The London School of Economics and Political Science

Quantitative Modelling Of Market Booms And Crashes

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A thesis submitted to the Department of Statistics of the London School of Economics for the degree of Doctor of Philosophy, London, September 2012

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

ABSTRACT

Multiple equilibria models are one of the main categories of theoretical models for stock market crashes. To the best of my knowledge, existing multiple equilibria models have been developed within a discrete time framework and only explain the intuition behind a single crash on the market.

The main objective of this thesis is to model multiple equilibria and demonstrate how market prices move from one regime into another in a continuous time framework. As a consequence of this, a multiple jump structure is obtained with both possible booms and crashes, which are defined as points of discontinuity of the stock price process.

I consider five different models for stock market booms and crashes, and look at their pros and cons. For all of these models, I prove that the stock price is a càdlàg semimartingale process and find conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump, given the public information available to market participants. Finally, I discuss the problem of model parameter estimation and conduct a number of numerical studies.

Acknowledgements

I would like to express my deepest gratitude and utmost respect to my supervisor Dr. Umut Cetin. This work would not have been done without his continued guidance and tremendous support. Even the words “deepest” and “tremendous” do not demonstrate in full how grateful I am to my supervisor.

I want to acknowledge my department for giving me a great opportunity to study and work in such a friendly and intellectually inspiring environment. I also wish to warmly thank my examiners Dr. Angelos Dassios and Prof. Nizar Touzi, as well as Prof. Pauline Barrieu, Dr. Erik Baurdoux, Mr. Ian Marshall, Prof. Antonio Mele, Dr. Irini Moustaki, Dr. Philippe Mueller, Prof. Dimitri Vayanos, Dr. Andrea Vedolin and Dr. Hao Xing for their numerous comments and suggestions.

Finally, I am immensely indebted to my mother and father for their love, support and sacrifice throughout all my life, and I dedicate my thesis to my beloved parents.

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1. INTRODUCTION

In literature, there are four major categories of models for stock market crashes: liquidity shortage models, multiple equilibria and sunspot models, bursting bubble models, and lumpy information aggregation models (see, e.g., Brunnermeier [9]). In liquidity shortage models, market price might plummet due to a temporary reduction in liquidity (see, e.g., Grossman [22]). According to multiple equilibria and sunspot models, several price levels exist and a market crash might occur for no fundamental reason (see, e.g., Gennotte and Leland [21], Krugman [31], Drazen [18], Barlevy and Veronesi [5,7], Yuan [48], Angeletos and Werning [4], Barlevy and Veronesi [6], Ozdenoren and Yuan [35], and Ganguli and Yang [20]). In bursting bubble models, all market participants realise an asset price is greater than its fundumental value, but they keep buying that asset since they believe others do not know that it is overpriced, and at some point the bubble bursts and market crashes (see, e.g., Abreu and Brunnermeier [2], Scheinkman and Xiong [42], Cox and Hobson [15], Jarrow et al. [28], O'Hara [34], Allen and Gale [3], Brunnermeier [10], Friedman and Abraham [19], Jarrow et al. [26,27,29], Kindleberger and Aliber [30], and Brunnermeier and Oehmke [11]). According to the lumpy information aggregation approach, the overpricing issue is not a common knowledge among the market participants, but at some point an additional relevant information is revealed and, combining that with the past price dynamics, less informed traders suddenly realise that this overpricing exists and the price sharply declines (see, e.g., Romer [40], Caplin and Leahy [14] and Hong and Stein [24]).

The main objective of this thesis is to develop a quantitative approach to the modelling of multiple equilibria which describes how market prices jump from one regime to another. As a starting point for the research, I take the one-period model in the paper of Gennotte and Leland [21] and study its extension into continuous time.

Gennotte and Leland [21] attempts to explain the market crash of 1987 by the presence of dynamic hedgers. In this model, two assets are traded: a single risky stock and risk-free bond. The future price of the risky security is assumed to be normally distributed and the current price is determined according to supply and demand. Net supply consists of a fixed amount, some normally distributed liquidity shocks and some dynamic hedgers component. Demand consists of uninformed and informed investors, who all maximise expected exponential utility of their wealth over a single period. According to this model, when hedging activity is unobserved the excess-demand curve can be backward-bending, and this creates multiple equilibria. It means that a small shift in information can lead to a market crash.

In Chapter 2, I develop three multiple equilibria models in a continuous time. It is assumed that two assets, a single risky stock and risk-free bond, are traded and three groups of agents are considered: rational investors, dynamic hedgers and noise traders. The first group of agents corresponds to the total demand, while the second and the third groups correspond to the total supply in Genotte and Leland [20]. For the sake of simplicity, it is supposed that there is no information asymmetry. In making their decisions, agents approximate the future stock price dynamics with an auxiliary Brownian motion with a drift process, and this makes it normally distributed. The first two models assume that the total number of dynamic hedgers stays constant over all of the time period. The difference between the two models is in alternative mechanisms for determining how the market price moves from one regime to another. The third model corresponds to the scenario of the number of dynamic hedgers being a jump stochastic process. For all three models, I prove that the stock price is a càdlàg semimartingale process and find conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump, given the information available to market participants.

Although all three models work in accordance with the main objective of this thesis, they have some drawbacks. First, they do not eliminate the possibility of negative prices. Second, actual price dynamics are different from the auxiliary Brownian motion with a drift approximation. Third, they do not have a solution in a closed form and, therefore, can be solved only numerically. Finally, the jump structure in the first two models is quite restrictive and does not allow for some frameworks; in particular more than two consecutive market booms or more than two consecutive market crashes. This provides the motivation to develop two alternative models that will be presented in Chapter 3. For both models, I prove that the stock price is a càdlàg semimartingale process and find conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump, given the information available to market participants. These models yield positive prices and closed-form solutions, but the pricing equation is given exogenously and a simple jump structure model does not allow two consecutive booms or crashes: any boom precedes a crash which in turn precedes a boom etc. The simple jump structure model is designed just to resemble the shape of the market microstructure models. The Markov chain jump structure model is an extension of the simple jump structure model and relaxes the construction that a crash can be followed only by a boom and a boom can be followed only by a crash.

The sequence of this thesis is organised as follows. In Chapter 2, three market microstructure models are introduced. In Chapter 3, two alternative models are considered. In Chapter 4, the problem of model parameter estimation is discussed. Chapter 5 contains numerical studies and Chapter 6 concludes.

2. MARKET MICROSTRUCTURE MODELS

2.1 Market microstructure framework

I will work on a filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. It is assumed that time horizon is $[0, T]$ and trading takes place continuously. In the models developed in this chapter, there are two underlying assets in the economy: risky stock and risk-free bond. Risk-free bonds are in perfectly elastic supply and grow at net return $r > 0$: one unit invested at time t returns $e^{r\Delta t}$ units at time $t + \Delta t$, $0 \leq t < t + \Delta t \leq T$. Stock is assumed to be in zero net supply.

In making their decisions, agents use their wealth $(W_s, 0 \leq s \leq t < T)$, the stock price process $(P_s, 0 \leq s \leq t < T)$ and an auxiliary process $(p_u, t \leq u \leq T)$ such that

$$p_u = P_t + \alpha_1 \times \beta_{u-t} + \alpha_2 \times (u - t), \tag{2.1}$$

where β is a standard Brownian motion that starts at 0, $\alpha_1 > 0$ and $\alpha_2 \in \mathbb{R}$. This process $(p_u, t \leq u \leq T)$ approximates the future dynamics of the stock price $(P_u, t \leq u \leq T)$.

Let $T_0 \in (0, T)$. It is assumed that agents estimate parameters in (2.1) based on the values $P_{t_i} - P_{t_{i-1}}$, $1 \leq i \leq k$, where $0 = t_0 < t_1 < ... < t_k < T_0$ and P_{t_i} stand for the end-of-day prices up to time T_0. Since Brownian motion has independent increments, they can use the following maximum likelihood estimates:

$$\hat{\alpha}_2 = \frac{\sum_{i=1}^{k}(P_{t_i} - P_{t_{i-1}})}{\sum_{i=1}^{k}(t_i - t_{i-1})} = \frac{P_{t_k} - P_0}{t_k}$$

and

$$\hat{\alpha}_1 = \sqrt{\frac{1}{k}\sum_{i=1}^{k}\frac{(P_{t_i} - P_{t_{i-1}} - \hat{\alpha}_2(t_i - t_{i-1}))^2}{t_i - t_{i-1}}}.$$

In the subsequent sections, I will analyse the stock price dynamics $(P_t, T_0 \leq t < T)$.

2.1.1 Rational investors' demand for stock

First, I start in the discrete framework and then take limits at the end. Following the methodology of Gennotte and Leland [21], each rational investor maximises the expected utility of time $t+\Delta t$ wealth $W_{t+\Delta t}$ with respect to the amount of shares of risky stock, given the information this investor has at time t, and assuming there is no trading between t and $t+\Delta t$ and that he or she invests in two underlying assets:

$$\mathbb{E}\Big[U\Big(W_{t+\Delta t}\Big) \mid \Big((W_s, P_s), 0 \leq s \leq t\Big)\Big] \to \max_x, \tag{2.2}$$

where

$$W_{t+\Delta t} = xp_{t+\Delta t} + e^{r\Delta t}(W_t - xP_t) \tag{2.3}$$

and utility function is assumed to exhibit constant absolute risk aversion with coefficient $a > 0$:

$$U\Big(W_{t+\Delta t}\Big) = -e^{-\frac{W_{t+\Delta t}}{a}}.$$

In view of (2.2) and (2.3), rational investors solve the following maximisation problem:

$$-e^{\frac{(e^{r\Delta t}-1)xP_t - \alpha_2 x\Delta t}{a}}\mathbb{E}\Big(e^{-\frac{\alpha_1 x\beta_{\Delta t}}{a}}\Big) \to \max_x .$$

The formula for the moment-generating function of a normal random variable yields the individual rational investor's demand for stock in the discrete framework is equal to

$$\frac{a(\alpha_2\Delta t - (e^{r\Delta t} - 1)P_t)}{\alpha_1^2\Delta t}.$$

As $\Delta t \downarrow 0$, it can be concluded that the cumulative demand for rational investors in the continuous framework is equal to

$$w^R \times \frac{a(\alpha_2 - rP_t)}{\alpha_1^2},$$

where w^R is the total number of rational investors, which is supposed to be constant.

2.1.2 Dynamic hedgers' demand for stock

It is assumed that the total number of dynamic hedgers follows some stochastic process w_t^D with the sole objective to replicate contingent claims of the following type:

$$F(P_T) = \max(P_T - K, 0).$$

Since at-the-money forward options attract the greatest amount of volume, which decreases dramatically as the option becomes deeper in-the-money forward or out-of-the-money forward, I normalise the total number of contingent claims for each hedger to 1 but assume that the number of contingent claims with strike $\in dK$ is equal to $\frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK$ for some small value of $\sigma_\kappa > 0$, where $\kappa = P_{s_k} e^{r(T-s_k)}$ and P_{t_k} is the most recent end-of-day price observation: $0 = t_0 < t_1 < ... < t_k < T_0$. In (2.18), an upper bound for σ_κ will be specified.

It is supposed that the dynamic hedgers believe that the stock price follows (2.1), thus, they value the claim at

$$P(t,x) = \mathbb{E}^{\mathbb{P}}\Big[e^{-r(T-t)} F\Big(e^{r(T-t)}(x + \alpha_1 \int_0^{T-t} e^{-rs} d\beta_s)\Big)\Big], \quad \text{for} \quad t \in [T_0, T).$$

Therefore,

$$\begin{aligned} P(t,x) &= \int_{Ke^{-r(T-t)}}^{\infty} (y - Ke^{-r(T-t)}) \frac{1}{\sqrt{2\pi\Sigma^2(t)}} e^{-\frac{(y-x)^2}{2\Sigma^2(t)}} dy \\ &= \Sigma(t) \times \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-Ke^{-r(T-t)})^2}{2\Sigma^2(t)}} + (x - Ke^{-r(T-t)})\Phi\Big(\frac{x - Ke^{-r(T-t)}}{\Sigma(t)}\Big), \end{aligned}$$

where

$$\Sigma(t) = \alpha_1 \sqrt{\frac{1 - e^{-2r(T-t)}}{2r}}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

is the cumulative distribution function of a standard normal distribution.
Hence, the dynamic hedgers component of demand at time $t \in [T_0, T)$ is equal to

$$\begin{aligned} \pi(t,x) &= w_t^D \int_{-\infty}^{\infty} \frac{\partial P(t,x)}{\partial x} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \\ &= w_t^D \int_{-\infty}^{\infty} \Phi\Big(\frac{x - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK. \end{aligned}$$

2.1.3 Noise traders' demand for stock

It is assumed that the noise traders component of demand is given by $w^N \times (\mu_N + \sigma_N B_t)$, $\sigma_N > 0$, where $(B_t, t \geq 0)$ is a standard Brownian motion starting at 0 and w^N is the total number of noise traders, which is supposed to be constant. Noise traders trade according to the rule that is independent of the stock price fundamental value and is exogenous to the model. The noise traders component of demand makes the dynamics of the stock price stochastic. Note that since Brownian motion is a continuous process, the noise traders component of demand is also continuous.

2.1.4 Pricing equation

The market clearing condition states that the total demand should be equal to 0:

$$w^R \times \frac{a(\alpha_2 - rP_t)}{\alpha_1^2} + w_t^D \times \int_{-\infty}^{\infty} \Phi\Big(\frac{P_t - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK + w^N \times (\mu_N + \sigma_N B_t) = 0.$$

Denote by

$$\gamma_1 = w^R \times \frac{ar}{\alpha_1^2}, \quad \gamma_2 = w^R \times \frac{a\alpha_2}{\alpha_1^2} + w^N \times \mu_N, \quad \gamma_3 = w^N \times \sigma_N, \tag{2.4}$$

and define function $H : [T_0, T) \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by

$$H(t, z, x) = \frac{\gamma_1 x - z \int_{-\infty}^{\infty} \Phi\Big(\frac{x - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK - \gamma_2}{\gamma_3}. \tag{2.5}$$

Thus, the pricing equation is given by

$$H(t, w_t^D, P_t) = B_t. \tag{2.6}$$

In the remaining part of this section, the properties of this equation will be discussed.

Remark 2.1. Since $0 \le z \int_{-\infty}^{\infty} \Phi\Big(\frac{x - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \le z$, it can be concluded that

$$\lim_{x \to -\infty} H(t, z, x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} H(t, z, x) = \infty.$$

Remark 2.2. Note that $H(t, z, x)$ is $C^{1,0,2}\Big([T_0, T) \times \mathbb{R}_+ \times \mathbb{R}\Big)$.

Differentiating $H(t, z, x)$ with respect to x, it can be shown that

$$\begin{aligned} H_x(t, z, x) &= \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{z}{\sqrt{2\pi\sigma_\kappa^2\Sigma^2(t)}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-Ke^{-r(T-t)})^2}{2\Sigma^2(t)}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK\Big) \\ &= \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{z}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}} e^{-\frac{(x-\kappa e^{-r(T-t)})^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}}\Big). \end{aligned} \tag{2.7}$$

If the total number of dynamic hedgers satisfies

$$w_t^D \leq \gamma_1 \sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}, \tag{2.8}$$

then $H_x(t, w_t^D, x) \geq 0$ for all x, that is, $H(t, w_t^D, x)$ is an increasing function of x. In virtue of Remark 2.1 and Remark 2.2, the pricing equation has a single solution which is denoted by

$$\bar{p}(t, w_t^D, B_t). \tag{2.9}$$

If the total number of dynamic hedgers is a continuous process, then, in obedience to the implicit function theorem, the stock price process is also continuous. Therefore, if w_t^D satisfies (2.8), the price jumps only through a jump in the number of dynamic hedgers w_t^D.

On the other hand, if w_t^D satisfies

$$w_t^D > \gamma_1 \sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}, \tag{2.10}$$

then $H_x(t, w_t^D, x)$ as a function of x changes its sign in $\bar{p}_1(t, w_t^D)$ and $\bar{p}_2(t, w_t^D)$:

$$H_x(t, w_t^D, P_t) \begin{cases} > 0 & \text{if } P_t < \bar{p}_1(t, w_t^D) \text{ or } P_t > \bar{p}_2(t, w_t^D) \\ = 0 & \text{if } P_t = \bar{p}_1(t, w_t^D) \text{ or } P_t = \bar{p}_2(t, w_t^D) \\ < 0 & \text{if } \bar{p}_1(t, w_t^D) < P_t < \bar{p}_2(t, w_t^D), \end{cases} \tag{2.11}$$

where

$$\bar{p}_1(t, w_t^D) = \kappa e^{-r(T-t)} - \sqrt{-2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t)) \ln\Big(\frac{\gamma_1}{w_t^D}\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}\Big)} \tag{2.12}$$

and

$$\bar{p}_2(t, w_t^D) = \kappa e^{-r(T-t)} + \sqrt{-2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t)) \ln\Big(\frac{\gamma_1}{w_t^D}\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}\Big)}. \tag{2.13}$$

Denote the local maximum and local minimum values by

$$H_1(t, w_t^D) = H(t, w_t^D, \bar{p}_1(t, w_t^D)) \quad \text{and} \quad H_2(t, w_t^D) = H(t, w_t^D, \bar{p}_2(t, w_t^D)). \tag{2.14}$$

In the market microstructure models developed in this chapter, the dynamic hedgers component of demand $\pi(t, P_t)$ is an increasing function of P_t, while the rational investors component of demand $w^R \times \frac{a(\alpha_2 - rP_t)}{\alpha_1^2}$ is a decreasing function of P_t. If the total number of dynamic hedgers w_t^D is large

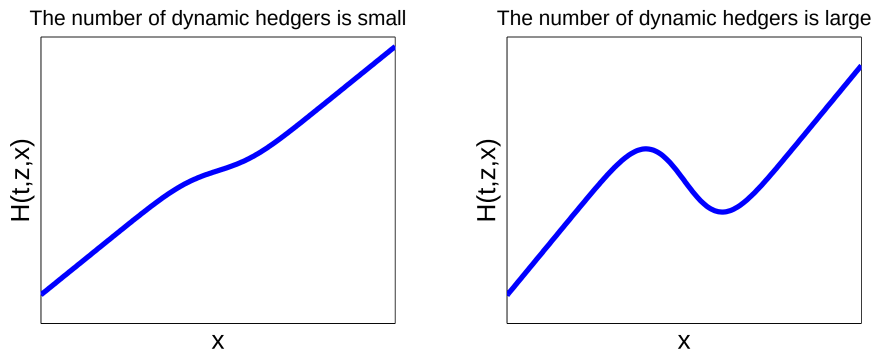

Fig. 2.1: Plot of $H(t,z,x)$ if the number of dynamic hedgers $w_t^D = z$ at time t is small and large

enough such that it satisfies (2.10), then the roots of the pricing equation (2.6) have the following structure:

$$\begin{cases} \bar{p}^l(t,w_t^D,B_t) & \text{if } B_t < H_2(t,w_t^D) \\ \bar{p}^l(t,w_t^D,H_2(t,w_t^D)) \text{ and } \bar{p}_2(t,w_t^D) & \text{if } B_t = H_2(t,w_t^D) \\ \bar{p}^l(t,w_t^D,B_t), \bar{p}^m(t,w_t^D,B_t) \text{ and } \bar{p}^u(t,w_t^D,B_t) & \text{if } H_2(t,w_t^D) < B_t < H_1(t,w_t^D) \\ \bar{p}_1(t,w_t^D) \text{ and } \bar{p}^u(t,w_t^D,H_1(t,w_t^D)) & \text{if } B_t = H_1(t,w_t^D) \\ \bar{p}^u(t,w_t^D,B_t) & \text{if } B_t > H_1(t,w_t^D), \end{cases} \tag{2.15}$$

where $\bar{p}^l(t,w_t^D,B_t)$, $\bar{p}^m(t,w_t^D,B_t)$ and $\bar{p}^u(t,w_t^D,B_t)$ are defined implicitly as the roots of (2.6) satisfying

$$\begin{cases} \bar{p}^l(t,w_t^D,B_t) \leq \bar{p}_1(t,w_t^D) & \text{and defined if} \quad B_t \leq H_1(t,w_t^D) \\ \bar{p}_1(t,w_t^D) \leq \bar{p}^m(t,w_t^D,B_t) \leq \bar{p}_2(t,w_t^D) & \text{and defined if} \quad H_2(t,w_t^D) \leq B_t \leq H_1(t,w_t^D) \\ \bar{p}^u(t,w_t^D,B_t) \geq \bar{p}_2(t,w_t^D) & \text{and defined if} \quad B_t \geq H_2(t,w_t^D). \end{cases} \tag{2.16}$$

Therefore, the system exhibits multiple equilibria if $H_2(t,w_t^D) \leq B_t \leq H_1(t,w_t^D)$. Market booms and crashes occur when the price moves from one regime into another, either through a jump into an alternative root according to (2.15) or through a jump in the total number of dynamic hedgers w_t^D.

2.2 *Constant number of dynamic hedgers models*

In this section, it is assumed that the total number of dynamic hedgers w^D is a constant satisfying condition

$$w^D > \max_{t\in[T_0,T)}\left(\gamma_1\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}\right). \tag{2.17}$$

Recall that the value of σ_κ should be quite small, hence, it can be specified that

$$0 < \sigma_\kappa^2 \leq \frac{\alpha_1^2}{2r}. \tag{2.18}$$

In view of (2.18), condition (2.17) is equivalent to

$$w^D > \gamma_1\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-T_0)} + \Sigma^2(T_0)\Big)}. \tag{2.19}$$

In virtue of (2.10), the system admits multiple equilibria which give rise to jumps during the whole interval $[T_0, T)$. To simplify the notation introduced in (2.5), (2.12) – (2.14) and (2.16), let

$$h(t,x) = H(t, w^D, x), \tag{2.20}$$

$$p_1(t) = \bar{p}_1(t, w^D), \quad p_2(t) = \bar{p}_2(t, w^D), \tag{2.21}$$

$$h_1(t) = H_1(t, w^D), \quad h_2(t) = H_2(t, w^D), \tag{2.22}$$

and

$$p^l(t,y) = \bar{p}^l(t, w_t^D, y), \quad p^m(t,y) = \bar{p}^m(t, w_t^D, y), \quad p^u(t,y) = \bar{p}^u(t, w_t^D, y). \tag{2.23}$$

Remark 2.3. According to (2.20) and Remark 2.1, it can be concluded that

$$\lim_{x\to-\infty} h(t,x) = -\infty \quad \text{and} \quad \lim_{x\to\infty} h(t,x) = \infty. \tag{2.24}$$

Remark 2.4. According to (2.20) and Remark 2.2, it can be shown that $h(t,x)$ is $C^{1,2}\Big([T_0,T)\times\mathbb{R}\Big)$.

In view of (2.11), (2.20) and (2.21), it can be concluded that

$$h_x(t,P_t)\begin{cases} > 0 & \text{if } P_t < p_1(t) \text{ or } P_t > p_2(t) \\ = 0 & \text{if } P_t = p_1(t) \text{ or } P_t = p_2(t) \\ < 0 & \text{if } p_1(t) < P_t < p_2(t). \end{cases} \tag{2.25}$$

The pricing equation (2.6) can be rewritten as

$$h(t, P_t) = B_t. \tag{2.26}$$

Roots of (2.26) have the following structure:

$$\begin{cases} p^l(t, B_t) & \text{if } B_t < h_2(t) \\ p^l(t, h_2(t)) \text{ and } p_2(t) & \text{if } B_t = h_2(t) \\ p^l(t, B_t), p^m(t, B_t) \text{ and } p^u(t, B_t) & \text{if } h_2(t) < B_t < h_1(t) \\ p_1(t) < p^u(t, h_1(t)) & \text{if } B_t = h_1(t) \\ p^u(t, B_t) & \text{if } B_t > h_1(t), \end{cases} \tag{2.27}$$

where $p^l(t, B_t)$, $p^m(t, B_t)$ and $p^u(t, B_t)$ satisfy

$$\begin{cases} p^l(t, B_t) \leq p_1(t) & \text{and defined if} \quad B_t \leq h_1(t) \\ p_1(t) \leq p^m(t, B_t) \leq p_2(t) & \text{and defined if} \quad h_2(t) \leq B_t \leq h_1(t) \\ p^u(t, B_t) \geq p_2(t) & \text{and defined if} \quad B_t \geq h_2(t). \end{cases} \tag{2.28}$$

Recall that the main goal of this thesis is to model how market prices move from one root to another within this multiple equilibria framework. To do that, define a state process S_t taking values in a state space $\mathbb{S}$ consisting of three different states: lower level equilibrium s_1, medium level equilibrium s_2 and upper level equilibrium s_3. If S_t is known, the stock price value can be assigned by

$$P_t = \begin{cases} p^l(t, B_t) & \text{if } S_t = s_1 \\ p^m(t, B_t) & \text{if } S_t = s_2 \\ p^u(t, B_t) & \text{if } S_t = s_3. \end{cases} \tag{2.29}$$

According to (2.28), $S_t = s_1$ for $B_t < h_2(t)$ and $S_t = s_3$ for $B_t > h_1(t)$ whereas S_t can take any value in $\mathbb{S}$ for $h_2(t) \leq B_t \leq h_1(t)$, that is, when the system exhibits multiple equilibria.

Remark 2.5 I would like to have a model that satisfies three basic conditions. First, it should not have infinite price oscillation. Second, the jump times should be random. Finally, the jump sizes and the price values at the time of the jump should depend not only on those jump times but also from some other source of randomness. Otherwise, it would be known at time t by how much or at what price level the stock price process could jump at time $u > t$, and this is not the case if discussing actual stock price dynamics.

Remark 2.6 The most intuitive and simple model would be the one that excludes state s_2 from consideration and defines S_t such that it switches from s_1 to s_3 (respectively from s_3 to s_1) when B_t crosses $h_1(t)$ (respectively $h_2(t)$). In virtue of Theorem 2.1, it can be concluded that an infinite price oscillation is not possible; but the problem is that, although the jump times are random, the size of positive (respectively negative) jump at time t is equal to $p^u(t, h_1(t)) - p_1(t)$ (respectively $p^l(t, h_1(t)) - p_2(t)$), that is, there is no other source of randomness aside from the jump time. For this reason, consideration is given to the models with state processes taking all three values in $\mathbb{S}$. In Section 2.2.1 and Section 2.2.2, two models are developed that satisfy all three conditions described in Remark 2.5.

Theorem 2.1 There exists some $\Delta > 0$ such that

$$h_1(t) - h_2(t) \geq \Delta, \ \forall t \in [T_0, T).$$

Proof The proof is provided in the Appendix. ■

2.2.1 Endogenous switching model

Suppose the system is in the lower level equilibrium s_1. If a simple rule is set $S_t = s_2$ or $S_t = s_3$ for $h_2(t) \leq B_t \leq h_1(t)$, the result would be an infinite price oscillation when Brownian motion B_t hits the boundary $h_2(t)$ since B_t would come back to $h_2(t)$ infinitely fast. To avoid this oscillation, it is necessary for S_t to stay in the state s_1 for a while if B_t hits $h_2(t)$. According to Remark 2.5, the rule to wait until B_t hits the boundary $h_1(t)$ does not work very well. In the endogenous switching model, it is assumed that there is some exogenous exponentially distributed random waiting period until B_t hits the boundary $h_1(t)$. After that random period expires, if the system is still in the state s_1, then instead of the boundary $h_1(t)$, a new boundary is necessary which is a convex combination of $h_1(t)$ and $h_2(t)$. When B_t hits that boundary, $h_2(t) < B_t < h_1(t)$, and the system switches from the lower level equilibrium to the upper or medium level equilibrium pursuant to the value of an independent Bernoulli random variable. If the system is in the upper level equilibrium s_3, then the switching procedure is similar. If the system is in the medium level equilibrium s_2, then it is necessary to wait until B_t hits one of the two boundaries $h_1(t)$ or $h_2(t)$ and then S_t switches to the corresponding regime.

Model setup

For any fixed $u \in [T_0, \infty)$ and $c \in \mathbb{R}_+$, define functions $h^l : [T_0, T) \to \mathbb{R}$ and $h^u : [T_0, T) \to \mathbb{R}$ by:

$$h^l(t; u) = \begin{cases} h_1(t) & \text{if } t \leq u \\ e^{-c(t-u)} h_1(t) + \Big(1 - e^{-c(t-u)}\Big) h_2(t) & \text{if } t > u \end{cases} \tag{2.30}$$

and

$$h^u(t; u) = \begin{cases} h_2(t) & \text{if } t \leq u \\ \Big(1 - e^{-c(t-u)}\Big) h_1(t) + e^{-c(t-u)} h_2(t) & \text{if } t > u. \end{cases} \tag{2.31}$$

Function h^l (respectively h^u) corresponds to a boundary the process B_t should hit to switch from the lower level equilibrium (respectively upper level equilibrium) to another equilibrium. In the models developed in the thesis, the distributions for the time of, the size of and the type of the next jump are calculated, and, for the market microstructure models, it can be seen that these probabilities can be expressed in terms of some functions of Brownian motion hitting time densities and probabilities of one-sided or two-sided curved boundaries. By construction, functions $h^l(t; u)$ and $h^u(t; u)$ are in the class of $C^2\Big([T_0, T)\Big)$, and this technical condition admits application of various numerical techniques that I discuss in Chapter 5.
Let the sequences of independent random variables $(T_i^l, i = 0, 1, ...)$, $(T_i^u, i = 0, 1, ...)$, $(\zeta_i^{lu}, i = 0, 1, ...)$ and $(\zeta_i^{ul}, i = 0, 1, ...)$, where

$$T_i^l \sim \text{Exp}(\lambda_l), \lambda_l > 0, \quad T_i^u \sim \text{Exp}(\lambda_u), \lambda_u > 0,$$

$$\zeta_i^{lu} = \begin{cases} 1 & \text{with probability } p_{lu} \\ 0 & \text{with probability } p_{lm} = 1 - p_{lu} \end{cases} \quad \text{and} \quad \zeta_i^{ul} = \begin{cases} 1 & \text{with probability } p_{ul} \\ 0 & \text{with probability } p_{um} = 1 - p_{ul}, \end{cases} \tag{2.32}$$

be $\mathcal{F}$-measurable and such that they are all independent of $(B_t, t \geq 0)$ and of each other.

Sequences $(T_i^l, i = 0, 1, ...)$ and $(T_i^u, i = 0, 1, ...)$ correspond to waiting times in state s_1 and s_3 until the Brownian motion hits the convex combination of h_1 and h_2 instead of just h_1 and h_2, and if this happens, then sequences of independent Bernoulli random variables ζ_i^{lu} and ζ_i^{ul} determine the new values of the state process S_t.

Definition 2.1 Define processes $(S_t, T_0 \leq t < T)$ and $(P_t, T_0 \leq t < T)$ according to the following construction mechanism.

Step 1 Set $i = 0$, $\tau_0 = T_0$ and the starting value of the state process

$$S_{\tau_0} = \begin{cases} s_1 & \text{if } B_{\tau_0} \leq h_2(\tau_0) \\ s_3 & \text{if } B_{\tau_0} \geq h_1(\tau_0) \\ s & \text{if } h_2(\tau_0) < B_{\tau_0} < h_1(\tau_0), \end{cases}$$

where $s \in \mathbb{S}$ is some known constant. If $h_2(\tau_0) < B_{\tau_0} < h_1(\tau_0)$, then all three states are possible and $S_{\tau_0} = s$ just for definiteness. Although the system exhibits multiple equilibria when $B_{\tau_0} = h_2(\tau_0)$ (respectively $B_{\tau_0} = h_1(\tau_0)$), assign value $S_{\tau_0} = s_1$ (respectively $S_{\tau_0} = s_3$) in order to avoid an infinite price oscillation. For this reason, it is assigned $S_t = s_1$ (respectively $S_t = s_3$) if $B_t \leq h_2(t)$ (respectively $B_t \geq h_1(t)$) for all $t \in [T_0, T)$.

Step 2 Set

$$\tau_{i+1} = \begin{cases} \inf\Big(t > \tau_i : B_t \geq h^l(t; \tau_i + T_i^l)\Big) \wedge T & \text{if } S_{\tau_i} = s_1 \\ \inf\Big(t > \tau_i : B_t \geq h_1(t) \text{ or } B_t \leq h_2(t)\Big) \wedge T & \text{if } S_{\tau_i} = s_2 \\ \inf\Big(t > \tau_i : B_t \leq h^u(t; \tau_i + T_i^u)\Big) \wedge T & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where $\inf \emptyset = \infty$ by convention.

If the system is in the lower (respectively upper) level state s_1, then it is necessary to wait until B_t hits the boundary h^l (respectively h^u). If the system is in the medium level state s_2, then it is necessary to wait until B_t hits either h_1 or h_2.

Step 3 Set $S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1})$.

Step 4 If $\tau_{i+1} = T$, then algorithm stops.

Step 5 Set

$$S_{\tau_{i+1}} = \begin{cases} s_1 & \text{if } B_{\tau_{i+1}} \le h_2(\tau_{i+1}) \\ s_3 & \text{if } B_{\tau_{i+1}} \ge h_1(\tau_{i+1}) \\ s_3 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_1 \text{ and } \zeta_i^{lu} = 1 \\ s_2 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_1 \text{ and } \zeta_i^{lu} = 0 \\ s_1 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_3 \text{ and } \zeta_i^{ul} = 1 \\ s_2 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_3 \text{ and } \zeta_i^{ul} = 0. \end{cases}$$

If, e.g., $S_{\tau_i} = s_1$ and $\tau_{i+1} > \tau_i + T_i^l$, then, at time τ_{i+1}, B_t hits a convex combination of h_1 and h_2, which means that $h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1})$. In this case, the system switches from the lower level to the upper or medium level according to the value of an independent Bernoulli random variable. If $B_{\tau_{i+1}} \le h_2(\tau_{i+1})$ (respectively $B_{\tau_{i+1}} \ge h_1(\tau_{i+1})$), then assign $S_{\tau_{i+1}} = s_1$ (respectively $S_{\tau_{i+1}} = s_3$) in concordance with the argument described in Step 1.

Step 6 Set $i = i + 1$ and go to Step 2.

Finally, define the stock price $(P_t, T_0 \le t < T)$ pursuant to (2.29). ■

Intensities λ_l and λ_u and parameter c control the frequency of the stock price jumps, while probabilities p_{lu} and p_{ul} control the proportion of small versus big market jumps corresponding to the scenarios where B_t hits a convex combination of h_1 and h_2.

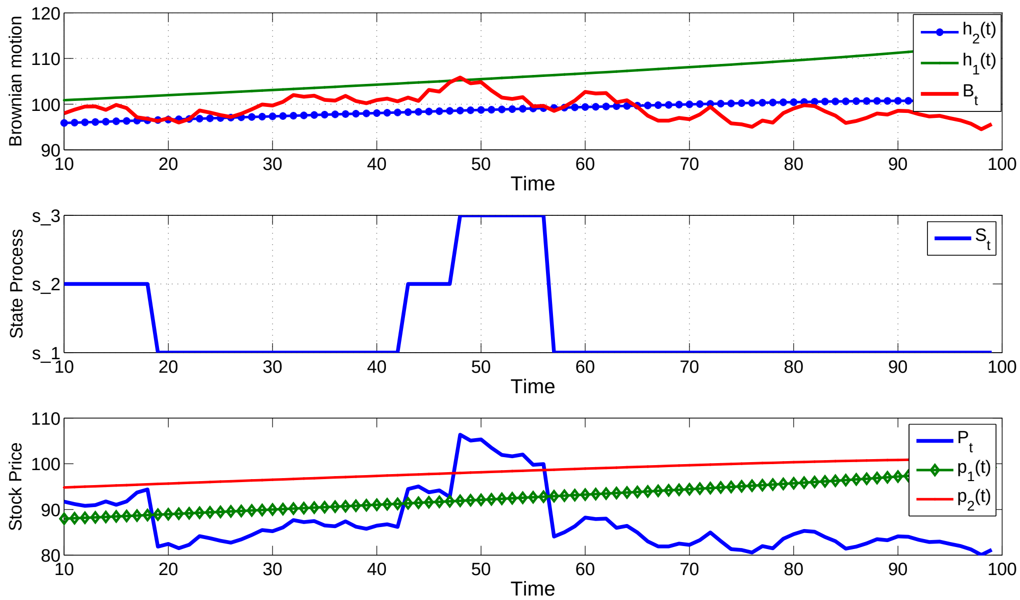

Fig. 2.2: Simulated stock price dynamics in the endogenous switching model computed for some set of parameters: $T_0 = 10$, $T = 100$, $\alpha_1 = 0.3$, $c = 0.025$, $\sigma_\kappa = 0.03$, $\kappa = 100$, $w^D = 30$, $\gamma_1 = 2$, $\gamma_2 = 1$, $\gamma_3 = 2$, $\zeta_1^{lu} = 0$; initial value of S_t is assumed to be equal to s_2; convex combination starts at $t = 39$ and $t = 72$; stock price jumps at $t = 19$, $t = 43$, $t = 48$, $t = 57$.

2.2.2 *Exogenous shocks model*

In the exogenous shocks model, like in the endogenous switching model, if $B_t \leq h_2(t)$ (respectively $B_t \geq h_1(t)$), then $S_t = s_1$ (respectively $S_t = s_3$), for all $t \in [T_0, T)$. If $h_2(t) < B_t < h_1(t)$, the system stays in its current state until there is a new arrival in an exogenous sunspot shock process which is assumed to be a Poisson process independent of B_t. The shock switches the state of the system to one of the other two states for no fundamental reason, and the new level value is determined in obedience to the value of an independent Bernoulli random variable with probability of success depending on the current state of the state process.

It is assumed that $(Z_t, t \geq 0)$ is a $\mathcal{F}$-measurable homogeneous Poisson process having some intensity λ_Z and this process is independent of $(B_t, t \geq 0)$. Let the sequences of independent Bernoulli random variables $(\zeta_i^{lu}, i = 0, 1, ...)$ and $(\zeta_i^{ul}, i = 0, 1, ...)$ be defined according to (2.32) and the sequence of independent Bernoulli random variables $(\zeta_i^{mu}, i = 0, 1, ...)$ be given by

$$\zeta_i^{mu} := \begin{cases} 1 & \text{with probability } p_{mu} \\ 0 & \text{with probability } p_{ml} = 1 - p_{mu}. \end{cases}$$

Suppose that all three sequences are in $\mathcal{F}$ and that they are all independent of $(B_t, t \geq 0)$, $(Z_t, t \geq 0)$ and of each other. These sequences determine new states of the state process S_t in case of shock arrivals when $h_2(t) < B_t < h_1(t)$.

Definition 2.2 Define processes $(S_t, T_0 \leq t < T)$ and $(P_t, T_0 \leq t < T)$ according to the following construction mechanism.

Step 1 Set $i = 0$, $\tau_0 = T_0$ and the starting value of the state process

$$S_{\tau_0} = \begin{cases} s_1 & \text{if } B_{\tau_0} \leq h_2(\tau_0) \\ s_3 & \text{if } B_{\tau_0} \geq h_1(\tau_0) \\ s & \text{if } h_2(\tau_0) < B_{\tau_0} < h_1(\tau_0), \end{cases}$$

where $s \in \mathbb{S}$ is some known constant. All the intuition is the same as in Step 1 of Definition 2.1.

Step 2 Set

$$\tau_{i+1} = \begin{cases} \inf\Big(t > \tau_i : B_t \geq h_1(t)\Big) \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_1 \\ \inf\Big(t > \tau_i : B_t \geq h_1(t) \text{ or } B_t \leq h_2(t)\Big) \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_2 \\ \inf\Big(t > \tau_i : B_t \leq h_2(t)\Big) \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where $\hat{\tau}_i$ is the first arrival after τ_i in Poisson process Z_t such that $h_2(\hat{\tau}_i) < B_{\hat{\tau}_i} < h_1(\hat{\tau}_i)$. If there are no such arrivals, then define $\hat{\tau}_i = \infty$. Recall that $\inf \emptyset = \infty$ by convention.

It is necessary to wait until B_t hits the corresponding one-sided or two-sided curved boundary, or until $\hat{\tau}_i$, or until time expires, whatever is earlier. Intensity λ_Z controls the frequency of jumps.

Step 3 Set $S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1})$.

Step 4 If $\tau_{i+1} = T$, then algorithm stops.

Step 5 Set

$$S_{\tau_{i+1}} = \begin{cases} s_1 & \text{if } B_{\tau_{i+1}} \leq h_2(\tau_{i+1}) \\ s_3 & \text{if } B_{\tau_{i+1}} \geq h_1(\tau_{i+1}) \\ s_3 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_1 \text{ and } \zeta_i^{lu} = 1 \\ s_2 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_1 \text{ and } \zeta_i^{lu} = 0 \\ s_1 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_3 \text{ and } \zeta_i^{ul} = 1 \\ s_2 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_3 \text{ and } \zeta_i^{ul} = 0 \\ s_3 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_2 \text{ and } \zeta_i^{mu} = 1 \\ s_1 & \text{if } h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1}) \text{ and } S_{\tau_i} = s_2 \text{ and } \zeta_i^{mu} = 0. \end{cases}$$

Recall that if $B_{\tau_{i+1}} \leq h_2(\tau_{i+1})$ (respectively $B_{\tau_{i+1}} \geq h_1(\tau_{i+1})$), then assign $S_{\tau_{i+1}} = s_1$ (respectively $S_{\tau_{i+1}} = s_3$) in view of the argument described in Step 1 of Definition 2.1.
If $h_2(\tau_{i+1}) < B_{\tau_{i+1}} < h_1(\tau_{i+1})$ and, e.g., the system is in the lower level state s_1, then it switches to the upper or the medium level state according to the value of an independent Bernoulli random variable ζ_i^{lu}.
Step 6 Set $i = i + 1$ and go to Step 2.
Finally, define the stock price $(P_t, t \in [T_0, T))$ pursuant to (2.29). ■

2.2.3 Main properties of constant number of dynamic hedgers models

In Theorem 2.2, it will be shown that construction mechanisms in Definition 2.1 and Definition 2.2 determine the stock market price $(P_t, T_0 \leq t < T)$, that is, for all $t \in [T_0, T)$, there is some finite i such that $t \in [\tau_i, \tau_{i+1})(\mathbb{P}\text{-a.s.})$.

Theorem 2.2 In Definition 2.1 and Definition 2.2,
(i) for all $i \geq 0$, if $\tau_i < T$, then $\tau_i < \tau_{i+1}$ ($\mathbb{P}$-a.s.)
(ii) construction mechanisms stop after a finite number of iterations ($\mathbb{P}$-a.s.).
Proof The first part of this theorem holds true due to Theorem 2.1, construction of τ_i and the facts that B_t is continuous and that exponential random variable is positive ($\mathbb{P}$-a.s.). The second part will be proved by contradiction. Suppose there is an infinite number of τ_i on $[T_0, T)$ with a positive probability. Then one or both of the following scenarios must occur. According to the first

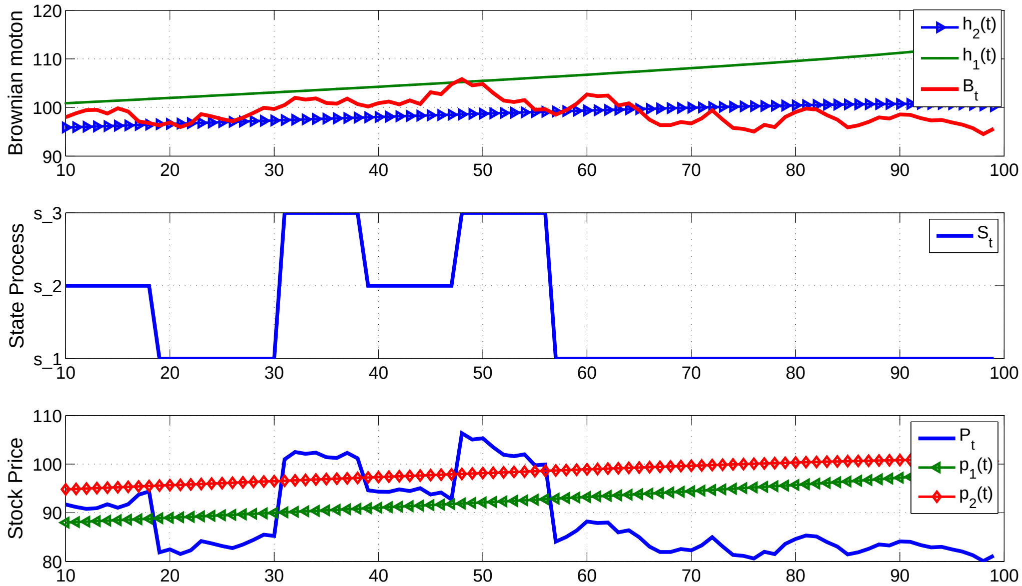

Fig. 2.3: Simulated stock price dynamics in the exogenous shocks model computed for some set of parameters: $T_0 = 10$, $T = 100$, $\alpha_1 = 0.3$, $\sigma_\kappa = 0.03$, $\kappa = 100$, $w^D = 30$, $\gamma_1 = 2$, $\gamma_2 = 1$, $\gamma_3 = 2$; initial value of S_t is assumed to be equal to s_2; shocks occur at times $t = 31$, $t = 39$, $t = 73$, $t = 78$ and $t = 95$; stock price jumps at $t = 19$, $t = 31$, $t = 39$, $t = 48$ and $t = 57$; state process jumps to s_3 and s_2 at times $t = 31$ and $t = 39$ according to the values of corresponding Bernoulli random variables.

scenario, there are infinitely many independent identically distributed exponential random variables such that their sum is less than $T - T_0$. According to the second scenario, for any $0 < \delta < T - T_0$, there exists an interval of length δ in $[T_0, T)$, and, in that interval, there are infinitely many points s such that $B_s \geq h_1(s)$ and infinitely many points s such that $B_s \leq h_2(s)$. If $(X_i, i = 1, 2, ...)$ is a sequence of independent exponential random variables with a rate parameter λ, then, for all $n \geq 0$, $\sum_{i=1}^{n} X_i$ is distributed according to Erlang distribution Erlang(n, λ) (see, e.g., Cox [16]). Because of this,

$$P(\sum_{i=1}^{\infty} X_i < T - T_0) \leq P(\sum_{i=1}^{n} X_i < T - T_0) = 1 - \sum_{i=0}^{n-1} \frac{(\lambda(T - T_0))^i}{i!} e^{-\lambda(T - T_0)} \to 0, \quad n \to \infty.$$

Therefore, the first scenario is impossible ($\mathbb{P}$-a.s.). The second scenario is impossible as well due to Theorem 2.1 and continuity of B_t ($\mathbb{P}$-a.s.). ■

Remark 2.7 Note that, according to the construction of the stock price process, for all $t \in [T_0, T)$, P_t can not be equal to $p_1(t)$ or $p_2(t)$ defined in accordance with (2.12), (2.13) and (2.21). Indeed, if it is assumed that P_t is equal to $p_1(t)$, then $B_t = h_1(t)$ and either $S_t = s_1$ or $S_t = s_2$, but it is known that, if $B_t \geq h_1(t)$, then $S_t = s_3$, which is the contradiction. The same argument applies to $p_2(t)$.

Remark 2.8 There is one-to-one correspondence between P_t and (B_t, S_t).
Indeed, in virtue of (2.26), Definition 2.1, Definition 2.2 and Remark 2.7, given P_t,

$$B_t = h(t, P_t) \quad \text{and} \quad S_t = \begin{cases} s_1 & \text{if } P_t < p_1(t) \\ s_2 & \text{if } p_1(t) < P_t < p_2(t) \\ s_3 & \text{if } P_t > p_2(t). \end{cases}$$

Conversely, if B_t and S_t are known, P_t can be determined according to (2.29).

Definition 2.3 Define a market crash as a point of discontinuity of $(P_t, 0 < t < T)$ such that $P_t < P_{t-}$ and a market boom as a point of discontinuity of $(P_t, 0 < t < T)$ such that $P_t > P_{t-}$, where $P_{t-} = \lim_{s \uparrow t} P_s$.

In virtue of Theorem 2.2 and Remark 2.4 applied to Definition 2.1 and Definition 2.2, there is no infinite price oscillation and $(\tau_i < T, i = 1, 2, ...)$ are the only jump points on $[T_0, T)$. I denote the value of the i-th jump by $J_i = \Delta P_{\tau_i} = P_{\tau_i} - P_{\tau_i -}$.

Definition 2.4 Define a big market crash (respectively a big market boom) as a transition of S_t from state s_3 (respectively s_1) to state s_1 (respectively s_3). A small market crash (respectively a small market boom) is a transition of S_t from state s_3 (respectively s_1) to state s_2 or from state s_2 to state s_1 (respectively s_3).

Note that Definition 2.1, Definition 2.2 and Remark 2.7 imply that

$$J_i = \begin{cases} J^u(\tau_i) = p^u(\tau_i, h_1(\tau_i)) - p_1(\tau_i) & \text{if } B_{\tau_i} = h_1(\tau_i) \\ J^l(\tau_i) = p^l(\tau_i, h_2(\tau_i)) - p_2(\tau_i) & \text{if } B_{\tau_i} = h_2(\tau_i) \\ J^{lu}(\tau_i, B_{\tau_i}) = p^u(\tau_i, B_{\tau_i}) - p^l(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_1 \text{ and } S_{\tau_{i+1}} = s_3 \\ J^{lm}(\tau_i, B_{\tau_i}) = p^m(\tau_i, B_{\tau_i}) - p^l(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_1 \text{ and } S_{\tau_{i+1}} = s_2 \\ J^{mu}(\tau_i, B_{\tau_i}) = p^u(\tau_i, B_{\tau_i}) - p^m(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_2 \text{ and } S_{\tau_{i+1}} = s_3 \\ J^{ml}(\tau_i, B_{\tau_i}) = p^l(\tau_i, B_{\tau_i}) - p^m(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_2 \text{ and } S_{\tau_{i+1}} = s_1 \\ J^{ul}(\tau_i, B_{\tau_i}) = p^l(\tau_i, B_{\tau_i}) - p^u(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_3 \text{ and } S_{\tau_{i+1}} = s_1 \\ J^{um}(\tau_i, B_{\tau_i}) = p^m(\tau_i, B_{\tau_i}) - p^u(\tau_i, B_{\tau_i}) & \text{if } h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i), S_{\tau_i} = s_3 \text{ and } S_{\tau_{i+1}} = s_2. \end{cases} \tag{2.33}$$

In view of (2.7) and (2.20), an increase in the number of dynamic hedgers w^D leads to an increase in the magnitude of booms and crashes. In Theorem 2.3, the uniform boundedness of jump sizes will be shown. This property will be applied in the proof of Theorem 2.6 that shows that the stock price process is a special semimartingale.

Theorem 2.3 Jump sizes $\mid \Delta P_{\tau_i} \mid$ of the stock price process are uniformly bounded by the ratio of the total number of dynamic hedgers w^D and γ_1:

$$\mid \Delta P_{\tau_i} \mid \leq \frac{w^D}{\gamma_1}.$$

Proof The pricing equation (2.26) and the continuity of Brownian motion yield that

$$h(\tau_i, P_{\tau_i}) = h(\tau_i, P_{\tau_i -}),$$

which means that

$$\gamma_1 \Delta P_{\tau_i} + w^D \int_{-\infty}^{\infty} \Big[\Phi\Big(\frac{K e^{-r(T-\tau_i)} - P_{\tau_i}}{\Sigma(\tau_i)} \Big) - \Phi\Big(\frac{K e^{-r(T-\tau_i)} - P_{\tau_i -}}{\Sigma(\tau_i)} \Big) \Big] \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK = 0.$$

As a consequence,

$$|\Delta P_{\tau_i}| \leq \frac{w^D}{\gamma_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK = \frac{w^D}{\gamma_1}$$

since the cumulative distribution function satisfies $0 \leq \Phi(x) \leq 1, \forall x \in \mathbb{R}$. ■

In Theorem 2.4, the càdlàg property of the stock price process will be proved.

Theorem 2.4 The stock price process P_t is càdlàg ($\mathbb{P}$-a.s.).

Proof By Theorem 2.2 and Step 3 in Definition 2.1 and Definition 2.2, process S_t is càdlàg ($\mathbb{P}$-a.s.). Recall that, in view of (2.29),

$$P_t = \begin{cases} p^l(t, B_t) & \text{if } S_t = s_1 \\ p^m(t, B_t) & \text{if } S_t = s_2 \\ p^u(t, B_t) & \text{if } S_t = s_3, \end{cases}$$

which means that P_t is càdlàg ($\mathbb{P}$-a.s.) as well due to Remark 2.4 and the implicit function theorem. ■

Let $\mathcal{F}_t^P$ be the natural filtration generated by the stock price process:

$$\mathcal{F}_t^P = \sigma\{P_s, T_0 \leq s \leq t\}. \tag{2.34}$$

I call this filtration the market filtration since this is the public information available to all market agents. In Theorem 2.5 and Theorem 2.6, it will be shown that the stock price jump times are $\mathcal{F}_t^P$-stopping times and the stock price dynamics on $[T_0, T)$ will be analysed.

Theorem 2.5 The sequence $(\tau_i < T, i = 1, 2, ...)$ is a sequence of $\mathcal{F}_t^P$-stopping times.

Proof By Theorem 2.4, the stock price process P_t is càdlàg ($\mathbb{P}$-a.s.). This process is adapted to its natural filtration, and the result follows from Proposition 1.32 in Jacod and Shiryaev [25], p.8. ■

Theorem 2.6 Stock price process is a special semimartingale such that

$$P_t = P_{T_0} + \int_{T_0}^t \theta_1(s, P_s)ds + \int_{T_0}^t \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i}, \quad \text{for } t \in [T_0, T), \tag{2.35}$$

where $N_t = \sum_{i \geq 1} \mathbb{I}(\tau_i \leq t)$ is the total number of jumps on $[T_0, t]$,

$$\theta_1(s, P_s) = -\frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s, P_s)})^2}{h_x(s, P_s)} \tag{2.36}$$

and

$$\theta_2(s, P_s) = \frac{1}{h_x(s, P_s)}. \tag{2.37}$$

Proof Consider the decomposition

$$P_t - P_{T_0} = P_t - P_{\tau_{N_t}} + \sum_{i=1}^{N_t}(P_{\tau_i -} - P_{\tau_{i-1}}) + \sum_{i=1}^{N_t} \Delta P_{\tau_i}. \tag{2.38}$$

According to Remark 2.4, the implicit function theorem and Theorem 32 (p.78) in Protter [38],

$$P_t - P_{\tau_{N_t}} = \int_{\tau_{N_t}}^{t} \theta_1^{(N_t)}(s, P_s)ds + \int_{\tau_{N_t}}^{t} \theta_2^{(N_t)}(s, P_s)dB_s,$$

for some functions $\theta_1^{(N_t)}$ and $\theta_2^{(N_t)}$. Applying Ito's lemma to the pricing equation (2.26), it can be shown that

$$h_t(t, P_t)dt + h_x(t, P_t)\theta_1^{(N_t)}(t, P_t)dt + h_x(t, P_t)\theta_2^{(N_t)}(t, P_t)dB_t + \frac{1}{2}h_{xx}(t, P_t)(\theta_2^{(N_t)}(t, P_t))^2 dt = dB_t.$$

As a consequence,

$$\theta_2^{(N_t)}(s, P_s) = \frac{1}{h_x(s, P_s)}, \quad \theta_1^{(N_t)}(s, P_s) = -\frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)},$$

and

$$P_t - P_{\tau_{N_t}} = -\int_{\tau_{N_t}}^{t} \frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)}ds + \int_{\tau_{N_t}}^{t} \frac{1}{h_x(s, P_s)}dB_s. \tag{2.39}$$

Similarly

$$P_{\tau_i -} - P_{\tau_{i-1}} = -\int_{\tau_{i-1}}^{\tau_i -} \frac{h_s(s, P_s) + \frac{1}{2}h_{xx}(s, P_s)(\frac{1}{h_x(s,P_s)})^2}{h_x(s, P_s)}ds + \int_{\tau_{i-1}}^{\tau_i -} \frac{1}{h_x(s, P_s)}dB_s, \quad i = 1, 2, ..., N_t. \tag{2.40}$$

In view of formulas (2.38) – (2.40), it can be concluded that formulas (2.35) – (2.37) hold.
Define processes $(P_t^{(k)}, k = 1, 2, ...)$ by

$$P_t^{(k)} = P_{T_0} + \int_{T_0}^{t\wedge\tau_k} \theta_1(s, P_s)ds + \int_{T_0}^{t\wedge\tau_k} \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t\wedge k} \Delta P_{\tau_i}.$$

By Remark 2.4, Theorem 32 (p.78) in Protter [38] and induction,

$$P_{T_0} + \int_{T_0}^{t\wedge\tau_k} \theta_1(s, P_s)ds + \int_{T_0}^{t\wedge\tau_k} \theta_2(s, P_s)dB_s$$

is a semimartingale. By Theorem 2.3, jumps of the stock price process are bounded, hence, processes $P_t^{(k)}$ are semimartingales as well. By Proposition 1.4.25c in Jacod and Shiryaev [25], p.44, and Theorem 2.2, the stock price process is a semimartingale. Proposition 1.4.24 in Jacod and Shiryaev [25], p.44, and Theorem 2.3 imply it is a special semimartingale, and the result follows. ■

In Theorem 2.9 and Corollary 2.7, the canonical decomposition of the special semimartingale process $(P_t, T_0 \le t < T)$ will be obtained.

2.2.4 Conditional distributions in the endogenous switching model

In this section, conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump in the endogenous switching model will be found, given that the stock price dynamics on $[T_0, t]$, $t \in [T_0, T)$, is observed. In Theorem 2.7, their joint conditional distribution, given $\mathcal{F}_t^P$, is computed. Based on this theorem, marginal conditional distributions can be found.

Theorem 2.7

Assume that $T_0 \leq t < u \leq T$, C_1 is any combination of elements in $\mathbb{S}$ and $C_2 \in \mathbb{B}(\mathbb{R})$. In the endogenous switching model, the joint conditional distribution for the time of the next jump, the type of the next jump and the size of the next jump, given the information $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \mid \mathcal{F}_t^P) = \begin{cases} F_1(t, \tau_{N_t}, R_t^l, B_t, u, C_1, C_2) & \text{if } S_t = s_1 \\ F_2(t, B_t, u, C_1, C_2) & \text{if } S_t = s_2 \\ F_3(t, \tau_{N_t}, R_t^u, B_t, u, C_1, C_2) & \text{if } S_t = s_3, \end{cases} \tag{2.41}$$

where expressions for F_1, F_2 and F_3 are given in the proof of this theorem in the Appendix.

Proof The proof is provided in the Appendix. ■

Distribution of the time of the next jump

Taking $C_1 = \mathbb{S}$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.7, the conditional cumulative distribution function of the time of the next jump, given the market filtration $\mathcal{F}_t^P$, can be obtained.

Corollary 2.1

Suppose that $T_0 \leq t < u \leq T$. Then the conditional cumulative distribution function of the time of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < u \mid \mathcal{F}_t^P) = \begin{cases} 1 - \int_{R_t^l}^{\infty} D^l(u, \tau_{N_t} + x, t, B_t)\lambda_l e^{-\lambda_l(x - R_t^l)}dx & \text{if } S_t = s_1 \\ 1 - D_m(u, t, B_t) & \text{if } S_t = s_2 \\ 1 - \int_{R_t^u}^{\infty} D^u(u, \tau_{N_t} + x, t, B_t)\lambda_u e^{-\lambda_u(x - R_t^u)}dx & \text{if } S_t = s_3 \end{cases}$$

$$= \begin{cases} 1 - \int_{0}^{\infty} D^l(u, \tau_{N_t} + R_t^l + x, t, B_t)\lambda_l e^{-\lambda_l x}dx & \text{if } S_t = s_1 \\ 1 - D_m(u, t, B_t) & \text{if } S_t = s_2 \\ 1 - \int_{0}^{\infty} D^u(u, \tau_{N_t} + R_t^u + x, t, B_t)\lambda_u e^{-\lambda_u x}dx & \text{if } S_t = s_3, \end{cases}$$

where D^l, D^u, R_t^l and R_t^u are defined in the proof of Theorem 2.7 in the Appendix and D_m is defined in (2.46).

Distribution of the next state of the state process

Let $t \in [T_0, T)$. Taking $u = T$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.7, the conditional cumulative distribution function of the next state of the state process, given $\mathcal{F}_t^P$, can be computed. On the set $[P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a small boom given $\mathcal{F}_t^P$ is equal to

$$F_4(t, \tau_{N_t}, R_t^l, B_t) = p_{lm}\Big[\int_{R_t^l}^{t-\tau_{N_t}} \Big(1 - D^l(T, \tau_{N_t} + x, t, B_t)\Big)\lambda_l e^{-\lambda_l(x-R_t^l)}dx + \int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big(D_1(\tau_{N_t} + x, t, B_t) - D^l(T, \tau_{N_t} + x, t, B_t)\Big)\lambda_l e^{-\lambda_l(x-R_t^l)}dx\Big],$$

while the conditional probability that there will be at least one more jump and the first jump will be a big boom given $\mathcal{F}_t^P$ is equal to

$$F_5(t, \tau_{N_t}, R_t^l, B_t) = p_{lu}\Big[\int_{R_t^l}^{t-\tau_{N_t}} \Big(1 - D^l(T, \tau_{N_t} + x, t, B_t)\Big)\lambda_l e^{-\lambda_l(x-R_t^l)}dx + \int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big(D_1(\tau_{N_t} + x, t, B_t) - D^l(T, \tau_{N_t} + x, t, B_t)\Big)\lambda_l e^{-\lambda_l(x-R_t^l)}dx\Big] + \int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big[1 - D_1(\tau_{N_t} + x, t, B_t)\Big]\lambda_l e^{-\lambda_l(x-R_t^l)}dx + e^{-\lambda_l(T-\tau_{N_t}-R_t^l)}(1 - D_1(T, t, B_t)),$$

where D^l, D_1 and R_t^l are defined in the proof of Theorem 2.7 in the Appendix.

On the set $[p_1(t) < P_t < p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a market boom given $\mathcal{F}_t^P$ is equal to $D_{m,1}(T, t, B_t)$, while the probability that there will be at least one more jump and the first jump will be a market crash is equal to $D_{m,2}(T,t,B_t)$, where $D_{m,1}(T,t,B_t)$ and $D_{m,2}(T,t,B_t)$ are defined in the proof of Theorem 2.7 in the Appendix.

On the set $[P_t > p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a small crash given $\mathcal{F}_t^P$ is equal to

$$\begin{aligned} F_6(t,\tau_{N_t},R_t^u,B_t) = p_{um}\Big[&\int_{R_t^u}^{t-\tau_{N_t}} \Big(1-D^u(T,\tau_{N_t}+x,t,B_t)\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx \\ &+\int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big(D_2(\tau_{N_t}+x,t,B_t)-D^u(T,\tau_{N_t}+x,t,B_t)\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx\Big], \end{aligned}$$

while the conditional probability that there will be at least one more jump and the first jump will be a big crash is equal to

$$\begin{aligned} &F_7(t,\tau_{N_t},R_t^u,B_t) \\ &= p_{ul}\Big[\int_{R_t^u}^{t-\tau_{N_t}} \Big(1-D^u(T,\tau_{N_t}+x,t,B_t)\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx \\ &\qquad +\int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big(D_2(\tau_{N_t}+x,t,B_t)-D^u(T,\tau_{N_t}+x,t,B_t)\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx\Big] \\ &+\int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big[1-D_2(\tau_{N_t}+x,t,B_t)\Big]\lambda_u e^{-\lambda_u(x-R_t^u)}dx + e^{-\lambda_u(T-\tau_{N_t}-R_t^u)}(1-D_1(T,t,B_t)), \end{aligned}$$

where D^u, D_2 and R_t^u are defined in the proof of Theorem 2.7 in the Appendix.

Combining these formulas all together, Corollary 2.2 can be obtained.

Corollary 2.2 Suppose that $T_0 \le t < T$. Then the conditional cumulative distribution function of the next state of the state process, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\begin{cases} \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_2 \mid \mathcal{F}_t^P) = F_4(t,\tau_{N_t},R_t^l,B_t) & \text{if } S_t = s_1 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_3 \mid \mathcal{F}_t^P) = F_5(t,\tau_{N_t},R_t^l,B_t) & \text{if } S_t = s_1 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_3 \mid \mathcal{F}_t^P) = D_{m,1}(T,t,B_t) & \text{if } S_t = s_2 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_1 \mid \mathcal{F}_t^P) = D_{m,2}(T,t,B_t) & \text{if } S_t = s_2 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_2 \mid \mathcal{F}_t^P) = F_6(t,\tau_{N_t},R_t^u,B_t) & \text{if } S_t = s_3 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_1 \mid \mathcal{F}_t^P) = F_7(t,\tau_{N_t},R_t^u,B_t) & \text{if } S_t = s_3. \end{cases}$$

Distribution of the size of the next jump

Let $t \in [T_0, T)$ and $C \in \mathcal{B}(\mathbb{R})$. Taking $u = T$ and $C_1 = \mathbb{S}$ in the formulas in Theorem 2.7, the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, can be computed.

On the set $[P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ is equal to

$$
\begin{aligned}
&F_8(t, \tau_{N_t}, R_t^l, B_t, C) \\
&= e^{-\lambda_l(T-\tau_{N_t}-R_t^l)} \int_t^T \mathbb{I}(J^u(y) \in C)\phi_1(y, t, B_t)dy \\
&+ \int_{t-\tau_{N_t}}^{T-\tau_{N_t}} \Big(\int_t^{\tau_{N_t}+x} \mathbb{I}(J^u(y) \in C)\phi_1(y, t, B_t)dy + \int_{\tau_{N_t}+x}^{T} (p_{lu}\mathbb{I}(J^{lu}(y, h^l(y; \tau_{N_t}+x)) \in C) \\
&\qquad + p_{lm}\mathbb{I}(J^{lm}(y, h^l(y; \tau_{N_t}+x)) \in C))\phi^l(y, \tau_{N_t}+x, t, B_t)dy \Big) \lambda_l e^{-\lambda_l(x-R_t^l)}dx \\
&+ \int_{R_t^l}^{t-\tau_{N_t}} \Big(\int_t^T \Big(p_{lu}\mathbb{I}(J^{lu}(y, h^l(y; \tau_{N_t}+x)) \in C) \\
&\qquad\qquad + p_{lm}\mathbb{I}(J^{lm}(y, h^l(y; \tau_{N_t}+x)) \in C) \Big) \phi^l(y, \tau_{N_t}+x, t, B_t)dy \Big) \lambda_l e^{-\lambda_l(x-R_t^l)})dx,
\end{aligned}
$$

where ϕ_1, ϕ^l and R_t^l are defined in the proof of Theorem 2.7 in the Appendix and J^u, J^{lu} and J^{lm} are defined in (2.33).

On the set $[p_2(t) < P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ is equal to

$$
F_9(t, B_t, C) = \int_t^T \Big[\mathbb{I}(J^u(y) \in C)\phi_{m,1}(y, t, B_t)dy + \mathbb{I}(J^l(y) \in C)\phi_{m,2}(y, t, B_t) \Big] dy,
$$

where $\phi_{m,1}$ and $\phi_{m,2}$ are defined in the proof of Theorem 2.7 in the Appendix and J^u and J^l are defined in (2.33).

On the set $[P_t > p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ is equal to

$$\begin{aligned}
&F_{10}(t,\tau_{N_t},R_t^u,B_t,C)\\
&= e^{-\lambda_u(T-\tau_{N_t}-R_t^u)}\int_t^T \mathbb{I}(J^l(y)\in C)\phi_2(y,t,B_t)dy\\
&\quad+\int_{t-\tau_{N_t}}^{T-\tau_{N_t}}\Big(\int_t^{\tau_{N_t}+x}\mathbb{I}(J^l(y)\in C)\phi_2(y,t,B_t)dy+\int_{\tau_{N_t}+x}^T(p_{ul}\mathbb{I}(J^{ul}(y,h^u(y;\tau_{N_t}+x))\in C)\\
&\qquad+p_{um}\mathbb{I}(J^{um}(y,h^u(y;\tau_{N_t}+x))\in C))\phi^u(y,\tau_{N_t}+x,t,B_t)dy\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx\\
&\quad+\int_{R_t^u}^{t-\tau_{N_t}}\Big(\int_t^T\Big(p_{ul}\mathbb{I}(J^{ul}(y,h^u(y;\tau_{N_t}+x))\in C)\\
&\qquad\qquad+p_{um}\mathbb{I}(J^{um}(y,h^u(y;\tau_{N_t}+x))\in C)\Big)\phi^u(y,\tau_{N_t}+x,t,B_t)dy\Big)\lambda_u e^{-\lambda_u(x-R_t^u)})dx,
\end{aligned}$$

where ϕ_2, ϕ^u and R_t^u are defined in the proof of Theorem 2.7 in the Appendix and J^l, J^{ul} and J^{um} are defined in (2.33).

Combining these formulas all together, Corollary 2.3 can be obtained.

Corollary 2.3 Suppose that $T_0 \le t < T$ and $C \in \mathcal{B}(\mathbb{R})$. Then the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1}<T, J_{\tau_{N_t+1}}\in C \mid \mathcal{F}_t^P)=\begin{cases} F_8(t,\tau_{N_t},R_t^l,B_t,C) & \text{if } S_t=s_1\\ F_9(t,B_t,C) & \text{if } S_t=s_2\\ F_{10}(t,\tau_{N_t},R_t^u,B_t,C) & \text{if } S_t=s_3.\end{cases}$$

2.2.5 Conditional distributions in the exogenous shocks model

In this section, conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump in the exogenous shocks model will be obtained, given the information about the stock price dynamics on $[T_0,t]$, $t\in[T_0,T)$. In Theorem 2.8, their joint conditional distribution, given $\mathcal{F}_t^P$, is computed. Based on this theorem, marginal conditional distributions can be derived.

Theorem 2.8

Assume that $T_0 \le t < u \le T$, C_1 is any combination of elements in $\mathbb{S}$ and $C_2 \in \mathbb{B}(\mathbb{R})$. In the exogenous shocks model, the joint conditional distribution for the time of the next jump, the type of the next jump and the size of the next jump given the information $\mathcal{F}_t^P$ is equal to

$$\mathbb{P}(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \mid \mathcal{F}_t^P) = \begin{cases} F_{11}(t, B_t, u, C_1, C_2) & \text{if } S_t = s_1 \\ F_{12}(t, B_t, u, C_1, C_2) & \text{if } S_t = s_2 \\ F_{13}(t, B_t, u, C_1, C_2) & \text{if } S_t = s_3, \end{cases}$$

where expressions for F_{11}, F_{12} and F_{13} are given in the proof of this theorem in the Appendix.

Proof The proof is provided in the Appendix. ■

Distribution of the time of the next jump

Taking $C_1 = \{s_1, s_2, s_3\}$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.8, conditional distribution for the time of the next jump, given the market filtration $\mathcal{F}_t^P$, can be computed.

Corollary 2.4 Suppose that $T_0 \leq t \leq u \leq T$. Then conditional distribution for the time of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < u \mid \mathcal{F}_t^P) = \begin{cases} F_{14}(t, B_t, u) & \text{if } S_t = s_1 \\ F_{15}(t, B_t, u) & \text{if } S_t = s_2 \\ F_{16}(t, B_t, u) & \text{if } S_t = s_3, \end{cases}$$

where F_{14}, F_{15} and F_{16} satisfy

$$F_{14}(t, B_t, u) = e^{-\lambda_Z(u-t)}\left(1 - D_1(u, t, B_t)\right) + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[(1 - D_1(t+r, t, B_t)) + \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{14}(t+r, x, u)dx + \Phi_1(t+r, t, B_t)\Big]dr,$$

$$F_{15}(t, B_t, u) = 1 - e^{-\lambda_Z(u-t)} D_m(u, t, B_t),$$

$$F_{16}(t, B_t, u) = e^{-\lambda_Z(u-t)}\left(1 - D_2(u, t, B_t)\right) + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[(1 - D_2(t+r, t, B_t)) + \int_{h_1(t+r)}^{\infty} q_2(x; r, t, B_t) F_{16}(t+r, x, u)dx + \Phi_2(t+r, t, B_t)\Big]dr,$$

and D_1 and D_2 are defined in the proof of Theorem 2.7 in the Appendix, D_m is defined in (2.46), q_1 and q_2 are defined in the proof of Theorem 2.8 and Φ_1 and Φ_2 are defined in (2.47) and (2.48).

Let $t \in [T_0, T)$. Taking $u = T$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.8, the conditional cumulative distribution function of the next state of the state process in the exogenous shocks model, given the market filtration $\mathcal{F}_t^P$, can be computed.

On the set $[P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a small boom given $\mathcal{F}_t^P$ satisfies

$$F_{17}(t, B_t) = \int_0^{T-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{17}(t+r, x) dx + p_{lm} \Phi_1(t+r, t, B_t) \Big] dr,$$

while the conditional probability that there will be at least one more jump and the first jump will be a big boom given $\mathcal{F}_t^P$ satisfies

$$\begin{aligned} F_{18}(t, B_t) = e^{-\lambda_Z (T-t)} \Big(1 - D_1(T, t, B_t) \Big) + \int_0^{T-t} \lambda_Z e^{-\lambda_Z r} \Big[\Big(1 - D_1(t+r, t, B_t) \Big) \\ + \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{18}(t+r, x) dx + p_{lu} \Phi_1(t+r, t, B_t) \Big] dr, \end{aligned}$$

where D_1 is defined in the proof of Theorem 2.7 in the Appendix, q_1 is defined in the proof of Theorem 2.8 in the Appendix and Φ_1 is defined in (2.47).

On the set $[p_1(t) < P_t < p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a market boom given $\mathcal{F}_t^P$ is equal to

$$F_{19}(t, B_t) = e^{-\lambda_Z (T-t)} D_{m,1}(T, t, B_t) + \int_0^{T-t} \lambda_Z e^{-\lambda_Z r} \Big[D_{m,1}(t+r, t, B_t) + p_{mu} D_m(t+r, t, B_t) \Big] dr,$$

while the probability that there will be at least one more jump and the first jump will be a market crash is equal to

$$F_{20}(t, B_t) = e^{-\lambda_Z (T-t)} D_{m,2}(T, t, B_t) + \int_0^{T-t} \lambda_Z e^{-\lambda_Z r} \Big[D_{m,2}(t+r, t, B_t) + p_{ml} D_m(t+r, t, B_t) \Big] dr,$$

where $D_{m,1}$ and $D_{m,2}$ are defined in the proof of Theorem 2.7 in the Appendix and D_m is defined in (2.46).

On the set $[P_t > p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump will be a small crash given $\mathcal{F}_t^P$ satisfies

$$F_{21}(t, B_t) = \int_0^{T-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_{h_1(t+r)}^{\infty} q_2(x; r, t, B_t) F_{21}(t+r, x) dx + p_{um} \Phi_2(t+r, t, B_t) \Big] dr,$$

while the conditional probability that there will be at least one more jump and the first jump will be a big crash given $\mathcal{F}_t^P$ satisfies

$$F_{22}(t,B_t)=e^{-\lambda_Z(T-t)}\Big(1-D_2(T,t,B_t)\Big)+\int_0^{T-t}\lambda_Z e^{-\lambda_Z r}\Big[\Big(1-D_2(t+r,t,B_t)\Big)\\+\int_{h_1(t+r)}^{\infty}q_2(x;r,t,B_t)F_{22}(t+r,x)dx+p_{ul}\Phi_2(t+r,t,B_t)\Big]dr,$$

where D_2 is defined in the proof of Theorem 2.7 in the Appendix, q_2 is defined in the proof of Theorem 2.8 in the Appendix and Φ_2 is defined in (2.48).

Combining these formulas all together, Corollary 2.5 can be obtained.

Corollary 2.5 Suppose that $T_0 \le t < T$. Then the conditional cumulative distribution function of the next state of the state process, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\begin{cases}\mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_2\mid\mathcal{F}_t^P)=F_{17}(t,B_t) & \text{if } S_t=s_1\\ \mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_3\mid\mathcal{F}_t^P)=F_{18}(t,B_t) & \text{if } S_t=s_1\\ \mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_3\mid\mathcal{F}_t^P)=F_{19}(t,B_t) & \text{if } S_t=s_2\\ \mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_1\mid\mathcal{F}_t^P)=F_{20}(t,B_t) & \text{if } S_t=s_2\\ \mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_2\mid\mathcal{F}_t^P)=F_{21}(t,B_t) & \text{if } S_t=s_3\\ \mathbb{P}(\tau_{N_t+1}<T,S_{\tau_{N_t+1}}=s_1\mid\mathcal{F}_t^P)=F_{22}(t,B_t) & \text{if } S_t=s_3.\end{cases}$$

Distribution of the size of the next jump

Let $t \in [T_0, T)$ and $C \in \mathcal{B}(\mathbb{R})$. Taking $u = T$ and $C_1 = \mathbb{S}$ in the formulas in Theorem 2.8, the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, can be computed.

On the set $[P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ satisfies

$$\begin{aligned}
&F_{23}(t, B_t, C) \\
&= e^{-\lambda_Z(T-t)} \int_t^T \mathbb{I}(J^u(y) \in C)\phi_1(y, t, B_t)dy \\
&+ \int_0^{T-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(J^u(y) \in C)\phi_1(y, t, B_t)dy + \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t)F_{23}(t+r, x, C)dx \\
&+ \int_{h_2(t+r)}^{h_1(t+r)} q_1(x; r, t, B_t)\Big(p_{lu}\mathbb{I}(J^{lu}(t+r, x) \in C) + p_{lm}\mathbb{I}(J^{lm}(t+r, x) \in C)\Big)dx\Big]dr,
\end{aligned}$$

where J^u, J^{lu} and J^{lm} are defined in (2.33), ϕ_1 is defined in the proof of Theorem 2.7 in the Appendix, and q_1 is defined in the proof of Theorem 2.8 in the Appendix.

On the set $[p_2(t) < P_t < p_1(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ is equal to

$$\begin{aligned}
&F_{24}(t, B_t, C) \\
&= e^{-\lambda_Z(T-t)} \int_t^T \Big[\mathbb{I}(J^u(y) \in C)\phi_{m,1}(y, t, B_t) + \mathbb{I}(J^l(y) \in C)\phi_{m,2}(y, t, B_t)\Big]dy \\
&+ \int_0^{T-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \Big[\mathbb{I}(J^u(y) \in C)\phi_{m,1}(y, t, B_t) + \mathbb{I}(J^l(y) \in C)\phi_{m,2}(y, t, B_t)\Big]dy \\
&\quad + \int_{h_2(t+r)}^{h_1(t+r)} q^m(x; r, t, B_t)\Big(p_{mu}\mathbb{I}(J^{mu}(t+r, x) \in C) + p_{ml}\mathbb{I}(J^{ml}(t+r, x) \in C)\Big)dx\Big]dr,
\end{aligned}$$

where J^u, J^l, J^{mu} and J^{ml} are defined in (2.33), $\phi_{m,1}$ and $\phi_{m,2}$ are defined in the proof of Theorem 2.7, and q_m is defined in the proof of Theorem 2.8 in the Appendix.

On the set $[P_t > p_2(t)]$ the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ satisfies

$$\begin{aligned}
&F_{25}(t, B_t, C) \\
&= e^{-\lambda_Z(T-t)} \int_t^T \mathbb{I}(J^l(y) \in C)\phi_2(y, t, B_t)dy \\
&+ \int_0^{T-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(J^l(y) \in C)\phi_2(y, t, B_t)dy + \int_{h_1(t+r)}^{\infty} q_2(x; r, t, B_t)F_{25}(t+r, x, C)dx \\
&+ \int_{h_2(t+r)}^{h_1(t+r)} q_2(x; r, t, B_t)\Big(p_{lu}\mathbb{I}(J^{ul}(t+r, x) \in C) + p_{lm}\mathbb{I}(J^{um}(t+r, x) \in C)\Big)dx\Big]dr,
\end{aligned}$$

where J^l, J^{ul} and J^{um} are defined in (2.33), ϕ_2 is defined in the proof of Theorem 2.7 in the Appendix, and q_2 is defined in the proof of Theorem 2.8 in the Appendix.

Combining these formulas all together, Corollary 2.6 can be obtained.

Corollary 2.6 Suppose that $T_0 \le t < T$ and $C \in \mathcal{B}(\mathbb{R})$. Then the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < T, J_{\tau_{N_t+1}} \in C \mid \mathcal{F}_t^P) = \begin{cases} F_{23}(t, B_t, C) & \text{if } S_t = s_1 \\ F_{24}(t, B_t, C) & \text{if } S_t = s_2 \\ F_{25}(t, B_t, C) & \text{if } S_t = s_3. \end{cases}$$

2.2.6 Canonical decomposition of the stock price process

In Theorem 2.6, it has been shown that, for both models, the stock price process is a special semimartingale. In this section, its canonical decomposition, that is, a decomposition to a local martingale and a predictable finite variation process starting at zero, will be computed.

Canonical decomposition in the endogenous switching model

Theorem 2.9 describes the canonical decomposition of the stock price process in the endogenous switching model. Lemma 2.1 and Lemma 2.2 will be used in the proof of Theorem 2.9.
Let

$$J_0 = 0 \quad \text{and} \quad Z_i^P = (P_{\tau_i}, J_i), i = 0, 1, ..., \tag{2.42}$$

then in view of Theorem 2.2 a double sequence (τ_i, Z_i^P) is a marked point process.
Denote by

$$\mathcal{F}_{\tau_i}^{Z^P} = \sigma\{(\tau_j, Z_j^P), 0 \le j \le i\} \tag{2.43}$$

and

$$g^{(i+1)}(u, C) = \frac{\partial \mathbb{P}(\tau_{i+1} \le u, Z_{i+1}^P \in C \mid \mathcal{F}_{\tau_i}^{Z^P})}{\partial u}, \quad u \in [\tau_i, T), \tag{2.44}$$

where $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$.

Lemma 2.1 In the endogenous switching model, suppose that $u \in [\tau_i, T)$ for some $i \ge 0$,

$C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then conditional distribution for the marked point process (τ_i, Z_i^P) given $\mathcal{F}_{\tau_i}^{Z^P}$ is equal to

$$\mathbb{P}(\tau_{i+1} \leq u, Z_{i+1}^P \in C \mid \mathcal{F}_{\tau_i}^{Z^P}) = \begin{cases} F_{26}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_1 \\ F_{27}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_2 \\ F_{28}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where F_{26}, F_{27} and F_{28} are defined in the proof of this lemma in the Appendix.

Proof The proof is provided in the Appendix. ■

Lemma 2.2

In the endogenous switching model, assume that $u \in [\tau_i, T)$ for some $i \geq 0$, $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then the function $g^{(i+1)}(u, C)$ satisfies

$$g^{(i+1)}(u, C) = \begin{cases} F_{29}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_1 \\ F_{30}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_2 \\ F_{31}(\tau_i, B_{\tau_i}, u, C) & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where F_{29}, F_{30} and F_{31} are defined in the proof of this lemma in the Appendix. In particular, for $E = \mathbb{R}^2$,

$$g^{(i+1)}(u, E) = \begin{cases} e^{-\lambda_l (u-\tau_i)} \phi_1(u, \tau_i, B_{\tau_i}) + \int_0^{u-\tau_i} \phi^l(u, \tau_i + x, \tau_i, B_{\tau_i}) \lambda_l e^{-\lambda_l x} dx & \text{if } S_{\tau_i} = s_1 \\ \phi_m(u, \tau_i, B_{\tau_i}) & \text{if } S_{\tau_i} = s_2 \\ e^{-\lambda_u (u-\tau_i)} \phi_2(u, \tau_i, B_{\tau_i}) + \int_0^{u-\tau_i} \phi^u(u, \tau_i + x, \tau_i, B_{\tau_i}) \lambda_u e^{-\lambda_u x} dx & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where

$$\phi_m(u, t, y) = -\frac{\partial D_m(u, t, y)}{\partial u} \tag{2.45}$$

and

$$D_m(u, t, y) = \mathbb{P}\Big(h_2(t+s) - y < B_s < h_1(t+s) - y, \forall s \in [0, u-t]\Big) \tag{2.46}$$

are Brownian motion hitting density and probability of a two-sided curved boundary.

Proof The proof is provided in the Appendix. ■

Theorem 2.9 Let $t \in [T_0, T)$. The canonical decomposition of $(P_t, T_0 \le t < T)$ in the endogenous switching model is given by

$$P_t = P_{T_0} + M_t + A_t, \quad M_{T_0} = 0, \quad A_{T_0} = 0,$$

where

$$M_t = \int_{T_0}^t \theta_2(s, P_s) dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i} - \int_{T_0}^t \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}) ds$$

is a local martingale,

$$A_t = \int_{T_0}^t \theta_1(s, P_s) ds + \int_{T_0}^t \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}) ds$$

is a predictable process with finite variation, $\theta_1(s, P_s)$ and $\theta_2(s, P_s)$ are defined in (2.36) and (2.37),

$$\theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}) = \begin{cases} F_{32}(s, \tau_{N_s}, B_{\tau_{N_s}}) & \text{if } S_{\tau_{N_s}} = s_1 \\ F_{33}(s, \tau_{N_s}, B_{\tau_{N_s}}) & \text{if } S_{\tau_{N_s}} = s_2 \\ F_{34}(s, \tau_{N_s}, B_{\tau_{N_s}}) & \text{if } S_{\tau_{N_s}} = s_3, \end{cases}$$

with

$$F_{32}(s, \tau_{N_s}, B_{\tau_{N_s}}) = \frac{1}{\int_0^\infty D^l(s, \tau_{N_s} + x, \tau_{N_s}, B_{\tau_{N_s}}) \lambda_l e^{-\lambda_l x} dx} \Big[J^u(s) e^{-\lambda_l (s - \tau_{N_s})} \phi_1(s, \tau_{N_s}, B_{\tau_{N_s}})$$
$$+ \int_0^{s-\tau_{N_s}} \Big(p_{lu} J^{lu}(s, h^l(s; \tau_{N_s} + x)) + p_{lm} J^{lm}(s, h^l(s; \tau_{N_s} + x)) \Big) \times$$
$$\times \phi^l(s, \tau_{N_s} + x, \tau_{N_s}, B_{\tau_{N_s}}) \lambda_l e^{-\lambda_l x} dx \Big],$$

$$F_{33}(s, \tau_{N_s}, B_{\tau_{N_s}}) = \frac{1}{D_m(s, \tau_{N_s}, B_{\tau_{N_s}})} \Big[J^u(s) \phi_{m,1}(s, \tau_{N_s}, B_{\tau_{N_s}}) + J^l(s) \phi_{m,2}(s, \tau_{N_s}, B_{\tau_{N_s}}) \Big],$$

$$F_{34}(s, \tau_{N_s}, B_{\tau_{N_s}}) = \frac{1}{\int_0^\infty D^u(s, x, \tau_{N_s}, B_{\tau_{N_s}}) \lambda_u e^{-\lambda_u x} dx} \Big[J^l(s) e^{-\lambda_u (s - \tau_{N_s})} \phi_2(s, \tau_{N_s}, B_{\tau_{N_s}})$$
$$+ \int_0^{s-\tau_{N_s}} \Big(p_{ul} J^{ul}(s, h^u(s; \tau_{N_s} + x)) + p_{um} J^{um}(s, h^u(s; \tau_{N_s} + x)) \Big) \times$$
$$\times \phi^u(s, \tau_{N_s} + x, \tau_{N_s}, B_{\tau_{N_s}}) \lambda_u e^{-\lambda_u x} dx \Big],$$

J^u, J^{lu}, J^{lm}, J^l, J^{ul} and J^{um} are defined in (2.33), D^l, ϕ_1, ϕ^l, $\phi_{m,1}$, $\phi_{m,2}$, D^u, ϕ_2 and ϕ^u are defined in the proof of Theorem 2.7 in the Appendix, and D_m is defined in (2.46).

Proof Applying Theorem T7 from Bremaud [8], p.238, to the counting process $N_t^Z(C)$ defined by

$$N_t^Z(C) = \sum_{i \ge 1} \mathbb{I}(Z_i^P \in C) \mathbb{I}(\tau_i \le t),$$

it can be concluded that the process $\int_{T_0}^t l_s(C)ds$ such that

$$l_s(C) = \frac{g^{(i+1)}(s,C)}{1 - \int_0^{s-\tau_i} g^{(i+1)}(\tau_i + y, E)dy} \quad \text{for } s \in [\tau_i, \tau_{i+1}), \quad i = 0, 1, ...,$$

is the compensator of $N_t^Z(C)$.

In view of Lemma 2.2,

$$l_s(C) = \begin{cases} \frac{F_{29}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{29}(\tau_i, B_{\tau_i}, \tau_i + y, E)dy} & \text{if } S_{\tau_i} = s_1 \\ \frac{F_{30}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{30}(\tau_i, B_{\tau_i}, \tau_i + y, E)dy} & \text{if } S_{\tau_i} = s_2 \\ \frac{F_{31}(\tau_i, B_{\tau_i}, s, C)}{1 - \int_0^{s-\tau_i} F_{31}(\tau_i, B_{\tau_i}, \tau_i + y, E)dy} & \text{if } S_{\tau_i} = s_3. \end{cases}$$

In virtue of the results of Theorem 2.6 and Chapter 8 in Bremaud [8], it can be shown that

$$M_t = \int_{T_0}^t \theta_2(s, P_s)dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i} - \int_{T_0}^t \int_E z_2 l_s(dz)ds$$

and

$$A_t = \int_{T_0}^t \theta_1(s, P_s)ds + \int_{T_0}^t \int_E z_2 l_s(dz)ds,$$

where $E = \mathbb{R}^2$ and $z = (z_1, z_2)$, and the result follows since

$$\int_E z_2 l_s(dz) = \theta_3(s, \tau_{N_s}, B_{\tau_{N_s}}).$$

■

Canonical decomposition in the exogenous shocks model

Define J_0, $(Z_i^P, i = 0, 1, ...)$, $(\mathcal{F}_{\tau_i}^{Z^P}, i = 0, 1, ...)$ and $((g^{(i+1)}(u, C), u \in [\tau_i, T)), i = 0, 1, ...)$, where $C = (C_1, C_2)$ and $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$, according to formulas (2.42) – (2.44). To find the canonical decomposition in the exogenous shocks model, the same methodology as the one applied in the endogenous switching model will be used.

Lemma 2.3 In the exogenous shocks model, assume that $u \in [\tau_i, T)$, $i = 0, 1, ...$, $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then conditional distribution for the marked point process (τ_i, Z_i^P)

given $\mathcal{F}^{Z^P}_{\tau_i}$ is equal to

$$\mathbb{P}(\tau_{i+1} \le u, Z^P_{i+1} \in C \mid \mathcal{F}^{Z^P}_{\tau_i}) = \begin{cases} F_{35}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_1 \\ F_{36}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_2 \\ F_{37}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where F_{35}, F_{36} and F_{37} are defined in the proof of this lemma in the Appendix.

Proof The proof is provided in the Appendix. ■

Lemma 2.4 In the exogenous shocks model, assume that $u \in [\tau_i, T)$, $i = 0, 1, ...$, $C = (C_1, C_2)$, $C_1 \in \mathcal{B}(\mathbb{R})$ and $C_2 \in \mathcal{B}(\mathbb{R})$. Then the function $g^{(i+1)}(u, C)$ is equal to

$$g^{(i+1)}(u,C) = \begin{cases} F_{38}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_1 \\ F_{39}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_2 \\ F_{40}(u,\tau_i,B_{\tau_i},C) & \text{if } S_{\tau_i} = s_3, \end{cases}$$

where F_{38}, F_{39} and F_{40} are defined in the proof of this lemma in the Appendix. In particular, for $E = \mathbb{R}^2$, $F_{38}(u, t, B_t, E)$ satisfies

$$\begin{aligned} F_{38}(u,t,B_t,E) &= e^{-\lambda_Z(u-t)}\phi_1(u,t,B_t) + \lambda_Z e^{-\lambda_Z(u-t)}\Phi_1(u,t,B_t) \\ &\quad + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h_2(t+r)} q_1(x;r,t,B_t)F_{38}(u,t+r,x,E)dx\Big]dr, \end{aligned}$$

where

$$\Phi_1(u,t,y) = \mathbb{P}\Big(B_s < h_1(t+s) - y, 0 \le s \le u-t; B_{u-t} > h_2(u) - y\Big), \tag{2.47}$$

$F_{39}(u, t, B_t, E)$ is equal to

$$F_{39}(u,t,B_t,E) = e^{-\lambda_Z(u-t)}\phi_m(u,t,B_t) + \lambda_Z e^{-\lambda_Z(u-t)}$$

and $F_{40}(u, t, B_t, E)$ satisfies

$$\begin{aligned} F_{40}(u,t,B_t,E) &= e^{-\lambda_Z(u-t)}\phi_2(u,t,B_t) + \lambda_Z e^{-\lambda_Z(u-t)}\Phi_2(u,t,B_t) \\ &\quad + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{h_1(t+r)}^{\infty} q_2(x;t,B_t,r)F_{40}(u,t+r,x,E)dx\Big]dr, \end{aligned}$$

where

$$\Phi_2(u,t,y) = \mathbb{P}\Big(B_s > h_2(t+s) - y, 0 \le s \le u-t; B_{u-t} < h_1(u) - y\Big). \tag{2.48}$$

Proof The proof is provided in the Appendix. ■

Applying the same argument as in the proof of Theorem 2.9, Corollary 2.7, which describes the canonical decomposition of the stock price process in the exogenous shocks model, can be obtained.

Corollary 2.7 The canonical decomposition of $(P_t, t \in [T_0, T))$ in the exogenous shocks model is given by

$$P_t = P_{T_0} + M_t + A_t, \quad M_{T_0} = 0, \quad A_{T_0} = 0,$$

where

$$M_t = \int_{T_0}^{t} \theta_2(s, P_s) dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i} - \int_{T_0}^{t} \theta_4(s, \tau_{N_s}, B_{\tau_{N_s}}) ds$$

is a local martingale,

$$A_t = \int_{T_0}^{t} \theta_1(s, P_s) ds + \int_{T_0}^{t} \theta_4(s, \tau_{N_s}, B_{\tau_{N_s}}) ds$$

is a predictable process with finite variation, $\theta_1(s, P_s)$ and $\theta_2(s, P_s)$ are defined in (2.36) and (2.37),

$$\theta_4(s, \tau_{N_s}, B_{\tau_{N_s}}) = \begin{cases} \frac{F_{41}(s,\tau_{N_s},B_{\tau_{N_s}})}{1-\int_0^{s-\tau_{N_s}} F_{38}(\tau_{N_s}+y,\tau_{N_s},B_{\tau_{N_s}},E)dy} & \text{if } S_{\tau_{N_s}} = s_1 \\ \frac{F_{42}(s,\tau_{N_s},B_{\tau_{N_s}})}{1-\int_0^{s-\tau_{N_s}} F_{39}(\tau_{N_s}+y,\tau_{N_s},B_{\tau_{N_s}},E)dy} & \text{if } S_{\tau_{N_s}} = s_2 \\ \frac{F_{43}(s,\tau_{N_s},B_{\tau_{N_s}})}{1-\int_0^{s-\tau_{N_s}} F_{40}(\tau_{N_s}+y,\tau_{N_s},B_{\tau_{N_s}},E)dy} & \text{if } S_{\tau_{N_s}} = s_3, \end{cases}$$

$F_{41}(u, t, B_t)$ satisfies

$$\begin{aligned} F_{41}(u, t, B_t) = e^{-\lambda_Z(u-t)} J^u(u)\phi_1(u, t, B_t) + \lambda_Z e^{-\lambda_Z(u-t)} \Big[\int_{h_2(u)}^{h_1(u)} q_1(x; u-t, t, B_t)\Big(p_{lu} J^{lu}(u, x) \\ + p_{lm} J^{lm}(u, x)\Big) dx\Big] + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{41}(u, t+r, x) dx\Big] dr, \end{aligned}$$

$$\begin{aligned} F_{42}(u, t, B_t) = e^{-\lambda_Z(u-t)} \Big[J^u(u)\phi_{m,1}(u, t, B_t) + J^l(u)\phi_{m,2}(u, t, B_t)\Big] \\ + \lambda_Z e^{-\lambda_Z(u-t)} \Big[\int_{h_2(u)}^{h_1(u)} q^m(x; u-t, t, B_t)\Big(p_{mu} J^{mu}(u, x) + p_{ml} J^{ml}(u, x)\Big) dx\Big], \end{aligned}$$

$F_{43}(u, t, B_t)$ satisfies

$$\begin{aligned} F_{43}(u, t, B_t) = e^{-\lambda_Z(u-t)} J^l(u)\phi_2(u, t, B_t) + \lambda_Z e^{-\lambda_Z(u-t)} \Big[\int_{h_2(u)}^{h_1(u)} q_2(x; u-t, t, B_t)\Big(p_{ul} J^{ul}(u, x) \\ + p_{um} J^{um}(u, x)\Big) dx\Big] + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r} \Big[\int_{h_1(t+r)}^{\infty} q_2(x; r, t, B_t) F_{43}(u, t+r, x) dx\Big] dr, \end{aligned}$$

J^u, J^l, J^{ul}, J^{um}, J^{lu}, J^{lm}, J^{mu}, J^{ml} are defined in (2.33), ϕ_1, $\phi_{m,1}$, $\phi_{m,2}$ and ϕ_2 are defined in the proof of Theorem 2.7 in the Appendix, q_1, q^m and q_2 are defined in the proof of Theorem 2.8 in the Appendix and $E = \mathbb{R}^2$.

2.3 Stochastic number of dynamic hedgers model

In this section, a model is developed with the number of dynamic hedgers as a piecewise constant positive stochastic process that jumps at random times by random amounts. Hence, if a model is constructed with no infinite price oscillation, then such a model would satisfy all the conditions mentioned in Remark 2.5. Since the model should be as simple as possible, it will be developed based on the most intuitive framework discussed in Remark 2.6.

Denote by

$$g^D(t) = \gamma_1 \sqrt{2\pi\Big(\frac{\alpha_1^2}{2r} + (\sigma_\kappa^2 - \frac{\alpha_1^2}{2r})e^{-2r(T-t)}\Big)}, \quad \text{for } t \in [T_0, T],$$

and assume that the value of σ_κ satisfies (2.18). Then conditions (2.8) and (2.10) imply that the system admits multiple equilibria if and only if $w_t^D > g^D(t)$. In view of (2.18), if the system admits multiple equilibria at $t \in [T_0, T)$, it should admit multiple equilibria all the time before the next jump in the number of dynamic hedgers process since function $g^D(t)$ is decreasing on its domain. Similar to the model discussed in Remark 2.6, the medium level equilibrium is excluded from consideration. If the state process is in the lower (respectively upper) level equilibrium, it is necessary to wait either until B_t crosses $H_1(t, w_t^D)$ (respectively $H_2(t, w_t^D)$) or until the number of dynamic hedgers changes, or until T, whatever happens first.

If the number of dynamic hedgers does not satisfy condition (2.10), then there are two possible scenarios. According to the first scenario,

$$w_t^D > g^D(T) = \gamma_1 \sqrt{2\pi\sigma_\kappa^2},$$

hence,

$$w_t^D \le g^D(u), \text{ for } u \in [t, T^D(w_t^D)],$$

and

$$w_t^D > g^D(u), \text{ for } u \in (T^D(w_t^D), T),$$

where

$$T^D(w_t^D) = T - \frac{\ln\left(\frac{\frac{\alpha_1^2}{2r} - \sigma_\kappa^2}{\frac{\alpha_1^2}{2r} - (\frac{w_t^D}{\gamma_1\sqrt{2\pi}})^2}\right)}{2r}. \tag{2.49}$$

In this case, it is necessary to wait either until time $T^D(w_t^D)$ or until the number of dynamic hedgers changes, whatever happens first. During this waiting period the pricing equation (2.6) has a single solution. If the number of dynamic hedgers stays constant on $[t, T^D(w_t^D)]$, it means that the system will admit multiple equilibria all the time after $T^D(w_t^D)$ until the number of dynamic hedgers changes, and the value of the state process is set to the lower level equilibrium if

$$B_{T^D(w_t^D)} < H_2(T^D(w_t^D), w_t^D) = H_1(T^D(w_t^D), w_t^D) = H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})$$

or upper level equilibrium if

$$B_{T^D(w_t^D)} > H_2(T^D(w_t^D), w_t^D) = H_1(T^D(w_t^D), w_t^D) = H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))}).$$

Then the system evolves in concordance with the mechanism that corresponds to the case when w_t^D satisfies condition (2.10). According to the second scenario, $w_t^D \leq g^D(T)$, and the pricing equation (2.6) has a single solution all the time until the number of dynamic hedgers changes.

For the sake of definiteness, it is postulated that if the number of dynamic hedgers changes in such a way that the system admits multiple equilibria and $H_2(t, w_t^D) < B_t < H_1(t, w_t^D)$, then if the system admitted multiple equilibria right before the jump, it will stay at the same equilibrium level. Otherwise, it switches to the upper or lower level equilibrium according to the value of an independent Bernoulli random variable.

2.3.1 Model setup

It is assumed that $(Z_t, t \geq 0)$ is a $\mathcal{F}$-measurable homogeneous Poisson process having some intensity λ_Z. It is supposed that the noise traders component of demand and the number of dynamic hedgers are independent, which means independence of stochastic processes $(B_t, t \geq 0)$ and $(Z_t, t \geq 0)$. It is assumed that a sequence of independent $\mathcal{F}$-measurable random variables $(\xi_i, i = 1, 2, ...)$ exists, such that they are also independent of both $(B_t, t \geq 0)$ and $(Z_t, t \geq 0)$. Each time Z_t changes its value, the number of dynamic hedgers is multiplied by a corresponding random variable ξ_i distributed according to a uniform law with density function $f_\xi(x) = \frac{1}{\xi^u - \xi^l}, x \in [\xi^l, \xi^u]$, where $0 \le \xi^l < 1 < \xi^u$, which means that both decreases and increases in the number of dynamic hedgers are possible. For the sake of determination, it is also supposed that the initial number of dynamic hedgers $w^D_{T_0}$ is given.

Denote by $\mathbb{S}$ a state space consisting of three different states: the lower level equilibrium s_1, the single equilibrium s_2 and the upper level equilibrium s_3. In Definition 2.5, a state process $(S_t, T_0 \le t < T)$ taking values in $\mathbb{S}$ is defined. Based on that process, the value of the stock price $(P_t, T_0 \le t < T)$ is determined.

It is also assumed that there exists a sequence of independent $\mathcal{F}$-measurable Bernoulli random variables $(\zeta_i, i = 1, 2, ...)$ with

$$\zeta_i := \begin{cases} 1 & \text{with probability } p_l \\ 0 & \text{with probability } p_u = 1 - p_l \end{cases}$$

such that this sequence is independent of $(B_t, t \ge 0)$, $(Z_t, t \ge 0)$ and the sequence of $(\xi_i, i = 1, 2, ...)$. If the system admits multiple equilibria, $H_2(t, w^D_t) < B_t < H_1(t, w^D_t)$ after a change in the number of dynamic hedgers and the system does not admit multiple equilibria right before the change, then S_t switches to the lower level equilibrium s_1 or the upper level equilibrium s_3 according to the value of the corresponding random variable ζ_i.

In Definition 2.5, an auxiliary process $(\hat{S}_t, T_0 \le t < T)$ taking values equal to 0 or 1, which means that the system is either in a normal or an abnormal state, will be defined. If the system gets to an abnormal state, it stays there forever, that is, this state is absorbing. In Section 2.3.2, it will be shown that, $\mathbb{P}$-a.s., the system will never get to an abnormal state and that if it is in a normal state over the whole interval $[T_0, T)$, then there is no infinite price oscillation. In Section 2.3.3, conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump on the set $[\hat{S}_t = 0]$ will be found, given the market information available.

Definition 2.5 Define processes $(S_t, T_0 \le t < T)$ and $(P_t, T_0 \le t < T)$ according to the following construction mechanism.

Step 1. Initially set the system to the normal state:

$$\hat{S}_t = 0, \forall t \in [T_0, T),$$

and let $i = 0$ and $\tau_i = T_0$.

If $w^D_{\tau_0} > g^D(\tau_0)$, that is, the system admits multiple equilibria, then set

$$S_{\tau_0} = \begin{cases} s_1 & \text{if } B_{\tau_0} \le H_2(\tau_0, w^D_{\tau_0}) \\ s_3 & \text{if } B_{\tau_0} \ge H_1(\tau_0, w^D_{\tau_0}) \\ s_0 & \text{if } H_2(\tau_0, w^D_{\tau_0}) < B_{\tau_0} < H_1(\tau_0, w^D_{\tau_0}), \end{cases}$$

where $s_0 \in \{s_1, s_3\}$ is some known constant. All the intuition in assigning the value for S_{τ_0} is the same as in Step 1 of Definition 2.1.

If $w^D_{\tau_0} \le g^D(\tau_0)$, that is, the pricing equation has a single solution, then set $S_{\tau_0} = s_2$.

Step 2. Denote by $\hat{\tau}_i$ the first time after τ_i when the number of dynamic hedgers changes, and if this number never changes after τ_i at all, define $\hat{\tau}_i = \infty$.

Then

$$\begin{cases} \text{go to Step 3} & \text{if } w^D_{\tau_i} > g^D(\tau_i) \\ \text{go to Step 4} & \text{if } w^D_{\tau_i} \le g^D(\tau_i) \quad \text{and} \quad w^D_{\tau_i} \le g^D(T) \\ \text{go to Step 5} & \text{if } w^D_{\tau_i} \le g^D(\tau_i) \quad \text{and} \quad w^D_{\tau_i} > g^D(T). \end{cases}$$

Step 3. Set

$$\tau_{i+1} = \begin{cases} \inf\Big(t > \tau_i : B_t \ge H_1(t, w^D_{\tau_i})\Big) \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_1 \\ \inf\Big(t > \tau_i : B_t \le H_2(t, w^D_{\tau_i})\Big) \wedge \hat{\tau}_i \wedge T & \text{if } S_{\tau_i} = s_3. \end{cases}$$

Recall that $\inf \emptyset = \infty$ by convention. Then assign $S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1})$, and go to Step 6.

The system gets to Step 3 if it admits multiple equilibria. The system stays in the current regime either until B_t hits the corresponding boundary, or until the number of dynamic hedgers changes, or until time elapses, whatever happens first. The state process value stays unchanged until that.

Step 4. Set $\tau_{i+1} = \hat{\tau}_i \wedge T$ and assign $S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1})$.

If $\tau_{i+1} < T$, set

$$S_{\tau_{i+1}} = \begin{cases} s_1 & \text{if} \quad w^D_{\tau_{i+1}} > g^D(\tau_{i+1}) \text{ and } B_{\tau_{i+1}} \le H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) \\ s_1 & \text{if} \quad w^D_{\tau_{i+1}} > g^D(\tau_{i+1}), \quad H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) < B_{\tau_{i+1}} < H_1(\tau_{i+1}, w^D_{\tau_{i+1}}) \quad \text{and} \quad \zeta_i = 1 \\ s_2 & \text{if } w^D_{\tau_{i+1}} \le g^D(\tau_{i+1}) \\ s_3 & \text{if} \quad w^D_{\tau_{i+1}} > g^D(\tau_{i+1}), \quad H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) < B_{\tau_{i+1}} < H_1(\tau_{i+1}, w^D_{\tau_{i+1}}) \quad \text{and} \quad \zeta_i = 0 \\ s_3 & \text{if} \quad w^D_{\tau_{i+1}} > g^D(\tau_{i+1}) \quad \text{and} \quad B_{\tau_{i+1}} \ge H_1(\tau_{i+1}, w^D_{\tau_{i+1}}), \end{cases} \tag{2.50}$$

assign $i = i + 1$ and go to Step 2. Otherwise, stop.

The system gets to Step 4 if the number of dynamic hedgers is so small that, with the current number of dynamic hedgers, the pricing equation has a single solution up to maturity T, therefore, it is necessary to wait either until the number of dynamic hedgers changes or time elapses, whatever happens first. The state process value stays unchanged until that. If the number of dynamic hedgers changes before the maturity, the system admits multiple equilibria and $B_{\tau_{i+1}} \leq H_2(\tau_{i+1}, w^D_{\tau_{i+1}})$ (respectively $B_{\tau_{i+1}} \geq H_1(\tau_{i+1}, w^D_{\tau_{i+1}})$), then assign $S_{\tau_{i+1}} = s_1$ (respectively $S_{\tau_{i+1}} = s_3$). If the number of dynamic hedgers changes before the maturity, the system admits multiple equilibria and $H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) < B_{\tau_{i+1}} < H_1(\tau_{i+1}, w^D_{\tau_{i+1}})$, assign the value for $S_{\tau_{i+1}}$ according to the value of the corresponding Bernoulli random variable ζ_i. If the number of dynamic hedgers changes before the maturity and the pricing equation still has a single solution, assign $S_{\tau_{i+1}} = s_2$. If the number of dynamic hedgers stays unchanged up to T, stop.

Step 5. If $\hat{\tau}_i \leq T^D(w^D_{\tau_i})$, then set

$$\tau_{i+1} = \hat{\tau}_i, \quad S_t = S_{\tau_i}, \forall t \in [\tau_i, \tau_{i+1}),$$

assign $S_{\tau_{i+1}}$ according to (2.50), set $i = i + 1$ and go to Step 2.

If $\hat{\tau}_i > T^D(w^D_{\tau_i})$ and $B_{T^D(w^D_{\tau_i})} = H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))})$, then set

$$S_t = S_{\tau_i}, \forall t \in [\tau_i, T^D(w^D_{\tau_i})), \quad \hat{S}_t = 1, \forall t \in [T^D(w^D_{\tau_i}), T),$$

and stop.

Otherwise, set $S_t = S_{\tau_i}, \forall t \in [\tau_i, T^D(w^D_{\tau_i}))$, assign τ_{i+1} and S_t to be equal to

$$\begin{cases} \inf\Big(t > T^D(w^D_{\tau_i}) : B_t \geq H_1(t, w^D_{\tau_i})\Big) \wedge \hat{\tau}_i \wedge T & \text{if } B_{T^D(w^D_{\tau_i})} < H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))}) \\ \inf\Big(t > T^D(w^D_{\tau_i}) : B_t \leq H_2(t, w^D_{\tau_i})\Big) \wedge \hat{\tau}_i \wedge T & \text{if } B_{T^D(w^D_{\tau_i})} > H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))}), \end{cases}$$

and

$$\begin{cases} s_1 & \forall t \in [T^D(w^D_{\tau_i}), \tau_{i+1}) \text{ if } B_{T^D(w^D_{\tau_i})} < H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))}) \\ s_3 & \forall t \in [T^D(w^D_{\tau_i}), \tau_{i+1}) \text{ if } B_{T^D(w^D_{\tau_i})} > H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))}), \end{cases}$$

and go to Step 6. Recall that $\inf \emptyset = \infty$ by convention.

The system gets to Step 5 if the number of dynamic hedgers is such that, with the current number of dynamic hedgers, the pricing equation has a single solution up to $T^D(w^D_{\tau_i})$ defined in (2.49). If the number of dynamic hedgers changes earlier than $T^D(w^D_{\tau_i})$, then the value of the state process stays unchanged until that and assign the value $S_{\tau_{i+1}}$ according to (2.50). If the number of dynamic hedgers stays unchanged until $T^D(w^D_{\tau_i})$, the system will start admitting multiple equilibria. If $B_{T^D(w^D_{\tau_i})}$ is greater or less than $H(T^D(w^D_{\tau_i}), w^D_{\tau_i}, \kappa e^{-r(T-T^D(w^D_{\tau_i}))})$, then the system switches to the corresponding upper or lower level equilibrium and evolves according to the mechanism described in Step 3. Otherwise, go to the abnormal state.

Step 6. If $\tau_{i+1} < T$ and $w^D_{\tau_{i+1}} > g^D(\tau_{i+1})$, that is, the system admits multiple equilibria, then set

$$S_{\tau_{i+1}} = \begin{cases} s_1 & \text{if } B_{\tau_{i+1}} \leq H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) \\ s_3 & \text{if } B_{\tau_{i+1}} \geq H_1(\tau_{i+1}, w^D_{\tau_{i+1}}) \\ S_{\tau_{i+1}-} & \text{if } H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) < B_{\tau_{i+1}} < H_1(\tau_{i+1}, w^D_{\tau_{i+1}}), \end{cases}$$

set $i = i+1$ and go to Step 2. Recall that, for the sake of definiteness, it is postulated that if $H_2(\tau_{i+1}, w^D_{\tau_{i+1}}) < B_{\tau_{i+1}} < H_1(\tau_{i+1}, w^D_{\tau_{i+1}})$, then the state process stays at its current level.

If $\tau_{i+1} < T$ and $w^D_{\tau_{i+1}} \leq g^D(\tau_{i+1})$, that is, the pricing equation has a single solution, then assign $S_{\tau_{i+1}} = s_2$, set $i = i+1$ and go to Step 2.

Otherwise, that is, if $\tau_{i+1} = T$, stop.

The system gets to Step 6 if it admits multiple equilibria and then either Brownian motion B_t hits the corresponding boundary $H_1(t, w^D_t)$ (and the state process jumps from the lower level equilibrium s_1 to the upper level equilibrium s_3) or $H_2(t, w^D_t)$ (and the state process jumps from the upper level equilibrium s_3 to the lower level equilibrium s_1), or the number of dynamic hedgers changes, or time elapses, whatever happens first.

Finally, define the stock price $(P_t, T_0 \leq t < T)$ pursuant to (2.9) and (2.16):

If $\hat{S}_t = 0$, then set

$$P_t = \begin{cases} \bar{p}^l(t, w^D_t, B_t) & \text{if } S_t = s_1 \\ \bar{p}(t, w^D_t, B_t) & \text{if } S_t = s_2 \\ \bar{p}^u(t, w^D_t, B_t) & \text{if } S_t = s_3. \end{cases}$$

If $\hat{S}_t = 1$, then define P_t as any (e.g., the smallest or the largest if there are more than one) solution of the pricing equation. ■

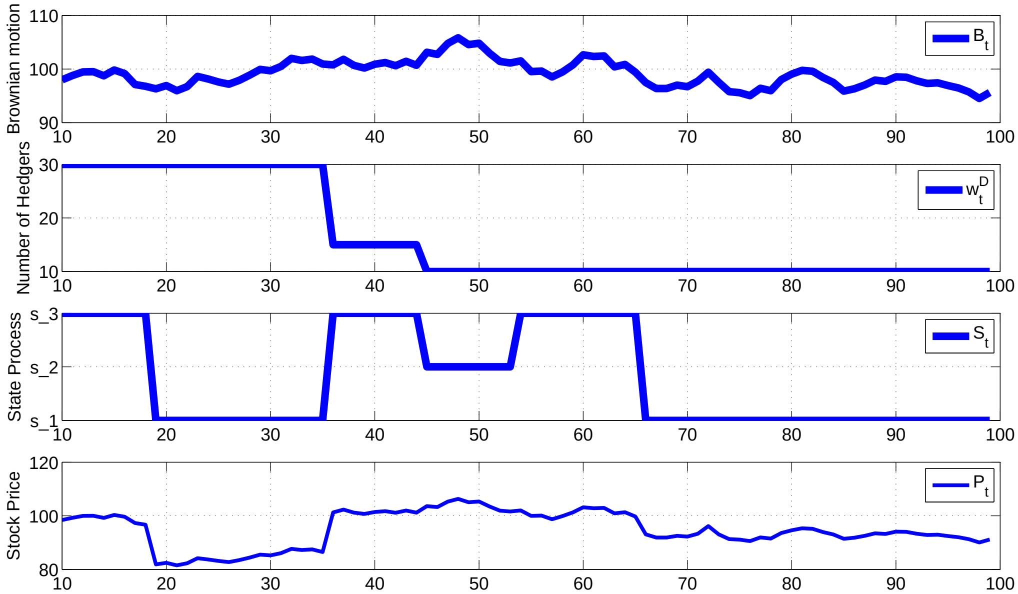

Fig. 2.4: Simulated stock price dynamics in the stochastic number of dynamic hedgers model computed for some set of parameters: $T_0 = 10$, $T = 100$, $\alpha_1 = 0.3$, $c = 0.025$, $\sigma_\kappa = 0.03$, $\kappa = 100$, $w_0^D = 30$, $\gamma_1 = 2$, $\gamma_2 = 1$, $\gamma_3 = 2$; initial value of S_t is assumed to be equal to s_3; the number of dynamic hedgers declines at $t = 36$ and $t = 45$; at time $t = 45$ it is equal to 10, which corresponds to $T^D(10) = 53.72$, and after time $t = T^D(10)$ the system admits multiple equilibria; stock price jumps at $t = 19$, $t = 36$, $t = 45$ and $t = 66$.

2.3.2 *Main properties*

In Theorem 2.10, it will be shown that the construction mechanism in Definition 2.5 determines the stock market price $(P_t, T_0 \le t < T)$, that is, for all $t \in [T_0, T)$, either $\hat{S}_t = 0$ and there is some finite i such that $t \in [\tau_i, \tau_{i+1})$ or $\hat{S}_t = 1$ ($\mathbb{P}$-a.s.). Moreover, it will be proved that the system does not get to the abnormal state ($\mathbb{P}$-a.s.).

Theorem 2.10 In Definition 2.5,

(i) for all $i \ge 0$, if $\tau_i < T$, then $\tau_i < \tau_{i+1}$ ($\mathbb{P}$-a.s.)

(ii) construction mechanism stops after a finite number of iterations ($\mathbb{P}$-a.s.)

(iii) $\mathbb{P}(\hat{S}_t = 0, \quad \forall t \in [T_0, T)) = 1$.

Proof The proof of the first statement follows from the construction since hitting times of continuous processes and exponential random variables that correspond to the inter-arrival times for homogeneous Poisson process are both positive ($\mathbb{P}$-a.s.).

Assume the second statement in this theorem does not hold. Since Z_t is a Poisson process, there is a finite number of times on $[T_0, T)$ when the number of dynamic hedgers changes ($\mathbb{P}$-a.s.). Hence, there should exist a time interval such that w_t^D is constant on that interval and such that there is an infinite number of iterations on that interval, and this leads to a contradiction due to Theorem 2.2 and Remark 2.4.

Second statement combined with the fact that B_t has a continuous distribution implies that the third statement also holds true. ■

Remark 2.9 If w_t^D satisfies (2.10), then $P_t < \bar{p}_1(t, w_t^D)$ or $P_t > \bar{p}_2(t, w_t^D)$, $t \in [T_0, T)$. This result follows from Definition 2.5, Remark 2.7 and the fact that, by construction, medium level equilibrium is excluded from consideration. If w_t^D satisfies (2.8), then $H(t, w_t^D, x)$ is also an increasing function of x.

Definition 2.6 Define a market crash as a point of discontinuity of $(P_t, 0 < t < T)$ such that $P_t < P_{t-}$ and a market boom as a point of discontinuity of $(P_t, 0 < t < T)$ such that $P_t > P_{t-}$.

This definition is the same as Definition 2.3 considered in the analysis of the constant number of dynamic hedgers models. In view of Theorem 2.10, Remark 2.4 and Definition 2.5, $(\tau_i < T, i = 1, 2, ...)$, are the only jump points on $[T_0, T)$ and there is no infinite price oscillation if the system stays in the normal state on $[T_0, T)$ ($\mathbb{P}$-a.s.), and probability that it stays in the normal state on $[T_0, T)$ is equal to 1. Denote the value of the i-th jump by $J_i = \Delta P_{\tau_i} = P_{\tau_i} - P_{\tau_i -}$. Similar to the constant number of dynamic hedgers models, it can be shown that the càdlàg property of the stock price process holds. Defining the market filtration $\mathcal{F}_t^P$ in accordance with (2.34), it can be concluded that the stock price jump times $(\tau_i < T, i = 1, 2, ...)$, are $\mathcal{F}_t^P$-stopping times. The proofs of these two properties are patterned after Theorem 2.4 and Theorem 2.5. Finally, Theorem 2.11 describes the stock price dynamics for $t \in [T_0, T)$.

Theorem 2.11 The stock price process is a semimartingale that follows

$$P_t = P_{T_0} + \int_{T_0}^{t} \theta_1(s, P_s, w_s^D)ds + \int_{T_0}^{t} \theta_2(s, P_s, w_s^D)dB_s + \sum_{i=1}^{N_t} \Delta P_{\tau_i}, \quad \text{for } t \in [T_0, T),$$

where $N_t = \sum_{i\geq 1} \mathbb{I}(\tau_i \leq t)$ is the total number of jumps on $[T_0, t]$,

$$\theta_1(s, P_s, w_s^D) = -\frac{H_s(s, P_s, w_s^D) + \frac{1}{2}H_{xx}(s, P_s, w_s^D)(\frac{1}{H_x(s,P_s,w_s^D)})^2}{H_x(s, P_s, w_s^D)}$$

and

$$\theta_2(s, P_s, w_s^D) = \frac{1}{H_x(s, P_s, w_s^D)}.$$

Proof The proof is patterned after Theorem 2.6. ■

2.3.3 Conditional distributions

Recall that $[\hat{S}_t = 0]$ means that the system is in the normal state at time $t \in [T_0, T)$. In this section, it is supposed that $[\hat{S}_t = 0]$ and find conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump, given that the stock price dynamics on $[T_0, t]$ is observed. In Theorem 2.12, their joint conditional distribution is found, given the market filtration $\mathcal{F}_t^P$. Based on this theorem, marginal conditional distributions can be derived.

Theorem 2.12 Assume that $T_0 \leq t < u \leq T$, $[\hat{S}_t = 0]$, C_1 is any combination of elements in $\mathbb{S}$ and $C_2 \in \mathbb{B}(\mathbb{R})$. In the stochastic number of dynamic hedgers model, the joint conditional distribution for the time of the next jump, the type of the next jump and the size of the next jump on the set $[\hat{S}_t = 0]$, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \mid \mathcal{F}_t^P) = \begin{cases} F_{44}(t, w_t^D, B_t, u, C_1, C_2) & \text{if } S_t = s_1 \\ F_{45}(t, w_t^D, T^D(w_t^D), B_t, u, C_1, C_2) & \text{if } S_t = s_2 \\ F_{46}(t, w_t^D, B_t, u, C_1, C_2) & \text{if } S_t = s_3, \end{cases}$$

where F_{44}, F_{45} and F_{46} are defined in the proof of Theorem 2.12 in the Appendix.

Proof The proof is provided in the Appendix. ■

Distribution of the time of the next jump

Taking $C_1 = \{s_1, s_2, s_3\}$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.12, the conditional cumulative distribution function of the time of the next jump, given the market filtration $\mathcal{F}_t^P$, can be computed.

Corollary 2.8 Suppose that $T_0 \leq t < u \leq T$ and $[\hat{S}_t = 0]$. Then the conditional cumulative distribution function of the time of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < u \mid \mathcal{F}_t^P) = \begin{cases} 1 - e^{-\lambda_Z(u-t)}\bar{D}_1(u,t,B_t,w_t^D) & \text{if } S_t = s_1 \\ F_{47}(t, w_t^D, T^D(w_t^D), B_t, u) & \text{if } S_t = s_2 \\ 1 - e^{-\lambda_Z(u-t)}\bar{D}_2(u,t,B_t,w_t^D) & \text{if } S_t = s_3, \end{cases}$$

where

$$F_{47}(t, w_t^D, T^D(w_t^D), B_t, u) = \Big(1 - e^{-\lambda_Z(u-t)}\Big) + \mathbb{I}\Big(T^D(w_t^D) < u\Big)e^{-\lambda_Z(u-t)} \times$$
$$\times \Big[\int_{-\infty}^{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} (1 - \bar{D}_1(u, T^D(w_t^D), x, w_t^D))dx$$
$$+ \int_{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})}^{\infty} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} (1 - \bar{D}_2(u, T^D(w_t^D), x, w_t^D))dx\Big]$$

and $\bar{D}_1$ and $\bar{D}_2$ are defined in the proof of Theorem 2.12 in the Appendix.

Distribution of the next state of the state process

Let $t \in [T_0, T)$ and suppose $[\hat{S}_t = 0]$. Taking $u = T$ and $C_2 = \mathbb{R}$ in the formulas in Theorem 2.12, the conditional cumulative distribution function of the next state of the state process, given the market filtration $\mathcal{F}_t^P$, can be computed. On the set $[P_t < \bar{p}_1(t, w_t^D)]$ the conditional probability that there will be at least one more jump and the first jump will be a small boom given $\mathcal{F}_t^P$ is equal to $F_{44}(t, w_t^D, B_t, T, s_2, \mathbb{R})$, while the conditional probability that there will be at least one more jump and the first jump will be a big boom given $\mathcal{F}_t^P$ is equal to $F_{44}(t, w_t^D, B_t, T, s_3, \mathbb{R})$. On the set $[\bar{p}_1(t, w_t^D) < P_t < \bar{p}_2(t, w_t^D)]$ the conditional probability that there will be at least one more jump and the first jump will be a market boom given $\mathcal{F}_t^P$ is equal to $F_{45}(t, w_t^D, T^D(w_t^D), B_t, T, s_3, \mathbb{R})$, while the probability that there will be at least one more jump and the first jump will be a market crash is equal to $F_{45}(t, w_t^D, T^D(w_t^D), B_t, T, s_1, \mathbb{R})$. Finally, on the set $[P_t > \bar{p}_2(t, w_t^D)]$ the conditional probability that there will be at least one more jump and the first jump will be a small crash given $\mathcal{F}_t^P$ is equal to $F_{46}(t, w_t^D, B_t, T, s_2, \mathbb{R})$, while the conditional probability that there will be at least one more jump and the first jump will be a big crash is equal to $F_{46}(t, w_t^D, B_t, T, s_1, \mathbb{R})$. Combining these formulas all together, Corollary 2.9 can be obtained.

Corollary 2.9 Suppose that $T_0 \le t < T$ and $[\hat{S}_t = 0]$. Then the conditional cumulative distribution function of the next state of the state process, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\begin{cases} \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_2 \mid \mathcal{F}_t^P) = F_{44}(t, w_t^D, B_t, T, s_2, \mathbb{R}) & \text{if } S_t = s_1 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_3 \mid \mathcal{F}_t^P) = F_{44}(t, w_t^D, B_t, T, s_3, \mathbb{R}) & \text{if } S_t = s_1 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_3 \mid \mathcal{F}_t^P) = F_{45}(t, w_t^D, T^D(w_t^D), B_t, T, s_3, \mathbb{R}) & \text{if } S_t = s_2 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_1 \mid \mathcal{F}_t^P) = F_{45}(t, w_t^D, T^D(w_t^D), B_t, T, s_1, \mathbb{R}) & \text{if } S_t = s_2 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_2 \mid \mathcal{F}_t^P) = F_{46}(t, w_t^D, B_t, T, s_2, \mathbb{R}) & \text{if } S_t = s_3 \\ \mathbb{P}(\tau_{N_t+1} < T, S_{\tau_{N_t+1}} = s_1 \mid \mathcal{F}_t^P) = F_{46}(t, w_t^D, B_t, T, s_1, \mathbb{R}) & \text{if } S_t = s_3, \end{cases}$$

where F_{44}, F_{45} and F_{46} are defined in Theorem 2.12 in the Appendix.

Distribution of the size of the next jump

Let $C \in \mathcal{B}(\mathbb{R})$ and suppose that $t \in [T_0, T)$ and $[\hat{S}_t = 0]$. Taking $u = T$ and $C_1 = \mathbb{S}$ in the formulas in Theorem 2.12, the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, can be obtained. On the set $[P_t < \bar{p}_1(t, w_t^D)]$ (respectively $[\bar{p}_1(t, w_t^D) < P_t < \bar{p}_2(t, w_t^D)]$, respectively $[P_t > \bar{p}_2(t, w_t^D)]$) the conditional probability that there will be at least one more jump and the first jump value will be in C given $\mathcal{F}_t^P$ is equal to $F_{44}(t, w_t^D, B_t, T, \mathbb{S}, C)$ (respectively $F_{45}(t, w_t^D, T^D(w_t^D), B_t, T, \mathbb{S}, C)$, respectively $F_{46}(t, w_t^D, B_t, T, \mathbb{S}, C)$). Combining these formulas all together, Corollary 2.10, can be obtained.

Corollary 2.10 Suppose that $T_0 \le t < T$, $[\hat{S}_t = 0]$ and $C \in \mathcal{B}(\mathbb{R})$. Then the conditional cumulative distribution function of the size of the next jump, given the market filtration $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}(\tau_{N_t+1} < T, J_{\tau_{N_t+1}} \in C \mid \mathcal{F}_t^P) = \begin{cases} F_{44}(t, w_t^D, B_t, T, \mathbb{S}, C) & \text{if } S_t = s_1 \\ F_{45}(t, w_t^D, B_t, T, \mathbb{S}, C) & \text{if } S_t = s_2 \\ F_{46}(t, w_t^D, B_t, T, \mathbb{S}, C) & \text{if } S_t = s_3, \end{cases}$$

where F_{44}, F_{45} and F_{46} are defined in the proof of Theorem 2.12 in the Appendix.

3. ALTERNATIVE MODELS

3.1 Motivation

In the previous chapter, three multiple equilibria and stock market booms and crashes models were developed based on the market microstructure framework: the model with a constant number of dynamic hedgers and endogenous switching, the model with a constant number of dynamic hedgers and exogenous switching and the model with a stochastic number of dynamic hedgers. For all of these models, the stock price process dynamics and conditional distribution formulas for time to, the type of and the size of the next jump were computed. Note that these models might yield negative prices and assume agents make their decisions based on a Brownian motion with a drift approximation of the stock price process, but its actual dynamics have a different form. According to the jump structure in the constant number of dynamic hedgers models, the stock price can not have more than two consecutive upward or downward jumps, and this is quite restrictive. If, for example, the stock price is in the lower level equilibrium, then the next jump type should be an upward jump. Similarly, if the price is in the upper level equilibrium, then the next jump type should be a downward jump. Moreover, distribution formulas in these models are given in terms of the functions of Brownian motion hitting probabilities and densities for one-sided and two-sided curved boundaries, and these probabilities and densities can be evaluated only numerically. To overcome these drawbacks, two alternative models are developed.

In the simple jump structure model, it is considered that the pricing equation pattern that resembles the shape of the one obtained within the market microstructure framework. This new pattern excludes negative prices and has a closed-form solution, but it assumes the stock price process is given exogenously. Similar to the stochastic number of dynamic hedgers model, for the sake of simplicity, it is assumed that the state process that corresponds to the price equilibrium levels can take only two values: the lower level equilibrium s_1 and the upper level equilibrium s_2.

In this model, any upward jump always precedes a downward jump, which, in turn, always precedes an upward jump. Even if the medium level equilibrium is incorporated, similar to the constant number of dynamic hedgers models, still it would not be possible to have, for example, three consecutive upward or downward jumps.

This observation is the motivating factor for the development of an alternative approach that could have any jump structure dynamics. The simple jump structure model, thus, can be considered as a transition model from the market microstructure models to the Markov chain jump structure model, in which the next jump type, market boom or market crash, is determined by a Markov chain with a 2×2 transition probabilities matrix. This model exhibits all the pros of the simple jump structure model: it excludes negative prices and has a closed-form solution. As in the simple jump structure model, the price in the Markov chain jump structure model is determined exogenously rather than by the law of supply and demand.

3.2 Alternative models framework

I will work on a filtered stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions. Assume that on this probability space there exists a standard Brownian motion $(B_t, t \geq 0)$ starting at 0.

In this chapter, framework will be developed which satisfies some properties. First, all the conditions mentioned in Remark 2.5 should hold. Second, it should avoid negative stock prices. Third, it is required to have conditional probabilities of the time of the next jump, the type of the next jump and the size of the next jump that can be found in a closed form. Fourth, the pricing equation should look like the one in the market microstructure models considered in Chapter 2. Finally, the model should be as simple as possible.

For the sake of simplicity, the preferred model will have the pricing equation that resembles the form of (2.26), which is the special case of the pricing equation (2.6) like in the constant number of dynamic hedgers models and excludes medium level equilibria from consideration like in the stochastic number of dynamic hedgers model. Recall that, according to Remark 2.3 and Remark 2.4, both lower and upper level branches of function $h(t, x)$ are in the class $C^{1,2}$ inside their domains and property (2.24) holds true. To exclude the possibility of negative stock prices arising from (2.24), the following modification of the market microstructure framework is considered.

Definition 3.1 Define the stock price process $(P_t, t \geq 0)$ taking values in $\mathbb{R}_+$ as the solution of equation

$$h(t, P_t - \eta_t) = B_t,$$

where an auxiliary stochastic piecewise-constant process $(\eta_t, t \geq 0)$ taking values in $\mathbb{R}_+$ is model-specific and will be defined in Definition 3.3 for the simple jump structure model and in Definition 3.4 for the Markov chain jump structure model and function $h(t, x) \in C^{1,2}(\mathbb{R}_+, \mathbb{R}_+)$ is known and satisfies the following properties:

(i) $h_x(t, x) > 0$ on its domain, that is, it is an increasing function of x

(ii) $\lim_{x \downarrow 0} h(t, x) = -\infty$ and $\lim_{x \to +\infty} h(t, x) = +\infty$.

Remark 3.1 By the implicit function theorem, for each $t \geq 0$ fixed, the inverse function $h^{-1}(y, t)$ exists and is twice continuously differentiable. Based on Definition 3.1, the stock price P_t satisfies

$$P_t = h^{-1}(B_t, t) + \eta_t. \tag{3.1}$$

Remark 3.2 If $h(t, x) = a_1 t + a_2 \ln(x)$ with some constants $a_1 \in \mathbb{R}$ and $a_2 > 0$, then a Geometric Brownian motion for the stock price can be obtained:

$$P_t = e^{-\frac{a_1}{a_2} t + \frac{1}{a_2} B_t} + \eta_t.$$

In the models developed in this chapter, the same definitions of market filtration and market crashes and booms are used as applied in the market microstructure models. Similar to (2.34), the market filtration $\mathcal{F}_t^P$ is defined by

$$\mathcal{F}_t^P = \sigma\{P_s, 0 \leq s \leq t\}.$$

Definition 3.2 determines market jumps based on Definition 2.3 (or, equivalently, Definition 2.6).

Definition 3.2 Define a market crash as a point of discontinuity of $(P_t, t > 0)$ such that $P_t < P_{t-}$ and a market boom as a point of discontinuity of $(P_t, t > 0)$ such that $P_t > P_{t-}$.

It is also assumed that on the probability space exist $(\zeta_i^l, i = 0, 1, ...)$ and $(\zeta_i^u, i = 0, 1, ...)$, the sequences of independent random variables distributed according to some laws with density functions $(f^l(x), x \in [0, 1])$ and $(f^u(x), x \geq 1)$, such that both sequences are independent of B_t and each other.

In the simple jump structure model, define an auxiliary state process $(S_t, t \geq 0)$ that takes values in the state space $\mathcal{S}$ consisting of two values: lower level state s_1 and upper level state s_2. If S_t is in the state s_1, then both S_t and η_t stay unchanged until the Brownian motion B_t hits some boundary and then S_t switches to the other state s_2 and the stock price jumps upwards by some random amount: at the time of the jump the value of η_t is multiplied by some corresponding random variable ζ_i^u, and, according to Remark 3.1, the jump size is equal to $\eta_t(\zeta_i^u - 1)$. Then both S_t and η_t stay unchanged until the Brownian motion B_t hits some other boundary and then S_t switches back to the state s_1 and price jumps downwards by some random amount: at the time of the jump the value of η_t is multiplied by some corresponding random variable ζ_i^l, and, according to Remark 3.1, the jump size is equal to $\eta_t(\zeta_i^l - 1)$. Then this mechanism iterates. Figure 3.1 shows the analogy between the market microstructure framework discussed in Chapter 2 and the simple jump structure model. In the simple jump structure model, each upward jump is followed by a downward jump which in turn is followed by an upward jump.

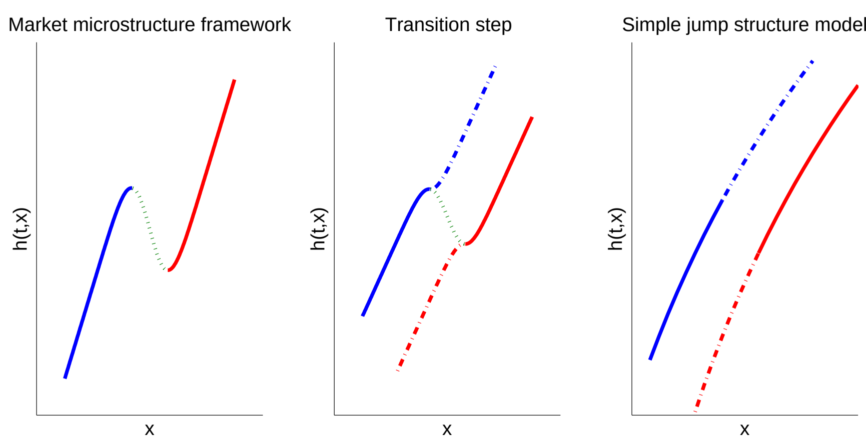

Fig. 3.1: Analogy between the market microstructure framework and the simple jump structure model

To make the jump structure not so restrictive, the Markov chain jump structure model is developed. It is assumed that the state of the asset space $\mathbb{S}$ consists of two states: lower level equilibrium state s_1 and upper level equilibrium state s_2, and the jump type state space $\mathbb{S}^{\mathbb{J}}$ consists of two states: market crash state s_1^J and market boom state s_2^J. Two auxiliary processes are defined: the state of the asset process $(S_t, t \geq 0)$ taking values in $\mathbb{S}$ and the jump type state process $(S_t^J, t \geq 0)$ taking values in $\mathbb{S}^{\mathbb{J}}$. If S_t is in the state s_1, then S_t, S_t^J and η_t stay unchanged until the Brownian motion

B_t hits some boundary and then S_t switches to the other state s_2. Similarly, if S_t is in the state s_2, then S_t, S_t^J and η_t stay unchanged until the Brownian motion B_t hits another boundary and then S_t switches to the other state s_1. Therefore, the state of the asset process has the same type of dynamics as the state process in the simple jump structure model. The difference between two models is in the structure of the jumps. In the Markov chain jump structure model the type of the next jump, market crash or market boom, which is described by the value of the jump type state process, is determined according to the Markov chain mechanism with a 2×2 transition probabilities matrix

$$\begin{pmatrix} p_c & 1 - p_c \\ 1 - p_b & p_b \end{pmatrix}, \tag{3.2}$$

where $0 < p_c < 1$ and $0 < p_b < 1$, and such that it is assumed to be independent of $(B_t, t \geq 0)$ and sequences $(\zeta_i^u, i = 0, 1, ...)$ and $(\zeta_i^l, i = 0, 1, ...)$. In this matrix, p_c denotes the probability that the next jump of the stock price process will be a market crash given the current jump is a market crash, $1 - p_c$ denotes the probability that the next jump will be a market boom given the current jump is a market crash, $1 - p_b$ denotes the probability that the next jump will be a market crash given the current jump is a market boom, and finally p_b denotes the probability that the next jump will be a market boom given the current jump is a market boom. If the next jump is a market crash, then at the time of the jump the value of η_t is multiplied by some corresponding random variable ζ_i^l, and like in the simple jump structure model, the jump size is equal to $\eta_t(\zeta_i^l - 1)$. If the next jump is a market boom, then at the time of the jump the value of η_t is multiplied by some corresponding random variable ζ_i^u, and like in the simple jump structure model, the jump size is equal to $\eta_t(\zeta_i^u - 1)$. Then the process is iterated. Note that if this transition probabilities matrix has identical rows, a special case of the jump structure is obtained where the probability of the next jump type, a market boom or a market crash, does not depend on the current state of the jump type state process.

The next question is how the boundary processes that move the stock prices from one regime to another should be modelled. Recall that an explicit form is required for the conditional probability of the time of the next jump, given the market information $\mathcal{F}_t^P$. To do that the appropriate boundary processes are required which Brownian motion should hit in order for the stock price to switch the regimes. A possible solution would be to use one of deterministic functions for which an explicit form exists (see examples of those boundaries in Salminen [41], Daniels [17] and Novikov [32]). A problem with this kind of modelling is that, in virtue of (3.1) and the fact that η_t stays unchanged between the stock price jumps, it can be known at time t at what value the stock price could jump at time $u > t$, and this is not the case if discussing actual stock price dynamics. For this reason, an example of stochastic boundaries will be considered that admit this conditional probability in a closed form.

Let a deterministic function $(\alpha(t), t \geq 0)$ and constants $a \in \mathbb{R}$ and $A \in \mathbb{R}$ such that $a < A$ be given. Assume that processes $(L_t, t \geq 0)$ and $(U_t, t \geq 0)$ satisfy

$$dL_t = \alpha(t)(B_t - L_t)dt, \quad L_0 = a, \tag{3.3}$$

and

$$dU_t = \alpha(t)(B_t - U_t)dt, \quad U_0 = A, \tag{3.4}$$

that is,

$$L_t = ae^{-\int_0^t \alpha(s)ds} + \int_0^t e^{-\int_s^t \alpha(r)dr}\alpha(s)B_s ds < Ae^{-\int_0^t \alpha(s)ds} + \int_0^t e^{-\int_s^t \alpha(r)dr}\alpha(s)B_s = U_t. \tag{3.5}$$

In Section 3.6, it will be shown that, for both models, if the boundary processes are given by L_t and U_t, then there is no infinite price oscillation and the conditional probability of the time of the next jump, the type of the next jump and the size of the next jump, given the market information $\mathcal{F}_t^P$, can be found in a closed form.

Remark 3.2 Consider the simple jump structure model. Suppose the current state of the state process is equal to s_1, which means that the next jump will be upwards. Denote the time of the next jump, which is the Brownian motion hitting time of the boundary U_t, by T. In Theorem 3.5 in Section 3.6, it will be shown that T is finite ($\mathbb{P}$-a.s.). Since process $U_t - B_t$ is continuous, T, which is its first hitting time of 0, is a predictable stopping time (see Protter [38], p.104). Therefore, there is a sequence of stopping times T_n increasing to T. Consider the sequence of trading strategies, $\mathbb{I}\Big(T_n < t \leq T\Big)$, which consist in buying the stock right after T_n and selling at T. The profit associated with this strategy is $P_T - P_{T_n}$. In Theorem 3.2 in Section 3.6, it will be shown that the stock price process P_t is càdlàg, hence, P_{T_n} converges to P_{T-}, so the profits converge to $P_T - P_{T-}$, which is strictly positive, and there would be an arbitrage in the limit. Similarly, if the current state of the state process is equal to s_2, there would also be an arbitrage in the limit. To avoid that arbitrage opportunity in the simple jump structure model, it is assumed that there is a sequence of independent exponential random variables $(\mu_i, i = 0, 1, ...)$ with a rate parameter λ_μ defined on the probability space such that this sequence is also independent of B_t and sequences $(\zeta_i^l, i = 0, 1, ...)$ and $(\zeta_i^u, i = 0, 1, ...)$. In Definition 3.3, boundary processes L_t and U_t are replaced by corresponding modified boundary processes $L_t^{(i)}$ and $U_t^{(i)}$ that depend on μ_i in accordance with formula (3.8). Agents do not know the corresponding value of μ_i before the jump happens, and this excludes the arbitrage opportunity. At the same time $L_t^{(i)}$ and $U_t^{(i)}$ satisfy all the pros of boundaries L_t and U_t defined in (3.3) and (3.4): there is no infinite price oscillation and corresponding conditional probabilities can be found in a closed form.

Remark 3.3 In contrast to the simple jump structure model, in the Markov chain jump structure model, it is never known whether the next jump will be upwards or downwards. Indeed, by assumption, $0 < p_c < 1$ and $0 < p_b < 1$, which means that both crash and boom are possible, regardless of the current state of the jump type state process, and the boundaries L_t and U_t are used since they do a good job. Note that all three market microstructure models have a finite time horizon, which means that with a positive probability there might be no next jump at all and such an arbitrage opportunity as the one described in Remark 3.2 does not exist.

In the subsequent sections, the simple jump structure and the Markov chain jump structure model setups will be discussed, including their main properties and conditional distributions for the time of, the type of and the size of the next jump, given the market filtration $\mathcal{F}_t^P$.

3.3 Simple jump structure model

Model setup

In Definition 3.3, the state process $(S_t, t \geq 0)$ and the process $(\eta_t, t \geq 0)$ taking values in $\mathbb{S}$ and $\mathbb{R}_+$ are determined.

Definition 3.3 Define state process $(S_t, t \geq 0)$ and the process $(\eta_t, t \geq 0)$ according to the following construction.

Step 1 Set $i = 0$ and $\tau_0 = 0$.

Step 2 Define boundary processes $(U_t^{(i)}, t \geq \tau_i)$ and $(L_t^{(i)}, t \geq \tau_i)$ by

$$dL_t^{(i)} = \alpha(t)(B_t - L_t^{(i)})dt, \quad L_{\tau_i}^{(i)} = L_{\tau_i} - \mu_i, \tag{3.6}$$

and

$$dU_t^{(i)} = \alpha(t)(B_t - U_t^{(i)})dt, \quad U_{\tau_i}^{(i)} = U_{\tau_i} + \mu_i, \tag{3.7}$$

that is,

$$L_t^{(i)} = L_t - \mu_i e^{-\int_{\tau_i}^t \alpha(r)dr} \leq L_t < U_t \leq U_t + \mu_i e^{-\int_{\tau_i}^t \alpha(r)dr} = U_t^{(i)}. \tag{3.8}$$

Step 3 If $i = 0$, then set initial values of η_t and S_t

$$\eta_{\tau_0} = c \quad \text{and} \quad S_{\tau_0} = \begin{cases} s_1 & \text{if } B_{\tau_0} \leq L_{\tau_0}^{(0)} \\ s_2 & \text{if } B_{\tau_0} \geq U_{\tau_0}^{(0)} \\ s_0 & \text{if } L_{\tau_0}^{(0)} < B_{\tau_0} < U_{\tau_0}^{(0)}, \end{cases}$$

where $c \in \mathbb{R}_+$ and $s_0 \in \mathbb{S}$ are some known constants. Assign value s_0 for the sake of definiteness since for $L_{\tau_0}^{(0)} < B_{\tau_0} < U_{\tau_0}^{(0)}$ both states s_1 and s_2 are possible. Note that, according to Step 2 and formulas (3.3) and (3.4), $L_{\tau_0}^{(0)} = a - \mu_0$ and $U_{\tau_0}^{(0)} = A + \mu_0$.

Step 4 Set

$$\tau_{i+1} = \begin{cases} \inf\Big(t > \tau_i : B_t = U_t^{(i)}\Big) & \text{if } S_{\tau_i} = s_1 \\ \inf\Big(t > \tau_i : B_t = L_t^{(i)}\Big) & \text{if } S_{\tau_i} = s_2. \end{cases}$$

Recall that $\inf \emptyset = \infty$ by convention.

Step 5 For $t \in [\tau_i, \tau_{i+1})$, set $S_t = S_{\tau_i}$ and $\eta_t = \eta_{\tau_i}$.

Step 6 Set the next state of the state process equal to the other state: $S_{\tau_{i+1}} = \mathbb{S} \setminus S_{\tau_i}$.

Step 7 Set

$$\eta_{\tau_{i+1}} = \begin{cases} \zeta_i^u \eta_{\tau_i} & \text{if } S_{\tau_i} = s_1 \\ \zeta_i^l \eta_{\tau_i} & \text{if } S_{\tau_i} = s_2. \end{cases}$$

Step 8 Set $i = i + 1$ and go to Step 2.

Finally, define the stock price $(P_t, t \geq 0)$ pursuant to (3.1). ■

3.4 Markov chain jump structure model

Model setup

In Definition 3.4, the state of the asset process $(S_t, t \geq 0)$, the jump type state process $(S_t^J, t \geq 0)$ and the process $(\eta_t, t \geq 0)$ taking values in $\mathbb{S}$, $\mathbb{S}^{\mathbb{J}}$ and $\mathbb{R}_+$ are determined.

Definition 3.4 Define the state of the asset process $(S_t, t \geq 0)$, the jump type state process $(S_t^J, t \geq 0)$ and the process $(\eta_t, t \geq 0)$ according to the following construction.
Step 1 Set $i = 0$, $\tau_0 = 0$ and starting values

$$\eta_{\tau_0} = c, \quad S_{\tau_0}^J = s_0^J \quad \text{and} \quad S_{\tau_0} = \begin{cases} s_1 & \text{if } B_{\tau_0} \leq a \\ s_2 & \text{if } B_{\tau_0} \geq A \\ s_0 & \text{if } a < B_{\tau_0} < A, \end{cases}$$

where $s_0^J \in \mathbb{S}^{\mathbb{J}}$ and $s_0 \in \mathbb{S}$ are some known constants. Assign values s_0^J and s_0 for the sake of definiteness, that is, when more than one state is possible.
Step 2 Set

$$\tau_{i+1} = \begin{cases} \inf\Big(t > \tau_i : B_t = U_t\Big) & \text{if } S_{\tau_i} = s_1 \\ \inf\Big(t > \tau_i : B_t = L_t\Big) & \text{if } S_{\tau_i} = s_2. \end{cases}$$

Recall that $\inf \emptyset = \infty$ by convention.
Step 3 For $t \in [\tau_i, \tau_{i+1})$, set $S_t = S_{\tau_i}$, $S_t^J = S_{\tau_i}^J$ and $\eta_t = \eta_{\tau_i}$.
Step 4 Set the next state of the state of the asset process: $S_{\tau_{i+1}} = \mathbb{S} \setminus S_{\tau_i}$.
Step 5 Set the next state of the jump type state process $S_{\tau_{i+1}}^J$ according to the Markov chain mechanism (3.2).
Step 6 Set $i = i + 1$ and go to Step 2.
Finally, define the stock price $(P_t, t \geq 0)$ pursuant to (3.1). ■

3.5 Main properties of alternative models

In Theorem 3.1, it will be shown that there is no infinite price oscillation ($\mathbb{P}$-a.s.).

Theorem 3.1 In both simple jump structure and Markov chain jump structure models,

(i) for all $i = 0, 1, ...$, there is $\tau_i < \tau_{i+1}$ ($\mathbb{P}$-a.s.),

(ii) for all $T > 0$, there is only a finite number of τ_i on $[0, T]$, hence, they are not accumulating ($\mathbb{P}$-a.s.).

Proof According to (3.5) and (3.8), for $t \in [0, T]$ and $i = 0, 1, ...,$

$$U^{(i)}(t) - L^{(i)}(t) \geq U_t - L_t \geq \delta(T),$$

where

$$\delta(T) = (A - a)\, e^{-\int_0^T |\alpha(r)| dr} > 0,$$

and the result follows from the continuity of Brownian motion and processes L_t and U_t. ■

Theorem 3.2 shows the càdlàg property of the stock price process.

Theorem 3.2 The stock price process is càdlàg ($\mathbb{P}$-a.s.).

Proof The result follows from Remark 3.1, Theorem 3.1 and the construction of $(\eta_t, t \geq 0)$ in Definition 3.3 and Definition 3.4. ■

By construction and (3.1), the set of $(\tau_i, i = 1, 2, ...)$ and the set of all the jumps in the stock price process are the same and the value of the i-th jump is equal to $J_i = \Delta P_{\tau_i} = P_{\tau_i} - P_{\tau_i -} = \eta_{\tau_i} - \eta_{\tau_{i-1}}$. Theorem 3.3 shows that jump times $(\tau_i, i = 1, 2...)$ are $\mathcal{F}_t^P$-stopping times and Theorem 3.4 shows that the stock price is a semimartingale.

Theorem 3.3 Jump times $(\tau_i, i = 1, 2...)$ are $\mathcal{F}_t^P$-stopping times.

Proof In virtue of Theorem 3.2 the proof patterns after Theorem 2.5. ■

Theorem 3.4 The stock price process is a semimartingale that follows the dynamics

$$P_t = h^{-1}(B_t, t) + \eta_t, \quad t \geq 0.$$

Proof Indeed, the result follows from Remark 3.1, Theorem 32 (p.78) in Protter [38], Theorem 3.1 and the construction of $(\eta_t, t \geq 0)$ in Definition 3.3 and Definition 3.4. ■

Denote by

$$N_t = \sum_{i=1}^{\infty} \mathbb{I}(\tau_i \leq t), \quad t \geq 0,$$

the total number of jumps on $[0, t]$ and let

$$D_t^S = \begin{cases} U_t^{(N_t)} - B_t & \text{if } S_t = s_1 \\ B_t - L_t^{(N_t)} & \text{if } S_t = s_2 \end{cases} \tag{3.9}$$

and

$$D_t^{MC} = \begin{cases} U_t - B_t & \text{if } S_t = s_1 \\ B_t - L_t & \text{if } S_t = s_2 \end{cases} \tag{3.10}$$

be the distances to the border processes corresponding to Step 4 in Definition 3.3 and Step 2 in Definition 3.4:

$$\tau_{N_t+1} = \begin{cases} \inf\Big(u > t : D_u^S = 0\Big) & \text{for the simple jump structure model} \\ \inf\Big(u > t : D_u^{MC} = 0\Big) & \text{for the Markov chain jump structure model.} \end{cases} \tag{3.11}$$

In virtue of the definition of D_t^{MC} for the Markov chain jump structure model, a similar process for the simple jump structure model can be defined:

$$d_t^S = \begin{cases} U_t - B_t & \text{if } S_t = s_1 \\ B_t - L_t & \text{if } S_t = s_2. \end{cases} \tag{3.12}$$

In view of (3.8), it can be concluded that

$$D_t^S = \gamma(d_t^S, \tau_{N_t}, t, \mu_{N_t}), \tag{3.13}$$

where

$$\gamma(d_t^S, \tau_{N_t}, t, x) = d_t^S + x e^{-\int_{\tau_{N_t}}^t \alpha(r)dr}. \tag{3.14}$$

If τ_{N_t+1} is finite, then values of $S_{\tau_{N_t+1}}$, $S^J_{\tau_{N_t+1}}$ and $J_{\tau_{N_t+1}}$ can be determined according to Definition 3.3 and Definition 3.4. For the sake of completeness, assign S_∞, S^J_∞ and J_∞ any value from $\mathbb{S}$, $\mathbb{S}^{\mathbb{J}}$ and $\mathbb{R}$. Theorem 3.5 shows that, for all $t \geq 0$, the next jump time is finite ($\mathbb{P}$-a.s.). In the subsequent sections, conditional distribution for the time of the next jump, the type of the next jump and the size of the next jump will be calculated, given the market information $\mathcal{F}_t^P$.

Theorem 3.5 For all $t \geq 0$, the next jump time is finite ($\mathbb{P}$-a.s.):

$$\mathbb{P}\left(\tau_{N_t+1} < \infty \mid \mathcal{F}_t^P\right) = 1.$$

Proof In view of (3.3), (3.4), (3.6), (3.7), (3.9) and (3.10),

$$\begin{cases} dD_t^S = -\alpha(t) D_t^S dt - dB_t & \text{if } S_t = s_1 \\ dD_t^S = -\alpha(t) D_t^S dt + dB_t & \text{if } S_t = s_2 \end{cases}$$

and

$$\begin{cases} dD_t^{MC} = -\alpha(t) D_t^{MC} dt - dB_t & \text{if } S_t = s_1 \\ dD_t^{MC} = -\alpha(t) D_t^{MC} dt + dB_t & \text{if } S_t = s_2, \end{cases}$$

which means that, on $[t, \tau_{N_t+1})$, distance to the border processes D^S and D^{MC} have an Ornstein-Uhlenbeck type dynamics and satisfy

$$D_u^S = \begin{cases} e^{-\int_t^u \alpha(s)ds}\left(D_t^S - \int_t^u e^{\int_t^s \alpha(r)dr} dB_s\right) & \text{if } S_t = s_1 \\ e^{-\int_t^u \alpha(s)ds}\left(D_t^S + \int_t^u e^{\int_t^s \alpha(r)dr} dB_s\right) & \text{if } S_t = s_2 \end{cases} \tag{3.15}$$

and

$$D_u^{MC} = \begin{cases} e^{-\int_t^u \alpha(s)ds}\left(D_t^{MC} - \int_t^u e^{\int_t^s \alpha(r)dr} dB_s\right) & \text{if } S_t = s_1 \\ e^{-\int_t^u \alpha(s)ds}\left(D_t^{MC} + \int_t^u e^{\int_t^s \alpha(r)dr} dB_s\right) & \text{if } S_t = s_2. \end{cases} \tag{3.16}$$

According to Revuz-Yor [39], p.181, one can obtain a representation of $\int_t^u e^{\int_t^s \alpha(r)dr} dB_s$ as a time changed standard Brownian motion $W = (W_t, t \geq 0)$ starting from 0 and such that

$$\int_t^u e^{\int_t^s \alpha(r)dr} dB_s = W_{T(t,u)}, \tag{3.17}$$

where

$$T(t,u) = \int_t^u e^{2\int_t^s \alpha(r)dr} ds. \tag{3.18}$$

Therefore, τ_{N_t+1} is finite ($\mathbb{P}$-a.s.) since hitting times of Brownian motion of a fixed level are finite.
■

3.6 Conditional distributions in the simple jump structure model

In this section, conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump in the simple jump structure model will be found, given that the stock price dynamics on $[0, t]$, $t \geq 0$, is observed.

Distribution of the time of the next jump

Theorem 3.6 Suppose that $0 \leq t < u$. Then conditional distribution for the time of the next jump, given the market information $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}\Big(\tau_{N_t+1} \leq u \mid \mathcal{F}_t^P\Big) = \frac{2}{\sqrt{2\pi}} \int_{R_t}^{\infty} \Big[\int_{\frac{\gamma(d_t^S, \tau_{N_t}, t, x)}{\sqrt{T(t,u)}}}^{\infty} e^{-\frac{y^2}{2}} dy\Big] \lambda_\mu e^{-\lambda_\mu (x - R_t)} dx,$$

where d_t^S, $\gamma(d_t^S, \tau_{N_t}, t, x)$ and $T(t, u)$ are defined in (3.12), (3.14) and (3.18), and

$$R_t = \sup_{s \in [\tau_{N_t}, t]} \left(-d_s^S e^{\int_{\tau_{N_t}}^s \alpha(r)dr}\right) = -\inf_{s \in [\tau_{N_t}, t]} \left(d_s^S e^{\int_{\tau_{N_t}}^s \alpha(r)dr}\right).$$

Proof According to (3.5) and (3.12) – (3.14), $d_t^S \in \mathcal{F}_t^P$ and $D_t^S \in \mathcal{F}_t^{P,\mu}$, where

$$\mathcal{F}_t^{P,\mu} = \sigma\Big((P_s, 0 \leq s \leq t), \mu_{N_t}\Big).$$

Therefore,

$$\begin{aligned}
\mathbb{P}\Big(\tau_{N_t+1} \leq u \mid \mathcal{F}_t^P\Big) &= \mathbb{E}^P\Big[\mathbb{E}^P\Big[\mathbb{I}\Big(\tau_{N_t+1} \leq u\Big) \mid \mathcal{F}_t^{P,\mu}\Big] \mid \mathcal{F}_t^P\Big] \\
&= \mathbb{E}^P\Big[\frac{2}{\sqrt{2\pi}} \int_{\frac{D_t^S}{\sqrt{T(t,u)}}}^{\infty} e^{-\frac{y^2}{2}} dy \mid \mathcal{F}_t^P\Big] \\
&= \mathbb{E}^P\Big[\frac{2}{\sqrt{2\pi}} \int_{\frac{\gamma(d_t^S, \tau_{N_t}, t, \mu_{N_t})}{\sqrt{T(t,u)}}}^{\infty} e^{-\frac{y^2}{2}} dy \mid \mathcal{F}_t^P\Big].
\end{aligned}$$

The first equality follows from the law of iterated expectations. The second equality is due to formulas (3.11) and (3.15), time-changed Brownian motion representation (3.17) and the cumulative distribution function for the maximum of Brownian motion (see, e.g., Shreve [43], p.113). Finally, the third equality holds true according to (3.13).

Then the result follows in view of the assumption that μ_{N_t} is an exponential random variable with parameter λ_μ and the fact that the condition

$$\Big[\gamma(d_s^S, \tau_{N_t}, s, \mu_{N_t}) > 0, \forall s \in [\tau_{N_t}, t]\Big]$$

is equivalent to the condition

$$\Big[\mu_{N_t} > R_t\Big].$$

■

Distribution of the next state of the state process

Remark 3.4 Suppose that $t \geq 0$ and $s \in \mathbb{S}$. According to Theorem 3.5 and Step 6 in Definition 3.3, the next state of the state process is equal to the other state ($\mathbb{P}$-a.s.), which means that

$$\mathbb{P}\Big(S_{\tau_{N_t+1}} = s \mid \mathcal{F}_t^P\Big) = 1 - \mathbb{I}(S_t = s).$$

Distribution of the size of the next jump

Remark 3.5 Suppose that $t \geq 0$ and $C \in \mathbb{B}(\mathbb{R})$. In virtue of Theorem 3.5 and Step 5 and Step 7 in Definition 3.3, the distribution of the size of the next jump is given by

$$\mathbb{P}\Big(J_{N_t+1} \in C \mid \mathcal{F}_t^P\Big) = \begin{cases} \int_1^\infty \mathbb{I}\Big(\eta_t(x-1) \in C\Big) f^u(x)dx & \text{if } S_t = s_1 \\ \int_0^1 \mathbb{I}\Big(\eta_t(x-1) \in C\Big) f^l(x)dx & \text{if } S_t = s_2. \end{cases}$$

Recall that $(f^u(x), x \geq 1)$ and $(f^l(x), x \in [0,1])$ are the density functions of random variables $\zeta^u_{N_t}$ and $\zeta^l_{N_t}$.

3.7 Conditional distributions in the Markov chain jump structure model

In this section, conditional distributions for the time of the next jump, the type of the next jump and the size of the next jump in the Markov chain jump structure model will be found, given that the stock price dynamics on $[0, t]$, $t \geq 0$, is observed.

Distribution of the time of the next jump

Theorem 3.7 Suppose that $0 \leq t < u$. Then conditional distribution for the time of the next jump, given the market information $\mathcal{F}_t^P$, is equal to

$$\mathbb{P}\Big(\tau_{N_t+1} \leq u \mid \mathcal{F}_t^P\Big) = \frac{2}{\sqrt{2\pi}} \int_{\frac{D_t^{MC}}{\sqrt{T(t,u)}}}^{\infty} e^{-\frac{y^2}{2}} dy. \tag{3.19}$$

Proof The proof is patterned after Theorem 3.6 by applying formulas (3.11) and (3.16), time-changed Brownian motion representation (3.17) and the cumulative distribution function for the maximum of Brownian motion. ■

Distribution of the type of the next jump

Remark 3.6 Suppose that $t \geq 0$. According to Theorem 3.5 and Step 5 in Definition 3.4

$$\mathbb{P}\Big(S^J_{\tau_{N_t+1}} = s_1^J \mid \mathcal{F}_t^P\Big) = \begin{cases} p_c & \text{if } S_t^J = s_1^J \\ 1 - p_b & \text{if } S_t^J = s_2^J \end{cases}$$

and

$$\mathbb{P}\Big(S^J_{\tau_{N_t+1}} = s_2^J \mid \mathcal{F}_t^P\Big) = \begin{cases} 1 - p_c & \text{if } S_t^J = s_1^J \\ p_b & \text{if } S_t^J = s_2^J. \end{cases}$$

Distribution of the size of the next jump

Remark 3.7 Suppose that $t \geq 0$ and $C \in \mathbb{B}(\mathbb{R})$. In virtue of Theorem 3.5 and Step 3 and Step 5 in Definition 3.4, the distribution of the size of the next jump is given by

$$\begin{aligned} &\mathbb{P}\Big(J_{N_t+1} \in C \mid \mathcal{F}_t^P\Big) \\ &= \begin{cases} p_c \int_0^1 \mathbb{I}\left(\eta_t(x-1) \in C\right) f^l(x)dx + (1-p_c) \int_1^\infty \mathbb{I}\left(\eta_t(x-1) \in C\right) f^u(x)dx & \text{if } S_t^J = s_1^J \\ (1-p_b) \int_0^1 \mathbb{I}\left(\eta_t(x-1) \in C\right) f^l(x)dx + p_b \int_1^\infty \mathbb{I}\left(\eta_t(x-1) \in C\right) f^u(x)dx & \text{if } S_t^J = s_2^J. \end{cases} \end{aligned}$$

4. ESTIMATION OF PARAMETERS

All the parameters can be divided into two groups. The first group is model-specific probabilities, rate parameters and intensities. In the subsequent sections, they will be estimated by assuming some prior distributions and obtaining posterior distributions according to the Bayesian inference approach. All other parameters and parameters of those prior distributions can be calibrated by doing a number of stock price simulations and finding a set of parameter values that fits some historical price dynamics.

4.1 Bayesian inference in the endogenous switching model

Estimation of λ_l

To estimate the rate parameter λ_l, assume it has some prior density $f_{\lambda_l}(\lambda)$ and let

$$N_t^{l,1} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_1\Big)\mathbb{I}\Big(B_{\tau_{i+1}} < h_1(\tau_{i+1})\Big)$$

$$\Big(\text{respectively } N_t^{l,2} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_1\Big)\mathbb{I}\Big(B_{\tau_{i+1}} = h_1(\tau_{i+1})\Big)\Big).$$

be the number of times up to time t when, at τ_i, the system starts from $S_{\tau_i} = s_1$ and then jumps after (respectively before) $\tau_i + T_i^l$.

Set $i_0^{l,1} < 0$ (respectively $i_0^{l,2} < 0$) and, for $j = 1, ..., N_t^{l,1}$ (respectively $j = 1, ..., N_t^{l,2}$), let

$$i_j^{l,1} = \min\Big(i > i_{j-1}^{l,1} : S_{\tau_i} = s_1 \text{ and } B_{\tau_{i+1}} < h_1(\tau_{i+1})\Big)$$

$$\Big(\text{respectively } i_j^{l,2} = \min\Big(i > i_{j-1}^{l,2} : S_{\tau_i} = s_1 \text{ and } B_{\tau_{i+1}} = h_1(\tau_{i+1})\Big)\Big)$$

be the indices of the corresponding jumps.

Therefore, information that is available is the following:

$$h^l(\tau_{i_j^{l,1}+1}; \tau_{i_j^{l,1}} + T^l_{i_j^{l,1}}) = B_{\tau_{i_j^{l,1}+1}},$$

that is, in view of (2.30),

$$T^l_{i_j^{l,1}} = x^l_{i_j^{l,1}} = \tau_{i_j^{l,1}+1} - \tau_{i_j^{l,1}} + \frac{1}{c}\ln\Big(\frac{B_{\tau_{i_j^{l,1}+1}} - h_2(\tau_{i_j^{l,1}+1})}{h_1(\tau_{i_j^{l,1}+1}) - h_2(\tau_{i_j^{l,1}+1})}\Big),$$

and

$$T^l_{i_j^{l,2}} \geq y^l_{i_j^{l,2}} = \tau_{i_j^{l,2}+1} - \tau_{i_j^{l,2}}.$$

Then by Bayes formula the posterior density

$$f_{\lambda_l}(\lambda \mid x^l_{i_1^{l,1}}, ..., x^l_{i^{l,1}_{N_t^{l,1}}}, y^l_{i_1^{l,2}}, ..., y^l_{i^{l,2}_{N_t^{l,2}}}) \propto f_{\lambda_l}(\lambda)\Pi_{j=1}^{N_t^{l,1}}\Big(\lambda\exp(-\lambda x^l_{i_j^{l,1}})\Big)\Pi_{j=1}^{N_t^{l,2}}\exp(-\lambda y^l_{i_j^{l,2}}).$$

Assuming the conjugate prior Gamma$(\lambda; a_l, b_l)$, where

$$\text{Gamma}(\lambda; a_l, b_l) = \frac{{b_l}^{a_l}}{\Gamma(a_l)}\lambda^{a_l-1}e^{-\lambda b_l}, \quad \lambda \geq 0,$$

and $\Gamma(a_l)$ denotes the Gamma function, it can be shown that

$$f_{\lambda_l}(\lambda \mid x^l_{i_1^{l,1}}, ..., x^l_{i^{l,1}_{N_t^{l,1}}}, y^l_{i_1^{l,2}}, ..., y^l_{i^{l,2}_{N_t^{l,2}}}) = \text{Gamma}(\lambda; a_l + N_t^{l,1}, b_l + \sum_{j=1}^{N_t^{l,1}} x^l_{i_j^{l,1}} + \sum_{j=1}^{N_t^{l,2}} y^l_{i_j^{l,2}}).$$

It can be concluded that an increase in one of the values of $x^l_{i_j^{l,1}}$ or $y^l_{i_j^{l,2}}$ leads to a decrease in the posterior mean of λ_l, while an increase in $N_t^{l,1}$, given that the number of observations $N_t^{l,1} + N_t^{l,1}$ and all the values $x^l_{i_j^{l,1}}$ and $y^l_{i_j^{l,2}}$ stay unchanged, causes the opposite effect. Indeed, if it is known that one of the values of T_i^l in the sample is greater than z_1 rather than greater than z_2, or one of the values of T_i^l is equal to z_1 rather than equal to z_2, or one of the values of T_i^l is equal to z_1 rather than greater than z_1, where $z_1 < z_2$, then the posterior mean of the rate parameter λ_l should increase.

Similarly, to estimate the rate parameter λ_u, assume it has some prior density $f_{\lambda_u}(\lambda)$, $\lambda \geq 0$, and let

$$N_t^{u,1} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_3\Big)\mathbb{I}\Big(B_{\tau_{i+1}} > h_2(\tau_{i+1})\Big)$$
$$\Big(\text{respectively } N_t^{u,2} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_3\Big)\mathbb{I}\Big(B_{\tau_{i+1}} = h_2(\tau_{i+1})\Big)\Big).$$

Set $i_0^{u,1} < 0$ (respectively $i_0^{u,2} < 0$) and, for $j = 1, ..., N_t^{u,1}$ (respectively $j = 1, ..., N_t^{u,2}$), let

$$i_j^{u,1} = \min\Big(i > i_{j-1}^{u,1} : S_{\tau_i} = s_3 \text{ and } B_{\tau_{i+1}} > h_2(\tau_{i+1})\Big)$$
$$\Big(\text{respectively } i_j^{u,2} = \min\Big(i > i_{j-1}^{u,2} : S_{\tau_i} = s_3 \text{ and } B_{\tau_{i+1}} = h_2(\tau_{i+1})\Big)\Big).$$

In view of (2.31), information that is available is the following:

$$\begin{aligned} T^u_{i_j^{u,1}} =& x^u_{i_j^{u,1}} = \tau_{i_j^{u,1}+1} - \tau_{i_j^{u,1}} + \frac{1}{c}\ln\Big(\frac{h_1(\tau_{i_j^{u,1}+1}) - B_{\tau_{i_j^{u,1}+1}}}{h_1(\tau_{i_j^{u,1}+1}) - h_2(\tau_{i_j^{u,1}+1})}\Big), \\ T^u_{i_j^{u,2}} \geq& y^u_{i_j^{u,2}} = \tau_{i_j^{u,2}+1} - \tau_{i_j^{u,2}}. \end{aligned}$$

Then by Bayes formula the posterior density

$$f_{\lambda_u}(\lambda \mid x^u_{i_1^{u,1}}, ..., x^u_{i^{l,1}_{N_t^{u,1}}}, y^u_{i_1^{u,2}}, ..., y^u_{i^{u,2}_{N_t^{u,2}}}) \propto f_{\lambda_u}(\lambda)\Pi_{j=1}^{N_t^{u,1}}\Big(\lambda \exp(-\lambda x^u_{i_j^{u,1}})\Big)\Pi_{j=1}^{N_t^{u,2}}\exp(-\lambda y^u_{i_j^{u,2}}).$$

Assuming the conjugate prior Gamma$(\lambda; a_u, b_u)$, it can be obtained that

$$f_{\lambda_u}(\lambda \mid x^u_{i_1^{u,1}}, ..., x^u_{i^{l,1}_{N_t^{u,1}}}, y^u_{i_1^{u,2}}, ..., y^u_{i^{u,2}_{N_t^{u,2}}}) = \text{Gamma}(\lambda; a_u + N_t^{u,1}, b_u + \sum_{j=1}^{N_t^{u,1}} x^u_{i_j^{u,1}} + \sum_{j=1}^{N_t^{u,2}} y^u_{i_j^{u,2}}).$$

Similar to the analysis of the posterior distribution for λ_l, an increase in one of the values of $x^u_{i_j^{u,1}}$ or $y^u_{i_j^{u,2}}$ leads to a decrease in the posterior mean of λ_u and an increase in $N_t^{u,1}$, given that the number of observations $N_t^{u,1} + N_t^{u,1}$ and all the values $x^u_{i_j^{u,1}}$ and $y^u_{i_j^{u,2}}$ stay unchanged, does the opposite.

Estimation of p_{lu}

To estimate the sunspot probability p_{lu}, assume it has some prior density $f_{lu}(p)$ and let

$$N_t^{l,3} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_1\Big)\mathbb{I}\Big(B_{\tau_{i+1}} < h_1(\tau_{i+1})\Big)\mathbb{I}\Big(S_{\tau_{i+1}} = s_3\Big)$$

$$\Big(\text{respectively } N_t^{l,4} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_1\Big)\mathbb{I}\Big(B_{\tau_{i+1}} < h_1(\tau_{i+1})\Big)\mathbb{I}\Big(S_{\tau_{i+1}} = s_2\Big)\Big)$$

be the number of times up to time t when, at τ_i, the system starts from $S_{\tau_i} = s_1$ and then jumps after $\tau_i + T_i^l$ to state s_3 (respectively s_2).

Then by Bayes formula the posterior density

$$f_{lu}(p \mid N_t^{l,3}, N_t^{l,4}) \propto f_{lu}(p)p^{N_t^{l,3}}(1-p)^{N_t^{l,4}}.$$

Assuming the conjugate prior $\mathrm{B}(p; x_1, y_1)$, where

$$\mathrm{B}(p; x_1, y_1) = \frac{p^{x_1-1}(1-p)^{y_1-1}}{B(x_1, y_1)}$$

and $B(x_1, y_1)$ denotes the Beta function, it follows that

$$f_{lu}(p \mid N_t^{l,3}, N_t^{l,4}) = \mathrm{B}(p; x_1 + N_t^{l,3}, y_1 + N_t^{l,4}).$$

It can be concluded that an increase in $N_t^{l,3}$, given that the number of observations $N_t^{l,3}+N_t^{l,4}$ stays unchanged, leads to an increase in the posterior mean of p_{lu}. Indeed, the greater the proportion of times when Bernoulli random variable is equal to 1, the greater the posterior mean of that probability to be equal to 1.

Estimation of p_{ul}

Similarly, to estimate the sunspot probability p_{ul}, assume it has some prior density $f_{ul}(p)$ and let

$$N_t^{u,3} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_3\Big)\mathbb{I}\Big(B_{\tau_{i+1}} > h_2(\tau_{i+1})\Big)\mathbb{I}\Big(S_{\tau_{i+1}} = s_1\Big)$$

$$\Big(\text{respectively } N_t^{u,4} = \sum_{i=0}^{N_t-1} \mathbb{I}\Big(S_{\tau_i} = s_3\Big)\mathbb{I}\Big(B_{\tau_{i+1}} > h_2(\tau_{i+1})\Big)\mathbb{I}\Big(S_{\tau_{i+1}} = s_2\Big)\Big).$$

Then by Bayes formula the posterior density

$$f_{ul}(p \mid N_t^{u,3}, N_t^{u,4}) \propto f_{ul}(p) p^{N_t^{u,3}} (1-p)^{N_t^{u,4}}.$$

Assuming the conjugate prior $\mathrm{B}(p; x_2, y_2)$, it can be shown that the posterior density

$$f_{ul}(p \mid N_t^{u,3}, N_t^{u,4}) = \mathrm{B}(p; x_2 + N_t^{u,3}, y_2 + N_t^{u,4}).$$

Similar to the analysis of p_{lu}, an increase in $N_t^{u,3}$, given that the number of observations $N_t^{u,3} + N_t^{u,4}$ stays unchanged, leads to an increase in the posterior mean of p_{ul}.

4.2 Bayesian inference in the exogenous shocks model

To estimate intensity λ_Z, assume it has some prior density $f_{\lambda_Z}(\lambda)$. According to Remark 2.8, at time $t \in [T_0, T)$, the Brownian motion past dynamics $(B_s,\ T_0 \leq s \leq t)$ is known, and the total number of exogenous shocks when the system admitted multiple equilibria is equal to

$$N_t^Z = \sum_{i \geq 1} \mathbb{I}\Big(\tau_i \leq t\Big) \mathbb{I}\Big(h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i)\Big).$$

The question is how the posterior distribution of λ_Z can be found, based on the information contained in the sigma-algebra

$$\mathcal{F}_t^{N^Z,B} = \sigma\{(B_s, N_s^Z), T_0 \leq s \leq t\}.$$

Denote by

$$\hat{\mathcal{F}}_t^{N^Z,B} = \mathcal{F}_t^{N^Z} \vee \mathcal{F}_\infty^B,$$

where $\mathcal{F}_\infty^B = \sigma(B_s, s \geq T_0)$ and $\mathcal{F}_t^{N^Z} = \sigma(N_s^Z, s \in [T_0, t])$, hence, $\mathcal{F}_t^{N^Z,B} \subset \hat{\mathcal{F}}_t^{N^Z,B}$. In Theorem 4.1, it will be shown that the process $A_t^Z = \lambda_Z \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big) ds$ is the compensator in the Doob-Meyer decomposition for $(N_t^Z, \hat{\mathcal{F}}_t^{N^Z,B})$, $t \in [T_0, T)$.

To compute the posterior distribution of λ_Z, the method of the reference probability described in Chapter VI in Bremaud [8] is applied. According to this method, a reference probability $\mathbb{Q}$ can be obtained by an absolutely continuous change of measure with the corresponding Radon-Nikodym derivative given by

$$L_t = \frac{d\mathbb{P}_t}{d\mathbb{Q}_t} = e^{\lambda_Z \int_{T_0}^t (1 - \mathbb{I}(h_2(s) < B_s < h_1(s))) ds},$$

where, for each $t \in [T_0, T)$, $\mathbb{P}_t$ and $\mathbb{Q}_t$ are the restrictions of $\mathbb{P}$ and $\mathbb{Q}$ respectively to $(\Omega, \mathcal{F}_t^{N^Z,B})$. By the results of Chapter VI in Bremaud [8], under the probability measure $\mathbb{Q}$, process N_t^Z is a Poisson process with intensity λ_Z and it is independent of Brownian motion B_t. For any Borel-measurable and bounded function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}^{\mathbb{P}}\Big(f(\lambda_Z) \mid \mathcal{F}_t^{N^Z,B}\Big) = \mathbb{E}^{\mathbb{Q}}\Big(L_t f(\lambda_Z) \mid \mathcal{F}_t^{N^Z,B}\Big) = L_t \mathbb{E}^{\mathbb{Q}}\Big(f(\lambda_Z) \mid N_t^Z\Big),$$

hence, it is required to calculate the posterior distribution of λ_Z based on the values of N_t^Z and $\int_{T_0}^t \mathbb{I}(h_2(s) < B_s < h_1(s))ds$, and it can be implemented by applying Bayes formula.

Theorem 4.1 Process $A_t^Z = \lambda_Z \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds$ is the compensator in the Doob-Meyer decomposition for $(N_t^Z, \hat{\mathcal{F}}_t^{N^Z,B})$, $t \in [T_0, T)$.

Proof First, since the expected total number of exogenous shocks on $[T_0, t]$ is equal to $\lambda_Z(t - T_0)$ and $0 \leq \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big) \leq 1$, for $s \in [T_0, t]$, it can be concluded that

$$\mathbb{E}^{\mathbb{P}} \mid N_t^Z - A_t^Z \mid \leq \mathbb{E}^{\mathbb{P}} N_t^Z + \mathbb{E}^{\mathbb{P}} A_t^Z \leq \lambda_Z(t - T_0) + \lambda_Z(t - T_0) < \infty.$$

Suppose that $s \in [T_0, t]$. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}}\Big(N_t^Z - A_t^Z \mid \hat{\mathcal{F}}_s^{N^Z,B}\Big) =& N_s^Z - A_s^Z + \mathbb{E}^{\mathbb{P}}\Big(\sum_{i\geq 1} \mathbb{I}\Big(s < \tau_i \leq t\Big)\mathbb{I}\Big(h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i)\Big) \mid \hat{\mathcal{F}}_s^{N^Z,B}\Big) \\ &- \lambda_Z \int_s^t \mathbb{I}\Big(h_2(r) < B_r < h_1(r)\Big)dr\end{aligned}$$

Pursuant to the monotone convergence theorem and the law of iterated expectations,

$$\begin{aligned}&\mathbb{E}^{\mathbb{P}}\Big(\sum_{i\geq 1} \mathbb{I}\Big(s < \tau_i \leq t\Big)\mathbb{I}\Big(h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i)\Big) \mid \hat{\mathcal{F}}_s^{N^Z,B}\Big) \\ &= \sum_{i\geq 1} \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big(s < \tau_i \leq t\Big)\mathbb{I}\Big(h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i)\Big) \mid \hat{\mathcal{F}}_s^{N^Z,B}\Big) \\ &= \sum_{i\geq 1} \mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big(s < \tau_i \leq t\Big)\mathbb{I}\Big(h_2(\tau_i) < B_{\tau_i} < h_1(\tau_i)\Big) \mid \tau_i, \hat{\mathcal{F}}_s^{N^Z,B}\Big) \mid \hat{\mathcal{F}}_s^{N^Z,B}\Big) \\ &= \sum_{i\geq 1} \int_0^{t-s} \mathbb{I}\Big(h_2(s+r) < B_{s+r} < h_1(s+r)\Big)\frac{\lambda_Z^i r^{i-1} e^{-\lambda_Z r}}{(i-1)!}dr \\ &= \int_0^{t-s} \mathbb{I}\Big(h_2(s+r) < B_{s+r} < h_1(s+r)\Big)\sum_{i\geq 1} \frac{\lambda_Z^i r^{i-1} e^{-\lambda_Z r}}{(i-1)!}dr \\ &= \lambda_Z \int_s^t \mathbb{I}\Big(h_2(r) < B_r < h_1(r)\Big)dr,\end{aligned}$$

and the martingale property holds true. ■

Bayes formula and independence of N^Z and B yield

$$\begin{aligned}
&\mathbb{P}\Big(\lambda_Z \in d\Lambda \mid N_t^Z = n, \int_{T_0}^t \mathbb{I}(h_2(s) < B_s < h_1(s))ds = x\Big)\\
&\propto \mathbb{P}\Big(N_t^Z = n, \int_{T_0}^t \mathbb{I}(h_2(s) < B_s < h_1(s))ds \in dx \mid \lambda_Z \in d\Lambda\Big)\mathbb{P}\Big(\lambda_Z \in d\Lambda\Big)\\
&\propto \mathbb{E}^{\mathbb{Q}}\Big(L_t \mathbb{I}(\int_{T_0}^t \mathbb{I}(h_2(s) < B_s < h_1(s))ds \in dx)\mathbb{I}(N_t^Z = n) \mid \lambda_Z \in d\Lambda\Big)\mathbb{P}(\lambda_Z \in d\Lambda)\\
&\propto e^{\Lambda(t-T_0)-\Lambda x} e^{-\Lambda(t-T_0)}\Lambda^n \mathbb{P}(\lambda_Z \in d\Lambda)\\
&\propto e^{-\Lambda x}\Lambda^n \mathbb{P}(\lambda_Z \in d\Lambda).
\end{aligned}$$

For the rate parameter λ_Z, it is assumed that the conjugate prior is given by Gamma$(\lambda; a, b)$, hence, the posterior density is equal to

$$f_{\lambda_Z}\Big(\lambda \mid N_t^Z, \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds\Big) = \text{Gamma}\Big(\lambda; a+N_t^Z, b+\int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds\Big).$$

An increase in N_t^Z, given that $\int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds$ stays unchanged, leads to an increase in the posterior mean of λ_Z, while an increase in $\int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds$ given N_t^Z stays unchanged does the opposite. It can be concluded that this posterior density coincides with the one obtained for a standard Poisson process taking value N_t^Z at time $\int_{T_0}^t \mathbb{I}(h_2(s) < B_s < h_1(s))ds$. The value of this integral is equal to the total amount of time Brownian motion spends in the interval where the system admits multiple equilibria since when the Brownian motion is outside this interval, new shocks can not be detected.

4.3 Bayesian inference in the stochastic number of dynamic hedgers model

Estimation of λ_Z

To estimate λ_Z, assume that it has some prior density $f_{\lambda_Z}(\lambda)$ and count the total number of stock price jumps caused by Poisson process Z_t:

$$N_t^Z = \sum_{i=1}^{N_t}\Big[\mathbb{I}\Big(\Delta P_{\tau_i} > 0\Big)\mathbb{I}\Big(H_1(\tau_i, w_{\tau_i-}^D) \neq B_{\tau_i}\Big) + \mathbb{I}\Big(\Delta P_{\tau_i} < 0\Big)\mathbb{I}\Big(H_2(\tau_i, w_{\tau_i-}^D) \neq B_{\tau_i}\Big)\Big]$$

Then by Bayes formula the posterior density

$$f_{\lambda_Z}(\lambda \mid N_t^Z) \propto f_{\lambda_Z}(\lambda) e^{-\lambda(t-T_0)} \lambda^{N_t^Z}.$$

Assuming the conjugate prior Gamma$(\lambda; a, b)$, it can be obtained that

$$f_{\lambda_Z}(\lambda \mid N_t^Z) = \text{Gamma}(\lambda; a + N_t^Z, b + (t - T_0)).$$

An increase in N_t^Z, given that $t - T_0$ stays unchanged, leads to an increase in the posterior mean of λ_Z, while an increase in $t - T_0$ given N_t^Z stays unchanged does the opposite.

Estimation of p_l

To estimate the probability p_l, assume it has some prior density $f_{p_l}(p)$ and let

$$N_t^l = \sum_{i=1}^{N_t} \mathbb{I}\Big(S_{\tau_{i-1}} = s_2\Big) \mathbb{I}\Big(\tau_i < \hat{\tau}_{i-1}\Big) \mathbb{I}\Big(H_2(\tau_i, w_{\tau_i}^D) < B_{\tau_i} < H_1(\tau_i, w_{\tau_i}^D)\Big) \mathbb{I}\Big(S_{\tau_i} = s_3\Big)$$

$$\left(\text{respectively } N_t^u = \sum_{i=1}^{N_t} \mathbb{I}\Big(S_{\tau_{i-1}} = s_2\Big) \mathbb{I}\Big(\tau_i < \hat{\tau}_{i-1}\Big) \mathbb{I}\Big(H_2(\tau_i, w_{\tau_i}^D) < B_{\tau_i} < H_1(\tau_i, w_{\tau_i}^D)\Big) \mathbb{I}\Big(S_{\tau_i} = s_1\Big)\right)$$

denote the total number of observable values of $(\xi_i, i = 1, 2, ...)$ such that $\xi_i = s_1$ (respectively $\xi_i = s_3$). Values of ξ_i can be observed if and only if the number of dynamic hedgers changes when the state process is in the state s_2 and $H_2(\tau_i, w_{\tau_i}^D) < B_{\tau_i} < H_1(\tau_i, w_{\tau_i}^D)$.
Then by Bayes formula the posterior density

$$f_{p_l}(p \mid N_t^l, N_t^u) \propto f_{p_l}(p) p^{N_t^l} (1-p)^{N_t^u}.$$

Assuming the conjugate prior B$(p; a, b)$, it can be concluded that

$$f_{p_l}(p \mid N_t^l, N_t^u) = \text{B}(p; a + N_t^l, b + N_t^u).$$

An increase in N_t^l, given that the number of observations $N_t^l + N_t^u$ stays unchanged, leads to an increase in the posterior mean of p_l, while an increase in N_t^u, given that the number of observations $N_t^l + N_t^u$ stays unchanged, does the opposite.

4.4 Bayesian inference in the simple jump structure model

To estimate the rate parameter λ_μ, assume it has some prior density $f_{\lambda_\mu}(\mu)$, $\mu \geq 0$. Based on the information $\mathcal{F}_t^P$, $t \geq 0$,

$$\mu_i = -d^S_{\tau_{i+1}} e^{\int_{\tau_i}^{\tau_{i+1}} \alpha(r)dr}, \quad i = 1, ..., N_t - 1,$$

can be calculated.

Then by Bayes formula the posterior density

$$f_{\lambda_\mu}\Big(\lambda \mid \mu_1, ..., \mu_{N_t-1}\Big) \propto \lambda^{N_t-1} e^{-\lambda \sum_{j=1}^{N_t-1} \mu_j} f_{\lambda_\mu}(\lambda)$$

Assuming the conjugate prior Gamma$(\lambda; a, b)$, it can be shown that

$$f_{\lambda_\mu}\Big(\lambda \mid \mu_1, ..., \mu_{N_t-1}\Big) = \text{Gamma}\Big(\lambda; a + (N_t - 1), b + \sum_{j=1}^{N_t-1} \mu_j\Big).$$

An increase in one of the values of μ_j causes an increase in the posterior mean of λ_μ.

4.5 Bayesian inference in the Markov chain jump structure model

To estimate probabilities p_c and p_b, assume they have some prior densities $f_{p_c}(p)$ and $f_{p_b}(p)$, $p \in [0, 1]$. Based on the information $\mathcal{F}_t^P$, $t \geq 0$,

$$s_c = \sum_{i=1}^{N^c_{\tau_{N_t-1}}} X_i^c \quad \text{and} \quad f_c = N^c_{\tau_{N_t-1}} - s_c,$$

can be calculated, where

$$l_0 = 0, \quad l_i = \min(i > l_{i-1} : J_i < 0), i = 1, 2, ..., N^c_{\tau_{N_t-1}}, \quad X_i^c = \begin{cases} 1, & \text{if } J_{l_i+1} < 0 \\ 0, & \text{if } J_{l_i+1} > 0, \end{cases}$$

and

$$s_b = \sum_{i=1}^{N^b_{\tau_{N_t-1}}} X_i^b \quad \text{and} \quad f_b = N^b_{\tau_{N_t-1}} - s_b,$$

where

$$k_0 = 0, \quad k_i = \min(i > k_{i-1} : J_i > 0), i = 1, 2, ..., N^b_{\tau_{N_t-1}}, \quad X_i^b = \begin{cases} 1, & \text{if } J_{k_i+1} > 0 \\ 0, & \text{if } J_{k_i+1} < 0. \end{cases}$$

Then s_c is the number of successes and f_c is the number of fails in the sample of a random variable which is a Bernoulli trial with unknown probability of success p_c and s_b is the number of successes and f_b is the number of fails in the sample of a random variable which is a Bernoulli trial with unknown probability of success p_b.

By Bayes formula, the posterior densities

$$f_{p_c}(p \mid s_c, f_c) = p^{s_c}(1-p)^{f_c} f_{p_c}(p) \quad \text{and} \quad f_{p_b}(p \mid s_b, f_b) = p^{s_b}(1-p)^{f_b} f_{p_b}(p).$$

Assuming conjugate priors $\mathrm{B}(p; a_1, b_1)$ and $\mathrm{B}(p; a_2, b_2)$, it can be shown that

$$f_{p_c}(p \mid s_c, f_c) = B(p; a_1 + s_c, b_1 + f_c) \quad \text{and} \quad f_{p_b}(p \mid s_b, f_b) = B(p; a_2 + s_b, b_2 + f_b).$$

It can be concluded that an increase in s_c given that the number of observations $s_c + f_c$ stays unchanged leads to an increase in the posterior mean of p_c. Similarly, an increase in s_b given that the number of observations $s_b + f_b$ stays unchanged causes an increase in the posterior mean of p_b.

5. NUMERICAL STUDIES

In this chapter, a number of numerical studies are conducted in C/C++ and MATLAB. Numerical techniques to find conditional probabilities discussed in Chapter 2 and Chapter 3 will be demonstrated by the example of the time of the next jump. Conditional probabilities of the type of the next jump and the size of the next jump can be computed applying similar numerical algorithms.

5.1 *Market microstructure models*

5.1.1 *A numerical algorithm for the endogenous switching model*

Owing to the results of Corollary 2.1 and Sections 4.1.1 and 4.1.2, it can be concluded that the conditional probability of the time of the next jump is equal to

$$\begin{cases} 1 - F_{51}(t, \tau_{N_t} + R_t^l, B_t, u, a_l + N_t^{l,1}, b_l + \sum_{j=1}^{N_t^{l,1}} x^l_{i_j^{l,1}} + \sum_{j=1}^{N_t^{l,2}} y^l_{i_j^{l,2}}) & \text{if } S_t = s_1 \\ 1 - D_m(u, t, B_t) & \text{if } S_t = s_2 \\ 1 - F_{52}(t, \tau_{N_t} + R_t^u, B_t, u, a_u + N_t^{u,1}, b_u + \sum_{j=1}^{N_t^{u,1}} x^l_{i_j^{u,1}} + \sum_{j=1}^{N_t^{u,2}} y^u_{i_j^{u,2}}) & \text{if } S_t = s_3, \end{cases}$$

where

$$F_{51}(t, z, y, u, a, b) = \int_0^\infty \left(\int_0^\infty D^l(u, z + x, t, y) \lambda e^{-\lambda x} dx \right) \text{Gamma}(\lambda; a, b) d\lambda$$

and

$$F_{52}(t, z, y, u, a, b) = \int_0^\infty \left(\int_0^\infty D^u(u, z + x, t, y) \lambda e^{-\lambda x} dx \right) \text{Gamma}(\lambda; a, b) d\lambda.$$

In Sections 5.1.4 and 5.1.5, numerical algorithms to compute corresponding probabilities D^l, D_m and D^u will be discussed. Conditional probabilities F_{51} and F_{52} can be numerically approximated by applying Gauss-Laguerre formula (see, e.g., Abramowitz and Stegun [1]).

Owing to the results of Corollary 2.4 and Section 4.2, it can be concluded that the conditional probability of the time of the next jump is equal to

$$\begin{cases} F_{53}(t, B_t, u, a + N_t^Z, b + \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds) & \text{if } S_t = s_1 \\ F_{54}(t, B_t, u, a + N_t^Z, b + \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds) & \text{if } S_t = s_2 \\ F_{55}(t, B_t, u, a + N_t^Z, b + \int_{T_0}^t \mathbb{I}\Big(h_2(s) < B_s < h_1(s)\Big)ds) & \text{if } S_t = s_3, \end{cases}$$

where $F_{53}(t, y, u, a, b)$ satisfies

$$F_{53}(t, y, u, a, b) = \Big(1 - D_1(u, t, y)\Big) \int_0^\infty e^{-\lambda(u-t)} \text{Gamma}(\lambda; a, b) d\lambda + \int_0^{u-t} \Big[\Big(1 - D_1(t + r, t, y)\Big) + \Phi_1(t + r, t, y) + \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, y) F_{53}(t + r, x, u) dx\Big] \Big[\int_0^\infty \lambda e^{-\lambda r} \text{Gamma}(\lambda; a, b) d\lambda\Big] dr,$$

$$F_{54}(t, y, u, a, b) = 1 - D_m(u, t, y) \int_0^\infty e^{-\lambda(u-t)} \text{Gamma}(\lambda; a, b) d\lambda \tag{5.1}$$

and $F_{55}(t, y, u, a, b)$ satisfies

$$F_{55}(t, y, u, a, b) = \Big(1 - D_2(u, t, y)\Big) \int_0^\infty e^{-\lambda(u-t)} \text{Gamma}(\lambda; a, b) d\lambda + \int_0^{u-t} \Big[\Big(1 - D_2(t + r, t, y)\Big) + \Phi_2(t + r, t, y) + \int_{h_1(t+r)}^{\infty} q_2(x; r, t, y) F_{55}(t + r, x, u) dx\Big] \Big[\int_0^\infty \lambda e^{-\lambda r} \text{Gamma}(\lambda; a, b) d\lambda\Big] dr,$$

The value of F_{53} can be approximated by finding F_{56}, where

$$\begin{aligned} F_{56}(t_i, y_m, t_{n_1}, a, b) &= \Big(1 - D_1(t_{n_1}, t_i, y_m)\Big) \int_0^\infty e^{-\lambda(t_{n_1}-t_i)} \text{Gamma}(\lambda; a, b) d\lambda + \\ &+ \Delta_1 \times \sum_{j=i+1}^{n_1} \Big(\int_0^\infty \lambda e^{-\lambda(t_j - t_i)} \text{Gamma}(\lambda; a, b) d\lambda \times \Big[\Big(1 - D_1(t_j, t_i, y_m)\Big) + \Phi_1(t_j, t_i, y_m) + \\ &+ \sum_{k=1}^{k_j} \mathbb{P}\Big(y_{k-1} - y_m < B_{t_j - t_i} \le y_k - y_m \mid B_s < h_1(t_i + s), \forall s \in [0, t_j - t_i]\Big) F_{56}(t_j, y_k, t_{n_1}) dx\Big]\Big), \end{aligned} \tag{5.2}$$

boundary condition is

$$F_{56}(t_{n_1}, y_m, t_{n_1}, a, b) = 0 \quad \text{for} \quad m = 0, 1, ..., k_{n_1},$$

$$k_j = \max\Big(0 \le k \le n_2 : y_k \le h_2(t_j)\Big), \quad j = 1, 2, ..., n_1,$$

and a mesh with uniform spacing is given by

$$t_i = t + i\Delta_1, i = 0, 1, ..., n_1, \quad \text{and} \quad y_m = C_1 + m\Delta_2, m = 0, 1, ..., n_2,$$

with

$$\Delta_1 = \frac{u-t}{n_1},\ n_1 \geq 1, \quad \text{and} \quad \Delta_2 = \frac{C_2 - C_1}{n_2},\ n_2 \geq 1.$$

Constants C_1 and C_2 are taken such that

$$\mathbb{P}(\min_{s\in[0,u-t]} B_s \leq C_1) = \mathbb{P}(\max_{s\in[0,u-t]} B_s \geq -C_1) = 2\Phi\Big(\frac{C_1}{\sqrt{u-t}}\Big) = \epsilon \tag{5.3}$$

for some small $\epsilon > 0$ and

$$C_2 \geq \max_{s\in[0,u-t]} h_1(t+s).$$

The value F_{56} can be computed applying backward induction to $i = 1, ..., n_1$ and Gauss-Laguerre formula for

$$\int_0^\infty e^{-\lambda(t_{n_1}-t_i)}\text{Gamma}(\lambda; a, b)d\lambda$$

and

$$\int_0^\infty \lambda e^{-\lambda(t_j-t_i)}\text{Gamma}(\lambda; a, b)d\lambda, \quad j = i+1, ..., n_1,$$

and F_{54} can be approximated by applying Gauss-Laguerre formula for

$$\int_0^\infty e^{-\lambda(u-t)}\text{Gamma}(\lambda; a, b)d\lambda.$$

Finally, F_{55} can be computed according to exactly the same procedure as the one applied for F_{53}, therefore, the details are omitted here.

In Sections 5.1.4 and 5.1.5, numerical algorithms to approximate corresponding Brownian motion probabilities in formulas (5.1) and (5.2) will be discussed.

5.1.3 A numerical algorithm for the stochastic number of dynamic hedgers model

Owing to the results of Corollary 2.8 and Section 4.3.1, it can be concluded that the conditional probability of the time of the next jump is equal to

$$\begin{cases} 1 - \bar{D}_1(u, t, B_t, w_t^D)\int_0^\infty e^{-\lambda(u-t)}\text{Gamma}(\lambda; a + N_t^Z, b + (t - T_0))d\lambda & \text{if } S_t = s_1 \\ F_{57}(u, t, B_t, w_t^D, a + N_t^Z, b + (t - T_0)) & \text{if } S_t = s_2 \\ 1 - \bar{D}_2(u, t, B_t, w_t^D)\int_0^\infty e^{-\lambda(u-t)}\text{Gamma}(\lambda; a + N_t^Z, b + (t - T_0))d\lambda & \text{if } S_t = s_3, \end{cases}$$

where

$$
\begin{aligned}
&F_{57}(u,t,y,w_t^D,a,b)\\
&=\Big(1-\int_0^\infty e^{-\lambda(u-t)}\mathrm{Gamma}(\lambda;a,b)d\lambda\Big)+\mathbb{I}\Big(T^D(w_t^D)<u\Big)\int_0^\infty e^{-\lambda(u-t)}\mathrm{Gamma}(\lambda;a,b)d\lambda\times\\
&\times\Big[\int_{-\infty}^{H(T^D(w_t^D),w_t^D,\kappa e^{-r(T-T^D(w_t^D))})}\frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}}e^{-\frac{(x-y)^2}{2(T^D(w_t^D)-t)}}(1-\bar{D}_1(u,T^D(w_t^D),x,w_t^D))dx\\
&\quad+\int_{H(T^D(w_t^D),w_t^D,\kappa e^{-r(T-T^D(w_t^D))})}^{\infty}\frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}}e^{-\frac{(x-y)^2}{2(T^D(w_t^D)-t)}}(1-\bar{D}_2(u,T^D(w_t^D),x,w_t^D))dx\Big].
\end{aligned}
$$

On the sets $[S_t=s_1]$ and $[S_t=s_3]$, this conditional probability can be numerically approximated applying Gauss-Laguerre formula for

$$\int_0^\infty e^{-\lambda(u-t)}\mathrm{Gamma}(\lambda;a+N_t^Z,b+(t-T_0))d\lambda$$

and

$$\int_0^\infty e^{-\lambda(u-t)}\mathrm{Gamma}(\lambda;a+N_t^Z,b+(t-T_0))d\lambda.$$

On the set $[S_t=s_2]$, one can apply Gauss-Laguerre formula

$$\int_0^\infty e^{-\lambda(u-t)}\mathrm{Gamma}(\lambda;a,b)d\lambda,$$

replace

$$\int_{-\infty}^{H(T^D(w_t^D),w_t^D,\kappa e^{-r(T-T^D(w_t^D))})}\frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}}e^{-\frac{(x-y)^2}{2(T^D(w_t^D)-t)}}(1-\bar{D}_1(u,T^D(w_t^D),x,w_t^D))dx$$

by

$$
\begin{aligned}
&\sum_{i=1}^n\frac{(1-\bar{D}_1(u,T^D(w_t^D),x_{i-1},w_t^D))+(1-\bar{D}_1(u,T^D(w_t^D),x_i,w_t^D))}{2}\times\\
&\qquad\times\int_{x_{i-1}}^{x_i}\frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}}e^{-\frac{(x-y)^2}{2(T^D(w_t^D)-t)}}dx\\
&=\sum_{i=1}^n\frac{(1-\bar{D}_1(u,T^D(w_t^D),x_{i-1},w_t^D))+(1-\bar{D}_1(u,T^D(w_t^D),x_i,w_t^D))}{2}\times\\
&\qquad\times\Big(\Phi\Big(\frac{x_i-y}{\sqrt{T^D(w_t^D)-t}}\Big)-\Phi\Big(\frac{x_{i-1}-y}{\sqrt{T^D(w_t^D)-t}}\Big)\Big),
\end{aligned}
$$

where

$$y+C_1=x_0<x_1<...<x_n=H(T^D(w_t^D),w_t^D,\kappa e^{-r(T-T^D(w_t^D))})$$

is an equally spaced grid on $[y+C_1, H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})]$ with a constant C_1 defined according to (5.3), and then, similarly, replace

$$\int_{H(T^D(w_t^D),w_t^D,\kappa e^{-r(T-T^D(w_t^D))})}^{\infty} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-y)^2}{2(T^D(w_t^D)-t)}} (1-\bar{D}_2(u, T^D(w_t^D), x, w_t^D))dx$$

by

$$\sum_{i=1}^{n} \frac{(1-\bar{D}_2(u,T^D(w_t^D),x_{i-1},w_t^D)) + (1-\bar{D}_2(u,T^D(w_t^D),x_i,w_t^D))}{2} \times$$
$$\times \left(\Phi\Big(\frac{x_i - y}{\sqrt{T^D(w_t^D)-t}}\Big) - \Phi\Big(\frac{x_{i-1} - y}{\sqrt{T^D(w_t^D)-t}}\Big)\right),$$

where

$$H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))}) = x_0 < x_1 < ... < x_n = y - C_1$$

is an equally spaced grid on $[H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))}), y - C_1]$.
In Section 5.1.5, numerical algorithms to compute corresponding probabilities $\bar{D}_1$ and $\bar{D}_2$ will be discussed.

5.1.4 *Examples of numerical techniques to calculate Brownian motion hitting probabilities and densities for two-sided curved boundaries*

In this section, the application of the numerical techniques developed by Skorohod [44], Novikov et al. [32], Poetzelberger and Wang [37] and Buonocore et al. [12] to calculating Brownian motion hitting probabilities

$$\mathbb{P}\Big(\tau > u, B_u \leq K\Big), \quad u \in [0,T], \tag{5.4}$$

and

$$\mathbb{P}\Big(\tau < u, B_\tau = f(\tau)\Big), \quad u \in [0,T], \tag{5.5}$$

will be discussed, where

$$\tau = \inf\Big(t \geq 0 : B_t = f(t) \text{ or } B_t = g(t)\Big),$$

deterministic functions f and g are in the class $C^2\Big([0,u]\Big)$ and satisfy $f(t) < g(t)$, $\forall t \in [0,u]$, and constant K is such that $f(u) \leq K \leq g(u)$.

To compute

$$\mathbb{P}\Big(\tau > u, K_1 \le B_u \le K_2\Big), \quad u \in [0,T],$$

it can be used that

$$\mathbb{P}\Big(\tau > u, K_1 \le B_u \le K_2\Big) = \mathbb{P}\Big(\tau > u, B_u \le K_2\Big) - \mathbb{P}\Big(\tau > u, B_u \le K_1\Big). \tag{5.6}$$

Based on these results applied for $K = g(u)$ and Brownian motions B and $-B$, the values of D_m, $D_{m,1}$ and $D_{m,2}$ can be derived. Values of q^m, ϕ_m, $\phi_{m,1}$ and $\phi_{m,2}$ can be calculated according to a rectangle rule. Note that other numerical methods can be applied as well.

PDE approach

According to Skorohod [44],

$$\mathbb{P}\Big(\tau > u, B_u \le K\Big) = v_1(0,0)$$

and

$$\mathbb{P}\Big(\tau < u, B_\tau = f(\tau)\Big) = v_2(0,0),$$

where, for $0 < t < u$ and $f(t) < x < g(t)$, functions $v_1(t,x)$ and $v_2(t,x)$ solve the backward linear heat equation

$$\frac{\partial v_i}{\partial t} + \frac{1}{2}\frac{\partial^2 v_i}{\partial x^2} = 0, \quad i = 1,2,$$

with corresponding boundary conditions

$$v_1(t, f(t)) = 0, \quad v_1(t, g(t)) = 0, \quad v_1(u,x) = \mathbb{I}\Big(x \le K\Big)$$

and

$$v_2(t, f(t)) = 1, \quad v_2(t, g(t)) = 0, \quad v_2(u,x) = 0.$$

To find $v_1(0,0)$ and $v_2(0,0)$, one can use 3-sigma and rectangle rules approximating function H from formula (2.5) with

$$\frac{\gamma_1 x - z \times \Delta \times \sum_{i=1}^{n} \Phi\Big(\frac{x - K_i e^{-r(T-t)}}{\Sigma(t)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K_i-\kappa)^2}{2\sigma_\kappa^2}} - \gamma_2}{\gamma_3} \tag{5.7}$$

where $\kappa - 3\sigma_\kappa = K_0 < K_1 < ... < K_n = \kappa + 3\sigma_\kappa$ and $\Delta = K_{i+1} - K_i$, $i = 0, 1, ..., n$, and then apply Crank-Nicolson finite difference method which is used for numerically solving the heat equation (see, e.g., Thomas [45] and Wilmott et al. [47]).

Approximation by piecewise linear boundaries

In this section, an alternative to PDE approach to evaluate probabilities and densities that correspond to formula (5.4) is considered.

Let $\hat{f}(t)$ and $\hat{g}(t)$ be piecewise linear approximations for $f(t)$ and $g(t)$ on the interval $[0, u]$, with nodes t_i, $t_0 = 0 < t_1 < t_2 < ... < t_n = u$, $\Delta t_i = t_{i+1} - t_i$, such that $\hat{f}(t_i) = f(t_i)$ and $\hat{g}(t_i) = g(t_i)$. Then Novikov et al. [32] refers to Hall [23] that calculated

$$\begin{aligned} p(i, \hat{f}, \hat{g} \mid x_i, x_{i+1}) &= \mathbb{P}\Big(\hat{f}(t) < B_t < \hat{g}(t), t_i \le t \le t_{i+1} \mid B_{t_i} = x_i, B_{t_{i+1}} = x_{i+1}\Big) \\ &= 1 - P(a_1, a_2, \hat{b}, x_i) - P(-a_2, -a_1, -\hat{b}, -x_i), \end{aligned}$$

where

$$P(a_1, a_2, \hat{b}, x_i) = \sum_{j=1}^{\infty} e^{2b(2j-1)(jc+a_2)} e^{\frac{2(jc+a_2)}{\Delta t_i}(\Delta x_i - \hat{b}\Delta t_i - (jc+a_2))} - \sum_{j=1}^{\infty} e^{4bj(2j-\hat{a})} e^{\frac{2}{\Delta t_i} jc(\Delta x_i - \hat{b}\Delta t_i - jc)},$$

with

$$a_1 = g(t_{i+1}) - x_i, \quad a_2 = f(t_{i+1}) - x_i, \quad b_1 = \frac{g(t_{i+1}) - g(t_i)}{\Delta t_i}, \quad b_2 = \frac{f(t_{i+1}) - f(t_i)}{\Delta t_i},$$

$$c = a_1 - a_2, \quad b = \frac{b_2 - b_1}{2}, \quad \hat{b} = \frac{b_2 + b_1}{2}, \quad \hat{a} = \frac{a_1 + a_2}{2}, \quad \Delta x_i = x_{i+1} - x_i,$$

and develops the recurrent algorithm to evaluate probability (5.4). Using that algorithm and approximation (5.7), one can compute

$$z_0(x) = p(0, \hat{f}, \hat{g} \mid 0, x) \frac{1}{\sqrt{2\pi t_1}} \exp(-\frac{x^2}{2t_1})$$

and

$$z_k(x) = \int_{\hat{f}(t_k)}^{\hat{g}(t_k)} z_{k-1}(y) p(k, \hat{f}, \hat{g} \mid y, x) \frac{1}{\sqrt{2\pi\Delta t_k}} \exp(-\frac{(x-y)^2}{2\Delta t_k}) dy, \quad k = 1, ..., n-1,$$

and then evaluate (5.4) by calculating

$$\int_{\hat{f}(t_n)}^{K} z_{n-1}(y) \frac{1}{\sqrt{2\pi u}} \exp(-\frac{y^2}{2u}) dy.$$

Alternatively, to evaluate (5.4), one can use (5.7) and the Monte Carlo simulation method developed in Poetzelberger and Wang [37] and generate a random sample $X_1,...,X_k$ from the multivariate normal distribution of $B_{t_1},...,B_{t_n}$ and estimate probability (5.4) by the sample mean

$$\frac{1}{k}\sum_{i=1}^{k} r_2(X_i; f(t_1), ..., f(t_{n-1}), f(t_n); g(t_1), ..., g(t_{n-1}), K),$$

where

$$r_2(x_1, ..., x_n; a_1, ..., a_n; b_1, ..., b_n)$$
$$= \Pi_{i=1}^n \mathbb{I}(a_i < x_i < b_i)\Big(1 - \exp\Big[-\frac{2}{\Delta t_{i-1}}(a_{i-1} - x_{i-1})(a_i - x_i)\Big] - \exp\Big[-\frac{2}{\Delta t_{i-1}}(b_{i-1} - x_{i-1})(b_i - x_i)\Big]\Big).$$

Volterra integral equations approach

Volterra integral equations approach is an alternative to PDE approach to calculate the probabilities and densities that correspond to formula (5.5). According to Buonocore et al. [12], densities $\phi_{m,1}$ and $\phi_{m,2}$ satisfy a system of Volterra integral equations of the second kind:

$$\begin{cases} \phi_{m,1}(t) & = -2m(g(t), t \mid 0, 0) + 2\int_0^t [\phi_{m,1}(s)m(g(t), t \mid g(s), s) + \phi_{m,2}(s)m(g(t), t \mid f(s), s)]ds \\ \phi_{m,2}(t) & = 2m(f(t), t \mid 0, 0) - 2\int_0^t [\phi_{m,1}(s)m(f(t), t \mid f(s), s) + \phi_{m,2}(s)m(f(t), t \mid g(s), s)]ds, \end{cases}$$

where for all $y \in \mathbb{R}$ and $s < t$ one has

$$m(f(t), t \mid y, s) = n(f(t), t \mid y, s)r(t, s, y),$$
$$n(x, t \mid y, s) = [2\pi(t-s)]^{-\frac{1}{2}} \exp\Big(-\frac{(x-y)^2}{2(t-s)}\Big),$$
$$r(t, s, y) = \frac{f'(t)}{2} - \frac{f(t) - y}{2(t-s)},$$

and density ϕ_m defined in (2.45) is equal to

$$\phi_m(t) = \phi_{m,1}(t) + \phi_{m,2}(t).$$

Buonocore et al. [12] has shown that if functions $f(t)$ and $g(t)$ are in the class $C^2\Big([0, \infty)\Big)$, then this system of Volterra integral equations possesses a unique continuous solution that can be found numerically, e.g., according to a composite trapezium rule. One can apply (5.7), set the integration step $\Delta > 0$ and $t = k\Delta$, $k = 1, 2, ...$, and use the following approximation:

$$\phi_{m,1}(\Delta) = -2m(g(\Delta), \Delta \mid 0, 0),$$

$$\phi_{m,1}(k\Delta) = -2m(g(k\Delta), k\Delta \mid 0, 0) + 2\Delta \sum_{j=1}^{k-1} [\phi_{m,1}(j\Delta) m(g(k\Delta), k\Delta \mid g(j\Delta), j\Delta) + \phi_{m,2}(j\Delta) m(g(k\Delta), k\Delta \mid f(j\Delta), j\Delta)], k \geq 2,$$

$$\phi_{m,2}(\Delta) = 2m(f(\Delta), \Delta \mid 0, 0),$$

$$\phi_{m,2}(k\Delta) = 2m(f(k\Delta), k\Delta \mid 0, 0) - 2\Delta \sum_{j=1}^{k-1} [\phi_{m,1}(j\Delta) m(f(k\Delta), k\Delta \mid g(j\Delta), j\Delta) + \phi_{m,2}(j\Delta) m(f(k\Delta), k\Delta \mid f(j\Delta), j\Delta)], k \geq 2.$$

The sum $\phi_{m,1} + \phi_{m,2}$ then provides an evaluation of ϕ_m. Finally, the values of D_m, $D_{m,1}$ and $D_{m,2}$ can be calculated by applying a rectangle rule.

5.1.5 Examples of numerical techniques to calculate Brownian motion hitting probabilities and densities for one-sided curved boundaries

In this section, it will be discussed how one can apply the numerical techniques developed by Skorohod [44], Novikov et al. [33] and Wang and Poetzelberger [46] to calculate Brownian motion hitting probability

$$\mathbb{P}\Big(\tau > u, B_u \leq K\Big), \quad u \in [0, T], \tag{5.8}$$

where

$$\tau = \inf\Big(t \geq 0 : B_t = g(t)\Big),$$

deterministic function g is in the class $C^2\Big([0, u]\Big)$ and satisfies $g(0) > 0$, and constant K is such that $K \leq g(u)$, and the numerical techniques developed by Buonocore et al. [13] and Peskir [36] to calculate the special case of formula (5.8), which corresponds to $K = g(u)$,

$$\mathbb{P}\Big(\tau > u\Big), \quad u \in [0, T]. \tag{5.9}$$

To compute

$$\mathbb{P}\Big(\tau > u, K_1 \leq B_u \leq K_2\Big), \quad u \in [0, T],$$

formula (5.6) can be applied.

Based on these results applied for Brownian motions B and $-B$, probabilities Φ_1, Φ_2, D_1, D_2, $\bar{D}_1$, $\bar{D}_2$, D^l and D^u can be found. To calculate densities q_1, q_2, ϕ_1, ϕ_2, $\bar{\phi}_1$, $\bar{\phi}_2$, ϕ^l and ϕ^u, a rectangle rule can be used. As in the two-sided boundary case, some other numerical methods can be applied as well.

PDE approach

Since

$$
\begin{aligned}
&\mathbb{P}\Big(B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) \\
&= \mathbb{P}\Big(C < B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) + \mathbb{P}\Big(\min_{t\in[0,u]} B_t \leq C, \quad B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) \\
&\leq \mathbb{P}\Big(C < B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) + \mathbb{P}\Big(\min_{t\in[0,u]} B_t \leq C\Big) \\
&\leq \mathbb{P}\Big(C < B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) + \mathbb{P}\Big(\max_{t\in[0,u]} B_t \geq -C\Big)
\end{aligned}
$$

and

$$
\mathbb{P}\Big(B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big) \geq \mathbb{P}\Big(C < B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big)
$$

for all $C < 0$, probability (5.8) can be approximated with

$$
\mathbb{P}\Big(C_1 < B_t < g(t), t \in [0,u], \text{ and } B_u \leq K\Big), \tag{5.10}
$$

where a constant C_1 is defined in (5.3). Probability (5.10) can be evaluated according to the PDE approach discussed in Section 5.1.4.

Approximation by piecewise linear boundaries

Approximation by piecewise linear boundaries is an alternative to PDE approach to evaluate probabilities and densities that correspond to formula (5.8).

Let $\hat{g}(t)$ be piecewise linear approximations for $g(t)$ on the interval $[0,u]$, with nodes t_i, $t_0 = 0 < t_1 < t_2 < ... < t_n = u$, $\Delta t_i = t_{i+1} - t_i$, such that $\hat{g}(t_i) = g(t_i)$.

Novikov et al. [33] calculates

$$p(i, \hat{g} \mid x_i, x_{i+1}) = \mathbb{P}\Big(B_t < \hat{g}(t), t_i \le t \le t_{i+1} \mid B_{t_i} = x_i, B_{t_{i+1}} = x_{i+1}\Big)$$
$$= \mathbb{I}\Big(\hat{g}(t_i) > x_i, \hat{g}(t_{i+1}) > x_{i+1}\Big)\Big[1 - e^{-\frac{2(\hat{g}(t_i)-x_i)(\hat{g}(t_{i+1})-x_{i+1})}{\Delta t_i}}\Big]$$

and develops the recurrent algorithm to evaluate probability (5.8). Applying (5.7) and that algorithm, one can compute

$$z_0(x) = p(0, \hat{g} \mid 0, x)\frac{1}{\sqrt{2\pi t_1}}\exp(-\frac{x^2}{2t_1})$$

and

$$z_k(x) = \int_{-\infty}^{\hat{g}(t_k)} z_{k-1}(y)p(k, \hat{g} \mid y, x)\frac{1}{\sqrt{2\pi\Delta t_k}}\exp(-\frac{(x-y)^2}{2\Delta t_k})dy, \quad k = 1, ...n-1,$$

and then evaluate (5.8) by calculating

$$\int_{-\infty}^{K} z_{n-1}(y)\frac{1}{\sqrt{2\pi u}}\exp(-\frac{y^2}{2u})dy.$$

Alternatively, to evaluate (5.8), one can use (5.7) and the Monte Carlo simulation method developed in Wang and Poetzelberger [46] and generate a random sample X_1,...,X_k from the multivariate normal distribution of B_{t_1},...,B_{t_n} and estimate probability (5.8) by the sample mean

$$\frac{1}{k}\sum_{i=1}^{k} r_1(X_i; g(t_1), ..., g(t_{n-1}), K),$$

where

$$r_1(x_1, ..., x_n; b_1, ..., b_n) = \Pi_{i=1}^{n}\mathbb{I}(x_i < b_i)\Big(1 - \exp\Big[-\frac{2}{\Delta t_{i-1}}(b_{i-1} - x_{i-1})(b_i - x_i)\Big]\Big).$$

Volterra integral equations

Volterra integral equations is an alternative to PDE approach to evaluate probabilities and densities that correspond to formula (5.9).

According to Buonocore et al. [13], the density ϕ of the first passage time of B over g can be determined implicitly from the integral equation

$$\phi(t) = -2m(g(t), t \mid 0, 0) + 2\int_0^t \phi(s)m(g(t), t \mid g(s), s)ds.$$

Buonocore et al. [13] has shown that if $g(t)$ is $C^2\Big([0,\infty)\Big)$-class function, then this integral equation possesses a unique continuous solution that can be found numerically applying a composite trapezium rule. One can apply (5.7), set the integration step $\Delta > 0$ and $t = k\Delta$, $k = 1, 2, ...,$ and use the following approximation:

$$\phi(\Delta) = -2m(g(\Delta), \Delta \mid 0, 0),$$

$$\phi(k\Delta) = -2m(g(k\Delta), k\Delta \mid 0, 0) + 2\Delta \sum_{j=1}^{k-1} \phi(j\Delta) m(g(k\Delta), k\Delta \mid g(j\Delta), j\Delta), k \geq 2.$$

Alternatively, Peskir [36] has shown that this density function ϕ also satisfies a linear Volterra integral equation of the first kind

$$\Psi(\frac{g(t)}{\sqrt{t}}) = \int_0^t \Psi(\frac{g(t) - g(s)}{\sqrt{t-s}})\phi(s)ds, \ t > 0,$$

where

$$\Psi(x) = 1 - \int_0^x \frac{1}{\sqrt{2\pi}} \exp(-\frac{z^2}{2})dz.$$

Applying (5.7) and setting $t_j = j\Delta t$ for $j = 0, 1, ..., n$, $\Delta t = \frac{t}{n}$ and $n \geq 1$, one can implement the following numerical approximation algorithm:

$$\Delta t \sum_{j=1}^{i-1} \Psi(\frac{g(t_i) - g(t_j)}{\sqrt{t_i - t_j}})\phi(t_j) = \Psi(\frac{g(t_i)}{\sqrt{t_i}}), \ i = 1, ..., n.$$

Finally, the cumulative distribution function of the first passage time of B over g can be determined applying a rectangle rule.

5.1.6 Numerical studies

In this section, conditional distribution for the time of the next jump is computed for some given set of parameters: $t = 1$, $T = 5$, $\alpha_1 = 1$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$. For the constant number of dynamic hedgers models, it is supposed that $w_t^D = 14$, which means that condition (2.10) holds true, and the dynamics of lower and upper boundaries h_2 and h_1 is illustrated by Figure 5.1. For the stochastic number of dynamic hedgers model, two different cases are considered. In the first case, it is assumed that $w_t^D = 14$ and, similar to the constant number of dynamic hedgers models, (2.10) holds true, therefore, the state process is either in the lower level state s_1 or in the upper level state s_3. In the second case, it is assumed that $w_t^D = 5$, hence, the system does not exhibit multiple equilibria and the state process is in the state s_2. According to (2.49) and (2.5), $T^D(w_t^D) = 2.01$ and $H(T^D(w_t^D), w_t^D, \kappa e^{-r(T - T^D(w_t^D))}) = 52.34$. Figures 5.2-5.7 plot probabilities of time to the next jump for different values of B_t.

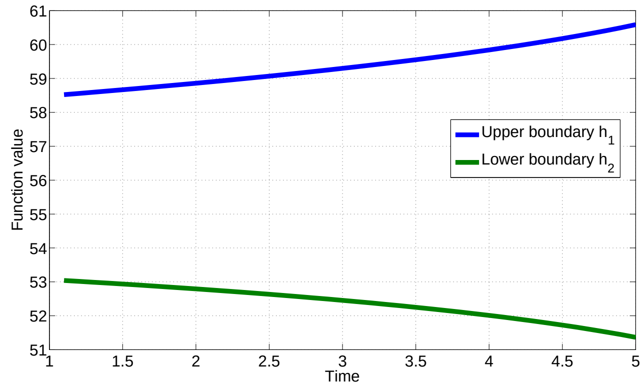

Fig. 5.1: Lower and upper boundaries: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$

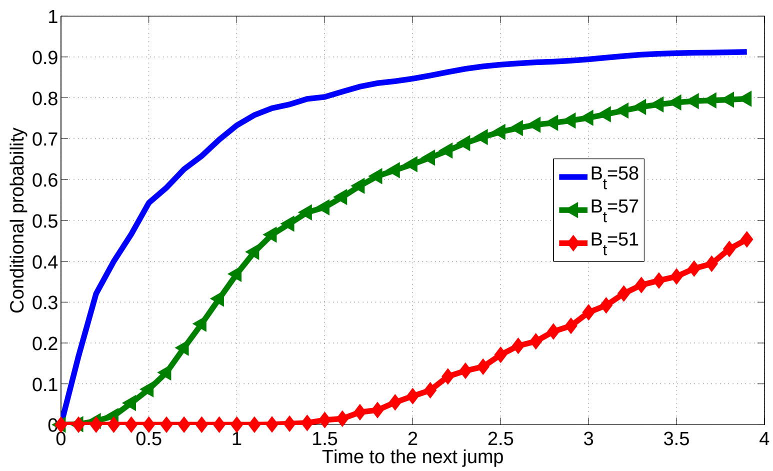

Fig. 5.2: Conditional probability of the time of the next jump given $S_t = s_1$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $c = 1$, $a = 4$, $b = 5$

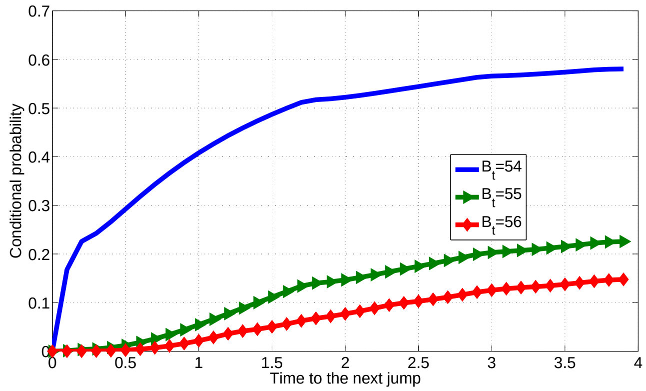

Fig. 5.3: Conditional probability of the time of the next jump given $S_t = s_2$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$

Numerical studies for the exogenous shocks model

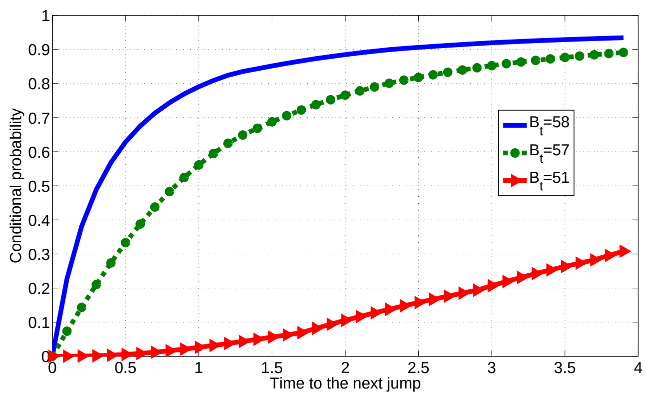

Fig. 5.4: Conditional probability of the time of the next jump given $S_t = s_1$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $a = 4$, $b = 5$

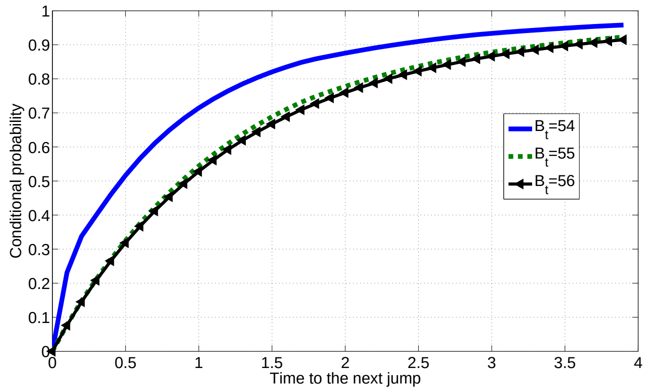

Fig. 5.5: Conditional probability of the time of the next jump given $S_t = s_2$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $a = 4$, $b = 5$

Numerical studies for the stochastic number of dynamic hedgers model

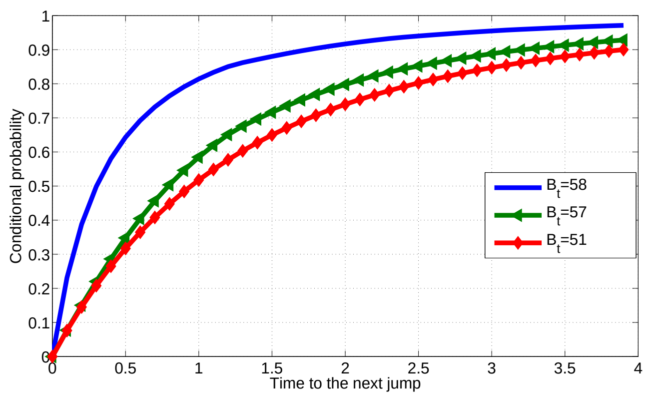

Fig. 5.6: Conditional probability of the time of the next jump given $S_t = s_1$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 14$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $a = 4$, $b = 5$

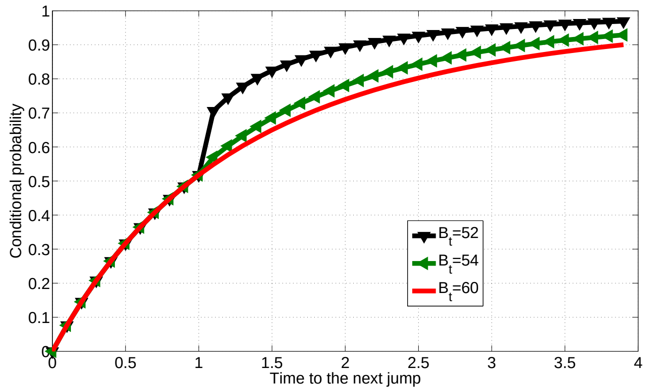

Fig. 5.7: Conditional probability of the time of the next jump given $S_t = s_2$ computed according to the PDE approach: $t = 1$, $T = 5$, $w_t^D = 5$, $\alpha_1 = 1$, $r = 0.001$, $\sigma_\kappa = 1$, $\kappa = 50$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 1$, $a = 4$, $b = 5$

5.2 *Alternative models*

5.2.1 *A numerical algorithm for the simple jump structure model*

Owing to the results of Theorem 3.6 and Section 4.4, the conditional probability for the time of the next jump can be numerically approximated by applying Gauss-Laguerre formula for

$$\frac{2}{\sqrt{2\pi}} \int_0^\infty \Big[\int_{R_t}^\infty \Big(\int_{\frac{\gamma(d_t^S, \tau_{N_t}, t, x)}{\sqrt{T(t,u)}}}^\infty e^{-\frac{y^2}{2}} dy\Big) \lambda e^{-\lambda(x-R_t)} dx\Big] \mathrm{Gamma}\left(\lambda; a + (N_t - 1), b + \sum_{j=1}^{N_t - 1} \mu_j\right) d\lambda.$$

5.2.2 *A numerical algorithm for the Markov chain jump structure model*

According to Theorem 3.7, the conditional probability of the time of the next jump can be numerically approximated by applying Gauss-Laguerre formula for (3.26).

5.2.3 *Numerical studies*

In this section, conditional distribution for the time of the next jump is calculated for two different examples of $(\alpha(s), s \geq 0)$: $\alpha(s) = 1$ and $\alpha(s) = \frac{1}{s}$. Suppose that current time is $t = 3$. In the simple jump structure model, it is also assumed that $\tau_{N_t} = 2$, $a + (N_t - 1) = 4$ and $b + \sum_{j=1}^{N_t-1} \mu_j = 5$. Figures 5.8-5.11 plot probabilities of time to the next jump for different values of d_t^S and D_t^{MC}.

Numerical studies for the simple jump structure model

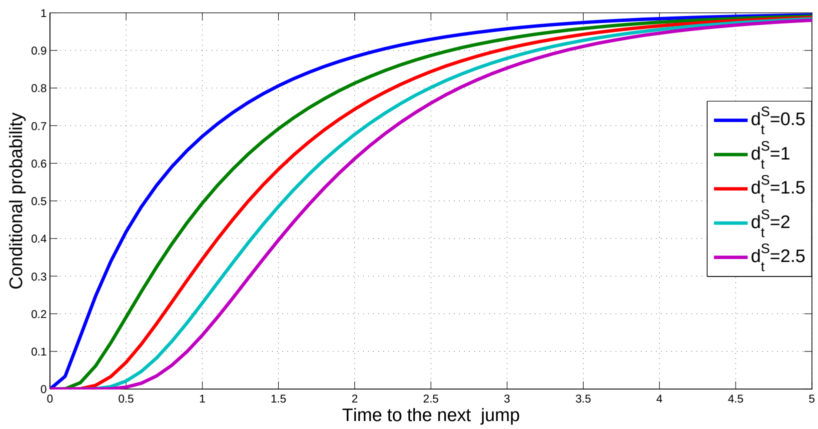

Fig. 5.8: Conditional probability of the time of the next jump: $t = 3$, $\tau_{N_t} = 2$, $\alpha(s) = 1$, $a + (N_t - 1) = 4$, $b + \sum_{j=1}^{N_t-1} \mu_j = 5$

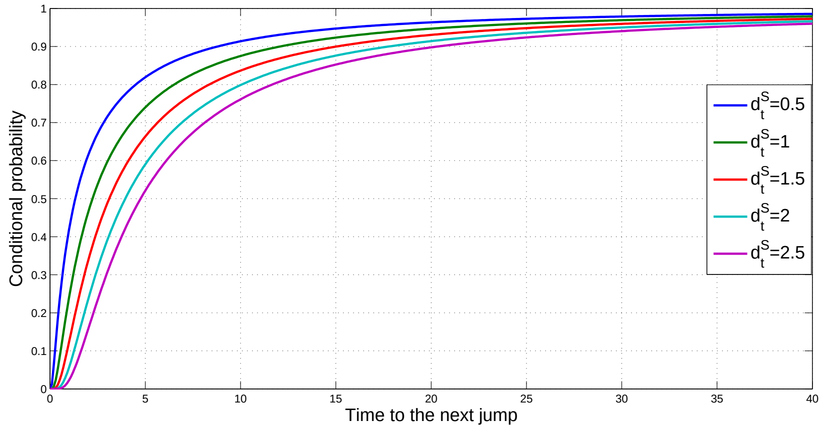

Fig. 5.9: Conditional probability of the time of the next jump: $t = 3$, $\tau_{N_t} = 2$, $\alpha(s) = \frac{1}{s}$, $a + (N_t - 1) = 4$, $b + \sum_{j=1}^{N_t-1} \mu_j = 5$

Numerical studies for the Markov chain jump structure model

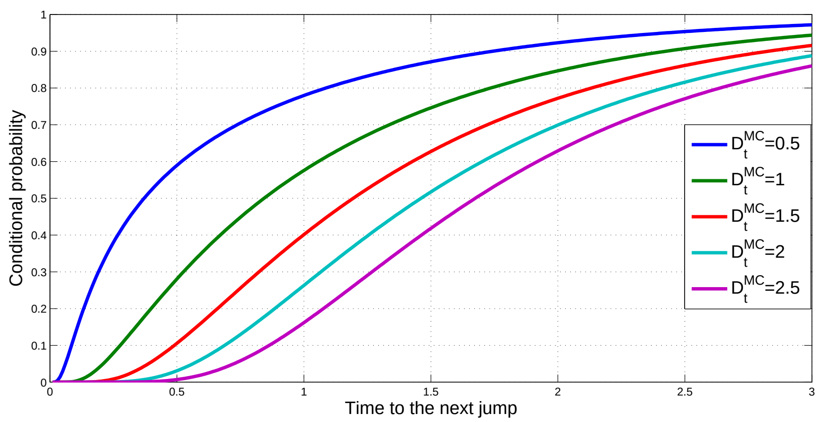

Fig. 5.10: Conditional probability of the time of the next jump: $t = 3$ and $\alpha(s) = 1$

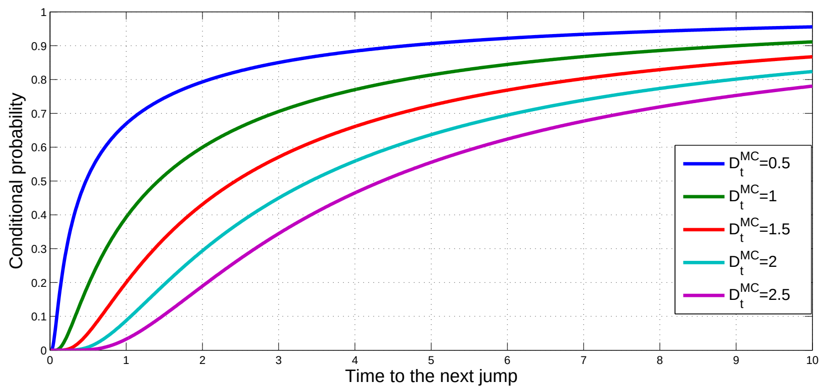

Fig. 5.11: Conditional probability of the time of the next jump: $t = 3$ and $\alpha(s) = \frac{1}{s}$

6. CONCLUSION

In this thesis, I present a quantitative approach to the modelling of market booms and crashes within a multiple equilibria continuous time framework. I consider five different multiple equilibria models describing how market prices fluctuate and move from one regime to another.

As a starting point for my research I used a one-period multiple equilibria model from Gennotte and Leland [21] and extended it into a continuous time framework. In the market microstructure models discussed in Chapter 2, price is determined pursuant to the law of supply and demand. In Chapter 3, I develop simple jump structure and Markov chain jump structure models within an alternative framework in which pricing equation is given exogenously, and this is basically the main drawback of this framework. For all the models presented in the thesis, I prove that the stock price process is a càdlàg semimartingale; find conditional distributions for the time of, the type of and the size of the next jump, which is defined as a point of discontinuity of this process; discuss the parameter estimation procedures; and conduct a number of numerical studies. I develop alternative models in order to overcome some drawbacks of the market microstructure models. For example, in contrast to the market microstructure models described in Chapter 2, alternative models exclude the possibility of negative prices and give expressions of conditional probabilities in explicit form.

It seems that this topic has a high potential for future research. It would be of an interest to calibrate the models and see how they work in different stock markets. Another direction is pricing and hedging of securities with underlying following the dynamics of stock price processes of the models presented here. Finally, it would be good to find a powerful framework that would possess all of the good features of the models discussed.

APPENDIX

Proof of Theorem 2.1 This theorem will be proved in several steps.

Step 1 First, it will be shown that there exist some $\delta_1 \in (0, T - T_0)$ and $\Delta_1 > 0$ such that

$$h_1(t) - h_2(t) \geq \Delta_1, \ \forall t \in (T - \delta_1, T).$$

According to (2.12), (2.13) and (2.21),

$$A_1 = \lim_{t \uparrow T} p_1(t) = \kappa - \sqrt{-2\sigma_\kappa^2 \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)}$$

and

$$A_2 = \lim_{t \uparrow T} p_2(t) = \kappa + \sqrt{-2\sigma_\kappa^2 \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)},$$

which means that $A_1 < A_2$.

Then

$$\begin{aligned}
&\lim_{t \uparrow T} \int_{-\infty}^{\infty} \Phi\Big(\frac{Ke^{-r(T-t)} - p_1(t)}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \\
&= \int_{-\infty}^{\infty} \Phi\Big(\lim_{t \uparrow T} \frac{Ke^{-r(T-t)} - p_1(t)}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \\
&= \int_{A_1}^{\infty} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{t \uparrow T} \int_{-\infty}^{\infty} \Phi\Big(\frac{Ke^{-r(T-t)} - p_2(t)}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \\
&= \int_{-\infty}^{\infty} \Phi\Big(\lim_{t \uparrow T} \frac{Ke^{-r(T-t)} - p_2(t)}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK \\
&= \int_{A_2}^{\infty} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{t \uparrow T}\Big(h_1(t) - h_2(t)\Big) &= \frac{1}{\gamma_3}\Big(w^D \int_{A_1}^{A_2} \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK - 2\gamma_1 \sqrt{-2\sigma_\kappa^2 \ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)}\Big) \\
&= \frac{2}{\gamma_3}\Big(\gamma_1 \sqrt{2\pi\sigma_\kappa^2} e^{\frac{z^2}{2}} \int_0^z \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy - \gamma_1 \sigma_\kappa z\Big) \\
&=: f(z),
\end{aligned}$$

where

$$z = \sqrt{-2\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi\sigma_\kappa^2}\Big)} > 0.$$

Since $f(0) = 0$ and $f'(z) = \frac{2\gamma_1\sqrt{2\pi\sigma_\kappa^2}ze^{\frac{z^2}{2}}\int_0^z \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy}{\gamma_3}$ is positive for $z > 0$ and 0 for $z = 0$, I obtain that

$$\lim_{t\uparrow T}\Big(h_1(t) - h_2(t)\Big) > 0.$$

Finally, one can take, e.g., $\Delta_1 = \frac{1}{2}\lim_{t\uparrow T}\Big(h_1(t) - h_2(t)\Big)$ and use the definition of the limit.

Step 2 Second, it will be proved that there exists some $\Delta_2 > 0$ such that

$$h_1(t) - h_2(t) \geq \Delta_2,\ \forall t \in [T_0, T - \delta_1].$$

Assume that $t \in [T_0, T - \delta_1]$. Then (2.12), (2.13) and (2.21) imply that

$$\begin{aligned} p_2(t) - p_1(t) &= 2\sqrt{-2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}\Big)} \\ &\geq 2\sqrt{-2(\sigma_\kappa^2 e^{-r(T-T_0)} + \alpha_1^2\frac{1-e^{-2r\delta_1}}{2r})\ln\Big(\frac{\gamma_1}{w^D}\sqrt{2\pi(\frac{\alpha_1^2}{2r} + (\sigma_\kappa^2 - \frac{\alpha_1^2}{2r})e^{-2r(T-T_0)})}\Big)} \\ &=: \delta_2 > 0, \end{aligned}$$

which means that, for all $y \in [-\frac{\delta_2}{2}, \frac{\delta_2}{2}]$,

$$p_1(t) \leq \kappa e^{-r(T-t)} + y \leq p_2(t)$$

and, hence,

$$h_1(t) \geq h(t, \kappa e^{-r(T-t)} + y) \geq h_2(t). \tag{.1}$$

Furthermore, in virtue of (2.7) and (2.20),

$$\begin{aligned} h_x(t, \kappa e^{-r(T-t)} + y) &= \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t)\Big)}}e^{-\frac{y^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}}\Big) \\ &\leq \frac{1}{\gamma_3}\Big(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-T_0)} + \Sigma^2(T_0)\Big)}}e^{-\frac{y^2}{2(\sigma_\kappa^2 e^{-r(T-T_0)} + \alpha_1^2\frac{1-e^{-2r\delta_1}}{2r})}}\Big) \end{aligned}$$

Condition (2.19) guarantees that there exists some positive $\delta_3 \leq \frac{\delta_2}{2}$ such that

$$\begin{aligned} h_x(t, \kappa e^{-r(T-t)} - \delta_3) &= h_x(t, \kappa e^{-r(T-t)} + \delta_3) \\ &\leq \frac{1}{\gamma_3}\Bigg(\gamma_1 - \frac{w^D}{\sqrt{2\pi\Big(\sigma_\kappa^2 e^{-2r(T-T_0)} + \Sigma^2(T_0)\Big)}} e^{-\frac{\delta_3^2}{2(\sigma_\kappa^2 e^{-r(T-T_0)} + \Sigma^2(T-\delta_1))}}\Bigg) \\ &=: -\delta_4 < 0. \end{aligned}$$

Taking the partial derivative with respect to x in (2.7) and using (2.20), it can be concluded that

$$h_{xx}(t,x) = \frac{w^D(x - \kappa e^{-r(T-t)})}{\gamma_3\sqrt{2\pi(\sigma_\kappa^2 e^{-2r(T-t)} + \Sigma^2(t))}(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))} e^{-\frac{(\kappa e^{-r(T-t)} - x)^2}{2(\sigma_\kappa^2 e^{-r(T-t)} + \Sigma^2(t))}},$$

that is, $h_x(t,x)$ is a decreasing function of x for $x \leq \kappa e^{-r(T-t)}$ and an increasing function of x for $x \geq \kappa e^{-r(T-t)}$.

It means that, for $x \in [\kappa e^{-r(T-t)} - \delta_3, \kappa e^{-r(T-t)} + \delta_3]$,

$$h_x(t,x) \leq \max\Big(h_x(t, \kappa e^{-r(T-t)} - \delta_3), h_x(t, \kappa e^{-r(T-t)} + \delta_3)\Big) \leq -\delta_4.$$

Thus, by the mean value theorem and in view of (.1),

$$h_1(t) - h_2(t) \geq h(t, \kappa e^{-r(T-t)} - \delta_3) - h(t, \kappa e^{-r(T-t)} + \delta_3) \geq 2\delta_3\delta_4 > 0.$$

Step 3 Finally, it will be shown that there exists some $\Delta > 0$ such that

$$h_1(t) - h_2(t) \geq \Delta, \ \forall t \in [T_0, T).$$

Indeed, one can take $\Delta = \min(\Delta_1, \Delta_2)$, and the result follows. ■

Proof of Theorem 2.7 The proof of this theorem will be done in several steps.

Step 1 Initial decomposition.

In virtue of Remark 2.8, $S_t \in \mathcal{F}_t^P$. Hence, the following decomposition can be considered:

$$\begin{aligned} &\mathbb{P}(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \mid \mathcal{F}_t^P) \\ &= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^P\Big) \\ &= \sum_{i=1}^{3} \mathbb{I}\Big[S_t = s_i\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^P\Big) \end{aligned} \tag{.2}$$

Step 2 Calculation of the conditional probability on the set $[S_t = s_1]$.

In view of Step 2 in Definition 2.1,

$$\tau_{N_t+1} = \inf\Big(s > t : B_s \geq h^l(s; \tau_{N_t} + T^l_{N_t})\Big),$$

where function h^l is defined by formula (2.30). To find conditional distribution for $T^l_{N_t}$ given $\mathcal{F}^P_t$, note that the information that is available about $T^l_{N_t}$ is that

$$B_s < h^l(s; \tau_{N_t} + T^l_{N_t}), \forall s \in [\tau_{N_t}, t],$$

and, in view of the continuity of the Brownian motion and function h^l, it is equivalent to

$$f^l(T^l_{N_t}) < 0,$$

where

$$f^l(T^l_{N_t}) = \max_{\tau_{N_t} \leq s \leq t}\Big(B_s - h^l(s; \tau_{N_t} + T^l_{N_t})\Big). \tag{.3}$$

Since $h_1(s) > h_2(s), \forall s \in [\tau_{N_t}, t]$, and $\psi(x) = e^{-cx}$ is a strictly decreasing function for $c > 0$, formula (2.30) implies that, if $0 \leq t_1 < t_2 \leq t - \tau_{N_t}$, then

$$h^l(s; \tau_{N_t} + t_1) = h^l(s; \tau_{N_t} + t_2) = h_1(s), \forall s \in [\tau_{N_t}, \tau_{N_t} + t_1],$$

and

$$h^l(s; \tau_{N_t} + t_1) < h^l(s; \tau_{N_t} + t_2), \forall s > \tau_{N_t} + t_1,$$

that is,

$$\begin{cases} f^l(t_1) \geq f^l(t_2) & \text{if } f^l(t_1) < 0 \\ f^l(t_1) > f^l(t_2) & \text{if } f^l(t_1) \geq 0. \end{cases} \tag{.4}$$

If $f^l(0) \leq 0$, then define R^l_t by

$$R^l_t = 0 \tag{.5}$$

and if $f^l(0) > 0$, define R^l_t implicitly as the solution of

$$f^l(R^l_t) = 0, \tag{.6}$$

which exists and is unique due to (.4), the fact that

$$f^l(t-\tau_{N_t}) = \max_{\tau_{N_t}\le s\le t}\Big(B_s - h^l(s;t)\Big) = \max_{\tau_{N_t}\le s\le t}\Big(B_s - h_1(s)\Big) < 0$$

and the continuity of function f^l.

Recall that $T^l_{N_t} \sim \text{Exp}(\lambda_l)$, and it means that conditional distribution for $T^l_{N_t}$ given $\mathcal{F}^P_t$ is the distribution of $T^l_{N_t}$ conditional on the set $[T^l_{N_t} > R^l_t]$, that is, its density function is given by

$$g^l(x) = \lambda_l e^{-\lambda_l(x-R^l_t)}, x \ge R^l_t. \tag{.7}$$

Let

$$\mathcal{F}^{P,T^l_{N_t}}_t = \sigma\{(P_s, T_0 \le s \le t), T^l_{N_t}\}.$$

Then, in view of the law of iterated expectations and the construction mechanism in Definition 2.1, the following decomposition can be considered:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^P_t\Big)\\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^{P,T^l_{N_t}}_t\Big) \mid \mathcal{F}^P_t\Big)\\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[T^l_{N_t+1} \ge u-\tau_{N_t}\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^{P,T^l_{N_t}}_t\Big) \mid \mathcal{F}^P_t\Big)\\
&\quad + \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[t-\tau_{N_t} < T^l_{N_t+1} < u-\tau_{N_t}\Big]\times\\
&\qquad\qquad \times \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} \le \tau_{N_t} + T^l_{N_t+1}, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^{P,T^l_{N_t}}_t\Big) \mid \mathcal{F}^P_t\Big)\\
&\quad + \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[t-\tau_{N_t} < T^l_{N_t+1} < u-\tau_{N_t}\Big]\times\\
&\qquad\qquad \times \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t} + T^l_{N_t+1} < \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^{P,T^l_{N_t}}_t\Big) \mid \mathcal{F}^P_t\Big)\\
&\quad + \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[T^l_{N_t+1} \le t-\tau_{N_t}\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}^{P,T^l_{N_t}}_t\Big) \mid \mathcal{F}^P_t\Big).
\end{aligned}$$

The first two terms in this decomposition correspond to the scenario in which Brownian motion hits h_1. By construction, the next state of the state process is s_3 and the jump size is equal to $J^u(\tau_{N_t+1})$ defined in accordance with (2.33).

The other two terms correspond to the scenario in which Brownian motion hits the convex combination of h_1 and h_2. The next state of the state process is equal to s_3 with probability p_{lu} and s_2 with probability p_{lm}. If the next state is equal to s_3 (respectively s_2), then J_{N_t+1} is equal to $J^{lu}\Big(\tau_{N_t+1}, h^l(\tau_{N_t+1}, \tau_{N_t} + T^l_{N_t})\Big)$ (respectively $J^{lm}\Big(\tau_{N_t+1}, h^l(\tau_{N_t+1}, \tau_{N_t} + T^l_{N_t})\Big)$) defined in accordance with (2.33).

Applying formula (.7), I obtain the expression for F_1 in terms of Brownian motion hitting densities ϕ_1 and ϕ^l:

$$
\begin{aligned}
&F_1(t, \tau_{N_t}, R_t^l, B_t, u, C_1, C_2) \\
&= e^{-\lambda_l(u-\tau_{N_t}-R_t^l)} \int_t^u \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t)dy \\
&+ \int_{t-\tau_{N_t}}^{u-\tau_{N_t}} \Big(\int_t^{\tau_{N_t}+x} \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t)dy \Big) \lambda_l e^{-\lambda_l(x-R_t^l)}dx \\
&+ \int_{t-\tau_{N_t}}^{u-\tau_{N_t}} \Big(\int_{\tau_{N_t}+x}^{u} (p_{lu}\mathbb{I}(s_3 \in C_1, J^{lu}(y, h^l(y; \tau_{N_t}+x)) \in C_2) \\
&\quad + p_{lm}\mathbb{I}(s_2 \in C_1, J^{lm}(y, h^l(y; \tau_{N_t}+x)) \in C_2))\phi^l(y, \tau_{N_t}+x, t, B_t)dy \Big) \lambda_l e^{-\lambda_l(x-R_t^l)}dx \\
&+ \int_{R_t^l}^{t-\tau_{N_t}} \Big(\int_t^u \Big[p_{lu}\mathbb{I}(s_3 \in C_1, J^{lu}(y, h^l(y; \tau_{N_t}+x)) \in C_2) \\
&\quad + p_{lm}\mathbb{I}(s_2 \in C_1, J^{lm}(y, h^l(y; \tau_{N_t}+x)) \in C_2) \Big] \phi^l(y, \tau_{N_t}+x, t, B_t)dy \Big) \lambda_l e^{-\lambda_l(x-R_t^l)}dx,
\end{aligned}
$$

where

$$
\phi_1(u, t, y) = -\frac{\partial D_1(u, t, y)}{\partial u}, \quad D_1(u, t, y) = \mathbb{P}\Big(B_s < h_1(t+s) - y, \forall s \in [0, u-t] \Big),
$$

$$
\phi^l(u, v, t, y) = -\frac{\partial D^l(u, v, t, y)}{\partial u}, \quad D^l(u, v, t, y) = \mathbb{P}\Big(B_s < h^l(t+s; v) - y, \forall s \in [0, u-t] \Big),
$$

are Brownian motion hitting densities and probabilities of one-sided curved boundaries and R_t^l is defined in accordance with formulas (.3), (.5) and (.6).

Step 3 Calculation of the conditional probability on the set $[S_t = s_2]$.

According to the first scenario, Brownian motion hits the upper boundary h_1 earlier than the lower boundary h_2, then the state process switches to the state s_3 and the jump size is equal to $J^u(\tau_{N_t+1})$ defined by (2.33). According to the other scenario, Brownian motion hits the lower boundary h_2 earlier than the upper boundary h_1, then the state process switches to the state s_1 and the jump size is equal to $J^l(\tau_{N_t+1})$ defined by (2.33). Therefore,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2 \Big] \mid \mathcal{F}_t^P \Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{\tau_{N_t+1}} = h_1(\tau_{N_t+1}), \tau_{N_t+1} < u, s_3 \in C_1, J^u(\tau_{N_t+1}) \in C_2 \Big] \mid \mathcal{F}_t^P \Big) \\
&\quad + \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{\tau_{N_t+1}} = h_2(\tau_{N_t+1}), \tau_{N_t+1} < u, s_1 \in C_1, J^l(\tau_{N_t+1}) \in C_2 \Big] \mid \mathcal{F}_t^P \Big),
\end{aligned}
$$

and I obtain the expression for F_2 in terms of Brownian motion hitting densities $\phi_{m,1}$ and $\phi_{m,2}$:

$$F_2(t, B_t, u, C_1, C_2) = \int_0^{u-t} \Big[\mathbb{I}(s_3 \in C_1, J^u(t+y) \in C_2)\phi_{m,1}(y, t, B_t) \\ + \mathbb{I}(s_1 \in C_1, J^l(t+y) \in C_2)\phi_{m,2}(y, t, B_t)\Big] dy,$$

where

$$\phi_{m,1}(u,t,y) = \frac{\partial D_{m,1}(u,t,y)}{\partial u}, \quad D_{m,1}(u,t,y) = \mathbb{P}\Big(\tau(t,y) \le u-t, B_{\tau(t,y)} = h_1(t+\tau(t,y)) - y\Big),$$

$$\phi_{m,2}(u,t,y) = \frac{\partial D_{m,2}(u,t,y)}{\partial u} \quad \text{and} \quad D_{m,2}(u,t,y) = \mathbb{P}\Big(\tau(t,y) \le u-t, B_{\tau(t,y)} = h_2(t+\tau(t,y)) - y\Big)$$

and

$$\tau(t,y) = \inf\{s \ge 0 : B_s = h_2(t+s) - y \quad \text{or} \quad B_s = h_1(t+s) - y\},$$

are Brownian motion hitting densities and probabilities of a two-sided curved boundary with $\tau(t,y)$ as the first hitting time of this boundary.

Step 4 Calculation of the conditional probability on the set $[S_t = s_3]$.

Calculation procedure is patterned after Step 2. Similar to the lower equilibrium scenario, denote by

$$f^u(T^u_{N_t}) = \min_{\tau_{N_t} \le s \le t} \Big(B_s - h^u(s; \tau_{N_t} + T^u_{N_t})\Big). \tag{.8}$$

If $f^u(0) \ge 0$, then define R^u_t by

$$R^u_t = 0 \tag{.9}$$

and if $f^u(0) < 0$, define R^u_t implicitly as the solution of

$$f^u(R^u_t) = 0. \tag{.10}$$

As a result, I obtain the expression for F_3 in terms of Brownian motion hitting densities ϕ_2 and ϕ^u:

$$
\begin{aligned}
&F_3(t,\tau_{N_t},R_t^u,B_t,u,C_1,C_2)\\
&= e^{-\lambda_u(u-\tau_{N_t}-R_t^u)}\int_t^u \mathbb{I}(s_1\in C_1, J^l(y)\in C_2)\phi_2(y,t,B_t)dy\\
&+\int_{t-\tau_{N_t}}^{u-\tau_{N_t}}\Big(\int_t^{\tau_{N_t}+x}\mathbb{I}(s_1\in C_1, J^l(y)\in C_2)\phi_2(y,t,B_t)dy\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx\\
&+\int_{t-\tau_{N_t}}^{u-\tau_{N_t}}\Big(\int_{\tau_{N_t}+x}^{u}(p_{ul}\mathbb{I}(s_1\in C_1, J^{ul}(y,h^u(y;\tau_{N_t}+x))\in C_2)\\
&\quad+p_{um}\mathbb{I}(s_2\in C_1, J^{um}(y,h^u(y;\tau_{N_t}+x))\in C_2))\phi^u(y,\tau_{N_t}+x,t,B_t)dy\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx\\
&+\int_{R_t^u}^{t-\tau_{N_t}}\Big(\int_t^u\Big[p_{ul}\mathbb{I}(s_1\in C_1, J^{ul}(y,h^u(y;\tau_{N_t}+x))\in C_2)\\
&+p_{um}\mathbb{I}(s_2\in C_1, J^{um}(y,h^u(y;\tau_{N_t}+x))\in C_2)\Big]\phi^u(y,\tau_{N_t}+x,t,B_t)dy\Big)\lambda_u e^{-\lambda_u(x-R_t^u)}dx,
\end{aligned}
$$

where

$$\phi_2(u,t,y)=-\frac{\partial D_2(u,t,y)}{\partial u},\quad D_2(u,t,y)=\mathbb{P}\Big(B_s>h_2(t+s)-y,\forall s\in[0,u-t]\Big),$$

$$\phi^u(u,v,t,y)=-\frac{\partial D^u(u,v,t,y)}{\partial u}\quad\text{and}\quad D^u(u,v,t,y)=\mathbb{P}\Big(B_s>h^u(t+s;v)-y,\forall s\in[0,u-t]\Big)$$

are Brownian motion hitting densities and probabilities of one-sided curved boundaries, and R_t^u is defined in accordance with formulas (.8) – (.10). ■

Proof of Theorem 2.8 The proof of this theorem will be done in several steps.

Step 1 Consider the initial decomposition described by (.2) and denote by τ the remaining time to the first arrival after t in the sunspot shock process Z_t. Recall that τ is independent of $\mathcal{F}_t^P$ and Z_t is a Poisson process with intensity λ_Z. Hence, τ has an exponential distribution with parameter λ_Z. Let

$$\mathcal{F}_t^{P,\tau}=\sigma\{(P_s,T_0\le s\le t),\tau\}.$$

Step 2 Calculation of the conditional probability on the set $[S_t = s_1]$.

By the law of iterated expectations,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^P\Big)\\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)\\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau \geq u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)\\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{N_t+1} < t+\tau, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)\\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \leq h_2(t+\tau), t+\tau \leq \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big)\\
&+ \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h_2(t+\tau), t+\tau \leq \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^P\Big).
\end{aligned}
$$

The first term in this decomposition corresponds to the scenario that there are no shock arrivals on $[t,u)$ at all and, hence, Brownian motion hits the boundary h_1 on (t,u). The new state of the state process is equal to s_3 and the jump size is $J^u(\tau_{N_t+1})$.

The second term corresponds to the scenario that the first shock arrival time is $t+\tau < u$ and Brownian motion hits the boundary h_1 on $(t, t+\tau)$. As in the first scenario, the process switches to s_3, the jump size is equal to $J^u(\tau_{N_t+1})$.

According to the third scenario, the first shock arrival time is $t+\tau < u$, the Brownian motion value stays smaller than the value of the boundary h_1 on $(t, t+\tau)$ and at the time of the shock $B_{t+\tau} \leq h_2(t+\tau)$. As a consequence, there is no jump at time $t+\tau$.

The fourth scenario is the same is the third one with the only difference that $B_{t+\tau} > h_2(t+\tau)$. Therefore, the price jumps at time $t+\tau$. With probability p_{lu}, the new state of the state process is s_3 and the jump size is $J^{lu}(t+\tau, B_{t+\tau})$. With probability $1-p_{lu}$, the new state of the state process is s_2 and the jump size is $J^{lm}(t+\tau, B_{t+\tau})$.

In view of the independence of τ and $\mathcal{F}_t^P$, the first and second terms are equal to

$$
e^{-\lambda_Z(u-t)}\int_t^u \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y,t,B_t)dy
$$

and

$$
\int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y,t,B_t)dy\Big]dr.
$$

The third term is equal to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h_2(t+\tau), t+\tau \le \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^{P}\Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h_2(t+\tau), (B_s < h_1(s), \forall s \in [t, t+\tau))\Big] \\
&\qquad\qquad \mathbb{I}\Big(\tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big) \mid \mathcal{F}_{t+\tau}^{P}\Big) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^{P}\Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} \le h_2(t+\tau), (B_s < h_1(s), \forall s \in [t, t+\tau))\Big] \\
&\qquad\qquad F_{11}(t+\tau, B_{t+\tau}, u, C_1, C_2) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^{P}\Big) \\
&= \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{11}(t+r, x, u, C_1, C_2) dx\Big] dr,
\end{aligned}
$$

where $q_1(x; r, t, y)$ is the density of B_r on the set $\Big[B_s < h_1(t+s) - y, \forall s \in [0, r]\Big]$, and the fourth term is equal to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h_2(t+\tau), t+\tau \le \tau_{N_t+1} < u, S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big] \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^{P}\Big) \\
&= \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau < u-t\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[B_{t+\tau} > h_2(t+\tau), (B_s < h_1(s), \forall s \in [t, t+\tau))\Big] \\
&\qquad\qquad\qquad \mathbb{I}\Big(S_{\tau_{N_t+1}} \in C_1, J_{N_t+1} \in C_2\Big) \mid \mathcal{F}_t^{P,\tau}\Big) \mid \mathcal{F}_t^{P}\Big) \\
&= \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{h_2(t+r)}^{h_1(t+r)} q_1(x; r, t, B_t)\Big(p_{lu}\mathbb{I}(s_3 \in C_1, J^{lu}(t+r, x) \in C_2) \\
&\qquad\qquad\qquad + p_{lm}\mathbb{I}(s_2 \in C_1, J^{lm}(t+r, x) \in C_2)\Big) dx\Big] dr.
\end{aligned}
$$

Combining all the terms together implies that

$$
\begin{aligned}
F_{11}(t, B_t, u, C_1, C_2) &= e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t) dy \\
&+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(s_3 \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t) dy \\
&+ \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t) F_{11}(t+r, x, u, C_1, C_2) dx \\
&+ \int_{h_2(t+r)}^{h_1(t+r)} q_1(x; r, t, B_t)\Big(p_{lu}\mathbb{I}(s_3 \in C_1, J^{lu}(t+r, x) \in C_2) \\
&\qquad + p_{lm}\mathbb{I}(s_2 \in C_1, J^{lm}(t+r, x) \in C_2)\Big) dx\Big] dr.
\end{aligned}
$$

Step 3 Calculation of conditional probability on the set $[S_t = s_2]$.

According to the first scenario, there are no shock arrivals on $[t, u)$ at all and, hence, Brownian motion hits one of the two boundaries h_1 or h_2 on (t, u). If it hits h_1 earlier than h_2, then the new state of the state process is s_3 and the jump size is equal to $J^u(t+\tau_{N_t+1})$. If it hits h_2 earlier than h_1, then the new state of the state process is s_1 and the jump size is equal to $J^l(t+\tau_{N_t+1})$. According to the second scenario, the first shock arrival time is $t+\tau<u$ and Brownian motion hits one of the two boundaries h_1 or h_2 on $(t,t+\tau)$, then the new state of the state process and the jump size are determined by the same mechanism as in the first scenario. Finally, according to the third scenario, the first shock arrival time is $t+\tau<u$ and Brownian motion stays between both boundaries h_1 and h_2 on $[t,t+\tau]$. With probability p_{mu}, the new state of the state process is s_3 and the jump size is $J^{mu}(t+\tau,B_{t+\tau})$. With probability $1-p_{mu}$, the new state of the state process is s_1 and the jump size is $J^{ml}(t+\tau,B_{t+\tau})$. Taking this decomposition, I obtain the formula for F_{12}:

$$
\begin{aligned}
&F_{12}(t,B_t,u,C_1,C_2)\\
&=e^{-\lambda_Z(u-t)}\int_t^u\Big[\mathbb{I}(s_3\in C_1,J^u(y)\in C_2)\phi_{m,1}(y,t,B_t)+\mathbb{I}(s_1\in C_1,J^l(y)\in C_2)\phi_{m,2}(y,t,B_t)\Big]dy\\
&+\int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r}\Big[\mathbb{I}(s_3\in C_1,J^u(y)\in C_2)\phi_{m,1}(y,t,B_t)+\mathbb{I}(s_1\in C_1,J^l(y)\in C_2)\phi_{m,2}(y,t,B_t)\Big]dy\\
&\qquad+\int_{h_2(t+r)}^{h_1(t+r)}q^m(x;r,t,B_t)\Big(p_{mu}\mathbb{I}(s_3\in C_1,J^{mu}(t+r,x)\in C_2)\\
&\qquad\qquad+p_{ml}\mathbb{I}(s_1\in C_1,J^{ml}(t+r,x)\in C_2)\Big)dx\Big]dr,
\end{aligned}
$$

where $q^m(x;r,t,y)$ is the density of B_r on the set $\Big[h_2(t+s)-y<B_s<h_1(t+s)-y,\forall s\in[0,r]\Big]$.

Step 4 Calculation of conditional probability on the set $[S_t=s_3]$.

The conditional probability on the set $[S_t=s_3]$ satisfies

$$
\begin{aligned}
F_{13}(t,B_t,u,C_1,C_2)&=e^{-\lambda_Z(u-t)}\int_t^u\mathbb{I}(s_1\in C_1,J^l(y)\in C_2)\phi_2(y,t,B_t)dy\\
&+\int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r}\mathbb{I}(s_1\in C_1,J^l(y)\in C_2)\phi_2(y,t,B_t)dy\\
&+\int_{h_1(t+r)}^{\infty}q_2(x;r,t,B_t)F_{13}(t+r,x,u,C_1,C_2)dx\\
&+\int_{h_2(t+r)}^{h_1(t+r)}q_2(x;r,t,B_t)\Big(p_{ul}\mathbb{I}(s_1\in C_1,J^{ul}(t+r,x)\in C_2)\\
&\qquad+p_{um}\mathbb{I}(s_2\in C_1,J^{um}(t+r,x)\in C_2)\Big)dx\Big]dr,
\end{aligned}
$$

where $q_2(x;r,t,y)$ is the density of B_r on the set $\Big[B_s>h_2(t+s)-y,\forall s\in[0,r]\Big]$. The calculation procedure is patterned after Step 2. ■

Proof of Lemma 2.1 First, the following decomposition is considered.

$$\mathbb{P}(\tau_{i+1} \leq u, Z_{i+1}^P \in C \mid \mathcal{F}_{\tau_i}^{Z^P}) = \mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{i+1} \leq u, Z_{i+1}^P \in C\Big] \mid \mathcal{F}_{\tau_i}^{Z^P}\Big)$$
$$= \sum_{j=1}^{3} \mathbb{I}\Big[S_{\tau_i} = s_j\Big]\mathbb{E}^{\mathbb{P}}\Big(\mathbb{I}\Big[\tau_{i+1} \leq u, Z_{i+1}^P \in C\Big] \mid \mathcal{F}_{\tau_i}^{Z^P}\Big).$$

Applying the same technique as in the proof of Theorem 2.7, I obtain that the conditional probabilities on the sets $[S_{\tau_i} = s_1]$, $[S_{\tau_i} = s_2]$ and $[S_{\tau_i} = s_3]$ are equal to

$$\begin{aligned}
&F_{26}(\tau_i, B_{\tau_i}, u, C)\\
&= e^{-\lambda_l(u-\tau_i)} \int_{\tau_i}^{u} \mathbb{I}(p^u(y, h_1(y)) \in C_1, J^u(y) \in C_2)\phi_1(y, \tau_i, B_{\tau_i})dy\\
&+ \int_0^{u-\tau_i} \Big(\int_{\tau_i}^{\tau_i+x} \mathbb{I}(p^u(y, h_1(y)) \in C_1, J^u(y) \in C_2)\phi_1(y, \tau_i, B_{\tau_i})dy\Big)\lambda_l e^{-\lambda_l x}dx\\
&+ \int_0^{u-\tau_i} \Big(\int_{\tau_i+x}^{u} (p_{lu}\mathbb{I}(p^u(y, h^l(y;\tau_i + x)) \in C_1, J^{lu}(y, h^l(y;\tau_i + x)) \in C_2)\\
&+ p_{lm}\mathbb{I}(p^m(y, h^l(y;\tau_i + x)) \in C_1, J^{lm}(y, h^l(y;\tau_i + x)) \in C_2))\phi^l(y, \tau_i + x, \tau_i, B_{\tau_i})dy\Big)\lambda_l e^{-\lambda_l x}dx,
\end{aligned}$$

$$\begin{aligned}
F_{27}(\tau_i, B_{\tau_i}, u, C) = \int_0^{u-\tau_i} &\Big[\mathbb{I}(p^u(\tau_i + y, h_1(\tau_i + y)) \in C_1, J^u(\tau_i + y) \in C_2)\phi_{m,1}(y, \tau_i, B_{\tau_i})\\
&+ \mathbb{I}(p^l(\tau_i + y, h_2(\tau_i + y)) \in C_1, J^l(\tau_i + y) \in C_2)\phi_{m,2}(y, \tau_i, B_{\tau_i})\Big]dy
\end{aligned}$$

and

$$\begin{aligned}
&F_{28}(\tau_i, B_{\tau_i}, u, C)\\
&= e^{-\lambda_u(u-\tau_i)} \int_{\tau_i}^{u} \mathbb{I}(p^l(y, h_2(y)) \in C_1, J^l(y) \in C_2)\phi_2(y, \tau_i, B_{\tau_i})dy\\
&+ \int_0^{u-\tau_i} \Big(\int_{\tau_i}^{\tau_i+x} \mathbb{I}(p^l(y, h_2(y)) \in C_1, J^l(y) \in C_2)\phi_2(y, \tau_i, B_{\tau_i})dy\Big)\lambda_u e^{-\lambda_u x}dx\\
&+ \int_0^{u-\tau_i} \Big(\int_{\tau_i+x}^{u} (p_{ul}\mathbb{I}(p^l(y, h^u(y;\tau_i + x)) \in C_1, J^{ul}(y, h^u(y;\tau_i + x)) \in C_2)\\
&+ p_{um}\mathbb{I}(p^m(y, h^u(y;\tau_i + x)) \in C_1, J^{um}(y, h^u(y;\tau_i + x)) \in C_2))\phi^u(y, \tau_i + x, \tau_i, B_{\tau_i})dy\Big)\lambda_u e^{-\lambda_u x}dx.
\end{aligned}$$

■

Proof of Lemma 2.2 Applying Leibniz's rule for differentiating integrals to F_{26}, F_{27} and F_{28}, I obtain

$$
\begin{aligned}
&F_{29}(\tau_i, B_{\tau_i}, s, C) = e^{-\lambda_l(u-\tau_i)}\mathbb{I}(p^u(u, h_1(u)) \in C_1, J^u(u) \in C_2)\phi_1(u, \tau_i, B_{\tau_i})\\
&\quad + \int_0^{u-\tau_i} \Big(p_{lu}\mathbb{I}(p^u(u, h^l(u; \tau_i + x)) \in C_1, J^{lu}(u, h^l(u; \tau_i + x)) \in C_2)\\
&\quad + p_{lm}\mathbb{I}(p^m(u, h^l(u; \tau_i + x)) \in C_1, J^{lm}(u, h^l(u; \tau_i + x)) \in C_2)\Big)\phi^l(u, \tau_i + x, \tau_i, B_{\tau_i})\lambda_l e^{-\lambda_l x}dx,
\end{aligned}
\tag{.11}
$$

$$
\begin{aligned}
F_{30}(\tau_i, B_{\tau_i}, u, C) &= \mathbb{I}(p^u(u, h_1(u)) \in C_1, J^u(u) \in C_2)\phi_{m,1}(u, \tau_i, B_{\tau_i})\\
&\quad + \mathbb{I}(p^l(u, h_2(u)) \in C_1, J^l(u) \in C_2)\phi_{m,2}(u, \tau_i, B_{\tau_i})
\end{aligned}
\tag{.12}
$$

and

$$
\begin{aligned}
&F_{31}(\tau_i, B_{\tau_i}, u, C) = e^{-\lambda_u(u-\tau_i)}\mathbb{I}(p^l(u, h_2(u)) \in C_1, J^l(u) \in C_2)\phi_2(u, \tau_i, B_{\tau_i})\\
&\quad + \int_0^{u-\tau_i} \Big(p_{ul}\mathbb{I}(p^l(u, h^u(u; \tau_i + x)) \in C_1, J^{ul}(u, h^u(u; \tau_i + x)) \in C_2)\\
&\quad + p_{um}\mathbb{I}(p^m(u, h^u(u; \tau_i + x)) \in C_1, J^{um}(u, h^u(u; \tau_i + x)) \in C_2)\Big)\phi^u(u, \tau_i + x, \tau_i, B_{\tau_i})\lambda_u e^{-\lambda_u x}dx.
\end{aligned}
\tag{.13}
$$

Finally, if $C = \mathbb{R}^2$, then indicator functions in (.11) – (.13) are equal to 1, and the result for $g^{(i+1)}(u, \mathbb{R}^2)$ follows. ■

Proof of Lemma 2.3 Calculations pattern after Theorem 2.8, and $F_{35}(u, t, B_t, C)$ satisfies

$$
\begin{aligned}
F_{35}(u, t, B_t, C) &= e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(p^u(y, h_1(y)) \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t)dy\\
&\quad + \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r} \mathbb{I}(p^u(y, h_1(y)) \in C_1, J^u(y) \in C_2)\phi_1(y, t, B_t)dy\\
&\quad + \int_{-\infty}^{h_2(t+r)} q_1(x; r, t, B_t)F_{35}(u, t + r, x, C)dx\\
&\quad + \int_{h_2(t+r)}^{h_1(t+r)} q_1(x; r, t, B_t)\Big(p_{lu}\mathbb{I}(p^u(t + r, x) \in C_1, J^{lu}(t + r, x) \in C_2)\\
&\qquad + p_{lm}\mathbb{I}(p^m(t + r, x) \in C_1, J^{lm}(t + r, x) \in C_2)\Big)dx\Big]dr,
\end{aligned}
$$

$$
\begin{aligned}
&F_{36}(u,t,B_t,C) \\
&= e^{-\lambda_Z(u-t)}\int_t^u \Big[\mathbb{I}(p^u(y,h_1(y))\in C_1, J^u(y)\in C_2)\phi_{m,1}(y,t,B_t) \\
&\qquad + \mathbb{I}(p^l(y,h_2(y))\in C_1, J^l(y)\in C_2)\phi_{m,2}(y,t,B_t)\Big]dy \\
&+ \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r}\Big[\mathbb{I}(p^u(y,h_1(y))\in C_1, J^u(y)\in C_2)\phi_{m,1}(y,t,B_t) \\
&\qquad + \mathbb{I}(p^l(y,h_2(y))\in C_1, J^l(y)\in C_2)\phi_{m,2}(y,t,B_t)\Big]dy \\
&\qquad + \int_{h_2(t+r)}^{h_1(t+r)}\Big(p_{mu}\mathbb{I}(p^u(t+r,x)\in C_1, J^{mu}(t+r,x)\in C_2) \\
&\qquad + p_{ml}\mathbb{I}(p^l(t+r,x)\in C_1, J^{ml}(t+r,x)\in C_2)\Big)q^m(x;r,t,B_t)dx\Big]dr
\end{aligned}
$$

and $F_{37}(u,t,B_t,C)$ satisfies

$$
\begin{aligned}
F_{37}(u,t,B_t,C) &= e^{-\lambda_Z(u-t)}\int_t^u \mathbb{I}(p^l(y,h_2(y))\in C_1, J^l(y)\in C_2)\phi_2(y,t,B_t)dy \\
&+ \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_t^{t+r}\mathbb{I}(p^l(y,h_2(y))\in C_1, J^l(y)\in C_2)\phi_2(y,t,B_t)dy \\
&+ \int_{h_1(t+r)}^{\infty} q_2(x;t,B_t,r)F_{37}(u,t+r,x,C)dx \\
&+ \int_{h_2(t+r)}^{h_1(t+r)} q_2(x;t,B_t,r)\Big(p_{ul}\mathbb{I}(p^l(t+r,x)\in C_1, J^{ul}(t+r,x)\in C_2) \\
&\qquad + p_{um}\mathbb{I}(p^m(t+r,x)\in C_1, J^{um}(t+r,x)\in C_2)\Big)dx\Big]dr.
\end{aligned}
$$

■

Proof of Lemma 2.4 Applying Leibniz's rule for differentiating integrals to F_{35}, F_{36} and F_{37}, I obtain that $F_{38}(u,t,B_t,C)$ satisfies

$$
\begin{aligned}
F_{38}(u,t,B_t,C) &= e^{-\lambda_Z(u-t)}\mathbb{I}(p^u(u,h_1(u))\in C_1, J^u(u)\in C_2)\phi_1(u,t,B_t) \\
&+ \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h_2(u)}^{h_1(u)} q_1(x;u-t,t,B_t)\Big(p_{lu}\mathbb{I}(p^u(u,x)\in C_1, J^{lu}(u,x)\in C_2) \\
&\qquad + p_{lm}\mathbb{I}(p^m(u,x)\in C_1, J^{lm}(u,x)\in C_2)\Big)dx\Big] \\
&+ \int_0^{u-t}\lambda_Z e^{-\lambda_Z r}\Big[\int_{-\infty}^{h_2(t+r)} q_1(x;r,t,B_t)F_{38}(u,t+r,x,C)dx\Big]dr, \qquad (.14)
\end{aligned}
$$

$$F_{39}(u,t,B_t,C) = e^{-\lambda_Z(u-t)}\Big[\mathbb{I}(p^u(u,h_1(u)) \in C_1, J^u(u) \in C_2)\phi_{m,1}(u,t,B_t)$$
$$+ \mathbb{I}(p^l(u,h_2(u)) \in C_1), J^l(u) \in C_2)\phi_{m,2}(u,t,B_t)\Big]$$
$$+ \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h_2(u)}^{h_1(u)} q^m(x;u-t,t,B_t)\Big(p_{mu}\mathbb{I}(p^u(u,x) \in C_1, J^{mu}(u,x) \in C_2)$$
$$+ p_{ml}\mathbb{I}(p^l(u,x) \in C_1, J^{ml}(u,x) \in C_2)\Big)dx\Big] \quad (.15)$$

and $F_{40}(u,t,B_t,C)$ satisfies

$$F_{40}(u,t,B_t,C) = e^{-\lambda_Z(u-t)}\mathbb{I}(p^l(u,h_2(u)) \in C_1, J^l(u) \in C_2)\phi_2(u,t,B_t)$$
$$+ \lambda_Z e^{-\lambda_Z(u-t)}\Big[\int_{h_2(u)}^{h_1(u)} q_2(x;t,B_t,u-t)\Big(p_{ul}\mathbb{I}(p^l(u,x) \in C_1, J^{ul}(u,x) \in C_2)$$
$$+ p_{um}\mathbb{I}(p^m(u,x) \in C_1, J^{um}(u,x) \in C_2)\Big)dx\Big]$$
$$+ \int_0^{u-t} \lambda_Z e^{-\lambda_Z r}\Big[\int_{h_1(t+r)}^{\infty} q_2(x;t,B_t,r)F_{40}(u,t+r,x,C)dx\Big]dr. \quad (.16)$$

In particular, for $C = \mathbb{R}^2$, indicator functions in (.14) – (.16) are equal to 1, and the result for $g^{(i+1)}(u,\mathbb{R}^2)$ follows. ■

Proof of Theorem 2.12 First, I prove that stochastic processes w_t^D, B_t and S_t are adapted to the filtration $\mathcal{F}_t^P$. By the pricing equation and continuity of B_t, for $i = 1, 2, ...$,

$$\frac{\gamma_1 P_{\tau_i} - w_{\tau_i}^D \int_{-\infty}^{\infty} \Phi\Big(\frac{P_{\tau_i} - Ke^{-r(T-\tau_i)}}{\Sigma(\tau_i)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK - \gamma_2}{\gamma_3} = B_{\tau_i}$$

and

$$\frac{\gamma_1 P_{\tau_i -} - w_{\tau_{i-1}}^D \int_{-\infty}^{\infty} \Phi\Big(\frac{P_{\tau_i -} - Ke^{-r(T-\tau_i)}}{\Sigma(\tau_i)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK - \gamma_2}{\gamma_3} = B_{\tau_i},$$

which means that

$$w_{\tau_i}^D = \frac{\gamma_1 \Delta P_{\tau_i} + w_{\tau_{i-1}}^D \int_{-\infty}^{\infty} \Phi\Big(\frac{P_{\tau_i -} - Ke^{-r(T-\tau_i)}}{\Sigma(\tau_i)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK}{\int_{-\infty}^{\infty} \Phi\Big(\frac{P_{\tau_i} - Ke^{-r(T-\tau_i)}}{\Sigma(\tau_i)}\Big)\frac{1}{\sqrt{2\pi\sigma_\kappa^2}}e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}}dK},$$

thus, since $(\tau_i < T, i = 1, 2, ...)$ are $\mathcal{F}_t^P$-stopping times and $w_{T_0}^D$ is known, it can be concluded that, by induction, $w_t^D = \sum_{i=0}^{\infty} w_{\tau_i}^D \mathbb{I}(\tau_i \le t < \tau_{i+1})$ is adapted to the filtration $\mathcal{F}_t^P$.

Hence,

$$B_t = \frac{\gamma_1 P_t + w_t^D \int_{-\infty}^{\infty} \Phi\Big(\frac{P_t - Ke^{-r(T-t)}}{\Sigma(t)}\Big) \frac{1}{\sqrt{2\pi\sigma_\kappa^2}} e^{-\frac{(K-\kappa)^2}{2\sigma_\kappa^2}} dK - w^R \times \frac{a\alpha_2}{\alpha_1^2} - w^N \times \mu_N}{\gamma_3}$$

is also adapted to the filtration $\mathcal{F}_t^P$.

Finally, S_t is adapted to the filtration $\mathcal{F}_t^P$ since $S_t = \sum_{i=0}^{\infty} S_{\tau_i} \mathbb{I}(\tau_i \le t < \tau_{i+1})$ and, for all $i = 0, 1, ...,$

$$S_{\tau_i} = \begin{cases} s_1 & \text{if } w_{\tau_i}^D > g^D(\tau_i) \text{ and } P_{\tau_i} < \bar{p}_1(\tau_i, w_{\tau_i}^D) \\ s_2 & \text{if } w_{\tau_i}^D \le g^D(\tau_i) \\ s_3 & \text{if } w_{\tau_i}^D > g^D(\tau_i) \text{ and } P_{\tau_i} > \bar{p}_2(\tau_i, w_{\tau_i}^D). \end{cases}$$

The rest of the proof is patterned after Theorem 2.7 and Theorem 2.8. In view of the fact that S_t is adapted to $\mathcal{F}_t^P$, one can apply the initial decomposition described by (.2) and then calculate conditional probabilities on the sets $[S_t = s_1]$, $[S_t = s_2]$ and $[S_t = s_3]$ considering all possible scenarios in accordance with the model construction. Recall that, when the number of dynamic hedgers changes, it is multiplied by a corresponding random variable ξ_i distributed according to a uniform law with density function $f_\xi(x) = \frac{1}{\xi^u - \xi^l}, x \in [\xi^l, \xi^u]$, where $0 \le \xi^l < 1 < \xi^u$.

It can be concluded that $F_{44}(t, w_t^D, B_t, u, C_1, C_2)$ is equal to

$$\begin{aligned} &F_{44}(t, w_t^D, B_t, u, C_1, C_2) \\ &= e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(s_3 \in C_1, \bar{p}^u(y, w_t^D, H_1(y, w_t^D)) - \bar{p}_1(y, w_t^D) \in C_2) \bar{\phi}_1(y, t, B_t, w_t^D) dy \\ &+ \int_0^{u-t} \Big(\int_t^{t+r} \mathbb{I}(s_3 \in C_1, \bar{p}^u(y, w_t^D, H_1(y, w_t^D)) - \bar{p}_1(y, w_t^D) \in C_2) \bar{\phi}_1(y, t, B_t, w_t^D) dy \Big) \lambda_Z e^{-\lambda_Z r} dr \\ &+ \int_0^{u-t} \Big(\int_{-\infty}^{H_1(t+r, w_t^D)} \Big[\int_{\xi^l}^{\xi^u} F_{48}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy \Big] \bar{q}_1(x; r, t, B_t, w_t^D) dx \Big) \lambda_Z e^{-\lambda_Z r} dr, \end{aligned}$$

where

$$\bar{\phi}_1(u, t, y, x) = -\frac{\partial \bar{D}_1(u, t, y, x)}{\partial u} \quad \text{and} \quad \bar{D}_1(u, t, y, x) = \mathbb{P}\Big(B_s < H_1(t+s, x) - y, 0 \le s \le u - t\Big)$$

are Brownian motion hitting density and probability of one-sided curved boundary, $\bar{q}_1(x; r, t, y, w_t^D)$ is the density of B_r on the set $\Big[B_s < H_1(t+s, w_t^D) - y, \forall s \in [0, r]\Big]$ and

$$\begin{aligned} &F_{48}(y, w_t^D, t+r, x, C_1, C_2) \\ &= \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big) \mathbb{I}\Big(x < H_1(t+r, yw_t^D)\Big) \mathbb{I}\Big(s_1 \in C_1, \bar{p}^l(t+r, yw_t^D, x) - \bar{p}^l(t+r, w_t^D, x) \in C_2\Big) \\ &+ \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big) \mathbb{I}\Big(x \ge H_1(t+r, yw_t^D)\Big) \mathbb{I}\Big(s_3 \in C_1, \bar{p}^u(t+r, yw_t^D, x) - \bar{p}^l(t+r, w_t^D, x) \in C_2\Big) \\ &+ \mathbb{I}\Big(yw_t^D \le g^D(t+r)\Big) \mathbb{I}\Big(s_2 \in C_1, \bar{p}(t+r, yw_t^D, x) - \bar{p}^l(t+r, w_t^D, x) \in C_2\Big). \end{aligned}$$

Similarly, $F_{46}(t, w_t^D, B_t, u, C_1, C_2)$ is equal to

$$
\begin{aligned}
&F_{46}(t, w_t^D, B_t, u, C_1, C_2) \\
&= e^{-\lambda_Z(u-t)} \int_t^u \mathbb{I}(s_1 \in C_1, \bar{p}^l(y, w_t^D, H_2(y, w_t^D)) - \bar{p}_2(y, w_t^D) \in C_2)\bar{\phi}_2(y, t, B_t, w_t^D)dy \\
&+ \int_0^{u-t} \Big(\int_t^{t+r} \mathbb{I}(s_1 \in C_1, \bar{p}^l(y, w_t^D, H_2(y, w_t^D)) - \bar{p}_2(y, w_t^D) \in C_2)\bar{\phi}_2(y, t, B_t, w_t^D)dy \Big) \lambda_Z e^{-\lambda_Z r} dr \\
&+ \int_0^{u-t} \Big(\int_{H_2(t+r, w_t^D)}^{\infty} \Big[\int_{\xi^l}^{\xi^u} F_{49}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy \Big] \bar{q}_2(x; r, t, B_t, w_t^D) dx \Big) \lambda_Z e^{-\lambda_Z r} dr,
\end{aligned}
$$

where

$$
\bar{\phi}_2(u, t, y, x) = -\frac{\partial \bar{D}_2(u, t, y, x)}{\partial u} \quad \text{and} \quad \bar{D}_2(u, t, y, x) = \mathbb{P}\Big(B_s > H_2(t+s, x) - y, 0 \le s \le u - t\Big)
$$

are Brownian motion hitting density and probability of one-sided curved boundary, $\bar{q}_2(x; r, t, y, w_t^D)$ is the density of B_r on the set $\Big[B_s > H_2(t+s, w_t^D) - y, \forall s \in [0, r]\Big]$ and

$$
\begin{aligned}
&F_{49}(y, w_t^D, t+r, x, C_1, C_2) \\
&= \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big)\mathbb{I}\Big(x > H_2(t+r, yw_t^D)\Big)\mathbb{I}\Big(s_3 \in C_1, \bar{p}^u(t+r, yw_t^D, x) - \bar{p}^u(t+r, w_t^D, x) \in C_2\Big) \\
&+ \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big)\mathbb{I}\Big(x \ge H_2(t+r, yw_t^D)\Big)\mathbb{I}\Big(s_1 \in C_1, \bar{p}^l(t+r, yw_t^D, x) - \bar{p}^u(t+r, w_t^D, x) \in C_2\Big) \\
&+ \mathbb{I}\Big(yw_t^D \le g^D(t+r)\Big)\mathbb{I}\Big(s_2 \in C_1, \bar{p}(t+r, yw_t^D, x) - \bar{p}^u(t+r, w_t^D, x) \in C_2\Big).
\end{aligned}
$$

Finally, if I denote by

$$
\begin{aligned}
&F_{50}(y, w_t^D, t+r, x, C_1, C_2) \\
&= \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big)\Big[\mathbb{I}\Big(x \le H_2(t+r, yw_t^D)\Big) + p_l \mathbb{I}\Big(H_2(t+r, yw_t^D) < x < H_1(t+r, yw_t^D)\Big)\Big] \\
&\qquad \times \mathbb{I}\Big(s_1 \in C_1, \bar{p}^l(t+r, yw_t^D, x) - \bar{p}(t+r, w_t^D, x) \in C_2\Big)\Big] \\
&+ \mathbb{I}\Big(yw_t^D > g^D(t+r)\Big)\Big[\mathbb{I}\Big(x \ge H_1(t+r, yw_t^D)\Big) + p_u \mathbb{I}\Big(H_2(t+r, yw_t^D) < x < H_1(t+r, yw_t^D)\Big)\Big] \\
&\qquad \times \mathbb{I}\Big(s_3 \in C_1, \bar{p}^u(t+r, yw_t^D, x) - \bar{p}(t+r, w_t^D, x) \in C_2\Big)\Big] \\
&+ \mathbb{I}\Big(yw_t^D \le g^D(t+r)\Big)\mathbb{I}\Big(s_2 \in C_1, \bar{p}(t+r, yw_t^D, x) - \bar{p}(t+r, w_t^D, x) \in C_2\Big),
\end{aligned}
$$

then $F_{45}(t, w_t^D, B_t, u, C_1, C_2)$ is equal to

$$
\begin{aligned}
&F_{45}(t, w_t^D, T^D(w_t^D), B_t, u, C_1, C_2)\\
&= \mathbb{I}\Big(T^D(w_t^D) \geq u\Big) \int_0^{u-t} \Big(\int_{-\infty}^{\infty} \Big[\int_{\xi^l}^{\xi^u} F_{50}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy\Big] \frac{1}{\sqrt{2\pi r}} e^{-\frac{(x-B_t)^2}{2r}} dx\Big) \lambda_Z e^{-\lambda_Z r} dr\\
&+ \mathbb{I}\Big(T^D(w_t^D) < u\Big) \times\\
&\times \Big[\int_0^{T^D(w_t^D)-t} \Big(\int_{-\infty}^{\infty} \Big[\int_{\xi^l}^{\xi^u} F_{50}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy\Big] \frac{1}{\sqrt{2\pi r}} e^{-\frac{(x-B_t)^2}{2r}} dx\Big) \lambda_Z e^{-\lambda_Z r} dr\\
&\quad + \int_{T^D(w_t^D)-t}^{u-t} \lambda_Z e^{-\lambda_z r} \Big(\int_{-\infty}^{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad \times [\int_{T^D(w_t^D)}^{t+r} \mathbb{I}\Big(s_3 \in C_1, \bar{p}^u(z, w_t^D, H_1(z, w_t^D)) - \bar{p}_1(z, w_t^D) \in C_2\Big) \bar{\phi}_1(z, T^D(w_t^D), x, w_t^D) dz] dx\\
&\qquad\qquad + \int_{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})}^{\infty} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad \times [\int_{T^D(w_t^D)}^{t+r} \mathbb{I}\Big(s_1 \in C_1, \bar{p}^l(z, w_t^D, H_2(z, w_t^D)) - \bar{p}_2(z, w_t^D) \in C_2\Big) \bar{\phi}_2(z, T^D(w_t^D), x, w_t^D) dz] dx\Big) dr\\
&\quad + \int_{T^D(w_t^D)-t}^{u-t} \lambda_Z e^{-\lambda_z r} \Big(\int_{-\infty}^{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad\qquad \times \Big[\int_{\xi^l}^{\xi^u} F_{48}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy\Big] \bar{D}_1(t+r, T^D(w_t^D), x, w_t^D) dx\\
&\qquad\qquad + \int_{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})}^{\infty} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad\qquad \times \Big[\int_{\xi^l}^{\xi^u} F_{49}(y, w_t^D, t+r, x, C_1, C_2) f_\xi(y) dy\Big] \bar{D}_2(t+r, T^D(w_t^D), x, w_t^D) dx\Big) dr\\
&\quad + e^{-\lambda_Z(u-t)} \Big(\int_{-\infty}^{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad \times [\int_{T^D(w_t^D)}^{u} \mathbb{I}\Big(s_3 \in C_1, \bar{p}^u(z, w_t^D, H_1(z, w_t^D)) - \bar{p}_1(z, w_t^D) \in C_2\Big) \bar{\phi}_1(z, T^D(w_t^D), x, w_t^D) dz] dx\\
&\qquad + \int_{H(T^D(w_t^D), w_t^D, \kappa e^{-r(T-T^D(w_t^D))})}^{\infty} \frac{1}{\sqrt{2\pi(T^D(w_t^D)-t)}} e^{-\frac{(x-B_t)^2}{2(T^D(w_t^D)-t)}} \times\\
&\qquad \times [\int_{T^D(w_t^D)}^{u} \mathbb{I}\Big(s_1 \in C_1, \bar{p}^l(z, w_t^D, H_2(z, w_t^D)) - \bar{p}_2(z, w_t^D) \in C_2\Big) \bar{\phi}_2(z, T^D(w_t^D), x, w_t^D) dz] dx\Big)\Big].
\end{aligned}
$$

■

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