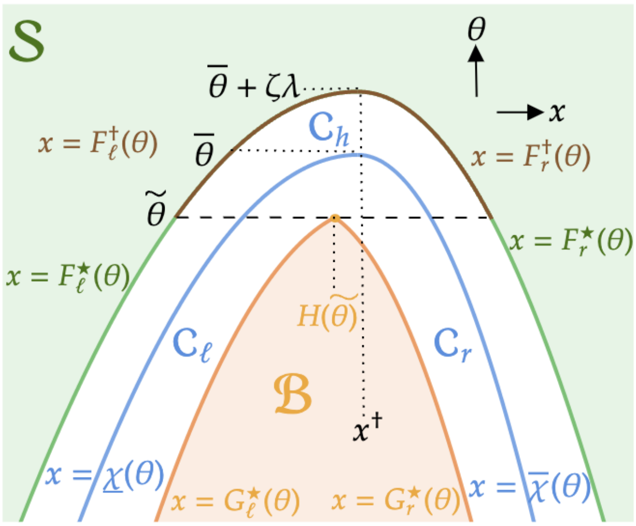

Figure 10: Free-boundaries when Θ strictly increasing and then strictly decreasing

In this context, we are faced with the solution v to the ODE (3.23) that is given by

$$v(\theta, x) = \begin{cases} A_\ell(\theta)\varphi(x) + B_\ell(\theta)\psi(x) - \frac{r}{\zeta}R_\Theta(x) + \frac{\theta}{\zeta}, & \text{if } (\theta, x) \in \operatorname{cl}\mathcal{C}_h \cup \operatorname{cl}\mathcal{C}_\ell, \\ A_r(\theta)\varphi(x) + B_r(\theta)\psi(x) - \frac{r}{\zeta}R_\Theta(x) + \frac{\theta}{\zeta}, & \text{if } (\theta, x) \in \operatorname{cl}\mathcal{C}_h \cup \operatorname{cl}\mathcal{C}_r, \\ v\big(\theta, \mathfrak{F}_\ell(\theta)\big) = v\big(\theta, \mathfrak{F}_r(\theta)\big) = \lambda, & \text{if } (\theta, x) \in \operatorname{int}\mathcal{S}, \\ v\big(\theta, \mathfrak{G}_\ell(\theta)\big) = v\big(\theta, \mathfrak{G}_r(\theta)\big) = -\lambda, & \text{if } (\theta, x) \in \operatorname{int}\mathcal{B}, \end{cases} \tag{3.115}$$

where, for all $\theta \in \left]-\infty, \overline{\theta} + \zeta\lambda\right]$,

$$A_\ell(\theta) = -\frac{r}{\zeta} \int_{-\infty}^{\mathfrak{F}_\ell(\theta)} \Psi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\, \mathrm{d}y, \tag{3.116}$$

$$A_r(\theta) = -\frac{r}{\zeta} \int_{-\infty}^{\mathfrak{F}_r(\theta)} \Psi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\, \mathrm{d}y, \tag{3.117}$$

$$B_\ell(\theta) = -\frac{r}{\zeta} \int_{\mathfrak{F}_\ell(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\, \mathrm{d}y \tag{3.118}$$

$$\text{and} \quad B_r(\theta) = -\frac{r}{\zeta} \int_{\mathfrak{F}_r(\theta)}^{\infty} \Phi(y)\big(\theta - \Theta(y) - \zeta\lambda\big)\, \mathrm{d}y, \tag{3.119}$$

where, for all $\theta \in \left[\widetilde{\theta}, \overline{\theta} + \zeta\lambda\right[$, $A_\ell(\theta) = A_r(\theta)$ and $B_\ell(\theta) = B_r(\theta)$, because $F_\ell^\dagger$ and $F_r^\dagger$ satisfy the system of equations (3.73).

Lemma 3.20. *The function v given by* (3.115) *is well-defined in the sense that the integrals in* (3.116)*,* (3.117)*,* (3.118) *and* (3.119) *are well-defined and real-valued. Furthermore, v is a $C^{1,2}$ solution to the ODE (3.23) that satisfies $|v(\theta, x)| < \lambda$ for all $(\theta, x) \in \mathcal{C}$.*

Proof. The proof that the integrals in (3.116), (3.117), (3.118) and (3.119) are well-defined and real-valued, as well as the proof that $|v(\theta, x)| < \lambda$ for all $(\theta, x) \in \mathcal{C}_\ell \cup \mathcal{C}_r$ are analogous to the proofs in Lemma 3.13. The proof that $|v(\theta, x)| < \lambda$ for all $(\theta, x) \in \mathcal{C}_h$ follows from Lemmas 3.16 and 3.17. □

3.5 The solution to the control problem

In this section, we construct the optimal trading strategies corresponding to the solution to the free-boundary Problem 3.8, in the case where Θ is strictly increasing, and the case where Θ is strictly increasing and then strictly decreasing.

3.5.1 Optimal trading strategy when Θ is strictly increasing

Suppose first that Θ is strictly increasing. In this case, the construction is exactly the same as in Løkka and Zervos [84] (similar constructions can also be found in Merhi and Zervos [92] and Kruk et al. [76]) - we reproduce the construction here for completeness of exposition. The free-boundary functions that we obtained previously suggests the trading strategy introduced by the following definition.

Definition 3.21. *Consider the functions $\mathfrak{F}$ and $\mathfrak{G}$ and the associated domains $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_m \cup \mathcal{C}_h$ appearing in Section 3.3.3. Given any initial condition $(\theta, x) \in \mathbb{R}^2$, we denote by $\hat{\vartheta}$ the trading strategy that instantaneously buys or sells shares to the closest boundary point of $\mathcal{C}$ if $(\theta, x) \notin \mathcal{C}$, and then takes minimal action so as to reflect the process $(\hat{\vartheta}, X)$ in the boundaries $\mathfrak{G}$ in the positive θ-direction and in the boundaries $\mathfrak{F}$ in the negative θ-direction. In particular, the process $\hat{\vartheta}$ has a positive jump of size $\mathfrak{G}^{-1}(x) - \theta$ if $\mathfrak{G}^{-1}(x) > \theta$, and a negative jump of size $\theta - \mathfrak{F}^{-1}(x)$ if $\theta > \mathfrak{F}^{-1}(x)$ at time 0, and satisfies*

$$\mathrm{d}\hat{\vartheta}_t = \big[\mathbf{1}_{\{\hat{\vartheta}_t = \mathfrak{G}^{-1}(X_t)\}} - \mathbf{1}_{\{\hat{\vartheta}_t = \mathfrak{F}^{-1}(X_t)\}}\big]\mathrm{d}\hat{\vartheta}_t \quad \textit{for all } t > 0. \tag{3.120}$$

The solution to Skorokhod's equation (see Karatzas and Shreve [71, Lemma 3.6.C.14]) inspires the iterative construction of the process $\hat{\vartheta}$ in a pathwise sense as follows. First, we fix a sample path $X(\omega)$ of any continuous real-valued stochastic process X and drop the argument "ω" for notational simplicity. We then define the times

$$\tau_0^+ = \inf\big\{t \geq 0 : X_t > \mathfrak{G}(\theta)\big\} \quad \text{and} \quad \tau_0^- = \inf\big\{t \geq 0 : X_t < \mathfrak{F}(\theta)\big\},$$

and we assume that $\tau_0^+ < \tau_0^-$ in what follows; if $\tau_0^- < \tau_0^+$, then only straightforward revisions of the arguments are required. We define

$$\vartheta_t^{(1)+} = \Big[\mathfrak{G}^{-1}\Big(\sup_{u \leq t} X_u\Big) - \theta\Big]^+ \mathbf{1}_{\{0<t\}}, \quad \vartheta_t^{(1)-} = 0, \quad \vartheta_t^{(1)} = \theta + \vartheta_t^{(1)+} - \vartheta_t^{(1)-}$$
$$\text{and} \quad \tau_1 = \inf\Big\{t \geq 0 : X_t < \mathfrak{F}\big(\vartheta_t^{(1)}\big)\Big\},$$

and we note that $(\vartheta^{(1)}, X)$ is reflecting in $\mathfrak{G}$ in the positive θ-direction,

$$(\vartheta^{(1)}, X) \in \operatorname{cl}\mathcal{C} \text{ for all } t \leq \tau_1 \quad \text{and} \quad \vartheta_{\tau_1}^{(1)} = \mathfrak{F}^{-1}(X_{\tau_1}).$$

If $\overline{\theta} < \infty$, $\underline{\theta} > -\infty$ and $\lambda \geq \frac{\overline{\theta} - \underline{\theta}}{2\zeta}$, then $\mathfrak{F}$ and $\mathfrak{G}$ identify with the functions $F^\dagger$

and $G^\dagger$ as depicted by Figure 6 and $\tau_1 = \infty$, and the construction is complete. On the other hand, if $\overline{\theta} < \infty$, $\underline{\theta} > -\infty$ and $\lambda < \frac{\overline{\theta}-\underline{\theta}}{2\zeta}$ (as depicted in Figure 5), or either $\overline{\theta} = \infty$ or $\underline{\theta} = -\infty$, then $\tau_1 < \infty$ and we continue the construction as follows. We define

$$\vartheta_t^{(2)+} = \vartheta_{t\wedge\tau_1}^{(1)+}, \quad \vartheta_t^{(2)-} = \Big[\vartheta_{\tau_1}^{(1)} - \mathfrak{F}^{-1}\Big(\inf_{\tau_1\le u\le t} X_u\Big)\Big]\mathbf{1}_{\{\tau_1\le t\}},$$
$$\vartheta_t^{(2)} = \theta + \vartheta_t^{(2)+} - \vartheta_t^{(2)-} \text{ and } \tau_2 = \inf\Big\{t \ge 0 : X_t > \mathfrak{G}\big(\vartheta_t^{(2)}\big)\Big\} > \tau_1.$$

An inspection of these definitions reveals that $(\vartheta^{(2)}, X)$ is reflecting in $\mathfrak{G}$ in the positive θ-direction and in $\mathfrak{F}$ in the negative θ-direction up to time τ_2,

$$\vartheta_t^{(2)} = \vartheta_t^{(1)} \text{ for all } t \le \tau_1, \quad (\vartheta_t^{(2)}, X_t) \in \operatorname{cl}\mathcal{C} \text{ for all } t \le \tau_2$$
$$\text{and} \quad \vartheta_{\tau_2}^{(2)} = \mathfrak{G}^{-1}(X_{\tau_2}).$$

We then iterate these constructions by defining

$$\vartheta_t^{(2n+1)+} = \vartheta_{t\wedge\tau_{2n}}^{(2n)+} + \Big[\mathfrak{G}^{-1}\Big(\sup_{\tau_{2n}\le u\le t} X_u\Big) - \vartheta_{\tau_{2n}}^{(2n)+}\Big]\mathbf{1}_{\{\tau_{2n}\le t\}},$$
$$\vartheta_t^{(2n+1)-} = \vartheta_{t\wedge\tau_{2n}}^{(2n)-}, \quad \vartheta_t^{(2n+1)} = \theta + \vartheta_t^{(2n+1)+} - \vartheta_t^{(2n+1)-},$$
$$\tau_{2n+1} = \inf\Big\{t \ge 0 : X_t < \mathfrak{F}\big(\vartheta_t^{(2n+1)}\big)\Big\} > \tau_{2n},$$
$$\vartheta_t^{(2n)-} = \vartheta_{t\wedge\tau_{2n-1}}^{(2n-1)-} + \Big[\vartheta_{\tau_{2n-1}}^{(2n-1)+} - \mathfrak{F}^{-1}\Big(\inf_{\tau_{2n-1}\le u\le t} X_u\Big)\Big]\mathbf{1}_{\{\tau_{2n-1}\le t\}},$$
$$\vartheta_t^{(2n)+} = \vartheta_{t\wedge\tau_{2n-1}}^{(2n-1)+}, \quad \vartheta_t^{(2n)} = \theta + \vartheta_t^{(2n)+} - \vartheta_t^{(2n)-}$$
$$\text{and} \quad \tau_{2n} = \inf\Big\{t \ge 0 : X_t > \mathfrak{G}\big(\vartheta_t^{(2n)}\big)\Big\} > \tau_{2n-1}$$

for $n \ge 1$, and we note that, given any $m, k \ge 1$,

$$\vartheta_t^{(m+k)} = \vartheta_t^{(m)} = \theta + \vartheta_t^{(m)+} - \vartheta_t^{(m)-} \quad \text{and} \quad (\vartheta_t^{(m)}, X_t) \in \operatorname{cl}\mathcal{C} \quad \text{for all } t \le \tau_m$$

and $\lim_{n\to\infty} \tau_n = \infty$. Therefore, we can define $\hat{\vartheta}^+$, $\hat{\vartheta}^-$ and $\hat{\vartheta}$ by

$$\hat{\vartheta}_t^+ = \vartheta_t^{(m)+}, \quad \hat{\vartheta}_t^- = \vartheta_t^{(m)-} \quad \text{and} \quad \hat{\vartheta}_t = \vartheta_t^{(m)}$$

for any $m \ge 1$ such that $t < \tau_m$. The finite-variation function $\hat{\vartheta}$ constructed satisfies (3.121) because this is true for all of the functions $\vartheta^{(n)}$. Indeed, an inspection of the iterative algorithm that we have developed reveals that $\vartheta^{(n)}$ increases (resp., decreases) on the set

$$\Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{G}^{-1}\Big(\sup_{0 \leq u \leq t} X_u\Big) \text{ and } X_t = \sup_{0 \leq u \leq t} X_u\Big\}$$
$$\Big(\text{resp., } \Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{F}^{-1}\Big(\inf_{0 \leq u \leq t} X_u\Big) \text{ and } X_t = \inf_{0 \leq u \leq t} X_u\Big\}\Big).$$

3.5.2 Optimal trading strategy when Θ is strictly increasing and then strictly decreasing

Suppose now that Θ is strictly increasing and then strictly decreasing. The construction of the corresponding optimal trading strategy is adapted from Løkka and Zervos [84]. The free-boundary functions that we obtained previously suggests the trading strategy introduced by the following definition.

Definition 3.22. *Consider the functions $\mathfrak{F}_\ell$, $\mathfrak{G}_\ell$, $\mathfrak{G}_r$ and $\mathfrak{F}_r$ and the associated domains $\mathcal{B}$, $\mathcal{S}$ and $\mathcal{C} = \mathcal{C}_\ell \cup \mathcal{C}_h \cup \mathcal{C}_r$ appearing in Section 3.4.3, and recall the definitions of $\widetilde{\theta}$ and $H(\widetilde{\theta})$ in Lemma 3.17. Given any initial condition $(\theta, x) \in \mathbb{R}^2$, we denote by $\hat{\vartheta}$ the trading strategy that instantaneously buys or sells shares to the closest boundary point of $\mathcal{C}$ if $(\theta, x) \notin \mathcal{C}$, and then takes minimal action so as to reflect the process $(\hat{\vartheta}, X)$ in the boundaries $\mathfrak{G}_\ell$ and $\mathfrak{G}_r$ in the positive θ-direction and in the boundaries $\mathfrak{F}_\ell$ and $\mathfrak{F}_r$ in the negative θ-direction. In particular, the process $\hat{\vartheta}$ has a positive jump of size*

$$\begin{cases} \mathfrak{G}_\ell^{-1}(x) - \theta, & \text{if } \mathfrak{G}_\ell^{-1}(x) > \theta \text{ and } x \leq H(\widetilde{\theta}), \\ \mathfrak{G}_r^{-1}(x) - \theta, & \text{if } \mathfrak{G}_r^{-1}(x) > \theta \text{ and } x \geq H(\widetilde{\theta}) \end{cases}$$

at time 0, has a negative jump of size

$$\begin{cases} \theta - \mathfrak{F}_\ell^{-1}(x), & \text{if } \theta > \mathfrak{F}_\ell^{-1}(x) \text{ and } x \leq x^\dagger, \\ \theta - \mathfrak{F}_r^{-1}(x), & \text{if } \theta > \mathfrak{F}_r^{-1}(x) \text{ and } x \geq x^\dagger \end{cases}$$

at time 0, and, for all $t > 0$, satisfies

$$\mathrm{d}\hat{\vartheta}_t = \big[\mathbf{1}_{\{\hat{\vartheta}_t=\mathfrak{G}_\ell^{-1}(X_t)\}} + \mathbf{1}_{\{\hat{\vartheta}_t=\mathfrak{G}_r^{-1}(X_t)\}} - \mathbf{1}_{\{\hat{\vartheta}_t=\mathfrak{F}_\ell^{-1}(X_t)\}} - \mathbf{1}_{\{\hat{\vartheta}_t=\mathfrak{F}_r^{-1}(X_t)\}}\big]\mathrm{d}\hat{\vartheta}_t. \tag{3.121}$$

Before we construct the trading strategy $\hat{\vartheta}$, we need to extend the functions $\mathfrak{F}_\ell^{-1}$, $\mathfrak{F}_r^{-1}$, $\mathfrak{G}_\ell^{-1}$ and $\mathfrak{G}_r^{-1}$ in the following way. Let

$$\overline{\mathfrak{F}_\ell}^{-1}(x) = \begin{cases} \mathfrak{F}_\ell^{-1}(x), & \text{if } x \leq x^\dagger, \\ \hat{\mathfrak{F}}_\ell^{-1}(x), & \text{if } x \geq x^\dagger, \end{cases} \quad \overline{\mathfrak{F}_r}^{-1}(x) = \begin{cases} \mathfrak{F}_r^{-1}(x), & \text{if } x \geq x^\dagger, \\ \hat{\mathfrak{F}}_r^{-1}(x), & \text{if } x \leq x^\dagger, \end{cases}$$

$$\overline{\mathfrak{G}_\ell}^{-1}(x) = \begin{cases} \mathfrak{G}_\ell^{-1}(x), & \text{if } x \leq H(\widetilde{\theta}), \\ \hat{\mathfrak{G}}_\ell^{-1}(x), & \text{if } x \geq H(\widetilde{\theta}), \end{cases} \text{ and } \overline{\mathfrak{G}_r}^{-1}(x) = \begin{cases} \mathfrak{G}_r^{-1}(x), & \text{if } x \geq H(\widetilde{\theta}), \\ \hat{\mathfrak{G}}_r^{-1}(x), & \text{if } x \leq H(\widetilde{\theta}), \end{cases}$$

where $\hat{\mathfrak{F}}_\ell^{-1} : [x^\dagger, \infty[\to [\overline{\theta} + \zeta\lambda, \infty[$ and $\hat{\mathfrak{G}}_\ell^{-1} : [H(\widetilde{\theta}), \infty[\to [\widetilde{\theta}, \infty[$ are strictly increasing functions and $\hat{\mathfrak{F}}_r^{-1} :]-\infty, x^\dagger] \to [\overline{\theta}+\zeta\lambda, \infty[$ and $\hat{\mathfrak{G}}_r^{-1} :]-\infty, H(\widetilde{\theta})] \to [\widetilde{\theta}, \infty[$ are strictly decreasing functions such that

$$\hat{\mathfrak{F}}_\ell^{-1}(x^\dagger) = \mathfrak{F}_\ell^{-1}(x^\dagger) = \mathfrak{F}_r^{-1}(x^\dagger) = \hat{\mathfrak{F}}_r^{-1}(x^\dagger) = \check{\theta},$$
$$\hat{\mathfrak{G}}_\ell^{-1}(H(\widetilde{\theta})) = \mathfrak{G}_\ell^{-1}(H(\widetilde{\theta})) = \mathfrak{G}_r^{-1}(H(\widetilde{\theta})) = \hat{\mathfrak{G}}_r^{-1}(H(\widetilde{\theta})) = \widetilde{\theta}.$$

Since $\mathfrak{F}_\ell^{-1}$ and $\mathfrak{G}_\ell^{-1}$ are strictly increasing functions and $\mathfrak{F}_r^{-1}$ and $\mathfrak{G}_r^{-1}$ are strictly decreasing functions, this implies that

$$\begin{cases} \overline{\mathfrak{F}_\ell}^{-1}(x) < \overline{\mathfrak{F}_r}^{-1}(x), & \text{if } x < x^\dagger, \\ \overline{\mathfrak{F}_\ell}^{-1}(x) > \overline{\mathfrak{F}_r}^{-1}(x), & \text{if } x > x^\dagger, \end{cases} \quad \text{and} \quad \begin{cases} \overline{\mathfrak{G}_\ell}^{-1}(x) < \overline{\mathfrak{G}_r}^{-1}(x), & \text{if } x < H(\widetilde{\theta}), \\ \overline{\mathfrak{G}_\ell}^{-1}(x) > \overline{\mathfrak{G}_r}^{-1}(x), & \text{if } x > H(\widetilde{\theta}), \end{cases}$$

as illustrated in Figure 11. As before, we perform a pathwise iterative construction of the process $\hat{\vartheta}$. For notational simplicity, we write $\mathfrak{F}_\ell^{-1}$,$\mathfrak{F}_r^{-1}$,$\mathfrak{G}_\ell^{-1}$ and $\mathfrak{G}_r^{-1}$ in place of $\overline{\mathfrak{F}_\ell}^{-1}$,$\overline{\mathfrak{F}_r}^{-1}$,$\overline{\mathfrak{G}_\ell}^{-1}$ and $\overline{\mathfrak{G}_r}^{-1}$. We define the times

$$\tau_0^+ = \inf\big\{t \geq 0 : \theta < \widetilde{\theta} \text{ and } \mathfrak{G}_\ell(\theta) < X_t < \mathfrak{G}_r(\theta)\big\}$$
$$\text{and} \quad \tau_0^- = \inf\big\{t \geq 0 : \theta < \overline{\theta} + \zeta\lambda \text{ and } X_t < \mathfrak{F}_\ell(\theta) \text{ or } X_t > \mathfrak{F}_r(\theta)\big\},$$

and we assume that $\tau_0^+ < \tau_0^-$ in what follows; if $\tau_0^- < \tau_0^+$, then only straight-

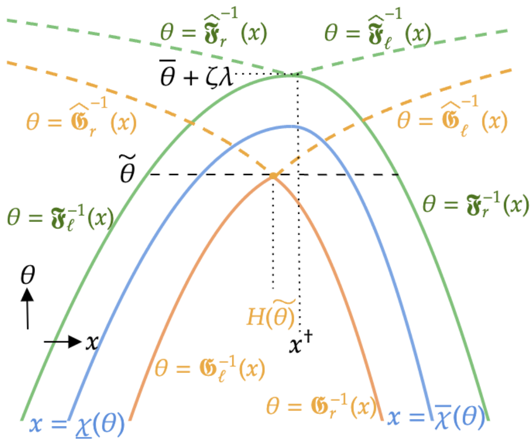

Figure 11: Extending functions

forward revisions of the arguments are required. Moreover, we note that after the initial jump, $\hat{\vartheta}_t \leq \overline{\theta} + \zeta\lambda$ for all $t \geq 0$, and once $\hat{\vartheta}$ hits $\widetilde{\theta}$, it will not exceed $\widetilde{\theta}$ again. We define

$$\vartheta_t^{(1)+} = \Big[\mathfrak{G}_\ell^{-1}\Big(\sup_{u\leq t} X_u\Big) \wedge \mathfrak{G}_r^{-1}\Big(\inf_{u\leq t} X_u\Big) - \theta\Big]^+ \mathbf{1}_{\{0<t\}}, \quad \vartheta_t^{(1)-} = 0,$$
$$\vartheta_t^{(1)} = \theta + \vartheta_t^{(1)+} - \vartheta_t^{(1)-},$$
$$\tau_1 = \inf\Big\{t \geq 0 : X_t < \mathfrak{F}_\ell\big(\vartheta_t^{(1)}\big) \text{ or } X_t > \mathfrak{F}_r\big(\vartheta_t^{(1)}\big)\Big\},$$

and we note that $(\vartheta^{(1)}, X)$ is reflecting in $\mathfrak{G}_\ell$ and $\mathfrak{G}_r$ in the positive θ-direction,

$$(\vartheta^{(1)}, X) \in \operatorname{cl}\mathcal{C} \text{ for all } t \leq \tau_1 \quad \text{and} \quad \vartheta_{\tau_1}^{(1)} = \mathfrak{F}_\ell^{-1}(X_{\tau_1}) \wedge \mathfrak{F}_r^{-1}(X_{\tau_1}).$$

Next, we define

$$\vartheta_t^{(2)-} = \Big[\vartheta_{\tau_1}^{(1)} - \mathfrak{F}_\ell^{-1}\Big(\inf_{\tau_1\leq u\leq t} X_u\Big) \wedge \mathfrak{F}_r^{-1}\Big(\sup_{\tau_1\leq u\leq t} X_u\Big)\Big]\mathbf{1}_{\{\tau_1\leq t\}},$$
$$\vartheta_t^{(2)+} = \vartheta_{t\wedge\tau_1}^{(1)+}, \quad \vartheta_t^{(2)} = \theta + \vartheta_t^{(2)+} - \vartheta_t^{(2)-}$$
$$\text{and} \quad \tau_2 = \inf\Big\{t \geq 0 : \vartheta_t^{(2)} < \widetilde{\theta} \text{ and } \mathfrak{G}_\ell\big(\vartheta_t^{(2)}\big) < X_t < \mathfrak{G}_r\big(\vartheta_t^{(2)}\big)\Big\} > \tau_1.$$

An inspection of these definitions reveals that $(\vartheta^{(2)}, X)$ is reflecting in $\mathfrak{G}_\ell$ and $\mathfrak{G}_r$ in the positive θ-direction and in $\mathfrak{F}_\ell$ and $\mathfrak{F}_r$ in the negative θ-direction up to time τ_2,

$$\vartheta_t^{(2)} = \vartheta_t^{(1)} \text{ for all } t \le \tau_1, \quad (\vartheta_t^{(2)}, X_t) \in \operatorname{cl}\mathcal{C} \text{ for all } t \le \tau_2 \\ \text{and} \quad \vartheta_{\tau_2}^{(2)} = \mathfrak{G}_\ell^{-1}(X_{\tau_2}) \wedge \mathfrak{G}_r^{-1}(X_{\tau_2}).$$

We then iterate these constructions by defining

$$\begin{gathered}
\vartheta_t^{(2n+1)+} = \vartheta_{t\wedge\tau_{2n}}^{(2n)+} + \Big[\mathfrak{G}_\ell^{-1}\Big(\sup_{\tau_{2n}\le u\le t} X_u\Big) \wedge \mathfrak{G}_r^{-1}\Big(\inf_{\tau_{2n}\le u\le t} X_u\Big) - \vartheta_{\tau_{2n}}^{(2n)+}\Big]\mathbf{1}_{\{\tau_{2n}\le t\}} \\
\vartheta_t^{(2n+1)-} = \vartheta_{t\wedge\tau_{2n}}^{(2n)-}, \quad \vartheta_t^{(2n+1)} = \theta + \vartheta_t^{(2n+1)+} - \vartheta_t^{(2n+1)-}, \\
\tau_{2n+1} = \inf\Big\{t \ge 0 : X_t < \mathfrak{F}_\ell\big(\vartheta_t^{(2n+1)}\big) \text{ or } X_t > \mathfrak{F}_r\big(\vartheta_t^{(2n+1)}\big)\Big\} > \tau_{2n}, \\
\vartheta_t^{(2n)-} = \vartheta_{t\wedge\tau_{2n-1}}^{(2n-1)-} + \Big[\vartheta_{\tau_{2n-1}}^{(2n-1)+} - \mathfrak{F}_\ell^{-1}\Big(\inf_{\tau_{2n-1}\le u\le t} X_u\Big) \wedge \mathfrak{F}_r^{-1}\Big(\sup_{\tau_{2n-1}\le u\le t} X_u\Big)\Big]\mathbf{1}_{\{\tau_{2n-1}\le t\}} \\
\vartheta_t^{(2n)+} = \vartheta_{t\wedge\tau_{2n-1}}^{(2n-1)+}, \quad \vartheta_t^{(2n)} = \theta + \vartheta_t^{(2n)+} - \vartheta_t^{(2n)-} \\
\text{and} \quad \tau_{2n} = \inf\Big\{t \ge 0 : \vartheta_t^{(2n)} < \widetilde{\theta} \text{ and } \mathfrak{G}_\ell\big(\vartheta_t^{(2n)}\big) < X_t < \mathfrak{G}_r\big(\vartheta_t^{(2n)}\big)\Big\} > \tau_{2n-1}
\end{gathered}$$

for $n \ge 1$, and we note that, given any $m, k \ge 1$,

$$\vartheta_t^{(m+k)} = \vartheta_t^{(m)} = \theta + \vartheta_t^{(m)+} - \vartheta_t^{(m)-} \quad \text{and} \quad (\vartheta_t^{(m)}, X_t) \in \operatorname{cl}\mathcal{C} \quad \text{for all } t \le \tau_m$$

and $\lim_{n\to\infty} \tau_n = \infty$. Therefore, we can define $\hat{\vartheta}^+$, $\hat{\vartheta}^-$ and $\hat{\vartheta}$ by

$$\hat{\vartheta}_t^+ = \vartheta_t^{(m)+}, \quad \hat{\vartheta}_t^- = \vartheta_t^{(m)-} \quad \text{and} \quad \hat{\vartheta}_t = \vartheta_t^{(m)}$$

for any $m \ge 1$ such that $t < \tau_m$. The finite-variation function $\hat{\vartheta}$ constructed satisfies (3.121) because this is true for all of the functions $\vartheta^{(n)}$. Indeed, an inspection of the iterative algorithm that we have developed reveals that $\vartheta^{(n)}$

increases (resp., decreases) on the set

$$\begin{gathered}\Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{G}_\ell^{-1}\Big(\sup_{0\leq u\leq t} X_u\Big) \text{ and } X_t = \sup_{0\leq u\leq t} X_u < H(\widetilde{\theta})\Big\} \\ \cup\Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{G}_r^{-1}\Big(\inf_{0\leq u\leq t} X_u\Big) \text{ and } X_t = \inf_{0\leq u\leq t} X_u > H(\widetilde{\theta})\Big\} \\ \Big(\text{resp., } \Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{F}_\ell^{-1}\Big(\inf_{0\leq u\leq t} X_u\Big) \text{ and } X_t = \inf_{0\leq u\leq t} X_u < x^\dagger\Big\} \\ \cup\Big\{t \geq 0 : \vartheta_t^{(n)} = \mathfrak{F}_r^{-1}\Big(\sup_{0\leq u\leq t} X_u\Big) \text{ and } X_t = \sup_{0\leq u\leq t} X_u > x^\dagger\Big\}\Big).\end{gathered}$$

3.5.3 Admissibility of optimal trading strategies

Either of the above constructions define operators $\mathbf{F}^+(\theta;\cdot)$, $\mathbf{F}^-(\theta;\cdot)$ and $\mathbf{F}(\theta;\cdot)$ mapping the set $C^r(\mathbb{R})$ of all continuous functions $g : \mathbb{R} \to \mathbb{R}$ into the set of all càglàd finite-varation functions that are continuous in $\mathbb{R}$. In particular, given an initial condition $(\theta, x) \in \mathbb{R}^2$ and the solution $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, B, X)$ of (3.2) that we have associated with it, a process $\hat{\vartheta}$ that is as in Definition 3.22 is given by

$$\hat{\vartheta}_t = \mathbf{F}_t(\theta; X) = \theta + \mathbf{F}_t^+(\theta; X) - \mathbf{F}_t^-(\theta; X) \quad \text{for all } t \geq 0,$$

where, e.g., $\mathbf{F}_t(\theta; g)$ is the evaluation of the function $\mathbf{F}(\theta; g)$ at t, for $g \in C^r(\mathbb{R})$.

By Theorem 3.9, we prove that the strategy $\hat{\vartheta}$ we constructed is optimal (that is, it satisfies (3.21)) if we prove that $\hat{\vartheta} \in \mathcal{A}(\theta)$. By Definition 3.5 and Remark 3.6, we only need to prove that $\hat{\vartheta}$ satisfies the integrability condition (3.17). If Θ is strictly increasing with $\underline{\theta} > -\infty$ and $\overline{\theta} < \infty$, then ϑ clearly satisfies (3.17), as ϑ_t is bounded for all $t \geq 0$. If $\underline{\theta} = -\infty$ and $\overline{\theta} = \infty$, then by Definition 3.21 and the growth estimates in Lemma 3.12,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\vartheta}_t^2 \,\mathrm{d}t\right] &\leq \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\Big(\big(\mathfrak{F}^{-1}(X_t)\big)^2 + \big(\mathfrak{G}^{-1}(X_t)\big)^2\Big)\,\mathrm{d}t\right] \\ &\leq \big(C_F^{-2p} + C_G^{-2p}\big)\,\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}|X_t|^{2p}\,\mathrm{d}t\right] < \infty.\end{aligned}$$

The cases where $\underline{\theta} = -\infty$ and $\overline{\theta} < \infty$ and $\underline{\theta} > -\infty$ and $\overline{\theta} = \infty$ are similar. If Θ is strictly increasing then strictly decreasing, then by Definition 3.22 and the growth estimates in Lemma 3.19,

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\hat{\vartheta}_t^2\,\mathrm{d}t\right] &\leq \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\Big((\overline{\theta}+\zeta\lambda)^2+\big(\mathfrak{G}_\ell^{-1}(X_t)\big)^2+\big(\mathfrak{G}_r^{-1}(X_t)\big)^2\Big)\,\mathrm{d}t\right]\\ &\leq \frac{(\overline{\theta}+\zeta\lambda)^2}{r}+\big(C_\ell^{-2p}+C_r^{-2p}\big)\,\mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}|X_t|^{2p}\,\mathrm{d}t\right]<\infty.\end{aligned}$$

Moreover, in either case, a stronger transversality condition than (3.19) is satisfied. By the growth estimates as well as (3.6) and (3.7), there exists a constant $\hat{C}>0$ such that

$$\lim_{T\uparrow\infty}\mathrm{e}^{-rT}\,\mathbb{E}\big[\vartheta_T^2\big]\leq \hat{C}\lim_{T\uparrow\infty}\mathrm{e}^{-rT}\,\mathbb{E}\big[|X_T|^{2p}\big]\leq \hat{C}2^{p-1}\big(1+|x|^{2p}\big)\lim_{T\uparrow\infty}\mathrm{e}^{-(r-2pC)T}=0.$$

3.6 Transaction cost asymptotics

We first derive small transaction cost asymptotics in the case where Θ is strictly increasing. We first note that when $\underline{\theta}>-\infty$ and $\overline{\theta}<\infty$, $\lambda<\frac{\overline{\theta}-\underline{\theta}}{2\zeta}$ for small enough λ and

$$\lim_{\lambda\downarrow 0}\underset{\sim}{\theta}=\lim_{\lambda\downarrow 0}(\underline{\theta}+\zeta\lambda)=\underline{\theta}\quad\text{and}\quad\lim_{\lambda\downarrow 0}\widetilde{\theta}=\lim_{\lambda\downarrow 0}(\overline{\theta}-\zeta\lambda)=\overline{\theta}.$$

In other words, the continuation regions $\mathcal{C}_\ell$ and $\mathcal{C}_h$ vanish in the limit as λ goes to 0. Therefore, we only derive asymptotics for the functions $F^\star$ and $G^\star$ which define the continuation region $\mathcal{C}_m$. The properties of the function $F^\star$ in (3.47) imply that for each θ, the function $\lambda\mapsto F^\star(\theta,\lambda)$ is invertible and its inverse λ is such that

$$\lambda_F(\theta,F)<0\ \text{ for all }\ F<\Theta^{-1}(\theta)\quad\text{and}\quad\lim_{F\uparrow\Theta^{-1}(\theta)}\lambda(\theta,F)=0.\tag{3.122}$$

Furthermore, if we define

$$\mathbb{G}(\theta,F)=G^\star\left(\theta,\lambda(\theta,F)\right)\ \text{ for }\ F<\Theta^{-1}(\theta),\tag{3.123}$$

then λ and $\mathbb{G}$ satisfy the system of equations

$$\begin{aligned}\mathcal{H}_1\big(\theta, F, \mathbb{G}(\theta,F), \lambda(\theta,F)\big) = &\int_F^{\mathbb{G}(\theta,F)} \Psi(y)\,(\theta - \Theta(y))\,\mathrm{d}y \\ &+ \zeta\lambda(\theta,F)\left(\int_{-\infty}^{F} \Psi(y)\,\mathrm{d}y + \int_{-\infty}^{\mathbb{G}(\theta,F)} \Psi(y)\,\mathrm{d}y\right) = 0,\end{aligned} \tag{3.124}$$

$$\begin{aligned}\mathcal{H}_2\big(\theta, F, \mathbb{G}(\theta,F), \lambda(\theta,F)\big) = &\int_F^{\mathbb{G}(\theta,F)} \Phi(y)\,(\theta - \Theta(y))\,\mathrm{d}y \\ &- \zeta\lambda(\theta,F)\left(\int_{F}^{\infty} \Phi(y)\,\mathrm{d}y + \int_{\mathbb{G}(\theta,F)}^{\infty} \Phi(y)\,\mathrm{d}y\right) = 0.\end{aligned} \tag{3.125}$$

Furthermore, the properties of $G^\star$ in (3.47) and (3.122) imply that

$$\lim_{F \uparrow \Theta^{-1}(\theta)} G^\star\left(\theta, \lambda(\theta,F)\right) = \Theta^{-1}(\theta). \tag{3.126}$$

Similarly, we can define the inverse $\widetilde{\lambda}$ of the function $G^\star$ in (3.47) satisfying

$$\widetilde{\lambda}_G(\theta,G) > 0 \;\text{ for all }\; G > \Theta^{-1}(\theta) \quad \text{and} \quad \lim_{G \downarrow \Theta^{-1}(\theta)} \widetilde{\lambda}(\theta,G) = 0, \tag{3.127}$$

as well as

$$\mathbb{F}(\theta,G) = F^\star\big(\theta, \widetilde{\lambda}(\theta,G)\big) \;\text{ for }\; G > \Theta^{-1}(\theta),$$

satisfying

$$\mathcal{H}_1\big(\theta, \mathbb{F}(\theta,G), G, \widetilde{\lambda}(\theta,G)\big) = \mathcal{H}_2\big(\theta, \mathbb{F}(\theta,G), G, \widetilde{\lambda}(\theta,G)\big) = 0. \tag{3.128}$$

The following result provides the asymptotic behaviour of the free-boundary functions $F^\star$ and $G^\star$ as $\lambda \downarrow 0$.

Theorem 3.23. *Suppose* Θ *is strictly increasing and fix any* $\theta \in \left]\underset{\sim}{\theta}, \widetilde{\theta}\right[$ *such that* $\Theta'\left(\Theta^{-1}(\theta)\right) > 0$. *Assuming the functions* h_i *are* C^1, *we have*

$$F^\star(\theta,\lambda) = \Theta^{-1}(\theta) - \left(\frac{3\beta^2\left(\Theta^{-1}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\Theta^{-1}(\theta)\right)}\right)^{1/3} \lambda^{1/3} + O\left(\lambda^{2/3}\right) \tag{3.129}$$

$$\text{and} \quad G^\star(\theta,\lambda) = \Theta^{-1}(\theta) + \left(\frac{3\beta^2\left(\Theta^{-1}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\Theta^{-1}(\theta)\right)}\right)^{1/3} \lambda^{1/3} + O\left(\lambda^{2/3}\right). \tag{3.130}$$

Proof. Differentiating the equations (3.124) and (3.125) with respect to F, we obtain

$$\begin{aligned} &\Psi\left(\mathbb{G}(\theta,F)\right) Q^{[1]}(\theta,F)\mathbb{G}_F(\theta,F) + \zeta Q^{\Psi}(\theta,F)\lambda_F(\theta,F) = \Psi(F)Q^{[2]}(\theta,F) \\ \text{and} \quad &\Phi\left(\mathbb{G}(\theta,F)\right) Q^{[1]}(\theta,F)\mathbb{G}_F(\theta,F) - \zeta Q^{\Phi}(\theta,F)\lambda_F(\theta,F) = \Phi(F)Q^{[2]}(\theta,F), \end{aligned}$$

where

$$Q^{[1]}(\theta,F) = \theta - \Theta\left(\mathbb{G}(\theta,F)\right) + \zeta\lambda(\theta,F), \quad Q^{[2]}(\theta,F) = \theta - \Theta(F) - \zeta\lambda(\theta,F),$$

$$\begin{aligned} Q^{\Psi}(\theta,F) &= \int_{-\infty}^{F} \Psi(y)\,\mathrm{d}y + \int_{-\infty}^{\mathbb{G}(\theta,F)} \Psi(y)\,\mathrm{d}y \\ \text{and} \quad Q^{\Phi}(\theta,F) &= \int_{F}^{\infty} \Phi(y)\,\mathrm{d}y + \int_{\mathbb{G}(\theta,F)}^{\infty} \Phi(y)\,\mathrm{d}y. \end{aligned}$$

This system of equations is equivalent to

$$\zeta Q^{[0]}(\theta,F)\lambda_F(\theta,F) = Q^{[2]}(\theta,F)Q^{[3]}(\theta,F) \tag{3.131}$$

$$\text{and} \;\; \Psi\left(\mathbb{G}(\theta,F)\right) Q^{[1]}(\theta,F)\mathbb{G}_F(\theta,F) = \Psi(F)Q^{[2]}(\theta,F) - \frac{Q^{[2]}(\theta,F)Q^{[3]}(\theta,F)}{Q^{[0]}(\theta,F)}, \tag{3.132}$$

where

$$\begin{aligned} Q^{[0]}(\theta,F) &= \varphi\left(\mathbb{G}(\theta,F)\right) Q^{\Psi}(\theta,F) + \psi\left(\mathbb{G}(\theta,F)\right) Q^{\Phi}(\theta,F) \\ \text{and} \quad Q^{[3]}(\theta,F) &= \varphi\left(\mathbb{G}(\theta,F)\right) \Psi(F) - \psi\left(\mathbb{G}(\theta,F)\right) \Phi(F). \end{aligned}$$

In view of the limits in (3.122) and (3.126), the identity (A.4) in the appendix and the definition (3.30) of ζ, we can see that

$$\begin{aligned}\zeta \lim_{F\uparrow\Theta^{-1}(\theta)} Q^{[0]}(\theta,F) &= \frac{2\zeta}{r}\frac{\left(\varphi\left(\Theta^{-1}(\theta)\right)\psi'\left(\Theta^{-1}(\theta)\right)-\psi\left(\Theta^{-1}(\theta)\right)\varphi'\left(\Theta^{-1}(\theta)\right)\right)}{Cp'\left(\Theta^{-1}(\theta)\right)}\\ &= \frac{2\zeta}{r} = \frac{4}{\sigma^2\left(\gamma_1+\gamma_2\right)}\end{aligned}$$

and

$$\lim_{F\uparrow\Theta^{-1}(\theta)} Q^{[1]}(\theta,F) = \lim_{F\uparrow\Theta^{-1}(\theta)} Q^{[2]}(\theta,F) = \lim_{F\uparrow\Theta^{-1}(\theta)} Q^{[3]}(\theta,F) = 0, \tag{3.133}$$

These limits imply that

$$\lim_{F\uparrow\Theta^{-1}(\theta)} \lambda_F(\theta,F) = 0. \tag{3.134}$$

Furthermore, we differentiate (3.131) to obtain

$$\begin{aligned}\zeta Q^{[0]}(\theta,F)\lambda_{FF}(\theta,F) = Q^{[2]}(\theta,F)Q^{[3]}_F(\theta,F) + Q^{[2]}_F(\theta,F)Q^{[3]}(\theta,F)\\ - \zeta Q^{[0]}_F(\theta,F)\lambda_F(\theta,F),\end{aligned}$$

where

$$\begin{aligned}Q^{[2]}_F(\theta,F) &= -\Theta'(F) - \zeta\lambda_F(\theta,F)\\ \text{and}\quad Q^{[3]}_F(\theta,F) &= \left(\frac{2}{C\beta^2(F)p'(F)}\right)'\left(\varphi\left(\mathbb{G}(\theta,F)\right)\psi(F)-\psi\left(\mathbb{G}(\theta,F)\right)\varphi(F)\right)\\ &\quad + 2\frac{\varphi\left(\mathbb{G}(\theta,F)\right)\psi'(F)-\psi\left(\mathbb{G}(\theta,F)\right)\varphi'(F)}{C\beta^2(F)p'(F)}\\ &\quad + 2\frac{\varphi'\left(\mathbb{G}(\theta,F)\right)\psi(F)-\psi'\left(\mathbb{G}(\theta,F)\right)\varphi(F)}{C\beta^2(F)p'(F)}\mathbb{G}_F(\theta,F).\end{aligned}$$

In light of (3.134), this expression implies that

$$\lim_{F\uparrow\Theta^{-1}(\theta)} \lambda_{FF}(\theta,F) = 0. \tag{3.135}$$

Differentiating (3.131) twice and taking the limit $F\uparrow\Theta^{-1}(\theta)$, and using the limits (3.134) and (3.135), we obtain

$$\begin{aligned}\frac{4\lim_{F\uparrow\Theta^{-1}(\theta)}\lambda_{FFF}(\theta,F)}{\sigma^2\left(\gamma_1+\gamma_2\right)} &= 2\lim_{F\uparrow\Theta^{-1}(\theta)} Q_F^{[2]}(\theta,F)Q_F^{[3]}(\theta,F) \\ &= -\frac{4\Theta'(\Theta^{-1}(\theta))}{\beta^2(\Theta^{-1}(\theta))}\Big(1-\lim_{F\uparrow\Theta^{-1}(\theta)}\mathbb{G}_F(\theta,F)\Big). \qquad (3.136)\end{aligned}$$

Moreover, differentiating (3.132) and using the limits (3.133) and (3.134), we obtain

$$\lim_{F\uparrow\Theta^{-1}(\theta)}\Theta'(\mathbb{G}(\theta,F))\mathbb{G}_F^2(\theta,F)=\Theta'(\Theta^{-1}(\theta)).$$

It follows that

$$\lim_{F\uparrow\Theta^{-1}(\theta)}\mathbb{G}_F^2(\theta,F)=1,$$

thanks to (3.126). Combining this result with the identity

$$\mathbb{G}_F(\theta,F)=G_\lambda^\star(\theta,\lambda(\theta,F))\lambda_F(\theta,F),$$

which follows from differentiation of (3.123) and the inequalities (3.47) and (3.122), we can see that

$$\lim_{F\uparrow\Theta^{-1}(\theta)}\mathbb{G}_F(\theta,F)=-1.$$

This limit and (3.136) implies that

$$\lim_{F\uparrow\Theta^{-1}(\theta)}\frac{\partial^3\lambda}{\partial F^3}(\theta,F)=-\frac{2\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'(\Theta^{-1}(\theta))}{\beta^2(\Theta^{-1}(\theta))}.$$

In light of (3.134) and (3.135), we obtain the Taylor expansion

$$\lambda(\theta,F)=-\frac{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\Theta^{-1}(\theta)\right)}{3\beta^2\left(\Theta^{-1}(\theta)\right)}\left(F-\Theta^{-1}(\theta)\right)^3+O\left(\left(F-\Theta^{-1}(\theta)\right)^4\right).$$

Inverting this expansion, we obtain (3.129). To obtain the higher order terms, the Taylor expansion of λ can be expanded further, and the $O(\lambda^{2/3})$ coefficients can be computed using the Lagrange-Bürmann formula (from the Lagrange inversion theorem). By symmetric calculations on (3.128), we obtain

$$\lim_{G \downarrow \Theta^{-1}(\theta)} \frac{\partial^3 \widetilde{\lambda}}{\partial G^3}(\theta, G) = \frac{2\sigma^2 (\gamma_1 + \gamma_2) \Theta' (\Theta^{-1}(\theta))}{\beta^2 (\Theta^{-1}(\theta))},$$

from which we obtain (3.130). □

Remark 3.24. *In [51], where X is a one-dimensional standard Brownian motion and $h_i(x) = \xi_i x$, the frictionless optimiser is a linear function given by $\Theta(x) = \delta x$, where $\delta = \frac{\gamma_2 \xi_2 - \gamma_1 \xi_1}{\sigma(\gamma_1 + \gamma_2)}$ and $F^\star$ and $G^\star$ are also linear:*

$$F^\star(\theta, \lambda) = \frac{\theta}{\delta} - \left(\frac{3\lambda}{\delta \sigma^2 (\gamma_1 + \gamma_2)} \right)^{1/3}$$
$$\text{and} \quad G^\star(\theta, \lambda) = \frac{\theta}{\delta} + \left(\frac{3\lambda}{\delta \sigma^2 (\gamma_1 + \gamma_2)} \right)^{1/3},$$

which is (3.129) *and* (3.130) *with $\beta = 1$ and without any higher order terms. In other words, $\vartheta - \delta X$ is a doubly reflected Brownian motion with constant end points $\pm \ell$, where*

$$\ell = \left(\frac{3\delta^2 \lambda}{\sigma^2 (\gamma_1 + \gamma_2)} \right)^{1/3} = \left(\frac{3\lambda (\gamma_2 \xi_2 - \gamma_1 \xi_1)^2}{\sigma^4 (\gamma_1 + \gamma_2)^3} \right)^{1/3}.$$

In our setting, $\vartheta - \Theta(X)$ is a doubly reflected diffusion but with non-constant end points that depend on the position of ϑ.

We now consider the case where Θ is strictly increasing and then strictly decreasing. Recalling Lemma 3.15 and (3.96) in Lemma 3.17, we observe that $\widetilde{\theta} \uparrow \overline{\theta}$ and $\overline{\theta} + \zeta\lambda \downarrow \overline{\theta}$ as λ goes to 0. In other words, the continuation region $\mathcal{C}_h$ vanishes in the limit. Economically, $\overline{\theta}$ represents the maximal number of shares for the frictionless optimiser, while $\widetilde{\theta}$ represents the maximal number of shares for the frictional optimiser once the joint process (ϑ, X) is in $\mathcal{C}_\ell \cup \mathcal{C}_r$ (we recall that, once (ϑ, X) reaches $\mathcal{C}_\ell \cup \mathcal{C}_r$, it will never return to the region $\mathcal{C}_h$). Indeed, as the proportional transaction costs decrease, the gap between the maximal shares for the frictionless and frictional optimiser should decrease. Therefore, we only derive asymptotics for the functions $F^\star_\ell$ and $G^\star_\ell$, which define the continuation region $\mathcal{C}_\ell$, as well as the functions $F_r^\star$ and $G_r^\star$, which define the continuation region $\mathcal{C}_r$. The asymptotics are derived analogously to the case where Θ is strictly increasing.

Theorem 3.25. *Suppose Θ is strictly increasing and then strictly decreasing and fix any $\theta \in \left]-\infty, \widetilde{\theta}\right[$ such that $\Theta'\left(\underline{\chi}(\theta)\right) > 0$ and $\Theta'\left(\overline{\chi}(\theta)\right) < 0$. Assuming the functions h_i are C^1, we have*

$$\begin{aligned}
F_\ell^\star(\theta,\lambda) &= \underline{\chi}(\theta) - \left(\frac{3\beta^2\left(\underline{\chi}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\underline{\chi}(\theta)\right)}\right)^{1/3}\lambda^{1/3} + O\left(\lambda^{2/3}\right),\\
G_\ell^\star(\theta,\lambda) &= \underline{\chi}(\theta) + \left(\frac{3\beta^2\left(\underline{\chi}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\underline{\chi}(\theta)\right)}\right)^{1/3}\lambda^{1/3} + O\left(\lambda^{2/3}\right),\\
F_r^\star(\theta,\lambda) &= \overline{\chi}(\theta) + \left(\frac{3\beta^2\left(\overline{\chi}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\overline{\chi}(\theta)\right)}\right)^{1/3}\lambda^{1/3} + O\left(\lambda^{2/3}\right)\\
\textit{and}\quad G_r^\star(\theta,\lambda) &= \overline{\chi}(\theta) - \left(\frac{3\beta^2\left(\overline{\chi}(\theta)\right)}{\sigma^2\left(\gamma_1+\gamma_2\right)\Theta'\left(\overline{\chi}(\theta)\right)}\right)^{1/3}\lambda^{1/3} + O\left(\lambda^{2/3}\right).
\end{aligned}$$

A

Appendix: Results on ODEs and one-dimensional diffusions

In this section, we review a number of results on the solution of the ODE

$$\mathscr{L}u(x) - ru(x) + h(x) = 0,$$

where $\mathscr{L}$ is the second order differential operator defined by $\mathscr{L} = \frac{1}{2}\beta^2(x)\frac{\partial^2}{\partial x^2} + \alpha(x)\frac{\partial}{\partial x}$ (the infinitesimal generator of the homogeneous diffusion X). All of the claims that we do not prove are standard and can be found in several references such as [11, Chapter II.1.10,II.1.11,II.4.24].

In the presence of Assumption (3.2), the general solution to the second order linear homogeneous ODE $\mathscr{L}u(x) - ru(x) = 0$ is given by

$$u(x) = A\varphi(x) + B\psi(x)$$

for some constants $A, B \in \mathbb{R}$. The functions φ and ψ are the unique, modulo multiplicative constants, C^2 functions such that

$$0 < \varphi(x) \quad \text{and} \quad \varphi'(x) < 0 \quad \text{for all } x \in \mathbb{R}, \tag{A.1}$$

$$0 < \psi(x) \quad \text{and} \quad \psi'(x) > 0 \quad \text{for all } x \in \mathbb{R}, \tag{A.2}$$

$$\lim_{x\downarrow -\infty} \varphi(x) = \lim_{x\uparrow\infty} \psi(x) = \infty \quad \text{and} \quad \lim_{x\uparrow\infty} \varphi(x) = \lim_{x\downarrow -\infty} \psi(x) = 0. \tag{A.3}$$

To simplify the notation, we assume that $\varphi(0) = \psi(0) = 1$. Also, these functions satisfy the identity

$$\varphi(x)\psi'(x) - \varphi'(x)\psi(x) = Cp'(x) \quad \text{for all } x \in \mathbb{R}, \tag{A.4}$$

where $C = \psi'(0) - \varphi'(0) > 0$ and p is the scale function of X defined by

$$p'(x) = \exp\biggl(-\int_0^x \frac{2\alpha(y)}{\beta^2(y)}\,\mathrm{d}y\biggr).$$

Combining (A.3) with the inequalities

$$0 < \frac{\varphi(x)\psi'(x)}{Cp'(x)} < 1 \quad \text{and} \quad 0 < -\frac{\varphi'(x)\psi(x)}{Cp'(x)} < 1,$$

which follow from (A.1), (A.2) and (A.4), we can see that

$$\lim_{x\downarrow-\infty} \frac{\psi'(x)}{p'(x)} = \lim_{x\uparrow\infty} \frac{\varphi'(x)}{p'(x)} = 0. \tag{A.5}$$

By Corollary 4.5 of Lamberton and Zervos [78], a Borel measurable function h satisfies

$$\mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}\bigl|h(X_t)\bigr|\,\mathrm{d}t\biggr] < \infty, \tag{A.6}$$

where X is the solution to the SDE

$$\mathrm{d}X_t = \alpha(X_t)\,\mathrm{d}t + \beta(X_t)\,\mathrm{d}B_t, \quad X_0 = x \in \mathbb{R},$$

if and only if it satisfies the integrability condition

$$\int_{-\infty}^x \Psi(y)\bigl|h(y)\bigr|\,\mathrm{d}y + \int_x^\infty \Phi(y)\bigl|h(y)\bigr|\,\mathrm{d}y < \infty \quad \text{for all } x \in \mathbb{R}, \tag{A.7}$$

where

$$\Phi(x) = \frac{2\varphi(x)}{C\beta^2(x)p'(x)} \quad \text{and} \quad \Psi(x) = \frac{2\psi(x)}{C\beta^2(x)p'(x)}. \tag{A.8}$$

Given such a function h, if we define

$$R_h(x) = \mathbb{E}\biggl[\int_0^\infty \mathrm{e}^{-rt}h(X_t)\,\mathrm{d}t\biggr] \tag{A.9}$$

then the function R_h admits the analytic representation

$$R_h(x) = \varphi(x) \int_{-\infty}^{x} \Psi(y) h(y) \, \mathrm{d}y + \psi(x) \int_{x}^{\infty} \Phi(y) h(y) \, \mathrm{d}y \tag{A.10}$$

and satisfies the ODE $\mathscr{L} R_h(x) - r R_h(x) + h(x) = 0$, Lebesgue-a.e..

In our analysis, we have used the following result.

Lemma A.1. *In the presence of Assumption 3.2,*

$$\lim_{x \downarrow -\infty} \int_{x}^{\infty} \Phi(y) \, \mathrm{d}y = \lim_{x \uparrow \infty} \int_{-\infty}^{x} \Psi(y) \, \mathrm{d}y = \infty, \tag{A.11}$$

$$\frac{\varphi'(x)}{p'(x)} = -rC \int_{x}^{\infty} \Phi(y) \, \mathrm{d}y \quad \textit{and} \quad \frac{\psi'(x)}{p'(x)} = rC \int_{-\infty}^{x} \Psi(y) \, \mathrm{d}y. \tag{A.12}$$

Furthermore, if h is a Borel measurable function satisfying the equivalent integrability conditions (A.6) and (A.7) such that $h(x) \leq -\varepsilon$ for all $x \geq x_\varepsilon$ (resp., $h(x) \geq \varepsilon$ for all $x \leq x_\varepsilon$), for some $\varepsilon > 0$ and $x_\varepsilon \in \mathbb{R}$, then

$$\lim_{x \uparrow \infty} \int_{-\infty}^{x} \Psi(y) h(y) \, \mathrm{d}y = -\infty \quad \left(\textit{resp.,} \ \lim_{x \downarrow -\infty} \int_{x}^{\infty} \Phi(y) h(y) \, \mathrm{d}y = \infty \right). \tag{A.13}$$

Proof. To establish the second limit in (A.11), we argue by contradiction. To this end, assume that

$$\lim_{x \uparrow \infty} \int_{-\infty}^{x} \Psi(y) \, \mathrm{d}y < \infty. \tag{A.14}$$

For $h \equiv 1$, the expressions (A.9) and (A.10) imply that

$$\varphi(x) \int_{-\infty}^{x} \Psi(y) \, \mathrm{d}y + \psi(x) \int_{x}^{\infty} \Phi(y) \, \mathrm{d}y = \frac{1}{r}.$$

Since $\lim_{x \uparrow \infty} \varphi(x) = 0$ and $\lim_{x \uparrow \infty} \psi(x) = \infty$, the hypothesis (A.14), L'Hôpital's lemma and the definitions in (A.8) imply that

$$\frac{1}{r} = \lim_{x \uparrow \infty} \frac{\int_{x}^{\infty} \Phi(y) \, \mathrm{d}y}{\psi^{-1}(x)} = \lim_{x \uparrow \infty} \frac{\Phi(x)}{\psi^{-2}(x) \psi'(x)} = \lim_{x \uparrow \infty} \frac{\varphi(x) \psi(x)}{\psi'(x)} \Psi(x).$$

These identities imply that, given any $\varepsilon \in \left]0, \frac{1}{r}\right[$, there exists $x_\varepsilon \in \mathbb{R}$ such that

$$\Psi(x) > \left(\frac{1}{r} - \varepsilon\right) \frac{\psi'(x)}{\varphi(x)\psi(x)} \quad \text{for all } x > x_\varepsilon.$$

It follows that

$$\begin{aligned}
\lim_{x\uparrow\infty} \int_{-\infty}^{x} \Psi(y)\,\mathrm{d}y &> \lim_{x\uparrow\infty} \int_{-\infty}^{x} \left(\frac{1}{r} - \varepsilon\right) \frac{\psi'(y)}{\varphi(y)\psi(y)}\,\mathrm{d}y \\
&\geq \left(\frac{1}{r} - \varepsilon\right) \lim_{x\uparrow\infty} \int_{x_\varepsilon}^{x} \frac{\psi'(y)}{\varphi(y)\psi(y)}\,\mathrm{d}y \\
&> \left(\frac{1}{r} - \varepsilon\right) \frac{1}{\varphi(x_\varepsilon)} \lim_{x\uparrow\infty} \int_{x_\varepsilon}^{x} \frac{\psi'(y)}{\psi(y)}\,\mathrm{d}y \\
&= \left(\frac{1}{r} - \varepsilon\right) \frac{1}{\varphi(x_\varepsilon)} \lim_{x\uparrow\infty} \ln\left(\frac{\psi(x)}{\psi(x_\varepsilon)}\right) = \infty.
\end{aligned}$$

However, this result contradicts (A.14). Using the same reasoning with symmetric calculations, we can show that the first limit in (A.11) is also equal to ∞. The calculation

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{p'(x)} = \frac{2\alpha(x)}{\beta^2(x)p'(x)}$$

and the fact that ψ solves the ODE $\mathscr{L}\psi(x) - r\psi(x) = 0$ imply that

$$\frac{\mathrm{d}}{\mathrm{d}x} \frac{\psi'(x)}{p'(x)} = \frac{2\alpha(x)\psi'(x)}{\beta^2(x)p'(x)} + \frac{\big(2r\psi(x) - 2\alpha(x)\psi'(x)\big)}{\beta^2(x)p'(x)} = \frac{2r\psi(x)}{\beta^2(x)p'(x)} = rC\Psi(x).$$

This result and (A.5) imply the second identity in (A.12). The proof of the first identity in (A.12) is similar.

If h is a Borel measurable function satisfying the equivalent integrability conditions (A.6) and (A.7) such that $h(x) \leq -\varepsilon$ for all $x \geq x_\varepsilon$, for some $\varepsilon > 0$ and $x_\varepsilon \in \mathbb{R}$, then

$$\begin{aligned}
\lim_{x\uparrow\infty} \int_{-\infty}^{x} \Psi(y)h(y)\,\mathrm{d}y &\leq \int_{-\infty}^{x_\varepsilon} \Psi(y)h(y)\,\mathrm{d}y - \varepsilon \lim_{x\uparrow\infty} \int_{x_\varepsilon}^{x} \Psi(y)\,\mathrm{d}y \\
&= \int_{-\infty}^{x_\varepsilon} \Psi(y)\big(h(y) + \varepsilon\big)\,\mathrm{d}y - \varepsilon \lim_{x\uparrow\infty} \int_{-\infty}^{x} \Psi(y)\,\mathrm{d}y = -\infty,
\end{aligned}$$

the last identity following from the second limit in (A.11). The other limit in (A.13) is proved similarly. $\square$

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