Topics in Graph Colouring and Graph Structures

David G. Ferguson

A thesis submitted for the degree of
Doctor of Philosophy

Department of Mathematics

London School of Economics
and Political Science

April 2013

Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work, with the following exceptions:

Chapter 3 is based on joint work with Daniel Král'.

Chapter 4 and Appendix A are based on joint work with Tomáš Kaiser and Daniel Král'.

Abstract

This thesis investigates problems in a number of different areas of graph theory. These problems are related in the sense that they mostly concern the colouring or structure of the underlying graph.

The first problem we consider is in Ramsey Theory, a branch of graph theory stemming from the eponymous theorem which, in its simplest form, states that any sufficiently large graph will contain a clique or anti-clique of a specified size. The problem of finding the minimum size of underlying graph which will guarantee such a clique or anti-clique is an interesting problem in its own right, which has received much interest over the last eighty years but which is notoriously intractable. We consider a generalisation of this problem. Rather than edges being present or not present in the underlying graph, each is assigned one of three possible colours and, rather than considering cliques, we consider cycles. Combining regularity and stability methods, we prove an exact result for a triple of long cycles.

We then move on to consider removal lemmas. The classic Removal Lemma states that, for n sufficiently large, any graph on n vertices containing $o(n^3)$ triangles can be made triangle-free by the removal of $o(n^2)$ edges. Utilising a coloured hypergraph generalisation of this result, we prove removal lemmas for two classes of multinomials.

Next, we consider a problem in fractional colouring. Since finding the chromatic number of a given graph can be viewed as a integer programming problem, it is natural to consider the solution to the corresponding linear programming problem. The solution to this LP-relaxation is called the fractional chromatic number. By a probabilistic method, we improve on the best previously known bound for the fractional chromatic number of a triangle-free graph with maximum degree at most three.

Finally, we prove a weak version of Vizing's Theorem for hypergraphs. We prove that, if H is an intersecting 3-uniform hypergraph with maximum degree Δ and maximum multiplicity μ, then H has at most $2\Delta + \mu$ edges. Furthermore, we prove that the unique structure achieving this maximum is μ copies of the Fano Plane.

Acknowledgements

This thesis could not exist without the help and support of numerous people over the last few years. I am hugely indebted to friends, family and colleagues.

Let me begin by thanking my supervisors, Jan van den Heuvel and Jozef Skokan. Thanks to Jan for consistently providing thoughtful feedback on the content and presentation of this thesis and for pointing me in the direction of many interesting problems. Thanks to Jozef for introducing me to stability and regularity in the context of Ramsey Theory and for the many hours spent discussing this thesis over the recent months.

Thanks also to everyone else in the Mathematics Department at the LSE. In particular for helpful discussions, insightful comments and congenial lunchtime chats. I am, of course, also grateful to the department and to the school for financial support received through the LSE research studentship scheme.

I would also like to express my gratitude to the Department of Applied Mathematics at Charles University in Prague and especially to Dan Král' for hosting me as a visitor. Thanks also to Dan for introducing me to fractional colouring and removal lemmas and thanks to my other co-author, Tomáš Kaiser, for sharing ideas and for his efforts in writing up the long case analysis for our fractional colouring paper.

Special thanks go to John Mackay and Peter Allen for taking the time to review an early draft of this thesis and to all my colleagues at the University of Buckingham for their kind support and understanding throughout the last eighteen months.

Too many friends deserve thanks for each to receive a specific mention — their unwavering belief in my abilities has sustained me through the most difficult moments of the past few years.

Finally, thank you to my parents for encouraging my interest in mathematics from an early age and giving me the best possible start.

To Sarah,
for everything.

Contents

Chapter 1

Introduction

This thesis considers a number of problems in graph theory. A graph is an abstract mathematical structure formed by a set of vertices and edges joining pairs of those vertices. Graphs can be used to model the connections between objects; for instance, a computer network can be modelled as a graph with each server represented by a vertex and the connections between those servers represented by edges.

Many problems in graph theory involve some sort of colouring, that is, assignment of labels or 'colours' to the edges or vertices of a graph. Such problems fall broadly into two categories: The first type of problem concerns the possibility of assigning colours to a graph while respecting some set of rules; the second concerns the existence of coloured structures in a graph whose colouring we do not control.

The field of graph colouring traces its origins to 1852, when Francis Guthrie observed that a map of the counties of England can be coloured using four colours in such a way that adjacent counties receive different colours. The question of whether this is the case for any such map became known as the Four Colour Problem and is, without doubt, the most well-known problem from the first category above. This problem received much attention over the following century (see, for instance, [Wil03]) before, finally, being answered in the affirmative by Appel and Haken [AH77, AHK77] in 1976.

The archetypal problem of the second type can also be phrased in a non-abstract form as follows: Suppose you were to invite multiple guests to a dinner-party and that those guests have not necessarily met each other previously. How many guests would you need to invite in order to guarantee that there will be three mutual acquaintances or three mutual strangers at the dinner table? Upon first reading, it is less than obvious that such a question should have a finite answer — perhaps, for any size of party, there is a possible list of acquaintances and strangers without such a triad. In fact, it can easily be shown that the answer is six and, as we will see later, that no matter how large a collection of mutual acquaintances or collection of mutual strangers we require, there is a finite size of gathering that will guarantee the existence of one or the other. However, finding the exact answer to this general problem is notoriously difficult.

An interesting feature of many problems in Graph Theory (including the two problems above) is the contrast between the ease with which they may be stated and the apparent difficulty of their solution. This contrast is also apparent in most of the problems considered in this thesis.

Before formally introducing the main themes and problems considered in this thesis, we must give a few key definitions:

1.1 Definitions and notation

The notation used in this thesis is mostly standard and can be found, for instance, in [Bol98], [BM08] and [Die05]. In this section, we give definitions of the objects and concepts we will use most frequently.

Abstractly, a graph is defined by its vertices (which we assume form a finite set) and its edges (each of which *joins* a pair of distinct vertices). In this thesis we sometimes allow multiple edges between the same pair of vertices. We refer to a graph without such *multi-edges* as a *simple graph* (but usually omit the prefix) and refer to the analogous object in which multi-edges are allowed as a *multigraph*.

For a given graph G, we use $V(G)$ to denote its vertex set and $E(G)$ to denote its edge set. When it is clear from context which graph is being discussed, we will refer to these sets as simply V and E respectively.

As is standard, we use K_n to denote the *complete graph* on n vertices, that is, the graph on n vertices including all possible edges, and use C_n to refer to the cycle on n vertices. Additionally, we use P_n to denote the path on n vertices but will refer to such a path as having length $n-1$, that is, the *length* of a path P, denoted $|P|$, will be equal to the number of its edges.

We say that a graph $G = (V, E)$ is *Hamiltonian* if it has, as a subgraph, a cycle which visits every vertex and call such a cycle a *Hamiltonian cycle*. Analogously, we call a path which visits every vertex a *Hamiltonian path*.

Given X, a subset of the vertex set of a graph G, we denote by $G[X]$ the graph with vertex set X and edge set $\{e \in E(G) : e \subseteq X\}$. Similarly, given a pair of disjoint subsets X, Y of the vertex set of a graph G, we use $G[X, Y]$ to denote the subgraph of G whose edges have one end in X and one end in Y.

For a multigraph G, we refer to edges joining the same pair of vertices as *copies* of each other. Then, given an edge e, we define the *multiplicity* of that edge $\mu(e)$ to be the number of copies of e present in G.

For a (multi)graph G, given a vertex v, we define the *degree* of that vertex $d(v)$ to be the number of edges (including copies) incident at v. We write $\delta(G)$ for the minimum degree, that is, the minimum of $d(v)$ over the vertices of G. Similarly, we write $\Delta(G)$ for the maximum degree and $d(G)$ for the average degree. We use $e(G)$ to denote $|E(G)|$ and $e(X, Y)$ to denote $|e(G[X, Y])|$. We write $d(X, Y)$ for the *density* of the pair (X, Y), that is, $e(X, Y)/|X||Y|$.

For a given graph G, we say a set of vertices $X \subseteq V(G)$ is *independent* if $G[X]$ contains no edges. Similarly, we define a *matching* to be a set of independent edges, that is, a collection of pairwise vertex-disjoint edges. Equivalently, for a given graph, a matching is a subgraph in which every vertex has degree one. For a matching M including an edge uv, we refer to v (resp. u) as the M-mate of u (resp. v).

For a given graph G, a *perfect matching* or 1-*factor* is a matching which spans all the vertices of G or, equivalently, a spanning subgraph in which every vertex has degree one. Analogously, for a given graph, a k-*factor* is a spanning subgraph in which every vertex has degree k.

We also consider hypergraphs, that is, structures analogous to graphs in which the edges are permitted to span any number of vertices. Most often, when doing so, we, in fact, consider r-*uniform* hypergraphs, that is, hypergraphs in which every edge spans exactly r vertices. Most of the definitions given above carry over from graphs to hypergraphs. In particular, we define $d(v)$, $\delta(H)$, $\Delta(H)$ in the same way.

Note that further definitions appear in each of the sections and chapters that follow.

1.2 Graph colouring

In the chapters which follow, we will make use of various notions of colourings of graphs (and hypergraphs). We now define some of these notions and discuss some fundamental results in (proper) graph colouring.

By a *colouring* of a graph, we mean an assignment of a *colour* (that is, a label from some list) to either each vertex (a *vertex-colouring*) or each edge (an *edge-colouring*). A vertex-colouring is called *proper* if no two adjacent vertices are assigned the same colour. An edge-colouring is called *proper* if no two edges of the same colour meet at a vertex. A (proper) k*-colouring* is a (proper) colouring using at most k colours. A *multicolouring* is a colouring where multiple colours may be assigned to each edge or vertex. Where context permits, we will omit these prefixes so that, for instance, we may refer to a proper k-edge-multicolouring simply as a colouring.

Note that, when a small number of colours are being used, it is usual to give them names. In this thesis, the first three colours will always be refered to as red, blue and green (in that order). When a larger number of colours are used, they will be referred to as $c_1, c_2, \ldots c_k$ or $1, 2, \ldots k$.

In Chapters 2 and 3 we will make use of colourings without requiring them to be proper. However, it is quite usual for references to graph (and hypergraph) colourings to be taken to refer to proper colourings and indeed we will make use of this notion of colouring in Chapters 4 and 5 and, also, in the remainder of this section.

When colouring a graph, one may ask,

"What is the minimum number of colours required to properly colour a given graph G?"

For vertex-colouring, this minimum is called the *chromatic number* $\chi(G)$ of G and, for edge-colouring, the *chromatic index* $\chi'(G)$ of G.

At this point, it is worth noting some alternative but equivalent definitions in terms of independent sets. For vertex-colouring, defining a *colour class* to be the set of vertices assigned a particular colour, we can see that each colour class forms an independent set of vertices and that a proper k-vertex-colouring is a partition of the vertices of a graph into k independent sets. Thus, we could define $\chi(G)$ to be the minimum k such that there exists a partition of the vertices of G into k independent sets. Similarly, defining a *colour class* for an edge-colouring to be the set of edges assigned a particular colour, we can see that each colour class forms a matching. Thus, we may view an edge-colouring as a partition and define $\chi'(G)$ to be the minimum k such that there exists a partition of the edges of G into k matchings.

For vertex-colouring, Brooks' Theorem [Bro41] tells us that we can colour any graph G using at most $\Delta(G) + 1$ colours and that, for most graphs, $\Delta(G)$ colours suffice. More precisely, it tells us that, if G is a connected graph with maximum degree Δ, then $\chi(G) \leq \Delta$, unless G is an odd cycle or a complete graph, in which case $\chi(G) = \Delta + 1$.

Given a graph $G = (V, E)$, we define its *line graph* $L(G)$ to be the graph with vertex set E whose edges are the pairs $\{e_1, e_2\} \subseteq E$ which intersect at a vertex in G. Then, considering the line graph $L(G)$ and applying Brooks' Theorem gives us the following upper bound for the chromatic index:

$$\chi'(G) \leq 2\Delta - 1.$$

However, this upper bound is, in general, not best possible.

Before proceeding, let us note that, when vertex-colouring, the addition of extra copies of any given edge does not alter the possibility or impossibility of colouring a given graph using a given number of colours, since any colouring that is proper for a graph will remain proper if edges are removed or if extra copies of an existing edge are added. However, the same is not true of edge-colourings, since multiple copies of an edge will each require a different colour. Therefore, when discussing edge-colourings, we must specify carefully whether or not to allow multiple copies of a given edge.

Shannon [Sha49] proved the following bound for multigraphs:

$$\chi'(G) \leq \tfrac{3}{2}\Delta.$$

Shannon's bound is the best possible bound of this form, as demonstrated by the Shannon multigraphs shown in Figure 1.1.

Defining $\mu(G)$ to be the maximum multiplicity of the edges of G, using a re-colouring argument, Vizing [Viz64] proved the following bound:

$$\chi'(G) \leq \Delta(G) + \mu(G),$$

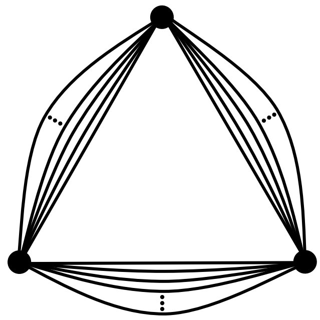


Figure 1.1: Shannon multigraph with $\chi'(G) = \frac{3}{2}\Delta$.

which, for simple graphs, reduces to

$$\chi'(G) \leq \Delta(G) + 1,$$

both of which are best possible.

Perhaps the best known, example of a simple graph which cannot be properly edge-coloured using only $\Delta(G)$ colours is the Petersen Graph shown in Figure 1.2, which has $\Delta(G) = 3$ but $\chi'(G) = 4$.

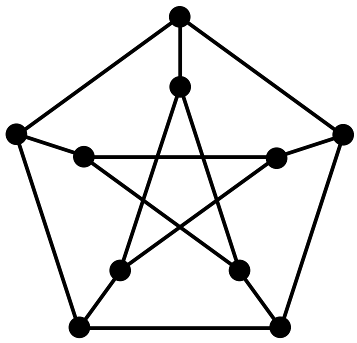


Figure 1.2: The Petersen Graph.

Note that, while Vizing tells us that any simple graph G has chromatic index $\Delta(G)$ or $\Delta(G) + 1$, the general problem of determining $\chi'(G)$ is NP-complete.

In Chapter 5, we prove a partial generalisation of Vizing's Theorem to hypergraphs.

1.3 Fractional colouring

When considering the chromatic number of certain graphs, one may notice colourings which are best possible (in that they use as few colours as possible) but which are in some sense wasteful. For instance, an odd cycle cannot be properly coloured with two colours but can be coloured using three colours in such a way that the third colour is used only once. Indeed, if C_7 has vertices $v_1, v_2, v_3, \ldots, v_7$, then we can colour v_1, v_3, v_5 red, v_2, v_4, v_6 blue and v_7 green.

If, however, our aim is instead to assign multiple colours to each vertex such that adjacent vertices receive disjoint lists of colours, then we could double-colour C_7 using five (rather than six) colours and triple-colour it using seven (rather than nine) colours in such a way that each colour is used exactly three times. Indeed, we could colour v_i with colours $3i, 3i+1, 3i+2 \pmod 7$. Thus, asking for the minimum of the ratio of colours required to the number of colours assigned to each vertex gives us a natural generalisation of the chromatic number.

Alternatively, for a graph $G = (V, E)$ we can consider a function w assigning to each independent set of vertices I a real number $w(I) \in [0, 1]$. We call such a function a *weighting*. The *weight* $w[v]$ of a vertex $v \in V$ with respect to w is then defined to be the sum of $w(I)$ over all independent sets containing v. A weighting w is a *fractional colouring* of G if, for each $v \in V$, $w[v] \geq 1$. The size $|w|$ of a fractional colouring is the sum of $w(I)$ over all independent sets I. The *fractional chromatic number* $\chi_f(G)$ is then defined to be the infimum of $|w|$ over all possible fractional colourings.

Thus, given a graph G, the problem of finding $\chi_f(G)$ can be viewed as the LP-relaxation of the problem of finding $\chi(G)$. Defining $\mathcal{I}(G)$ to be the set of independent sets of G, finding $\chi(G)$ is equivalent to solving the following:

$$\begin{aligned} \text{minimise} \quad & \sum_{I \in \mathcal{I}} w(I), \\ \text{subject to} \quad & \sum_{I \ni v} w(I) \geq 1 \quad \text{for each } v \in V(G), \\ & w(I) \in \{0, 1\} \quad \text{for each } I \in \mathcal{I}(G), \end{aligned}$$

and finding $\chi_f(G)$ is equivalent to solving the same problem but with the second constraint replaced by $w(I) \in [0, 1]$ for each $I \in \mathcal{I}(G)$.

Thus, we can see that, for any graph G, we have

$$\chi_f(G) \leq \chi(G).$$

Also, since $\chi_f(G)$ can be found by solving a linear programming problem with integer coefficients, for any graph G, we know that $\chi_f(G) \in \mathbb{Q}$ and that there exists a colouring w with $|w| = \chi_f(G)$ such that $w(I) \in \mathbb{Q} \cap [0,1]$ for every $I \in \mathcal{I}$. That is, for every graph G, the infimum in the definition of the fractional chromatic number is attained by a colouring with rational weights.

It can easily be shown that the above two definitions of the fractional chromatic number are equivalent to each other and to a third, probabilistic, definition. It is this third definition which we will make most use of in Chapter 4:

Lemma 1.3.1. *Let G be a graph and q a positive rational number. The following are equivalent:*

(i) $\chi_f(G) \leq q$;

(ii) there exists an integer N and a multi-set $\mathcal{W}$ of at most qN independent sets in G such that each vertex is contained in exactly N sets from $\mathcal{W}$;

(iii) there exists a probability distribution π on the independent sets of G such that, for each vertex v, the probability that v is contained in a random independent set (with respect to π) is at least $1/q$.

Proof.

(i)⇒(ii): Suppose that $\chi_f(G) \leq q$ for some $q \in \mathbb{Q}$. Then, there exists a weighting

$$w : \mathcal{I} \to [0,1]$$

such that $\sum_{I \ni v} w(I) \geq 1$ for every $v \in V(G)$ and $\sum_{I \in I} w(I) \leq q$. As remarked above, we may assume that $w(I) \in \mathbb{Q} \cap [0,1]$ for every $I \in \mathcal{I}$.

Then, there exists an integer N such that $Nw(I) \in \mathbb{N}$ for every $I \in \mathcal{I}$. We define $\mathcal{W}$ to include $Nw(I)$ copies of each independent set. Thus, $\mathcal{W}$ includes $N \sum_{I \in \mathcal{I}} w(I) \leq Nq$ independent sets with each vertex belonging to at least N of these sets. To complete the proof, we arbitrarily remove any vertex v belonging to too many of the members of $\mathcal{W}$

from sufficiently many of those independent sets so as to have every vertex belong to exactly N sets from $\mathcal{W}$.

(ii)⇒(iii): Suppose there exists an integer N and a multiset $\mathcal{W}$ of at most qN independent sets from $\mathcal{I}(G)$ such that each vertex belongs to exactly N of the sets from $\mathcal{W}$. Then, define a probability distribution $\pi : \mathcal{I} \to [0, 1]$ by

$$\pi(I) = \begin{cases} 1/qN & \text{if } I \in \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, since every vertex $v \in V(G)$ belongs to exactly N members of $\mathcal{W}$,

$$\sum_{I \ni v} \pi(I) = 1/q, \quad \text{and} \quad \sum_{I \in \mathcal{I}} \pi(I) = 1,$$

as required.

(iii)⇒(i): Suppose there exists a probability distribution $\pi : \mathcal{I} \to [0, 1]$ such that $\sum_{I \ni v} \pi(I) \geq 1/q$ and $\sum_{I \in \mathcal{I}} \pi(I) = 1$. Then, define a weighting $w : \mathcal{I} \to [0, 1]$ by

$$w(I) = \min\{q\pi(I), 1\}.$$

Then

$$\sum_{I \ni v} w(I) \geq 1$$

for every $v \in V(G)$ and

$$\sum_{I \in \mathcal{I}} w(I) = \sum_{I \in \mathcal{I}} \min\{q\pi(I), 1\} \leq \sum_{I \in \mathcal{I}} q\pi(I) = q$$

so $\chi_f(G) \leq q$. □

In Chapter 4, we consider the problem of bounding the fractional chromatic number of a graph with maximum degree at most three which contains no triangles. Brooks' Theorem asserts that such graphs have chromatic number at most three and, thus, have fractional chromatic number at most three. On the other hand, there exist such graphs with fractional chromatic number equal to 2.8. The main result of Chapter 4 is a probabilistic proof that triangle-free graphs with maximum degree at most three have fractional chromatic number at most $32/11 \approx 2.909$.

We refer the interested reader to the book of Scheinerman and Ullman [SU97] for more information on fractional colouring.

1.4 Ramsey Theory

Consider the complete graph on N vertices. Suppose we were to colour each of its edges either red or blue and to ask whether this can be done in such a way as to avoid structure in the monochromatic subgraphs induced by the edges of each colour. It is tempting to think that this is possible and that we could find such a colouring which, upon interrogation, would appear to lack structure.

However, this is not the case. For instance, any such *red-blue colouring* of the complete graph on six or more vertices will result in either a red or blue triangle (that is, three vertices, say, u, v, w such that uv, vw, uw are coloured identically). Indeed, consider any vertex v in such a coloured graph along with five of its neighbours, $u_1, u_2, \dots u_5$. By the pigeonhole principle, at least three of the edges connecting v to its neighbours, say, vu_1,vu_2 and vu_3 must have the same colour as each other, say, red. Then, consider u_1u_2, u_2u_3 and u_1u_3. Either one of these three edges is red (giving a red triangle of the form vu_iu_j), or they are all blue (giving a blue triangle $u_1u_2u_3$).

Ramsey's Theorem [Ram30], essentially tells us that, no matter what structure we require a coloured graph to have in one of its colours, we can guarantee that it will have that structure provided the graph has sufficiently many vertices. We begin by looking at the two-coloured version of the result:

Theorem 1.4.1 ([Ram30])**.** *Given integers n and m, there exists an integer $N_r(n, m)$ such that, for every integer $N \geq N_r(n, m)$, every red-blue colouring of the complete graph on N vertices results in the coloured graph containing either a red K_n or a blue K_m.*

We call the minimum such integer the *Ramsey Number of* (n, m), written $R(n, m)$.

Notice that our earlier discussion provides a proof that $R(3, 3) \leq 6$ and that the red-blue colouring of K_5 shown in Figure 1.3 below shows that $R(3, 3) > 5$, thus completing a proof that $R(3, 3) = 6$. Similarly, noting that K_2 consists of a single edge, we can see that $R(2, k) = k$ for all k.

In 1935, Erdős and Szekeres [ES35] rediscovered Ramsey's Theorem and, by considering the red and blue neighbourhoods of a given vertex, gave a new inductive proof and an

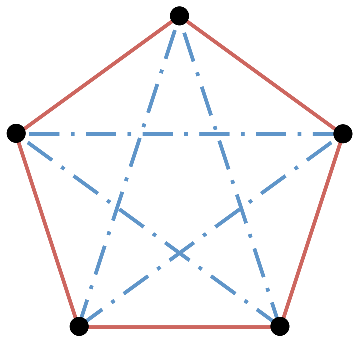


Figure 1.3: A red-blue colouring of K_5 containing no monochromatic triangles.

improved bound for $R(n, m)$. The key argument shows that

$$R(n, m) \leq R(n-1, m) + R(n, m-1)$$

and proceeds as follows: Let G be a graph on at least $R(n-1, m) + R(n, m-1)$ vertices. Consider a given vertex v and $R(n-1, m) + R(n, m-1) - 1$ of its neighbours. Then, either there is a set U of at least $R(n-1, m)$ neighbours of v with vu coloured red for every $u \in U$ or there is a set W of at least $R(n, m-1)$ neighbours of v with vw coloured blue for every $w \in W$. Without loss of generality, assume the latter. Since W contains at least $R(n, m-1)$ vertices, it contains either a red K_n or a blue K_{m-1} which, together with v, forms a blue K_m.

Combined with induction and the fact that $R(2, k) = k$, this gives an upper bound of

$$R(n, m) \leq \binom{n+m-2}{n-1} \leq 2^{n+m-2}.$$

For small values of n and m, the problem of finding the exact value of $R(n, m)$ is tractable. However, for larger values, things become increasingly more difficult with exact results only known for $(n, m) = (2, k), (3, 3), (3, 4), (3, 5), (3, 6), (3, 7), (3, 8), (3, 9), (4, 4)$ and $(4, 5)$ [Rad94].

Indeed, Joel Spencer [Spe94] recounts some advice given by Erdős:

> "...Imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) [then] we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6) [then] we should [instead] attempt to destroy

the aliens..."

Given the difficulty of finding the exact value of $R(n, m)$, results tend to fall into one of two categories: partial results for small values of n, m and asymptotic results. For a comprehensive overview of results of the first type, see [Rad94].

In terms of asymptotic results, the best known upper bound is due to Conlon [Con09], who proved that there exists a constant $C > 0$ such that

$$R(n+1, n+1) \leq n^{-C\frac{\log n}{\log\log n}} \binom{2n}{n},$$

whereas the best known lower bound is due to Spencer [Spe75], who, improving upon a probabilistic argument of Erdős [Erd47], proved that

$$R(n, n) \geq \tfrac{n\sqrt{2}}{e} 2^{n/2}.$$

Theorem 1.4.1 can be generalised in a number of ways but we will restrict our attention to those extensions which are considered in Chapter 2 of this thesis.

The multicolour Ramsey number $R(n_1, n_2, \ldots, n_r)$ is defined to be the minimum N such that every r-edge-colouring of the complete graph on at least N vertices results in the graph having, as a subgraph, a copy of K_{n_i} coloured with colour i, for some i.

Theorem 1.4.2. *For every $n_1, n_2, ..., n_r$, $R(n_1, n_2, \ldots, n_r)$ is finite. Moreover,*

$$R(n_1, n_2, \ldots, n_r) \leq R\left(R(n_1, n_2), n_3, \ldots n_r\right).$$

Proof. Consider the complete graph on $R\left(R(n_1, n_2), n_3, \ldots n_r\right)$ whose vertices are coloured with colours $1, 2, \ldots r$. Now, temporarily, cease to distinguish between colours 1 and 2 so that we have an $r-1$ coloured graph on $R\left(R(n_1, n_2), n_3, \ldots n_r\right)$, which contains either a complete graph on n_i vertices coloured with colour i for some $3 \leq i \leq r$ or contains a copy of the complete graph on $R(n_1, n_2)$ vertices coloured with colours 1 and 2, which (distinguishing again between colours 1 and 2) contains either a K_{n_1} coloured with colour 1 or a K_{n_2} coloured with colour 2. □

We may also generalise from complete graphs to general graphs as follows: For graphs $G_1, G_2, \ldots, G_r$, the Ramsey number $R(G_1, G_2, \ldots, G_r)$ is the smallest integer such that every edge-colouring of K_N, the complete graph on N vertices, with up to r colours, results in the graph having, as a subgraph, a copy of G_i coloured with colour i, for some i.

Suppose each G_i has n_i vertices. Then, since G_i is a subgraph of K_{n_i}, $R(G_1, G_2, ..., G_r)$ is well defined and

$$R(G_1, G_2, \ldots, G_r) \leq R(n_1, n_2, \ldots, n_r).$$

One of the first results in this direction was due to Gerencsér and Gyárfás [GG67] who considered the problem of finding the Ramsey number of a pair of paths and proved that, for $n \leq m$,

$$R(P_n, P_m) = m + \left\lfloor \tfrac{1}{2}n \right\rfloor - 1.$$

The survey of Radziszowski [Rad94] lists a wealth of results in this direction but we will only mention those which relate directly to the topic of Chapter 2, namely, the Ramsey number of cycles.

The problem of finding the Ramsey number of a pair of cycles was considered in the early 1970s by Bondy and Erdős [BE73], Rosta [Ros73] and Faudree and Schelp [FS74] with the second and third sets of authors independently proving the following exact result:

$$R(C_n, C_m) = \begin{cases} 6 & (n,m) = (3,3), (4,4), \\ 2n-1 & n \geq m \geq 3,\ m \text{ odd } (n \neq 3), \\ n + \tfrac{1}{2}m - 1 & n \geq m \geq 4,\ n, m \text{ even } (n \neq 4), \\ \max\{n + \tfrac{1}{2}m - 1, 2m-1\} & n \geq m \geq 4,\ n \text{ odd},\ m \text{ even}. \end{cases}$$

Bondy and Erdős noted the difficulty in finding multicolour Ramsey numbers in general, whilst suggesting that, for cycles, the problem should be tractable. They noted that they were

> "not able to evaluate $R(G_1, G_2, \ldots, G_r)$ for $k > 2$ even in the case of cycles."

They did, however, give the following bounds for the r-colour Ramsey number in the case when n is odd:

$$2^{r-1}(n-1) + 1 \leq R(C_n, C_n, \ldots, C_n) \leq (r+2)!n$$

and conjectured (see, for instance, [Erd81]) that this lower-bound gives the true value of the Ramsey number.

Recently, there has been renewed interest in these problems. Károlyi and Rosta [KR01] and Nikiforov and Schelp [NS08] provided new proofs for the two-coloured case. The latter pair utilised stability with the idea being, essentially, to consider the complete graph on slightly fewer than $R(C_n, C_m)$ vertices, to assume that this has a red-blue colouring with no red C_n or blue C_m and to show that this forces a particular structure which can then be exploited to give a cycle in the larger graph. This idea of a stability proof will be of great use to us in Chapter 2, where we will look at the analogous result for three colours.

1.5 Szemerédi's Regularity Lemma and its applications

Szemerédi's Regularity Lemma tells us essentially that any sufficiently large graph can be approximated by the union of a bounded number of random-like bipartite graphs.

The earliest version of the result appeared in 1975 in [Sze75] where it was used as a tool in the proof a conjecture of Erdős and Turán that, for any $d > 0$ and any integer k, any set of dN integers from $\{0, 1, \ldots, N\}$ contains a k-term arithmetic progression, provided N is sufficiently large. The most well known version of the Regularity Lemma for general graphs was later proved in [Sze78].

Recalling the definition of the *density* of a pair of sets of vertices, that is,

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|},$$

we define the concept of a regular pair:

Definition 1.5.1. *A pair of disjoint subsets (A, B) of the vertex set of a graph G is (ϵ, G)-regular for some $\epsilon > 0$ if, for every pair (A', B') with $A' \subseteq A$, $|A'| \geq \epsilon|A|$, $B' \subseteq B$, $|B'| \geq \epsilon|B|$, we have*

$$\big|d(A', B') - d(A, B)\big| < \epsilon.$$

Note that, when clear from context, we will use ϵ-regular or even just regular to mean (ϵ, G)-regular. The 'classic' version of the Regularity Lemma given below states that, given any graph on sufficiently many vertices, we can partition the vertices into a bounded number of *clusters* such that most pairs of clusters are regular.

Theorem 1.5.2 (Szemerédi's Regularity Lemma [Sze78])**.** *Given $\epsilon > 0$ and k_0 a positive integer, there exists $N_{1.5.2} = N_{1.5.2}(\epsilon, k_0)$ such that the following holds: For all graphs G with $|V(G)| \geq N_{1.5.2}$, there exists a partition $\Pi = (V_0, V_1, \ldots, V_K)$ of V such that*

(i) $k_0 \leq K \leq N_{1.5.2}$*;*

(ii) $|V_0| \leq \epsilon |V|$*;*

(iii) $|V_1| = |V_2| = \cdots = |V_K|$*; and*

(iv) all but at most $\epsilon\binom{K}{2}$ *of the pairs* (V_i, V_j) *are* (ϵ, G)*-regular.*

Note that, for for a given graph G and a value of ϵ, we call a partition $(V_1, V_2, \ldots, V_K)$ satisfying (ii)–(iv) above an (ϵ, G)*-regular partition.*

In Chapter 2 , we make use of a coloured version of the Regularity Lemma to define, for a given graph, its reduced graph whose vertices correspond to the clusters arising from the Regularity Lemma and whose edges correspond to the regular pairs. We also make use of a related blow-up lemma, which tells us that an edge in the reduced graph can be 'blown up' to a long path in the original graph.

In Chapter 3, we will look at Removal Lemmas, the first and most famous of which is the Triangle Removal Lemma of Ruzsa and Szemerédi [RS78], which states that a graph on n vertices containing $o(n^3)$ triangles can be made triangle-free by the removal of $o(n^2)$ edges (where we say graph G contains k copies of graph H if G has, as subgraphs, k graphs isomorphic to H).

A more precise formulation of this result, which was one of the earliest applications of the Regularity Lemma, follows. We also include its proof in full, since the counting required parallels that found in many places in Chapter 2.

Theorem 1.5.3 (The Triangle Removal Lemma [RS78])**.** *For every $\epsilon > 0$, there exists $N_{1.5.3} = N_{1.5.3}(\epsilon)$ and $\delta = \delta_{1.5.3}(\epsilon)$ such that, if G is a graph on $n \geq N$ vertices with at most δn^3 triangles, then one can remove from G at most ϵn^2 edges to obtain a graph that contains no triangles.*

Proof. Given $\epsilon > 0$, we set $k_0 = 4/\epsilon$. By the Regularity Lemma there exists $N = N_{1.5.2}(\epsilon/4, 4\epsilon^{-1})$ such that, given a graph G on $n \geq N$ vertices (with fewer than δn^3 triangles), there exists a partition Π of $V(G)$ into $K+1$ clusters $V_0, V_1, \ldots, V_K$ such that

(i) $4/\epsilon \leq K \leq N$;

(ii) $|V_0| \leq \frac{1}{4}\epsilon|V|$;

(iii) $|V_1| = |V_2| = \cdots = |V_K|$; and

(iv) all but at most $\frac{1}{4}\epsilon\binom{K}{2}$ of the pairs (V_i, V_j) are $(\frac{1}{4}\epsilon, G)$-regular.

We then remove from G any edges $(x, y) \in X_i \times X_j$ such that at least one of the following conditions hold:

(i) (X_i, X_j) is not $\frac{1}{4}\epsilon$-regular;

(ii) $e(X_i, X_j) < \frac{1}{2}\epsilon|X_i||X_j|$;

(iii) $x \in X_0$;

(iv) $X_i = X_j$.

Observe that there are at most $\frac{1}{2}(\frac{1}{4}\epsilon)K^2$ non-regular pairs, giving at most

$$\tfrac{1}{8}\epsilon K^2|V_1|^2 \leq \tfrac{1}{8}\epsilon n^2$$

edges of the first type. By definition, there are at most $\frac{1}{2}\epsilon n^2$ edges of the second type and at most $\frac{1}{4}\epsilon n^2$ edges of the third type. Finally, there are at most

$$K\binom{|V_1|}{2} \leq \tfrac{1}{2}K\left(\frac{n}{K}\right)^2 = \frac{n^2}{2K}$$

edges of the fourth type. Thus, since $K \geq k_0 \geq 4/\epsilon$, we have deleted at most ϵn^2 vertices in total.

Now, suppose that there remains a triangle in G. Since we have removed all edges from within each cluster and all edges intersecting V_0, the three vertices of the triangle must belong to distinct clusters X, Y, Z from $\{V_1, V_2, \ldots, V_K\}$. Also, since we have removed all edges from non-regular pairs and from pairs of low density, we know that $(X, Y), (Y, Z), (X, Z)$ are each $\frac{1}{4}\epsilon$-regular with density at least $\frac{1}{2}\epsilon$.

We claim that at most $\frac{1}{4}\epsilon|X|$ of the vertices in X have fewer than $\frac{1}{4}\epsilon|Y|$ neighbours in Y. Indeed, if this were not the case, these vertices would define $X' \subseteq X$ with $|X'| \geq \frac{1}{4}\epsilon|X|$ and $e(X',Y) < \frac{1}{4}\epsilon|X'||Y|$. However, we know that $e(X,Y) \geq \frac{1}{2}\epsilon|X||Y|$ and by the definition of a regular pair have

$$\left|d(X',Y) - d(X,Y)\right| \leq \tfrac{1}{4}\epsilon,$$

giving a contradiction. Similarly, we may assume that there are at most $\frac{1}{4}\epsilon|X|$ vertices in X with fewer than $\frac{1}{4}\epsilon|Z|$ neighbours in Z.

Thus, there are at least $(1-\frac{1}{2}\epsilon)|X|$ vertices in X with at least $\frac{1}{4}\epsilon|Y|$ neighbours in Y and at least $\frac{1}{4}\epsilon|Z|$ neighbours in Z. Given such a vertex, we consider the edges in $G[Y,Z]$ between $N(x) \cap Y$ and $N(x) \cap Z$. There are at least

$$\left(\frac{\epsilon}{2} - \frac{\epsilon}{4}\right)\left(\frac{\epsilon}{4}\right)\left(\frac{\epsilon}{4}\right)|Y||Z|$$

such edges. Since there are at least $(1-\frac{1}{2}\epsilon)|X|$ such vertices in X, these give rise to at least

$$\left(1-\frac{\epsilon}{2}\right)\left(\frac{\epsilon}{4}\right)^3|X||Y||Z| \geq \left(1-\frac{\epsilon}{2}\right)\left(\frac{\epsilon}{4}\right)^3\left(\frac{(1-\epsilon)n}{K}\right)^3$$

triangles, giving rise to a contradiction provided

$$\left(1-\frac{\epsilon}{2}\right)\left(\frac{\epsilon}{4}\right)^3\left(\frac{(1-\epsilon)n}{K}\right)^3 > \delta.$$

If $\epsilon > \frac{1}{2}$, the result is trivial. Hence, we may assume that $\epsilon \leq \frac{1}{2}$, in which case

$$\left(1-\frac{\epsilon}{2}\right)\left(\frac{\epsilon}{4}\right)^3\left(\frac{(1-\epsilon)n}{K}\right)^3 \geq \frac{3}{4}\left(\frac{\epsilon}{4}\right)^3\frac{1}{2^3}\left(\frac{n}{K}\right)^3 \geq \frac{1}{2^{10}}\frac{\epsilon^3}{K^3}.$$

Thus, we may set

$$\delta_{1.5.3}(\epsilon) = \left(\frac{1}{2^{10}}\right)\frac{\epsilon^3}{K^3}$$

in order to complete the proof. □

The above result generalises in a number of ways, for instance, from triangles to general graphs (see [EFR86]) and to hypergraphs (see for instance [AT10], [Ish09]). We will return to the topic of removal lemmas in Chapter 3, where we consider analogues for equations over finite fields.

1.6 Thesis outline

In Chapter 2, combining regularity and stability methods we prove an exact result in Ramsey Theory. For a triple of long cycles of particular parities, we provide an exact answer to the question of how large an underlying three-coloured graph must be in order to guarantee a monochromatic cycle of specified length.

We then move on, in Chapter 3, to consider removal lemmas for equations over finite fields. Utilising a coloured hypergraph generalisation of the Graph Removal Lemma, we prove a removal lemma for two specific classes of multinomials. Specifically, for $X_1, X_2, \dots X_m$ subsets of a finite field of of order q, we prove that, if a multinomial of order m of a particular form has $o(q^{m-1})$ solutions $(x_1, x_2, x_3, \dots, x_m)$ with $x_i \in X_i$, then we can delete $o(q)$ elements from each X_i so that no solutions remain.

Next, in Chapter 4, we consider a problem in fractional colouring. By a probabilistic method, we prove that, if G is a triangle-free graph with maximum degree at most three, then the fractional chromatic number of G is at most $32/11 \approx 2.909$, improving on the best previously known bound. Note that the proof includes a long case analysis which is postponed to the Appendix of this Thesis.

Finally, in Chapter 5, we prove a weak version of Vizing's Theorem for hypergraphs. We prove that, if H is an intersecting 3-uniform hypergraph with maximum degree Δ and maximum multiplicity μ, then, H has at most $2\Delta + \mu$ edges. Furthermore, we prove that the unique structure achieving this maximum is μ copies of the Fano Plane.

Chapter 2

The Ramsey number of cycles

Recall that, for graphs G_1, G_2, G_3, the Ramsey number $R(G_1, G_2, G_3)$ is the smallest integer N such that every edge-colouring of the complete graph on N vertices with up to three colours, results in the graph having, as a subgraph, a copy of G_i coloured with colour i for some i. In this chapter, we will consider the case when G_1, G_2 and G_3 are cycles.

In 1973, Bondy and Erdős [BE73] conjectured that, if $n > 3$ is odd, then

$$R(C_n, C_n, C_n) = 4n - 3.$$

Later, Łuczak [Łuc99] proved, that for n odd, $R(C_n, C_n, C_n) = 4n + o(n)$ as $n \to \infty$. Kohayakawa, Simonovits and Skokan [KSS09a], expanding upon the work of Łuczak, confirmed the Bondy-Erdős conjecture for sufficiently large odd values of n by proving that there exists a positive integer n_0 such that, for all odd $n, m, \ell > n_0$,

$$R(C_n, C_m, C_\ell) = 4 \max\{n, m, \ell\} - 3.$$

In the case when all three cycles are of even length, Figaj and Łuczak [FŁ07a] proved the following asymptotic. Defining $\langle\langle x \rangle\rangle$ to be the largest even integer not greater than x, they proved that, for all $\alpha_1, \alpha_2, \alpha_3 > 0$,

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle\langle \alpha_3 n \rangle\rangle}) = \tfrac{1}{2}\big(\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\}\big)n + o(n),$$

as $n \to \infty$.

Thus, in particular, for even n,

$$R(C_n, C_n, C_n) = 2n + o(n), \text{ as } n \to \infty.$$

Independently, Gyárfás, Ruszinkó, Sárközy and Szemeredi [GRSS07] proved a similar, but more precise, result for paths, namely that there exists a positive integer n_1 such that, for $n > n_1$,

$$R(P_n, P_n, P_n) = \begin{cases} 2n-1, & n \text{ odd}, \\ 2n-2, & n \text{ even}. \end{cases}$$

More recently, Benevides and Skokan [BS09] proved that there exists n_2 such that, for even $n > n_2$,

$$R(C_n, C_n, C_n) = 2n.$$

In this chapter, we look at the mixed-parity case, for which, defining $\langle x \rangle$ to be the largest odd number not greater than x, Figaj and Łuczak [FŁ07b] proved that, for all $\alpha_1, \alpha_2, \alpha_3 > 0$,

(i) $R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{2\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}n + o(n)$,

(ii) $R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\}n + o(n)$,

as $n \to \infty$.

Improving on their result, in the case when exactly one of the cycles is of odd length, we prove the following, which is the main result of this chapter:

Theorem A. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, there exists a positive integer $n_A = n_A(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_A$,*

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{2\langle\langle \alpha_1 n \rangle\rangle + \langle\langle \alpha_2 n \rangle\rangle - 3,\ \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle + \langle \alpha_3 n \rangle - 2\}.$$

Additionally, in Section 2.13, we give an outline of the proof of the corresponding result for the case when one of the cycles is of even length and the other two are of odd length.

Theorem C. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_2 \geq \alpha_3$, there exists a positive integer $n_C = n_C(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_C$,*

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\langle\langle \alpha_1 n \rangle\rangle - 3, \langle\langle \alpha_1 n \rangle\rangle + 2\langle \alpha_2 n \rangle - 3\}.$$

2.1 Lower bounds

Our first step in proving Theorem A is to exhibit three-edge-colourings of the complete graph on

$$\max\left\{2\langle\langle\alpha_1 n\rangle\rangle + \langle\langle\alpha_2 n\rangle\rangle - 4,\ \tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle + \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle + \langle\alpha_3 n\rangle - 3\right\}$$

vertices which do not contain any of the relevant coloured cycles, thus proving that

$$R(C_{\langle\langle\alpha_1 n\rangle\rangle}, C_{\langle\langle\alpha_2 n\rangle\rangle}, C_{\langle\alpha_3 n\rangle}) \geq \max\{2\langle\langle\alpha_1 n\rangle\rangle + \langle\langle\alpha_2 n\rangle\rangle - 3,\ \tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle + \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle + \langle\alpha_3 n\rangle - 2\}.$$

For this purpose, the well-known colourings shown in Figures 2.1 and 2.2 suffice.

The graph shown in Figure 2.1 has $2\langle\langle\alpha_1 n\rangle\rangle + \langle\langle\alpha_2 n\rangle\rangle - 4$ vertices divided into four classes V_1, V_2, V_3 and V_4, with

$$|V_1| = |V_2| = \langle\langle\alpha_1 n\rangle\rangle - 1, \qquad |V_3| = |V_4| = \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle - 1,$$

such that all edges in $G[V_1]$ and $G[V_2]$ are coloured red; all edges in $G[V_1, V_3]$ and $G[V_2, V_4]$ are coloured blue; all edges in $G[V_1 \cup V_3, V_2 \cup V_4]$ are coloured green; and all edges in $G[V_3]$ and $G[V_4]$ are coloured red or blue.

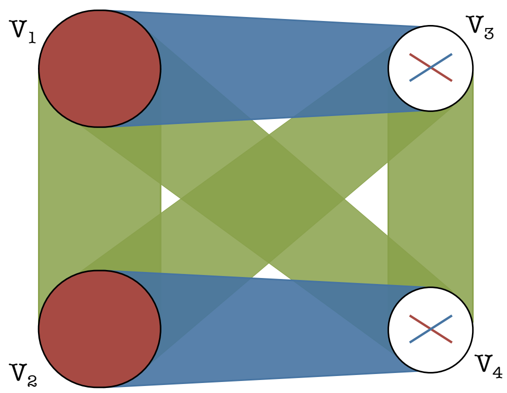


Figure 2.1: First extremal colouring for Theorem A.

Since there are no red edges between the vertex classes and each class contains fewer than $\langle\langle\alpha_1 n\rangle\rangle$ vertices, the graph has no red cycles of length $\langle\langle\alpha_1 n\rangle\rangle$. Also, since there are are no blue edges in $G[V_1] \cup G[V_2] \cup G[V_1 \cup V_3, V_2 \cup V_4]$, any blue cycle must belong to either $G[V_1,V_3]\cup G[V_3]$ (and, thus, have at least half its vertices in V_3) or $G[V_2,V_4]\cup G[V_4]$ (and, thus, have at least half its vertices in V_4). Thus, since $|V_3|,|V_4|=\frac{1}{2}\langle\langle\alpha_2 n\rangle\rangle-1$, the graph has no blue cycles of length $\langle\langle\alpha_2 n\rangle\rangle$. Finally, since the only green edges belong to $G[V_1\cup V_3, V_2\cup V_4]$, all green cycles in the graph are of even length. Thus, the graph has no green cycles of odd length and, in particular, no green cycles of length $\langle\alpha_3 n\rangle$.

The graph shown in Figure 2.2 has $\frac{1}{2}\langle\langle\alpha_1 n\rangle\rangle+\frac{1}{2}\langle\langle\alpha_2 n\rangle\rangle+\langle\alpha_3 n\rangle-3$ vertices, divided into three classes V_1, V_2 and V_3, with

$$|V_1|=\tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle-1,\qquad |V_2|=\tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle-1,\qquad |V_3|=\langle\alpha_3 n\rangle-1.$$

such that all edges in $G[V_1]\cup G[V_1,V_3]$ are coloured red; all edges in $G[V_2]\cup G[V_2,V_3]$ are coloured blue; and all edges in $G[V_1,V_2]\cup G[V_3]$ are coloured green.

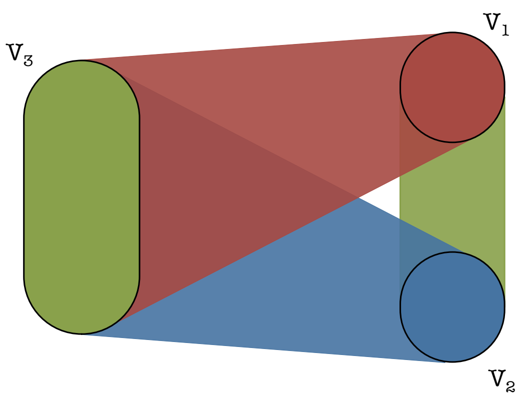


Figure 2.2: Second extremal colouring for Theorem A.

Similarly, this graph has no red cycles of length $\langle\langle\alpha_1 n\rangle\rangle$, no blue cycles of length $\langle\langle\alpha_2 n\rangle\rangle$ and no green cycles of length $\langle\alpha_3 n\rangle$.

Thus, it remains to prove the corresponding upper-bound. To do so, we combine regularity (as used in [Łuc99], [FŁ07a], [FŁ07b]) with stability methods using a simillar approach to [GRSS07], [BS09], [KSS09a], [KSS09b].

Note that all references to colouring in the remainder of this chapter should be understood as referring to edge-colouring and, where appropriate, to edge-multicolouring.

2.2 Key steps in the proof

In order to complete the proof of Theorem A, we must show that, for n sufficiently large, any three-colouring of G, the complete graph on

$$N = \max\left\{2\langle\langle\alpha_1 n\rangle\rangle + \langle\langle\alpha_2 n\rangle\rangle - 3,\ \tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle + \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle + \langle\alpha_3 n\rangle - 2\right\}$$

vertices, will result in either a red cycle on $\langle\langle\alpha_1 n\rangle\rangle$ vertices, a blue cycle on $\langle\langle\alpha_2 n\rangle\rangle$ or a green cycle on $\langle\alpha_3 n\rangle$ vertices.

The main steps of the proof are as follows: Firstly, we apply a version of the Regularity Lemma (Theorem 2.3.1) to give a partition $V_0 \cup V_1 \cup \cdots \cup V_K$ of the vertices which is simultaneously regular for the red, blue and green spanning subgraphs of G. Given this partition, we define the three-multicoloured reduced-graph $\mathcal{G}$ on vertex set $V_1, V_2, \ldots V_K$ whose edges correspond to the regular pairs. We colour the edges of the reduced-graph with all those colours for which the corresponding pair has density above some threshold. Łuczak [Łuc99] showed that, if the threshold is chosen properly, then the existence of a matching in a monochromatic connected-component of the reduced-graph implies the existence of a monochromatic cycle of the corresponding length in the original graph.

Thus, the key step in the proof of Theorem A will be to prove a Ramsey-type stability result for so-called connected-matchings (Theorem B). Defining a *connected-matching* to be a matching with all its edges belonging to the same component, this result essentially says that, for every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$ and every sufficiently large k, every three-multicolouring of a graph $\mathcal{G}$ on slightly fewer than $K = \max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}k$ vertices with sufficiently large minimum degree results in either a connected-matching on at least $\alpha_1 k$ vertices in the red subgraph of $\mathcal{G}$, a connected-matching on at least $\alpha_2 k$ vertices in the blue subgraph of $\mathcal{G}$, a connected-matching on at least $\alpha_3 k$ vertices in a non-bipartite component of the green subgraph of $\mathcal{G}$ or one of a list of particular structures which will be defined later.

In the case that $\mathcal{G}$ contains a suitably large connected-matching in one of its coloured subgraphs, a blow-up result of Figaj and Łuczak (see Theorem 2.3.4) can be used to give a monochromatic cycle of the same colour in G. If $\mathcal{G}$ does not contain such a connected-matching, then the stability result gives us information about the structure of $\mathcal{G}$. We then show that G has essentially the same structure which we exploit to force the existence of a monochromatic cycle.

In the next section, given a three-colouring of the complete graph on N vertices, we will define its three-multicoloured reduced-graph. We will also state and prove a version of the blow-up lemma of Figaj and Łuczak, which motivates our whole approach.

In Section 2.4, we will deal with some notational formalities before proceeding in Section 2.5 to define the structures we need and to give a precise formulation of the connected-matching stability result which we shall call Theorem B.

In Section 2.6, we give a number of technical lemmas needed for the proofs of Theorem A and Theorem B. Among these is a decomposition result of Figaj and Łuczak which provides insight into the structure of the reduced-graph.

The hard work is done in Sections 2.7–2.8, where we prove Theorem B, and in Sections 2.9–2.12, where we translate this result for connected-matchings into one for cycles, thus completing the proof of Theorem A.

The proof of Theorem B is divided into two parts according to the relative sizes of α_1 and α_3. Section 2.7 deals with the case when $\alpha_1 \geq \alpha_3$, that is, the case when the longest cycle has even length. In that case, a combination of the decomposition lemma of Figai and Łuczak and some careful counting of edges allows for a reasonably short proof. Section 2.8 deals with the opposite case, which requires a longer proof utilising an alternative decomposition and extensive case analysis.

The final part of the proof of Theorem A is divided into four sub-parts, one dealing with the general setup and three further sections, each dealing with one of the structures that can occur.

Note that Sections 2.3–2.7, 2.9 and 2.10 together give a complete proof for the case where the longest cycle is of even length, allowing the reader to omit sections 2.8, 2.11 and 2.12, while still getting a good flavour of the method of proof.

2.3 Cycles, Matchings and the Regularity Lemma

Recall that Szemerédi's Regularity Lemma [Sze78] asserts that any sufficiently large graph can be approximated by the union of a bounded number of random-like bipartite graphs. Recall also that, given a pair (A, B) of disjoint subsets of the vertex set of a graph G, we write $d(A, B)$ for the *density* of the pair, that is, $d(A, B) = e(A, B)/|A||B|$.

Finally, recall that we say such a pair is (ϵ, G)-*regular* for some $\epsilon > 0$ if, for every pair (A', B') with $A' \subseteq A$, $|A'| \geq \epsilon|A|$, $B' \subseteq B$, $|B'| \geq \epsilon|B|$, we have $|d(A', B') - d(A, B)| < \epsilon$.

In this chapter, we will make use of a generalised version of Szemerédi's Regularity Lemma in order to move from considering monochromatic cycles to considering monochromatic connected-matchings, the version below being a slight modification of one found, for instance, in [KS96]:

Theorem 2.3.1. *For every $\epsilon > 0$ and every positive integer k_0, there exists $K_{2.3.1} = K_{2.3.1}(\epsilon, k_0)$ such that the following holds: For all graphs G_1, G_2, G_3 with $V(G_1) = V(G_2) = V(G_3) = V$ and $|V| \geq K_{2.3.1}$, there exists a partition $\Pi = (V_0, V_1, \ldots, V_K)$ of V such that*

(i) $k_0 \leq K \leq K_{2.3.1}$*;*

(ii) $|V_0| \leq \epsilon|V|$*;*

(iii) $|V_1| = |V_2| = \cdots = |V_K|$*; and*

(iv) for each i, all but at most ϵK of the pairs (V_i, V_j), $1 \leq i < j \leq K$, are simultaneously (ϵ, G_r)-regular for $r = 1, 2, 3$.

Note that, given $\epsilon > 0$ and graphs G_1, G_2 and G_3 on the same vertex set V, we call a partition $\Pi = (V_0, V_1, \ldots, V_K)$ satisfying (ii)–(iv) $(\epsilon, G_1, G_2, G_3)$-*regular*.

In what follows, given a three-coloured graph G, we will use G_1, G_2, G_3 to refer to its monochromatic spanning subgraphs. That is G_1 (resp. G_2, G_3) has the same vertex set as G and includes, as an edge, any edge which in G is coloured red (resp. blue, green).

Then, given a three-coloured graph G, we can use Theorem 2.3.1 to define a partition which is simultaneously regular for G_1, G_2, G_3 and then define the three-multicoloured reduced-graph $\mathcal{G}$ as follows:

Definition 2.3.2. *Given $\epsilon > 0$, $\xi > 0$, a three-coloured graph $G = (V, E)$ and an $(\epsilon, G_1, G_2, G_3)$-regular partition $\Pi = (V_0, V_1, \ldots, V_K)$, we define the three-multicoloured (ϵ, ξ, Π)-reduced-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ by:*

$$\mathcal{V} = \{V_1, V_2, \ldots, V_K\},$$
$$\mathcal{E} = \{V_iV_j : (V_i, V_j) \text{ is simultaneously } (\epsilon, G_r)\text{-regular for } r = 1, 2, 3\},$$

where V_iV_j is coloured with all colours r such that $d_{G_r}(V_i, V_j) \geq \xi$.

One well known fact about regular pairs is that they contain long paths. This is summarised in the following lemma, which is a slight modification of one found in [Łuc99]:

Lemma 2.3.3. *For every ϵ such that $0 \leq \epsilon < 1/600$ and every $k \geq 1/\epsilon$, the following holds: Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1|, |V_2| \geq k$, the pair (V_1, V_2) is ϵ-regular and $e(V_1, V_2) \geq \epsilon^{1/2}|V_1||V_2|$. Then, for every integer ℓ such that $1 \leq \ell \leq k - 2\epsilon^{1/2}k$ and every $v' \in V_1$, $v'' \in V_2$ such that $d(v'), d(v'') \geq \frac{2}{3}\epsilon^{1/2}k$, G contains a path of length $2\ell + 1$ between v' and v''.*

Proof. We begin by considering the case when $1 \leq \ell \leq \frac{1}{2}\epsilon^{1/2}k$.

Suppose there exists $U_1 \subseteq V_1$ of size at least ϵk such that $d(u) \leq \frac{2}{3}\epsilon^{1/2}k$ for every $u \in U_1$. By regularity, $d(U_1, V_2)$ is within ϵ of $d(V_1, V_2)$.

But

$$d(V_1, V_2) \geq \epsilon^{1/2} \text{ and } d(U_1, V_2) < \tfrac{2}{3}\epsilon^{1/2},$$

which, since $\epsilon < 1/600$, gives a contradiction.

Thus, we can discard at most ϵk vertices from each of V_1, V_2 to obtain $\widehat{V}_1, \widehat{V}_2$ such that $|\widehat{V}_1|, |\widehat{V}_2| \geq (1 - \epsilon)k$ and the subgraph H induced in G by $\widehat{V}_1 \cup \widehat{V}_2$ has minimum degree at least $\frac{2}{3}\epsilon^{1/2}k - \epsilon k \geq \frac{1}{2}\epsilon^{1/2}k + \epsilon k + 1$ (provided $k \geq 1/\epsilon$). We can then greedily construct a path $P = v_0 v_1 \dots v_{2\ell-2}$ of length $2\ell - 2$ from $v_0 = v' \in \widehat{V}_1$ to $v_{2\ell-2} \in \widehat{V}_1$ such that $v'' \notin P$. Then, defining $W_1 \subseteq V_1$ to be the set of neighbours of v'' in $\widehat{V}_1 \backslash P$ and $W_2 \subseteq V_2$ to be the set of neighbours of $v_{2\ell-2}$ in $\widehat{V}_2 \backslash P$, we have $|W_1|, |W_2| \geq \epsilon k$. Then, by regularity, $d(W_1, W_2) \geq d(V_1, V_2) - \epsilon \geq \epsilon^{1/2} - \epsilon > 0$. Thus, there exists an edge $w_1 w_2$ between W_1 and W_2 which can be used along with $v_{2\ell-2}w_1$ and $w_2 v''$ to extend the path to length $2\ell + 1$.

Now, suppose that $\frac{1}{2}\epsilon^{1/2}k \leq \ell \leq k - 2\epsilon^{1/2}k$ and that we have constructed a path $P = v_0 v_1 v_2 \dots v_{2\ell-1}$ from $v_0 = v' \in \widehat{V}_1$ to $v_{2\ell-1} = v'' \in \widehat{V}_2$. Consider $V_1 \cap P$, $V_2 \cap P$ and suppose we have $W_1 \subseteq V_1 \cap P$ such that $|W_1| \geq \epsilon k$ and every $w \in W_1$ has fewer than ϵk neighbours in $V_2 \backslash P$.

Then, by regularity,

$$\begin{aligned} |d(V_1 \cap P, V_2 \backslash P) - d(W_1, V_2 \backslash P)| \leq\ & |d(V_1 \cap P, V_2 \backslash P) - d(V_1, V_2)| \\ & + |d(W_1, V_2 \backslash P) - d(V_1, V_2)| \leq 2\epsilon \end{aligned}$$

and

$$d(V_1 \cap P, V_2 \backslash P) > d(V_1, V_2) - \epsilon = (\epsilon^{1/2} - \epsilon),$$

but

$$d(W_1, V_2 \backslash P) = \frac{e(W_1, V_2 \backslash P)}{|W_1||V_2 \backslash P|} \leq \frac{\epsilon k |W_1|}{|W_1||V_2 \backslash P|} \leq \frac{\epsilon k}{|V_2 \backslash P|} \leq \tfrac{1}{2}\epsilon^{1/2},$$

which, since $\epsilon \leq 1/600$, gives rise to a contradiction.

So, all but at most ϵk vertices in $V_1 \cap P$ have at least ϵk neighbours in $V_2 \backslash P$ and, similarly, all but at most ϵk vertices in $V_2 \cap P$ have at least ϵk neighbours in $V_1 \backslash P$. Since $|V_1 \cup P|, |V_2 \cup P| \geq \frac{1}{2}\epsilon^{1/2} k \geq 2\epsilon k$, there exists i such that $v_i \in V_1 \cap P$ has at least ϵk neighbours in $V_2 \backslash P$ (call the set of these neighbours X) and $v_{i+1} \in V_2 \cap P$ has at least ϵk neighbours in $V_1 \backslash P$ (call the set of these neighbours Y). The density of (X, Y) is within ϵ of the density of (V_1, V_2) and so is non-zero. Therefore, there exists an edge xy such that $x \in X$ and $y \in Y$, which can be used to give a path $v_0 v_1 \dots v_i x y v_{i+1} \dots v_{2\ell-1}$ of length $2\ell + 1$. □

Recall that we call a matching with all its vertices in the same component of G a *connected-matching* and note that we say a connected-matching is *odd* if the component containing the matching also contains an odd cycle.

The following theorem makes use of the Lemma above to blow up large connected-matchings in the reduced-graph to cycles (of appropriate length and parity) in the original. This facilitates our approach to proving Theorem A in that it allows us to shift our attention away from cycles to connected-matchings, which turn out to be somewhat easier to find.

Figaj and Łuczak [FŁ07b, Lemma 3] proved a more general version of this theorem in a slightly different context (they considered any number of colours and any combination of parities and used a different threshold for colouring the reduced-graph):

Theorem 2.3.4. *For all $c_1, c_2, c_3, d, \eta > 0$ such that $0 < \eta < \min\{0.01, (64c_1 + 64c_2 + 64c_3)^{-1}\}$, there exists $n_{2.3.4} = n_{2.3.4}(c_1, c_2, c_3, d, \eta)$ such that, for $n > n_{2.3.4}$, the following holds:*

Given $\alpha_1, \alpha_2, \alpha_3$ such that $0 < \alpha_1, \alpha_2, \alpha_3 \leq 2$, and ξ such that $\eta \leq \xi \leq \frac{1}{3}$, a complete three-coloured graph $G = (V, E)$ on

$$N = c_1 \langle\langle \alpha_1 n \rangle\rangle + c_2 \langle\langle \alpha_2 n \rangle\rangle + c_3 \langle \alpha_3 n \rangle - d$$

vertices and an (η^4, G_1, G_2, G_3)-regular partition $\Pi = (V_0, V_1, \ldots, V_K)$ for some $K > 8(c_1+c_2+c_3)^2/\eta$, letting $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the three-multicoloured (η^4, ξ, Π)- reduced-graph of G on K vertices, and letting k be an integer such that

$$c_1\alpha_1 k + c_2\alpha_2 k + c_3\alpha_3 k - \eta k \leq K \leq c_1\alpha_1 k + c_2\alpha_2 k + c_3\alpha_3 k - \tfrac{1}{2}\eta k,$$

(i) if $\mathcal{G}$ contains a red connected-matching on at least $\alpha_1 k$ vertices, then G contains a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices;

(ii) if $\mathcal{G}$ contains a blue connected-matching on at least $\alpha_2 k$ vertices, then G contains a blue cycle on $\langle\langle \alpha_2 n \rangle\rangle$ vertices;

(iii) if $\mathcal{G}$ contains a green odd connected-matching on at least $\alpha_3 k$ vertices, then G contains a green cycle on $\langle \alpha_3 n \rangle$ vertices.

Proof. Consider $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the (η^4, ξ, Π)- reduced-graph of G on K vertices. By the definition of a regular partition, we have $|V_0| \leq \eta^4 N$. Thus, letting $c = c_1 + c_2 + c_3$, we have

$$\begin{aligned}|V_1 \cup V_2 \cup \cdots \cup V_K| &\geq c_1\langle\langle \alpha_1 n \rangle\rangle + c_2\langle\langle \alpha_2 n \rangle\rangle + c_3\langle \alpha_3 n \rangle - d - 2c\eta^4 n \\ &\geq (c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - 2c\eta^4)n - d - 2c.\end{aligned}$$

Then, since $\eta \leq (1/16c)^{1/3}$, provided $n \geq 8(2c+d)/\eta$, we have at least $(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - \frac{1}{4}\eta)n$ vertices in $V_1 \cup V_2 \cup \cdots \cup V_K$. So, recalling that $\alpha_1, \alpha_2, \alpha_3 \leq 2$ and letting $w = |V_1| = |V_2| = \cdots = |V_K|$, we have

$$w \geq \frac{(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - \frac{1}{4}\eta)n}{(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - \frac{1}{2}\eta)k} = \frac{n}{k} + \frac{\frac{1}{4}\eta n}{(c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 - \frac{1}{2}\eta)k} \geq \left(1 + \tfrac{1}{8c}\eta\right)\frac{n}{k}.$$

Suppose this three-multicolouring of $\mathcal{G}$ results in a green odd connected-matching on at least $\alpha_3 k$ vertices. Then, $\mathcal{G}$ contains a connected green component $\mathcal{F}$, which contains a matching $\mathcal{M} = \{e_1, e_2, \ldots, e_q\}$ for some q such that $\alpha_3 k \leq 2q \leq \alpha_3 k + 2$, and also an odd cycle $\mathcal{D}$.

Notice that the minimum connected subgraph of $\mathcal{F}$ containing $\mathcal{M}$ and a vertex from $\mathcal{D}$ is a tree $\mathcal{T}$. There is a *cyclic-walk* with an even number of edges, which traverses each edge of this tree exactly twice. This can be extended using the edges of $\mathcal{D}$ to give a cyclic-walk $\mathcal{C}$ in $\mathcal{F}$ with an odd number of edges including every edge of $\mathcal{M}$. Label the vertices of this cyclic-walk $\widehat{V}_1, \widehat{V}_2, \widehat{V}_3, \ldots, \widehat{V}_p$ and observe that $p \leq 3K$.

Consider the green graph G_3 and recall that each pair $(\widehat{V}_i, \widehat{V}_{i+1})$ is (η^4, G_3)-regular and that, by the definition of the colouring of $\mathcal{G}$, $d(\widehat{V}_i, \widehat{V}_{i+1}) \geq \xi$. Now, suppose that, for some i, there exists $X_i \subseteq \widehat{V}_i$ with $|X_i| \geq \eta^4|\widehat{V}_i|$ such that every vertex in X_i has degree at most $\frac{4}{5}\xi w$ in $G_3[\widehat{V}_i, \widehat{V}_{i+1}]$. In that case, we have $d(X_i, \widehat{V}_{i+1}) \leq \frac{4}{5}\xi$ but, as noted above, we have $d(\widehat{V}_i, \widehat{V}_{i+1}) \geq \xi$ and $|d(X_i, \widehat{V}_{i+1}) - d(\widehat{V}_i, \widehat{V}_{i+1})| \leq \eta^4$, which, since $\eta \leq 0.01$ and $\xi \geq \eta$, gives rise to a contradiction.

Similarly, for each i, there can be at most $\eta^4|\widehat{V}_i|$ vertices in $\widehat{V}_i$ with degree less than $\frac{4}{5}\xi w$ in $G_3[\widehat{V}_{i-1}, \widehat{V}_i]$. Thus, for each i, there exists $U_i \subseteq \widehat{V}_i$ such that $|U_i| \geq (1 - 2\eta^4)|\widehat{V}_i|$ and every vertex in U_i has degree at least $\frac{4}{5}\xi w$ in $G_3[\widehat{V}_i, \widehat{V}_{i+1}]$ and in $G_3[\widehat{V}_{i-1}, \widehat{V}_i]$. Note, then, that every vertex in U_i has degree at least $\frac{4}{5}\xi w - 2\eta^4 w \geq \frac{1}{2}\xi w$ in $G_3[U_i, U_{i+1}]$ and in $G_3[U_{i-1}, U_i]$.

Thus, we can then greedily construct a path $v_1, v_2, \ldots, v_{p-2}$ such that $v_i \in U_i$. Notice that v_{p-2} has at least $\frac{1}{2}\xi w$ neighbours in U_{p-1} (call the set of these neighbours X) and v_1 has at least $\frac{1}{2}\xi w$ neighbours in U_p (call the set of these neighbours Y). Then, since $|X|, |Y| \geq \frac{1}{2}\xi w \geq \eta^4 w$, by regularity, the density of the pair (X, Y) is within η^4 of the density of $(\widehat{V}_{p-1}, \widehat{V}_p)$ and so is non-zero. Therefore, there exists an edge $v_{p-1}v_p$ such that $v_{p-1} \in X \subseteq U_{p-1} \subseteq \widehat{V}_{p-1}$ and $v_p \in X \subseteq U_p \subseteq \widehat{V}_p$. This edge can then be used to extend the path to an odd cycle $C = v_1 v_2, \ldots, v_p$ such that, for each i, $v_i \in U_i$. Observe, also, that $|V(C)| = p \leq 3K$.

Let $\mathcal{I}$ be the set of i such that $\widehat{V}_i\widehat{V}_{i+1}$ corresponds to the first time the cyclic-walk visits a given edge of $\mathcal{M}$. Then, for each $i \in \mathcal{I}$, we may use Lemma 2.3.3 to replace $v_i v_{i+1}$ by a suitably long path not containing any other vertices of C.

Indeed, for each $i \in \mathcal{I}$, define $W_i = (\widehat{V}_i \backslash C) \cup \{v_i\}$ and $W_{i+1} = (\widehat{V}_{i+1} \backslash C) \cup \{v_{i+1}\}$. Then, since $|V(C)| \leq 3K$, we have

$$|W_i|, |W_{i+1}| \geq |\widehat{V}_1| - |C| \geq \left(1 + \tfrac{1}{8c}\eta\right) \frac{n}{k} - 3K \geq \left(1 + \tfrac{1}{16c}\eta\right) \frac{n}{k} \geq \frac{1}{2\eta^4},$$

provided that $n \geq \max\{3K^2/8c\eta, K/2\eta^4\}$. Observe also that the pairs (W_i, W_{i+1}) are each $2\eta^4$-regular. Now, since each v_i has degree at least $\frac{4}{5}\xi w$ in each of $G[\widehat{V}_i, \widehat{V}_{i+1}]$ and $G[\widehat{V}_{i-1}, \widehat{V}_i]$, provided $n \geq 5K^2/\eta$, each v_i has at degree at least $\frac{2}{3}(2\eta^4)^{1/2}w$ in each of $G[W_i, W_{i+1}]$ and $G[W_{i-1}, W_i]$ and so, since $\eta \leq 0.01$, we may use Lemma 2.3.3 to replace the edge v_iv_{i+1} with a path of length ℓ from v_i to v_{i+1} for any ℓ such that

$$3 \leq \ell \leq (1-2\eta^2)\,(2\min\{|W_i|,|W_{i+1}|\}) + 1.$$

Replacing each such edge with a path in this way, we can extend C to any length up to

$$\begin{aligned}|C| + 2(1-2\eta^2)\left(\sum_{i\in\mathcal{I}} \min\{|W_i|,|W_{i+1}|\}\right) &\geq |C| + 2\left(1-2\eta^2\right)\left(1+\tfrac{1}{16c}\eta\right)\left(\tfrac{n}{k}\right)q \\ &\geq |C| + (1-2\eta^2)\left(1+\tfrac{1}{16c}\eta\right)\alpha_3 n.\end{aligned}$$

Thus, since $\eta \leq 1/64c$, we can obtain a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices.

If the three-multicolouring of $\mathcal{G}$ results in a red (resp. blue) connected-matching on $\alpha_1 k$ (resp. $\alpha_2 k$) vertices, then G contains a red (resp. blue) cycle on $\langle\langle \alpha_1 n \rangle\rangle$ (resp. $\langle\langle \alpha_2 n \rangle\rangle$) vertices with the proof being simpler in that the cyclic-walk does not need to be extended to become odd. □

2.4 Definitions and notation

Recall that, given a three-coloured graph G, we use G_1, G_2, G_3 to refer to its monochromatic spanning subgraphs. That is, G_1 (resp. G_2, G_3) has the same vertex set as G and includes, as an edge, any edge which (in G) is coloured red (resp. blue, green). If G_1 contains the edge uv, we say that u and v are *red neighbours* of each other in G. Similarly, if $uv \in E(G_2)$, we say that u and v are *blue neighbours* and, if $uv \in E(G_3)$, we say that that u and v are *green neighbours.*

Given a graph G, we say $u, v \in V(G)$ are *connected* (in G) if there exists a path in G between u and v. The graph itself is said to be *connected* if any pair of vertices are connected. By extension, given a subgraph H of G, we say H is *connected* if, given any pair $u, v \in V(H)$, there exists a path in H between u and v and say that H is *effectively-connected* if, given any pair $u, v \in V(H)$, there exists a path in G between u and v.

A *connected-component* of a graph G is a maximal connected subgraph. A subgraph of H, a subgraph of G, is an *effectively-connected-component* or *effective-component* of H if it is a maximal effectively-connected subgraph of H. Thus the effective-components of H are restrictions of the components of G to H.

Given a multicoloured graph G, we say that two vertices u and v belong to the same *monochromatic component* of G if they belong to the same component of G_i for some i. Given a subgraph H of a multicoloured graph G, we say that two vertices u and v belong to the same *monochromatic effective-component* of H if they belong to the same effective-component of G_i for some i. We can thus talk about, for instance, the *red components* of a graph G or the *red effective-components* of a subgraph H of G.

We say that a graph $G = (V, E)$ on N vertices is a-*almost-complete* for $0 \leq a \leq N-1$ if its minimum degree $\delta(G)$ is at least $(N-1)-a$. Observe that, if G is a-almost-complete and $X \subseteq V$, then $G[X]$ is also a-almost-complete.

We say that a graph G on N vertices is $(1-c)$-*complete* for $0 \leq c \leq 1$ if it is $c(N-1)$-almost-complete, that is, if $\delta(G) \geq (1-c)(N-1)$. Observe that, for $c \leq \frac{1}{2}$, any $(1-c)$-complete graph is connected.

We say that a bipartite graph $G = G[U, W]$ is a-*almost-complete* if every $u \in U$ has degree at least $|W| - a$ and every $w \in W$ has degree at least $|U| - a$. Notice that, if $G[U, W]$ is a-almost-complete and $U_1 \subseteq U, W_1 \subseteq W$, then $G[U_1, W_1]$ is a-almost-complete.

We say that a bipartite graph $G = G[U, W]$ is $(1-c)$-*complete* if every $u \in U$ has degree at least $(1-c)|W|$ and every $w \in W$ has degree at least $(1-c)|U|$. Again, notice that, for $c < \frac{1}{2}$, any $(1-c)$-complete bipartite graph $G[U, W]$ is connected, provided that $U, W \neq \emptyset$.

We say that a graph G on N vertices is c-*sparse* for $0 < c < 1$ if its maximum degree is at most $c(N-1)$. We say a bipartite graph $G = G[U, W]$ is c-*sparse* if every $u \in U$ has degree at most $c|W|$ and every vertex $w \in W$ has degree at most $c|U|$.

For vertices u and v in a graph G, we will say that the edge uv is *missing* if $uv \notin E(G)$.

Recall that, given a graph $G = (V, E)$, we define a *matching* in that graph to be a collection of edges such that no two edges are incident at the same vertex. We will sometimes abuse terminology and, where appropriate, refer to a matching by its vertex set rather than its edge set. Recall also that we call a matching with all its vertices in the same component of G a *connected-matching* and that a connected-matching is called *odd* if the component containing the matching also contains an odd cycle. Note that we call a connected-matching with all its edges contained in a monochromatic component of G a *monochromatic connected-matching*.

2.5 Connected-matching stability result

Before proceeding to state Theorem B, we define the coloured structures we will need.

Definition 2.5.1. *For x_1, x_2, c_1, c_2 positive, γ_1, γ_2 colours, let $\mathcal{H}(x_1, x_2, c_1, c_2, \gamma_1, \gamma_2)$ be the class of edge-multicoloured graphs defined as follows:*

A given two-multicoloured graph $H = (V, E)$ belongs to $\mathcal{H}$ if its vertex set can be partitioned into $X_1 \cup X_2$ such that

(i) $|X_1| \geq x_1, |X_2| \geq x_2$*;*

(ii) H *is* c_1*-almost-complete; and*

(iii) defining H_1 *to be the spanning subgraph induced by the colour* γ_1 *and* H_2 *to be the subgraph induced by the colour* γ_2*,*

(a) $H_1[X_1]$ *is* $(1 - c_2)$*-complete and* $H_2[X_1]$ *is* c_2*-sparse,*

(b) $H_2[X_1, X_2]$ *is* $(1 - c_2)$*-complete and* $H_1[X_1, X_2]$ *is* c_2*-sparse.*

Figure 2.3: $H \in \mathcal{H}(x_1, x_2, c_1, c_2, red, blue)$.

Definition 2.5.2. *For x_1, x_2, x_3, c positive, let $\mathcal{K}(x_1, x_2, x_3, c)$ be the class of edge-multicoloured graphs defined as follows:*

A given three-multicoloured graph $H = (V, E)$ belongs to $\mathcal{K}$ if its vertex set can be partitioned into $X_1 \cup X_2 \cup X_3$ such that

(i) $|X_1| \geq x_1, |X_2| \geq x_2, |X_3| \geq x_3$*;*

(ii) H *is* c*-almost-complete;*

(iii) (a) all edges present in $H[X_1, X_3]$ *are red,*

(b) all edges present in $H[X_2, X_3]$ *are blue,*

(c) all edges present in $H[X_3]$ *are green.*

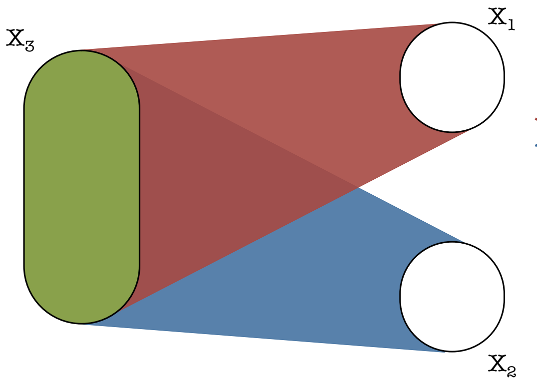


Figure 2.4: $H \in K(x_1, x_2, x_3, c)$.

Definition 2.5.3. *For x_1, x_2, y_1, y_2, z, c positive, let $\mathcal{K}^*(x_1, x_2, y_1, y_2, z, c)$ be the class of edge-multicoloured graphs defined as follows:*

A given three-multicoloured graph $H = (V, E)$ belongs to $\mathcal{K}^*$, if its vertex set can be partitioned into $X_1 \cup X_2 \cup Y_1 \cup Y_2$ such that

(i) $|X_1| \geq x_1, |X_2| \geq x_2, |Y_1| \geq y_2, |Y_2| \geq y_2, |Y_1| + |Y_2| \geq z$*;*

(ii) H is c-almost-complete;

(iii) (a) all edges present in $H[X_1, Y_1]$ and $H[X_2, Y_2]$ are red,

(b) all edges present in $H[X_1, Y_2]$ and $H[X_2, Y_1]$ are blue,

(c) all edges present in $H[X_1, X_2]$ and $H[Y_1, Y_2]$ are green.

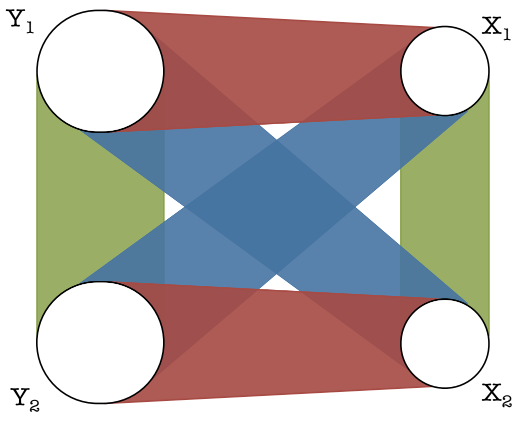


Figure 2.5: $H \in K^*(x_1, x_2, y_1, y_2, c)$.

Having defined the coloured structures, we are in a position to state the main technical result of this chapter, that is, the connected-matching stability result. The proof of this result, which follows in Sections 2.7–2.8, takes up the majority of this chapter.

Theorem B. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, letting*

$$c = \max\{2\alpha_1 + \alpha_2, \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3\},$$

there exists $\eta_B = \eta_B(\alpha_1, \alpha_2, \alpha_3)$ and $k_B = k_B(\alpha_1, \alpha_2, \alpha_3, \eta)$ such that, for every $k > k_B$ and every η such that $0 < \eta < \eta_B$, every three-multicolouring of G, a $(1 - \eta^4)$-complete graph on

$$(c - \eta)k \leq K \leq (c - \tfrac{1}{2}\eta)k$$

vertices, results in the graph containing at least one of the following:

(i) *a red connected-matching on at least $\alpha_1 k$ vertices;*

(ii) *a blue connected-matching on at least $\alpha_2 k$ vertices;*

(iii) *a green odd connected-matching on at least $\alpha_3 k$ vertices;*

(iv) *two disjoint subgraphs H_1, H_2 from $\mathcal{H}_1 \cup \mathcal{H}_2$, where*

$$\begin{aligned}\mathcal{H}_1 =& \mathcal{H}\left((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \textit{red, blue}\right),\\ \mathcal{H}_2 =& \mathcal{H}\left((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \textit{blue, red}\right);\end{aligned}$$

(v) *a subgraph from*

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 14000\eta^{1/2})k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4 k\right);$$

(vi) *a subgraph from $\mathcal{K}_1^* \cup \mathcal{K}_2^*$, where*

$$\begin{aligned}\mathcal{K}_1^* = \mathcal{K}^*\big(&(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k,\\ &(\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big),\end{aligned}$$

$$\begin{aligned}\mathcal{K}_2^* = \mathcal{K}^*\big(&(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\tfrac{3}{4}\alpha_3 - 140\eta^{1/2})k,\\ &100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big).\end{aligned}$$

Furthermore,

(iv) *occurs only if $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 14\eta^{1/2}$ with $H_1, H_2 \in \mathcal{H}_1$ unless $\alpha_2 \geq \alpha_1 - \eta^{1/8}$;*

(v) *and (vi) occur only if $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$.*

This result forms a partially strengthened analogue of the main technical result of the paper of Figaj and Łuczak [FŁ07b]. In that paper, Figaj and Łuczak considered a similar graph but on slightly more than $\max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}k$ vertices and proved the existence of a connected-matching, whereas we consider a graph on slightly fewer vertices and prove the existence of either a monochromatic connected-matching or a particular structure.

2.6 Tools

In this section, we summarise results that we shall use later in our proofs beginning with some results on Hamiltonicity including Dirac's Theorem, which gives us a minimum-degree condition for Hamiltonicity:

Theorem 2.6.1 (Dirac's Theorem [Dir52])**.** *If G is a graph on $n \geq 3$ vertices such that every vertex has degree at least $\frac{1}{2}n$, then G is Hamiltonian.*

Observe then that, by Dirac's Theorem, any c-almost-complete graph on n vertices is Hamiltonian, provided that $c \leq \frac{1}{2}n - 1$. Then, since almost-completeness is a hereditary property, we may prove the following corollary:

Corollary 2.6.2. *If G is a c-almost-complete graph on n vertices, then, for any integer m such that $2c + 2 \leq m \leq n$, G contains a cycle of length m.*

Proof. Given G, a c-almost-complete graph on n vertices, let $X \subseteq V(G)$ be such that $|X| = m \geq 2c + 2$. Then, $G[X]$ is a c-almost-complete graph on $|X|$ vertices so every vertex in $G[X]$ has degree at least $|X| - 1 - c = \frac{1}{2}|X| + (\frac{1}{2}|X| - 1 - c) \geq \frac{1}{2}|X|$. Thus, $G[X]$ satisfies the conditions in Dirac's Theorem and therefore contains a cycle on $|X| = m$ vertices. □

Dirac's Theorem may be used to assert the existence of Hamiltonian paths in a given graph as follows:

Corollary 2.6.3. *If $G = (V, E)$ is a simple graph on $n \geq 4$ vertices such that every vertex has degree at least $\frac{1}{2}n + 1$, then any two vertices of G are joined by a Hamiltonian path.*

Proof. Given any two vertices $x_1, x_2 \in V$, let $W = V \backslash \{x_1, x_2\}$. Then, $G[W]$ has $n-2$ vertices and has minimum degree at least $\frac{1}{2}(n-2)$ and so has a Hamiltonian cycle H. Since x_1, x_2 each have degree at least $\frac{1}{2}n$ to W, we can find u, v in W such that $ux_1, vx_2 \in E$ and u, v are consecutive vertices in H. Thus, we can construct a Hamiltonian path in G from x_1 to x_2. □

For balanced bipartite graphs, we make use of the following result of Moon and Moser:

Theorem 2.6.4 ([MM63])**.** *If $G = G[X, Y]$ is a simple bipartite graph on n vertices such that $|X| = |Y| = \frac{1}{2}n$ and $d(x) + d(y) \geq \frac{1}{2}n + 1$ for every $xy \notin E(G)$, then G is Hamiltonian.*

Observe that, by the above, any c-almost-complete balanced bipartite graph on n vertices is Hamiltonian, provided that $c \leq \frac{1}{4}n - \frac{1}{2}$. Then, since almost-completeness is a hereditary property, we may prove the following corollary:

Corollary 2.6.5. *If $G = G[X, Y]$ is c-almost-complete bipartite graph, then, for any even integer m such that $4c + 2 \leq m \leq 2\min\{|X|, |Y|\}$, G contains a cycle on m vertices.*

Proof. Given $G = G[X, Y]$, a c-almost-complete bipartite graph, let $U \subseteq X$, $V \subseteq Y$ be such that $|U| = |V| = \frac{1}{2}m \geq 2c+1$. Then, $G[U, V]$ is a c-almost-complete bipartite graph so, for any $u \in U$ and $v \in V$, we have $d(x) + d(y) \geq |U| + |V| - 2c \geq \frac{1}{2}(|U| + |V|) + 1$. Thus, $G[U, V]$ satisfies the conditions for Theorem 2.6.4 and therefore contains a cycle on $|U| + |V| = m$ vertices. □

For bipartite graphs which are not balanced, we make use of the Lemma below:

Lemma 2.6.6. *If $G = G[X_1, X_2]$ is a simple bipartite graph on $n \geq 4$ vertices such that $|X_1| > |X_2| + 1$ and every vertex in X_2 has degree at least $\frac{1}{2}n + 1$, then any two vertices x_1, x_2 in X_1 such that $d(x_2) \geq 2$ are joined by a path which visits every vertex of X_2.*

Proof. Observe that $\frac{1}{2}n + 1 = \frac{1}{2}|X_1| + \frac{1}{2}|X_2| + 1 = |X_1| - (\frac{1}{2}|X_1| - \frac{1}{2}|X_2| - 1)$ so any pair of vertices in X_2 have at least $|X_1| - (|X_1| - |X_2| - 2)$ common neighbours and, thus, at least $|X_1| - (|X_1| - |X_2|) \geq |X_2|$ common neighbours distinct from x_1, x_2.

Then, ordering the vertices of X_2 such that the first vertex is a neighbour of x_1 and the last is a neighbour of x_2, greedily construct the required path from x_1 to x_2. □

For graphs with a few vertices of small degree, we make use of the following result of Chvátal:

Theorem 2.6.7 ([Chv72])**.** *If G is a simple graph on $n \geq 3$ vertices with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$ such that*

$$d_k \leq k \leq \frac{n}{2} \implies d_{n-k} \geq n-k,$$

then G is Hamiltonian.

We also make extensive use of the theorem of Erdős and Gallai:

Theorem 2.6.8 ([EG59])**.** *Any graph on K vertices with at least $\frac{1}{2}(m-1)(K-1)+1$ edges, where $3 \leq m \leq K$, contains a cycle of length at least m.*

Observing that a cycle on m vertices contains a connected-matching on at least $m-1$ vertices, the following is an immediate consequence of the above.

Corollary 2.6.9. *For any graph G on K vertices and any m such that $3 \leq m \leq K$, if the average degree $d(G)$ is at least m, then G contains a connected-matching on at least m vertices.*

The following decomposition lemma of Figaj and Łuczak [FŁ07b] also follows from the theorem of Erdős and Gallai and is crucial in establishing the structure of a graph not containing large connected-matchings of the appropriate parities:

Lemma 2.6.10 ([FŁ07b, Lemma 9])**.** *For any graph G on K vertices and any m such that $3 \leq m \leq K$, if no odd component of G contains a matching on at least m vertices, then there exists a partition $V = V' \cup V''$ such that*

(i) $G[V']$ is bipartite;

(ii) every component of $G'' = G[V'']$ is odd;

(iii) $G[V'']$ has at most $\frac{1}{2}m|V(G'')|$ edges; and

(iv) there are no edges in $G[V', V'']$.

We recall two more results of Figaj and Łuczak. The first, is the main technical result from [FŁ07a]. The second from [FŁ07b], allows us to deal with graphs with a *hole*, that is, a subset $W \subseteq V(G)$ such that no edge of G lies inside W. Note that both of these results can be immediately extended to multicoloured graphs:

Theorem 2.6.11 ([FŁ07a, Lemma 8])**.** *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ and η such that $0 < \eta < 0.002\min\{\alpha_1^2, \alpha_2^2, \alpha_3^2\}$, there exists $k_{2.6.11} = k_{2.6.11}(\alpha_1, \alpha_2, \alpha_3, \eta)$ such that the following holds:*

For every $k > k_{2.6.11}$ and every $(1-\eta^4)$-complete graph G on

$$K \geq \tfrac{1}{2}\left(\alpha_1 + \alpha_2 + \alpha_3 + \max\{\alpha_1, \alpha_2, \alpha_3\} + 18\eta^{1/2}\right)k$$

vertices, for every three-colouring of the edges of G, there exists a colour $i \in \{1, 2, 3\}$ such that G_i contains a connected-matching on at least $(\alpha_i + \eta)k$ vertices.

Lemma 2.6.12 ([FŁ07b, Lemma 12])**.** *For every $\alpha, \beta > 0$, $v \geq 0$ and η such that $0 < \eta < 0.01\min\{\alpha, \beta\}$, there exists $k_{2.6.12} = k_{2.6.12}(\alpha, \beta, v, \eta)$ such that, for every $k > k_{2.6.12}$, the following holds:*

Let $G = (V, E)$ be a graph obtained from a $(1-\eta^4)$-complete graph on

$$K \geq \tfrac{1}{2}\left(\alpha + \beta + \max\{2v, \alpha, \beta\} + 6\eta^{1/2}\right)k$$

vertices by removing all edges contained within a subset $W \subseteq V$ of size at most vk. Then, every two-multicolouring of the edges of G results in either a red connected-matching on at least $(\alpha + \eta)k$ vertices or a blue connected-matching on at least $(\beta + \eta)k$ vertices.

The following pair of lemmas allow us to find large connected-matchings in almost complete bipartite graphs:

Lemma 2.6.13 ([FŁ07b, Lemma 10])**.** *Let $G = G[V_1, V_2]$ be a bipartite graph with bipartition (V_1, V_2), where $|V_1| \geq |V_2|$, which has at least $(1-\epsilon)|V_1||V_2|$ edges for some ϵ such that $0 < \epsilon < 0.01$. Then, G contains a connected-matching on at least $2(1-3\epsilon)|V_2|$ vertices.*

Notice that, if G is a $(1-\epsilon)$-complete bipartite graph with bipartition (V_1, V_2), then we may immediately apply the above to find a large connected-matching in G.

Lemma 2.6.14. *Let $G = G[V_1, V_2]$ be a bipartite graph with bipartition (V_1, V_2). If ℓ is a positive integer such that $|V_1| \geq |V_2| \geq \ell$ and G is a-almost-complete for some a such that $0 < a/\ell < 0.5$, then G contains a connected-matching on at least $2|V_2| - 2a$ vertices.*

Proof. Observe that G is $(1-a/\ell)$-complete. Therefore, since $a/\ell < 0.5$, G is connected. Thus, it suffices to find a matching of the required size. Suppose that we have found a matching with vertex set M such that $|M| = 2k$ for some $k < |V_2| - a$ and consider a vertex $v_2 \in V_2 \backslash M$. Since G is a-almost-complete, v_2 has at least $|V_1| - a$ neighbours in $|V_1|$ and thus at least one neighbour in $v_1 \in V_1 \backslash M$. Then, the edge v_1v_2 can be added to the matching and thus, by induction, we may obtain a matching on $2|V_2| - 2a$ vertices. □

We also make use of the following Lemma from [KSS09b], which is an extension of the two-colour Ramsey result for even cycles and which allows us to find, in any almost-complete two-multicoloured graph on K vertices, either a large matching or a particular structure.

Lemma 2.6.15 ([KSS09b])**.** *For every η such that $0 < \eta < 10^{-20}$, there exists $k_{2.6.15} = k_{2.6.15}(\eta)$ such that, for every $k > k_{2.6.15}$ and every $\alpha, \beta > 0$ such that $\alpha \geq \beta \geq 100\eta^{1/2}\alpha$, if $K > (\alpha + \frac{1}{2}\beta - \eta^{1/2}\beta)k$ and $G = (V, E)$ is a two-multcoloured $\beta\eta^2k$-almost-complete graph on K vertices, then at least one of the following occurs:*

(i) G contains a red connected-matching on at least $(1 + \eta^{1/2})\alpha k$ vertices;

(ii) G contains a blue connected-matching on at least $(1 + \eta^{1/2})\beta k$ vertices;

(iii) the vertices of G can be partitioned into three sets W, V', V'' such that

(a) $|V'| < (1 + \eta^{1/2})\alpha k$, $|V''| \leq \frac{1}{2}(1 + \eta^{1/2})\beta k$, $|W| \leq \eta^{1/16}k$,

(b) $G_1[V']$ is $(1 - \eta^{1/16})$-complete and $G_2[V']$ is $\eta^{1/16}$-sparse,

(c) $G_2[V', V'']$ is $(1 - \eta^{1/16})$-complete and $G_1[V', V'']$ is $\eta^{1/16}$-sparse;

(iv) we have $\beta > (1 - \eta^{1/8})\alpha$ and the vertices of G can be partitioned into sets W, V' and V'' such that

(a) $|V'| < (1 + \eta^{1/2})\beta k$, $|V''| \leq \frac{1}{2}(1 + \eta^{1/8})\alpha k$, $|W| \leq \eta^{1/16}k$,

(b) $G_2[V']$ is $(1 - \eta^{1/16})$-complete and $G_1[V']$ is $\eta^{1/16}$-sparse,

(c) $G_1[V', V'']$ is $(1 - \eta^{1/16})$-complete and $G_2[V', V'']$ is $\eta^{1/16}$-sparse.

Furthermore, if $\alpha + \frac{1}{2}\beta \geq 2(1 + \eta^{1/2})\beta$, then we can replace (i) with

(i') G contains a red odd connected-matching on $(1 + \eta^{1/2})\alpha k$ vertices.

We also make use of the following corollary of Lemma 2.6.15:

Corollary 2.6.16. *For every $0 < \epsilon < 10^{-12}$, there exists $k_{2.6.16} = k_{2.6.16}(\epsilon)$ such that, for every $k \geq k_{2.6.16}$, if $K > (1-\epsilon)k$ and $G = (V, E)$ is a two-multicoloured $\frac{27}{8}\epsilon^4 k$-almost-complete graph, then G contains at least one of the following:*

(i) a red connected-matching on $(\frac{2}{3} - 7\epsilon^{1/8})k$ vertices;

(ii) a blue connected-matching on $(\frac{2}{3} - 7\epsilon^{1/8})k$ vertices.

Proof. Setting $\eta = (\frac{3}{2}\epsilon)^2$, $\alpha = \beta = 2/3$, provided $k \geq k_{2.6.15}(\eta)$, we may apply Lemma 2.6.15, which results in at least one of the following occurring:

(i) G contains a red connected-matching on at least $(\frac{2}{3} + \epsilon)k$ vertices;

(ii) G contains a blue connected-matching on at least $(\frac{2}{3} + \epsilon)k$ vertices;

(iii) the vertices of G can be partitioned into three sets W, V', V'' such that

  (a) $|V'| < (\frac{2}{3} + \epsilon)k$, $|V''| \leq (\frac{1}{3} + \frac{1}{2}\epsilon)k$, $|W| \leq (\frac{3}{2}\epsilon)^{1/8}k$,
  (b) $G_1[V']$ is $(1 - (\frac{3}{2}\epsilon)^{1/8})$-complete and $G_2[V']$ is $(\frac{3}{2}\epsilon)^{1/8}$-sparse,
  (c) $G_2[V', V'']$ is $(1 - (\frac{3}{2}\epsilon)^{1/8})$-complete and $G_1[V', V'']$ is $(\frac{3}{2}\epsilon)^{1/8}$-sparse;

(iv) the vertices of G can be partitioned into three sets W, V', V'' such that

  (a) $|V'| < (\frac{2}{3} + \epsilon)k$, $|V''| \leq (\frac{1}{3} + \frac{1}{2}\epsilon)k$, $|W| \leq (\frac{3}{2}\epsilon)^{1/8}k$,
  (b) $G_2[V']$ is $(1 - (\frac{3}{2}\epsilon)^{1/8})$-complete and $G_1[V']$ is $(\frac{3}{2}\epsilon)^{1/8}$-sparse,
  (c) $G_1[V', V'']$ is $(1 - (\frac{3}{2}\epsilon)^{1/8})$-complete and $G_2[V', V'']$ is $(\frac{3}{2}\epsilon)^{1/8}$-sparse.

In the first two cases, the result is immediate.

In the third case, recalling that $K = |V'| + |V''| + |W|$, simple algebra yields $|V'| \geq (\frac{2}{3} - 2\epsilon^{1/8})$ and $|V''| \geq (\frac{1}{3} - 2\epsilon^{1/8})$. Then, since $G_2[V', V'']$ is $(1 - (\frac{3}{2}\epsilon)^{1/8})$-complete, there are at least $(1 - (\frac{3}{2}\epsilon)^{1/8})(\frac{1}{3} - 2\epsilon^{1/8})(\frac{2}{3} - 2\epsilon^{1/8})k^2$ edges in $G_2[V', V'']$. Thus, by Lemma 2.6.13, $G[V', V'']$ contains a blue connected-matching on at least $2(1 - 3(\frac{3}{2}\epsilon)^{1/8})(\frac{1}{3} - 2\epsilon^{1/8})k \geq (\frac{2}{3} - 7\epsilon^{1/8})k$ vertices.

In the fourth case, exchanging the roles of red and blue, an identical argument yields a red connected-matching on $(\frac{2}{3} - 7\epsilon^{1/8})k$ vertices. □

It is a well-known fact that either a graph is connected or its complement is. We now prove three simple extensions of this fact for two-coloured almost complete graphs, all of which can be immediately extended to two-multicoloured almost-complete graphs.

Lemma 2.6.17. *For every η such that $0 < \eta < 1/3$ and every $K \geq 1/\eta$, if $G = (V, E)$ is a two-coloured $(1-\eta)$-complete graph on K vertices and F is its largest monochromatic component, then $|F| \geq (1-3\eta)K$.*

Proof. If the largest monochromatic (say, red) component in G has at least $(1-3\eta)K$ vertices, then we are done. Otherwise, we may partition the vertices of G into sets A and B such that $|A|, |B| \geq 3\eta K \geq 2$ such that there are no red edges between A and B. Since G is $(1-\eta)$-complete, any two vertices in A have a common neighbour in B, and any two vertices in B have a common neighbour in A. Thus, $A \cup B$ forms a single blue component. □

The following lemmas form analogues of the above, the first concerns the structure of two-coloured almost complete graphs with one hole and the second concerns the structure of two-coloured almost complete graphs with two holes, that is, bipartite graphs.

Lemma 2.6.18. *For every η such that $0 < \eta < 1/20$ and every $K \geq 1/\eta$, the following holds. For W, any subset of V such that $|W|, |V \backslash W| \geq 4\eta^{1/2}K$, let $G_W = (V, E)$ be a two-coloured graph obtained from G, a $(1-\eta)$-complete graph on K vertices with vertex set V by removing all edges contained entirely within W. Let F be the largest monochromatic component of G_W and define the following two sets:*

$$W_r = \{w \in W : w \text{ has red edges to all but at most } 3\eta^{1/2}K \text{ vertices in } V\backslash W\};$$
$$W_b = \{w \in W : w \text{ has blue edges to all but at most } 3\eta^{1/2}K \text{ vertices in } V\backslash W\}.$$

Then, at least one of the following holds:

(i) $|F| \geq (1 - 2\eta^{1/2})K$;

(ii) $|W_r|, |W_b| > 0$.

Proof. Consider $G[V\backslash W]$. Since G is $(1-\eta)$-complete, $|V\backslash W| \geq 4\eta^{1/2}K$ and $\eta < 1/20$, we see that every vertex in $G[V\backslash W]$ has degree at least $|V\backslash W| - \eta(K-1) \geq (1 - \frac{1}{4}\eta^{1/2})(|V\backslash W| - 1)$, that is, $G[V\backslash W]$ is $(1-\frac{1}{4}\eta^{1/2})$-complete. Thus, provided $4\eta^{1/2}K \geq$

$1/(\frac{1}{4}\eta^{1/2})$, that is, provided $K \geq 1/\eta$, we can apply Lemma 2.6.17, which tells us that the largest monochromatic component in $G[V \backslash W]$ contains at least $|V \backslash W| - \eta^{1/2}K$ vertices. We assume, without loss of generality, that this large component is red and call it R.

Now, G is $(1-\eta)$-complete so either every vertex in W has a red edge to R (giving a monochromatic component of the required size) or there is a vertex $w \in W$ with at least $|R| - 2\eta K$ blue neighbours in R, that is, a vertex $w \in W_b$. Denote by B the set of $u \in R$ such that uw is blue. Then, $|B| \geq |V \backslash W| - 2\eta^{1/2}K$ and either every point in W has a blue edge to B, giving a blue component of size at least $|B \cup W| > (1 - 2\eta^{1/2})K$, or there is a vertex $w_1 \in W_r$. □

Lemma 2.6.19. *For every η such that $0 < \eta < 0.1$ and $K \geq 2/\eta$, the following holds: Suppose $G = (V, E)$ is a two-multicoloured graph obtained from an $(1-\eta)$-complete graph on K vertices with $V = A \cup B$ and $|A|, |B| \geq 6\eta K$ by removing all edges contained completely within A and all edges contained completely within B. Let F be the largest monochromatic component of G and define the following sets:*

$$A_r = \{a \in A : a \text{ has red edges to all but at most } 4\eta K \text{ vertices in } B\};$$
$$A_b = \{a \in A : a \text{ has blue edges to all but at most } 4\eta K \text{ vertices in } B\};$$
$$B_r = \{b \in B : b \text{ has red edges to all but at most } 4\eta K \text{ vertices in } A\};$$
$$B_b = \{b \in B : b \text{ has blue edges to all but at most } 4\eta K \text{ vertices in } A\}.$$

Then, at least one of the following occurs:

(i) $|F| \geq (1 - 7\eta)K$;

(ii) A, B *can be partitioned into* $A_1 \cup A_2, B_1 \cup B_2$ *such that* $|A_1|, |A_2|, |B_1|, |B_2| \geq 3\eta K$ *and all edges present between* A_i *and* B_j *are red for* $i = j$*, blue for* $i \neq j$;

(iii) $|A_r|, |A_b| > 0$;

(iv) $|B_r|, |B_b| > 0$.

Proof. Suppose $|F| < (1 - 7\eta)K$. Then, without loss of generality, A can be partitioned into $A_1 \cup A_2$, with $|A_1|, |A_2| \geq 3\eta K$, such that A_1 and A_2 are in different red components. Then, there exists no triple (a_1, b, a_2) with $a_1 \in A_1, b \in B, a_2 \in A_2$ and both a_1b and ba_2 coloured red. Thus, we may partition B into $B_1 \cup B_2$ such that there are no red edges present in $G[A_1, B_1]$ or $G[A_2, B_2]$.

Since G is $(1-\eta)$-complete, given any subsets $A' \subseteq A, B' \subseteq B$ every vertex in A' has degree at least $|B'|-\eta K$ in $G[A', B']$ and every vertex in B' has degree at least $|A'|-\eta K$ in $G[A', B']$. Thus, if $|B_1|, |B_2| \geq 3\eta K$, $G[A_1, B_1]$ and $G[A_2, B_2]$ each have a single blue component. Therefore, there can be no blue edges present in $G[A_1, B_2]$ or $G[A_2, B_1]$, giving rise to case (ii).

Thus, without loss of generality, we may assume that $|B_1| < 3\eta K$. Then, every vertex in A_2 is a vertex of A_b, in which case either every vertex $a \in A_1$ has a blue edge to B, leading to case (i), or there exists some $a \in A_1$ such that $a \in A_r$, giving rise to case (iii), thus completing the proof.

Note that exchanging the roles of A and B above leads to case (iv) in place of case (iii). □

2.7 Proof of the stability result – Part I

We begin with the case when $\alpha_1 \geq \alpha_2, \alpha_3$ and wish to prove that any three-multicoloured graph on slightly fewer than $(2\alpha_1 + \alpha_2)k$ vertices with sufficiently large minimum degree will contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices or a green odd connected-matching on at least $\alpha_3 k$ vertices, or will have a particular structure.

Thus, given $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 \geq \alpha_2, \alpha_3$, we choose

$$\eta < \eta_{B1} = \min \left\{ \frac{\alpha_2}{10^{20}}, \left(\frac{\alpha_2}{120}\right)^8, \left(\frac{\alpha_2}{800\alpha_1}\right)^2, \frac{\alpha_3}{10^4} \right\}$$

and consider $G = (V, E)$, a $(1-\eta^4)$-complete graph on $K \geq 72/\eta$ vertices, where

$$(2\alpha_1 + \alpha_2 - \eta)k \leq K \leq (2\alpha_1 + \alpha_2 - \tfrac{1}{2}\eta)k$$

for some integer $k > k_{B1}$, where $k_{B1} = k_{B1}(\alpha_1, \alpha_2, \alpha_3, \eta)$ will be defined implicitly during the course of this section, in that, on a finite number of occasions, we will need to bound k below in order to apply results from Section 2.6.

Note that, since $\alpha_1 \geq \alpha_2, \alpha_3$, the largest forbidden connected-matching is red and need not be odd. Note that, by scaling, we may assume that $\alpha_1 \leq 1$. Notice, then, that G is $3\eta^4 k$-almost-complete and, thus, for any $X \subset V$, $G[X]$ is also $3\eta^4 k$-almost-complete.

In this section, we seek to prove that G contains at least one of the following:

(i) a red connected-matching on at least $\alpha_1 k$ vertices;

(ii) a blue connected-matching on at least $\alpha_2 k$ vertices;

(iii) a green odd connected-matching on at least $\alpha_3 k$ vertices;

(iv) two disjoint subgraphs H_1, H_2 from $\mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\mathcal{H}_1 = \Big((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{red}, \text{blue}\Big),$$
$$\mathcal{H}_2 = \Big((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{blue}, \text{red}\Big).$$

We begin by noting that, if G has a green odd connected-matching on at least $\alpha_3 k$ vertices, then we are done. Thus, provided that $\alpha_3 k \geq 3$, since $\alpha_3 \leq \alpha_1$, by Lemma 2.6.10, we may partition the vertices of G into W, X and Y such that

(i) X and Y contain only red and blue edges;

(ii) W has at most $\frac{1}{2}\alpha_3 k|W| \leq \frac{1}{2}\alpha_1 k|W|$ green edges; and

(iii) there are no green edges between W and $X \cup Y$.

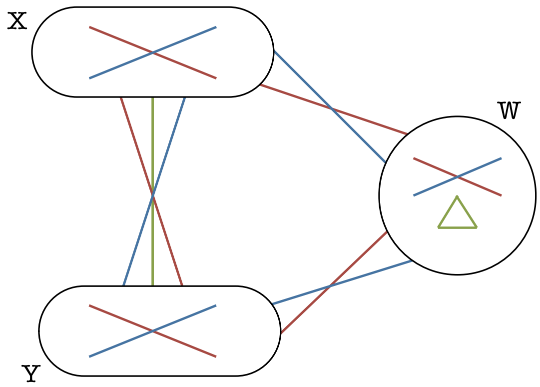


Figure 2.6: Decomposition of the green graph.

By this decomposition, writing w for $|W|/k$ and noticing that $e(G[X, Y])$ is maximised when X and Y are equal in size, we have

$$e(G_3) = e(G_3[W]) + e(G_3[X, Y]) \leq \tfrac{1}{2}\alpha_1 w k^2 + \tfrac{1}{4}(2\alpha_1 + \alpha_2 - w)^2 k^2. \qquad (2.1)$$

Now, consider the average degree of the green graph $d(G_3)$. Note that, since $K \leq 3k$, $\eta^4 K \leq \eta k$. Thus, the average number of missing edges at each vertex is at most ηk. Thus, if $d(G_3) \leq (\alpha_1 - 2\eta)k$, then either $d(G_1) \geq \alpha_1 k$ or $d(G_2) \geq \alpha_2 k$. If $d(G_1) \geq \alpha_1 k$, then, by Corollary 2.6.9, G contains a red connected-matching on at least $\alpha_1 k$ vertices. Similarly, if $d(G_2) \geq \alpha_2 k$, then G contains a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, we may assume that $d(G_3) > (\alpha_1 - 2\eta)k$, in which case

$$e(G_3) > \tfrac{1}{2}(\alpha_1 - 2\eta)(2\alpha_1 + \alpha_2 - \eta)k^2. \tag{2.2}$$

Comparing (2.1) and (2.2), we obtain

$$0 < w^2 + w(-2\alpha_1 - 2\alpha_2) + 2\alpha_1\alpha_2 + {\alpha_2}^2 + \eta(10\alpha_1 + 4\alpha_2).$$

Since $1 \geq \alpha_1 \geq \alpha_2$ and $\eta < \alpha_1/100$, this results in two cases:

(A) $w > \alpha_1 + \alpha_2 + \sqrt{{\alpha_1}^2 - (10\alpha_1 + 4\alpha_2)\eta} > 2\alpha_1 + \alpha_2 - 10\eta$;

(B) $w < \alpha_1 + \alpha_2 - \sqrt{{\alpha_1}^2 - (10\alpha_1 + 4\alpha_2)\eta} < \alpha_2 + 10\eta$.

Case A: $w > 2\alpha_1 + \alpha_2 - 10\eta$.

In this case, almost all the vertices of G are contained in the odd green component. Since $\eta < \eta_{B1}$, we have $2\alpha_1 + \alpha_2 - 10\eta > \alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + 9\eta^{1/2}$ and may apply Theorem 2.6.11 to obtain a connected-matching on $(\alpha_i + \eta)k$ vertices, provided that $k > k_{2.6.11}(\alpha_1, \alpha_2, \alpha_3, \eta)$. Furthermore, the nature of the decomposition means that, if the connected-matching is green, then it is odd, thus completing the case.

Case B: $w < \alpha_2 + 10\eta$.

We assume that $|X| \geq |Y|$ and consider the subgraph $G_1[X \cup W] \cup G_2[X \cup W]$, that is, the subgraph of G on $X \cup W$ induced by the red and blue edges.

Since $\eta \leq \eta_{B1}$, provided that $k > k_{2.6.12}(\alpha_1, \alpha_2, w, \eta)$, by Lemma 2.6.12 (regarding W as the hole), if

$$|X| + |W| \geq \tfrac{1}{2}\left(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1\} + 6\eta^{1/2}\right)k,$$

then we can obtain a red connected-matching on at least $(\alpha_1 + \eta)k$ vertices or a blue connected-matching on at least $(\alpha_2 + \eta)k$ vertices.

We may therefore assume that

$$|X| + |W| < \tfrac{1}{2}\left(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1\} + 6\eta^{1/2}\right)k. \tag{2.3}$$

Since $K = |X| + |Y| + |W|$ and $|X| \geq |Y|$,

$$|X| + |W| \geq \frac{K - |W|}{2} + |W| = \frac{(2\alpha_1 + \alpha_2 - \eta)k + wk}{2} = \left(\alpha_1 + \frac{\alpha_2}{2} - \frac{\eta}{2} + \frac{w}{2}\right)k. \tag{2.4}$$

Now, suppose that $w \leq \frac{1}{2}\alpha_1$, in which case, by (2.3) and (2.4), we have

$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\eta + \tfrac{1}{2}w)k \leq |W| + |X| < (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k,$$

which results in a contradiction, unless $w < 6\eta^{1/2} + \eta$, in which case almost all the vertices of G belong to $X \cup Y$.

Eliminating $|W|$, we find that

$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 4\eta^{1/2})k \leq |X| < (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k.$$

Since $\eta \leq \eta_{B1}$, provided $k > k_{2.6.15}(16\eta)$, we may apply Lemma 2.6.15 (with $\alpha = \alpha_1, \beta = \alpha_2$) to find that $G[X]$ contains either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices or a subgraph H_1 from $\mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\begin{aligned}
\mathcal{H}_1 =& \mathcal{H}\left((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{red}, \text{blue}\right), \\
\mathcal{H}_2 =& \mathcal{H}\left((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{blue}, \text{red}\right).
\end{aligned}$$

Furthermore, unless $\alpha_2 \geq \alpha_1 - \eta^{1/8}$, $H_1 \in \mathcal{H}_1$.

Now, consider Y. Since $|G| = |X| + |Y| + |W|$ and $|X| \geq |Y|$, we obtain

$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2})k \leq |Y| < (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k.$$

Then, provided $k \geq k_{2.6.15}(100\eta)$, we may apply Lemma 2.6.15 to $G[Y]$ to find that $G[Y]$

contains either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices or a two-coloured subgraph H_2 belonging to $\mathcal{H}_1 \cup \mathcal{H}_2$. Furthermore, unless $\alpha_2 \geq \alpha_1 - \eta^{1/8}$, $H_2 \in \mathcal{H}_1$ which would be sufficient to complete the proof in this case.

We may, therefore, assume that $\frac{1}{2}\alpha_1 < w < \alpha_2 + 10\eta$, in which case, from (2.3) and (2.4), we have

$$\left(\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}w - \tfrac{1}{2}\eta\right) k \leq |W| + |X| < \left(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + w + 3\eta^{1/2}\right) k.$$

Since $\alpha_2 > w - 10\eta$, we obtain $|W| + |X| \geq (\alpha_1 + w - 6\eta)k$.

Then, since $K = |X| + |Y| + |W|$ and $|X| \geq |Y|$, it follows that

$$\left.\begin{aligned}(\alpha_1 - \eta^{1/2})k &\leq |X| < \left(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2}\right) k,\\ (\alpha_1 - 4\eta^{1/2})k &< |Y| < \left(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2}\right) k,\\ (\alpha_1 - 8\eta^{1/2})k &\leq |W| < \left(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2}\right) k.\end{aligned}\right\} \quad (2.5)$$

The bounds for $|X|$ in (2.5) lead to a contradiction unless $\alpha_1 - \alpha_2 \leq 8\eta^{1/2}$. Therefore, we may only concern ourselves with the case where $\alpha_2 \leq \alpha_1 \leq \alpha_2 + 8\eta^{1/2}$ and, therefore, X, Y and W each contain about a third of the vertices of G.

Recall that there are no green edges contained within X or Y. Then, since G is $3\eta^4 k$-almost-complete, provided $k > k_{2.6.16}(\eta)$, we may apply Corollary 2.6.16 to $G[X], G[Y]$ to find that each contains a monochromatic connected-matching on at least $(\frac{2}{3}\alpha_1 - 8\eta^{1/8})k$ vertices. Thus, provided $\eta < (\alpha_1/120)^8$, we may assume that each of X and Y contain a monochromatic connected-matching on at least $\frac{3}{5}\alpha_1 k$ vertices. Referring to these matchings as $M_1 \subseteq G[X]$ and $M_2 \subseteq G[Y]$, we consider three subcases:

Case B.i: M_1 and M_2 are both red.

Suppose that there exists $r \in M_1, w \in W$ and $s \in M_2$ such that rw and ws are red. This would give a red connected-matching on at least $\frac{6}{5}\alpha_1 k$ vertices. Therefore, we may assume that every $w \in W$ has either rw blue or missing for all $r \in M_1$, or ws blue or missing for all $s \in M_2$.

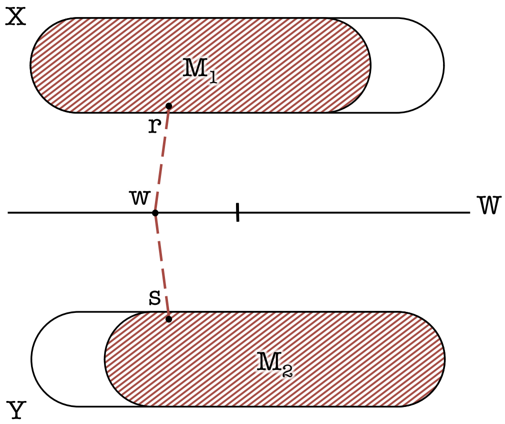


Figure 2.7: Red Matchings M_1 and M_2 and red edges rw and ws.

Thus, we may partition W into $W_1 \cup W_2$, where W_1, W_2 are defined as follows:

$$W_1 = \{w \in W \text{ such that } w \text{ has no red edges to } M_1\};$$
$$W_2 = \{w \in W \text{ such that } w \text{ has no red edges to } M_2\}.$$

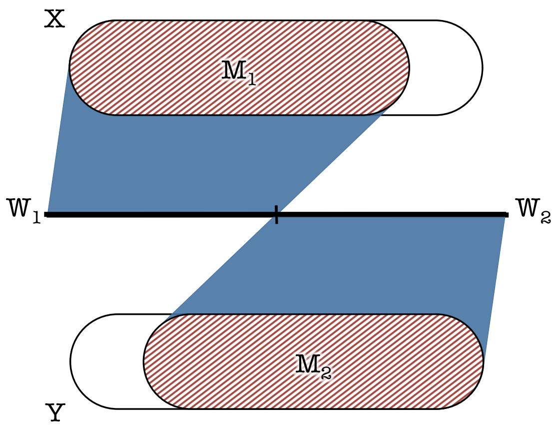


Figure 2.8: Partition of W into $W_1 \cup W_2$.

Suppose that $|W_1| \geq (\frac{1}{2}\alpha_2 + 6\eta^4)k$. Then, since G is $3\eta^4 k$-almost-complete, so is $G[W, M_1]$. Since $\eta < (\alpha_2/100)^2 \leq (\alpha_2/60)^{1/4}$, $|M_1| \geq \frac{3}{5}\alpha_1 k \geq \frac{\alpha_2}{2}k + 6\eta^4 k$ and we may apply Lemma 2.6.14 with $\ell = (\frac{1}{2}\alpha_2 + 6\eta^4)k$ and $a = 3\eta^4 k$ to give a blue connected-matching on at least $\alpha_2 k$ vertices. The result is the same in the event that $|W_2| \geq (\frac{1}{2}\alpha_2 + 6\eta^4)k$.

Therefore, we may assume that $|W_1|, |W_2| \leq (\frac{1}{2}\alpha_2 + 6\eta^4)k$. In that case, we have $|W_1| = |W| - |W_2| \geq (\frac{\alpha_1}{2} - 9\eta^{1/2})k$ and, likewise, $|W_2| \geq (\frac{\alpha_1}{2} - 9\eta^{1/2})k$. Thus, since $\eta < (\alpha_1/100)^2$, Lemma 2.6.14 gives a blue connected-matching on at least $(\alpha_2 - 20\eta^{1/2})k$

vertices in each of $G[M_1, W_1]$ and $G[M_2, W_2]$.

Then, suppose there exists a blue edge rw for $r \in M_1$, $w \in W_2$. This would connect these two blue connected-matchings, giving one on at least $(2\alpha_1 - 40\eta^{1/2})k \geq \alpha_2 k$ vertices. Thus, all edges present in $G[M_1, W_2]$ are coloured exclusively red. By the same argument, so are all edges present in $G[M_2, W_1]$.

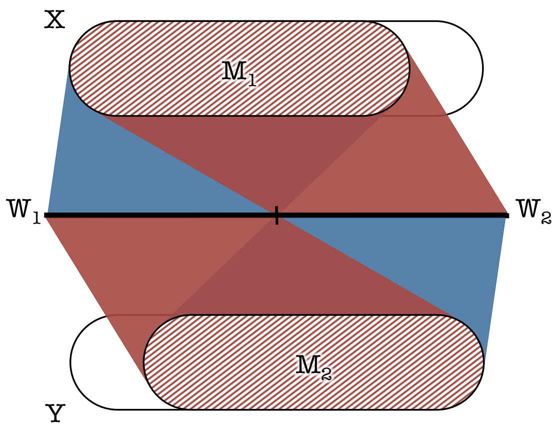


Figure 2.9: Colouring of the edges of $G[M_1, W_2] \cup G[M_2, W_1]$.

Now, choose any set R_1 of $10\eta^{1/2}k$ of the edges from the matching M_1, let $M_1' = M \backslash R_1$ and consider $G[V(M_1'), W_2]$. Since $\eta < (\alpha_1/100)^2$, we have $|V(M_1')| \geq (\frac{1}{2}\alpha_1 - 9\eta^{1/2})k$ and thus may apply Lemma 2.6.14 to $G[V(M_1'), W_2]$ to obtain a collection R_2 of edges from $G[V(M_1'), W_2]$ which form a red connected-matching on at least $(\alpha_1 - 20\eta^{1/2})k$ vertices. Since R_1 and R_2 do not share any vertices but do belong to the same red-component of G, the collection of edges $R_1 \cup R_2$ forms a red connected-matching on at least $\alpha_1 k$ vertices, completing this case.

Case B.ii: M_1 and M_2 are both blue.

Exchanging the roles of red and blue (and where necessary α_1 and α_2), the proof follows the same steps as in Case B.i above.

Case B.iii: M_1 and M_2 are different colours.

Without loss of generality, consider the case where M_1 is red and M_2 is blue. Since $\eta < (\alpha_1/100)^2$, by (2.5), we have $|X| + |W|, |Y| + |W| \geq \frac{1}{2}K$, so $G[X \cup W]$ and $G[Y \cup W]$

are each $(1-2\eta^4)$-complete. Additionally, $|W|, |V\backslash W| \geq 4(2\eta^4)^{1/2}|X \cup W|$. Thus, provided that $\frac{1}{2}K \geq 1/2\eta^4$, we may apply Lemma 2.6.18 separately to $G[X \cup W]$ and $G[Y \cup W]$ (regarding W as the hole in each case) with the result being that at least one of the following occurs:

(a) $X \cup W$ has a connected red component F on at least $|X \cup W| - \eta k$ vertices;

(b) $X \cup W$ has a connected blue component F on at least $|X \cup W| - \eta k$ vertices;

(c) $Y \cup W$ has a connected red component F on at least $|Y \cup W| - \eta k$ vertices;

(d) $Y \cup W$ has a connected blue component F on at least $|Y \cup W| - \eta k$ vertices;

(e) there exist points $w_1, w_2, w_3, w_4 \in W$ such that the following hold:

  (i) w_1 has red edges to all but at most ηk vertices in X,

  (ii) w_2 has blue edges to all but at most ηk vertices in X,

  (iii) w_3 has red edges to all but at most ηk vertices in Y,

  (iv) w_4 has blue edges to all but at most ηk vertices in Y.

In case (a), we discard from W the, at most ηk, vertices not contained in F and consider $G[W, Y]$. Either there are at least $\frac{1}{5}\alpha_1 k$ mutually independent red edges present in $G[W, Y]$ (which can be used to augment M_1) or we may obtain $W' \subset W$, $Y' \subset Y$ with $|W'|, |Y'| \geq (\frac{4}{5}\alpha_1 - 10\eta^{1/2})k$ such that all the edges present in $G[W', Y']$ are coloured exclusively blue. Notice that, since G is $3\eta^4 k$-almost-complete, so is $G_2[W', Y']$ and, since $\eta < (\alpha_1/100)^2$, we may apply Lemma 2.6.14 (with $a = 3\eta^4 k$ and $\ell = \frac{3}{5}\alpha_1 k$) to obtain a blue connected-matching on at least $\alpha_1 k \geq \alpha_2 k$ vertices.

In case (b), suppose there exists a blue edge in $G[M_2, F]$. Then, at least $|M_2 \cup W| - \eta k$ of the vertices of $M_2 \cup W$ would belong to the same blue component in G. We could then consider $G[W, X]$ and, by the same argument used in case (a) (with the roles of the colours reversed), find either a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, we may instead, after discarding at most ηk vertices from W, assume that all edges present in $G[M_2, W]$ are coloured exclusively red and apply Lemma 2.6.14 to obtain a red connected-matching on at least $\alpha_1 k$ vertices.

In case (c), the same argument as given in case (b) gives either a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices.

In case (d), the same argument as given in case (a) gives gives either a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices.

In case (e), there exist points $w_1, w_2, w_3, w_4 \in W$ such that w_1 has red edges to all but at most ηk vertices in X, w_2 has blue edges to all but at most ηk vertices in X, w_3 has red edges to all but at most ηk vertices in Y, and w_4 has blue edges to all but at most ηk vertices in Y.

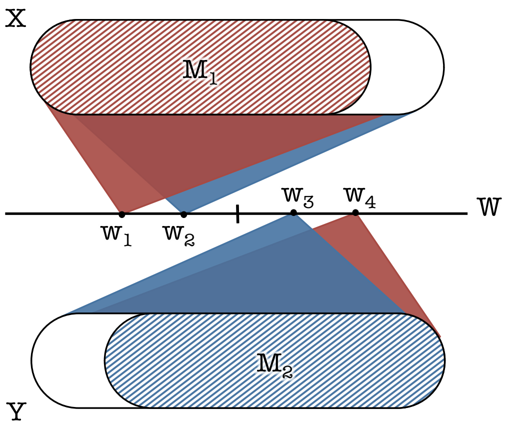


Figure 2.10: Vertices w_1, w_2, w_3 and w_4 in case (e).

Thus, defining

$$X_S = \{x \in X \text{ such that } xw_1 \text{ is red and } xw_2 \text{ is blue}\},$$
$$Y_S = \{y \in Y \text{ such that } yw_3 \text{ is red and } yw_4 \text{ is blue}\},$$

by (2.5), we have $|X_S|, |Y_S| \geq (\alpha_1 - 5\eta^{1/2})k$. Suppose there exists $x \in X_S$, $w \in W$, $y \in Y_S$ such that xw and wy are red. In that case, $X_S \cup Y_S$ belong to the same red component of G. Recall that M_1 contains a red matching on $\frac{3}{5}\alpha_1 k$ vertices and consider $G[W, Y_S]$. Either we can find $\frac{1}{5}\alpha_1 k$ mutually independent red edges in $G[W, Y_S]$ (which together with M_1 give a red connected-matching on at least $\alpha_1 k$ vertices) or we may obtain $W' \subset W$, $Y' \subset Y_S$ with $|W'|, |Y'| \geq (\frac{4}{5}\alpha_1 - 10\eta^{1/2})k$ such that all the edges present in $G[W', Y']$ are coloured exclusively blue. Then, as in case (a), we may apply Lemma 2.6.14 to obtain a blue connected-matching on at least $\alpha_1 k \geq \alpha_2 k$ vertices.

Thus, we assume no such triple exists and, similarly, we may assume there exists no triple $x \in X_S$, $w \in W$, $y \in Y_S$ such that xw and wy are blue. Thus, we may partition W into $W_1 \cup W_2$ such that all edges present in $G[W_1, X_S]$ and $G[W_2, Y_S]$ are coloured exclusively red and all edges present in $G[W_1, Y_S]$ and $G[W_2, X_S]$ are coloured exclusively blue.

Thus, we may assume that $|W_1|, |W_2| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k$ (else Lemma 2.6.14 could be used to give a red connected-matching on at least $\alpha_1 k$ vertices) and therefore also that $|W_1|, |W_2| \geq (\frac{1}{2}\alpha_1 - 9\eta^{1/2})k$, in which case, the same argument as in the last paragraph of case (a) gives a red connected-matching on $\alpha_1 k$ vertices.

This concludes Case B and, thus, Part I of the proof of Theorem B.

Note that the preceding section together with Section 2.9 and Section 2.10 forms a complete proof of Theorem A in the case that $\alpha_1 \geq \alpha_2, \alpha_3$. Therefore, a reader wishing to get a flavour of the overall proof method may like to skip over Section 2.8 and read Section 2.9 next.

2.8 Proof of the stability result – Part II

We now consider the case when $\alpha_3 \geq \alpha_1 \geq \alpha_2$. We wish to prove that any three-multicoloured $(1-\eta^4)$-complete graph on slightly fewer than $\max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}k$ vertices will have a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or will have a particular coloured structure.

Thus, given $\alpha_1, \alpha_2, \alpha_3$ such that $\alpha_3 \geq \alpha_1 \geq \alpha_2$, we set

$$c = \max\{2\alpha_1 + \alpha_2, \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3\},$$

choose

$$\eta < \eta_{B2} = \min\left\{\frac{1}{10^5}, \frac{\alpha_2}{10^{24}}, \left(\frac{\alpha_2}{100}\right)^8, \left(\frac{\alpha_2}{1200\alpha_1}\right)^2\right\}$$

and consider $G = (V, E)$, a $(1-\eta^4)$-complete graph on $K \geq 72/\eta$ vertices, where

$$(c - \eta)k \leq K \leq (c - \tfrac{\eta}{2})k$$

for some integer $k > k_{B2}$, where $k_{B2} = k_{B2}(\alpha_1, \alpha_2, \alpha_3, \eta)$ will be defined implicitly during the course of the proof, in that, on a finite number of occasions, we will need to bound k below in order to apply results from Section 2.6.

Note that, since $\alpha_3 \geq \alpha_1 \geq \alpha_2$, the largest forbidden connected-matching is green and odd. By scaling, we may assume that $2 \geq \alpha_3 \geq 1 \geq \alpha_1 \geq \alpha_2$. Thus, G is $3\eta^4 k$-almost-complete, as is $G[X]$, for any $X \subset V$.

We begin by noting that we can use Theorem 2.6.11 to obtain either a red connected-matching on $\alpha_1 k$ vertices, a blue connected-matching on $\alpha_2 k$ vertices or a green connected-matching of almost the required size. Note, however, that this green connected-matching need not be odd. Indeed, the graph has

$$|V| = vk \geq (c-\eta)k = (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + (v - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 9\eta^{1/2}) + 9\eta^{1/2})k$$

vertices and, since $\alpha_1 \geq \alpha_2$ and $\eta \leq (\frac{\alpha_2}{20})^2$, we have

$$\begin{aligned}(v - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 9\eta^{1/2}) &\geq (c - \eta - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 9\eta^{1/2}) \\ &\geq \max\{2\alpha_1 + \alpha_2, \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3\} - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 10\eta^{1/2} \\ &\geq \max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2} \geq \alpha_1 \geq \alpha_2.\end{aligned}$$

Thus, since $\eta \leq 0.002 \min\{\alpha_1^2, \alpha_2^2, \alpha_3^2\}$, by Theorem 2.6.11, provided $k \geq k_{2.6.11}(\alpha_1, \alpha_2, v - \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 9\eta^{1/2}, \eta)$, G contains either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices or a green connected-matching on at least

$$|V| - (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 9\eta^{1/2})k \geq (\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \qquad (2.6)$$

vertices.

Lemma 2.6.10 gives a decomposition of the green-graph G_3 into its bipartite and non-bipartite parts and in doing so gives a decomposition of the vertices of G into $X \cup Y \cup W$ such that there are no green edges between $X \cup Y$ and W or within X or Y. Choosing such a decomposition which maximises $|X \cup Y|$, results in $G_3[X \cup Y]$ being the union of the bipartite green components of G and $G_3[W]$ being the union of the non-bipartite green components of G. In what follows, we consider the vertices of G to have been thus partitioned. We will also assume that $|X| \geq |Y|$ and will write $\widehat{V}$ for $X \cup W$ and w for $|W|/k$. By (2.6), we may assume that the largest green connected-matching in G spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and distinguish three cases:

(C) the largest green connected-matching F is not odd and $w \geq 7\eta^{1/2}$;

(D) the largest green connected-matching F is not odd and $w \leq 7\eta^{1/2}$;

(E) the largest green connected-matching F is odd.

Within each case, we will, when necessary, distinguish between the two possible forms taken by c, that is, between $c = 2\alpha_1 + \alpha_2$ and $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$:

(II†) The first possibility arises only when $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, in which case we may assume that F spans at least $(\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k$ vertices.

(II‡) The second possibility arises only when $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, in which case we may assume that that F spans at least $(\alpha_3 - 10\eta^{1/2})k$ vertices.

Case C: Largest green connected-matching is not odd and $w \geq 7\eta^{1/2}$.

Suppose that we have $c = 2\alpha_1 + \alpha_2$. Then F spans at least $(\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2})k$ vertices and is assumed to not be contained in an odd component of G. Thus, by the decomposition, we have

$$|X| \geq |Y| \geq (\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2})k, \tag{2.7a}$$

$$|X| \geq \frac{K - |W|}{2} \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\eta - \tfrac{1}{2}w)k, \tag{2.7b}$$

$$|W| = K - |X| - |Y| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k. \tag{2.7c}$$

From (2.7a) and (2.7b) we obtain

$$|\widehat{V}| = |X| + |W| \geq (\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 + w - 5\eta^{1/2})k \tag{2.8a}$$

and

$$|\widehat{V}| = |X| + |W| \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}w - \tfrac{1}{2}\eta)k. \tag{2.8b}$$

Now, suppose that $|\widehat{V}| \geq \frac{1}{2}(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2})k$. In that case, since $\eta \leq 0.01\alpha_2$, provided $k > k_{2.6.12}(\alpha_1, \alpha_2, w, \eta)$, we may apply Lemma 2.6.12 to obtain either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices. Therefore, we may assume that

$$|\widehat{V}| \leq \tfrac{1}{2}\left(\alpha_1 + \alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2}\right)k. \tag{2.8c}$$

If $w \leq \frac{1}{2}\alpha_1$, then together (2.8b) and (2.8c) contradict our assumption that $w \geq 7\eta^{1/2}$.

Thus, we may assume that $w \geq \frac{1}{2}\alpha_1$. In that case, (2.8a) and (2.8c) give

$$(\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 + w - 5\eta^{1/2})k \leq |\widehat{V}| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + w + 3\eta^{1/2})k,$$

which together with (2.7a) and (2.7c) gives

$$\begin{aligned}
(\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2})k \leq |X| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k,\\
(\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2})k \leq |Y| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k,\\
(\alpha_1 - 7\eta^{1/2})k \leq |W| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k.
\end{aligned}$$

The condition for $|X|$ above gives a contradiction unless $\alpha_1 \leq \alpha_2 + 32\eta^{1/2}$. So, recalling that $\alpha_2 \leq \alpha_1$, we may obtain

$$\left.\begin{aligned}
(\alpha_1 - 13\eta^{1/2})k \leq |X| &\leq (\alpha_1 + 3\eta^{1/2})k,\\
(\alpha_1 - 13\eta^{1/2})k \leq |Y| &\leq (\alpha_1 + 3\eta^{1/2})k,\\
(\alpha_1 - 7\eta^{1/2})k \leq |W| &\leq (\alpha_1 + 10\eta^{1/2})k.
\end{aligned}\right\} \tag{2.9}$$

Suppose instead that $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, by (2.6), F spans at least $(\alpha_3 - 10\eta^{1/2})k$ vertices. Recall that we assume that F is not contained in an odd component of G, thus, by the decomposition, we have

$$|X| \geq |Y| \geq (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k, \tag{2.10a}$$

$$|X| \geq \frac{K - |W|}{2} \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 - \tfrac{1}{2}\eta - \tfrac{1}{2}w)k, \tag{2.10b}$$

$$|W| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k. \tag{2.10c}$$

From (2.10a) and (2.10b), we obtain

$$|\widehat{V}| = |X| + |W| \geq (\tfrac{1}{2}\alpha_3 + w - 5\eta^{1/2})k \tag{2.11a}$$

and

$$|\widehat{V}| = |X| + |W| \geq (\alpha_1 + \tfrac{1}{2}\alpha_2 + \tfrac{1}{2}w - \tfrac{1}{2}\eta)k. \tag{2.11b}$$

Again, we may assume that

$$|\widehat{V}| \leq \tfrac{1}{2}\left(\alpha_1 + \tfrac{1}{2}\alpha_2 + \max\{2w, \alpha_1, \alpha_2\} + 6\eta^{1/2}\right)k \tag{2.11c}$$

since, otherwise, we may apply Lemma 2.6.12 to obtain either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices. Again, we may assume that $w \geq \frac{1}{2}\alpha_1$ since, otherwise, together (2.11b) and (2.11c) contradict our assumption that $w \geq 7\eta^{1/2}k$. Then, (2.11a) and (2.11c) give

$$(\tfrac{1}{2}\alpha_3 + w - 5\eta^{1/2})k \leq |\widehat{V}| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + w + 3\eta^{1/2})k,$$

which, together with (2.10a) and (2.10c) gives

$$\begin{aligned} (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k \leq |X| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k \leq |Y| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k, \\ (\alpha_3 - \tfrac{1}{2}\alpha_1 - \tfrac{1}{2}\alpha_2 - 7\eta^{1/2})k \leq |W| &\leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k. \end{aligned}$$

Then, recalling that $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$ and that $\alpha_2 \leq \alpha_1$, in order to avoid a contradiction, we have $\alpha_1 \leq \alpha_2 + 32\eta^{1/2}$ and obtain

$$\left.\begin{aligned} (\alpha_1 - 13\eta^{1/2})k \leq |X| &\leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 13\eta^{1/2})k \leq |Y| &\leq (\alpha_1 + 3\eta^{1/2})k, \\ (\alpha_1 - 7\eta^{1/2})k \leq |W| &\leq (\alpha_1 + 10\eta^{1/2})k. \end{aligned}\right\} \qquad (2.12)$$

Considering (2.9) and (2.12), we see that we have obtained the same set of bounds irrespective of the form taken by c. Thus, in what follows, we consider both possibilities together.

Recall that, under the decomposition, there are no green edges contained within X or Y. Then, since G is $3\eta^4 k$-almost-complete, provided $k > k_{2.6.16}(\eta)$, we may apply Corollary 2.6.16 to each of $G[X]$ and $G[Y]$, thus finding that each contains a monochromatic connected-matching on at least $(\frac{2}{3}\alpha_1 - 8\eta^{1/8})k$ vertices. Thus, provided $\eta < (\alpha_1/120)^8$, we may assume that each of X and Y contain a monochromatic connected-matching on at least $\frac{3}{5}\alpha_1 k$ vertices. Referring to these matchings as $M_1 \subseteq G[X]$ and $M_2 \subseteq G[Y]$, we consider three subcases:

(i) M_1 and M_2 are both red;

(ii) M_1 and M_2 are both blue;

(iii) M_1 and M_2 are different colours.

The proof in the first subcase is identical to that of Case B.i, the proof in the second subcase is identical to that of Case B.ii and the proof in the third subcase is identical to that of Case B.iii with the overall result being that G contains either a red connected-matching on $\alpha_1 k$ vertices or a blue connected-matching on $\alpha_2 k$ vertices.

Case D: Largest green connected-matching is not odd and $w \leq 7\eta^{1/2}$.

Suppose that $c = 2\alpha_2 + \alpha_2$. Then, since $\eta \leq \eta_{B2}$, provided $k \geq k_{2.6.12}(\alpha_1, \alpha_2, w, \eta)$, we obtain bounds on the sizes of X and Y as follows:

$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 4\eta^{1/2})k \leq |X| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k,$$
$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 12\eta^{1/2})k \leq |Y| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k.$$

Suppose instead that $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$. Then, since $\eta \leq \eta_{B2}$, provided $k \geq k_{2.6.12}(\alpha_1, \alpha_2, w, \eta)$, we obtain bounds on the sizes of X and Y as follows:

$$(\tfrac{1}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 + \tfrac{1}{2}\alpha_3 - 4\eta^{1/2})k \leq |X| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k, \tag{2.13a}$$
$$(\alpha_3 - \tfrac{1}{2}\alpha_1 - 12\eta^{1/2})k \leq |Y| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k. \tag{2.13b}$$

Note that the inequalities in (2.13a) give a contradiction unless $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 14\eta^{1/2}$. Since $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, we obtain

$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 4\eta^{1/2})k \leq |X| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k,$$
$$(\alpha_1 + \tfrac{1}{2}\alpha_2 - 12\eta^{1/2})k \leq |Y| \leq (\alpha_1 + \tfrac{1}{2}\alpha_2 + 3\eta^{1/2})k.$$

Then, in either case, since $\eta \leq \eta_{B2}$, provided $k > k_{2.6.15}(144\eta)$, we may apply Lemma 2.6.15 (with $\alpha = \alpha_1, \beta = \alpha_2$) to each of X and Y to find that each contains a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices or has a structure belonging to one of the following classes as a subgraph:

$$\mathcal{H}_1 = \mathcal{H}\left((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{red}, \text{blue}\right);$$
$$\mathcal{H}_2 = \mathcal{H}\left((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{blue}, \text{red}\right),$$

with the latter case occurring only if $\alpha_2 \geq \alpha_1 - \eta^{1/8}$.

Case E: Largest green connected-matching is odd.

Recall, from (2.6), that F, the largest green connected-matching in G, spans at least $(\max\{\frac{3}{2}\alpha_1+\frac{1}{2}\alpha_2, \alpha_3\}-10\eta^{1/2})k$ vertices. We now consider the case when this connected-matching is contained in an odd component of G.

Thus far, we have made extensive use of the decomposition of Figaj and Łuczak described in Lemma 2.6.10. However, in this case, it is necessary to consider an alternative (and somewhat more complicated) decomposition:

We begin by partitioning the vertices of G into $L \cup P \cup Q$ as follows. We let L be the vertex set of F. Then, for each $v \in V \backslash L$, if there exists a green edge between v and L, we assign v to P; otherwise, we assign v to Q.

Suppose there exists a green edge mn in F and distinct vertices $p_1, p_2 \in P$ such that mp_1 and np_2 are both coloured green. Then, we can replace mn with mp_1 and np_2, contradicting the maximality of F. Thus, after discarding at most one edge from $G[L, P]$ for each edge of F, we may assume that given an edge uv in the matching, at most one of u or v has a green edge to P. We may therefore partition L into $M \cup N$ such that each edge of the matching belongs to $G[M, N]$ and there are no green edges in $G[N, P]$. Observe also that, by maximality of F, there can be no green edges within $G[P]$ or $G[P, Q]$.

In summary, we have a partition $M \cup N \cup P \cup Q$ such that

(E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;

(E2) every vertex in P has a green edge to M;

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

Note that, since G was assumed to be $(1-\eta^4)$-complete and also $3\eta^4 k$-almost-complete, having discarded the green edges described above, provided $k \geq 1/\eta^4$, we may now assume that the (new) graph is $(1-\frac{3}{2}\eta^4)$-complete and also $4\eta^4 k$-almost-complete. In what follows, on a number of occasions, we will discard vertices from $M \cup N \cup P \cup Q$ but will continue to refer the parts of the partition as M, N, P and Q. The discarded vertices remain in the graph and will be considered later. We need to take care to account of this when considering the sizes of $V(G), M, N, P, Q$, etc.

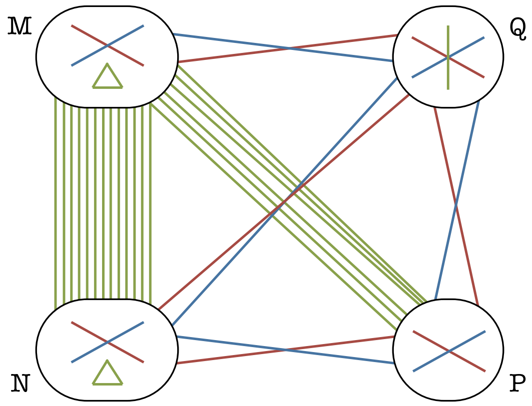


Figure 2.11: Decomposition into $M \cup N \cup P \cup Q$.

Recalling (II†), in the case that $c = 2\alpha_1 + \alpha_2$, we have $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, since $V(F) = M \cup N$ and $|M| + |N| + |P| + |Q| = K$, we have

$$(\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2})k \leq |M|\,, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{2.14a}$$

$$(2\alpha_1 + \alpha_2 - \alpha_3 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{2.14b}$$

The inequalities in (2.14a) give a contradiction unless $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$. Then, since $\alpha_3 \leq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$, we may re-write (2.14b) as

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{2.14b$'$}$$

Recalling (II‡), in the case that $c = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3$, we have $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2$. Then, since $V(F) = M \cup N$ and $|M| + |N| + |P| + |Q| = K$, we have

$$(\tfrac{1}{2}\alpha_3 - 5\eta^{1/2})k \leq |M|\,, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{2.15a}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{2.15b}$$

We will proceed considering the two possible situations together, assuming that

$$\alpha_3 \geq \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2}. \tag{2.16}$$

Comparing (2.14a) to (2.15a) and (2.14b$'$) to (2.15b), we will assume that

$$(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|\,, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{E4a}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b}$$

and distinguish between three possibilities:

(i) $|P| \leq 95\eta^{1/2}k$;

(ii) $|Q| \leq 95\eta^{1/2}k$;

(iii) $|P|, |Q| \geq 95\eta^{1/2}k$.

Case E.i: $|P| \leq 95\eta^{1/2}k$.

In this case, we disregard P, and, recalling that $L = M \cup N$, consider $G[L \cup Q]$. From (E4a) and (E4b), we have

$$(\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \tag{E4a$'$}$$
$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b$'$}$$

By (E3), we know that all edges in $G[L, Q]$ are coloured red or blue.

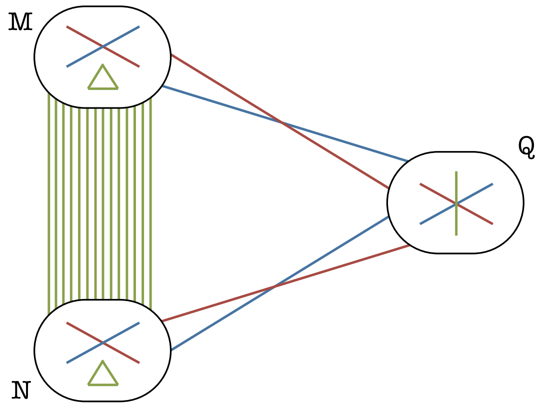


Figure 2.12: Decomposition in Case E.i.

Observe that, provided $\eta < (1/200)^2$, we have

$$|L| + |Q| \geq \left(\max\{2\alpha_1 + \alpha_2, \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + \alpha_3\} - 106\eta^{1/2}\right) k \geq \tfrac{3}{4}K.$$

Thus, since G is $(1 - \frac{3}{2}\eta^4)$-complete, $G[L \cup Q]$ is $(1 - 2\eta^4)$-complete. Also, provided $\eta < 10^{-5}$, we have $|L|, |Q| \geq 18\eta^{1/2}(|L| + |Q|)$. Thus, since $2\eta^4 \leq 3\eta^{1/2}$, provided $K \geq 2/\eta^{1/2}$, we may apply Lemma 2.6.19, giving rise to four cases:

(a) $G[L, Q]$ contains a monochromatic component on at least $(1 - 21\eta^{1/2})|L \cup Q| \geq |L \cup Q| - 63\eta^{1/2}k$ vertices;

(b) L, Q can be partitioned into $L_1 \cup L_2, Q_1 \cup Q_2$ such that $|L_1|, |L_2|, |Q_1|, |Q_2| \geq 9\eta^{1/2}|L \cup Q| \geq 9\eta^{1/2}k$ and all edges present between L_i and Q_j are red for $i = j$, blue for $i \neq j$;

(c) there exist vertices $v_r, v_b \in L$ such that v_r has red edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in Q and v_b has blue edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in Q;

(d) there exist vertices $v_r, v_b \in Q$ such that v_r has red edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in L and v_b has blue edges to all but $12\eta^{1/2}|L \cup Q| \leq 36\eta^{1/2}k$ vertices in L.

Case E.i.a: $G[L \cup Q]$ has a large monochromatic component.

Recall that we assume that F, the largest green connected-matching in G, spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G. We have a partition of $V(G)$ into $L \cup P \cup Q$ such that $|P| \leq 95\eta^{1/2}k$,

$$(\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \tag{E4a$'$}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b$'$}$$

Recalling that $L = M \cup N$, by (E3), all edges present in $G[L, Q]$ are coloured red or blue. Additionally, in this case, we assume that $G[L, Q]$ contains a monochromatic component on at least $|L \cup Q| - 63\eta^{1/2}k$ vertices. Suppose this large monochromatic component is red, then

(E5) $G[L, Q]$ has a red component on at least $|L \cup Q| - 63\eta^{1/2}k$ vertices.

We consider the largest red matching R in $G[L, Q]$ and, thus, partition L into $L_1 \cup L_2$ and Q into $Q_1 \cup Q_2$ where $L_1 = L \cap V(R)$, $L_2 = L \backslash L_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \backslash Q_1$.

By maximality of R, all edges present in $G[L_2, Q_2]$ are coloured exclusively blue. Notice that, since $\eta \leq (\alpha_1/100)^2$, we have, by (E4a$'$) and (E4b$'$), $|L| \geq |Q|$. Thus, since

$|L_1| = |Q_1|$, we have $|L_2| \geq |Q_2|$ and, so, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, by Lemma 2.6.14, we have $|Q_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$ and therefore, also, $|L_1| = |Q_1| = |Q| - |Q_2| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k$.

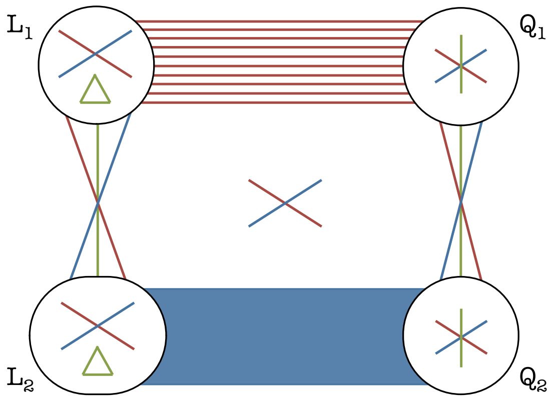


Figure 2.13: Decomposition into $L_1 \cup L_2 \cup Q_1 \cup Q_2$ in Case E.i.a.

Also, by (E5), in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices, we may assume that $|L_1| = |Q_1| \leq (\frac{1}{2}\alpha_1 + 64\eta^{1/2})k$. Finally, we have $|Q_2| = |Q| - |Q_1| \geq (\frac{1}{2}\alpha_2 - 160\eta^{1/2})k$.

In summary, we have $|L_1| = |Q_1|$,

$$(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |Q_1| \leq (\tfrac{1}{2}\alpha_1 + 64\eta^{1/2})k, \tag{2.20a}$$

$$(\tfrac{1}{2}\alpha_2 - 160\eta^{1/2})k \leq |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k. \tag{2.20b}$$

Note that, since $\eta \leq (\alpha_1/10000)^2$, by (E4a′), (E4b′), we have $|L| \geq |Q| + 3000\eta^{1/2}k$ and thus, since $|Q_1| = |L_1|$, also have

$$|L_2| \geq |Q_2| + 3000\eta^{1/2}k. \tag{2.20c}$$

Recalling that $|L_2| = |L| - |L_1| = |L| - |Q_1|$, since $\eta \leq (\alpha_1/10000)^2$, considering (E4a′) and (2.20a), we also have

$$|L_2| \geq |Q_1| + 3000\eta^{1/2}k. \tag{2.20d}$$

Equations (2.20c) and (2.20d) are crucial to the argument that follows since they provide us with the spare vertices we will need in order to establish the coloured structure of G. In what follows, we will show that after possibly discarding some vertices from each of L_1, L_2, Q_1 and Q_2, we may assume that all edges present in $G[L, Q_1]$ are coloured exclusively red and that all edges present in $G[L, Q_2]$ are coloured exclusively blue. This is done in three steps, the first dealing with $G[L_2, Q_1]$, the second dealing with $G[L_1, Q_2]$ and the third dealing with $G[L_1, Q_1]$. Similar arguments will appear many times throughout the remainder of the proof of Theorem B. Note that, in what follows, we mostly omit floors and ceilings for the sake of clarity of presentation, we may do this since we are free to increase k where necessary.

Claim 2.8.1. *We may discard at most $842\eta^{1/2}k$ vertices from each of L_1 and Q_1, at most $161\eta^{1/2}k$ vertices from L_2 and at most $260\eta^{1/2}k$ vertices from Q_2 such that all remaining in $G[L, Q_1]$ are coloured exclusively red and all edges present in $G[L, Q_2]$ are coloured exclusively blue.*

Proof. We begin by considering the blue graph. Observing that $G[L_2, Q_2]$ contains a blue connected-matching of size close to $\alpha_2 k$ and that there are 'spare' vertices in L_2. Thus, we note that there can only be few blue edges in $G[L_2, Q_1]$. Indeed, suppose there exists a blue matching B_S on at least $322\eta^{1/2}k$ vertices in $G[L_2, Q_1]$.

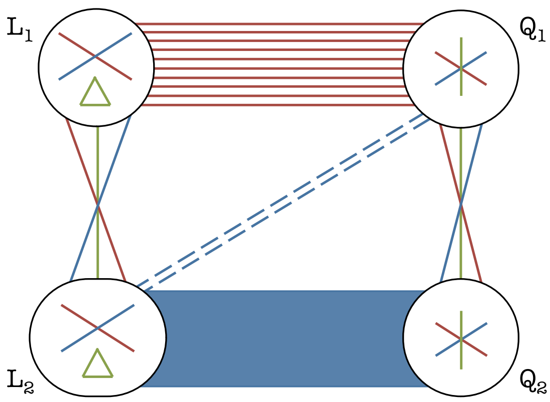


Figure 2.14: The blue matching B_S.

Then, by (2.20d), letting $\widetilde{L} = L_2 \backslash V(B_S)$, we have $|\widetilde{L}| \geq |Q_2| \geq (\frac{1}{2}\alpha_2 - 160\eta^{1/2})k$. Thus, by Lemma 2.6.14, there exists a blue connected-matching B_L on at least $(\alpha_2 - 322\eta^{1/2})k$ vertices in $G[\widetilde{L}, Q_2]$ which shares no vertices with B_S. Notice that, since $G_2[L_2, Q_2]$ is $4\eta^4k$-almost-complete, $L_2 \cup Q_2$ forms a single blue component in G and, thus, $B_S \cup B_L$ forms a blue connected-matching on $\alpha_2 k$ vertices. Therefore, no such matching as B_S can exist. So, after discarding at most $161\eta^{1/2}k$ vertices from each of Q_1 and L_2, we may assume that all edges present in $G[L_2, Q_1]$ are coloured exclusively red.

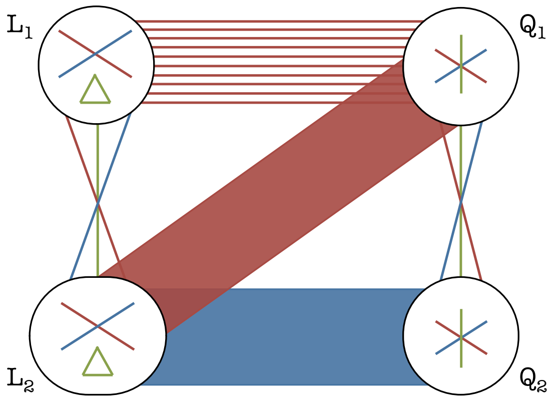


Figure 2.15: Resulting colouring of the edges of $G[L_2, Q_1]$.

In order to retain the equality $|L_1| = |Q_1|$ and the property that every vertex in L_1 belongs to an edge of R, we discard from L_1 each vertex whose R-mate in Q_1 has already been discarded. Recalling (2.20a)–(2.20d), we now have

$$|L_2| \geq |Q_1| + 2800\eta^{1/2}k, \qquad |L_1| = |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 258\eta^{1/2})k,$$
$$|L_2| \geq |Q_2| + 2800\eta^{1/2}k, \qquad |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 160\eta^{1/2})k.$$

We now consider the red graph. Since all edges in $G[L_2, Q_1]$ are coloured exclusively red, any two vertices in Q_1 have a common red neighbour in L_2. Thus, since every vertex in L_1 has a red neighbour in Q_1, we know that $G[L_1 \cup Q_1]$ has a single effective red component. Suppose, then, that there exists a red matching R_S on at least $520\eta^{1/2}k$ vertices in $G[L_1, Q_2]$. Then, recalling that the matching R spans all the vertices of $G[L_1, Q_1]$, we may construct a red connected-matching on at least $\alpha_1 k$ vertices as follows.

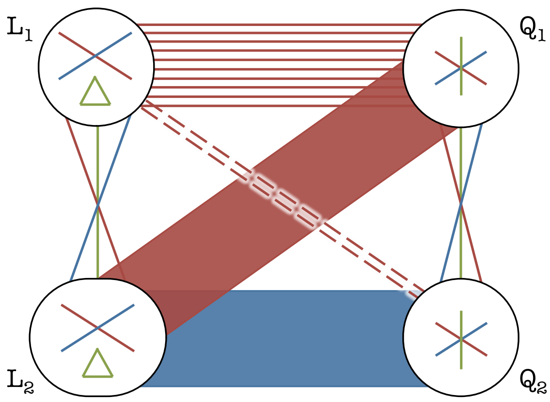


Figure 2.16: The red matching R_S.

Observe that there exists a set R^- of $260\eta^{1/2}k$ edges belonging to R such that $L_1 \cap V(R_S) = L_1 \cap V(R^-)$. Define $R^* = R \backslash R^-$ and $\widetilde{Q} = Q_1 \cap V(R^-)$ and consider $G[L_2, \widetilde{Q}]$. Since $|L_2|, |\widetilde{Q}| \geq 260\eta^{1/2}k$ and $G[L_2, \widetilde{Q}]$ is $4\eta^4 k$-almost-complete, by Lemma 2.6.14, there exists a red connected-matching R_T on at least $518\eta^{1/2}k$ vertices in $G[L_2, \widetilde{Q}]$. Then, $R^* \cup R_S \cup R_T$ is a red-connected-matching in $G[L_1, Q_1] \cup G[L_1, Q_2] \cup G[L_2, Q_1]$ on at least $2(|Q_1| - 261\eta^{1/2}k) + 520\eta^{1/2}k + 518\eta^{1/2}k \geq \alpha_1 k$ vertices.

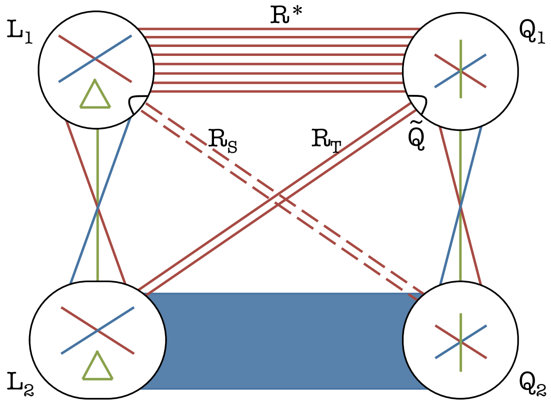


Figure 2.17: Construction of a red connected-matching on $\alpha_1 k$ vertices.

Therefore, a matching such as R_S cannot exist. Thus, after discarding at most $260\eta^{1/2}k$ vertices from each of L_1 and Q_2, we may assume that all edges present in $G[L_1, Q_2]$ are coloured exclusively blue.

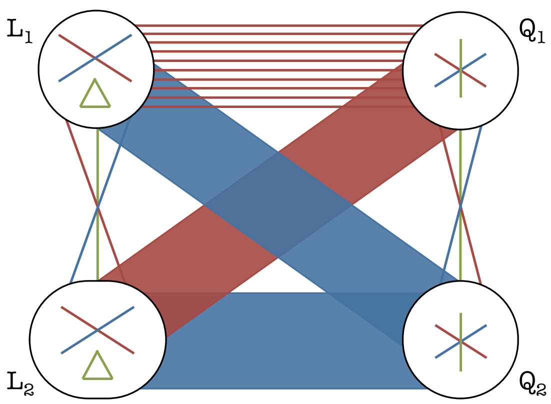


Figure 2.18: Resulting colouring of $G[L_1, Q_2]$.

Note that, in order to retain the equality $|L_1| = |Q_1|$, we also discard from Q_1 each vertex whose R-mate in L_1 has already been discarded. After discarding vertices, we have

$$|L_1| = |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 518\eta^{1/2}), \qquad |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 420\eta^{1/2}).$$

To complete the claim, we return to the blue graph. Since all edges present in $G[L, Q_2]$ are coloured exclusively blue and G is $4\eta^4 k$-almost-complete, $L \cup Q_2$ forms a single blue component. Also, since $|L_2|, |Q_2| \geq (\frac{1}{2}\alpha_2 - 420\eta^{1/2})k$, by Lemma 2.6.14, there exists a blue connected-matching B_2 on at least $(\alpha_2 - 842\eta^{1/2})k$ vertices in $G[L_2, Q_2]$. Thus, if there existed a blue matching B_1 on at least $842\eta^{1/2}k$ vertices in $G[L_1, Q_1]$, then $B_1 \cup B_2$ would form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $421\eta^{1/2}k$ vertices from each of L_1, Q_1, we may assume that all edges present in $G[L_1, Q_1]$ are coloured exclusively red. Thus completing the proof of the claim. □

In summary, having proved Claim 2.8.1, we know that all edges present in $G[L, Q_1]$ are coloured exclusively red and that all edges present in $G[L, Q_2]$ are coloured exclusively blue. Additionally, we have

$$\left.\begin{aligned} &|L_2| \geq |Q_1| + 2800\eta^{1/2}k, && |L_1| = |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 939\eta^{1/2})k, \\ &|L_2| \geq |Q_2| + 2800\eta^{1/2}k, && |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 420\eta^{1/2})k. \end{aligned}\right\} \quad (2.21)$$

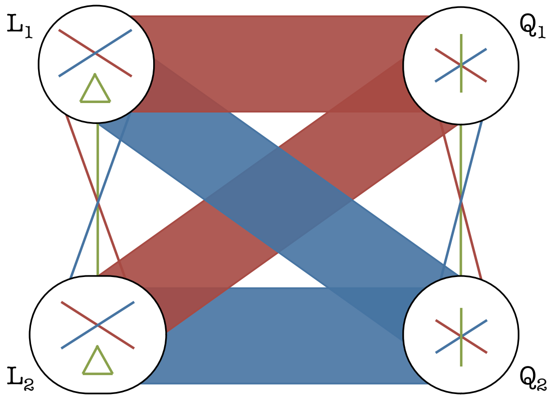


Figure 2.19: Colouring of $G[L, Q]$ after Claim 2.8.1.

Now, suppose that there exists a red matching R^+ on $1880\eta^{1/2}k$ vertices in L. Then, by (2.21), we have $|L \backslash V(R^+)| \geq |Q_1| \geq (\frac{1}{2}\alpha_1 - 939\eta^{1/2})k$. So, by Lemma 2.6.14, there exists a red connected-matching on at least $(\alpha_1 - 1880\eta^{1/2})k$ vertices in $G[L \backslash V(R^+), Q_1]$, which can be augmented with edges from R^+ to give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $1880\eta^{1/2}k$ vertices from L, we may assume that there are no red edges present in $G[L]$ and that we then have

$$|L| \geq |Q_1| + 900\eta^{1/2}k, \qquad |L| \geq |Q_2| + 900\eta^{1/2}k.$$

Finally, suppose that there exists a blue connected-matching B^+ on $842\eta^{1/2}k$ vertices in L. Then, $|L\backslash V(B^+)| \geq |Q_2| \geq (\frac{1}{2}\alpha_2 - 420\eta^{1/2})k$, so, by Lemma 2.6.14, there exists a blue connected-matching on at least $(\alpha_2 - 842\eta^{1/2})k$ vertices in $G[L\backslash V(B^+), Q_2]$, which can be augmented with edges from B^+ to give a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most a further $842\eta^{1/2}$ vertices from L, we may assume that all edges present in $G[L]$ are coloured exclusively green.

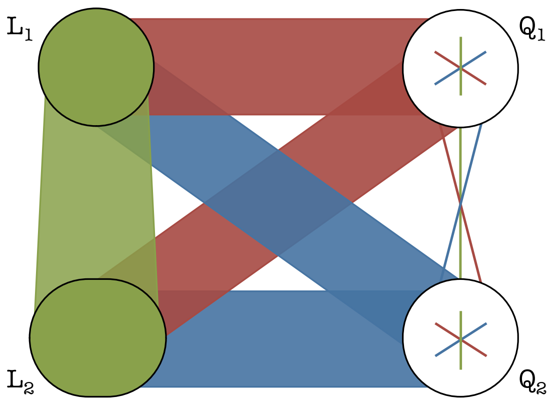


Figure 2.20: Final colouring in Case E.i.a.

In summary, having discarded at most $4828\eta^{1/2}k$ vertices, we have

$$|Q_1| \geq (\tfrac{1}{2}\alpha_1 - 939\eta^{1/2})k, \quad |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 420\eta^{1/2})k, \quad |L| \geq (\tfrac{1}{2}\alpha_3 - 3750\eta^{1/2})k,$$

and know that all edges present in $G[Q_1, L]$ are coloured exclusively red, all edges present in $G[Q_2, L]$ are coloured exclusively blue and all edges present in $G[L]$ are coloured exclusively green.

Thus, we have found, as a subgraph of G, a graph belonging to

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 1000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 1000\eta^{1/2})k, (\alpha_3 - 4000\eta^{1/2})k, 4\eta^4 k\right).$$

At the beginning of Case E.i.a, we assumed that the largest monochromatic component in $G[L, Q]$ was red. If, instead, that monochromatic component is blue, then the proof proceeds exactly as above, following the same steps with the roles of red and blue exchanged and with α_1 and α_2 exchanged. The result is identical.

Case E.i.b: $L \cup Q$ has a non-trivial partition with 'cross' colouring.

Recall that we assume that F, the largest green connected-matching in G, spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G. We have a partition of $V(G)$ into $L \cup P \cup Q$, such that $|P| \leq 95\eta^{1/2}k$,

$$(\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k \leq |L| \leq \alpha_3 k, \tag{E4a$'$}$$
$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 96\eta^{1/2})k \leq |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b$'$}$$

Additionally, in this subcase, we assume that L and Q can be partitioned into $L_1 \cup L_2$ and $Q_1 \cup Q_2$ such that $|L_1|, |L_2|, |Q_1|, |Q_2| \geq 9\eta^{1/2}k$ and all edges present in $G[L_i, Q_j]$ are coloured exclusively red for $i = j$, and exclusively blue for $i \neq j$.

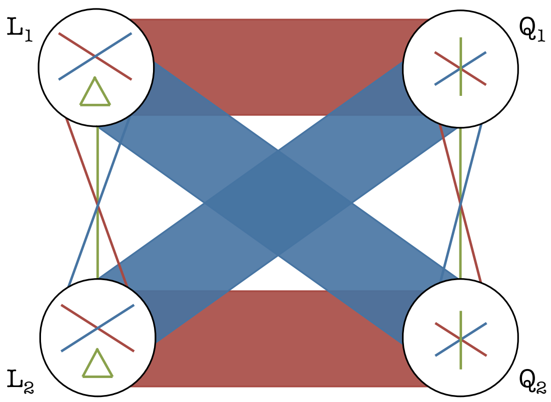


Figure 2.21: Decomposition into $L_1 \cup L_2 \cup Q_1 \cup Q_2$ in Case E.i.b.

Then, by Lemma 2.6.14, there exist red connected-matchings

$$M_{11} \text{ on at least } 2\,(\min\{|L_1|, |Q_1|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_1, Q_1], \tag{2.23a}$$
$$M_{22} \text{ on at least } 2\,(\min\{|L_2|, |Q_2|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_2, Q_2], \tag{2.23b}$$

and blue connected-matchings

$$M_{12} \text{ on at least } 2\,(\min\{|L_1|, |Q_2|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_1, Q_2], \tag{2.23c}$$
$$M_{21} \text{ on at least } 2\,(\min\{|L_2|, |Q_1|\}) - 2\eta^{1/2}k \text{ vertices in } G[L_2, Q_1]. \tag{2.23d}$$

Thus, in order to avoid a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that $|V(M_{11})|, |V(M_{22})| \leq \alpha_1 k$ and $|V(M_{12})|, |V(M_{21})| \leq \alpha_2 k$. Thus, (2.23a)–(2.23d), above can be used to obtain bounds on the sizes of L_1, L_2, Q_1 and Q_2 as follows:

$$\text{Since } |V(M_{11})|, |V(M_{22})| \leq \alpha_1 k, \text{ we have} \quad \min\{|L_1|, |Q_1|\} \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \tag{2.24a}$$
$$\text{and} \quad \min\{|L_2|, |Q_2|\} \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k. \tag{2.24b}$$
$$\text{Since } |V(M_{12})|, |V(M_{21})| \leq \alpha_2 k, \text{ we have} \quad \min\{|L_1|, |Q_2|\} \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \tag{2.24c}$$
$$\text{and} \quad \min\{|L_2|, |Q_1|\} \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k. \tag{2.24d}$$

Observe that, since $\eta \leq (\alpha_1/15)^2$, by (E4a′) and (E4b′), we have

$$\begin{aligned}|L_1| + |L_2| = |L| &\geq (\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2})k \\ &\geq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k + (\alpha_1 - 15\eta^{1/2})k \geq |Q| = |Q_1| + |Q_2|.\end{aligned}$$

Thus, it is not possible to have, for instance, $|Q_1|, |Q_2| \geq |L_1|, |L_2|$. Without loss of generality, we therefore consider four possibilities

(i) $|L_1|, |L_2| \geq |Q_1|, |Q_2|$;

(ii) $|L_1| \geq |Q_1|, |Q_2| \geq |L_2|$;

(iii) $|L_1| \geq |Q_i| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$;

(iv) $|Q_i| \geq |L_1| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Case E.i.b.i: $|L_1|, |L_2| \geq |Q_1|, |Q_2|$.

In this case, by (2.24c) and (2.24d), we may assume that $|Q_1|, |Q_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$ and, consequently, by (E4b′), we have

$$|Q_1|, |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k.$$

Thus, by (2.23a)–(2.23d), we have red connected-matchings M_{11}, M_{22}, each on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices and blue connected-matchings M_{12}, M_{21}, each on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices. (Also, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that $\alpha_1 \leq \alpha_2 + 196\eta^{1/2}$.)

Notice that there can be no red edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$ since any such edge would mean M_{11} and M_{22} being in the same red-component and so $M_{11} \cup M_{22}$ would form a red connected-matching on at least $\alpha_1 k$ vertices. Similarly, there can be no blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$. Therefore, all edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$ are coloured exclusively green.

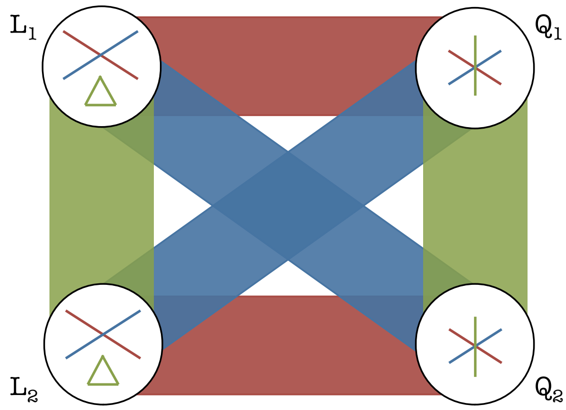


Figure 2.22: Colouring of $G[L_1 \cup L_2 \cup Q_1 \cup Q_2]$ in Case E.i.b.i.

Thus, we have found, as a subgraph of G, a graph belonging to

$$\mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, \\ (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big) \subseteq \mathcal{K}_1^* \cup \mathcal{K}_2^*.$$

Case E.i.b.ii: $|L_1| \geq |Q_1|, |Q_2| \geq |L_2|$.

In this case, by (2.24a) and (2.24c), we have

$$|Q_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \qquad |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \tag{2.25}$$

and so, by (E4b′),

$$|Q_1| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, \qquad |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k. \tag{2.26}$$

Thus, by (2.23a), we have a red connected-matching M_{11} on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices in $G[L_1, Q_1]$ and, by (2.23c), have a blue connected-matching M_{12} on at least $(\alpha_2 - 196\eta^{1/2})k$ vertices in $G[L_1, Q_2]$.

Suppose then that $|L_2| \geq 100\eta^{1/2}k$. Then, by Lemma 2.6.14, there exists a red connected-matching R_S on at least $198\eta^{1/2}k$ vertices in $G[L_2, Q_2]$ and a blue connected-matching B_S on at least $198\eta^{1/2}k$ vertices in $G[L_2, Q_1]$. Thus, there can be no red edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$, since otherwise M_{11} and R_S would belong to the same red component and together span at least $\alpha_1 k$ vertices. Likewise, there can be no blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$, since then M_{12} and B_S would then belong to the same blue component and together span at least $\alpha_2 k$ vertices. Therefore, we have, as a subgraph of G, a graph belonging to

$$\mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - \tfrac{1}{2}\alpha_2 - 12\eta^{1/2})k, \\ 100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big) \subset \mathcal{K}_2^*.$$

Thus, we may assume that $|L_2| \leq 100\eta^{1/2}k$, in which case we have

$$|L_1| \geq (\max\{\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2, \alpha_3\} - 110\eta^{1/2})k \tag{2.27}$$

and know that all edges present in $G[Q_1, L_1]$ are coloured exclusively red and all edges present in $G[Q_2, L_1]$ are coloured exclusively blue. In this case, we disregard L_2 and consider $G[L_1]$.

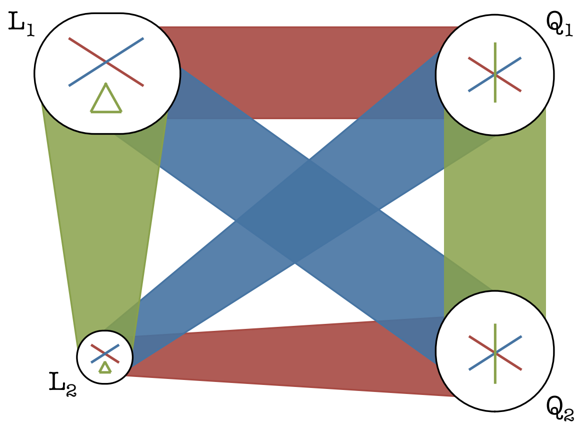


Figure 2.23: Colouring of $G[L \cup Q]$ in Case E.i.b.ii.

Since $\eta \leq (\alpha_1/1000)^2$, by (2.25) and (2.27),

$$|L_1| \geq |Q_1| + 200\eta^{1/2}k, \qquad |L_1| \geq |Q_2| + 200\eta^{1/2}k. \tag{2.28}$$

Suppose there exists a red matching R_S on $196\eta^{1/2}k$ vertices in $G[L_1]$. Then, by (2.26) and (2.28), we have

$$|L_1\backslash V(R_S)|, |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k.$$

Thus, by Lemma 2.6.14, there exists a red connected-matching R_L on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices in $G[L_2, Q_1]$ which shares no vertices with R_S. Since all edges present in $G[L_1, Q_1]$ are coloured exclusively red and G is $4\eta^4k$-almost-complete, R_S and R_L belong to the same red component and, thus, together form a red connected-matching on at least $\alpha_1 k$ vertices. Therefore, the largest red matching in $G[L_1]$ spans at most $196\eta^{1/2}k$ vertices.

Similarly, the largest blue connected-matching in $G[L_1]$ spans at most $196\eta^{1/2}k$ vertices. Thus, after discarding at most $392\eta^{1/2}k$ vertices from L_1, we may assume that all edges in $G[L]$ are coloured green. Thus, we have obtained, as a subgraph of G, a graph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - 502\eta^{1/2})k, 4\eta^4k\right).$$

Case E.i.b.iii: $|L_1| \geq |Q_i| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Suppose that $|L_1| \geq |Q_1| \geq |L_2| \geq |Q_2|$. Then by (2.24a)–(2.24d) and (E4b′), we obtain

$$(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |Q_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,$$
$$(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k.$$

Then, given the sizes of M_{11}, M_{12}, M_{21} and M_{22}, there can be no red or blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$ and we may assume that $\alpha_1 \leq \alpha_2 + 196\eta^{1/2}k$.

Thus, we have found, as a subgraph of G, a graph in

$$\mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - \tfrac{1}{2}\alpha_1 - 12\eta^{1/2})k,$$
$$(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4k\big) \subset \mathcal{K}_2^*.$$

Suppose instead that $|L_1| \geq |Q_2| \geq |L_2| \geq |Q_1|$. Then, by (2.24a)–(2.24d) and (E4b′), we obtain

$$(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |Q_1| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k,$$
$$(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k.$$

Again, there can be no red or blue edges present in $G[L_1, L_2] \cup G[Q_1, Q_2]$. Thus, we have found, as a subgraph of G, a graph in

$$\mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\alpha_3 - \tfrac{1}{2}\alpha_2 - 12\eta^{1/2})k,$$
$$(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big) \subset \mathcal{K}_2^*.$$

Case E.i.b.iv: $|Q_i| \geq |L_1| \geq |L_2| \geq |Q_j|$ for $\{i, j\} = \{1, 2\}$.

Suppose $|Q_1| \geq |L_1| \geq |L_2| \geq |Q_2|$. Then, $|Q_1| \geq \frac{1}{2}|L_1| + \frac{1}{2}|L_2| = \frac{1}{2}|L|$ and, since $\eta \leq (\alpha_2/100)^2$, by (E4a′), we have

$$|Q_1| \geq (\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2})k \geq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 5\eta^{1/2})k \geq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k.$$

Also, by (E4a′), we have

$$|L| \geq (\tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2})k \geq (\alpha_1 + \alpha_2 - 10\eta^{1/2})k.$$

Thus, either

$$|L_1| \geq (\alpha_1 - 5\eta^{1/2})k, \quad \text{or} \quad |L_2| \geq (\alpha_2 - 5\eta^{1/2})k.$$

Recall that all edges present in $G[L_1, Q_1]$ are coloured exclusively red and all edges present in $G[L_2, Q_1]$ are coloured exclusively blue. Thus, by Lemma 2.6.14, there exists either a red connected-matching on at least $\alpha_1 k$ vertices in $G[L_1, Q_1]$ or a blue connected-matching on at least $\alpha_2 k$ vertices in $G[L_1, Q_1]$.

The result is the same in the case that $|Q_2| \geq |L_1| \geq |L_2| \geq |Q_1|$.

Case E.i.c: $G[L, Q]$ contains red and blue 'stars' centred in L.

We continue to assume that F, the largest green connected-matching in G, spans at least $(\max\{\frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2, \alpha_3\} - 10\eta^{1/2})k$ vertices and is contained in an odd component of G and that we have a partition of $V(G)$ into $L \cup P \cup Q$, satisfying (E4a′) and (E4b′) such that all edges present in $G[L, Q]$ are coloured red or blue. Additionally, in this case, we have vertices $v_r, v_b \in L$ such that v_r has red edges to all but $36\eta^{1/2}k$ vertices in Q and v_b has blue edges to all but $36\eta^{1/2}k$ vertices in Q. Observe that the existence of v_r means that

(E5′) $G[Q]$ has a red effective-component on at least $|Q| - 36\eta^{1/2}k$ vertices.

The proof then proceeds exactly as in Case E.i.a but with (E5) replaced by (E5′). The result is the same.

Case E.i.d: $G[L,Q]$ contains red and blue 'stars' centred in Q.

We continue to assume that F, the largest green connected-matching in G, spans at least $(\max\{\frac{3}{2}\alpha_1+\frac{1}{2}\alpha_2,\alpha_3\}-10\eta^{1/2})k$ vertices and is contained in an odd component of G and that we have a partition of $V(G)$ into $L\cup P\cup Q$, satisfying (E4a′) and (E4b′) such that all edges present in $G[L,Q]$ are coloured red or blue. Additionally, in this case, we have vertices $v_r, v_b\in Q$ such that v_r has red edges to all but $36\eta^{1/2}k$ vertices in L and v_b has blue edges to all but $36\eta^{1/2}k$ vertices in L. Observe that the existence of v_r means that

(E5″) $G[L]$ has a red effective-component on at least $|L|-36\eta^{1/2}k$ vertices.

The proof then proceeds exactly as in Case E.i.a but with (E5) replaced by (E5″). The result is the same.

Case E.ii: $|Q|\leq 95\eta^{1/2}k$.

Recall that we have a decomposition of $V(G)$ into $M\cup N\cup P\cup Q$ such that:

(E1) $M\cup N$ is the vertex set of F and every edge of F belongs to $G[M,N]$;

(E2) every vertex in P has a green edge to M;

(E3) there are no green edges in $G[N,P]$, $G[M,Q]$, $G[N,Q]$, $G[P,Q]$ or $G[P]$.

Recall also that

$$(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-5\eta^{1/2})k\leq|M|\,,\,|N|\leq\tfrac{1}{2}\alpha_3 k, \tag{E4a}$$
$$(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2-\eta)k\leq|P|+|Q|\leq(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2+5\eta^{1/2})k. \tag{E4b}$$

Here, we consider the case when Q is sufficiently small to be disregarded. In that case, provided $|Q|\leq 95\eta^{1/2}k$, from (E4b), we have

$$(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2-96\eta^{1/2})k\leq|P|\leq(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2+5\eta^{1/2})k. \tag{E4b″}$$

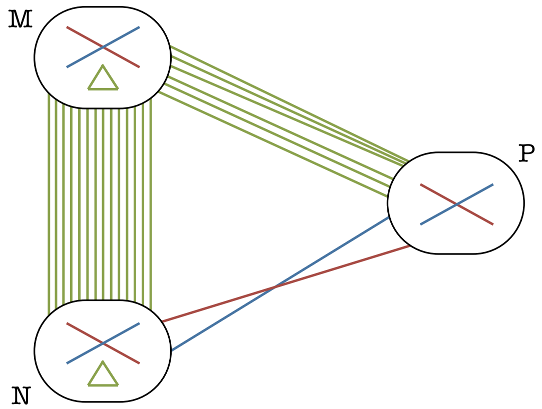


Figure 2.24: Decomposition in Case E.ii.

By (E3), every edge in $G[P]$ is red or blue. Then, since G is $4\eta^4k$-almost-complete, by Lemma 2.6.17, the largest monochromatic component F_P in $G[P]$ contains at least $|P| - \eta^{1/2}k$ vertices. Suppose that that this component is red.

We consider the largest red matching R in $G[N, P]$ and partition N into $N_1 \cup N_2$ and P into $P_1 \cup P_2$, where $N_1 = N \cap V(R)$, $N_2 = N \backslash N_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \backslash Q_1$. Then, by maximality of R, all edges present in $G[N_2, P_2]$ are coloured exclusively blue.

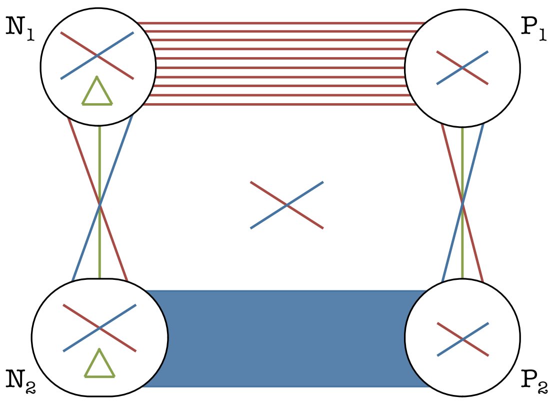


Figure 2.25: Decomposition of $N \cup P$ into $N_1 \cup N_2 \cup P_1 \cup P_2$ in Case E.ii.

Since P has a large red connected-component, all but $\eta^{1/2}k$ of the edges of R belong to the same red-component and thus form a red connected-matching. Thus, in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices, we have $|P_1| \leq (\frac{1}{2}\alpha_1 + \eta^{1/2})k$.

Suppose that $|N_2| \geq |P_2|$. Then, we have $|P_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$, since otherwise, by Lemma 2.6.14, $G[N_2, P_2]$ would contain a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, by (E4a) and (E4b''), we have

$$\left.\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |N_2|.
\end{aligned}\right\} \quad (2.29)$$

If instead $|N_2| \leq |P_2|$, then, by Lemma 2.6.14, we have $|N_2| \leq (\frac{1}{2}\alpha_2 + \eta^{1/2})k$ since, otherwise, by Lemma 2.6.14, $G[N_2, P_2]$ contains a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, by (E4a) and (E4b''), we have

$$\left.\begin{aligned}
(\tfrac{3}{4}\alpha_1 - \tfrac{1}{4}\alpha_2 - 6\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |P_2| \leq (\tfrac{3}{4}\alpha_1 - \tfrac{1}{4}\alpha_2 + 11\eta^{1/2})k,\\
(\tfrac{3}{4}\alpha_1 - \tfrac{1}{4}\alpha_2 - 6\eta^{1/2})k \leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 6\eta^{1/2})k \leq |N_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k,
\end{aligned}\right\} \quad (2.30)$$

which yields a contradiction unless $\alpha_1 \leq \alpha_2 + 28\eta^{1/2}$.

Six of the eight bounds obtained in (2.30) are stronger than the corresponding bounds obtained in (2.29). The seventh is weaker but can be written in a similar form. Thus, we will combine the two cases and continue under the assumption that

$$\left.\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |N_2|.
\end{aligned}\right\} \quad (2.31)$$

Suppose there exists a blue matching B_1 on $198\eta^{1/2}k$ vertices in $G[N_2, P_1]$ and a blue matching B_2 on $198\eta^{1/2}k$ vertices in $G[N_1, P_2]$. Then, defining $\widetilde{N} = N_2 \backslash V(B_1)$ and $\widetilde{P} = P_2 \backslash V(B_2)$, we have $|\widetilde{N}|, |\widetilde{P}| \geq (\frac{1}{2}\alpha_2 - 197\eta^{1/2})k$ and, thus, by Lemma 2.6.14, there exists a blue connected-matching B_3 on at least $(\alpha_2 - 396\eta^{1/2})k$ vertices in $G[\widetilde{N}, \widetilde{P}]$. Since $G_2[N_2, P_2]$ is $4\eta^4 k$-almost-complete, all vertices in $N_2 \cup P_2$ belong to the same blue component. Thus, $B_1 \cup B_2 \cup B_3$ forms a blue connected-matching on at least $\alpha_2 k$ vertices.

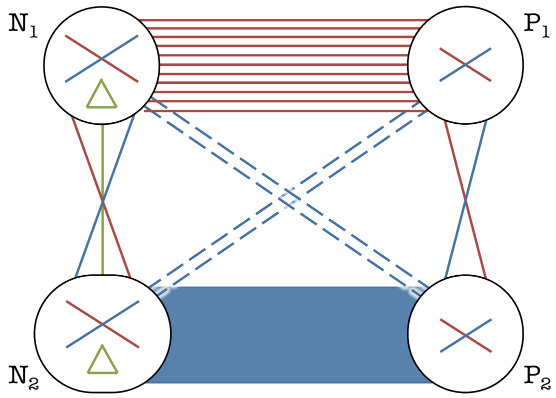


Figure 2.26: Colouring of the edges of $G[L_1, Q_2], G[L_2, Q_1]$.

Therefore, we proceed considering the following two subcases:

(a) the largest blue matching in $G[N_2, P_1]$ spans at most $198\eta^{1/2}k$ vertices;

(b) the largest blue matching in $G[N_1, P_2]$ spans at most $198\eta^{1/2}k$ vertices.

Case E.ii.a: Most edges in $G_2[N_2, P_1]$ are red.

Since the largest blue matching in $G[N_2, P_1]$ spans at most $198\eta^{1/2}k$ vertices, we can discard at most $99\eta^{1/2}k$ vertices from each of N_2 and P_1 so that all edges present in $G[N_2, P_1]$ are coloured exclusively red.

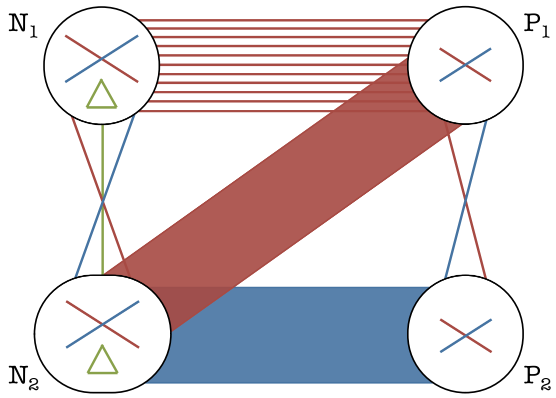


Figure 2.27: Initial colouring in Case E.ii.a.

In order to retain the equality $|N_1| = |P_1|$ and the property that every vertex in N_1 belongs to an edge of R, we discard from N_1 each vertex whose R-mate in P_1 has already been discarded. Recalling (2.31), we then have

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\le |P_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k &\le |P_2| \le (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\le |N_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\le |N_2|.
\end{aligned}$$

Now, suppose there exists a red matching R_U on $396\eta^{1/2}k$ vertices in $G[N_1, P_2]$. Observe that there exists a set R^- of $198\eta^{1/2}k$ edges belonging to R such that $N_1 \cap V(R_S) = N_1 \cap V(R^-)$. Then, we have $|N_2|, |P_1 \cap V(R^-)| \ge 198\eta^{1/2}k$ so, by Lemma 2.6.14, there exists a red connected-matching R_V on at least $394\eta^{1/2}k$ vertices in $G[N_2, P_1 \cap V(R^-)]$. Then, since all edges present in $G[N_2, P_1]$ are coloured exclusively red and every vertex in N_1 has a red neighbour in P_1, $(R \backslash R^-)$, R_U and R_V belong to the same red-component and, thus, together form a red-connected-matching on at least $2(\frac{1}{2}\alpha_1 - 196\eta^{1/2}k - 199\eta^{1/2}k) + 396\eta^{1/2}k + 394\eta^{1/2}k \ge \alpha_1 k$ vertices. Thus, we can discard at most $198\eta^{1/2}k$ vertices from each of N_1 and P_2 so that all edges present in $G[N_1, P_2]$ are coloured exclusively blue. We then have

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k &\le |P_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k &\le |P_2| \le (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 394\eta^{1/2})k &\le |N_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k &\le |N_2|.
\end{aligned}$$

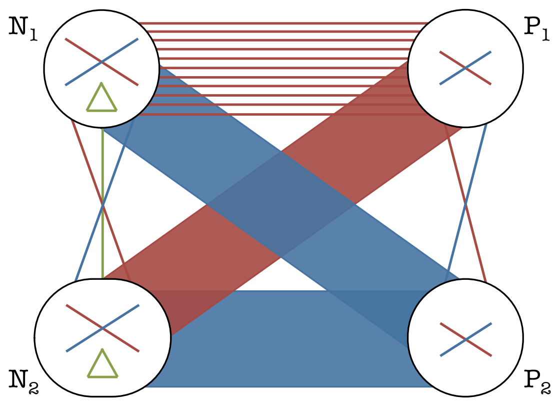


Figure 2.28: Colouring of the edges of $G[N_1, P_2]$.

Observe then that, since $|N_2|, |P_2| \geq (\frac{1}{2}\alpha_2 - 295\eta^{1/2})k$, by Lemma 2.6.14, there exists a blue connected-matching on at least $(\alpha_2 - 592\eta^{1/2})k$ vertices in $G[N_2, P_2]$. Thus, since all edges present in $G[N, P_2]$ are coloured exclusively blue, if there existed a blue matching on $592\eta^{1/2}k$ vertices in $G[N_1, P_1]$, we would have a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $296\eta^{1/2}k$ vertices from each of P_1 and N_1, we may assume that all edges in $G[P_1, N_1]$ are coloured exclusively red and that

$$\left.\begin{aligned} &(\tfrac{1}{2}\alpha_1 - 492\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_1 - 690\eta^{1/2})k \leq |N_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k \leq |N_2|. \end{aligned}\right\} \quad (2.32)$$

Then, since $\eta \leq (\alpha_2/20000)^2$, by (2.32), we have

$$|N| = |N_1| + |N_2| \geq |P_1| + 5000\eta^{1/2}k, \quad (2.33)$$

$$|N| = |N_1| + |N_2| \geq |P_2| + 5000\eta^{1/2}k. \quad (2.34)$$

In particular, $|N| \geq |P_2|$ so, by Lemma 2.6.14, we have

$$|P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k. \quad (2.35)$$

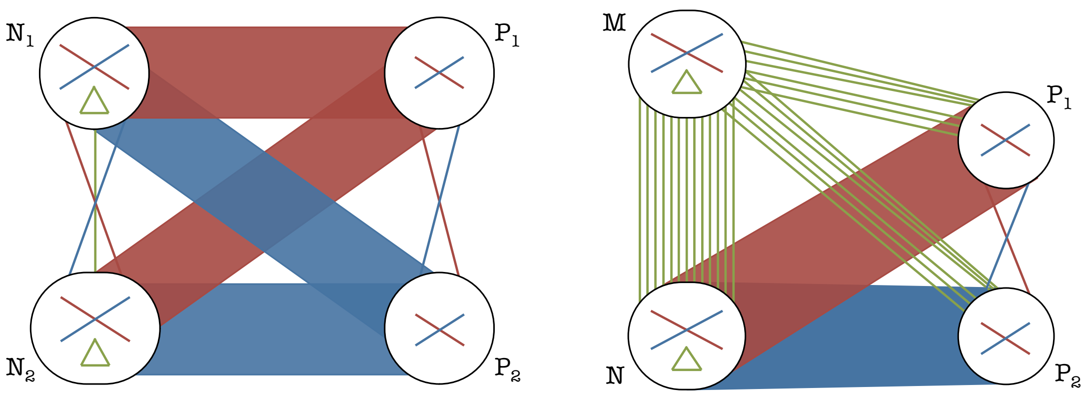


Figure 2.29: Colouring after three rounds of discarding vertices.

Having determined the colouring of the red-blue graph $G[N, P]$, we now expand our sights to $G[M \cup N]$ and $G[M, P]$, each of which can include green edges:

Suppose there exists a red matching R_A on $986\eta^{1/2}k$ vertices in $G[N] \cup G[M, N]$. Then, by (2.32) and (2.33), we have $|N\backslash V(R_A)| \geq |P_1| \geq (\frac{1}{2}\alpha_1 - 492\eta^{1/2})k$ so, by Lemma 2.6.14, there exists a red connected-matching R_B on at least $(\alpha_1 - 986\eta^{1/2})k$ in $G[N\backslash V(R_A), P_1]$. Since all edges in $G[N, P_1]$ are coloured exclusively red, R_A and R_B belong to the same red component and, thus, together form a red connected-matching on at least $\alpha_1 k$ vertices. Similarly, if there exists a blue matching on at least $594\eta^{1/2}k$ vertices in $G[N] \cup G[M, N]$, then this can be used along with $G[N, P_2]$ to give a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $790\eta^{1/2}k$ vertices from M and at most $2370\eta^{1/2}k$ vertices from N, we may assume that all edges present in $G[N]$ and $G[M \cup N]$ are green.

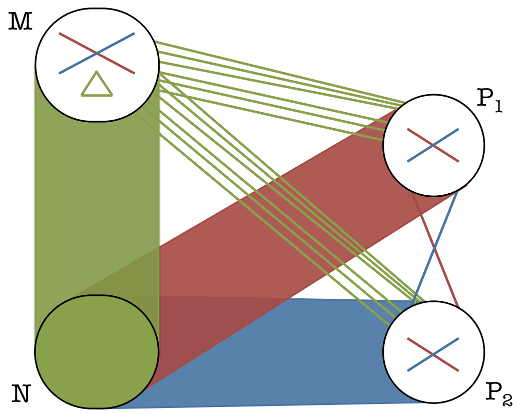


Figure 2.30: Colouring of the edges of $G[N] \cup G[N, M]$.

After discarding these vertices, we have

$$\left.\begin{aligned} (\tfrac{1}{2}\alpha_3 - 800\eta^{1/2})k \leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ (\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k \leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned}\right\} \quad (2.36)$$

Next, suppose there exists a green matching G_S on $8240\eta^{1/2}k$ vertices in $G[M, P]$. By (2.36), we have $|M\backslash V(G_S)| \geq (\frac{1}{2}\alpha_3 - 4920\eta^{1/2})k$. Then, taking $\widetilde{N}$ to be any subset of $(\frac{1}{2}\alpha_3 - 4920\eta^{1/2})k$ vertices in N, by Lemma 2.6.14, there exists a green connected-matching G_L on at least $(\alpha_3 - 9842\eta^{1/2})k$ vertices in $G[M\backslash V(G_S), \widetilde{N}]$. Also, since $|N\backslash \widetilde{N}| \geq 1618\eta^{1/2}k$, by Theorem 2.6.1, there exists a green matching G_T on $1610\eta^{1/2}k$ vertices in $G[N\backslash \widetilde{N}]$. Then, together G_S, G_L and G_T form a green connected-matching on at least $\alpha_3 k$ vertices which is odd by the definition of the decomposition. Thus, we may discard at most $4120\eta^{1/2}k$ vertices from each of M and P such that none of the edges present in $G[P, M \cup N]$ are coloured green.

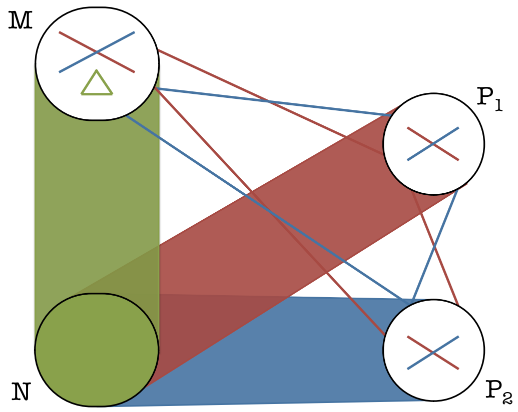


Figure 2.31: Colouring of the edges of $G[M, P]$.

Then, recalling (2.32), (2.35) and (2.36), we have

$$\left.\begin{aligned} &(\tfrac{1}{2}\alpha_1 - 4612\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_2 - 4415\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_3 - 4920\eta^{1/2})k \leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ &(\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k \leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned}\right\} \quad (2.37)$$

Now, suppose there exists a red matching R_S on $9230\eta^{1/2}k$ vertices in $G[M, P_2]$. By (2.33) and (2.37), we have $|N| \geq |P_1| \geq (\frac{1}{2}\alpha_1 - 4612\eta^{1/2})k$. So, by Lemma 2.6.14, there exists a red connected-matching R_L on at least $2|P_1| - 2\eta^{1/2}k \geq (\alpha_1 - 9226\eta^{1/2})k$ vertices in $G[N, P_1]$. Since $G[P]$ has a red effective-component on at least $|P| - \eta^{1/2}k$, R_L belongs to the same red component as at least $4614\eta^{1/2}k$ of the edges of R_S, thus giving a red connected-matching on at least $\alpha_1 k$ vertices in $G[M, P_2] \cup G[N, P_1]$. Therefore, after discarding at most $4615\eta k$ vertices from each of M and P_2, we may assume that all edges present in $G[M, P_2]$ are coloured exclusively blue.

We then have

$$\left.\begin{aligned} &(\tfrac{1}{2}\alpha_1 - 4612\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_2 - 9030\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k, \\ &(\tfrac{1}{2}\alpha_3 - 9335\eta^{1/2})k \leq |M| \leq \tfrac{1}{2}\alpha_3 k, \\ &(\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k \leq |N| \leq \tfrac{1}{2}\alpha_3 k. \end{aligned}\right\} \quad (2.38)$$

Similarly, suppose there exists a blue matching B_S on $18062\eta^{1/2}k$ vertices in $G[M, P_1]$. By (2.34) and (2.38), $|N| \geq |P_2| \geq (\frac{1}{2}\alpha_1 - 9030\eta^{1/2})k$. So, by Lemma 2.6.14, there exists a blue connected-matching B_L on at least $2|P_2| - 2\eta^{1/2}k \geq (\alpha_1 - 18062\eta^{1/2})k$ vertices in $G[N, P_2]$. Then, since G is $4\eta^4k$-almost-complete and all edges present in $G[M, P_2]$ are coloured exclusively blue, B_L and B_S belong to the same blue component of G, and thus, together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $9031\eta k$ vertices from each of M and P_1, we may assume that all edges present in $G[M, P_1]$ are coloured exclusively red. We then have

$$\left.\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 13643\eta^{1/2})k \leq |P_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 9030\eta^{1/2})k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_3 - 18366\eta^{1/2})k \leq |M| \leq \tfrac{1}{2}\alpha_3 k,\\
(\tfrac{1}{2}\alpha_3 - 3300\eta^{1/2})k \leq |N| \leq \tfrac{1}{2}\alpha_3 k.
\end{aligned}\right\} \quad (2.39)$$

Finally, since $|N| \geq |P_1|, |P_2|$, if there existed a red matching on at least $27288\eta^{1/2}k$ vertices in $G[M]$ or a blue matching on at least $18062\eta^{1/2}k$ vertices in $G[M]$, then we could obtain a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $45350\eta^{1/2}k$ further vertices from M, we may assume that all edges present in $G[M \cup N]$ are green.

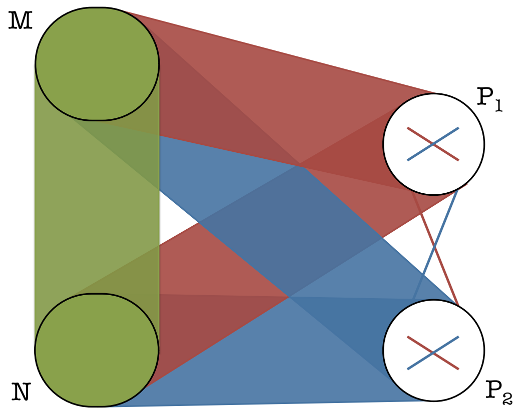


Figure 2.32: Final colouring in Case E.ii.a.

In summary, we now have

$$|P_1| \geq (\tfrac{1}{2}\alpha_1 - 13643\eta)k, \quad |P_2| \geq (\tfrac{1}{2}\alpha_2 - 9030\eta)k, \quad |M \cup N| \geq (\alpha_3 - 67216\eta^{1/2})k,$$

and know that all edges present in $G[M \cup N, P_1]$ are coloured exclusively red, all edges present $G[M \cup N, P_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green.

We have thus found, as a subgraph of G, a graph belonging to

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 14000\eta^{1/2})k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4 k\right).$$

Case E.ii.b: Most edges in $G_2[N_1, P_2]$ are red.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ such that:

(E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;

(E2) every vertex in P has a green edge to M;

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

Recall also that

$$(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|\,,\, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{E4a}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b}$$

Furthermore, recall that the largest red matching R in $G[N, P]$ defines a partition of N into $N_1 \cup N_2$ and P into $P_1 \cup P_2$ such that the edges of R belong $G[N_1, P_1]$, all edges present in $G[N_2, P_2]$ are coloured exclusively blue and that

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |P_1| &\leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |P_2| &\leq (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |N_1| &\leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |N_2|&.
\end{aligned}$$

Additionally, in this case, we assume that the largest blue matching in $G[N_1, P_2]$ spans at most $198\eta^{1/2}k$ vertices. Thus, we can discard at most $99\eta^{1/2}k$ vertices from each of N_1 and P_2 so that all edges present in $G[N_1, P_2]$ are coloured exclusively red.

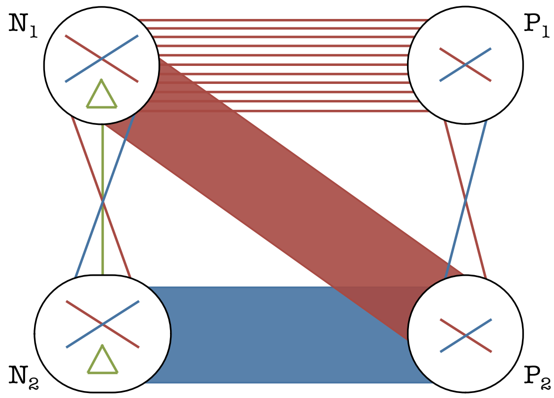


Figure 2.33: Initial colouring in Case E.ii.b.

In order to retain the equality $|N_1| = |P_1|$ and the property that every vertex in P_1 belongs to an edge of R, we discard from P_1 each vertex whose R-mate in N_1 has already been discarded. We then have

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k \le |P_1| &\le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k \le |P_2| &\le (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k \le |N_1| &\le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \le |N_2|&.
\end{aligned}$$

Then, suppose there exists a red matching on $396\eta^{1/2}k$ vertices in $G[N_2, P_1]$. Such a matching could be used together with $G[N_1, P_1]$ and $G[N_1, P_2]$ to give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, we can discard at most $198\eta^{1/2}k$ vertices from each of N_2 and P_1 so that all edges present in $G[N_2, P_1]$ are coloured exclusively blue.

We then have

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - 394\eta^{1/2})k \le |P_1| &\le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k \le |P_2| &\le (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_1 - 196\eta^{1/2})k \le |N_1| &\le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\
(\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k \le |N_2|&.
\end{aligned}$$

Then, if there existed a blue matching on $592\eta^{1/2}k$ vertices in $G[P_1, N_1]$, this could be used along with $G[P, N_2]$ to give a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most $296\eta^{1/2}k$ vertices from each of N_1 and P_1, we may assume that all edges present in $G[N_1, P_1]$ are coloured exclusively red and that

$$\left.\begin{aligned}(\tfrac{1}{2}\alpha_1 - 690\eta^{1/2})k \le |P_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\(\tfrac{1}{2}\alpha_2 - 196\eta^{1/2})k \le |P_2| \le (\tfrac{1}{2}\alpha_2 + 32\eta^{1/2})k,\\(\tfrac{1}{2}\alpha_1 - 492\eta^{1/2})k \le |N_1| \le (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k,\\(\tfrac{1}{2}\alpha_2 - 295\eta^{1/2})k \le |N_2|.\end{aligned}\right\} \quad (2.40)$$

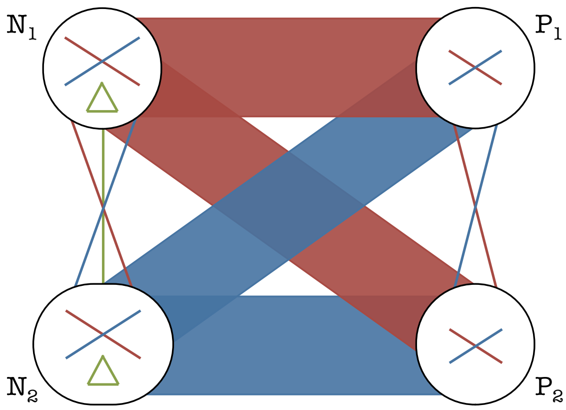


Figure 2.34: Colouring of the edges of $G[N, P]$.

Observe, now, that, given any subset P' of $(\frac{1}{2}\alpha_1 - 492\eta^{1/2})k$ vertices from P, by Lemma 2.6.14, there exists a red connected-matching on at least $(\alpha_1 - 986\eta^{1/2})k$ vertices in $G[N_1, P']$. Thus, the largest red matching in $G[P]$ spans at most $986\eta^{1/2}k$ vertices. Similarly, given any subset P'' of $(\frac{1}{2}\alpha_1 - 295\eta^{1/2})k$ vertices from P, we can find a blue connected-matching on at least $(\alpha_1 - 592\eta^{1/2})k$ vertices in $G[N_2, P'']$. Thus, the largest blue matching in $G[P]$ spans at most $592\eta^{1/2}k$ vertices. Thus, since there are no green edges within $G[P]$, we have $|P| \le 1600\eta^{1/2}k$, which, since $\eta \le (\alpha_1/10000)^2$, contradicts (2.40), completing Case E.ii.b.

At the beginning of Case E.ii, we assumed that F_P, the largest monochromatic component in $G[P]$, was red. If instead that monochromatic component is blue, then the proof proceeds exactly as above following the same steps with the roles of red and blue exchanged and with α_1 and α_2 exchanged. The result is identical.

Case E.iii: $|P|, |Q| \geq 95\eta^{1/2}k$.

We now consider the case when neither P nor Q is trivially small. This case is fairly involved, combining elements of Case E.i and Case E.ii with new arguments. However, because neither P nor Q is trivially small, we can exploit the structure of the two coloured graph $G[P \cup Q]$.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ such that:

(E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;

(E2) every vertex in P has a green edge to M;

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$.

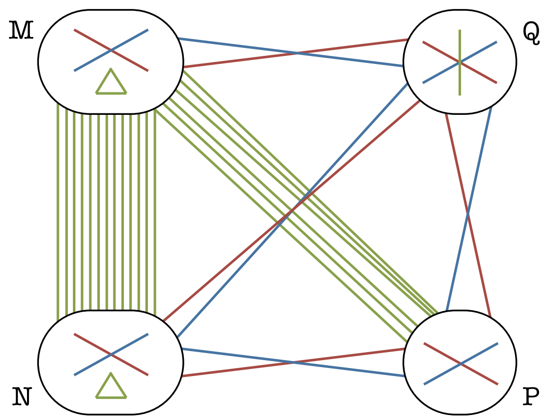


Figure 2.35: Decomposition in Case E.iii.

Recall also that

$$(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 5\eta^{1/2})k \leq |M|\,, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{E4a}$$
$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \leq |P| + |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \tag{E4b}$$

In this case, we assume that $|P|, |Q| \geq 95\eta^{1/2}k$. Recall that $G[P \cup Q]$ is $4\eta^4 k$-almost-complete. Then, since $|P| + |Q| \geq 190\eta^{1/2}k$, we have $4\eta^4 k \leq \eta^2(|P| + |Q| - 1)$ and, thus, $G[P \cup Q]$ is $(1 - \eta^2)$-complete. Since $\alpha_2 \leq \alpha_1 \leq 1$, recalling (E4b), we have $|P| + |Q| \leq (\frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 5\eta^{1/2})k \leq (1 + 5\eta^{1/2})k$ and so

$$|P|, |Q| \geq 95\eta^{1/2}k \geq 4\eta^{1/2}(1 + 5\eta^{1/2}) \geq 4\eta^{1/2}(|P| + |Q|) \geq 4\eta(|P| + |Q|).$$

Thus, provided $k \geq 1/(190\eta^{5/2})$, we may apply Lemma 2.6.18 to $G[P \cup Q]$, giving rise to two possibilities:

(a) $P \cup Q$ contains a monochromatic component F on at least $|P \cup Q| - 8\eta k$ vertices;

(b) there exist vertices $w_r, w_b \in Q$ such that w_r has red edges to all but $8\eta k$ vertices in P and w_b has blue edges to all but $8\eta k$ vertices in P.

Case E.iii.a: $P \cup Q$ has a large monochromatic component.

Suppose that F, the largest monochromatic component in $P \cup Q$ is red. We consider $G[M, Q]$ and $G[N, P]$, both of which have only red and blue edges, and let R be the largest red matching in $G[M, Q] \cup G[N, P]$. We partition each of M, N, P, Q into two parts such that $M_1 = M \cap V(R)$, $M_2 = M \backslash M_1$, $N_1 = N \cap V(R)$, $N_2 = N \backslash N_1$, $P_1 = P \cap V(R)$, $P_2 = M \backslash P_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \backslash Q_1$. Observe that, by maximality of R, all edges present in $G[P_2, N_2]$ and $G[Q_2, N_2]$ are blue.

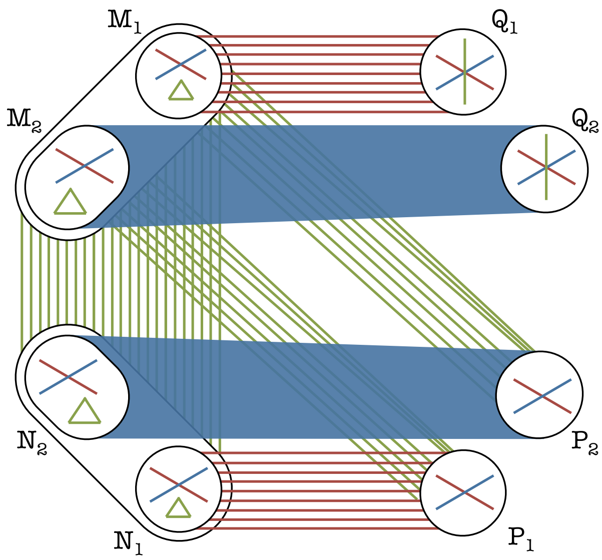


Figure 2.36: Decomposition into eight parts.

Notice that, since G is $4\eta^4 k$-almost-complete, $G[M_2 \cup Q_2]$ and $G[N_2 \cup P_2]$ each have a single blue component. We then consider two subcases:

(i) P_2 and Q_2 belong to the same blue component of G;

(ii) P_2 and Q_2 belong to different blue components of G.

Case E.iii.a.i: P_2 and Q_2 belong to the same blue component.

Since F, the largest red component in $P \cup Q$, contains at least $|P \cup Q| - 8\eta k$ vertices, all but at most $8\eta k$ of the edges of R belong to F. Thus, since $|M_1| = |Q_1|$ and $|N_1| = |P_1|$, we have a red connected-matching on at least $2(|P_1| + |Q_1| - 8\eta k)$ vertices and so we may assume that

$$|P_1| + |Q_1| \leq (\tfrac{1}{2}\alpha_1 + 8\eta)k \tag{2.41}$$

in order to avoid having a red connected-matching on at least $\alpha_1 k$ vertices.

Recalling (E4a) and (E4b), since $\eta \leq (\alpha_1/100)^2$, we have

$$|M| + |N| \geq |P| + |Q| + 80\eta^{1/2}k$$

and also

$$|M|, |N| \geq \tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 5\eta^{1/2}k \geq \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 5\eta^{1/2}k \geq |P| + |Q| - 10\eta^{1/2}.$$

Thus, since $|M_1| = |Q_1|$, $|N_1| = |P_1|$ and $|P|, |Q| \geq 95\eta^{1/2}k$, we have

$$|M_2| \geq |P| + |Q_2| - 10\eta^{1/2}k \geq |Q_2| + 85\eta^{1/2}k, \tag{2.42a}$$

$$|N_2| \geq |P_2| + |Q| - 10\eta^{1/2}k \geq |P_2| + 85\eta^{1/2}k. \tag{2.42b}$$

In particular, we have $|M_2| \geq |Q_2|$ and $|N_2| \geq |P_2|$. Thus, since P_2 and Q_2 belong to the same effective-blue component, by Lemma 2.6.14, there exists a blue connected-matching on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N_2, P_2] \cup G[M_2, Q_2]$. Thus, we may assume that

$$|P_2| + |Q_2| \leq (\tfrac{1}{2}\alpha_2 + 2\eta)k, \tag{2.43}$$

in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices.

Then, by (E4a), (E4b), (2.41) and (2.43), we have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 3\eta)k, \qquad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 9\eta)k. \tag{2.44}$$

We now proceed to determine the coloured structure of G. We begin by proving the following claim whose proof follows the same three steps as that of Claim 2.8.1. Considering in parallel $G[M, Q]$ and $G[N, P]$, the first step in the proof is to show that, after possibly discarding some vertices, all edges contained in $G[M_2, Q_1] \cup G[N_2, P_1]$ are coloured exclusively red, the second is to show that, after possibly discarding further vertices, all edges contained in $G[M_1, Q_2] \cup G[N_1, P_2]$ are coloured exclusively blue and the third is to show that, after possibly discarding still more vertices, all edges contained in $G[M_1, Q_1] \cup G[N_1, P_1]$ are coloured exclusively red.

Claim 2.8.2. *We may discard at most $70\eta k$ vertices from $P_1 \cup Q_1$, at most $48\eta^{1/2}k$ vertices from $P_2 \cup Q_2$, at most $94\eta^{1/2}k$ vertices from each of M_1 and N_1 and at most $11\eta k$ vertices from $M_2 \cup N_2$ such that, in what remains, all edges present in $G[M, Q_1] \cup G[N, P_1]$ are coloured exclusively red and all edges present in $G[M, Q_2] \cup G[N, P_2]$ are coloured exclusively blue.*

Proof. Suppose there exists a blue matching B_S on at least $22\eta k$ vertices in $G[M_2, Q_1] \cup G[N_2, P_1]$. By (2.42a) and (2.42b), we have $|M_2 \backslash V(B_S)| \geq |Q_2|$ and $|N_2 \backslash V(B_S)| \geq |P_2|$. Thus, since P_2 and Q_2 belong to the same blue component of G, applying Lemma 2.6.14 to each of $G[M_2 \backslash V(B_S), Q_2]$ and $G[N_2 \backslash V(B_S), P_2]$ gives a blue connected-matching B_L in $G[M_2 \backslash V(B_S), Q_2] \cup G[N_2 \backslash V(B_S), P_2]$ on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices which belongs to the same blue component of G as B_S but shares no vertices with it. Thus, $B_L \cup B_S$ forms a blue connected-matching on at least $2|P_2| + 2|Q_2| + 18\eta k \geq \alpha_2 k$ vertices. Therefore, after discarding at most $11\eta k$ vertices from each of $M_2 \cup N_2$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M_2, Q_1] \cup G[N_2, P_1]$ are coloured exclusively red.

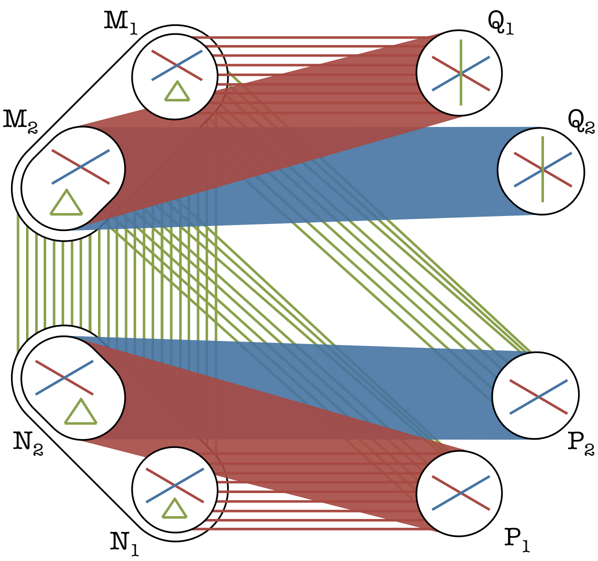


Figure 2.37: Colouring of $G[M_2, Q_1] \cup G[N_2, P_1]$.

Recalling (2.44), we then have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 14\eta)k, \qquad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 9\eta)k.$$

In order to retain the equalities $|M_1| = |Q_1|$ and $|N_1| = |P_1|$ and the property that every vertex in $M_1 \cup N_1$ belongs to an edge of R, we discard from $M_1 \cup N_1$ each vertex whose R-mate in $P_1 \cup Q_1$ has already been discarded. Thus, we discard at most a further $11\eta k$ vertices.

Then, if either $G[N_1, P_2]$ or $G[M_1, Q_2]$ contains a red matching on at least $48\eta k$ vertices in $G[N_1, P_2]$, then we may construct a red connected-matching on at least $\alpha_1 k$ vertices as follows:

Suppose that there exists a red matching R_P on $48\eta k$ vertices in $G[N_1, P_2]$. Then, observe that there exists a set R^- of $24\eta k$ edges belonging to R such that $N_1 \cap V(R_P) = N_1 \cap V(R^-)$. Define $\widetilde{P} = P_1 \cap V(R^-)$ and consider $G[N_2, \widetilde{P}]$. Since $|N_2|, |\widetilde{P}| \geq 24\eta k$ and $G[N_2, \widetilde{P}]$ is $4\eta^4 k$-almost-complete, we may apply Lemma 2.6.14 to find a red connected-matching R_S on at least $46\eta k$ vertices in $G[N_2, \widetilde{P}]$. Then, recalling that $P \cup Q$ has a red effective-component including all but $8\eta k$ of its vertices, we have a red connected-matching $R^\star \subseteq (R \backslash R^-) \cup R_S \cup R_P$ on at least

$$\begin{aligned}2\left(|P_1 \backslash V(R^-)| + |Q_1| + |P_1 \cap\ V(R_S)| + |P_2 \cap V(R_P)| - 8\eta k\right)& \\ \geq 2\left(\tfrac{1}{2}\alpha_1 - 39\eta + 23\eta + 24\eta - 8\eta\right)k \geq \alpha_1 k \text{ vertices}&\end{aligned}$$

in $G[N_1 \backslash V(R_P), P_1 \backslash V(R_S)] \cup G[M_1, Q_1] \cup G[N_2, P_1 \cap V(R_S)] \cup G[N_1 \cap V(R_P), P_2]$.

Similarly, suppose that there exists a red matching R_Q on $48\eta k$ vertices in $G[M_1, Q_2]$. Then, observe that there exists a set $R^=$ of $24\eta k$ edges belonging to R such that $M_1 \cap V(R_Q) = M_1 \cap V(R^=)$. Define $\widetilde{Q} = Q_1 \cap V(R^=)$ and consider $G[M_2, \widetilde{Q}]$. Since $|M_2|, |\widetilde{Q}| \geq 24\eta k$ and $G[M_2, \widetilde{Q}]$ is $4\eta^4 k$-almost-complete, we may apply Lemma 2.6.14 to find a red connected-matching R_T on at least $46\eta k$ vertices in $G[M_2, \widetilde{Q}]$. Then, recalling that $P \cup Q$ has a red effective-component including all but $8\eta k$ of its vertices, we have a red connected-matching $R^* \subseteq (R \backslash R^=) \cup R_T \cup R_Q$ on at least

$$2\left(|P_1| + |Q_1 \backslash V(R^=)| + |Q_1 \cap V(R_S)| + |Q_2 \cap V(R_Q)| - 8\eta k\right) \geq \alpha_1 k \text{ vertices}$$

in $G[N_1, P_1] \cup G[M_1 \backslash V(R_Q), Q_1 \backslash V(R_T)] \cup G[M_2, Q_1 \cap V(R_T)] \cup G[M_1 \cap V(R_Q), Q_2]$.

Thus, after discarding at most $24\eta k$ vertices from each of M_1, N_1, P_2 and Q_2, we may assume that all edges present in $G[M_1, Q_2] \cup G[N_1, P_2]$ are coloured exclusively blue. We then have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 14\eta)k, \qquad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 57\eta)k.$$

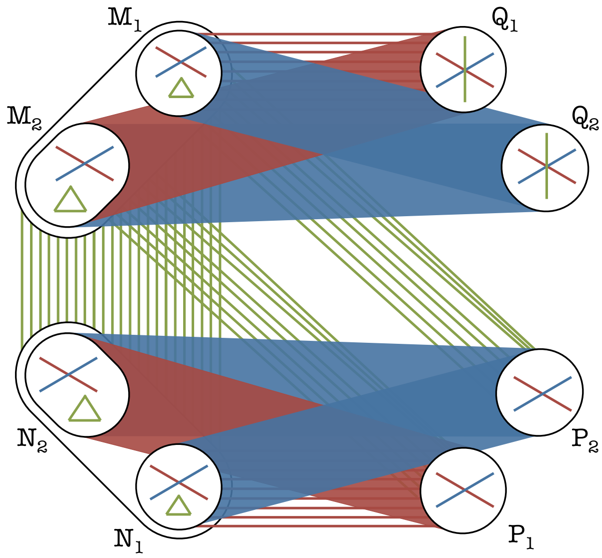


Figure 2.38: Colouring of $G[M_1, Q_2] \cup G[N_1, P_2]$.

Recalling (2.42a) and (2.42a), given that, so far, we have discarded at most $11\eta k$ vertices from $M_2 \cup N_2$, we have

$$|M_2| \geq |Q_2| + 14\eta^{1/2}k, \qquad |N_2| \geq |P_2| + 14\eta^{1/2}k.$$

Thus, by Lemma 2.6.14, there exist blue connected-matchings B_1 spanning at least $2|Q_2| - 2\eta k$ vertices in $G[M_2, Q_2]$, and B_2 spanning at least $|P_2| - 2\eta k$ vertices in $G[N_2, Q_2]$. Recall that we assume that P_2 and Q_2 belong to the same blue effective-component. Then, since all edges present in $G[M, Q_2]$ and $G[N, P_2]$ are coloured blue, all vertices in $M \cup N \cup P_2 \cup Q_2$ belong to the same blue component of G and $B_1 \cup B_2$ forms a connected-matching on at least $2(|P_2| + |Q_2|) - 4\eta k \geq (\alpha_2 - 118\eta)k$ vertices in that component.

Thus, the largest blue matching in $G[M_1, Q_1] \cup G[N_1, P_1]$ spans at most than $118\eta k$ vertices. Therefore, after discarding at most $59\eta k$ vertices from each of $M_1 \cup N_1$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M_1, Q_1] \cup G[N_1, P_1]$ are coloured exclusively red, completing the proof of Claim 2.8.2. □

In summary, we now know that all edges in $G[M, Q_1] \cup G[N, P_1]$ are coloured exclusively red and that all edges in $G[M, Q_2] \cup G[N, P_2]$ are coloured exclusively blue.

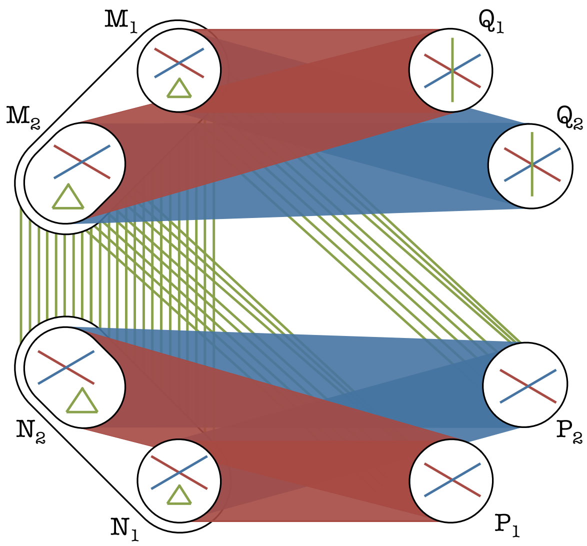


Figure 2.39: Colouring of $G[M, Q] \cup G[N, P]$.

Additionally, we have

$$\left.\begin{aligned} |M_1| + |M_2| &\geq |Q_1| + |Q_2| + 13\eta^{1/2}k, \qquad & |P_1| + |Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 73\eta)k, \\ |N_1| + |N_2| &\geq |P_1| + |P_2| + 13\eta^{1/2}k, \qquad & |P_2| + |Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 57\eta)k. \end{aligned}\right\} \quad (2.45)$$

We now move on to consider $G[M, P] \cup G[N, Q]$, taking the same approach as we did for $G[M, Q] \cup G[N, P]$ but recalling the possibility of green edges in $G[M, P]$. We prove the following claim:

Claim 2.8.3. *We may discard at most $145\eta k$ vertices from $P_1 \cup Q_1$, at most $84\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ and at most $229\eta k$ vertices from $M \cup N$ such that, in what remains, there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$ and no blue edges present in $G[M, P_1] \cup G[N, Q_1]$.*

Proof. Suppose that there exists a red matching $R^\times$ on at least $168\eta k$ vertices in $G[M, P_2] \cup G[N, Q_2]$. Then, by (2.45), we have $|M \backslash V(R^\times)| \geq |Q_1|$ and $|N \backslash V(R^\times)| \geq |P_1|$. Thus, since all but at most $8\eta k$ vertices of $P \cup Q$ belong to the same red component of G, $R^\times$ can be used along with edges from $G[N\backslash V(R^\times), P_1]$ and $G[M\backslash V(R^\times), Q_1]$ to form a red connected-matching on

$$(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k) + (168\eta k - 16\eta k) \geq \alpha_1 k$$

vertices. Thus, after discarding at most $84\eta k$ vertices from each of $M \cup N$ and $P_2 \cup Q_2$, we may assume that there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$. In particular, since there are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_2]$ are coloured exclusively blue. We then have

$$\left.\begin{aligned} |M_1| + |M_2| \geq |Q_1| + |Q_2| + 12\eta^{1/2}k, \quad & |P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 73\eta)k, \\ |N_1| + |N_2| \geq |P_1| + |P_2| + 12\eta^{1/2}k, \quad & |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 142\eta)k. \end{aligned}\right\} \quad (2.46)$$

Next, suppose that there exists a blue matching $B^\times$ on at least $290\eta k$ vertices in $G[M, P_1] \cup G[N, Q_1]$. Then, by (2.46), we have $|M\backslash(V(B^\times)| \geq |Q_2|$ and $|N\backslash(V(B^\times)| \geq |P_2|$. Thus, since $M_2 \cup N_2 \cup P_2 \cup Q_2$ belong to the same component of G, $B^\times$ can be used along with edges from $G[N\backslash V(B^\times), P_2]$ and $G[M\backslash V(B^\times), Q_2]$ to give a blue connected-matching on at least

$$(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k) + 290\eta k \geq \alpha_2 k$$

vertices. Thus, after discarding at most $145\eta k$ vertices from each of $P_1 \cup Q_1$ and $N \cup M$, we may assume that there are no blue edges present in $G[M, P_1] \cup G[N, Q_1]$. In particular, since there are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_1]$ are coloured exclusively red.

In summary, we have discarded at most $145\eta k$ vertices from $P_1 \cup Q_1$, at most $84\eta k$ vertices from $P_2 \cup Q_2$ and at most $229\eta k$ vertices from $M \cup N$. Having done so, we now know that there are no red edges present in $G[M, P_2] \cup G[N, Q_2]$ and that there are no blue edges present in $G[M, P_1] \cup G[N, Q_1]$. In particular, since we already knew that are no green edges present in $G[N, Q]$, we know that all edges present in $G[N, Q_1]$ are coloured exclusively red and that all edges present in $G[N, Q_2]$ are coloured exclusively blue, thus completing the proof of Claim 2.8.3. □

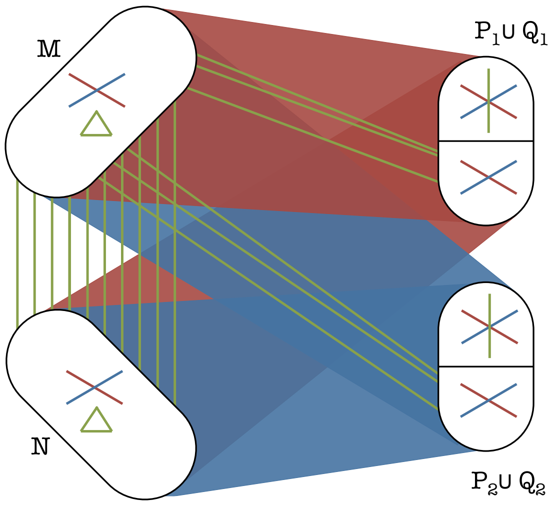


Figure 2.40: Colouring of $G[M, P] \cup G[N, Q]$ after Claim 2.8.3.

We now have

$$\left.\begin{array}{ll} |M| \geq |Q_1| + |Q_2| + 11\eta^{1/2}k, & |P_1| + |Q_1| \geq (\frac{1}{2}\alpha_1 - 218\eta)k, \\ |N| \geq |P_1| + |P_2| + 12\eta^{1/2}k, & |P_2| + |Q_2| \geq (\frac{1}{2}\alpha_2 - 142\eta)k. \end{array}\right\} \quad (2.47)$$

Finally, we turn our attention to $G[M \cup N]$, proving the following claim.

Claim 2.8.4. *We may discard at most $732\eta k$ vertices from $M \cup N$, such that, in what remains, all edges present in $G[M \cup N]$ are coloured exclusively green.*

Proof. Suppose that there exists a red matching $R^\dagger$ on $442\eta k$ vertices in $G[M \cup N]$. Then, by (2.47), we have $|M\backslash V(R^\dagger)| \geq |Q_1|$, $|N\backslash V(R^\dagger)| \geq |P_1|$. Also, since all edges present in $G[M \cup N, Q_1]$ are coloured exclusively red, $M \cup N$ belongs to a single red component of G. Thus, by Lemma 2.6.14, there exists a red connected-matching $R^\ddagger$ in $G[N\backslash V(R^\dagger), P_1] \cup G[M\backslash V(R^\dagger), Q_1]$ on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k)$ vertices. Then, since $R^\dagger$ and $R^\ddagger$ belong to the same red component but share no vertices, together they form a red connected-matching on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k) + 442\eta k \geq \alpha_1 k$ vertices.

Similarly, suppose that there exists a blue matching $B^\dagger$ on $290\eta k$ vertices in $G[M \cup N]$. Then, by (2.47), we have $|M\backslash V(B^\dagger)| \geq |Q_2|$, $|N\backslash V(B^\dagger)| \geq |P_2|$. Since all edges present in $G[M \cup N, Q_2]$ are coloured exclusively blue, $M \cup N$ belongs to a single blue component of G. Then, by Lemma 2.6.14, there exists a blue connected-matching $B^\ddagger$

in $G[N\backslash V(B^\dagger), P_2] \cup G[M\backslash V(B^\dagger), Q_2]$ on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices. Since $B^\dagger$ and $B^\ddagger$ belong to the same blue component but share no vertices, together they form a blue connected-matching on at least $(2|P_2| - 2\eta k) + (2|Q_2| - 2\eta k) + 290\eta k \geq \alpha_1 k$ vertices.

Thus, after discarding at most $732\eta k$ vertices from $M \cup N$, we can assume that all edges present in $G[M \cup N]$ are coloured exclusively green, completing the proof of Claim 2.8.4.

□

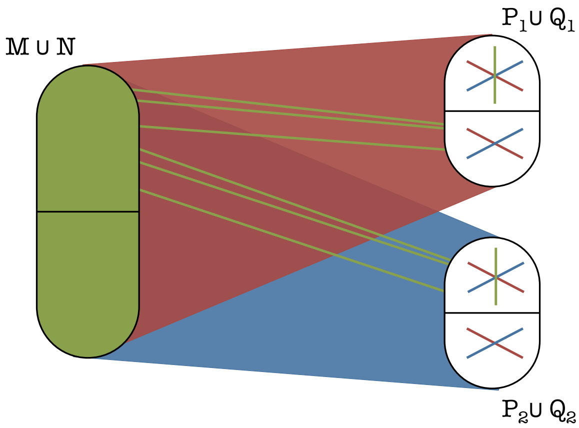


Figure 2.41: Colouring of $G[M \cup N]$.

Thus far, we have discarded at most $1200\eta k$ vertices from $M \cup N$. Recalling (E4), we now have $|M \cup N| \geq (\alpha_3 - 9\eta^{1/2})k$. Suppose there exists a green matching $G^\dagger$ on $20\eta^{1/2}k$ vertices in $G[M \cup N, P \cup Q]$. Then, we have $|(M \cup N)\backslash V(G^\dagger)| \geq (\alpha_3 - 19\eta^{1/2})k$. By Theorem 2.6.1, since G is $4\eta^4 k$-almost-complete, $G[(M \cup N)\backslash V(G^\dagger)]$ contains a green connected-matching on all but at most one of its vertices. Thus, provided $k \geq 1/\eta^{1/2}$, there exists a connected-green matching $G^\ddagger$ on at least $(\alpha_3 - 20\eta^{1/2})k$ vertices in $G[(M \cup N)\backslash V(G^\dagger)]$. Since G is $4\eta^4 k$-almost-complete and all edges present in $G[M \cup N]$ are coloured exclusivly green, all vertices of $M \cup N$ belong to the same green component of G. Thus, together, $G^\dagger$ and $G^\ddagger$ form a green connected-matching on at least $\alpha_3 k$ vertices. Thus, we may, after discarding at most $10\eta^{1/2}k$ vertices from each of $M \cup N$ and $P \cup Q$, assume that there are no green edges in $G[M \cup N, P \cup Q]$.

In summary, we now have

$$|P_1| + |Q_1| \geq (\tfrac{1}{2}\alpha_1 - 218\eta)k, \quad |P_2| + |Q_2| \geq (\tfrac{1}{2}\alpha_2 - 142\eta)k, \quad |M \cup N| \geq (\alpha_3 - 20\eta^{1/2})k,$$

and know that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red, all edges present $G[M \cup N, P_2 \cup Q_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green.

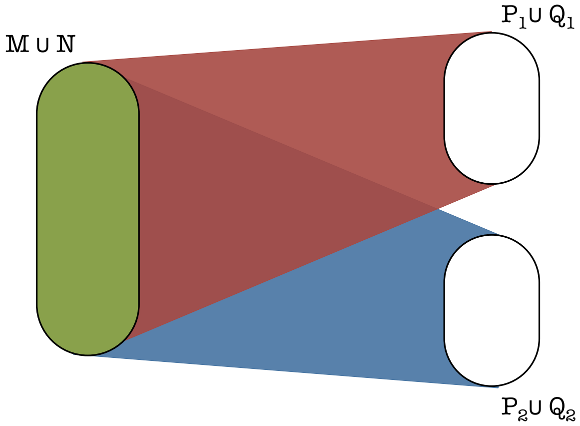


Figure 2.42: Final colouring in Case E.iii.a.i.

Thus, we have found, as a subgraph of G, a graph belonging to

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 10\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 10\eta^{1/2})k, (\alpha_3 - 20\eta^{1/2})k, 4\eta^4 k\right),$$

completing Case E.iii.a.i.

Case E.iii.a.ii: P_2 and Q_2 belong to the different components.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ satisfying (E1)–(E3) such that

$$(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 5\eta^{1/2})k \le |M|, |N| \le \tfrac{1}{2}\alpha_3 k \qquad \text{(E4a)}$$

$$(\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - \eta)k \le |P| + |Q| \le (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 5\eta^{1/2})k. \qquad \text{(E4b)}$$

Recall also that F, the largest monochromatic component in $P \cup Q$, is red and spans at least $|P \cup Q| - 8\eta k$ vertices and that each of M, N, P and Q have been subdivided into two parts such that $M_1 = M \cap V(R)$, $M_2 = M \backslash M_1$, $N_1 = N \cap V(R)$, $N_2 = N \backslash N_1$, $P_1 = P \cap V(R)$, $P_2 = M \backslash P_1$, $Q_1 = Q \cap V(R)$, $Q_2 = Q \backslash Q_1$, where R is the largest red matching in $G[M, Q] \cup G[N, P]$. By maximality of R, all edges present in $G[M_2, Q_2]$ or $G[N_2, P_2]$ are blue.

Additionally, in this case, we assume that P_2 and Q_2 belong to different blue components of G. Thus, in particular, all edges present in $G[N_2, Q_2]$ and $G[P_2, Q_2]$ are coloured red.

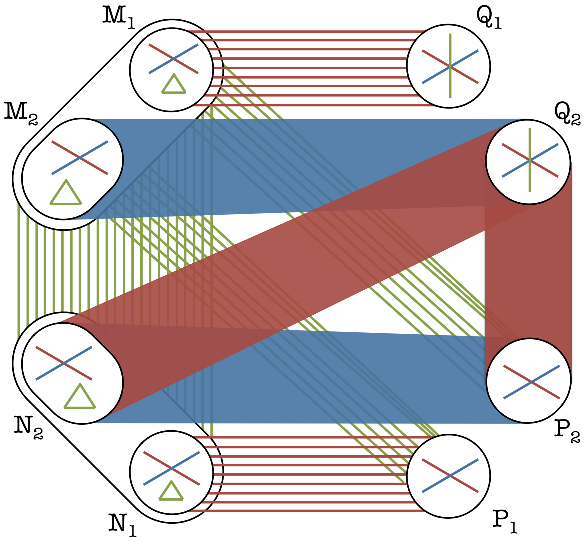


Figure 2.43: Initial colouring in Case E.iii.a.ii.

By Lemma 2.6.14, there exist red connected-matchings

R_{PQ} on at least $2\min\{|P_2|,|Q_2|\}-2\eta k$ vertices in $G[P_2,Q_2]$

R_{NQ} on at least $2\min\{|N_2|,|Q_2|\}-2\eta k$ vertices in $G[N_2,Q_2]$.

Then, since F includes all but at most $8\eta k$ of the vertices of $P\cup Q$, these connected-matchings can be augmented with edges from R to give the red connected-matchings illustrated in Figure 2.44:

R_1 on at least $2|P_1|+2|Q_1|+2\min\{|P_2|,|Q_2|\}-18\eta k$ vertices in $G[M_1,Q_1]\cup G[N_1,P_1]\cup G[P_2,Q_2]$,

R_2 on at least $2|P_1|+2|Q_1|+2\min\{|N_2|,|Q_2|\}-18\eta k$ vertices in $G[M_1,Q_1]\cup G[N_1,P_1]\cup G[N_2,Q_2]$,

Given the existence of these matchings, we can obtain bounds on the sizes of the various sets identified:

Since $|P|,|Q|\geq 95\eta^{1/2}k$, by (E4b), $|P|,|Q|\leq(\frac{1}{2}\alpha_1+\frac{1}{2}\alpha_2-90\eta^{1/2})k$. But, then,

$$|M|,|N|\geq(\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2-5\eta^{1/2})k\geq\max\{|P|,|Q|\}+85\eta^{1/2}k.$$

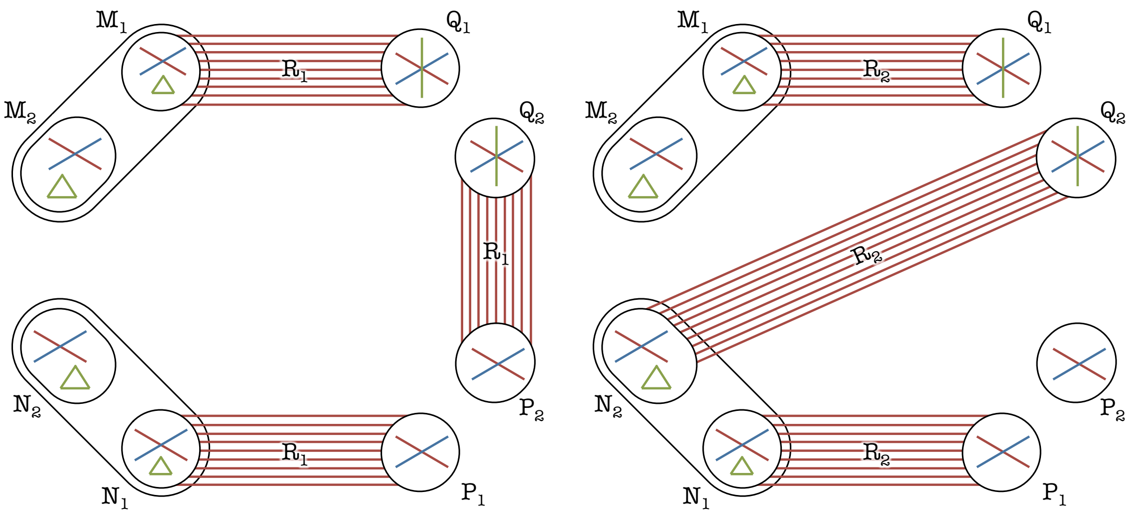


Figure 2.44: The red connected-matchings R_1 and R_2.

So, since $|M_1| = |Q_1|$ and $|N_1| = |P_1|$, we have

$$|M_2| \geq |Q_2| + 85\eta^{1/2}k, \qquad\qquad |N_2| \geq |P_2| + 85\eta^{1/2}k. \tag{2.48}$$

In particular, $|M_2| \geq |Q_2|$ and $|N_2| \geq |P_2|$. Thus, by Lemma 2.6.14, we may obtain a blue connected-matching on $2|P_2| - 2\eta k$ vertices in $G[N_2, P_2]$ and one on $2|Q_2| - 2\eta k$ vertices in $G[M_2, Q_2]$. Thus, in order to avoid a blue connected-matching on at least $\alpha_2 k$ vertices, we may assume that

$$|P_2|, |Q_2| \leq (\tfrac{1}{2}\alpha_2 + \eta)k. \tag{2.49}$$

Suppose, for now, that $|Q_2| \geq |N_2|$. Then, recalling, that $|N_2| \geq |P_2|$, we have $|Q_2| \geq |P_2|$ and, therefore, R_1 spans at least $2|P_1| + 2|Q_1| + 2|P_2| - 18\eta k$ vertices in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[P_2, Q_2]$. Thus, in order to avoid a red connected-matching on at least $\alpha_1 k$ vertices, we may assume that

$$|P_1| + |Q_1| + |P_2| \leq (\tfrac{1}{2}\alpha_1 + 10\eta)k.$$

Also, since $|Q_2| \leq (\frac{1}{2}\alpha_2 + \eta)k$, by (E4b), we have

$$|P_1| + |Q_1| + |P_2| \geq (\tfrac{1}{2}\alpha_1 - 2\eta)k.$$

Thus, R_1 spans at least $(\alpha_1 - 22\eta)k$ vertices. Now, since $|Q_2| \geq |N_2|$, by (2.48), we have $|Q_2| \geq |P_2| + 15\eta^{1/2}k$. Thus, there exists $\widetilde{Q} \subseteq Q_2 \backslash V(R_1)$ such that $|\widetilde{Q}| \geq 15\eta^{1/2}k$. Therefore, by Lemma 2.6.14, we may find a red connected-matching M_3 on at least $28\eta^{1/2}k$ vertices in $G[N_2, \widetilde{Q}]$ which belongs to the same red component as R_1. Thus, together R_1 and R_3 form a red-connected-matching on $\alpha_1 k$ vertices, completing the proof in this case.

Therefore, we may instead assume that $|N_2| \geq |Q_2|$. In that case, R_2 spans at least $2|P_1| + 2|Q_1| + |Q_2| - 20\eta k$ vertices in $G[N_1, P_1] \cup G[M_1, Q_1] \cup G[N_2, Q_2]$. Thus, $|P_1| + |Q_1| + |Q_2| \leq (\frac{1}{2}\alpha_1 + 10\eta)k$. Then, by (E4b) and (2.49), we obtain

$$\left.\begin{aligned} (\tfrac{1}{2}\alpha_1 - 2\eta)k \leq |P_1| + |Q_1| + |Q_2| \leq (\tfrac{1}{2}\alpha_1 + 10\eta)k, \\ (\tfrac{1}{2}\alpha_1 - 12\eta)k \leq |P_2| \leq (\tfrac{1}{2}\alpha_2 + \eta)k. \end{aligned}\right\} \quad (2.50)$$

Observe that, by (2.49) and (2.50), we may assume that

$$|Q_2| \leq |P_2| + \eta^{1/2}k. \quad (2.51)$$

We are now in a position to examine the coloured structure of $G[N, P]$. We show that, after possibly discarding some vertices, we may assume that all edges contained in $G[N, P_1]$ are coloured exclusively red and all edges contained in $G[N, P_2]$ are coloured exclusively blue. Following the same steps as in the proofs of Claim 2.8.1 and Claim 2.8.2 we prove:

Claim 2.8.5. *We may discard at most $67\eta k$ vertices from N_1, $14\eta k$ vertices from N_2, $54\eta k$ vertices from P_1 and at most $27\eta k$ vertices from P_2 such that, in what remains, all edges present in $G[N, P_1]$ are coloured exclusively red and all edges present in $G[N, P_2]$ are coloured exclusively blue.*

Proof. Given any $\widetilde{N}_2 \subseteq N$ such that $|\widetilde{N}_2| \geq |P_2| \geq (\frac{1}{2}\alpha_2 - 12\eta)k$, by Lemma 2.6.14, we can obtain a blue connected-matching on $(\alpha_2 - 26\eta)k$ vertices in $G[\widetilde{N}_2, P_2]$. Then, since $|N_2| \geq |P_2| + 15\eta^{1/2}k$, the existence a blue matching B_S on at least $28\eta k$ vertices $G[N_2, P_1]$ would allow us to obtain a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most $14\eta k$ vertices from each of P_1 and N_2, we may assume that all edges present in $G[N_2, P_1]$ are coloured exclusively red.

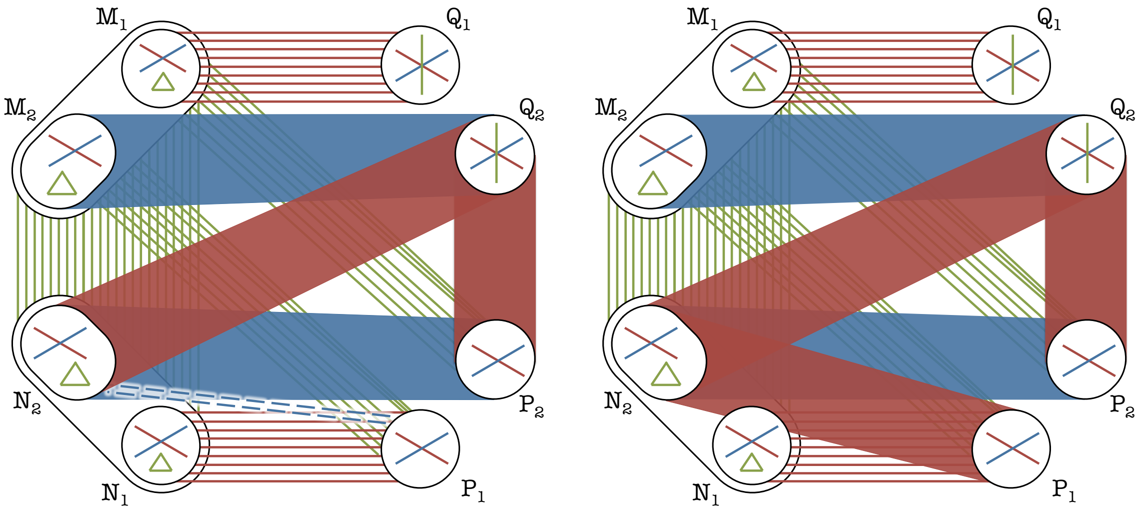


Figure 2.45: Colouring of the edges of $G[N_2, P_1]$.

After discarding these vertices, we have

$$|P_1| + |Q_1| + |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 16\eta)k$$

and, thus, may assume that R_2 spans at least $(\alpha_1 - 50\eta)k$ vertices in $G[M_1, Q_1] \cup G[N_1, P_1] \cup G[N_2, Q_2]$. Also, recalling (2.48) and (2.51), we have

$$|N_2| \geq |P_2| + 14\eta^{1/2}k \geq |Q_2| + 13\eta^{1/2}k.$$

Thus,

$$|N_2 \backslash V(R_2)| \geq |N_2| - |Q_2| \geq 13\eta^{1/2}k.$$

Suppose there exists a matching R_S on $54\eta k$ vertices in $G[N_1, P_2]$, then we can obtain a red connected-matching on at least $\alpha_1 k$ vertices as follows:

Observe that there exists a set R^- of $27\eta k$ edges belonging to R_2 such that $N_1 \cap V(R_S) = N_1 \cap V(R^-)$. Define $R^* = R_2 \backslash R^-$ and $\widetilde{P} = P_1 \cap V(R^-)$, let $\widetilde{N}$ be any set of $27\eta k$ vertices in $N_2 \backslash V(R_2)$ and consider $G[\widetilde{N}, \widetilde{P}]$. Since $|\widetilde{N}|, |\widetilde{P}| \geq 27\eta k$, we may apply Lemma 2.6.14 to find a red connected-matching R_T on at least $52\eta k$ vertices in $G[\widetilde{N}, \widetilde{P}]$. Since all edges present in $G[N_2, Q_2] \cup G[P_2, Q_2]$ are coloured exclusively red, R_S and R_T belong to the same red component as R_2. Then, $R^* \cup R_S \cup R_T$ is a red connected-matching in

$$G[M_1, Q_1] \cup G[N_1, P_1] \cup G[N_2, Q_2] \cup G[N_2, P_1] \cup G[N_1, P_2]$$

on at least $(\alpha_1 - 50\eta - 55\eta)k + 54\eta k + 52\eta k \geq \alpha_1 k$ vertices.

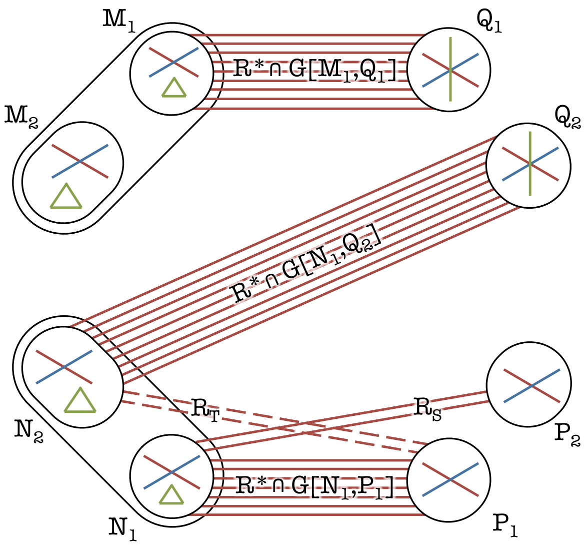


Figure 2.46: Enlarging the matching R_2 with edges from $G[N_1, P_2]$.

Thus, after discarding at most $27\eta k$ vertices from each of N_1 and P_2, we may assume that all edges present in $G[N_1, P_2]$ are coloured exclusively blue and, recalling (2.50), that $|P_2| \geq (\frac{1}{2}\alpha_2 - 38\eta)k$.

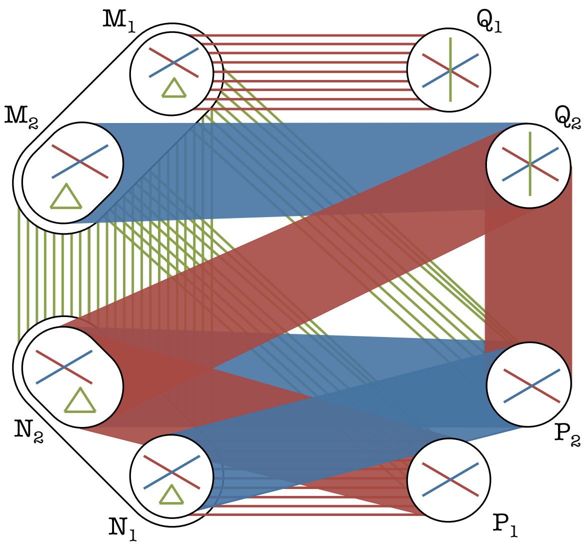


Figure 2.47: Resultant colouring of the edges of $G[N_1, P_2]$.

Finally, suppose that there exists a blue matching B_S on at least $80\eta k$ vertices in $G[N_1, P_1]$. Then, we could obtain a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most $40\eta k$ vertices from each of N_1 and P_1, we may assume that all edges present in $G[N_1, P_1]$ are coloured exclusively red, thus completing the proof of the claim. □

In summary, recalling (2.50), we now have

$$|P_1| + |Q_1| + |Q_2| \geq (\tfrac{1}{2}\alpha_1 - 56\eta)k,$$
$$|P_2| \geq (\tfrac{1}{2}\alpha_2 - 40\eta)k,$$

and know that all edges present in $G[N, P_1]$ are coloured exclusively red and that all edges present in $G[N, P_2]$ are coloured exclusively blue. Observe, also, that there can be no blue edges present in $G[N_1, Q_2]$ since then $M_2 \cup Q_2$ and $N_2 \cup P_2$ would belong to the same blue component of G.

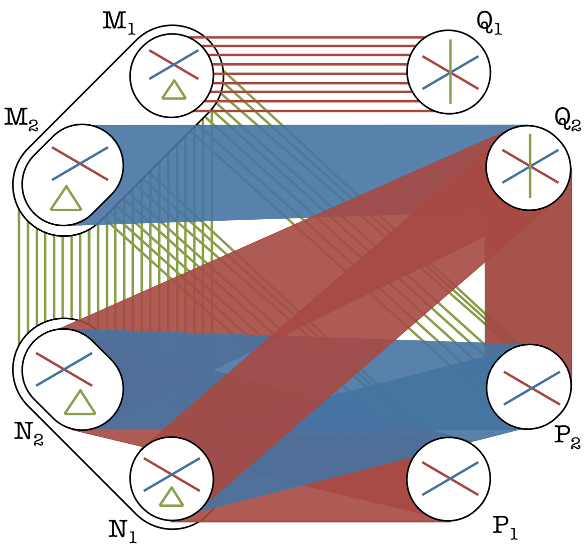


Figure 2.48: Colouring after Claim 2.8.5.

Next, we consider, in turn $G[M_2, N \cup P_2]$, $G[N]$ and $G[M_1, N]$ showing that after discarding a few vertices, we may assume that all edges remaining in each are coloured exclusively green:

Suppose there exists a red matching R_M on at least $136\eta k$ vertices in $G[M_2, N \cup P_2]$. Then, since $|N| \geq |P_1| + |Q_2| + 12\eta^{1/2}k$, we have $|N \backslash (R_M)| \geq |P_1| + |Q_2|$. Thus, by Lemma 2.6.14, there exists a red connected-matching R_N on at least $(2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N, P_1 \cup Q_2]$ sharing no vertices with R_M. Since all edges present in $G[N \cup P_2, Q_2] \cup G[N, P_1]$ are coloured red, R_M and R_N belong to the same red component of G.

Since $P \cup Q$ has a red effective-component on F on at least $|P \cup Q| - 8\eta k$ vertices, all but at most $8\eta k$ of the edges of R contained in $G[M_1, Q_1]$ belong to the same red component as $R_M \cup R_N$. Thus, defining R_Q to be the subset of R belonging to $G[M, Q_1 \cap F]$, we have a red connected-matching $R_M \cup R_N \cup R_Q$ on at least $136\eta k + (2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k) + (2|Q_1| - 16\eta k) \geq \alpha_1 k$ vertices in $G[M_2, N \cup P_2] \cup G[N, P_1 \cup Q_2] \cup G[M_1, Q_1]$.

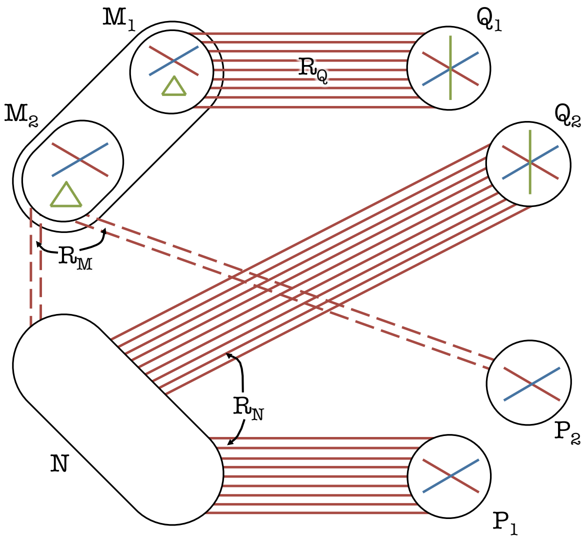


Figure 2.49: Construction of red connected-matching $R_M \cup R_N \cup R_Q$.

Thus, discarding at most $68\eta k$ vertices from each of M_2 and $N \cup P_2$, we may assume that there are no red edges $G[M_2, N \cup P_2]$. Thus, recalling that we assume that P_2 and Q_2 are in different blue components, all edges present in $G[M_2, N \cup P_2]$ are coloured exclusively green and have

$$\left.\begin{aligned} |N| &\geq |P_1| + |Q_2| + 11\eta^{1/2}k, & |P_1| + |Q_1| + |Q_2| &\geq (\tfrac{1}{2}\alpha_1 - 57\eta)k, \\ |N| &\geq |P_2| + 11\eta^{1/2}k, & |P_2| &\geq (\tfrac{1}{2}\alpha_2 - 108\eta)k. \end{aligned}\right\} \quad (2.52)$$

Next, suppose there exists a red matching R_A on $136\eta k$ vertices in $G[N]$. Then, by (2.52), we have $|N \backslash V(R_A)| \geq |P_1| + |Q_2|$. So, by Lemma 2.6.14, there exists a red connected-matching R_B on at least $(2|P_1| - 2\eta k) + (2|Q_2| - 2\eta k)$ vertices in $G[N \backslash V(R_A), P_1 \cup Q_2]$. Since all edges in $G[N, P_1]$ are coloured red, all edges of R_A and R_B belong to the same red component. Also, since the red component F spans all but at most $8\eta k$ vertices of $P \cup Q$, there exists a red-matching $R_C \subseteq R$ in $G[M_1, Q_1]$ on at least

$2|Q_1| - 16\eta k$ vertices belonging to the same red component as R_A and R_B. Then, together R_A, R_B and R_C form a red connected-matching on at least $\alpha_1 k$ vertices in $G[N] \cup G[N, P_1 \cup Q_2] \cup G[M_1 \cup Q_1]$.

Similarly, if there exists a blue matching B_A on $218\eta k$ vertices in $G[N]$, then we can construct a blue connected-matching on at least $\alpha_2 k$ vertices as follows. By (2.52), we have $|N \backslash V(R_B)| \geq |P_2|$. So, by Lemma 2.6.14, there exists a blue connected-matching B_B on at least $2|P_2| - 2\eta k$ vertices in $G[N \backslash V(B_A), P_2]$. Since all edges in $G[N, P_2]$ are coloured blue, all edges in B_A and B_B belong to the same blue component. Thus, together, B_A and B_B form a blue connected-matching on at least $2|P_2| + 216\eta k \geq \alpha_2 k$ vertices in $G[N] \cup G[N, P_1 \cup Q_2] \cup G[M_1 \cup Q_1]$. Thus, discarding at most $354\eta k$ vertices from N, we have

$$|N| \geq |P_1| + |Q_2| + 10\eta^{1/2} k, \qquad |N| \geq |P_2| + 10\eta^{1/2} k \tag{2.53}$$

and may assume that all edges in $G[N]$ are coloured exclusively green.

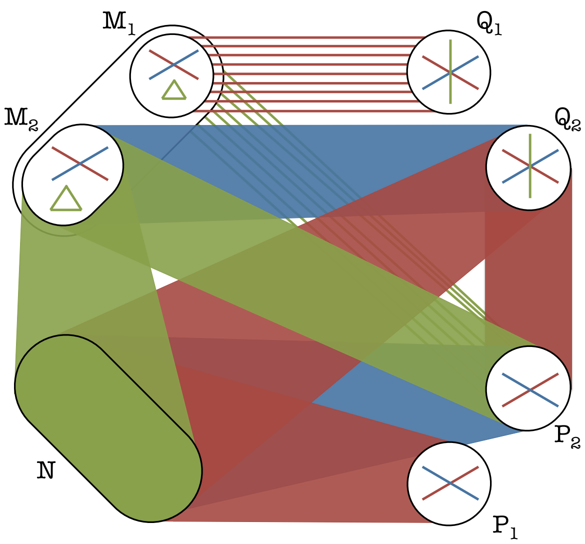


Figure 2.50: Colouring of $G[M_2, P_2] \cup G[N]$.

Finally, we consider $G[M_1, N]$. Suppose there exists a red matching R_D on $138\eta k$ vertices in $G[M_1, N]$. Then, by (2.53), we have $|N \backslash V(R_D)| \geq |P_1| + |Q_2|$. Therefore, by Lemma 2.6.14, there exists a red connected-matching R_E on at least $(2|P_1| - 2\eta k) + (2|Q_1| - 2\eta k)$ vertices in $G[N, P_1 \cup Q_2]$ which shares no vertices with R_D. Then, since all edges present in $G[N, P_1]$ are coloured red, R_D and R_E belong to the same red component. Since F, the largest red component in $G[P \cup Q]$ includes all but at most $8\eta k$

of the vertices of $P \cup Q$, there exists a matching $R_F \subseteq R$ in $G[M_1, Q_1]$ on at least $2(|Q_1| - |M_1 \cap V(R_D)| - 8\eta k)$ vertices which shares no vertices with R_D but belongs to the same red component as it. Thus, together, R_D, R_E and R_F form a red connected-matching on at least $2(|P_1| + |Q_1| + |Q_2| + |N \cap V(R_D)|) - 20\eta k \geq \alpha_1 k$ vertices.

Similarly, if there exists a blue matching B_D on $220\eta k$ vertices in $G[M_1, N]$. Then, by (2.53), we have $|N \backslash V(B_D)| \geq |P_2|$. Therefore, by Lemma 2.6.14, there exists a blue connected-matching B_E on at least $(2|P_2| - 2\eta k)$ vertices in $G[N, P_2]$ which shares no vertices with B_D. Since all edges present in $G[N, P_2]$ are coloured blue, B_D and B_E belong to the same blue component. Thus together B_D and B_E form a blue connected-matching on at least $2|P_2| - 2\eta k + 220\eta k \geq \alpha_2 k$ vertices. Therefore, after discarding at most $174\eta k$ vertices from each of M_1 and N, we may assume that all edges present in $G[M_1, N]$ are coloured exclusively green.

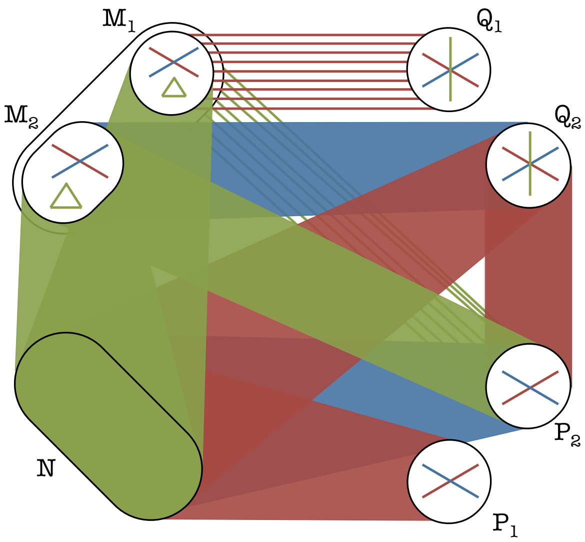


Figure 2.51: Final Colouring in Case E.iii.a.ii.

Given the colouring found so far, we now show that we may obtain a green connected-matching on at least $\alpha_3 k$ vertices: Having discarded at most $1000\eta k$ vertices from $M \cup N$, recalling (E4a), since $\eta < 10^{-7}$, we have $|M|, |N| \geq (\frac{1}{2}\alpha_3 - 6\eta^{1/2})k$. Recalling (2.48), we have $|M_2| \geq |Q_2| + 14\eta^{1/2}k \geq 14\eta^{1/2}$ and, by (2.52), have $|P_2| \geq (\frac{1}{2}\alpha_2 - 108\eta)k \geq 14\eta^{1/2}k$. Letting M' be a subset of M_2 and N' a subset of N such that $14\eta^{1/2}k \leq |M'| = |N'| \leq 15\eta^{1/2}k$, by Lemma 2.6.14, there exist green matchings G_{MP} on at least $2|M'| - 2\eta k$ vertices in $G[M', P_2]$ and G_{MN} on at least $2\min\{|M \backslash M'|, |N \backslash N'|\} - 2\eta k$ vertices in $G[M \backslash M', N \backslash N']$. Finally, by Theorem 2.6.1, provided $k \geq 1/\eta^2$, there exists a connected-matching G_N on at least $|N'|-1$ vertices in $G[N']$. Then, since all edges present in $G[M,N]$ are coloured green, G_{NP}, G_{MN} and G_N belong to the same green component and, since they share no vertices, form a green connected-matching on at least

$$\begin{aligned}(2|M'|-2\eta k)+(2\min\{|M\backslash M'|,|N\backslash N'|\}-2\eta k)+|N'|-1\\ \geq \alpha_3 k+\eta^{1/2}k-4\eta k-1\geq \alpha_3 k\end{aligned}$$

vertices. By the definition of the decomposition $M\cup N\cup P\cup Q$, this connected-matching is odd, thus completing Case E.iii.a.ii.

At the begining of Case E.iii.a, we made the assumption that F, the largest monochromatic component in $G[P\cup Q]$ was red. If, instead, F is blue, then the proof is essentially identical to the above with the roles of red and blue reversed. The result is the same, that is, G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1-10\eta^{1/2})k,(\tfrac{1}{2}\alpha_1-10\eta^{1/2})k,(\alpha_3-20\eta^{1/2})k,4\eta^4k\right),$$

thus completing Case E.iii.a.

Case E.iii.b: $G[P,Q]$ contains red and blue stars centred in Q.

Recall that we have a decomposition of $V(G)$ into $M\cup N\cup P\cup Q$ satisfying (E1)–(E3) with $|P|,|Q|\geq 95\eta^{1/2}k$ and

$$(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-5\eta^{1/2})k\leq |M|\,,\,|N|\leq \tfrac{1}{2}\alpha_3 k \tag{E4a}$$

$$(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2-\eta)k\leq |P|+|Q|\leq (\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2+5\eta^{1/2})k. \tag{E4b}$$

Additionally, in this case, we assume that the sets

$$W_r=\{q\in Q:\, q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b=\{q\in Q:\, q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\},$$

are both non-empty.

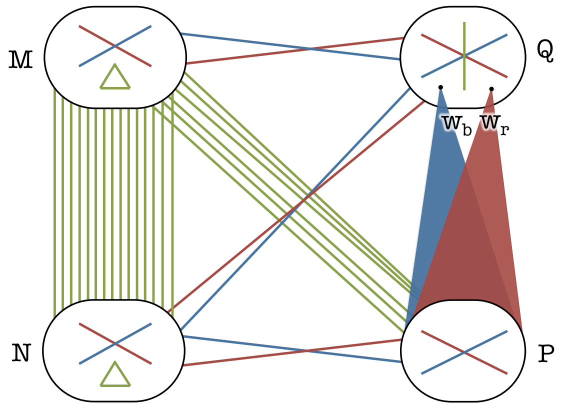


Figure 2.52: Red and blue stars centred in Q.

We define two further sets which will be useful in what follows:

$$P_r = \{p \in P : p \text{ has a red edge to some vertex } q \in W_r\},$$

$$P_b = \{p \in P : p \text{ has a blue edge to some vertex } q \in W_b\}.$$

Observe that, since $|P|, |Q| \geq 95\eta^{1/2}k$, by (E4b), we have

$$95\eta^{1/2}k \leq |Q| \leq (\tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 90\eta^{1/2})k.$$

Thus, considering (E4a) and (E4b), we have

$$|N| \geq |Q| \geq 95\eta^{1/2} \geq 6(2\eta^{1/2})|N \cup Q| \geq 6(2\eta)|N \cup Q|.$$

Recall that, at the start of Case E, after discarding some edges, G was assumed to be $(1 - \frac{3}{2}\eta^4)$-complete. Recall also that, from (2.16), we have $\alpha_3 \geq \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$. Thus, considering (E4a), since $|Q| \geq 95\eta^{1/2}k$, we have

$$|N \cup Q| \geq \tfrac{1}{2}\alpha_3 k \geq \tfrac{1}{4}(\alpha_3 + \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2})k \geq \tfrac{1}{4}(\alpha_3 + \tfrac{1}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 10\eta^{1/2})k \geq \tfrac{1}{4}K.$$

Then, since $\eta < 0.1$, we have $6\eta^4 \leq 2\eta$ so $G[N \cup Q]$ is $(1 - 2\eta)$-complete and, provided $|N \cup Q| \geq 1/\eta$, we may apply Lemma 2.6.19 to $G[N, Q]$ and distinguish four cases:

(i) $G[N \cup Q]$ has a monochromatic component E on at least $|N \cup Q| - 28\eta k$ vertices;

(ii) N, Q can be partitioned into $N_1 \cup N_2$, $Q_1 \cup Q_2$ such that $|N_1|, |N_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between N_i and Q_j are red for $i = j$ and blue for $i \neq j$;

(iii) there exist vertices $n_r, n_b \in N$ such that n_r has red edges to all but $16\eta k$ vertices in Q and n_b has blue edges to all but $16\eta k$ vertices in Q;

(iv) there exist vertices $q_r, q_b \in Q$ such that q_r has red edges to all but $16\eta k$ vertices in N and q_b has blue edges to all but $16\eta k$ vertices in N.

Case E.iii.b.i: $G[N \cup Q]$ has a large monochromatic component.

Suppose that E, the largest monochromatic component in $G[N \cup Q]$, is red. In that case, if $G[Q \cap E, P_r]$ contains a red edge, then $G[P \cup Q]$ has a red effective-component on at least $|P \cup Q| - 36\eta k$ vertices. Alternatively, every edge in $G[Q \cap E, P_r]$ is blue, in which case, $G[P \cup Q]$ has a blue connected-component on at least $|P \cup Q| - 36\eta k$ vertices.

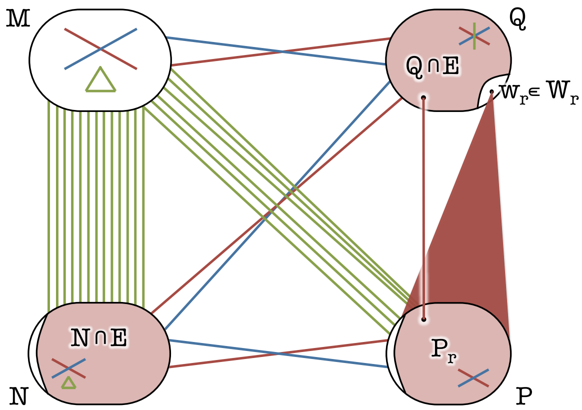


Figure 2.53: Large red effective-component.

Suppose instead that E is blue. In that case, if $G[Q \cap E, P_b]$ contains a blue edge, then $G[P \cup Q]$ has a blue effective-component on at least $|P \cup Q| - 36\eta k$ vertices. Alternatively, every edge present in $G[Q \cap E, P_b]$ is red, in which case, $G[P \cup Q]$ has a red connected-component on at least $|P \cup Q| - 36\eta k$ vertices.

In either case the proof proceeds via exactly the same steps as Case E.iii.a with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k\right),$$

thus completing Case E.iii.b.i.

Case E.iii.b.ii: $N \cup Q$ has a non-trivial partition with 'cross' colouring.

In this case, we assume that N and Q can be partitioned into $N_1 \cup N_2$, $Q_1 \cup Q_2$ such that $|N_1|, |N_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between N_i and Q_j are red for $i = j$ and blue for $i \neq j$.

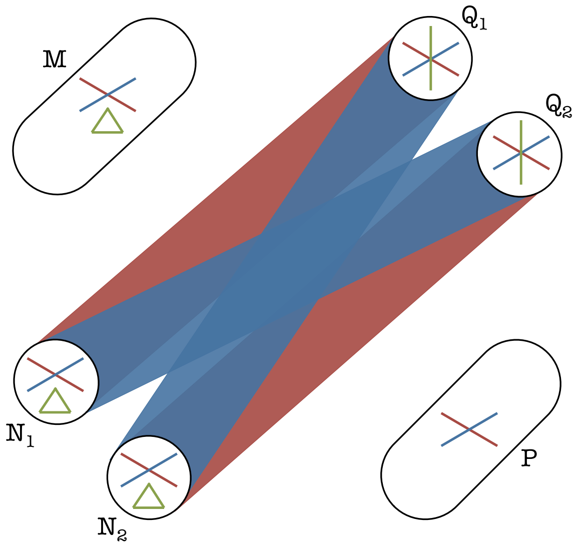


Figure 2.54: 'Cross' colouring of $G[N, Q]$.

Recall that W_r and W_b are both non-empty, that is, there exist vertices $w_r, w_b \in Q$ such that w_r has red edges to all but $8\eta k$ vertices in P and w_b has blue edges to all but $8\eta k$ vertices in P.

Observe that, we may assume that $P \cup Q$ does not have a monochromatic effective-component on at least $|P \cup Q| - 16\eta k$ vertices. Indeed, otherwise, the proof proceeds via the same steps as Case E.iii.a, with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k\right).$$

Without loss of generality, we assume $w_r \in Q_1$. Observe then that the existence of a red edge in $G[Q_2, P_r]$ would result in $P \cup Q$ having a red effective-component on at least $|P \cup Q| - 8\eta k$ vertices. Thus, we assume that every edge present in $G[Q_2, P_r]$ is blue. Similarly, we may assume every edge present in $G[Q_1, P_b]$ is red. Since W_r and W_b are non-empty, $|P_r|, |P_b| \geq |P| - 8\eta k$. Then, as $|P| \geq 95\eta^{1/2} k$, we have $|P_r \cap P_b| \geq$

$|P| - 16\eta k > 0$ and know that all edges present in $G[Q_2, P_r \cap P_b]$ are coloured exclusively red and all edges present in $G[Q_1, P_r \cap P_b]$ are coloured exclusively blue.

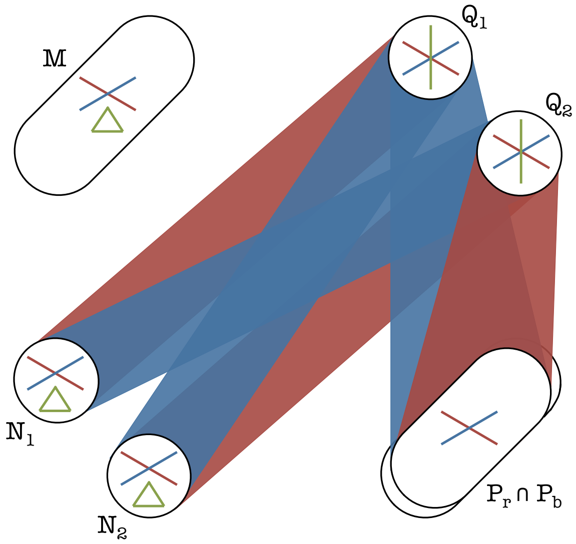


Figure 2.55: Red and blue edges in $G[N, Q] \cup G[P, Q]$.

Recall that, by (E3), there are no green edges in $G[P, N]$ and notice that, if there existed a vertex $p \in P$ with red edges to both N_1 and N_2, then $P \cup Q$ would have an effective red-component on at least $|P \cup Q| - 16\eta k$ vertices. Thus, we assume that there is no such vertex. Similarly, we assume there does not exist a vertex in P with blue edges to both N_1 and N_2. Thus, P can be partitioned into $P_1 \cup P_2$ such that all edges present between P_i and N_j are red for $i = j$ and blue for $i \neq j$.

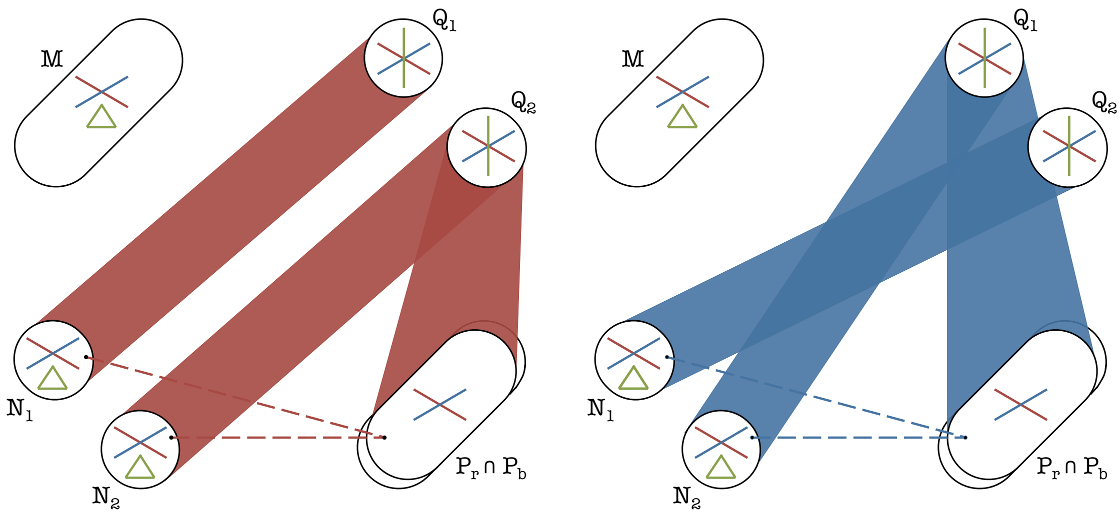


Figure 2.56: Partitioning of P.

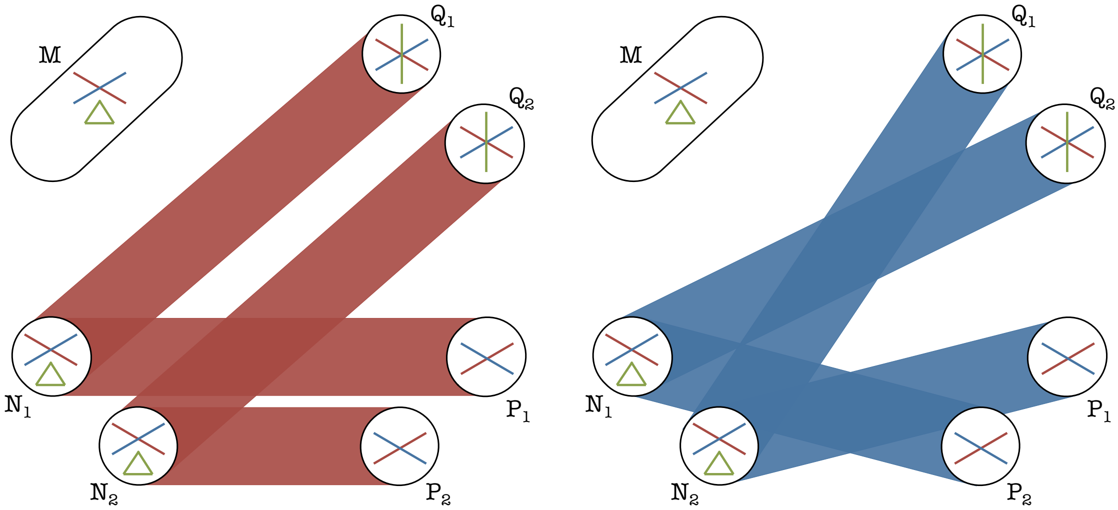


Figure 2.57: Resultant colouring of $G[N, P]$.

Recall, however, that all edges present in $G[Q_1, P_r]$ are red and that all edges present in $G[Q_2, P_b]$ are blue. Thus, in order to avoid $G[P, Q]$ having a red effective-component on at least $|P \cup Q| - 16\eta k$ vertices, we must have $P_r \subseteq P_1$ but then, since $|P_r \cap P_b| > 0$, there exists a blue edge in $G[Q_2, P_1]$, giving rise to a blue effective-component on at least $|P \cup Q| - 16\eta k$ vertices, completing Case E.iii.a.ii.

Case E.iii.b.iii: $G[N, Q]$ contains red and blue stars centred in N.

Recall that we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ satisfying (E1)–(E4b) and that there exists vertices $w_r \in W_r$ and $w_b \in W_b$, where

$$W_r = \{q \in Q : q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b = \{q \in Q : q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\}.$$

Additionally, in this case, we assume that there exist vertices $n_r \in N_r$ and $n_b \in N_b$, where

$$N_r = \{n \in N : n \text{ has red edges to all but at most } 16\eta k \text{ vertices in } Q\},$$

$$N_b = \{n \in N : n \text{ has blue edges to all but at most } 16\eta k \text{ vertices in } Q\}.$$

Recall that

$$P_r = \{p \in P : p \text{ has a red edge to some vertex } q \in W_r\},$$

$$P_b = \{p \in P : p \text{ has a blue edge to some vertex } q \in W_b\},$$

and define $Q_N = \{q \in Q : q \text{ has a red edge to some vertex } n \in N_r\}$.

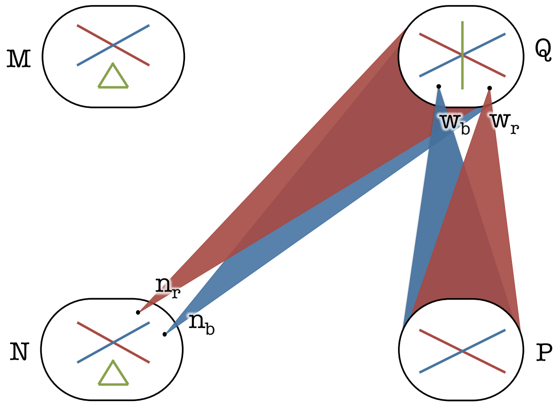


Figure 2.58: 'Stars' centred at w_r, w_b, n_r and n_b.

Notice that, since $|W_r|, |W_b| > 0$, we have $|P_r|, |P_b| \geq |P| - 8\eta k$ and, since $|N_r| > 0$, we have $|Q_N| \geq |Q| - 16\eta k$. So, if there exists a red edge in $G[P_r, Q_N]$, then $G[P \cup Q]$ has a red effective-component on at least $|P \cup Q| - 24\eta k$ vertices. Alternatively, every edge present in $G[P_r, Q_N]$ is blue. Then $G[P \cup Q]$ is has a blue component on at least $|P \cup Q| - 24\eta k$ vertices. In either case, the proof then follows the same steps has in Case E.iii.a with the result being that G will either contain a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 100\eta^{1/2})k, (\alpha_3 - 200\eta^{1/2})k, 4\eta^4 k\right),$$

thus completing case E.iii.b.iii.

Case E.iii.b.iv: $G[N, Q]$ contains red and blue stars centred in Q.

Recall that we have a decomposition of $V(G)$ into four parts $M \cup N \cup P \cup Q$ satisfying

(E1) $M \cup N$ is the vertex set of F and every edge of F belongs to $G[M, N]$;

(E2) every vertex in P has a green edge to M;

(E3) there are no green edges in $G[N,P]$, $G[M,Q]$, $G[N,Q]$, $G[P,Q]$ or $G[P]$;

such that the sizes of the four parts satisfy

$$(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\}-5\eta^{1/2})k \leq |M|\,,\, |N| \leq \tfrac{1}{2}\alpha_3 k, \tag{E4a}$$

$$(\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2-\eta)k \leq |P|+|Q| \leq (\tfrac{1}{2}\alpha_1+\tfrac{1}{2}\alpha_2+5\eta^{1/2})k. \tag{E4b}$$

Recall, also, that there exists vertices $w_r \in W_r$ and $w_b \in W_b$, where

$$W_r = \{q \in Q : q \text{ has red edges to all but at most } 8\eta k \text{ vertices in } P\},$$

$$W_b = \{q \in Q : q \text{ has blue edges to all but at most } 8\eta k \text{ vertices in } P\}.$$

Additionally, in this case, we assume that there exist vertices $q_r \in Q_r$ and $q_b \in Q_b$, where

$$Q_r = \{n \in Q : n \text{ has red edges to all but at most } 16\eta k \text{ vertices in } N\},$$

$$Q_b = \{n \in Q : n \text{ has blue edges to all but at most } 16\eta k \text{ vertices in } N\}.$$

We consider $G[Q \cup M]$. Since all edges present in $G[Q,M]$ are coloured red or blue, Lemma 2.6.19 can be applied to $G[M \cup Q]$ as it was previously applied to $G[N \cup Q]$ with the same four possible outcomes:

(i) $G[M \cup Q]$ has a monochromatic component on at least $|M \cup Q| - 28\eta k$ vertices;

(ii) M, Q can be partitioned into $M_1 \cup M_2$, $Q_1 \cup Q_2$ such that $|M_1|, |M_2|, |Q_1|, |Q_2| \geq 3\eta k$ and all edges present between M_i and Q_j are red for $i = j$ and blue for $i \neq j$;

(iii) there exist vertices $m_r, m_b \in N$ such that m_r has red edges to all but $16\eta k$ vertices in Q and m_b has blue edges to all but $16\eta k$ vertices in Q;

(iv) there exist vertices $q_r, q_b \in Q$ such that q_r has red edges to all but $16\eta k$ vertices in M and q_b has blue edges to all but $16\eta k$ vertices in M.

For possibilities (i)–(iii), the proof proceeds via exactly the same steps as in the corresponding cases above with the same possible outcomes.

Thus, we consider possibility (iv). That is, we assume that (in addition to w_r, w_b, q_r, q_b) there exist vertices $v_r, v_b \in Q$ such that v_r has red edges to all but $16\eta k$ vertices in M and v_b has blue edges to all but $16\eta k$ vertices in M.

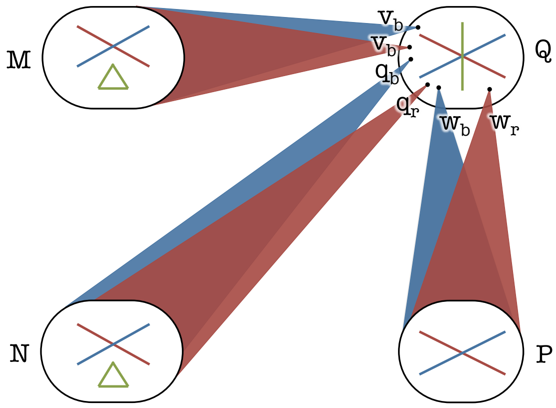


Figure 2.59: ‘Stars’ centred in Q.

Consider the largest red matching R in $G[N, P \cup Q]$ and partition each of N, P and Q into two parts such that $N_1 = N \cap V(R)$, $N_2 = N \backslash N_1$, $P_1 = P \cap V(R)$, $P_2 = P \backslash P_1$, $Q_1 = Q \cap V(R)$ and $Q_2 = Q \backslash Q_1$. By maximality of R, all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

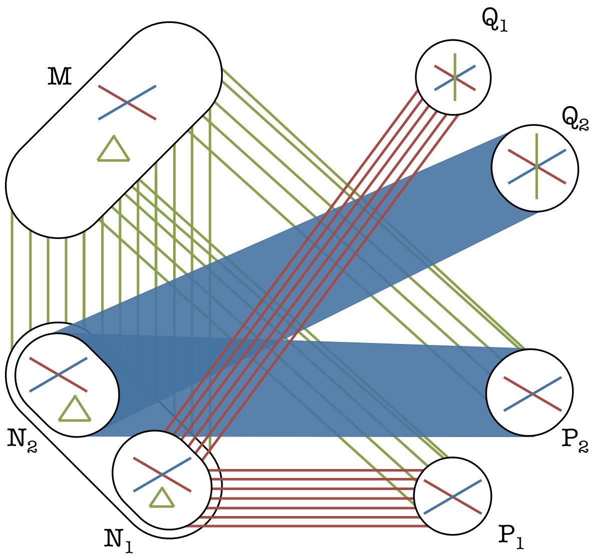


Figure 2.60: Partition of N, P and Q.

Notice that, in order to avoid having a blue connected-matching on at least $\alpha_2 k$ vertices, by Lemma 2.6.14, we may assume that

$$\min\{|P_2 \cup Q_2|, |N_2|\} \leq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k.$$

Thus, by (E4a) and (E4b), we may thus assume that

$$|N_1| = |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 6\eta^{1/2})k. \tag{2.54a}$$

Since q_r has red edges to all but at most $16\eta k$ vertices in N, all but at most $16\eta k$ of the edges of R belong to the same red component and we have a red connected-matching on $2(|N_1| - 16\eta k)$ vertices in $G[N_1, P_1 \cup Q_1]$. Thus, we may assume that

$$|N_1| = |P_1 \cup Q_1| \leq (\tfrac{1}{2}\alpha_1 + \eta^{1/2})k$$

and, therefore, by (E4a) and (E4b), that

$$|N_2|, |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 6\eta^{1/2}). \tag{2.54b}$$

Recall that v_r has red edges to all but $16\eta k$ vertices in M and v_b has blue edges to all but $16\eta k$ vertices in M. Thus, we may, after discarding at most $32\eta k$ vertices from M, assume that $G[M]$ is effectively red-connected and effectively blue-connected.

Similarly, we may, after discarding at most $16\eta k$ vertices from $G[P]$, assume that P is effectively red-connected and effectively blue-connected and, after discarding at most $32\eta k$ vertices from N, assume that N is effectively red-connected and effectively blue-connected. In order to maintain the equality $|N_1| = |P_1|$, we also discard from $N_1 \cup P_1$ any vertex whose R-mate has already been discarded.

Then, in summary, having discarded some vertices, we have a decomposition of $V(G)$ into $M \cup N \cup P \cup Q$ and a refinement into $M \cup N_1 \cup N_2 \cup P_1 \cup P_2 \cup Q_1 \cup Q_2$ such that

(E3) there are no green edges in $G[N, P]$, $G[M, Q]$, $G[N, Q]$, $G[P, Q]$ or $G[P]$;

(E6a) $G[M]$, $G[N]$ and $G[P]$ each have a single red and a single blue effective-component;

(E6b) $G[N_1, P_1 \cup Q_1]$ contains a red matching utilising every vertex in $G[N_1 \cup P_1 \cup Q_1]$;

(E6c) all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

Having discarded some vertices, recalling (E4a), (2.54a) and (2.54b), we have

$$\left.\begin{aligned} |M| &\geq \big(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 6\eta^{1/2}\big)k, \quad |N_1| = |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N| &\geq \big(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 7\eta^{1/2}\big)k, \quad |N_2|, |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 8\eta^{1/2})k. \end{aligned}\right\} \tag{E7}$$

and also

$$|P| \geq 90\eta^{1/2}k, \qquad \max\{|Q_1|, |Q_2|\} \geq 45\eta^{1/2}k. \tag{E8}$$

For the final time, we distinguish between three cases:

(a) $G[M, N \cup P \cup Q_1]$ contains a red edge;

(b) $G[M, N \cup P \cup Q_2]$ contains a blue edge;

(c) $G_1[M, N \cup P \cup Q_1]$ and $G_2[M, N \cup P \cup Q_2]$ each contain no edges.

Case E.iii.b.iv.a: $G[M, N \cup P \cup Q_1]$ contains a red edge.

Given the existence of a red edge in $G[M, N \cup P \cup Q_1, M]$, then the existence of a red matching on at least $14\eta^{1/2}k$ vertices in $G[P_2 \cup Q_2, M]$ would give a red connected-matching on at least $\alpha_1 k$ vertices. Thus, we may, after discarding at most $7\eta^{1/2}k$ vertices from each of M and $P_2 \cup Q_2$, assume that all present edges in $G[M, Q_2]$ are coloured exclusively blue and that there are no red edges in $G[M, P_2]$.

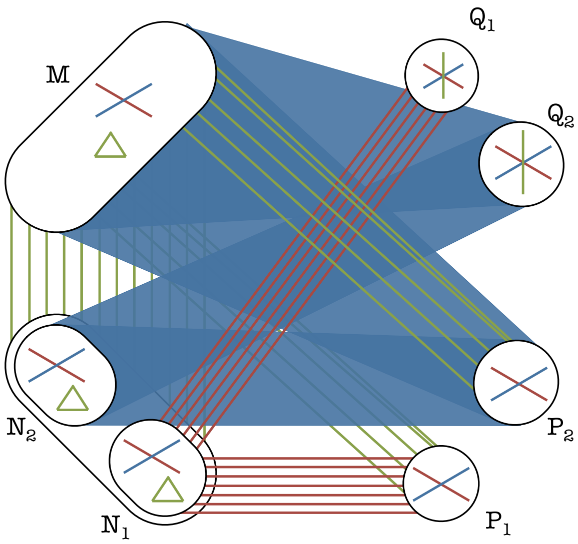


Figure 2.61: Colouring immediately before Claim 2.8.6.

We then have

$$\left.\begin{aligned} |N_1| = |P_1 \cup Q_1| &\geq (\tfrac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N_2|, |P_2 \cup Q_2| &\geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k. \end{aligned}\right\} \tag{2.55}$$

The following pair of claims establish the coloured structure of $G[M \cup N, P \cup Q]$:

Claim 2.8.6.a. *If $|Q_1| \geq 45\eta^{1/2}k$, we may discard at most $43\eta^{1/2}k$ vertices from N_1, at most $43\eta^{1/2}k$ vertices from N_2, at most $16\eta^{1/2}k$ vertices from M, at most $59\eta^{1/2}k$ vertices from $P_1 \cup Q_1$ and at most $27\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ such that, in what remains, there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$.*

Claim 2.8.6.b. *If $|Q_2| \geq 45\eta^{1/2}k$, we may discard at most $45\eta^{1/2}k$ vertices from N_1, at most $18\eta^{1/2}k$ vertices from N_2, at most $18\eta^{1/2}k$ vertices from M, at most $18\eta^{1/2}k$ vertices from $P_1 \cup Q_1$ and at most $27\eta^{1/2}k$ vertices from $P_2 \cup Q_2$ such that, in what remains, there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$.*

Proof. (a) We begin by considering $G[M \cup N_1, P_1 \cup Q_1]$. Suppose there exists a blue matching B_S on $32\eta^{1/2}k$ vertices in $G[M \cup N_1, P_1 \cup Q_1]$. Observe that, since $|N_2|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 15\eta^{1/2})k$, by Lemma 2.6.14, there exists a blue connected-matching B_L on at least $(\alpha_2 - 32\eta^{1/2})k$ vertices in $G[N_2, P_2 \cup Q_2]$. Then, since all edges present in $G[M \cup N_2, Q_2]$ are coloured blue and $G[N]$ is blue effectively-connected, B_S and B_L belong to the same blue component and thus form a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $16\eta^{1/2}k$ vertices from each of $M \cup N_1$ and $P_1 \cup Q_1$, we may assume that there are no blue edges present in $G[M \cup N_1, P_1 \cup Q_1]$. Notice then, in particular, that all edges present in $G[M, Q_1] \cup G[N_1, P_1 \cup Q_1]$ are coloured exclusively red.

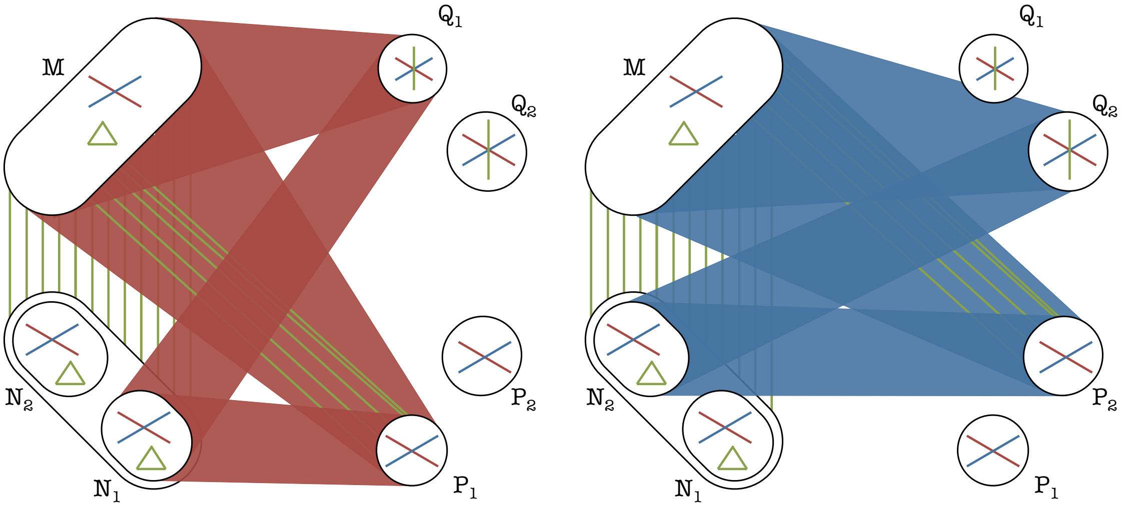


Figure 2.62: Colouring of the edges of $G[M \cup N_1, P_1 \cup Q_1]$ in Claim 2.8.6.a.

Having discarding these vertices, we have

$$|Q_1| \geq 29\eta^{1/2}k, \qquad\qquad |N_1| \geq |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 24\eta^{1/2})k.$$

Now, suppose there exists a red matching R_S on at least $54\eta^{1/2}k$ vertices in $G[N_1, P_2 \cup Q_2]$. Since $|P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 24\eta^{1/2})k$ and $|Q_2| \geq 29\eta^{1/2}k$, there exists $\widetilde{Q} \subseteq Q_1$ such that $|P_1 \cup \widetilde{Q}| \geq (\frac{1}{2}\alpha_1 - 52\eta^{1/2})k$ and $|Q_1 \backslash \widetilde{Q}| \geq 27\eta^{1/2}k$. Then, since $|N_1 \backslash V(R_S)|, |P_1 \cup \widetilde{Q}| \geq (\frac{1}{2}\alpha_1 - 52\eta^{1/2})k$ and $|M|, |Q_1 \backslash \widetilde{Q}| \geq 27\eta^{1/2}k$, by Lemma 2.6.14, there exist red connected-matchings R_L on at least $(\alpha_1 - 106\eta^{1/2})k$ vertices in $G[N_1, P_1 \cup Q_1]$ and R_T on at least $52\eta^{1/2}k$ vertices in $G[Q_1, M]$ sharing no vertices with each other or R_S. Then, since all edges in $G[N_1, Q_1]$ are coloured red, R_L, R_S and R_T belong to the same red component and, thus, together, form a red connected-matching on at least $\alpha_1 k$ vertices. Therefore, after discarding at most $27\eta^{1/2}k$ vertices from each of N_1 and $P_2 \cup Q_2$, we may assume that all edges in $G[N_1, P_2 \cup Q_2]$ are coloured exclusively blue.

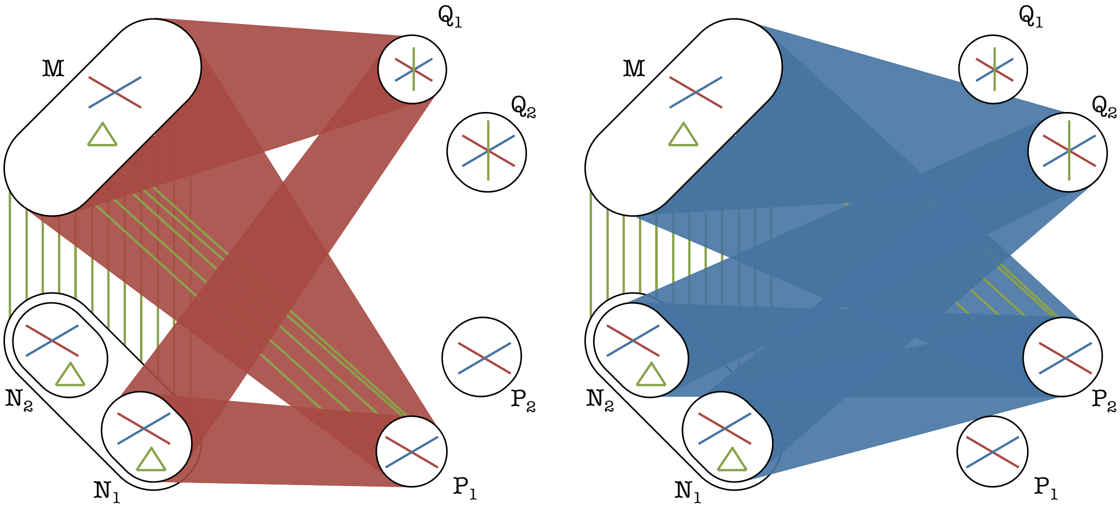


Figure 2.63: Colouring of the edges of $G[N_1, P_2 \cup Q_2]$ in Claim 2.8.6.a.

We then have

$$|N_1| \geq (\tfrac{1}{2}\alpha_1 - 51\eta^{1/2})k, \qquad\qquad |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 24\eta^{1/2})k,$$
$$|N_2| \geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k, \qquad\qquad |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 42\eta^{1/2})k.$$

Suppose now that there exists a blue matching B_U on $86\eta^{1/2}k$ vertices in $G[N_2, P_1 \cup Q_1]$. Then, since $|N \backslash V(B_U)|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 42\eta^{1/2})k$, by Lemma 2.6.14, there exists a blue connected-matching B_V on at least $(\alpha_2 - 86\eta^{1/2})k$ vertices in $G[N \backslash V(B_U), P_2 \cup Q_2]$.

Therefore, $B_U \cup B_V$ forms a blue connected-matching on at least $\alpha_2 k$ vertices in $G[N, P \cup Q]$. Thus, after discarding at most $43\eta^{1/2}k$ vertices from each of N_2 and $P_1 \cup Q_1$, we may assume that all edges present in $G[N_1, P_1 \cup Q_1]$ are coloured exclusively red, thus completing the proof of Claim 2.8.6.a.

(b) Suppose that G is coloured as in Figure 2.61 and that $|Q_2| \geq 45\eta^{1/2}k$. We begin by considering $G[M \cup N, P_1 \cup Q_1]$. Suppose there exists a blue matching B_S on $36\eta^{1/2}k$ vertices in $G[M \cup N, P_1 \cup Q_1]$, then we can obtain a blue connected-matching on at least $\alpha_1 k$ vertices as follows: Recalling (2.55), we have $|P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 15\eta^{1/2})k$. Then, since $|Q_2| \geq 45\eta^{1/2}k$, there exists $\widetilde{Q} \subseteq Q_2$ such that $|P_2 \cup \widetilde{Q}| \geq (\frac{1}{2}\alpha_2 - 34\eta^{1/2})k$ and $|Q_2 \backslash \widetilde{Q}| \geq 18\eta^{1/2}k$. Then, we have $|N_2 \backslash V(B_S)|, |P_2 \cup \widetilde{Q}| \geq (\frac{1}{2}\alpha_2 k - 34\eta^{1/2})k$ and $|M \backslash V(B_S)|, |Q_2 \backslash \widetilde{Q}| \geq 18\eta^{1/2}k$. Thus, by Lemma 2.6.14, there exist blue connected-matchings B_L on at least $(\alpha_2 k - 70\eta^{1/2})k$ vertices in $G[N \backslash V(B_S), P_2 \cup \widetilde{Q}]$ and B_T on at least $34\eta^{1/2}k$ vertices in $G[M \backslash V(B_S), Q_2 \backslash \widetilde{Q}]$. Since M and N are each blue effectively-connected and all edges present in $G[M \cup N_2, P_2]$ are coloured blue, B_L, B_S and B_T belong to the same blue component in G and, thus, together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $18\eta^{1/2}k$ vertices from each of $M \cup N$ and $P_1 \cup Q_1$, we may assume that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red.

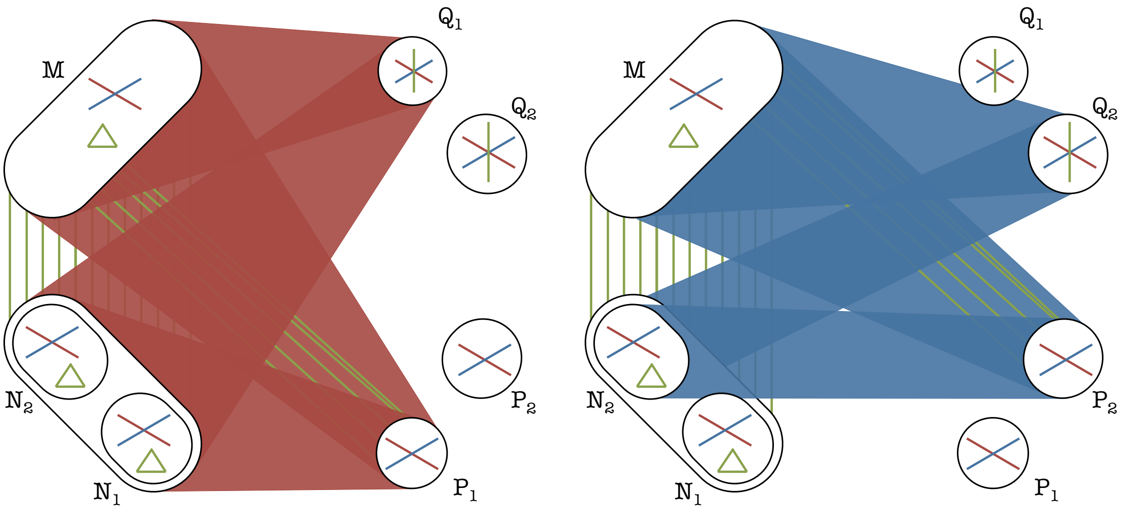


Figure 2.64: Colouring of $G[M \cup N, P_1 \cup Q_1]$ in Claim 2.8.6.b.

Recalling (2.55), we then have

$$|N_1| \geq (\tfrac{1}{2}\alpha_1 - 26\eta^{1/2})k, \qquad |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 26\eta^{1/2})k,$$
$$|N_2| \geq (\tfrac{1}{2}\alpha_2 - 33\eta^{1/2})k, \qquad |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 15\eta^{1/2})k.$$

Finally, suppose there exists a red matching R_S on $54\eta^{1/2}k$ vertices in $G[N_1, P_2 \cup Q_2]$. Then, $|N \backslash V(R_S)|, |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 26\eta^{1/2})k$ so, by Lemma 2.6.14, there exists a red connected-matching R_L on at least $(\alpha_1 - 54\eta^{1/2})k$ vertices in $G[N\backslash V(R_S), P_1 \cup Q_1]$. The red matchings R_S and R_L share no vertices and, since $G[N]$ is red effectively-connected, belong to the same red component of G, thus, together, they form a red connected-matching on at least $\alpha_1 k$ vertices. Therefore, after discarding at most $27\eta^{1/2}k$ vertices from each of N_1 and $P_2 \cup Q_2$, we may assume that all edges in $G[N, P_2 \cup Q_2]$ are coloured blue, thus completing the proof of Claim 2.8.6.b. □

In summary, combining the two cases above, we may now assume that there are no blue edges present in $G[M \cup N, P_1 \cup Q_1]$ and no red edges present in $G[M \cup N, P_2 \cup Q_2]$. In particular, we may assume that all all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and that all edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.

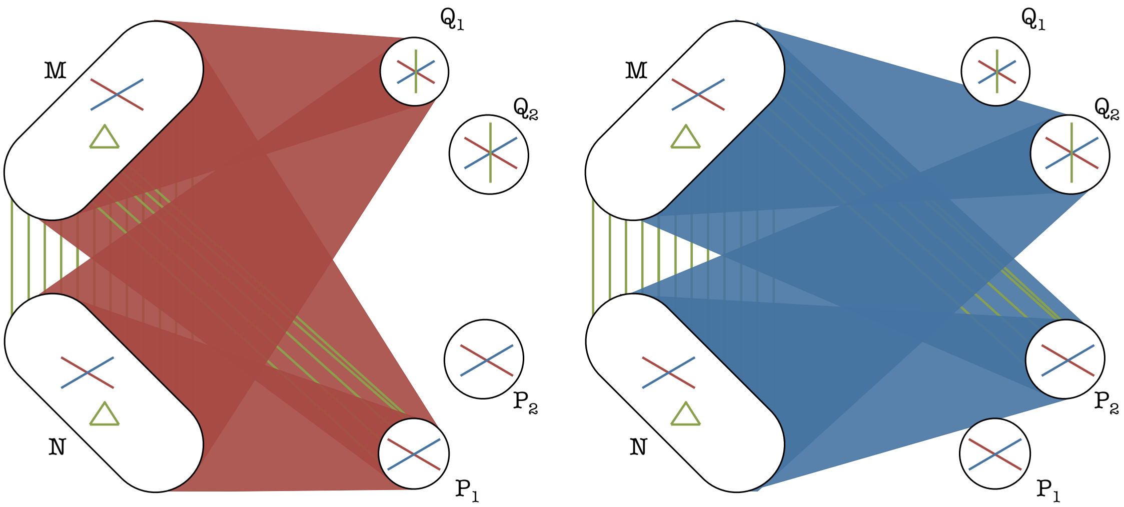


Figure 2.65: Colouring after Claim 2.8.6.

We then have

$$\left.\begin{array}{ll} |N_1| \geq (\frac{1}{2}\alpha_1 - 54\eta^{1/2})k, & |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 67\eta^{1/2})k, \\ |N_2| \geq (\frac{1}{2}\alpha_2 - 58\eta^{1/2})k, & |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 42\eta^{1/2})k. \end{array}\right\} \quad (2.56)$$

Recalling (E7), having discarded at most $86\eta^{1/2}k$ vertices from N and at most $25\eta^{1/2}k$ vertices from M, we have

$$|M| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 31\eta^{1/2}\right)k, \quad |N| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 93\eta^{1/2}\right)k.$$

Thus, provided that $\eta \leq (\alpha_2/600)^2$, we have

$$|N| \geq |P_1 \cup Q_1| + 400\eta^{1/2}k, \qquad |N| \geq |P_2 \cup Q_2| + 400\eta^{1/2}k.$$

Therefore, if there existed either a red matching on $136\eta^{1/2}k$ vertices or a blue matching on $86\eta^{1/2}k$ vertices in $G[N] \cup G[N, M]$, then these could be used together with edges from $G[N, P \cup Q]$ to obtain a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. Thus, after discarding at most $111\eta^{1/2}k$ vertices from M and $333\eta^{1/2}k$ vertices from N, we may assume that all edges present in $G[N] \cup G[M, N]$ are coloured exclusively green.

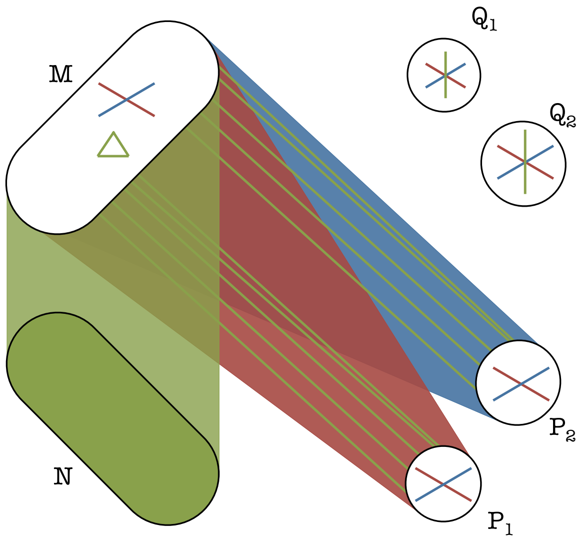


Figure 2.66: The green graph after considering $G[N] \cup G[M, N]$.

We then have

$$|M| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 142\eta^{1/2}\right)k,$$
$$|N| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 426\eta^{1/2}\right)k.$$

Now, suppose there exists a green matching G_{MP} on $1146\eta^{1/2}k$ vertices in $G[M, P]$. Letting $\widetilde{N}$ be any subset of $290\eta^{1/2}k$, by Lemma 2.6.1, provided $k \geq 1/\eta^2$, there exists a green connected-matching G_N on at least $289\eta^{1/2}k$ vertices in $G[\widetilde{N}]$. We then have $|M \backslash V(G_{MP})|, |N \backslash V(G_N)| \geq (\frac{1}{2}\alpha_3 - 716\eta^{1/2})k$ and, thus, by Lemma 2.6.14, we have a green connected-matching G_{MN} on at least $(\alpha_3 - 1434\eta^{1/2})k$ vertices in $G[M, N]$, which shares no vertices with G_{MP} or G_N. Then, since all edges present in $G[M, N]$ are coloured green, together, G_{MP}, G_{MN} and G_N form a green connected-matching on at least $\alpha_3 k$ vertices which, since all edges in $G[N]$ are green, is odd. Thus, after discarding at most $574\eta^{1/2}k$ vertices from each of M and P, we may assume that there are no green edges in $G[M, P \cup Q]$. In particular, since earlier we found that there could be no blue edges in $G[M, P_1]$ and no red edges in $G[M, P_2]$, we now know that all edges present in $G[M, P_1]$ are coloured exclusively red and all edges present in $G[M, P_2]$ are coloured exclusively blue.

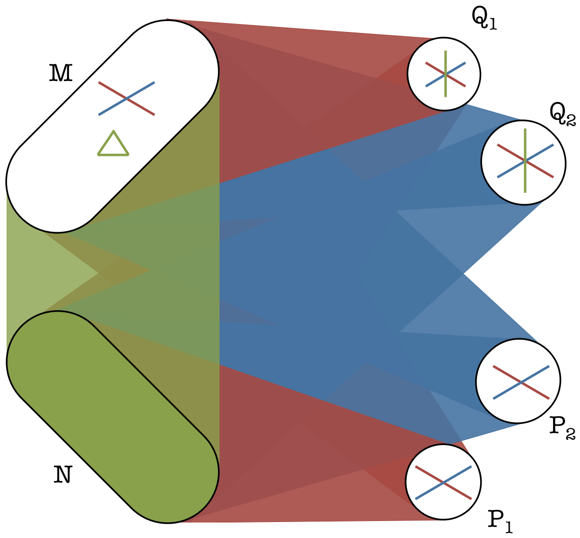


Figure 2.67: Colouring of $G[M, P]$.

In summary, having discarded these vertices, we may assume that all edges present in $G[M\cup N, P_1\cup Q_1]$ are coloured exclusively red, that all edges present in $G[M\cup N, P_2\cup Q_2]$ are coloured exclusively blue, that all edges in $G[N] \cup G[M, N]$ are coloured exclusively green and that we have

$$|M| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 716\eta^{1/2}\right)k, \qquad |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 642\eta^{1/2})k,$$
$$|N| \geq \left(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 426\eta^{1/2}\right)k, \qquad |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_1 - 616\eta^{1/2})k.$$

Notice that, provided that $\eta \leq (\alpha_2/5000)^2$, we have $|M| \geq |P_1 \cup Q_1| + 2200\eta^{1/2}k$. Thus, there cannot exist in $G[M]$ a red matching on $1286\eta^{1/2}k$ vertices or a blue matching on $1234\eta^{1/2}k$ vertices. Therefore, after discarding at most $2520\eta^{1/2}k$ vertices from M, we may assume that all edges present in $G[M]$ are coloured exclusively green and that

$$|M \cup N| \geq (\alpha_3 - 3662\eta^{1/2})k.$$

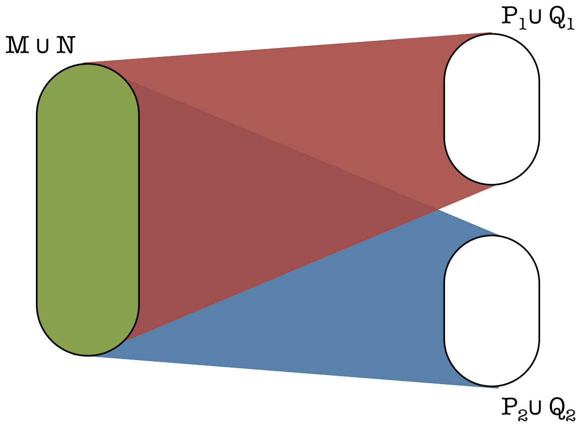


Figure 2.68: Final colouring in Case E.iii.b.iv.a.

In summary, we know that all edges present in $G[M \cup N, P_1 \cup Q_1]$ are coloured exclusively red, all edges present $G[M \cup N, P_2 \cup Q_2]$ are coloured exclusively blue and all edges in $G[M \cup N]$ are coloured exclusively green.

Thus, we have found, as a subgraph of G, a graph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 700\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 700\eta^{1/2})k, (\alpha_3 - 4000\eta^{1/2})k, 4\eta^4 k\right),$$

thus completing Case E.iii.b.iv.a.

Case E.iii.b.iv.b: $G[M, N \cup P \cup Q_2]$ contains a blue edge.

In this case, following similar steps as in Case E.iii.b.iv.a will result in either a red connected-matching on at least $\alpha_1 k$ vertices, a blue connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or a subgraph in

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 700\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 700\eta^{1/2})k, (\alpha_3 - 4000\eta^{1/2})k, 2\eta^4\right),$$

thus completing Case E.iii.b.iv.a.

Case E.iii.b.iv.c: $G[M, N \cup P \cup Q_1]$ contains no red edges.
and $G[M, N \cup P \cup Q_2]$ contains no blue edges.

Recall that we that we have a decomposition $V(G)$ into $M \cup N \cup P \cup Q$ and a refinement into $M \cup N_1 \cup N_2 \cup P_1 \cup P_2 \cup Q_1 \cup Q_2$, such that

(E3) there are no green edges in $G[N,P]$, $G[M,Q]$, $G[N,Q]$, $G[P,Q]$ or $G[P]$;

(E6a) $G[M]$, $G[N]$ and $G[P]$ each have a single red and a single blue effective-component;

(E6b) $G[N_1, P_1 \cup Q_1]$ contains a red matching utilising every vertex in $G[N_1 \cup P_1 \cup Q_1]$;

(E6c) all edges present in $G[N_2, P_2 \cup Q_2]$ are coloured exclusively blue.

This decomposition also satisfies

$$\left.\begin{array}{ll} |M| \geq \left(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 6\eta^{1/2}\right)k, & |N_1| = |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 8\eta^{1/2})k, \\ |N| \geq \left(\max\{\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \frac{1}{2}\alpha_3\} - 7\eta^{1/2}\right)k, & |N_2|, |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_2 - 8\eta^{1/2})k. \end{array}\right\} \quad \text{(E7)}$$

$$|P| \geq 90\eta^{1/2}k, \qquad \max\{|Q_1|, |Q_2|\} \geq 45\eta^{1/2}k. \qquad \text{(E8)}$$

Additionally, in this case, we may assume that

(E9a) all edges present in $G[Q_2, M]$ are coloured exclusively red;

(E9b) all edges present in $G[Q_1, M]$ are coloured exclusively blue; and

(E9c) all edge present in $G[M, P_1 \cup P_2] \cup G[M, N]$ are green.

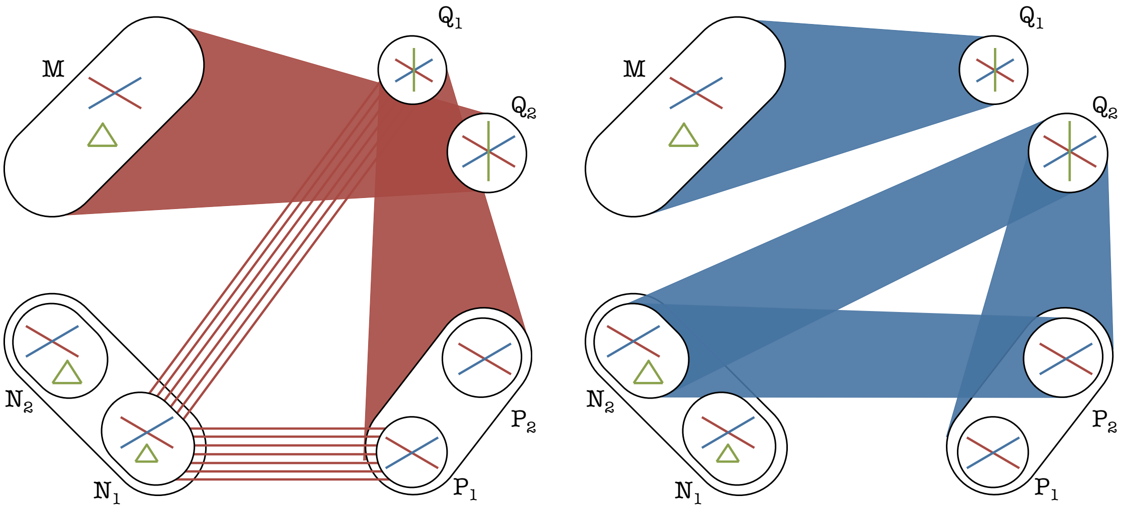


Figure 2.69: Initial red and blue graphs in Case E.iii.b.iv.c.

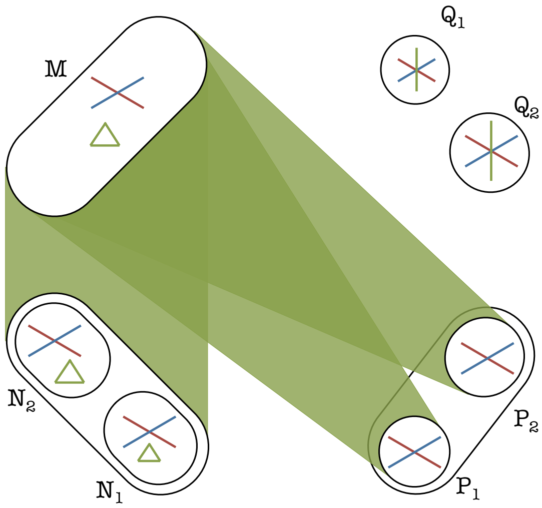


Figure 2.70: Initial green graph in Case E.iii.b.iv.c.

We begin by proving the following claim which concerns the structure of the red and blue graphs:

Claim 2.8.7.a. *If $|Q_1| \leq 38\eta^{1/2}k$, we may discard at most $71\eta^{1/2}k$ vertices from N, at most $12\eta^{1/2}k$ vertices from P_1, at most $59\eta^{1/2}k$ vertices from P_2 and at most $38\eta^{1/2}k$ vertices from Q_1 so that, in what remains, all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.*

Claim 2.8.6.b. *If $|Q_1| \geq 38\eta^{1/2}k$, we may discard at most $57\eta^{1/2}k$ vertices from N, at most $19\eta^{1/2}k$ vertices from P_1, at most $38\eta^{1/2}k$ vertices from P_2 and at most $30\eta^{1/2}k$ vertices from Q_1 so that, in what remains, all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.*

Proof. (a) Observe that, since $|Q_1| \leq 38\eta^{1/2}k < 45\eta^{1/2}k$, by (E8), we have $|Q_2| \geq 45\eta^{1/2}k$. Considering the red graph, we show that all edges present in $G[N \cup P, Q_2]$ are blue as follows: Given (E9a), since $|M|, |Q_2| \geq 45\eta^{1/2}k$, by Lemma 2.6.14, there exists a red connected-matching R_S on at least $88\eta^{1/2}k$ vertices in $G[M, Q_2]$. Then, since $|N| = |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 8\eta^{1/2})k$, recalling (E6a), if $G[N \cup P, Q_2]$ contains a red edge, then $R \cup R_S$ forms a red connected-matching on at least $\alpha_1 k$ vertices. Thus, recalling (E3), all edges present in $G[N \cup P, Q_2]$ are coloured exclusively blue (see Figure 2.71).

Now, suppose there exists a blue matching B_T on $24\eta^{1/2}k$ vertices in $G[N, P_1]$ (see Figure 2.72). Then, since $|Q_2| \geq 45\eta^{1/2}k$, there exist subsets $\widetilde{N}_2 \subseteq N_2 \backslash V(B_T)$ and $\widetilde{Q}_2 \subseteq Q_2$ such that $|\widetilde{N}_2| = |P_2 \cup \widetilde{Q}_2| \geq (\frac{1}{2}\alpha_2 - 20\eta^{1/2})k$ and $|Q_2 \backslash \widetilde{Q}_2| \geq 11\eta^{1/2}k$.

Then, by Lemma 2.6.14, there exist blue connected-matchings B_U on at least $(\alpha_2 - 42\eta^{1/2})k$ vertices in $G[N_2\backslash V(B_T), P_2 \cup \widetilde{Q}_2]$ and B_V on at least $20\eta^{1/2}k$ vertices in $G[N_1\backslash V(B_T), Q_2\backslash\widetilde{Q}_2]$. Since all edges present in $G[N, Q_2]$ are coloured blue, B_T, B_U and B_V belong to the same blue component and, thus, together, form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, there can not exist such a matching as B_T and, after discarding at most $12\eta^{1/2}k$ vertices from each of N and P_1, we may assume that all edges present in $G[N, P_1]$ are coloured exclusively red.

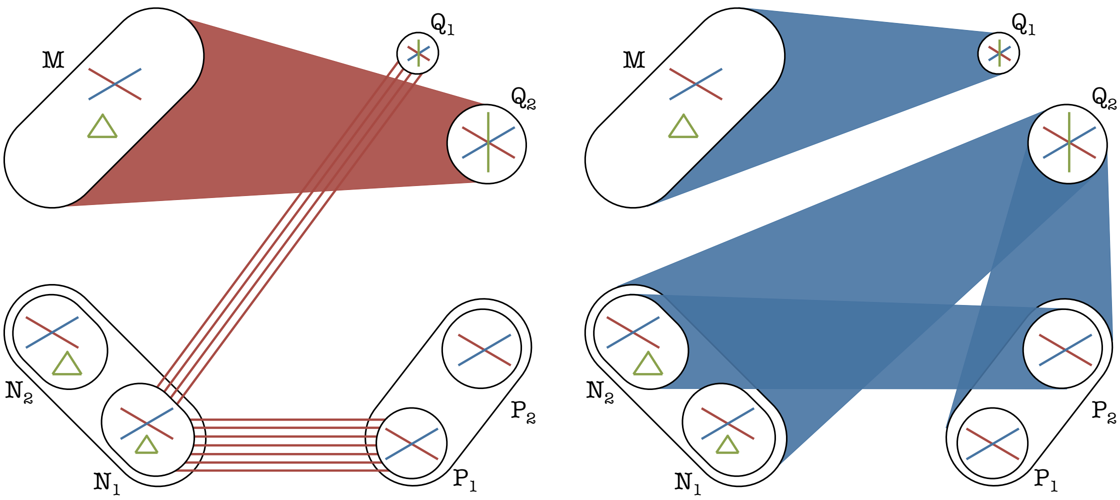


Figure 2.71: Red and blue subgraphs of G, provided $|Q_1| \leq 38\eta^{1/2}k$.

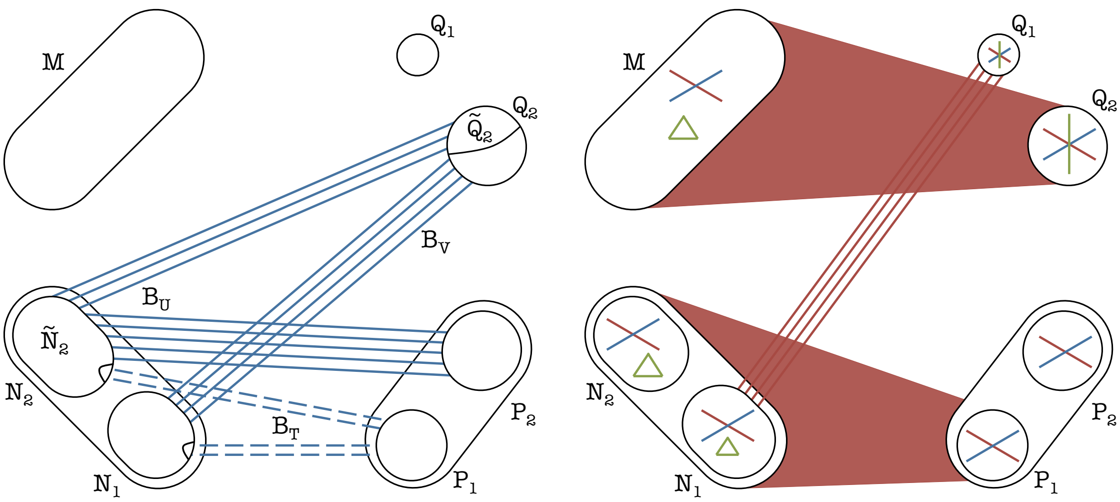


Figure 2.72: B_T and the resultant colouring of the edges of $G[N, P_2]$.

Having discarded these vertices, we have

$$\left.\begin{aligned} |M| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\}-6\eta^{1/2}\right)k, & |N_1|, |P_1\cup Q_1| &\geq (\tfrac{1}{2}\alpha_1-20\eta^{1/2})k,\\ |N| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\}-19\eta^{1/2}\right)k, & |P_2\cup Q_2| &\geq (\tfrac{1}{2}\alpha_2-8\eta^{1/2})k,\\ & & |N_2| &\geq (\tfrac{1}{2}\alpha_2-20\eta^{1/2})k. \end{aligned}\right\} \quad (2.57)$$

Since $|Q_1| \leq 38\eta^{1/2}k$ and $\eta < (\alpha_1/1000)^2$, by (E4b) and (2.57), we have

$$|N| \geq |P_1| + 200\eta^{1/2}k, \qquad |P_1| \geq (\tfrac{1}{2}\alpha_1 - 58\eta^{1/2})k.$$

Thus, if there existed a red matching R_S on $118\eta^{1/2}k$ vertices in $G[N, P_2]$, we could obtain a red connected-matching on at least $\alpha_1 k$. Indeed, since $|N\backslash V(R_S)|, |P_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k$, by Lemma 2.6.14, there exists a red connected-matching R_L on $118\eta^{1/2}k$ vertices in $G[N\backslash V(R_S), P_1]$. Then, since $G[N]$ has a single red effective-component, R_L and R_S belong to the same red component of G and, therefore, together, form a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $59\eta^{1/2}k$ vertices from each of N and P_2, we may assume that all edges present in $G[P_2, N]$ are coloured exclusively blue.

Finally, we discard all vertices from Q_1. Having done so, we have $P_1 = P_1 \cup Q_1$ and know that all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue.

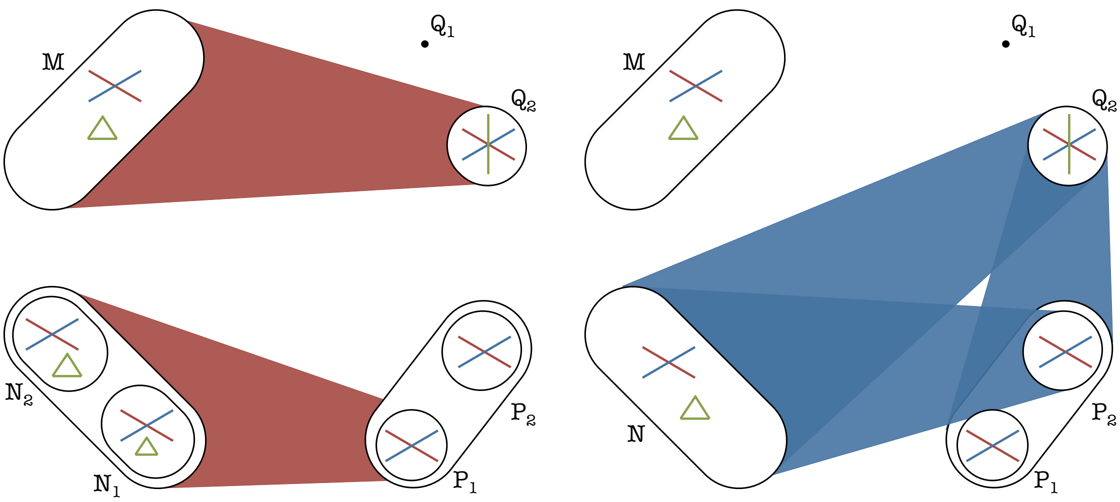


Figure 2.73: The final red and blue graphs in Claim 2.8.7.a.

We then have

$$\left.\begin{aligned} |M| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-6\eta^{1/2}\right)k, & |P_1\cup Q_1| &\geq (\tfrac{1}{2}\alpha_1-58\eta^{1/2})k,\\ |N| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-78\eta^{1/2}\right)k, & |P_2\cup Q_2| &\geq (\tfrac{1}{2}\alpha_2-67\eta^{1/2})k. \end{aligned}\right\} \quad (2.58)$$

thus completing the proof of Claim 2.8.7.a.

(b) Suppose G is coloured as shown in Figures 2.69–2.70 and that we have $|Q_1|\geq 38\eta^{1/2}k$. Considering the blue graph, we are able to show that all edges present in $G[N\cup P,Q_1]$ are red as follows: Given (E9b), since $|M|,|Q_1|\geq 38\eta^{1/2}k$, by Lemma 2.6.14, there exists a blue connected-matching B_S on at least $74\eta^{1/2}k$ vertices in $G[M,Q_1]$. Then, since $|N_2|,|P_2\cup Q_2|\geq(\frac{1}{2}\alpha_2-8\eta^{1/2})k$, if $G[N\cup P,Q_1]$ contains a blue edge, we can obtain a blue connected-matching on at least $\alpha_2 k$ vertices utilising edges from B_S and from $G[N_2,P_2\cup Q_2]$. Thus, recalling (E3), all edges present in $G[N\cup P,Q_1]$ are coloured exclusively red.

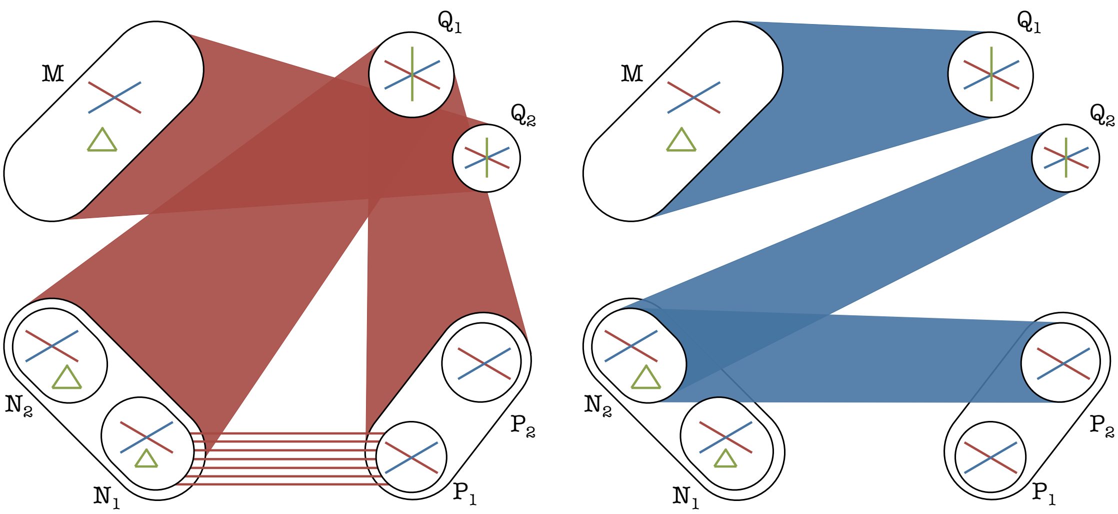


Figure 2.74: Red and blue subgraphs of G when $|Q_1|\geq 38\eta^{1/2}k$.

Now, let $N_{11}=N_1\cap V(R)$ and $N_{12}=N_1\backslash V(R)$. Then, since $|Q_1|\geq 38\eta^{1/2}k$ and $|N_{11}|=|P_1|$, we have $|N_{12}|\geq 38\eta^{1/2}k$. Suppose there exists a red matching R_S on $18\eta^{1/2}k$ vertices in $G[N_{12},P_2]$. Then, since $|N_2|,|Q_1|\geq(\frac{1}{2}\alpha_1-8\eta^{1/2})k-|P_1|$ and all edges in $G[N,Q_1]$ are coloured red, by Lemma 2.6.14, there exists a red connected-matching R_T on at least $(\alpha_1-18\eta^{1/2})k-2|P_1|$ vertices in $G[N_2,Q_1]$. Observe that, since $|N_{11}|=|P_1|$, $R_U=R\cap G[N_{11},P_1]$ is a red matching on $2|P_1|$ vertices. Thus, since $G[N]$ has a single red effective-component, $R_S\cup R_T\cup R_U$ forms a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $9\eta^{1/2}k$ vertices from each of N_{12}

and P_2, we may assume that all edges in $G[N_{12}, P_2]$ are coloured exclusively blue. We then have $|N_{12}| \geq 28\eta^{1/2}k$ and $|P_2 \cup Q_2| \geq (\alpha_2 k - 17\eta^{1/2})k$.

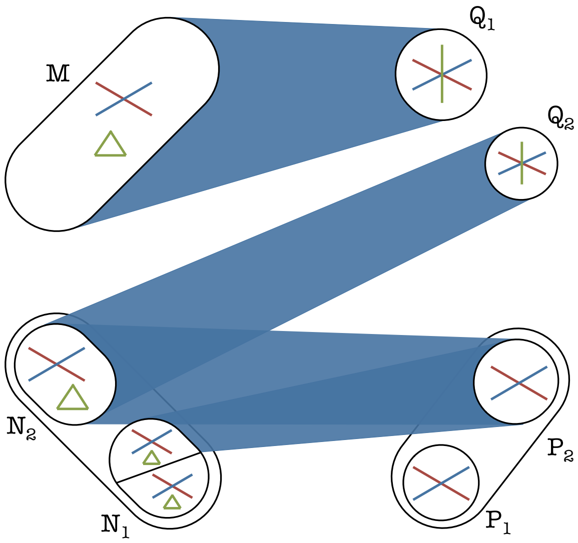


Figure 2.75: Colouring of $G[N_{12}, P_2]$.

Then, suppose there exists a blue matching B_S on $38\eta^{1/2}k$ vertices in $G[N, P_1]$. Then

$$|(N_2 \cup N_{12})\backslash V(B_S)| \geq (\tfrac{1}{2}\alpha_2 - 8\eta^{1/2})k + 28\eta^{1/2}k - 19\eta^{1/2}k$$
$$\geq (\tfrac{1}{2}\alpha_2 + \eta^{1/2})k \geq |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 17\eta^{1/2})k.$$

Thus, by Lemma 2.6.14, there exists a blue connected-matching on at least $(\alpha_2 - 38\eta^{1/2})k$ vertices belonging to the same component as B_S. Thus, together, these matchings form a blue connected-matching on at least $\alpha_2 k$ vertices. Therefore, after discarding at most $19\eta^{1/2}k$ vertices from each of N and P_1, we may assume that all edges present in $G[N, P_1]$ are coloured exclusively red and that $|P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 27\eta^{1/2})k$.

Penultimately, suppose there exists a red matching R_V on $58\eta^{1/2}k$ vertices in $G[N_{11}, P_2]$. Then, since $|N\backslash V(R_V)|, |P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 27\eta^{1/2})k$, by Lemma 2.6.14, there exists a red connected-matching R_W on at least $(\alpha_1 - 56\eta^{1/2})k$ vertices in $G[N\backslash V(R_V), P_1 \cup P_2]$, which together with R_V gives a red connected-matching on at least $\alpha_1 k$ vertices. Thus, after discarding at most $29\eta^{1/2}k$ vertices from each of N_{11} and P_2, we may assume that all edges in $G[N, P_2]$ are coloured exclusively blue.

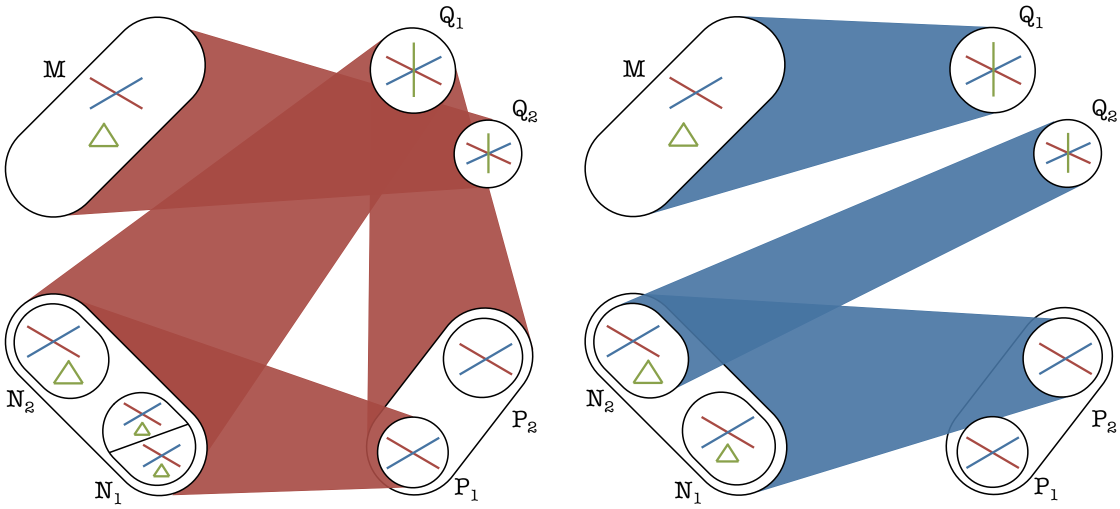


Figure 2.76: Colouring of $G[N, P]$.

Finally, observe that, if $|Q_2| \geq 30\eta^{1/2}k$, there can be no red edges in $G[N \cup P, Q_2]$. Indeed, in that case, since $|M|, |Q_2| \geq 30\eta^{1/2}k$ and $|N|, |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 27\eta^{1/2})k$, there exist red connected-matchings R_W in $G[M, Q_2]$ and R_X in $G[P_1 \cup Q_1, N]$ which belong to the same component and together span at least $\alpha_1 k$ vertices. Alternatively, if $|Q_2| \leq 30\eta^{1/2}k$, we can discard every vertex in Q_2, rendering the graph $G[N, Q_2]$ trivial. Thus, in either case, all edges in $G[N, Q_2]$ are coloured exclusively blue.

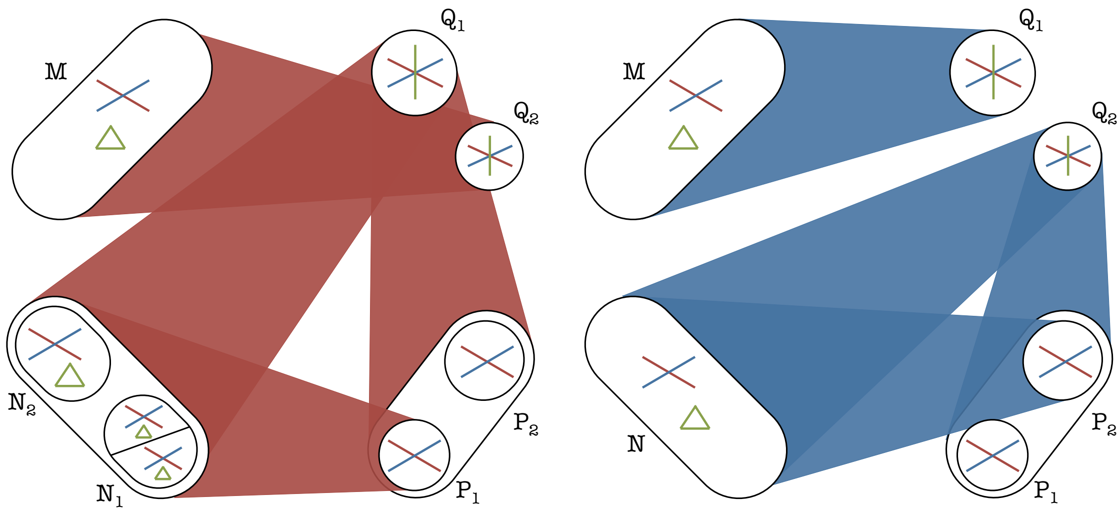


Figure 2.77: The final red and blue graphs in Claim 2.8.7.b. if $|Q_2| \geq 30\eta^{1/2}k$.

In summary, having discarded some vertices, we may assume that all edges present in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges present in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue and that

$$\left.\begin{aligned} |M| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-6\eta^{1/2}\right)k, \quad |P_1\cup Q_1|\geq (\tfrac{1}{2}\alpha_1-27\eta^{1/2})k,\\ |N| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-64\eta^{1/2}\right)k, \quad |P_2\cup Q_2|\geq (\tfrac{1}{2}\alpha_2-76\eta^{1/2})k, \end{aligned}\right\} \quad (2.59)$$

thus, completing the proof of Claim 2.8.7.b. □

Having proved the claim, we know that all edges in $G[N, P_1 \cup Q_1]$ are coloured exclusively red and all edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively blue. Combining (2.59) and (2.58), we have

$$\left.\begin{aligned} |M| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-6\eta^{1/2}\right)k, \quad |P_1\cup Q_1|\geq (\tfrac{1}{2}\alpha_1-58\eta^{1/2})k,\\ |N| &\geq \left(\max\{\tfrac{3}{4}\alpha_1+\tfrac{1}{4}\alpha_2,\tfrac{1}{2}\alpha_3\}-78\eta^{1/2}\right)k, \quad |P_2\cup Q_2|\geq (\tfrac{1}{2}\alpha_2-76\eta^{1/2})k. \end{aligned}\right\} \quad (2.60)$$

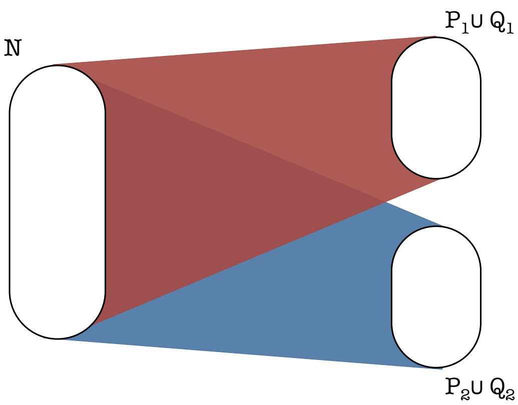


Figure 2.78: $G_1[N \cup P \cup Q] \cup G_2[N \cup P \cup Q]$ after Claim 2.8.7.

We now consider the green graph (see Figure 2.70). Recall that all edges present in $G[M, N \cup P]$ are coloured green. Now, suppose there exists a green matching G_S on $17\eta^{1/2}k$ vertices in $G[N]$. Since $|M| \geq |P| \geq 90\eta^{1/2}k$, by Lemma 2.6.14, there exists a green connected-matching G_T on $178\eta^{1/2}k$ vertices in $G[M, P]$. Then, by (2.60), we have

$$|M\backslash V(G_T)|, |N\backslash V(G_S)| \geq (\tfrac{1}{2}\alpha_3 - 96\eta^{1/2})k.$$

Thus, by Lemma 2.6.14, there exists a green connected-matching G_L on at least $(\alpha_3 - 194\eta^{1/2})k$ in $G[M\backslash V(G_T), N\backslash V(G_S)]$. Then, since all edges present in $G[M, N]$ are coloured exclusively green, G_S, G_T and G_L belong to the same green component of G

and, thus, together form an green odd connected-matching on at least $\alpha_3 k$ vertices. Therefore, after discarding at most $17\eta^{1/2}k$ vertices from N, we may assume that there are no green edges in $G[N]$ and that

$$|N| \geq \big(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 95\eta^{1/2}\big)k. \tag{2.61}$$

We continue to consider $G[N]$. Recall that we know that all edges in $G[N, P_1 \cup Q_1]$ are coloured exclusively red, that all edges in $G[N, P_2 \cup Q_2]$ are coloured exclusively red and that. From (2.60), we have

$$|P_1 \cup Q_1| \geq (\tfrac{1}{2}\alpha_1 - 58\eta^{1/2})k, \qquad\qquad |P_2 \cup Q_2| \geq (\tfrac{1}{2}\alpha_2 - 76\eta^{1/2})k.$$

Recalling (E4b) and (2.61), provided that $\eta \leq (\alpha_1/2000)^2$, we have

$$|N| \geq |P_1 \cup Q_1| + 300\eta^{1/2}k.$$

Then, suppose that there exists a red-matching R_A on at least $118\eta^{1/2}k$ vertices in $G[N]$. Then, since $|N \backslash V(R_A)| \geq |P_1 \cup Q_1| \geq (\frac{1}{2}\alpha_1 - 58\eta^{1/2})k$, by Lemma 2.6.14, there exists a red connected-matching R_B on at least $(\alpha_1 - 118\eta^{1/2})k$ vertices in $G[N\backslash V(R_A), P_1 \cup Q_1]$. Then, by (E6a), R_A and R_B belong to the same red component of G and thus, together, form a red connected-matching on at least $\alpha_1 k$ vertices.

Likewise, if there exists a blue-matching B_A on at least $154\eta^{1/2}k$ vertices in $G[N]$, then we can obtain a blue connected-matching on at least $\alpha_2 k$ vertices, as follows: Since $|N\backslash V(R_B)| \geq |P_2 \cup Q_2| \geq (\frac{1}{2}\alpha_1 - 76\eta^{1/2})k$, by Lemma 2.6.14, there exists a blue connected-matching B_B on at least $(\alpha_1 - 154\eta^{1/2})k$ vertices in $G[N\backslash V(B_A), P_2 \cup Q_2]$. Then, since all edges present in $G[N, Q_2]$ are coloured exclusively blue, B_A and B_B, together, form a blue connected-matching on at least $\alpha_2 k$ vertices.

Thus, after discarding at most a further $272\eta^{1/2}k$ vertices from N, we can assume that there are no edges of any colour in $G[N]$ and that

$$|N| \geq \big(\max\{\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2, \tfrac{1}{2}\alpha_3\} - 367\eta^{1/2}\big)k \geq 10\eta^{1/2}k.$$

This contradicts the fact that G is $4\eta^4 k$-almost-complete, thus completing Case E.iii.b.iv.c. and the proof of Theorem B.

2.9 Proof of the main result – Setup

For $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, we set $c = \max\{2\alpha_1 + \alpha_2, \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3\}$,

$$\eta = \frac{1}{2} \min \left\{ \eta_{B1}(\alpha_1, \alpha_2, \alpha_3), \eta_{B2}(\alpha_1, \alpha_2, \alpha_3), 10^{-50}, \left(\frac{\alpha_2}{2500}\right)^2, \left(\frac{\alpha_2}{200}\right)^{64} \right\}$$

and let k_0 be the smallest integer such that

$$k_0 \geq \max \left\{ \left(c - \tfrac{1}{2}\eta\right) k_{B1}(\alpha_1, \alpha_2, \alpha_3, \eta), \left(c - \tfrac{1}{2}\eta\right) k_{B2}(\alpha_1, \alpha_2, \alpha_3, \eta), \frac{72}{\eta} \right\}.$$

We let

$$N = \max \left\{ 2\langle\langle \alpha_1 n \rangle\rangle + \langle\langle \alpha_2 n \rangle\rangle - 3,\ \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle + \langle \alpha_3 n \rangle - 2 \right\},$$

for some integer n such that $N \geq K_{2.3.1}(\eta^4, k_0)$ and

$$n > \max\{n_{2.3.4}(2, 1, 0, \eta), n_{2.3.4}(\tfrac{1}{2}, \tfrac{1}{2}, 1, \eta)\}$$

and consider a three-colouring of $G = (V, E)$, the complete graph on N vertices.

In order to prove Theorem A, we must prove that G contains either a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices, a blue cycle on $\langle\langle \alpha_2 n \rangle\rangle$ vertices or a green cycle on $\langle \alpha_3 n \rangle$ vertices.

We will use G_1, G_2, G_3 to refer to the monochromatic spanning subgraphs of G. That is, G_1 (resp. G_2, G_3) has the same vertex set as G and includes as an edge any edge which in G is coloured red (resp. blue, green).

By Theorem 2.3.1, there exists an (η^4, G_1, G_2, G_3)-regular partition $\Pi = (V_0, V_1, \dots, V_K)$ for some K such that $k_0 \leq K \leq K_{2.3.1}(\eta^4, k_0)$. Given such a partition, we define the (η^4, η, Π)-reduced-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ on K vertices as in Definition 2.3.2. The result is a three-multicoloured graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with

$$\mathcal{V} = \{V_1, V_2, \dots, V_K\}, \qquad \mathcal{E} = \{V_iV_j : (V_i, V_j) \text{ is } (\eta^4, G_r)\text{-regular for } r = 1, 2, 3\},$$

such that a given edge V_iV_j of $\mathcal{G}$ is coloured with every colour for which there are at least $\eta|V_i||V_j|$ edges of that colour between V_i and V_j in G.

Note that, by scaling, we may assume that either $\alpha_2, \alpha_3 \leq \alpha_1 = 1$ or $\alpha_2 \leq \alpha_1 \leq 1 \leq \alpha_3 \leq 2$. Thus, since $K \geq k_0 \geq 72/\eta$, we may fix an integer k such that

$$(c-\eta)\,k \le K \le \left(c-\tfrac{1}{2}\eta\right)k. \tag{2.62}$$

and may assume that $k \le K \le 3k$, $n \le N \le 3n$.

Notice, also, that since the partition is η^4-regular, we have $|V_0| \le \eta^4 N$ and, for $1 \le i \le K$,

$$(1-\eta^4)\frac{N}{K} \le |V_i| \le \frac{N}{K}. \tag{2.63}$$

Applying Theorem B, we find that $\mathcal{G}$ contains at least one of the following:

(i) a red connected-matching on at least $\alpha_1 k$ vertices;

(ii) a blue connected-matching on at least $\alpha_2 k$ vertices;

(iii) a green odd connected-matching on at least $\alpha_3 k$ vertices;

(iv) two disjoint subgraphs H_1, H_2 from $\mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\begin{aligned}
\mathcal{H}_1 =& \mathcal{H}\left((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{red}, \text{blue}\right),\\
\mathcal{H}_2 =& \mathcal{H}\left((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{blue}, \text{red}\right);
\end{aligned}$$

(v) a subgraph from

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 14000\eta^{1/2})k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4 k\right);$$

(vi) a subgraph from $\mathcal{K}_1^* \cup \mathcal{K}_2^*$, where

$$\begin{aligned}
\mathcal{K}_1^* = \mathcal{K}^*\big(&(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k,\\
&(\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big),\\
\mathcal{K}_2^* = \mathcal{K}^*\big(&(\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\tfrac{3}{4}\alpha_3 - 140\eta^{1/2})k,\\
&100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big).
\end{aligned}$$

Furthermore,

(iv) occurs only if $\alpha_3 \le \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 + 14\eta^{1/2}$, with $H_1, H_2 \in \mathcal{H}_1$ unless $\alpha_2 \ge \alpha_1 - \eta^{1/8}$;

(v) and (vi) occur only if $\alpha_3 \ge \frac{3}{2}\alpha_1 + \frac{1}{2}\alpha_2 - 10\eta^{1/2}$.

Since $n > \max\{n_{2.3.4}(2,1,0,\eta), n_{2.3.4}(\frac{1}{2},\frac{1}{2},1,\eta)\}$ and $\eta < 10^{-20}$, in cases (i)–(iii), Theorem 2.3.4 gives a cycle of appropriate length, colour and parity to complete the proof.

Thus, we need only concern ourselves with cases (iv)–(vi). We divide the remainder of the proof into three parts, each corresponding to one of the possible coloured structures.

2.10 Proof of the main result – Part I – Case (iv)

Suppose that $\mathcal{G}$ contains two disjoint subgraphs H_1, H_2 from $\mathcal{H}_1 \cup \mathcal{H}_2$, where

$$\mathcal{H}_1 = \Big((\alpha_1 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{red}, \text{blue}\Big),$$
$$\mathcal{H}_2 = \Big((\alpha_2 - 2\eta^{1/16})k, (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k, 3\eta^4 k, \eta^{1/16}, \text{blue}, \text{red}\Big).$$

In this case, additionally, from Theorem B we may assume that

$$\alpha_3 \le \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 + 14\eta^{1/2} \le 2\alpha_1 + 14\eta^{1/2}. \tag{2.64}$$

We divide the proof that follows into three sub-parts depending on the colouring of the subgraphs H_1 and H_2, that is, whether H_1 and H_2 belong to $\mathcal{H}_1$ or $\mathcal{H}_2$:

Part I.A: $H_1, H_2 \in \mathcal{H}_1$.

In this case, defining $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ to be the monochromatic spanning subgraphs of the reduced-graph $\mathcal{G}$, the vertex set $\mathcal{V}$ of $\mathcal{G}$ has a partition into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ with

$$\left.\begin{aligned} (\alpha_1 - 2\eta^{1/16})k \le |\mathcal{X}_1| = |\mathcal{Y}_1| = p \le \alpha_1 k, \\ (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k \le |\mathcal{X}_2| = |\mathcal{Y}_2| = q \le \tfrac{1}{2}\alpha_2 k, \end{aligned}\right\} \tag{2.65}$$

such that

(HA1) $\mathcal{G}_1[\mathcal{X}_1], \mathcal{G}_1[\mathcal{Y}_1]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HA2) $\mathcal{G}_2[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}_2[\mathcal{Y}_1, \mathcal{Y}_2]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HA3) $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2]$ is $3\eta^4 k$-almost-complete (and thus connected);

(HA4) $\mathcal{G}[\mathcal{X}_1], \mathcal{G}[\mathcal{Y}_1]$ are each $2\eta^{1/16}$-sparse in blue and contain no green edges; and

(HA5) $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2]$ are each $2\eta^{1/16}$-sparse in red and contain no green edges.

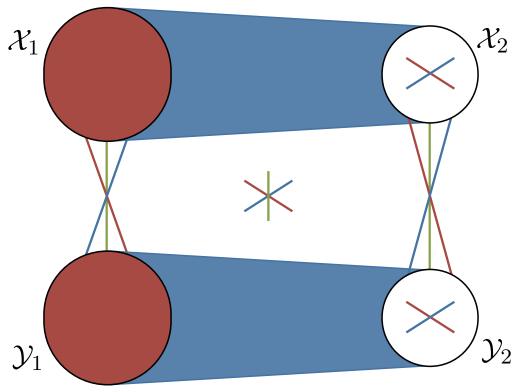


Figure 2.79: Coloured structure of the reduced-graph in Part I.A.

The remainder of this section focuses on showing that the original graph must have a similar structure which can then be exploited in order to force a cycle of appropriate length, colour and parity.

By definition, each vertex V_i of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents a class of vertices of $G = (V, E)$. In what follows, we will refer to these classes as *clusters* (of vertices of G). Additionally, recall, from 2.63, that

$$(1-\eta^4)\frac{N}{K} \leq |V_i| \leq \frac{N}{K}.$$

Since $n > \max\{n_{2.3.4}(2,1,0,\eta), n_{2.3.4}(\frac{1}{2},\frac{1}{2},1,\eta)\}$, we can (as in the proof of Theorem 2.3.4) prove that

$$|V_i| \geq \left(1+\frac{\eta}{24}\right)\frac{n}{k} > \frac{n}{k}.$$

We partition the vertices of G into sets X_1, X_2, Y_1, Y_2 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{W}$. Then, X_1, Y_1 each contain p clusters of vertices and X_2, Y_2 each contain q clusters and, recalling (2.65), we have

$$\left.\begin{aligned} |X_1|, |Y_1| = p|V_1| \geq (\alpha_1 - 2\eta^{1/16})n, \\ |X_2|, |Y_2| = q|V_1| \geq (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})n. \end{aligned}\right\} \quad (2.66)$$

In what follows, we will *remove* vertices from $X_1 \cup X_2 \cup Y_1 \cup Y_2$ by moving them into W in order to show that, in what remains, $G[X_1 \cup X_2 \cup Y_1 \cup Y_2]$ has a particular coloured structure. We begin by proving the below claim which essentially tells us that G has similar coloured structure to $\mathcal{G}$:

Claim 2.10.1. *We can remove at most $9\eta^{1/32}n$ vertices from each of X_1 and Y_1 and at most $4\eta^{1/32}n$ vertices from each of X_2 and Y_2 so that the following pair of conditions hold.*

(HA6) $G_1[X_1]$ and $G_1[Y_1]$ are each $8\eta^{1/32}n$-almost-complete; and

(HA7) $G_2[X_1, X_2]$ and $G_2[Y_1, Y_2]$ are each $4\eta^{1/32}n$-almost-complete.

Proof. Consider the complete three-coloured graph $G[X_1]$ and recall, from (HA1), (HA3) and (HA4), that $\mathcal{G}[\mathcal{X}_1]$ contains only red and blue edges and is $3\eta^4k$-almost-complete. Given the structure of $\mathcal{G}$, we can bound the number of non-red edges in $G[X_1]$ as follows:

Since regularity provides no indication as to the colours of the edges contained within each cluster, these could potentially all be non-red. There are p clusters in X_1, each with at most N/K vertices. Thus, there are at most

$$p\binom{N/K}{2}$$

non-red edges in X_1 within the clusters.

Now, consider a pair of clusters (U_1, U_2) in X_1. If (U_1, U_2) is not η^4-regular, then we can only trivially bound the number of non-red edges in $G[U_1, U_2]$ by $|U_1||U_2| \leq (N/K)^2$. However, by (HA3), there are at most $3\eta^4|\mathcal{X}_1|k$ such pairs in $\mathcal{G}$. Thus, we can bound the number of non-red edges coming from non-regular pairs by

$$3\eta^4 pk \left(\frac{N}{K}\right)^2.$$

If the pair is regular and U_1 and U_2 are joined by a blue edge in the reduced-graph, then, again, we can only trivially bound the number of non-red edges in $G[U_1, U_2]$ by $(N/K)^2$. However, by (HA4), $\mathcal{G}_2[X_1]$ is $2\eta^{1/16}$-sparse, so there are at most $2\eta^{1/16}\binom{p}{2}$ blue edges in $\mathcal{G}[X_1]$ and, thus, there are at most

$$2\eta^{1/16}\binom{p}{2}\left(\frac{N}{K}\right)^2$$

non-red edges in $G[X_1]$ corresponding to such pairs of clusters.

Finally, if the pair is regular and U_1 and U_2 are not joined by a blue edge in the reduced-graph, then the blue density of the pair is at most η (since, if the density were higher, they would be joined by a blue edge). Likewise, the green density of the pair is at most η (since there are no green edges in $\mathcal{G}[X_1]$). Thus, there are at most

$$2\eta\binom{p}{2}\left(\frac{N}{K}\right)^2$$

non-red edges in $G[X_1]$ corresponding to such pairs of clusters.

Summing the four possibilities above gives an upper bound of

$$p\binom{N/K}{2} + 3\eta^4 pk\left(\frac{N}{K}\right)^2 + 2\eta^{1/16}\binom{p}{2}\left(\frac{N}{K}\right)^2 + 2\eta\binom{p}{2}\left(\frac{N}{K}\right)^2$$

non-red edges in $G[X_1]$.

Since $K \geq k, \eta^{-1}$, $N \leq 3n$ and $p \leq \alpha_1 k \leq k$, we obtain

$$e(G_2[X_1]) + e(G_3[X_1]) \leq [4.5\eta + 27\eta^4 + 18\eta^{1/16} + 9\eta]n^2 \leq 24\eta^{1/16}n^2.$$

Since $G[X_1]$ is complete and contains at most $24\eta^{1/16}n^2$ non-red edges, there are at most $6\eta^{1/32}n$ vertices with red degree at most $|X_1| - 8\eta^{1/32}n$. Removing these vertices from X_1, that is, re-assigning these vertices to W gives a new X_1 such that every vertex in $G[X_1]$ has red degree at least $|X_1| - 8\eta^{1/32}n$. The same argument works for $G[Y_1]$, thus completing the proof of (HA6).

Now, consider $G[X_1, X_2]$. In a similar way to the above, we can bound the number of non-blue edges in $G[X_1, X_2]$ by

$$3\eta^4 pk\left(\frac{N}{K}\right)^2 + 2\eta^{1/16}pq\left(\frac{N}{K}\right)^2 + 2\eta pq\left(\frac{N}{K}\right)^2.$$

Where the first term bounds the number of non-blue edges between non-regular pairs, the second bounds the number of non-blue edges between pairs of clusters that are joined by a red edge in the reduced-graph and the second bounds the number of non-blue edges between pairs of clusters that are not joined by a red edge in the reduced-graph.

Since $K \geq k$, $N \leq 3n$, $p \leq \alpha_1 k \leq k$ and $q \leq \frac{1}{2}\alpha_2 k \leq \frac{1}{2}k$, we obtain

$$e(G_1[X_1, X_2]) + e(G_3[X_1, X_2]) \leq 16\eta^{1/16}n^2.$$

Since $G[X_1, X_2]$ is complete and contains at most $16\eta^{1/16}n^2$ non-blue edges, there are at most $4\eta^{1/32}n$ vertices in X_1 with blue degree to X_2 at most $|X_2| - 4\eta^{1/32}n$ and at most $4\eta^{1/32}n$ vertices in X_2 with blue degree to X_1 at most $|X_1| - 4\eta^{1/32}n$. Removing these vertices results in every vertex in X_1 having degree in $G_2[X_1, X_2]$ at least $|X_2| - 4\eta^{1/32}n$ and every vertex in X_2 having degree in $G_2[X_1, X_2]$ at least $|X_1| - 4\eta^{1/32}n$.

We repeat the above for $G[Y_1, Y_2]$, removing vertices such that every (remaining) vertex in Y_1 has degree in $G_2[Y_1, Y_2]$ at least $|Y_2| - 4\eta^{1/32}n$ and every (remaining) vertex in Y_2 has degree in $G_2[Y_1, Y_2]$ at least $|Y_1| - 4\eta^{1/32}n$, thus completing the proof of (HA7). □

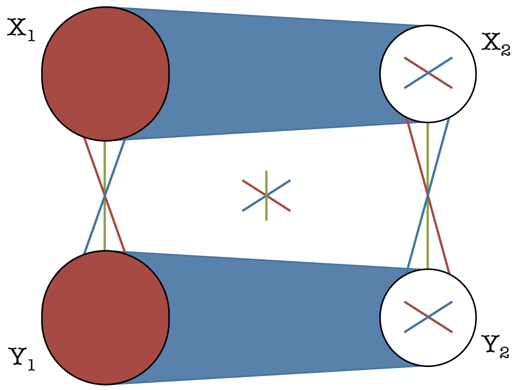


Figure 2.80: Colouring of G after Claim 2.10.1.

Having discarded some vertices, recalling (2.66), we have

$$|X_1|, |Y_1| \geq (\alpha_1 - 10\eta^{1/32})n, \qquad |X_2|, |Y_2| \geq (\tfrac{1}{2}\alpha_2 - 5\eta^{1/32})n. \tag{2.67}$$

We now proceed to the end-game: Notice that, given the colouring found thus far, $G[X_1]$ and $G[Y_1]$ each contain a red Hamiltonian cycle. Similarly, $G[X_1, X_2]$ and $G[Y_1, Y_2]$ each contain a blue cycle of length twice the size of the smaller part. Essentially, we will show that it is possible to augment each of X_1, X_2, Y_1, Y_2 with vertices from W while maintaining these properties and, then, considering the sizes of each part, show that there must, in fact, be a cycle of appropriate length, colour and parity to complete the proof.

The following pair of claims tell us that after removing a small number of vertices from X_1 and Y_1, we may assume that all edges in $G[X_1, Y_1]$ are coloured green:

Claim 2.10.2.a. *If there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured red, then G contains a red cycle of length exactly $\langle\langle \alpha_1 n \rangle\rangle$.*

Claim 2.10.2.b. *If there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured blue, then G contains a blue cycle of length exactly $\langle\langle \alpha_2 n \rangle\rangle$.*

Proof. (a) Suppose there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that the edges x_1y_1 and x_2y_2 are coloured red. Then, let $\widetilde{X}_1$ be any set of $\frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ vertices in X_1 such that $x_1, x_2 \in \widetilde{X}_1$.

By (HA6), every vertex in $\widetilde{X}_1$ has degree at least $|\widetilde{X}_1| - 8\eta^{1/32}n$ in $G_1[\widetilde{X}_1]$. Since $\eta \leq (\alpha_1/100)^{32}$, we have $|\widetilde{X}_1| - 8\eta^{1/32}n \geq \frac{1}{2}|\widetilde{X}_1| + 2$. So, by Corollary 2.6.3, there exists a Hamiltonian path in $G_1[\widetilde{X}_1]$ between x_1, x_2, that is, there exists a red path between x_1 and x_2 in $G[X_1]$ on exactly $\frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ vertices.

Likewise, given any two vertices y_1, y_2 in Y_1, there exists a red path between y_1 and y_2 in $G[Y_1]$ on exactly $\frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ vertices. Combining the edges x_1y_1 and x_2y_2 with the red paths gives a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices.

(b) Suppose there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured blue. Then, let $\widetilde{X}_2$ be any set of

$$\ell_1 = \left\lfloor \frac{\langle\langle \alpha_2 n \rangle\rangle - 2}{4} \right\rfloor \geq 4\eta^{1/32}n + 2$$

vertices from X_2. By (HA7), x_1 and x_2 each have at least two neighbours in $\widetilde{X}_2$ and, since $\eta \leq (\alpha_1/100)^{32}$, every vertex in $\widetilde{X}_2$ has degree at least $|X_1| - 4\eta^{1/32}n \geq \frac{1}{2}|X_1| + \frac{1}{2}|\widetilde{X}_2| + 1$ in $G[X_1, \widetilde{X}_2]$. Since $|X_1| > \ell_1 + 1$, by Lemma 2.6.6, $G_2[X_1, \widetilde{X}_2]$ contains a path on exactly $2\ell_1 + 1$ vertices from x_1 to x_2.

Likewise, given $y_1, y_2 \in Y_1$, for any set $\widetilde{Y}_2$ of

$$\ell_2 = \left\lceil \frac{\langle\langle \alpha_2 n \rangle\rangle - 2}{4} \right\rceil \geq 4\eta^{1/32}n + 2$$

vertices from Y_2, $G_2[Y_1, \widetilde{Y}_2]$ contains a a path on exactly $2\ell_2 + 1$ vertices from y_1 to y_2.

Combining the edges x_1y_1, x_2y_2 with the blue paths found gives a blue cycle on exactly $2\ell_1 + 2\ell_2 + 2 = \langle\langle \alpha_2 n \rangle\rangle$ vertices, completing the proof of the claim. □

The existence of red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices or a blue cycle on $\langle\langle \alpha_2 n \rangle\rangle$ vertices would be sufficient to complete the proof of Theorem A. Thus, there cannot exist such a pair of vertex-disjoint red edges or such a pair of vertex-disjoint blue edges in $G[X_1, Y_1]$.

Similarly, there cannot be a pair of vertex-disjoing blue edges in $G[X_1, Y_2]$ or in $G[X_2, Y_1]$.

Thus, after removing at most three vertices from each of X_1 and Y_1 and one vertex from from each of X_2 and Y_2, we may assume that

(HA8a) the green graph $G_3[X_1, Y_1]$ is complete; and

(HA8b) the are no blue edges in $G[X_1, Y_2] \cup G[X_2, Y_1]$.

Then, recalling (2.67), we have

$$|X_1|, |Y_1| \geq (\alpha_1 - 11\eta^{1/32})n, \qquad |X_2|, |Y_2| \geq (\tfrac{1}{2}\alpha_2 - 6\eta^{1/32})n. \tag{2.68}$$

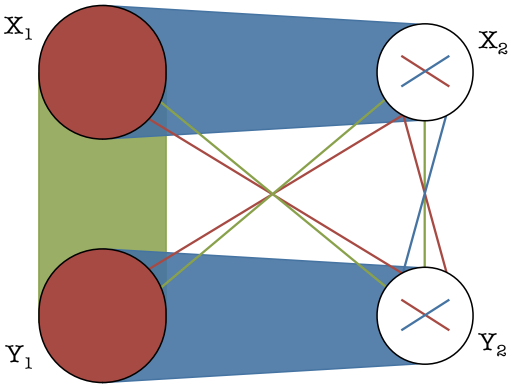


Figure 2.81: Colouring of G after Claim 2.10.2.

We now consider W. Defining W_G to be the set of vertices in W having a green edge to both X_1 and Y_1, we prove the following claim which allows us to assume that W_G is empty:

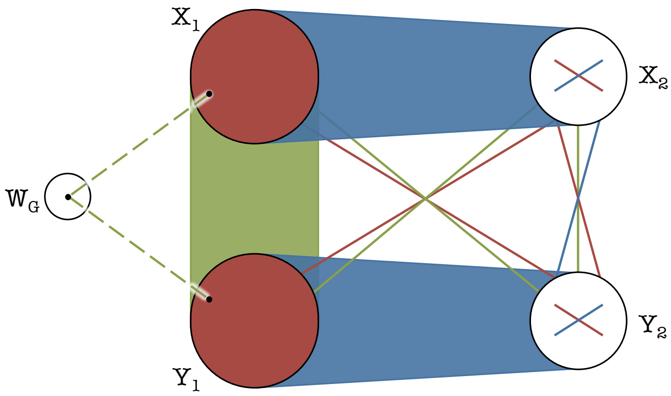


Figure 2.82: Using $w \in W_G$ to construct an odd green cycle.

Claim 2.10.3. *If W_G is non empty, then G contains either a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices or a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices.*

Proof. Suppose that W_G is non-empty. Then, there exists $w \in W$, $x \in X_1$, $y \in Y_1$ such that wx and wy are both coloured green. Recalling (HA8a), $G_3[X_1, Y_1]$ is complete, thus, we may obtain a cycle of any odd length up to $|X_1| + |Y_1| + 1$ in $G[W_G \cup X \cup Y]$. Therefore, to avoid having a a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices, we may assume that $|X_1| + |Y_1| + 1 < \langle \alpha_3 n \rangle$.

Then, considering (2.64) and (2.68), we have

$$2\alpha_1 - 22\eta^{1/32} \leq \alpha_3 \leq 2\alpha_1 + 2\eta^{1/32},$$

and

$$|X_1| + |Y_1| \geq (2\alpha_1 - 22\eta^{1/32})n \geq (\alpha_3 - 24\eta^{1/32})n.$$

In that case, suppose that, for some $x_a, x_b \in X_1$ and $y_a, y_b \in Y_1$, there exist green paths

$$P_1 \text{ from } x_a \in X_1 \text{ to } x_b \in X_1 \quad \text{on } 2\lceil 12\eta^{1/32} n \rceil + 1 \text{ vertices in } G[X_1, Y_2],$$
$$P_2 \text{ from } y_a \in Y_1 \text{ to } y_b \in Y_1 \quad \text{on } 2\lceil 12\eta^{1/32} n \rceil + 1 \text{ vertices in } G[Y_1, Y_2].$$

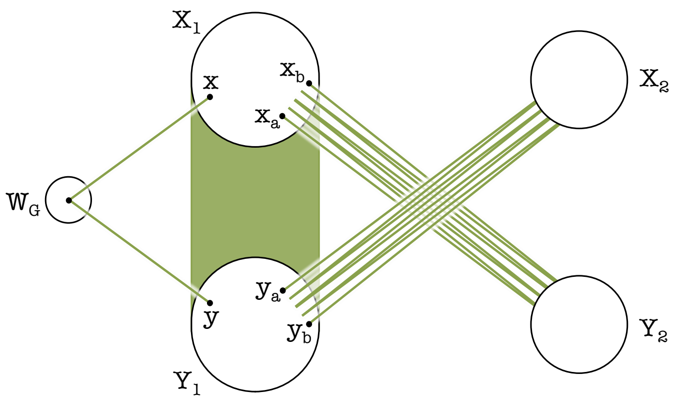


Figure 2.83: Construction of a long odd green cycle.

Then, since $G_3[X_1, Y_1]$ is complete and W_G is non-empty, P_1 and P_2 could be used along with edges from $G[X_1, Y_1]$ and $G[w, X_1 \cup Y_1]$ to give an odd green cycle on exactly $\langle \alpha_3 n \rangle$ vertices. Therefore, without loss of generality, we may assume that $G[X_1, Y_2]$ does not contain a green path on $2\lceil 12\eta^{1/32} n \rceil + 1$ vertices. Thus, by Theorem 2.6.8, $G[X_1, Y_2]$ contains at most $16\eta^{1/32} n^2$ green edges.

Recalling (HA8b), we know that $G[X_1, Y_2]$ is complete, contains no blue edges and contains at most $16\eta^{1/32}n^2$ green edges. Thus, there are at most $4\eta^{1/64}n$ vertices in X_1 with red degree to Y_2 at most $|Y_2| - 4\eta^{1/64}n$ and at most $4\eta^{1/64}n$ vertices in Y_2 with red degree to X_1 at most $|X_1| - 4\eta^{1/64}n$. Removing these vertices from $X_1 \cup Y_2$ results in every vertex in X_1 having red degree at least $|Y_2| - 4\eta^{1/64}n$ in $G[X_1, Y_2]$ and every vertex in Y_2 having red degree at least $|X_1| - 4\eta^{1/64}n$ in $G[X_1, Y_2]$. Thus, $G_1[X_1, Y_2]$ is $4\eta^{1/64}n$-almost-complete.

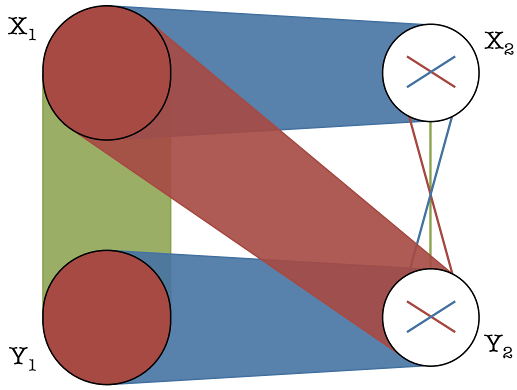


Figure 2.84: Colouring of $G[X_1, Y_2]$ in Claim 2.10.3.

Recalling (2.68), since $\eta < 10^{-20}$, having discarded these vertices, we have

$$|X_1| \geq (\alpha_1 - 7\eta^{1/64})n, \qquad |Y_2| \geq (\tfrac{1}{2}\alpha_2 - 7\eta^{1/64})n. \tag{2.69}$$

Then, given this bound for X_1, there exist disjoint subsets $X_L, X_S \subseteq X_1$ such that

$$|X_L| = \langle\langle \alpha_1 n \rangle\rangle - 2\lfloor 8\eta^{1/64}n \rfloor - 1, \qquad |X_S| = \lfloor 8\eta^{1/64}n \rfloor. \tag{2.70}$$

Let x_1, x_2 be distinct vertices in X_L. By (HA6), every vertex in X_1 has red degree at least $(|X_1| - 1) - 8\eta^{1/32}n$ in $G[X_1]$. Thus, since $\eta \leq (\alpha_1/100)^{64}$, every vertex in X_L has red degree in $G[X_L]$ at least $\frac{1}{2}|X_L| + 1$ and so, by Corollary 2.6.3, there exists a red Hamiltonian path R_1 in $G[X_L]$ from x_1 to x_2.

We now consider $G[Y_2, X_S]$. Since $G_1[X_1, Y_2]$ is $4\eta^{1/64}n$-almost-complete, there exist disjoint vertices $y_1, y_2 \in Y_2$, such that x_1y_1 and x_2y_2 are red and $d(y_1), d(y_2) \geq 2$. Since $\eta \leq (\alpha_2/100)^{64}$, considering (2.69) and (2.70), we have $|Y_2| > |X_S| + 1$ and know that every vertex in X_S has red degree at least $|Y_2| - 4\eta^{1/64}n \geq \frac{1}{2}(|Y_2| + \frac{1}{2}\alpha_2 n - 13\eta^{1/64}n) \geq \frac{1}{2}(|Y_2| + |X_S| + 1)$ in $G[Y_2, X_S]$. Thus, by Lemma 2.6.6, there exists a red path R_2

in $G[Y_2, X_S]$ from y_1 to y_2 which visits every vertex of X_S.

Together, the paths R_1, R_2 and the edges x_1y_1 and x_2y_2 form a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices in $G[X_1] \cup G[X_1, Y_2]$, thus completing the proof of the claim. □

The existence of a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices or a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices as offered by Claim 2.10.3 would be sufficient to complete the proof of Theorem A. We may, therefore, assume that W_G is empty.

Thus, defining W_X to be the set of vertices in W having no green edges to X_1 and W_Y to be the set of vertices in W having no green edges to Y_1, we see that $W_X \cup W_Y$ is a partition of W.

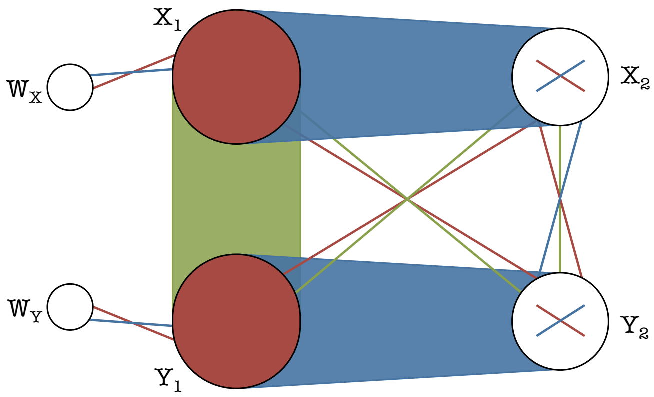


Figure 2.85: Partition of W into $W_X \cup W_Y$.

We thus have a partition of $V(G)$ into $X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup W_X \cup W_Y$. Then, since

$$|V(G)| \geq 2\langle\langle \alpha_1 n \rangle\rangle + \langle\langle \alpha_2 n \rangle\rangle - 3,$$

without loss of generality, we may assume that

$$|X_1 \cup X_2 \cup W_X| \geq \langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$$

since, if not, then $Y_1 \cup Y_2 \cup W_Y$ is that large instead.

Given (HA6) and (HA7), we can obtain upper bounds on $|X_1|$, $|X_2|$, $|Y_1|$ and $|Y_2|$ as follows: By Corollary 2.6.2, for every integer m such that $16\eta^{1/32}n + 2 \leq m \leq |X_1|$, we know that $G[X_1]$ contains a red cycle of length m. Thus, in order to avoid having a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices, we may assume that $|X_1| < \langle\langle \alpha_1 n \rangle\rangle$. By Corollary 2.6.5, for every even integer m such that $16\eta^{1/32}n + 2 \leq m \leq 2\min\{|X_1|, |X_2|\}$, we know

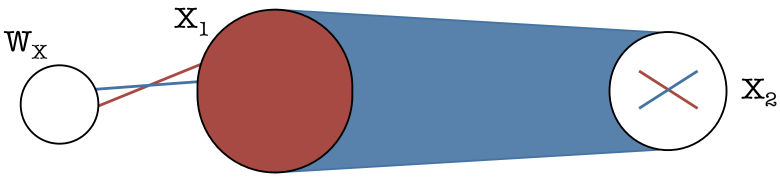


Figure 2.86: Colouring of $G[W_X \cup X_1 \cup X_2]$.

that $G[X_1, X_2]$ contains a blue cycle on m vertices. Recalling (2.68), we have $|X_1| \geq (\alpha_1 - 11\eta^{1/32})n \geq \frac{1}{2}\alpha_2 n$, thus, in order to avoid having a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices, we may assume that $|X_2| < \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$. In summary, we then have

$$\left.\begin{aligned} (\alpha_1 - 9\eta^{1/32})n \leq |X_1| < \langle\langle \alpha_1 n \rangle\rangle, \\ (\tfrac{1}{2}\alpha_2 - 6\eta^{1/32})n \leq |X_2| < \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle. \end{aligned}\right\} \quad (2.71)$$

Letting

$$W_B = \{w \in W_X \text{ such that } w \text{ has at least } |X_1| - 64\eta^{1/32} \text{ blue neighbours in } X_1\}; \text{ and}$$
$$W_R = W \backslash W_B = \{w \in W_X \text{ such that } w \text{ has at least } 64\eta^{1/32} \text{ red neighbours in } X_1\},$$

we have $W = W_R \cup W_B$. Thus, either $|X_1 \cup W_R| \geq \langle\langle \alpha_1 n \rangle\rangle$ or $|X_2 \cup W_B| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$.

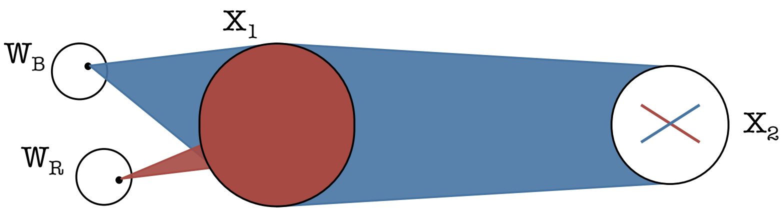


Figure 2.87: Partition of W_X into $W_B \cup W_R$.

If $|X_1 \cup W_R| \geq \langle\langle \alpha_1 n \rangle\rangle$, then we show that $G_1[X_1 \cup W_R]$ contains a long red cycle as follows: Let X be any set of $\langle\langle \alpha_1 n \rangle\rangle$ vertices from $X_1 \cup W_R$ consisting of every vertex from X_1 and $\langle\langle \alpha_1 n \rangle\rangle - |X_1|$ vertices from W_R. By (HA6) and (2.71), the red graph $G_1[X]$ has at least $\langle\langle \alpha_1 n \rangle\rangle - 11\eta^{1/32}n$ vertices of degree at least $|X| - 20\eta^{1/32}n$ and at most $11\eta^{1/32}$ vertices of degree at least $64\eta^{1/32}n$. Thus, by Theorem 2.6.7, $G[X]$ contains a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices.

Thus, we may, instead, assume that $|X_2 \cup W_B| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$, in which case, we consider the blue graph $G_2[X_1, X_2 \cup W_B]$. Given the relative sizes of X_1 and $X_2 \cup W_B$ and the large minimum-degree of the graph, we can use Theorem 2.6.4 to give a blue cycle on exactly $\langle\!\langle \alpha_2 n \rangle\!\rangle$ vertices. Indeed, by (2.71), we have $|X_1| \geq \frac{1}{2}\langle\!\langle \alpha_2 n \rangle\!\rangle$ and may choose subsets $\widetilde{X}_1 \subseteq X_1$, $\widetilde{X}_2 \subseteq X_2 \cup W_B$ such that $\widetilde{X}_2$ includes every vertex of X_2 and

$$|\widetilde{X}_1| = |\widetilde{X}_2| = \tfrac{1}{2}\langle\!\langle \alpha_2 n \rangle\!\rangle, \qquad\qquad |\widetilde{X}_2 \cup W_B| \leq 6\eta^{1/32} n.$$

Recall that $G_2[X_1, X_2]$ is $4\eta^{1/32}k$-almost-complete and that, all vertices in W_B have blue degree at least $|\widetilde{X}_1| - 64\eta^{1/32}n$ in $G[\widetilde{X}_1, \widetilde{X}_2]$. Thus, since $|\widetilde{X}_2 \cup W_B| \leq 6\eta^{1/32}n$ and $\eta \leq (\alpha_2/200)^{32}$, for any pair of vertices $x_1 \in \widetilde{X}_1$ and $x_2 \in \widetilde{X}_2$, we have

$$d(x_1) + d(x_2) \geq |\widetilde{X}_1| + |\widetilde{X}_2| - 74\eta^{1/32}n \geq \tfrac{1}{2}\langle\!\langle \alpha_2 n \rangle\!\rangle + 1.$$

Therefore, by Theorem 2.6.4, $G_2[\widetilde{X}_1, \widetilde{X}_2]$ contains a blue cycle on exactly $\langle\!\langle \alpha_2 n \rangle\!\rangle$ vertices, thus completing this part of the proof.

Part I.B: $H_1, H_2 \in \mathcal{H}_2$.

By Theorem B, this case only occurs when

$$\alpha_2 \leq \alpha_1 \leq \alpha_2 + \eta^{1/8}. \tag{HB0}$$

Recalling that $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are the monochromatic spanning subgraphs of the reduced-graph $\mathcal{G}$, the vertex set $\mathcal{V}$ of $\mathcal{G}$ has a partition into $\mathcal{W} \cup \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2$ with

$$\left.\begin{aligned} (\alpha_2 - 2\eta^{1/16})k \leq |\mathcal{X}_1| = |\mathcal{Y}_1| = p \leq \alpha_2 k, \\ (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k \leq |\mathcal{X}_2| = |\mathcal{Y}_2| = q \leq \tfrac{1}{2}\alpha_1 k, \end{aligned}\right\} \tag{2.72}$$

such that

(HB1) $\mathcal{G}_2[\mathcal{X}_1], \mathcal{G}_2[\mathcal{Y}_1]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HB2) $\mathcal{G}_1[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}_1[\mathcal{Y}_1, \mathcal{Y}_2]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HB3) $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2]$ is $3\eta^4 k$-almost-complete (and thus connected);

(HB4) $\mathcal{G}[\mathcal{X}_1], \mathcal{G}[\mathcal{Y}_1]$ are each $2\eta^{1/16}$-sparse in red and contain no green edges; and

(HB5) $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2]$ are each $2\eta^{1/16}$-sparse in blue and contain no green edges.

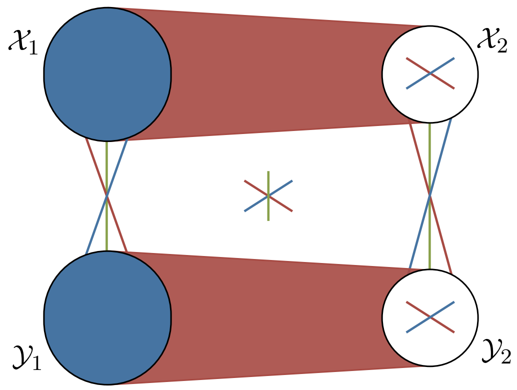


Figure 2.88: Coloured structure of the reduced-graph in Part I.B.

The proof in this case is essentially identical to that in Part I.A. However we include the key steps here for completeness. As in Part I.A, each vertex V_i of $\mathcal{G}$ represents a class of vertices of G. We partition the vertices of G into sets X_1, X_2, Y_1, Y_2 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{W}$. Then X_1, Y_1 each contain p clusters of vertices, X_2, Y_2 each contain q clusters and we have

$$|X_1|, |Y_1| = p|V_1| \geq (\alpha_2 - 2\eta^{1/16})n, \qquad |X_2|, |Y_2| = q|V_1| \geq (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})n. \qquad (2.73)$$

By the following claim, G has essentially the same colouring as the reduced-graph:

Claim 2.10.4. *Given G as described, we can remove at most $9\eta^{1/32}n$ vertices from each of X_1 and Y_1 and at most $4\eta^{1/32}n$ vertices from each of X_2 and Y_2 so that the following pair of conditions hold.*

(HB6) $G_2[X_1]$ and $G_2[Y_1]$ are each $8\eta^{1/32}n$-almost-complete;

(HB7) $G_1[X_1, X_2]$ and $G_1[Y_1, Y_2]$ are each $4\eta^{1/32}n$-almost-complete.

Proof. Identical to that of Claim 2.10.1 with the roles of red and blue exchanged. □

Having discarded some vertices, recalling (2.73), we have

$$|X_1|, |Y_1| \geq (\alpha_2 - 10\eta^{1/32})n, \qquad |X_2|, |Y_2| \geq (\tfrac{1}{2}\alpha_1 - 5\eta^{1/32})n, \qquad (2.74)$$

and can proceed to the end-game.

The following pair of claims allow us to determine the colouring of $G[X_1, X_2]$:

Claim 2.10.5.a. *If there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured blue, then G contains a blue cycle of length exactly $\langle\langle \alpha_2 n \rangle\rangle$.*

Claim 2.10.5.b. *If there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured red, then G contains a red cycle of length exactly $\langle\langle \alpha_1 n \rangle\rangle$.*

Proof. Identical to that of Claim 2.10.2 with the roles of red and blue exchanged and also the roles of α_1 and α_2. Note that, when needed in (b), the fact that $|X_1| > \ell_1 + 1, |Y_1| > \ell_2 + 1$, follows from (HB0). □

The existence of a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices or a blue cycle on $\langle\langle \alpha_2 n \rangle\rangle$ vertices would be sufficient to complete the proof of Theorem A. Thus, there cannot exist such a pair of vertex-disjoint red edges or such a pair of vertex-disjoint blue edges in $G[X_1, Y_1]$. Similarly, there cannot be a pair of vertex-disjoint red edges in $G[X_1, Y_2]$ or in $G[X_2, Y_1]$. Thus, after removing at most three vertices from each of X_1 and Y_1 and one vertex from from each of X_2 and Y_2, we may assume that

(HB8a) the green graph $G_3[X_1, Y_1]$ is complete; and

(HB8b) the are no red edges in $G[X_1, Y_2] \cup G[X_2, Y_1]$.

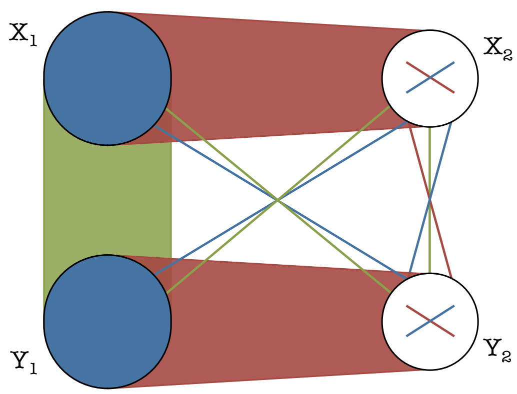


Figure 2.89: Colouring of G after Claim 2.10.5.

Then, recalling (2.74), we have

$$|X_1|, |Y_1| \geq (\alpha_2 - 11\eta^{1/32})n, \qquad |X_2|, |Y_2| \geq (\tfrac{1}{2}\alpha_1 - 6\eta^{1/32})n. \tag{2.75}$$

We now consider W. Defining W_G to be the set of vertices in W having a green edge to both X_1 and Y_1, the following claim allows us to assume that W_G is empty:

Claim 2.10.6. *If W_G is non empty, then G contains either a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices or a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices.*

Proof. Identical to that of Claim 2.10.2 with the roles of red and blue and also the roles of α_1 and α_2 exchanged. □

Since W_G is empty, defining W_X to be the set of vertices in W having no green edges to X_1 and W_Y to be the set of vertices in W having no green edges to Y_1, we see that $W_X \cup W_Y$ is a partition of W. We may assume, without loss of generality that

$$|X_1 \cup X_2 \cup W_X| \geq \langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1.$$

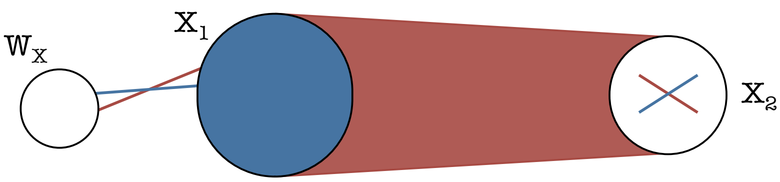


Figure 2.90: Colouring of $G[X_1 \cup X_2 \cup W_X]$.

Given (HB6) and (HB7), we can obtain upper bounds on $|X_1|$, $|X_2|$, $|Y_1|$ and $|Y_2|$ as follows: By Corollary 2.6.2, for every integer m such that $16\eta^{1/32}n + 2 \leq m \leq |X_1|$, we know that $G_2[X_1]$ contains a blue cycle of length m. Thus, in order to avoid having a blue cycle of length $\langle\langle \alpha_2 n \rangle\rangle$, we may assume that $|X_1| < \langle\langle \alpha_2 n \rangle\rangle$. By Corollary 2.6.5, for every even integer m such that $16\eta^{1/32}n + 2 \leq m \leq 2\min\{|X_1|, |X_2|\}$, we know that $G_1[X_1, X_2]$ contains a red cycle of length m. Recalling (HB0) and (2.75), we have $|X_1| \geq (\alpha_2 - 11\eta^{1/32})n \geq \frac{1}{2}\alpha_1 n$. Thus, in order to avoid having a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices, we may assume that $|X_2| < \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. In summary, we have

$$\left.\begin{aligned} (\alpha_2 - 1\eta^{1/32})n \leq |X_1| < \langle\langle \alpha_2 n \rangle\rangle, \\ (\tfrac{1}{2}\alpha_1 - 6\eta^{1/32})n \leq |X_2| < \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle. \end{aligned}\right\} \quad (2.76)$$

Let

$$W_R = \{w \in W_X \text{ such that } w \text{ has at least } |X_1| - 64\eta^{1/32} \text{ red neighbours in } X_1\}; \text{ and}$$
$$W_B = W \backslash W_R = \{w \in W_X \text{ such that } w \text{ has at least } 64\eta^{1/32} \text{ blue neighbours in } X_1\}.$$

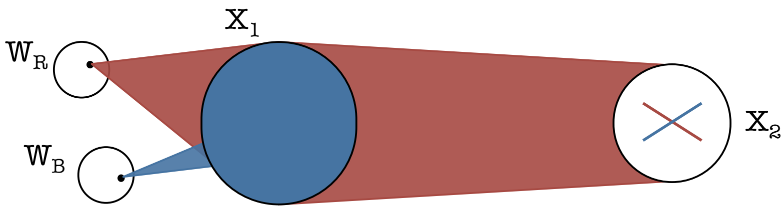


Figure 2.91: Partition of W_X into $W_R \cup W_B$.

Then, we have $W = W_B \cup W_R$. Thus, either $|X_1 \cup W_B| \geq \langle\langle \alpha_2 n \rangle\rangle$ or $|X_2 \cup W_R| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$.

If $|X_1 \cup W_B| \geq \langle\langle \alpha_2 n \rangle\rangle$, then we show that $G_2[X_1 \cup W_B]$ contains a long blue cycle as follows: Let X be any set of $\langle\langle \alpha_2 n \rangle\rangle$ vertices from $X_1 \cup W_B$ consisting of every vertex from X_1 and $\langle\langle \alpha_2 n \rangle\rangle - |X_1|$ vertices from W_B. By (HB6) and (2.76), the blue graph $G_2[X]$ has at least $\langle\langle \alpha_2 n \rangle\rangle - 11\eta^{1/32}n$ vertices of degree at least $|X| - 20\eta^{1/32}n$ and at most $11\eta^{1/32}n$ vertices of degree at least $64\eta^{1/32}n$. Thus, by Theorem 2.6.7, $G[X]$ contains a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices.

Thus, we may, instead, assume that $|X_2 \cup W_R| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$, in which case, we consider the red graph $G_2[X_1, X_2 \cup W_R]$. Given the relative sizes of X_1 and $X_2 \cup W_R$ and the large minimum-degree of the graph, we can use Theorem 2.6.4 to give a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices as follows: By (HB0) and (2.76), we have $|X_1| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and may choose subsets $\widetilde{X}_1 \subseteq X_1$, $\widetilde{X}_2 \subseteq X_2 \cup W_R$ such that $\widetilde{X}_2$ includes every vertex of X_2, $|\widetilde{X}_1| = |\widetilde{X}_2| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and $|\widetilde{X}_2 \cap W_R| \leq 6\eta^{1/32}n$. Recall, from (HB7), that $G_1[X_1, X_2]$ is $4\eta^{1/32}k$-almost-complete and that, by definition, all vertices in W_R have red degree at least $|\widetilde{X}_1| - 64\eta^{1/32}n$ in $G[\widetilde{X}_1, \widetilde{X}_2]$. Thus, since $|\widetilde{X}_2 \cap W_R| \leq 6\eta^{1/32}n$, for any pair of vertices $x_1 \in \widetilde{X}_1$ and $x_2 \in \widetilde{X}_2$, we have $d(x_1) + d(x_2) \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + 1$. Therefore, by Theorem 2.6.4, $G_1[\widetilde{X}_1, \widetilde{X}_2]$ contains a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices, thus completing this part of the proof.

Part I.C: $H_1 \in \mathcal{H}_1, H_2 \in \mathcal{H}_2$**.**

By Theorem B, this case only occurs when

$$\alpha_2 \leq \alpha_1 \leq \alpha_2 + \eta^{1/8}. \tag{HC0}$$

Recalling that $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are the monochromatic spanning subgraphs of the reduced-graph $\mathcal{G}$, the vertex set $\mathcal{V}$ of $\mathcal{G}$ has a partition into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ with

$$(\alpha_1 - 2\eta^{1/16})k \leq |\mathcal{X}_1| = p \leq \alpha_1 k, \qquad (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})k \leq |\mathcal{X}_2| = q \leq \tfrac{1}{2}\alpha_2 k,$$
$$(\alpha_2 - 2\eta^{1/16})k \leq |\mathcal{Y}_1| = r \leq \alpha_2 k, \qquad (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})k \leq |\mathcal{Y}_2| = q \leq \tfrac{1}{2}\alpha_1 k$$

such that

(HC1) $\mathcal{G}_1[\mathcal{X}_1], \mathcal{G}_2[\mathcal{Y}_1]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HC2) $\mathcal{G}_2[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}_1[\mathcal{Y}_1, \mathcal{Y}_2]$ are each $(1 - 2\eta^{1/16})$-complete (and thus connected);

(HC3) $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2]$ is $3\eta^4 k$-almost-complete (and thus connected);

(HC4) $\mathcal{G}[\mathcal{X}_1]$ is $2\eta^{1/16}$-sparse in blue and contains no green edges,
$\mathcal{G}[\mathcal{Y}_1]$ is $2\eta^{1/16}$-sparse in red and contains no green edges; and

(HC5) $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2]$ is $2\eta^{1/16}$-sparse in red and contains no green edges,
$\mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2]$ is $2\eta^{1/16}$-sparse in blue and contains no green edges.

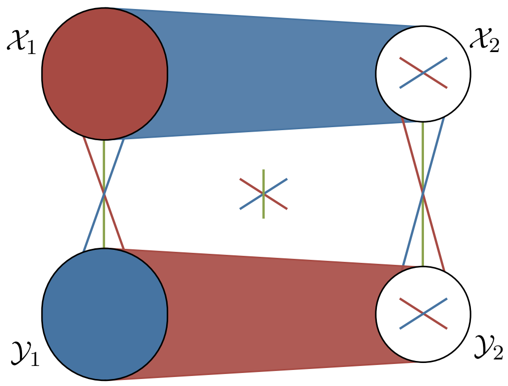


Figure 2.92: Coloured structure of the reduced-graph in Part I.C.

The proof that follows is essentially similar to those in Parts I.A and I.B but with some additional complications.

Again, each vertex V_i of $\mathcal{G}$ represents a class of vertices of G. We partition the vertices of G into sets X_1, X_2, Y_1, Y_2 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{W}$. Then X_1 contains p clusters, X_2 contains q clusters Y_1 contains r clusters, Y_2 contains s clusters and we have

$$\left.\begin{aligned} |X_1| = p|V_1| \geq (\alpha_1 - 2\eta^{1/16})n, \qquad |X_2| = q|V_1| \geq (\tfrac{1}{2}\alpha_2 - 2\eta^{1/16})n, \\ |Y_1| = r|V_1| \geq (\alpha_2 - 2\eta^{1/16})n, \qquad |Y_2| = s|V_1| \geq (\tfrac{1}{2}\alpha_1 - 2\eta^{1/16})n. \end{aligned}\right\} \quad (2.77)$$

We may then show that G has similar coloured structure to the reduced-graph $\mathcal{G}$:

Claim 2.10.7. *Given G as described, we can remove at most $9\eta^{1/32}n$ vertices from each of X_1 and Y_1 and at most $4\eta^{1/32}n$ vertices from each of X_2 and Y_2 so that the following pair of conditions hold:*

(HC6) $G_1[X_1]$ and $G_2[Y_1]$ are each $8\eta^{1/32}n$-almost-complete;

(HC7) $G_2[X_1, X_2]$ and $G_1[Y_1, Y_2]$ are each $4\eta^{1/32}n$-almost-complete.

Proof. The proof for $G_1[X_1]$ and $G_2[X_1, X_2]$ is identical to that of Claim 2.10.1. The proof for $G_2[Y_1]$ and $G_1[Y_1, Y_2]$ is identical but with the roles of red and blue exchanged. □

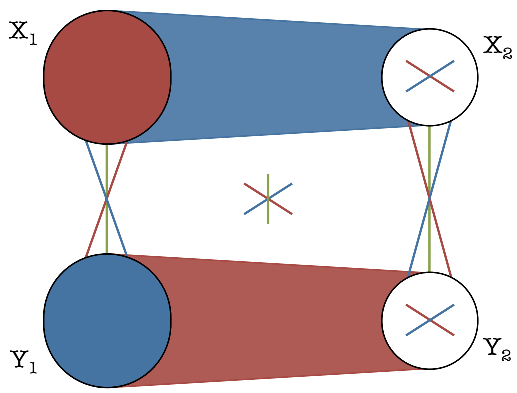


Figure 2.93: Colouring of G after Claim 2.10.7.

Having discarded some vertices, recalling (2.77), we now have

$$\left.\begin{aligned} |X_1| &\geq (\alpha_1 - 10\eta^{1/32})n, & |X_2| &\geq (\tfrac{1}{2}\alpha_2 - 5\eta^{1/32})n, \\ |Y_1| &\geq (\alpha_2 - 10\eta^{1/32})n, & |Y_2| &\geq (\tfrac{1}{2}\alpha_1 - 5\eta^{1/32})n. \end{aligned}\right\} \quad (2.78)$$

and proceed to consider $G[X, Y]$:

Claim 2.10.8. *If there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_1$ such that x_1y_1 and x_2y_2 are coloured red, then G contains a red cycle of length exactly $\langle\langle \alpha_1 n \rangle\rangle$.*

Proof. Suppose there exist distinct vertices $x_1, x_2 \in X_1$ and $y_1, y_2 \in Y_2$ such that the edges x_1y_1 and x_2y_2 are coloured red. Then, letting $\widetilde{X}_1$ be any set of of $2\lceil \frac{1}{4}\langle\langle \alpha_1 n \rangle\rangle \rceil - 1$ vertices in X_1 such that $x_1, x_2 \in \widetilde{X}_1$, by (HC6), every vertex in $\widetilde{X}_1$ has degree at least $|\widetilde{X}_1| - 8\eta^{1/32}n$ in $G[\widetilde{X}_1]$. Since $\eta \leq (\alpha_1/100)^{32}$, we have $|\widetilde{X}_1| - 8\eta^{1/32}n \geq \frac{1}{2}|\widetilde{X}_1| + 2$, so, by Corollary 2.6.3, there exists a red path from x_1 and x_2 on exactly $2\lceil \frac{1}{4}\langle\langle \alpha_1 n\rangle\rangle \rceil - 1$ vertices in X_1.

Let $\widetilde{Y}_2$ be any set of $\lfloor \frac{1}{4}\langle\langle \alpha_1 n\rangle\rangle \rfloor \geq 4\eta^{1/32}n + 2$ vertices from Y_2. By (HC7), y_1 and y_2 each have at least two neighbours in $\widetilde{Y}_2$. Also, by (HC0) and (HC7), every vertex in $\widetilde{Y}_2$ has degree at least $\frac{1}{2}|Y_1| + \frac{1}{2}|\widetilde{Y}_2| + 1$ in $G[Y_1, \widetilde{Y}_2]$. Finally, by (HC0), we have $|Y_1| \geq \ell + 1$. Thus, by Lemma 2.6.6, $G_2[Y_1, \widetilde{Y}_2]$ contains a path on exactly $2\lfloor \frac{1}{4}\langle\langle \alpha_1 n\rangle\rangle \rfloor + 1$ vertices from y_1 to y_2. Then, combining the red edges x_1y_1 and x_2y_2 with the red paths found in $G[X_1]$ and $G[Y_1, Y_2]$ gives a red cycle on exactly $\langle\langle \alpha_1 n\rangle\rangle$ vertices. □

The existence of a red cycle on $\langle\langle \alpha_1 n\rangle\rangle$ vertices would be sufficient to complete the proof. Thus, there cannot exist such a pair of vertex-disjoint red edges. Similarly, there cannot exist a pair of vertex-disjoint red edges in $G[X_1, Y_2]$ or a pair of vertex-disjoint blue edges in $G[X_1, Y_1]$ or $G[X_2, Y_1]$.

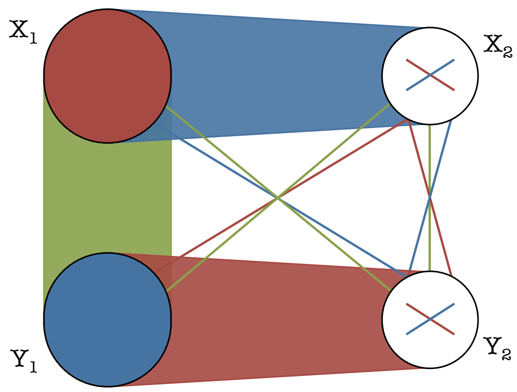


Figure 2.94: Colouring of G after Claim 2.10.8.

Thus, after removing at most three vertices from each of X_1, Y_1 and at most one vertex from each of X_2, Y_2, we may assume that

(HC8a) the green graph $G_3[X_1, Y_1]$ is complete; and

(HC8b) the are no red edges in $G[X_1, Y_2]$ and no blue edges in $G[X_2, Y_1]$.

Then, recalling (2.78), we have

$$\left.\begin{aligned} |X_1| &\geq (\alpha_1 - 11\eta^{1/32})n, & |X_2| &\geq (\tfrac{1}{2}\alpha_2 - 6\eta^{1/32})n, \\ |Y_1| &\geq (\alpha_2 - 11\eta^{1/32})n, & |Y_2| &\geq (\tfrac{1}{2}\alpha_1 - 6\eta^{1/32})n. \end{aligned}\right\} \quad (2.79)$$

We now consider W. Defining W_G to be the set of vertices in W having a green edge to both X_1 and Y_1, the following claim allows us to assume that W_G is empty:

Claim 2.10.9. *If W_G is non empty, then G contains either a red cycle on exactly $\langle\langle\alpha_1 n\rangle\rangle$ vertices, a blue cycle on exactly $\langle\langle\alpha_2 n\rangle\rangle$ vertices or a green cycle on exactly $\langle\alpha_3 n\rangle$ vertices.*

Proof. Suppose that W_G is non-empty. Then, there exists $w \in W$, $x \in X_1$, $y \in Y_1$ such that wx and wy are both coloured green. Recalling (HC8a), $G_3[X_1, Y_1]$ is complete, thus we may obtain an odd cycle of any odd length up to $|X_1| + |Y_1| + 1$ in $G[W_G, X_1 \cup Y_1] \cup G[X_1, Y_1]$. Therefore, to avoid having a a green cycle on exactly $\langle\alpha_3 n\rangle$ vertices, we assume that $|X_1| + |Y_1| + 1 < \langle\alpha_3 n\rangle$.

Then, considering, (2.64), (HC0) and (2.79), we have

$$|X_1| + |Y_1| \geq (\alpha_1 + \alpha_2 - 22\eta^{1/2})n \geq (\alpha_3 - 24\eta^{1/32})n.$$

In that case, suppose that, for some $x_a, x_b \in X_1$ and $y_a, y_b \in Y_1$, there exist green paths

$$\begin{aligned} &P_1 \text{ from } x_a \in X_1 \text{ to } x_b \in X_1 \quad \text{on } 2\lceil 12\eta^{1/32} n\rceil + 1 \text{ vertices in } G[X_1, Y_2],\\ &P_2 \text{ from } y_a \in Y_1 \text{ to } y_b \in Y_1 \quad \text{on } 2\lceil 12\eta^{1/32} n\rceil + 1 \text{ vertices in } G[Y_1, Y_2]. \end{aligned}$$

Then, since $G_3[X_1, Y_1]$ is complete and W_G is non-empty, P_1 and P_2 could be used along with edges from $G[X_1, Y_1]$ and $G[w, X_1 \cup Y_1]$ to give an odd green cycle on exactly $\langle\alpha_3 n\rangle$ vertices.

Thus, at most one of $G[X_1, Y_2]$, $G[X_2, Y_1]$ contains a green path on $2\lceil 12\eta^{1/32} n\rceil + 1$ vertices. Thus, by Theorem 2.6.8, either (a) $G[X_1, Y_2]$ contains at most $16\eta^{1/32}n^2$ green edges or (b) $G[X_2, Y_1]$ contains at most $16\eta^{1/32}n^2$ green edges.

(a) If $G[X_1, Y_2]$ contains at most $16\eta^{1/32}n^2$ green edges, then recalling (HC8b), we know that $G[X_1, Y_2]$ is complete, contains no red edges and contains at most $16\eta^{1/32}n^2$ green edges. Thus, after removing at most $4\eta^{1/64}$ vertices from each of X_1 and Y_2 we may assume that $G_2[X_1, Y_2]$ is $4\eta^{1/64}n$-almost-complete. Recall from (HC6) that $G_2[X_1, X_2]$ is $4\eta^{1/32}$-almost-complete. Thus, $G_2[X_1, X_2 \cup Y_2]$ is $6\eta^{1/64}$-almost-complete.

Having discarded these vertices, recalling (2.79), since $\eta < 10^{-20}$, we have

$$\left.\begin{aligned} |X_1| &\geq (\alpha_1 - 7\eta^{1/64})n, & |X_2| &\geq (\tfrac{1}{2}\alpha_2 - 2\eta^{1/64})n,\\ |Y_1| &\geq (\alpha_2 - 2\eta^{1/64})n, & |Y_2| &\geq (\tfrac{1}{2}\alpha_1 - 7\eta^{1/64})n. \end{aligned}\right\} \quad (2.80)$$

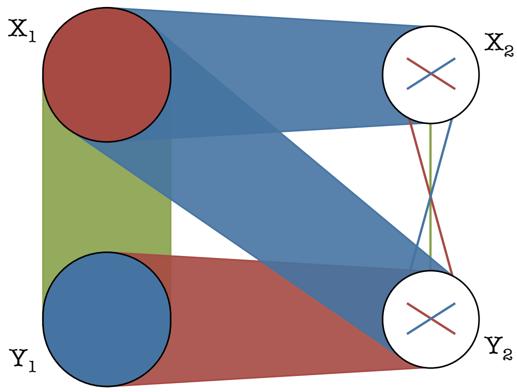


Figure 2.95: Colouring of $G[X_1, Y_2]$ in Claim 2.10.9(a).

Given the bounds in (2.80), there exist subsets $\widetilde{X}_1 \subseteq X_1$ and $\widetilde{X}_2 \subseteq X_2 \cup Y_2$ such that $|\widetilde{X}_1| = |\widetilde{X}_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$. Then, by Theorem 2.6.4, $G_2[\widetilde{X}_1, \widetilde{X}_2]$ is Hamiltonian and, thus, provides a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices.

(b) If instead $G[X_2, Y_1]$ contains at most $16\eta^{1/32}n^2$ green edges, then, after removing at most $4\eta^{1/32}n$ vertices from each of X_2 and Y_1, we may assume that $G_1[X_2, Y_1]$ is $4\eta^{1/64}n$-almost-complete. Recall from (HC6) that $G_2[Y_1, Y_2]$ is $4\eta^{1/32}$-almost-complete. Thus, $G_1[Y_1, X_2 \cup Y_2]$ is $6\eta^{1/64}$-almost-complete. Having discarded these vertices, recalling (2.79), since $\eta < 10^{-20}$, we have

$$\left.\begin{aligned} |X_1| &\geq (\alpha_1 - 2\eta^{1/64})n, & |X_2| &\geq (\tfrac{1}{2}\alpha_2 - 7\eta^{1/64})n, \\ |Y_1| &\geq (\alpha_2 - 7\eta^{1/64})n, & |Y_2| &\geq (\tfrac{1}{2}\alpha_1 - 2\eta^{1/64})n. \end{aligned}\right\} \quad (2.81)$$

Given these bounds, there exist subsets $\widetilde{Y}_1 \subseteq X_1$ and $\widetilde{Y}_2 \subseteq X_2 \cup Y_2$ such that $|\widetilde{Y}_1| = |\widetilde{Y}_2| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. Then, by Theorem 2.6.4, $G_1[\widetilde{Y}_1, \widetilde{Y}_2]$ is Hamiltonian and, thus, provides a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices. □

The existence of a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices, a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices or a green cycle on exactly $\langle \alpha_3 n \rangle$ vertices, as offered by Claim 2.10.9, would be sufficient to complete the proof of Theorem A. We may, therefore, instead assume that W_G is empty.

Then, defining W_X to be the set of vertices in W having no green edges to X_1 and W_Y to be the set of vertices in W having no green edges to Y_1, we see that $W_X \cup W_Y$ is a partition of W. Thus either

$$|X_1 \cup X_2 \cup W_X| \geq \langle\!\langle \alpha_1 n \rangle\!\rangle + \tfrac{1}{2}\langle\!\langle \alpha_2 n \rangle\!\rangle - 1 \quad \text{or} \quad |Y_1 \cup Y_2 \cup W_Y| \geq \langle\!\langle \alpha_1 n \rangle\!\rangle + \tfrac{1}{2}\langle\!\langle \alpha_2 n \rangle\!\rangle - 1.$$

In the first case, the proof proceeds exactly as in Part I.A. In the second case, the proof proceeds exactly as in Part I.B. Thus, we obtain either a red cycle on exactly $\langle\!\langle \alpha_1 n \rangle\!\rangle$ or a blue cycle on exactly $\langle\!\langle \alpha_1 n \rangle\!\rangle$, completing Part I of the proof of Theorem A.

Notice that, in particular, since the graph providing the corresponding lower bound has already been seen, we have proved that, given $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2, \alpha_3$, there exists $n_{A'} = n_{A'}(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_{A'}$,

$$R(C_{\langle\!\langle \alpha_1 n \rangle\!\rangle}, C_{\langle\!\langle \alpha_2 n \rangle\!\rangle}, C_{\langle \alpha_3 n \rangle}) = 2\langle\!\langle \alpha_1 n \rangle\!\rangle + \langle\!\langle \alpha_2 n \rangle\!\rangle - 3.$$

2.11 Proof of the main result – Part II – Case (v)

Suppose that $\mathcal{G}$ contains a subgraph from

$$\mathcal{K}\left((\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 14000\eta^{1/2})k, (\alpha_3 - 68000\eta^{1/2})k, 4\eta^4 k\right).$$

In that case, by Theorem B, we may assume that

$$\alpha_3 \geq \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2}.$$

Recalling that $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are the monochromatic spanning subgraphs of the reduced-graph $\mathcal{G}$, since $\eta < 10^{-20}$, the vertex set $\mathcal{V}$ of $\mathcal{G}$ has partition into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 \cup \mathcal{W}$ with

$$\begin{aligned}
(\tfrac{1}{2}\alpha_1 - \eta^{1/4})k \leq |\mathcal{X}_1| = p \leq \tfrac{1}{2}\alpha_1 k,\\
(\tfrac{1}{2}\alpha_2 - \eta^{1/4})k \leq |\mathcal{X}_2| = q \leq \tfrac{1}{2}\alpha_2 k,\\
(\alpha_3 - \eta^{1/4})k \leq |\mathcal{X}_3| = r \leq \alpha_3 k.
\end{aligned}$$

such that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_3]$ are coloured exclusively red, all edges present in $\mathcal{G}[\mathcal{X}_2, \mathcal{X}_3]$ are coloured exclusively blue, all edges present in $\mathcal{G}[\mathcal{X}_3]$ are coloured exclusively green and $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3]$ is $4\eta^4 k$-almost-complete.

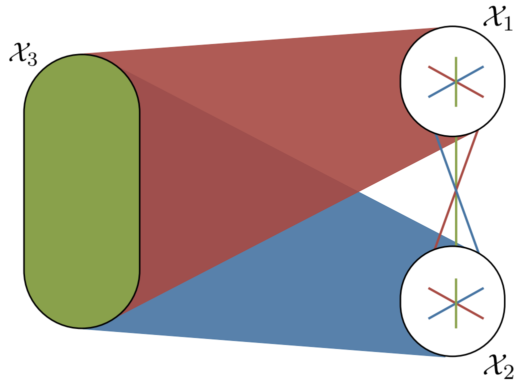


Figure 2.96: Coloured structure of the reduced-graph in Part II.

Again, each vertex V_i of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents a cluster of vertices of $G = (V, E)$ with

$$(1 - \eta^4)\frac{N}{K} \leq |V_i| \leq \frac{N}{K}$$

and, since $n > \max\{n_{2.3.4}(2, 1, 0, \eta), n_{2.3.4}(\frac{1}{2}, \frac{1}{2}, 1, \eta)\}$, we have

$$|V_i| \geq \left(1 + \frac{\eta}{24}\right)\frac{n}{k} > \frac{n}{k}.$$

We partition the vertices of G into sets X_1, X_2, X_3 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{W}$. Then X_1 contains p clusters of vertices, X_2 contains q clusters, X_3 contains r clusters and we have

$$\left.\begin{aligned} &|X_1| = p|V_1| \geq (\tfrac{1}{2}\alpha_1 - \eta^{1/4})n, && |X_3| = r|V_1| \geq (\alpha_3 - \eta^{1/4})n, \\ &|X_2| = q|V_1| \geq (\tfrac{1}{2}\alpha_2 - \eta^{1/4})n. && \end{aligned}\right\} \quad (2.82)$$

In what follows, we will remove vertices from X_1, X_2, X_3, by moving them into W. We prove the below claim which essentially tells us that G has a similar coloured structure to $\mathcal{G}$.

Claim 2.11.1. *We can remove at most $7\eta^{1/2}n$ vertices from X_1, $7\eta^{1/2}n$ vertices from X_2, and $24\eta^{1/2}n$ vertices from X_3 such that the following holds:*

(i) $G_1[X_1, X_3]$ *is* $7\eta^{1/2}n$*-almost-complete;*

(ii) $G_2[X_2, X_3]$ *is* $7\eta^{1/2}n$*-almost-complete;*

(iii) $G_3[X_3]$ *is* $10\eta^{1/2}n$*-almost-complete.*

Proof. Consider the complete three-coloured graph $G[X_3]$ and recall that $\mathcal{G}[\mathcal{X}_3]$ is $4\eta^4k$-almost-complete and has only green edges. Given the structure of $\mathcal{G}$, we can bound the number of non-green edges in $G[X_3]$ as follows:

Since regularity provides no indication as to the colours of the edges contained within each cluster, these could potentially all be non-green. There are r clusters, each with at most N/K vertices. Thus, there are at most

$$r\binom{N/K}{2}$$

non-green edges in X_3 within the clusters of $\mathcal{X}_3$.

Now, consider a pair of clusters (U_1, U_2) in X_3. If (U_1, U_2) is not η^4-regular, then we can only trivially bound the number of non-green edges in $G[U_1, U_2]$ by $|U_1||U_2| \leq (N/K)^2$. However, there are at most $4\eta^4|\mathcal{X}_3|k$ such pairs in $\mathcal{G}$. Thus, we can bound the number of non-red edges coming from non-regular pairs by

$$4\eta^4 rk\left(\frac{N}{K}\right)^2.$$

If the pair is regular, then U_1 and U_2 are joined, in the reduced-graph, by an edge which is coloured exclusively green. The red and blue densities of the pair are at most η (since a higher density would result in an edge of that colour in the reduced-graph). Thus, there are at most

$$2\eta\binom{r}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[X_3]$ corresponding to such pairs of clusters.

Summing the three possibilities above gives an upper bound of

$$r\binom{N/K}{2} + 4\eta^4 rk\left(\frac{N}{K}\right)^2 + 2\eta\binom{r}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[X_3]$.

Since $K \geq k, \eta^{-1}$, $N \leq 3n$ and $r \leq \alpha_3 k \leq 2k$, we obtain

$$e(G_1[X_3]) + e(G_2[X_3]) \leq [9\eta + 72\eta^4 + 36\eta]n^2 \leq 50\eta n^2.$$

Since $G[X_3]$ is complete and contains at most $50\eta n^2$ non-green edges, there are at most

$10\eta^{1/2}n$ vertices with green degree at most $|X_3| - 10\eta^{1/2}n$. After re-assigning these vertices to W, every vertex in $G[X_3]$ has red degree at least $|X_3| - 10\eta^{1/2}n$.

Now, consider $G[X_1, X_3]$. In a similar way to above, we can bound the the number of non-red edges in $G[X_1, X_3]$ by

$$4\eta^4 rk\left(\frac{N}{K}\right)^2 + 2\eta pr\left(\frac{N}{K}\right)^2.$$

Where the first term bounds the number of non-red edges between non-regular pairs, the second bounds the number of non-red edges between pairs of clusters that are joined by a red edge in the reduced-graph.

Since $K \geq k$, $N \leq 3n$, $p \leq \frac{1}{2}\alpha_1 k \leq \frac{1}{2}k$ and $r \leq \alpha_3 k \leq 2k$, we obtain

$$e(G_2[X_1, X_3]) + e(G_3[X_1, X_3]) \leq (72\eta^4 + 18\eta) \leq 40\eta n^2.$$

Since $G[X_1, X_3]$ is complete and contains at most $40\eta n^2$ non-red edges, there are at most $7\eta^{1/2}$ vertices in X_1 with red degree to X_3 at most $|X_3| - 7\eta^{1/2}n$ and at most $7\eta^{1/2}$ vertices in X_3 with red degree to X_1 at most $|X_1| - 7\eta^{1/2}n$. Re-assigning these vertices to W results in every vertex in X_1 having degree in $G_1[X_1, X_3]$ at least $|X_3| - 7\eta^{1/2}n$ and every vertex in X_3 having degree in $G_1[X_1, X_3]$ at least $|X_1| - 7\eta^{1/2}n$.

We repeat the above for $G[X_2, X_3]$, removing vertices such that every (remaining) vertex in X_2 has degree in $G_2[X_2, X_3]$ at least $|X_3| - 7\eta^{1/2}n$ and every (remaining) vertex in X_3 has degree in $G_2[X_1, X_3]$ at least $|X_2| - 7\eta^{1/2}n$, thus completing the proof of Claim 2.11.1. □

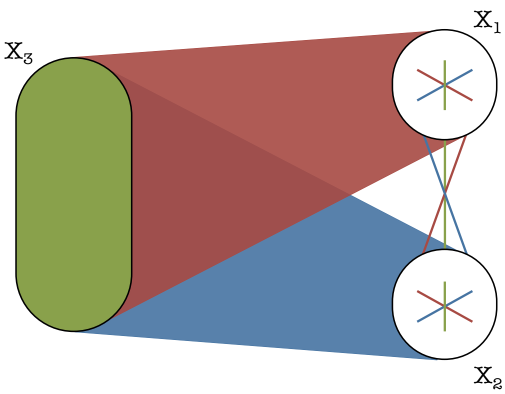


Figure 2.97: Colouring of G after Claim 2.11.1

We now proceed to the end-game: Observe that, by Corollary 2.6.5 there exist red cycles in $G[X_1, X_3]$ of every (non-trivial) even length up to twice the size of the smaller part and blue cycles in $G[X_2, X_3]$ of every (non-trivial) even length up to twice the size of the smaller part. Similarly, by Corollary 2.6.2, there exist green cycles in $G[X_3]$ of every (non-trivial) length up to $|X_3|$. We will show that it is possible to augment each of X_1, X_2, X_3 with vertices from W while maintaining this property. Then, considering the sizes of each part, there must, in fact, be a cycle of appropriate length, colour and parity to complete the proof:

Since $24\eta^{1/2} \leq \eta^{1/4}$, recalling (2.82), having discarded some vertices while proving Claim 2.11.1, we have

$$(\tfrac{1}{2}\alpha_1 - 2\eta^{1/4})n \leq |X_1| < \tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle,$$
$$(\tfrac{1}{2}\alpha_2 - 2\eta^{1/4})n \leq |X_2| < \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle,$$
$$(\alpha_3 - 2\eta^{1/4})n \leq |X_3| < \langle\alpha_3 n\rangle,$$

and know that $G_1[X_1, X_3]$, $G_2[X_2, X_3]$ and $G_3[X_3]$ are each $\eta^{1/4}n$-almost-complete.

Now, let

$$W_G = \{w \in W : w \text{ has at least } 4\eta^{1/4}n \text{ green edges to } X_3\}.$$

Suppose that $|X_3 \cup W_G| \geq \langle\alpha_3 n\rangle$. Then, since $|X_3| \leq \langle\alpha_3 n\rangle$, we may choose a subset X of size $\langle\alpha_3 n\rangle$ from $X_3 \cup W_G$ which includes every vertex from X_3 and $\langle\alpha_3 n\rangle - |X_3|$ vertices from W_G. Then, $G[X]$ has at least $(\alpha_3 - 2\eta^{1/4})n$ vertices of degree at least $(\alpha_3 - 4\eta^{1/4})n$ and at most $2\eta^{1/4}$ vertices of degree at least $4\eta^{1/4}n$, so, by Theorem 2.6.7, $G[X]$ is Hamiltonian and, thus, contains a green cycle of length exactly $\langle\alpha_3 n\rangle$. The existence of such a cycle would be sufficient to complete the proof of Theorem A in this case, so we may assume, instead that $|X_3 \cup W_G| < \langle\alpha_3 n\rangle$.

Thus, letting $W_{RB} = W \backslash W_G$, we may assume that

$$|X_1| + |X_2| + |W_{RB}| \geq \tfrac{1}{2}\langle\langle\alpha_1 n\rangle\rangle + \tfrac{1}{2}\langle\langle\alpha_2 n\rangle\rangle - 1,$$

and, defining

$$W_R = \{w \in W : w \text{ has at least } \tfrac{1}{2}|X_3| - 2\eta^{1/4}n \text{ red edges to } X_3\},$$
$$W_B = \{w \in W : w \text{ has at least } \tfrac{1}{2}|X_3| - 2\eta^{1/4}n \text{ blue edges to } X_3\},$$

may assume, without loss of generality, that $|X_1 \cup W_R| \geq \langle\langle \alpha_1 n \rangle\rangle$.

In that case, let $\widetilde{W}_R \subseteq W_R$ be such that $|X_1| + |\widetilde{W}_R| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. Then, observing that any $w \in W_R$ and $x \in X_1$ have at least $\frac{1}{2}|X_3| - 3\eta^{1/4}n \geq |\widetilde{W}_R|$ common neighbours and that any $x, y \in X_1$ have at least $(\alpha_3 - 2\eta^{1/4})n \geq \alpha_1 n$ common neighbours, we can greedily construct a red cycle of length $\langle\langle \alpha_1 n \rangle\rangle$ using all the vertices of $X_1 \cup \widetilde{W}_R$, thus completing this part of the proof of Theorem A.

2.12 Proof of the main result – Part III – Case (vi)

We now consider the final case, thus, we suppose that $\mathcal{G}$ contains a subgraph K^* from $\mathcal{K}_1^* \cup \mathcal{K}_2^*$, where

$$\begin{aligned}\mathcal{K}_1^* = \mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k,\\ (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big),\end{aligned}$$

$$\begin{aligned}\mathcal{K}_2^* = \mathcal{K}^*\big((\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k, (\tfrac{3}{4}\alpha_3 - 140\eta^{1/2})k,\\ 100\eta^{1/2}k, (\alpha_3 - 10\eta^{1/2})k, 4\eta^4 k\big).\end{aligned}$$

Recalling, Theorem B, we may assume that

$$\alpha_3 \geq \tfrac{3}{2}\alpha_1 + \tfrac{1}{2}\alpha_2 - 10\eta^{1/2}. \tag{K0}$$

Recalling that $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$ are the monochromatic spanning subgraphs of the reduced-graph, we have a partition of the vertex set $\mathcal{V}$ of $\mathcal{G}$ into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ such that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_2]$ are coloured exclusively red, all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_2] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_1]$ are coloured exclusively blue and all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2] \cup \mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2]$ are coloured exclusively green. Also, for any $\mathcal{Z} \subseteq \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2$, $\mathcal{G}[\mathcal{Z}]$ is $4\eta^4 k$-almost-complete.

In each case, before proceeding to consider G, we we must determine more about the the coloured structure of the reduced-graph $\mathcal{G}$. In the process, we will discard further vertices from $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$. As in Section 2.8, these discarded vertices are considered as having been re-assigned to $\mathcal{W}$.

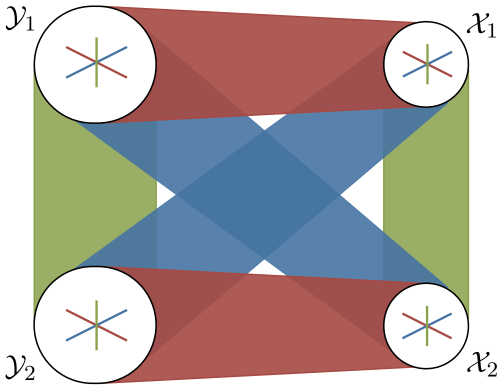


Figure 2.98: Initial coloured structure of the reduced-graph in Part III.

Part III.A: $K^* \in \mathcal{K}_1^*$.

In this case, we have a partition of the vertex set $\mathcal{V}$ of $\mathcal{G}$ into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ with

$$|\mathcal{X}_1|, |\mathcal{X}_2| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k,$$
$$|\mathcal{Y}_1|, |\mathcal{Y}_2| \geq (\tfrac{1}{2}\alpha_1 + 102\eta^{1/2})k,$$
$$|\mathcal{Y}_1| + |\mathcal{Y}_2| \geq (\alpha_3 - 10\eta^{1/2})k.$$

Observe that, since $|\mathcal{X}_1|, |\mathcal{Y}_1| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k$ and $\mathcal{G}_1[\mathcal{X}_1 \cup \mathcal{Y}_1]$ is $4\eta^4k$-complete, $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{Y}_1]$ has a single red component. Similarly, $\mathcal{G}[\mathcal{X}_2 \cup \mathcal{Y}_2]$ has a single red component and each of $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{Y}_2]$ and $\mathcal{G}[\mathcal{X}_2 \cup \mathcal{Y}_1]$ has a single blue component.

Consider $\mathcal{G}[\mathcal{Y}_1]$ and suppose that there exists a red matching $\mathcal{R}_1$ on $198\eta^{1/2}k$ vertices in $G[\mathcal{Y}_1]$. Then, we have $|\mathcal{Y}_1 \backslash \mathcal{V}(\mathcal{R}_1)|, |\mathcal{X}_1| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k$, so, by Lemma 2.6.14, $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1 \backslash \mathcal{V}(\mathcal{R}_1)]$ contains a red connected-matching on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices, which combined with $\mathcal{R}_1$ gives a red connected-matching on at least $\alpha_1 k$ vertices. Thus, there can be no such red matching in $\mathcal{G}[\mathcal{Y}_1]$. Similarly, $\mathcal{G}[\mathcal{Y}_1]$ cannot contain a blue matching on $198\eta^{1/2}k$ vertices.

Thus, after discarding at most $396\eta^{1/2}k$ vertices from $\mathcal{Y}_1$, we may assume that all edges present in $\mathcal{G}[\mathcal{Y}_1]$ are coloured exclusively green. Similarly, after discarding at most $396\eta^{1/2}k$ vertices from $\mathcal{Y}_2$, we may assume that all edges present in $\mathcal{G}[\mathcal{Y}_2]$ are coloured exclusively green.

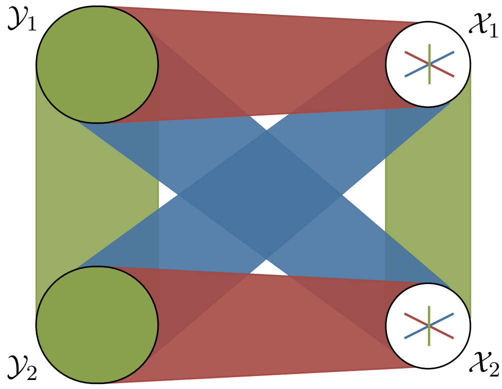


Figure 2.99: Colouring of $\mathcal{G}[\mathcal{Y}_1] \cup \mathcal{G}[\mathcal{Y}_2]$.

Thus, we may assume that we have a partition $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ such that

$$\left.\begin{aligned} (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |\mathcal{X}_1| = |\mathcal{X}_2| = p \leq \tfrac{1}{2}\alpha_1 k, \\ (\tfrac{1}{2}\alpha_1 - 294\eta^{1/2})k \leq |\mathcal{Y}_1| = r, \\ (\tfrac{1}{2}\alpha_1 - 294\eta^{1/2})k \leq |\mathcal{Y}_2| = s. \end{aligned}\right\} \quad (2.83\text{a})$$

Additionally, writing $\mathcal{Y}$ for $\mathcal{Y}_1 \cup \mathcal{Y}_2$, we have

$$(\alpha_3 - 802\eta^{1/2})k \leq |\mathcal{Y}| = r + s \leq \alpha_3 k \quad (2.83\text{b})$$

and know that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_2]$ are coloured exclusively red, all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_2] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_1]$ are coloured exclusively blue and all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2] \cup \mathcal{G}[\mathcal{Y}_1 \cup \mathcal{Y}_2]$ are coloured exclusively green.

Thus far, we have obtained information about the structure of the reduced-graph $\mathcal{G}$. The remainder of this section focuses on showing that the original graph must have a similar structure which can then be exploited to force a cycle of appropriate length, colour and parity. Again, each vertex V_i of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents a cluster of vertices of $G = (V, E)$ with

$$(1 - \eta^4)\frac{N}{K} \leq |V_i| \leq \frac{N}{K}$$

and that, since $n > \max\{n_{2.3.4}(2, 1, 0, \eta), n_{2.3.4}(\frac{1}{2}, \frac{1}{2}, 1, \eta)\}$, we have

$$|V_i| \geq \left(1 + \frac{\eta}{24}\right)\frac{n}{k} > \frac{n}{k}.$$

We partition the vertices of G into sets X_1, X_2, Y_1, Y_2 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{W}$. Then X_1 and X_2 contain p clusters of vertices, Y_1 contains q clusters and Y_2 contains r clusters. Note that, we write Y for $Y_1 \cup Y_2$ and $\mathcal{Y}$ for $\mathcal{Y}_1 \cup \mathcal{Y}_2$. Thus, we have

$$\left.\begin{aligned} |X_1| = |X_2| = p|V_1| &\geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})n, \\ |Y_1| = r|V_1| &\geq (\tfrac{1}{2}\alpha_1 - 294\eta^{1/2})n, \\ |Y_2| = s|V_1| &\geq (\tfrac{1}{2}\alpha_1 - 294\eta^{1/2})n, \\ |Y| = |Y_1| + |Y_2| = (r+s)|V_1| &\geq (\alpha_3 - 802\eta^{1/2})n. \end{aligned}\right\} \quad (2.84)$$

Again, we will remove vertices from $X_1 \cup X_2 \cup Y_1 \cup Y_2$, by moving them into W.

The following claim tells us that the graph G has essentially the same coloured structure as the reduced-graph $\mathcal{G}$:

Claim 2.12.1. *We can remove at most $14\eta^{1/2}n$ vertices from X_1, $14\eta^{1/2}n$ vertices from X_2 and $38\eta^{1/2}n$ vertices from Y such that each of the following holds:*

(KA1) $G_1[X_1, Y_1]$ and $G_1[X_2, Y_2]$ are each $7\eta^{1/2}n$-almost-complete;

(KA2) $G_2[X_1, Y_2]$ and $G_2[X_2, Y_1]$ are each $7\eta^{1/2}n$-almost-complete;

(KA3) $G_3[Y]$ is $10\eta^{1/2}n$-almost-complete.

Proof. Consider the complete three-coloured graph $G[Y]$ and recall that $\mathcal{G}[\mathcal{Y}]$ is $4\eta^4k$-almost-complete and that all edges present in $\mathcal{G}[\mathcal{Y}]$ are coloured exclusively green. Given the construction of $\mathcal{G}$, we can bound the number of non-green edges in $G[Y]$ as follows:

Since regularity provides no indication as to the colours of the edges contained within each cluster, these could potentially all be non-green. There are $r+s$ clusters, each with at most N/K vertices. Thus, there are at most

$$(r+s)\binom{N/K}{2}$$

non-green edges in Y within the clusters of $\mathcal{Y}$.

Now, consider a pair of clusters (U_1, U_2) in Y. If (U_1, U_2) is not η^4-regular, then we can only trivially bound the number of non-green edges in $G[U_1, U_2]$ by $|U_1||U_2| \leq (N/K)^2$.

However, there are at most $4\eta^4|\mathcal{Y}|k$ such pairs in $\mathcal{G}[\mathcal{Y}]$. Thus, we can bound the number of non-red edges coming from non-regular pairs by

$$4\eta^4(r+s)k\left(\frac{N}{K}\right)^2.$$

If the pair is regular, then U_1 and U_2 are joined by an edge in the reduced-graph which is coloured exclusively green. The red density of the pair is at most η (since, if the density were higher, they would be joined by a red edge) and likewise the blue density is at most η. Thus, there are at most

$$2\eta\binom{r+s}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[Y]$ corresponding to such pairs of clusters.

Summing the three possibilities above gives an upper bound of

$$(r+s)\binom{N/K}{2}+4\eta^4(r+s)k\left(\frac{N}{K}\right)^2+2\eta\binom{r+s}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[Y]$.

Since $K\geq k,\eta^{-1}$, $N\leq 3n$ and $r+s\leq\alpha_3 k\leq 2k$, we obtain

$$e(G_1[Y])+e(G_2[Y])\leq[9\eta+72\eta^4+36\eta]n^2\leq 50\eta n^2.$$

Since $G[Y]$ is complete and contains at most $50\eta n^2$ non-green edges, there are at most $10\eta^{1/2}n$ vertices with green degree at most $|Y|-10\eta^{1/2}n$. Re-assigning these vertices to W gives a new Y such that every vertex in $G[Y]$ has red degree at least $|Y|-10\eta^{1/2}n$.

Next, we consider $G[X_1,Y_1]$, bounding the the number of non-red edges in $G[X_1,Y_1]$ by

$$4\eta^4pk\left(\frac{N}{K}\right)^2+2\eta pr\left(\frac{N}{K}\right)^2.$$

Where the first term bounds the number of non-red edges between non-regular pairs and the second bounds the number of non-red edges between regular pairs.

Since $K\geq k$, $N\leq 3n$, $p\leq\frac{1}{2}\alpha_1\leq\frac{1}{2}k$ and $r\leq\alpha_3 k\leq 2k$, we obtain

$$e(G_2[X_1,Y_1])+e(G_3[X_1,Y_1])\leq(18\eta^4+18\eta)n^2\leq 40\eta n^2.$$

Since $G[X_1, Y_1]$ is complete and contains at most $40\eta n^2$ non-red edges, there are at most $7\eta^{1/2}n$ vertices in X_1 with red degree to Y_1 at most $|Y_1| - 7\eta^{1/2}n$ and at most $7\eta^{1/2}n$ vertices in Y_1 with red degree to X_1 at most $|X_1| - 7\eta^{1/2}n$. Re-assigning these vertices to W results in every vertex in X_1 having degree in $G_1[X_1, Y_1]$ at least $|Y_1| - 7\eta^{1/32}n$ and every vertex in Y_1 having degree in $G_1[X_1, Y_1]$ at least $|X_1| - 7\eta^{1/32}n$.

We repeat the above for each of $G[X_1, Y_2], G[X_2, Y_1]$ and $G[X_2, Y_2]$, thus completing the proof of the claim. □

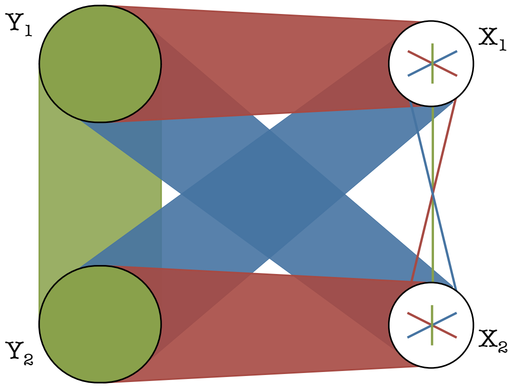


Figure 2.100: Colouring of G after Claim 2.12.1.

Having removed some vertices from X_1, X_2, Y_1 and Y_2, recalling (2.84), we have

$$\left.\begin{aligned} |X_1| = |X_2| &\geq (\tfrac{1}{2}\alpha_1 - 112\eta^{1/2})n, & |Y_1| &\geq (\tfrac{1}{2}\alpha_1 - 332\eta^{1/2})n, \\ |Y| &\geq (\alpha_3 - 840\eta^{1/2})n, & |Y_2| &\geq (\tfrac{1}{2}\alpha_1 - 332\eta^{1/2})n. \end{aligned}\right\} \quad (2.85)$$

Notice also, that, since $\eta \leq (\alpha_2/2500)^2$, without loss of generality, we have

$$\left.\begin{aligned} |Y_1| = \max\{|Y_1|, |Y_2|\} &\geq \tfrac{1}{2}(\alpha_3 - 840\eta^{1/2})n \\ &\geq (\tfrac{3}{4}\alpha_1 + \tfrac{1}{4}\alpha_2 - 430\eta^{1/2})n \geq (\tfrac{1}{2}\alpha_1 + 100\eta^{1/2})n. \end{aligned}\right\} \quad (2.86)$$

Then, recalling (KA1) and (KA2), by Corollary 2.6.5, in order to avoid having a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices in $G[X_1, Y_1]$ or a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices in $G[X_2, Y_1]$, we may assume that $|X_1| < \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and $|X_2| < \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$. Also, recalling (KA3), by Corollary 2.6.2, in order to avoid having a green cycle on exactly $\langle \alpha_3 n \rangle$, we may assume that $|Y| < \langle \alpha_3 n \rangle$.

Now, let

$$W_G = \{w \in W : w \text{ has at least } 850\eta^{1/2}n \text{ green edges to } Y\}.$$

Suppose that $|Y \cup W_G| \geq \langle \alpha_3 n \rangle$. Then, we may choose a set Y' of vertices from $Y \cup W_G$, including every vertex from Y and at most $840\eta^{1/2}n$ vertices from W_G. In that case, $G[Y']$ has at least $(\alpha_3 - 840\eta^{1/2})n$ vertices of degree at least $(\alpha_3 - 850\eta^{1/4})n$ and at most $840\eta^{1/2}n$ vertices of degree at least $850\eta^{1/2}n$, so, by Theorem 2.6.7, $G[Y']$ is Hamiltonian and thus contains a green cycle of length exactly $\langle \alpha_3 n \rangle$. The existence of such a cycle would be sufficient to complete the proof of Theorem A. Thus, we may assume, instead that $|Y \cup W_G| < \langle \alpha_3 n \rangle$.

Thus, letting $W_{RB} = W \backslash W_G$, we may assume that

$$|X_1| + |X_2| + |W_{RB}| \geq \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1. \tag{2.87}$$

By definition every vertex in W_{RB} has at least $|Y| - 850\eta^{1/2}n$ red or blue edges to Y.

Observe that, given any pair of vertices y_{11}, y_{12} in Y_1, we can use Lemma 2.6.6 to establish the existence of a long red path in $G[X_1, Y_1]$ from y_{11} to y_{12}. Likewise, given y_{21}, y_{22} in Y_2, we can use Lemma 2.6.6 to establish the existence of a long red path in $G[X_2, Y_2]$ from y_{21} to y_{22}. Thus, we may prove the following claim:

Claim 2.12.2. *If there exist distinct vertices $y_{11}, y_{12} \in Y_1$, $y_{21}, y_{22} \in Y_2$ and $w_1, w_2 \in W$ such that the edges $w_1y_{11}, w_2y_{12}, w_1y_{21}$ and w_2y_{22} are all coloured red, then G contains a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$.*

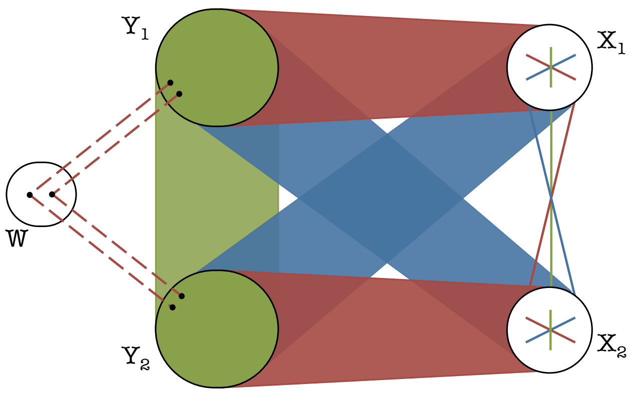


Figure 2.101: Existance of two red vertex-disjoint paths between X_1 and Y_1.

Proof. Suppose there exist distinct vertices $y_{11}, y_{12} \in Y_1$, $y_{21}, y_{22} \in Y_2$ and $w_1, w_2 \in W$ such that the edges $w_1y_{11}, w_2y_{12}, w_1y_{21}$ and w_2y_{22} are all coloured red.

Then, let $\widetilde{X}_1$ be any set of

$$\ell_1 = \left\lfloor \frac{\langle\langle \alpha_1 n \rangle\rangle - 4}{4} \right\rfloor \geq 7\eta^{1/2}n + 2$$

vertices from X_1.

By (KA1), y_{11} and y_{12} each have at least two neighbours in $\widetilde{X}_1$ and, since $\eta \leq (\alpha_1/100)^2$, every vertex in $\widetilde{X}_1$ has degree at least $|Y_1| - 7\eta^{1/2}n \geq \frac{1}{2}|Y_1| + \frac{1}{2}|\widetilde{X}_1| + 1$ in $G[\widetilde{X}_1, Y]$. Then, Since $|Y_1| > \ell_1 + 1$, by Lemma 2.6.6, $G_1[\widetilde{X}_1, Y_1]$ contains a red path R_1 on exactly $2\ell_1 + 1$ vertices from y_{11} to y_{12}.

Similarly, letting $\widetilde{X}_2$ be any set of

$$\ell_2 = \left\lceil \frac{\langle\langle \alpha_1 n \rangle\rangle - 4}{4} \right\rceil \geq 7\eta^{1/2}n + 2$$

vertices from X_2, by Lemma 2.6.6, $G_1[\widetilde{X}_2, Y_2]$ contains a red path R_2 on exactly $2\ell_2 + 1$ vertices from y_{21} to y_{22}. Then, combining R_1 and R_2 with $y_{11}w_1y_{21}$ and $y_{12}w_2y_{22}$ gives a red cycle on exactly $2\ell_1 + 2\ell_2 + 4 = \langle\langle \alpha_1 n \rangle\rangle$ vertices. □

Similarly, the existence of two such vertex-disjoint paths blue paths from Y_1 to Y_2 via W would result in a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices. The existence of such a red or blue cycle would be sufficient to complete the proof of Theorem A. Therefore, we may assume that there can be at most two vertices in W_{RB} with red edges to both Y_1 and Y_2 or blue edges to both Y_1 and Y_2.

We denote by W^* the (possibly empty) set of vertices having either red edges to both Y_1 and Y_2 or blue edges to both Y_1 and Y_2. Then, letting $W^*_{RB} = W_{RB} \backslash W^*$, observe that we may partition W^*_{RB} into $W_1 \cup W_2$ such that there are no blue edges in $G[W_1, Y_1] \cup G[W_2, Y_2]$ and no red edges in $G[W_1, Y_2] \cup G[W_2, Y_1]$. Then, recalling that every vertex in W_{RB} has at most $850\eta^{1/2}n$ green edges to Y, we know that

(KA4a) every vertex in W_1 has red degree at least $|Y_1| - 850\eta^{1/2}n$ in $G[W_1, Y_1]$;

(KA4b) every vertex in W_1 has blue degree at least $|Y_2| - 850\eta^{1/2}n$ in $G[W_1, Y_2]$;

(KA4c) every vertex in W_2 has red degree at least $|Y_2| - 850\eta^{1/2}n$ in $G[W_2, Y_2]$;

(KA4d) every vertex in W_2 has blue degree at least $|Y_1| - 850\eta^{1/2}n$ in $G[W_2, Y_1]$.

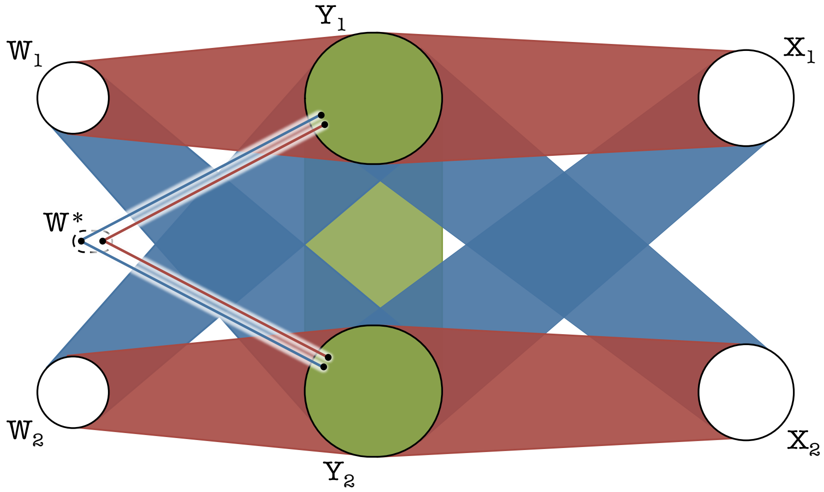


Figure 2.102: Partition of W_{RB} into $W_1 \cup W_2 \cup W^*$.

Then, since $W_{RB} = W_1 \cup W_2 \cup W^*$, by (2.87), we have

$$|X_1| + |X_2| + |W_1| + |W_2| + |W^*| \geq \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1.$$

Thus, one of the following must occur:

(i) $|X_1| + |W_1| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$;

(ii) $|X_2| + |W_2| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$;

(iii) $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 1$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$, and $|W^*| = 1$;

(iv) $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 2$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$, and $|W^*| = 2$;

(v) $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 1$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 2$, and $|W^*| = 2$.

In each case, we can show that G contains either a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices or a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices as follows:

(i) Suppose $|X_1| + |W_1| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and recall that, by (2.86), $|Y_1| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. Then, by (2.85), we may choose $\widetilde{X}_1 \in X_1 \cup W_1$ and $\widetilde{Y}_1 \in Y_1$ such that $|\widetilde{X}_1| = |\widetilde{Y}_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and $|\widetilde{X}_1 \cap W_1| \leq 114\eta^{1/2}n$. By (KA1), every vertex in $\widetilde{Y}_1$ has at least $|\widetilde{X}_1 \cap X_1| - 7\eta^{1/2}n$ red neighbours in $\widetilde{X}_1 \cap X_1$, that is, at least $|\widetilde{X}_1| - 121\eta^{1/2}n$ red neighbours in $\widetilde{X}_1$. By (KA1) and (KA4a), every vertex in $\widetilde{X}_1$ has at least $|\widetilde{Y}_1| - 850\eta^{1/2}n$ red neighbours in $\widetilde{Y}_1$. Thus, for any $x \in \widetilde{X}_1, y \in \widetilde{Y}_1$, $d(x) + d(y) \geq |\widetilde{X}_1| + |\widetilde{Y}_1| - 971\eta^{1/2}n \geq \frac{1}{2}|\widetilde{X}_1| + \frac{1}{2}|\widetilde{Y}_1| + 1$. So, by Theorem 2.6.4, there exists a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices in $G[\widetilde{X}_1, \widetilde{Y}_1]$.

(ii) Suppose $|X_2| + |W_2| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$ and recall that, by (2.86), $|Y_1| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$. Then, by (2.85), we may choose $\widetilde{X}_2 \in X_2 \cup W_2$ and $\widetilde{Y}_1 \in Y_1$ such that $|\widetilde{X}_2| = |\widetilde{Y}_1| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$ and $|\widetilde{X}_2 \cap W_2| \leq 114\eta^{1/2}n$. By (KA2), every vertex in $\widetilde{Y}_1$ has at least $|\widetilde{X}_2 \cap X_2| - 7\eta^{1/2}n$ blue neighbours in $\widetilde{X}_2 \cap X_2$, that is, at least $|\widetilde{X}_2| - 121\eta^{1/2}n$ neighbours in $\widetilde{X}_2$. By (KA2) and (KA4d), every vertex in $\widetilde{X}_2$ has at least $|\widetilde{Y}_1| - 850\eta^{1/2}n$ blue neighbours in $\widetilde{Y}_1$. Thus, for any $x \in \widetilde{X}_2, y \in \widetilde{Y}_1$, $d(x) + d(y) \geq |\widetilde{X}_2| + |\widetilde{Y}_1| - 971\eta^{1/2}n \geq \frac{1}{2}|\widetilde{X}_2| + \frac{1}{2}|\widetilde{Y}_1| + 1$. So, by Theorem 2.6.4, there exists a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices in $G[\widetilde{X}_2, \widetilde{Y}_2]$.

(iii) Suppose that $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 1$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$ and $|W^*| = 1$. Consider $w \in W^*$. Since $W^* \subset W_{RB}$, we know that w has green edges to at most $850\eta^{1/2}n$ of the vertices of $Y = Y_1 \cup Y_2$. Thus, w either has red edges to at least two vertices in Y_1 or blue edges to at least two vertices in Y_1. We denote two of these as y_1 and y_2.

In the former case, by (KA1), y_1 and y_2 each have at least two red neighbours in X_1. By (2.86), we have $|Y_1| > |W_1| + |X_1| + 1$. By (KA1), (KA4a) and (2.86), since $\eta \leq (\alpha_2/5000)^2$, every vertex in $W_1 \cup X_1$ has at least $|Y_1| - 850\eta^{1/2}n \geq \frac{1}{2}(|W_1| + |X_1| + |Y_1|) + 1$ red neighbours in Y_1. Thus, by Lemma 2.6.6, there exists a red path in $G[X_1 \cup W_1, Y_1]$ from y_1 to y_2 which visits every vertex of $X_1 \cup W_1$. This path, together with the red edges wy_1 and wy_2, forms a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices.

In the latter case, by (KA2), y_1 and y_2 each have at least two blue neighbours in X_2. By (2.86), we have $|Y_1| > |W_2| + |X_2| + 1$. By (KA2), (KA4d) and (2.86), since $\eta \leq (\alpha_2/5000)^2$, every vertex in $W_2 \cup X_2$ has at least $|Y_1| - 850\eta^{1/2}n \geq \frac{1}{2}(|W_2| + |X_2| + |Y_1|) + 1$ blue neighbours in Y_1. Thus, by Lemma 2.6.6, there exists a blue path in $G[X_2 \cup W_2, Y_1]$ from y_1 to y_2 which visits every vertex of $X_2 \cup W_2$. This path, together with the blue edges wy_1 and wy_2, forms a blue cycle on exactly $\langle\langle \alpha_2 n \rangle\rangle$ vertices.

(vi) Suppose that $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 2$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$ and $|W^*| = 2$. Then, considering $w_1, w_2 \in W^*$, since $W^* \subset W_{RB}$, we know that w_1 and w_2 each have green edges to at most $850\eta^{1/2}n$ of the vertices of $Y = Y_1 \cup Y_2$. Thus, either one of w_1, w_2 has blue edges to two distinct vertices in Y_1 or both have at least $|Y_1| - 900\eta^{1/2}n$ red neighbours in Y_1. In the former case, the situation is identical to one already considered in (iii): We have $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$ and know of the existence of a vertex in W

with blue edges to two distinct vertices in Y_1.

In the latter case, the situation is similar to the one considered in (i): We have $|X_1| + |W_1| + |W^*| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and $|Y_1| \geq \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. By (2.85), we may choose $\widetilde{X}_1 \in X_1 \cup W_1 \cup W^*$ and $\widetilde{Y}_1 \in Y_1$ such that $|\widetilde{X}_1| = |\widetilde{Y}_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ and $|\widetilde{X}_1 \cap (W_1 \cup W^*)| \leq 114\eta^{1/2}n$. By (KA1) and (KA4a), every vertex in $\widetilde{Y}_1$ at least $|\widetilde{X}_1| - 121\eta^{1/2}n$ red neighbours in $\widetilde{X}_1$ and every vertex in $\widetilde{X}_1$ has at least $|\widetilde{Y}_1| - 900\eta^{1/2}n$ red neighbours in $\widetilde{Y}_1$. Thus, for any $x \in \widetilde{X}_1, y \in \widetilde{Y}_1$, $d(x) + d(y) \geq |\widetilde{X}_1| + |\widetilde{Y}_1| - 1021\eta^{1/2}n \geq \frac{1}{2}|\widetilde{X}_1| + \frac{1}{2}|\widetilde{Y}_1| + 1$. So, by Theorem 2.6.4, there exists a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices in $G[\widetilde{X}_1, \widetilde{Y}_1]$.

(v) Suppose that $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 1$, $|X_2| + |W_2| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 2$ and $|W^*| = 2$. Then, considering $w_1, w_2 \in W^*$, since $W^* \subset W_{RB}$, we know that w_1 and w_2 each have green edges to at most $850\eta^{1/2}n$ of the vertices of Y. Thus, either one of w_1, w_2 has red edges to two distinct vertices in Y_1 or both have at least $|Y_1| - 900\eta^{1/2}n$ blue neighbours in Y_1. In the former case, the situation is identical to one already considered in (iii): We have $|X_1| + |W_1| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle - 1$ and know of the existence of a vertex in W with red edges to two distinct vertices in Y_1.

In the latter case, the situation is similar to the one considered in (ii): We have $|X_2| + |W_2| + |W^*| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$ and $|Y_1| \geq \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$. By (2.85), we may choose $\widetilde{X}_2 \in X_2 \cup W_2 \cup W^*$ and $\widetilde{Y}_1 \in Y_1$ such that $|\widetilde{X}_2| = |\widetilde{Y}_1| = \frac{1}{2}\langle\langle \alpha_2 n \rangle\rangle$ and $|\widetilde{X}_2 \cap (W_2 \cup W^*)| \leq 114\eta^{1/2}n$. By (KA2) and (KA4d), every vertex in $\widetilde{Y}_1$ at least $|\widetilde{X}_2| - 121\eta^{1/2}n$ blue neighbours in $\widetilde{X}_2$ and every vertex in $\widetilde{X}_2$ has at least $|\widetilde{Y}_1| - 900\eta^{1/2}n$ blue neighbours in $\widetilde{Y}_1$. Thus, for any $x \in \widetilde{X}_2, y \in \widetilde{Y}_1$, $d(x) + d(y) \geq |\widetilde{X}_1| + |\widetilde{Y}_1| - 1021\eta^{1/2}n \geq \frac{1}{2}|\widetilde{X}_2| + \frac{1}{2}|\widetilde{Y}_1| + 1$. So, by Theorem 2.6.4, there exists a red cycle on exactly $\langle\langle \alpha_1 n \rangle\rangle$ vertices in $G[\widetilde{X}_2, \widetilde{Y}_1]$.

The existence of such a red or blue cycle would be sufficient to complete the proof of Theorem A. Thus, we have completed Part III.A.

Part III.B: $K^* \in \mathcal{K}_2^*$**.**

In this case, we have a partition the vertex set $\mathcal{V}$ of $\mathcal{G}$ into $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ with

$$|\mathcal{X}_1| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k, \qquad |\mathcal{X}_2| \geq (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \tag{2.88a}$$

and, writing $\mathcal{Y}$ for $\mathcal{Y}_1 \cup \mathcal{Y}_2$,

$$|\mathcal{Y}_1| \geq (\tfrac{3}{4}\alpha_3 - 140\eta^{1/2})k, \qquad |\mathcal{Y}_2| \geq 100\eta^{1/2}k, \qquad |\mathcal{Y}| \geq (\alpha_3 - 10\eta^{1/2})k, \tag{2.88b}$$

such that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_2]$ are coloured exclusively red, all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_2] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_1]$ are coloured exclusively blue and all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2] \cup \mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2]$ are coloured exclusively green (see Figure 2.98). Also, for any $\mathcal{Z} \subseteq \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2$, $\mathcal{G}[\mathcal{Z}]$ is $4\eta^4 k$-almost-complete.

Observe that, since $\mathcal{G}_1[\mathcal{X}_1 \cup \mathcal{Y}_1]$ is $4\eta^4 k$-complete, it has a single red component. Similarly, $\mathcal{G}[\mathcal{X}_2 \cup \mathcal{Y}_2]$ has a single red component and each of $\mathcal{G}[\mathcal{X}_1 \cup \mathcal{Y}_2]$ and $\mathcal{G}[\mathcal{X}_2 \cup \mathcal{Y}_1]$ has a single blue component.

Consider $\mathcal{G}[\mathcal{Y}_1]$ and suppose that there exists a red matching $\mathcal{R}_1$ on $198\eta^{1/2}k$ vertices in $G[\mathcal{Y}_1]$. Then, we have $|\mathcal{Y}_1 \backslash \mathcal{V}(\mathcal{R}_1)|, |\mathcal{X}_1| \geq (\frac{1}{2}\alpha_1 - 97\eta^{1/2})k$, so, by Lemma 2.6.14, $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1 \backslash \mathcal{V}(\mathcal{R}_1)]$ contains a red connected-matching on at least $(\alpha_1 - 196\eta^{1/2})k$ vertices, which combined with $\mathcal{R}_1$ gives a red connected-matching on at least $\alpha_1 k$ vertices. Thus, there can be no such red matching in $\mathcal{G}[\mathcal{Y}_1]$. Similarly, $\mathcal{G}[\mathcal{Y}_1]$ cannot contain a blue matching on $198\eta^{1/2}k$ vertices. Thus, after discarding at most $396\eta^{1/2}k$ vertices from $\mathcal{Y}_1$, we may assume that all edges present in $\mathcal{G}[\mathcal{Y}_1]$ are coloured exclusively green.

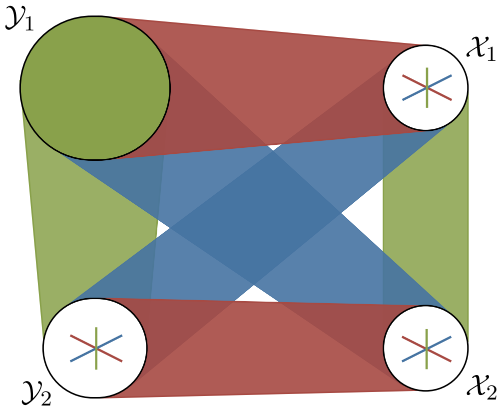


Figure 2.103: Colouring of $\mathcal{G}[\mathcal{Y}_1]$.

After discarding these vertices, recalling (2.88a) and (2.88b), we may assume that we have a partition $\mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{W}$ such that

$$\left.\begin{aligned} (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})k \leq |\mathcal{X}_1| = p \leq \tfrac{1}{2}\alpha_1 k, \\ (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})k \leq |\mathcal{X}_2| = q \leq \tfrac{1}{2}\alpha_2 k, \\ (\tfrac{3}{4}\alpha_3 - 536\eta^{1/2})k \leq |\mathcal{Y}_1| = r \leq (\alpha_3 - 100\eta^{1/2})k, \\ 100\eta^{1/2}k \leq |\mathcal{Y}_2| = s \leq (\tfrac{1}{4}\alpha_3 + 536\eta^{1/2})k. \end{aligned}\right\} \quad (2.89a)$$

Additionally, writing $\mathcal{Y}$ for $\mathcal{Y}_1 \cup \mathcal{Y}_2$, we have

$$(\alpha_3 - 406\eta^{1/2})k \leq |\mathcal{Y}| = r + s \leq \alpha_3 k \tag{2.89b}$$

and know that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_1] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_2]$ are coloured exclusively red, all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{Y}_2] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{Y}_1]$ are coloured exclusively blue and all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2] \cup \mathcal{G}[\mathcal{Y}_1, \mathcal{Y}_2] \cup \mathcal{G}[\mathcal{Y}_1]$ are coloured exclusively green.

The remainder of this section focuses on showing that the original graph must have a similar structure which can then be exploited to force a cycle of appropriate length, colour and parity. Again, each vertex V_i of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ represents a cluster of vertices of $G = (V, E)$ with

$$(1 - \eta^4)\frac{N}{K} \leq |V_i| \leq \frac{N}{K}$$

and, since $n > \max\{n_{2.3.4}(2, 1, 0, \eta), n_{2.3.4}(\frac{1}{2}, \frac{1}{2}, 1, \eta)\}$, we have

$$|V_i| \geq \left(1 + \frac{\eta}{24}\right)\frac{n}{k} > \frac{n}{k}.$$

Simillarly, we may show that

$$|V_i| \leq \left(1 + \eta\right)\frac{n}{k}.$$

We partition the vertices of G into sets X_1, X_2, Y_1, Y_2, and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{W}$. Then X_1 contains p clusters of vertices, X_2 contains p clusters, Y_1 contains r clusters and Y_2 contains s clusters.

Writing Y for $Y_1 \cup Y_2$ and recalling (2.89a) and (2.89b), we have

$$\left.\begin{aligned} &|X_1| = p|V_1| \geq (\tfrac{1}{2}\alpha_1 - 97\eta^{1/2})n, \qquad &&|Y_1| = r|V_1| \geq (\tfrac{3}{4}\alpha_3 - 536\eta^{1/2})n, \\ &|X_2| = p|V_1| \geq (\tfrac{1}{2}\alpha_2 - 97\eta^{1/2})n, \qquad &&|Y_2| = s|V_1| \geq 100\eta^{1/2}n, \\ & &&|Y| = (r + s)|V_1| \geq (\alpha_3 - 406\eta^{1/2})n. \end{aligned}\right\} \tag{2.90}$$

In what follows, we will remove vertices from $X_1 \cup X_2 \cup Y_1 \cup Y_2$ by moving them into W. The following claim tells us that the graph G has essentially the same coloured structure as the reduced-graph $\mathcal{G}$:

Claim 2.12.3. *We can remove at most $7\eta^{1/2}n$ vertices from X_1, $7\eta^{1/2}n$ vertices from X_2, $31\eta^{1/2}n$ vertices from Y_1 and $7\eta^{1/2}n$ vertices from Y_2 such that each of the following holds:*

(KB1) $G_1[X_1, Y_1]$ is $7\eta^{1/2}n$-almost-complete;

(KB2) $G_2[X_2, Y_1]$ is $7\eta^{1/2}n$-almost-complete;

(KB3) $G_3[Y_1]$ is $10\eta^{1/2}n$-almost-complete and $G_3[Y_1, Y_2]$ is $7\eta^{1/2}n$-almost-complete.

Proof. Consider the complete three-coloured graph $G[Y_1]$ and recall that $\mathcal{G}[\mathcal{Y}_1]$ is $4\eta^4k$-almost-complete and that all edges present in $\mathcal{G}[\mathcal{Y}_1]$ are coloured exclusively green. Given the construction of $\mathcal{G}$, we can bound the number of non-green edges in $G[Y_1]$ as follows:

Since regularity provides no indication as to the colours of the edges contained within each cluster, these could potentially all be non-green. There are r clusters, each with at most N/K vertices. Thus, there are at most

$$r\binom{N/K}{2}$$

non-green edges in Y within the clusters of $\mathcal{Y}_1$.

Now, consider a pair of clusters (U_1, U_2) in Y_1. If (U_1, U_2) is not η^4-regular, then we can only trivially bound the number of non-green edges in $G[U_1, U_2]$ by $|U_1||U_2| \leq (N/K)^2$. However, there are at most $4\eta^4|\mathcal{Y}_1|k$ such pairs in $\mathcal{G}[\mathcal{Y}_1]$. Thus, we can bound the number of non-red edges coming from non-regular pairs by

$$4\eta^4 rk\left(\frac{N}{K}\right)^2.$$

If the pair is regular, then U_1 and U_2 are joined by an edge in the reduced-graph which is coloured exclusively green. The red density of the pair is at most η (since, if the density were higher, the edge would also be coloured red). Likewise the blue density is at most η. Thus, there are at most

$$2\eta\binom{r}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[Y_1]$ corresponding to such pairs of clusters.

Summing the three possibilities above gives an upper bound of

$$r\binom{N/K}{2} + 4\eta^4 rk\left(\frac{N}{K}\right)^2 + 2\eta\binom{r}{2}\left(\frac{N}{K}\right)^2$$

non-green edges in $G[Y_1]$.

Since $K \geq k, \eta^{-1}$, $N \leq 3n$ and $r \leq \alpha_3 k \leq 2k$, we obtain

$$e(G_1[Y]) + e(G_2[Y]) \leq [9\eta + 72\eta^4 + 36\eta]n^2 \leq 50\eta n^2.$$

Since $G[Y_1]$ is complete and contains at most $50\eta n^2$ non-green edges, there are at most $10\eta^{1/2}n$ vertices with green degree at most $|Y_1| - 10\eta^{1/2}n$. Re-assigning these vertices to W gives a new Y_1 such that every vertex in $G[Y_1]$ has red degree at least $|Y_1| - 10\eta^{1/2}n$.

Now, consider $G[X_1, Y_1]$, bounding the the number of non-red edges in $G[X_1, Y_1]$ by

$$4\eta^4 pk \left(\frac{N}{K}\right)^2 + 2\eta pr \left(\frac{N}{K}\right)^2.$$

Where the first term bounds the number of non-red edges between non-regular pairs and the second bounds the number of non-red edges between regular pairs.

Since $K \geq k$, $N \leq 3n$, $p \leq \frac{1}{2}\alpha_1 k \leq \frac{1}{2}k$ and $r \leq \alpha_3 k \leq 2k$, we obtain

$$e(G_2[X_1, Y_1]) + e(G_3[X_1, Y_1]) \leq (18\eta^4 + 18\eta)n^2 \leq 40\eta n^2.$$

Since $G[X_1, Y_1]$ is complete and contains at most $40\eta n^2$ non-red edges, there are at most $7\eta^{1/2}n$ vertices in X_1 with red degree to Y_1 at most $|Y_1| - 7\eta^{1/2}n$ and at most $7\eta^{1/2}n$ vertices in Y_1 with red degree to X_1 at most $|X_1| - 7\eta^{1/2}n$. Re-assigning these vertices to W results in every vertex in X_1 having degree in $G_1[X_1, Y_1]$ at least $|Y_1| - 7\eta^{1/32}n$ and every vertex in Y_1 having degree in $G_1[X_1, Y_1]$ at least $|X_1| - 7\eta^{1/32}n$.

By the same argument we can show that each of $G_2[X_2, Y_1]$ and $G_3[Y_1, Y_2]$, are $7\eta^{1/2}n$-almost-complete, thus completing the proof of the claim. □

Following the removals in Claim 2.12.3, recalling (2.90), we have

$$\left.\begin{aligned} |X_1| &\geq (\tfrac{1}{2}\alpha_1 - 104\eta^{1/2})n, & |Y_1| &\geq (\tfrac{3}{4}\alpha_3 - 567\eta^{1/2})n, \\ |X_2| &\geq (\tfrac{1}{2}\alpha_2 - 104\eta^{1/2})n, & |Y_2| &\geq 93\eta^{1/2}n, \\ & & |Y| &\geq (\alpha_3 - 444\eta^{1/2})n. \end{aligned}\right\} \quad (2.91)$$

and know that $G_1[X_1, Y_1]$, $G_2[X_2, Y_1]$ and $G_3[Y_1, Y_2]$ are each $7\eta^{1/2}n$-almost-complete and $G_3[Y_1]$ is $10\eta^{1/2}n$-almost-complete.

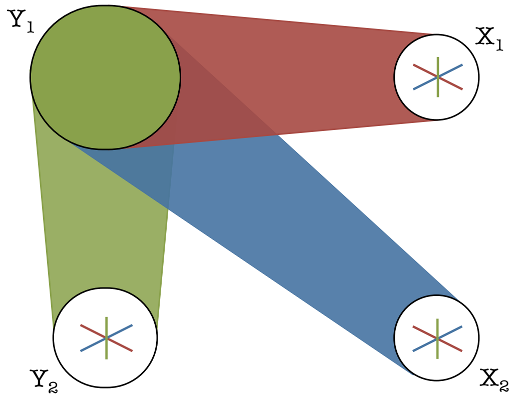


Figure 2.104: Colouring of G after Claim 2.12.3.

Observe that, by Corollary 2.6.5, there exist red cycles in $G[X_1, Y_1]$ of every (non-trivial) even length up to twice the size of the smaller part and blue cycles in $G[X_2, Y_1]$ of every (non-trivial) even length up to twice the size of the smaller part. Additionally, we may use use Corollary 2.6.3 and Lemma 2.6.6 to obtain a green cycle in $G[Y]$ of any length up to $|Y| = |Y_1| + |Y_2|$ as follows:

Recalling (2.89a), we have $s \leq (\frac{1}{4}\alpha_3 + 536\eta^{1/2})k$. Then, $|Y_2| = s|V_1| \leq (1+\eta)(\frac{1}{4}\alpha_3 + 536\eta^{1/2})n \leq (\frac{1}{4}\alpha_3 + 538\eta^{1/2})n$. Thus, we have $|Y_1| \geq (\frac{3}{4}\alpha_3 - 567\eta^{1/2}) \geq |Y_2| + 14\eta^{1/2}n + 2$. Thus, since $G_3[Y_1]$ is $10\eta^{1/2}n$-almost-complete and $G_3[Y_1, Y_2]$ is $7\eta^{1/2}n$-almost-complete, every vertex in Y_1 has degree at least two in $G_3[Y_1, Y_2]$ and every vertex in Y_2 has degree at least $\frac{1}{2}(|Y_1| + |Y_2|) + 1$ in $G_3[Y_1, Y_2]$. Therefore, by Lemma 2.6.6, given any two vertices y_1 and y_2 in Y_1, there exists a green path P_1 on $2|Y_2| + 1$ vertices from y_1 to y_2 in $G[Y_1, Y_2]$.

Let Y_1' be a subset of $\big(Y_1 \backslash V(P_1)\big) \cup \{y_1, y_2\}$ such that $y_1, y_2 \in Y_1'$ and $|Y_1'| \geq 20\eta^{1/2}n + 2$. Then, since $G_3[Y_1]$ is $10\eta^{1/2}n$-almost-complete, every vertex in Y_1' has degree at least $\frac{1}{2}|Y_1'| + 1$ in $G_3[Y_1']$ and so, by Corollary 2.6.3, there exists a green path P_2 on $|Y_1'|$ vertices from y_1 to y_2 in $G[Y_1']$. Together, the green paths P_1 and P_2 form a green cycle on exactly $|Y_1'| + |Y_2|$ vertices.

Thus, we may assume that

$$|X_1| \leq \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle, \qquad |X_2| \leq \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle, \qquad |Y| \leq \langle \alpha_3 n \rangle. \tag{2.92}$$

We will show that it is possible to augment each of X_1, X_2, Y with vertices from W and that, considering the sizes of each part, there must in fact be a cycle of appropriate length, colour and parity to complete the proof.

Observe that, by (KB3) and (2.91), every vertex in Y_1 has degree at least

$$(\tfrac{3}{4}\alpha_3 - 577\eta^{1/2})n \geq \langle \alpha_3 n \rangle - (\tfrac{1}{4}\alpha_3 + 577\eta^{1/2})n$$

in $G_3[Y_1] \subseteq G_3[Y]$ and every vertex in Y_2 has degree at least

$$(\tfrac{3}{4}\alpha_3 - 574\eta^{1/2})n \geq \langle \alpha_3 n \rangle - (\tfrac{1}{4}\alpha_3 + 574\eta^{1/2})n$$

in $G_3[Y_1, Y_2] \subseteq G_3[Y]$. Then, let

$$W_G = \{w \in W : w \text{ has at least } (\tfrac{1}{4}\alpha_3 + 578\eta^{1/2})n \text{ green edges to } Y_1\}$$

and suppose that $|W_G \cup Y_1 \cup Y_2| \geq \langle \alpha_3 n \rangle$.

In that case, since $|Y| \leq \langle \alpha_3 n \rangle$, we we may choose a subset $\widetilde{Y}$ of size $\langle \alpha_3 n \rangle$ from $W_G \cup Y_1 \cup Y_2$ which includes every vertex of $Y_1 \cup Y_2$ and $\langle \alpha_3 n \rangle - |Y|$ vertices from W_G. Then, by (2.91), $\widetilde{Y}$ includes at least $(\alpha_3 - 444\eta^{1/2})n$ vertices from $Y_1 \cup Y_2$ and at most $445\eta^{1/2}n$ vertices from W_G. Thus, $G[Y]$ has at least $(\alpha_3 - 444\eta^{1/2})n$ vertices of degree at least $\langle \alpha_3 n \rangle - (\frac{1}{4}\alpha_3 + 577\eta^{1/2})n$ and at most $445\eta^{1/2}$ vertices of degree at least $(\frac{1}{4}\alpha_3 + 578\eta^{1/2})n$. Therefore, by Theorem 2.6.7, $G[Y]$ is Hamiltonian and thus contains a green cycle of length exactly $\langle \alpha_3 n \rangle$. The existence of such a cycle would be sufficient to complete the proof of Theorem A in this case so we may assume instead that $|X_3 \cup W_G| < \langle \alpha_3 n \rangle$.

Thus, letting $W_{RB} = W \backslash W_G$, we may assume that

$$|X_1| + |X_2| + |W_{RB}| \geq \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle - 1$$

and, defining

$$W_R = \{w \in W : w \text{ has at least } (\tfrac{1}{2}\alpha_1 - 575\eta^{1/2})n \text{ red edges to } Y_1\},$$
$$W_B = \{w \in W : w \text{ has at least } (\tfrac{1}{2}\alpha_2 - 575\eta^{1/2})n \text{ blue edges to } Y_1\},$$

may assume, without loss of generality, that $|X_1 \cup W_R| \geq \langle\langle \alpha_1 n \rangle\rangle$.

In that case, by (2.92), we may choose a subset $\widetilde{W_R} \subseteq W_R$ such that $|X_1| + |\widetilde{W_R}| = \frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$. By (2.91), we have $|\widetilde{W_R}| \leq 106\eta^{1/2}n$. Then, observing that any $w \in W_R$ and $x \in X_1$ have at least $(\frac{1}{2}\alpha_1 - 582\eta^{1/2})n \geq |\widetilde{W_R}|$ common neighbours and that any

$x, y \in X_1$ have at least $(\frac{3}{4}\alpha_3 - 581\eta^{1/2})n \geq \alpha_1 n$ common neighbours, we can greedily construct a red cycle of length $\langle\langle \alpha_1 n \rangle\rangle$ using all the vertices of $X_1 \cup \widetilde{W}_R$ and $\frac{1}{2}\langle\langle \alpha_1 n \rangle\rangle$ vertices from Y_1, completing Part III of the proof of Theorem A.

Observing that we have exhausted all the possibilities arising from Theorem B and that the graphs providing the corresponding lower bounds have already been seen (in Section 2.1), we have proved that, given $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_1 \geq \alpha_2$, there exists $n_A = n_A(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_A$,

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle\langle \alpha_2 n \rangle\rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{2\langle\langle \alpha_1 n \rangle\rangle + \langle\langle \alpha_2 n \rangle\rangle - 3,\ \tfrac{1}{2}\langle\langle \alpha_1 n \rangle\rangle + \tfrac{1}{2}\langle\langle \alpha_2 n \rangle\rangle + \langle \alpha_3 n \rangle - 2\},$$

thus completing the the proof of Theorem A. □

2.13 The even-even-odd case

In this section, we consider the complementary mixed-parity case, for which Figaj and Łuczak [FŁ07b] proved that, for all $\alpha_1, \alpha_2, \alpha_3 > 0$,

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\}n + o(n),$$

as $n \to \infty$.

Improving on their result, we can prove the following:

Theorem C. *For every $\alpha_1, \alpha_2, \alpha_3 > 0$ such that $\alpha_2 \geq \alpha_3$, there exists a positive integer $n_C = n_C(\alpha_1, \alpha_2, \alpha_3)$ such that, for $n > n_C$,*

$$R(C_{\langle\langle \alpha_1 n \rangle\rangle}, C_{\langle \alpha_2 n \rangle}, C_{\langle \alpha_3 n \rangle}) = \max\{4\langle\langle \alpha_1 n \rangle\rangle - 3, \langle\langle \alpha_1 n \rangle\rangle + 2\langle \alpha_2 n \rangle - 3\}.$$

The full proof of this result is, necessarily, reasonably long. As a standalone proof, it would be comparable in length to this chapter, that is, circa 160 pages. It can, however, be shortened significantly, to about 35 pages, by quoting results (including Theorem B) from this chapter and a three-colour stability result from [KSS09a]. For the sake of space, time and sanity, we postpone the full proof of this result to [Fer13] and offer only an outline here.

The well-known structures shown in Figure 2.105 and 2.106 provide the lower bound:

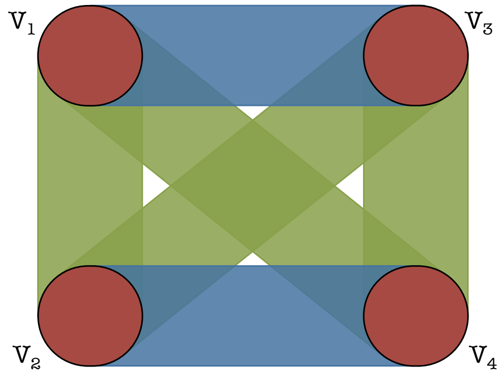


Figure 2.105: First extremal colouring for Theorem C.

The graph shown in Figure 2.105 has $4\langle\langle\alpha_1 n\rangle\rangle - 4$ vertices, divided into four equally-sized classes V_1, V_2, V_3 and V_4 such that all edges in $G[V_1]$, $G[V_2]$, $G[V_3]$ and $G[V_4]$ are coloured red, all edges in $G[V_1, V_3]$ and $G[V_2, V_4]$ are coloured blue and all edges in $G[V_1 \cup V_3, V_2 \cup V_4]$ are coloured green.

The graph shown in Figure 2.106 has $\langle\langle\alpha_1 n\rangle\rangle + 2\langle\alpha_2 n\rangle - 4$ vertices, divided into four classes V_1, V_2, V_3 and V_4 with $|V_1| = |V_2| = \langle\alpha_2 n\rangle - 1$ and $|V_3| = |V_4| = \frac{1}{2}\langle\langle\alpha_1 n\rangle\rangle - 1$ such that all edges in $G[V_1, V_3]$ and $G[V_2, V_4]$ are coloured red, all edges in $G[V_1]$ and $G[V_3]$ are coloured blue, all edges in $G[V_1 \cup V_3, V_2 \cup V_4]$ are coloured green and all edges in $G[V_3]$ and $G[V_4]$ are coloured red or blue.

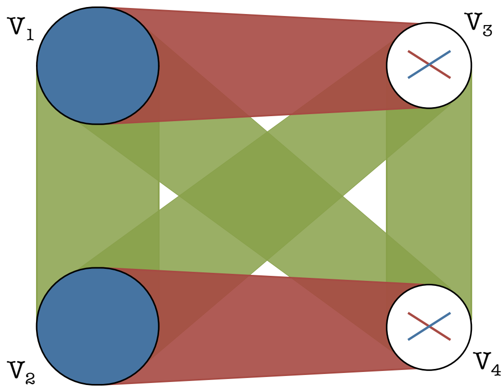


Figure 2.106: Second extremal colouring for Theorem C.

For the upper bound, the key steps are the same as in the Proof of Theorem A. Again, much of the work is concerned with proving a connected-matching stability result. The stability result says that, for every $\alpha_1, \alpha_2, \alpha_3 > 0$ and every k sufficiently large, every three-multicolouring of G, a graph on slightly fewer than $K = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\}k$ vertices with sufficiently many edges, results in G containing a red connected-matching on at least $\alpha_1 k$ vertices, a blue odd connected-matching on at least $\alpha_2 k$ vertices or a green odd connected-matching on at least $\alpha_3 k$ vertices, or results in G having a particular coloured structure.

The proof of the connected-matching stability results follows the same pattern as Part I of the proof of Theorem B (as found in Section 2.7):

Given $\alpha_1, \alpha_2, \alpha_3 > 0$, we set

$$c = \max\{4\alpha_1, \alpha_1 + 2\alpha_2, \alpha_1 + 2\alpha_3\},$$

choose η sufficiently small and consider G, a $(1-\eta^4)$-complete graph on K vertices, where $(c-\eta)k \leq K \leq (c - \frac{\eta}{2})k$ for some sufficiently large k. As in Section 2.7, we begin by considering the average degree of the monochromatic spanning subgraphs. If $d(G_1) \geq \alpha_1 k$, then, by Corollary 2.6.9, G contains a red connected-matching on $\alpha_1 k$ vertices. Thus, since the number of missing edges at each vertex can be bounded above, we see that either $d(G_2) > \frac{1}{2}(c - \alpha_1 - 2\eta)k$ or $d(G_3) > \frac{1}{2}(c - \alpha_1 - 2\eta)k$. Without loss of generality, we assume the former and, thus, have

$$e(G_2) > \tfrac{1}{4}(c - \alpha_1 - 2\eta)(c-\eta)k^2.$$

We then use the decomposition from Lemma 2.6.10 applied to the blue graph to partition the vertices into $W \cup X \cup Y$ such that there are no blue edges in $G[X] \cup G[Y] \cup G[W, X \cup Y]$ and few blue edges in W. By this decomposition, writing wk for $|W|$, we find that

$$e(G_2) \leq \tfrac{1}{2}\alpha_2 wk^2 + \tfrac{1}{4}(c-w)^2 k^2.$$

Comparing the upper and lower bounds obtained for $e(G_2)$, we obtain a quadratic inequality in w. Solving this results in two possibilities:

(F) $w > c - 4\eta$;

(G) $w < \alpha_1 + 4\eta$.

In Case F, almost all of the vertices of G belong to W. Since $G[W]$ is the union of the odd blue-components of G, any blue matching found there is, by definition, odd. Thus, in $G[W]$, any result which provides a blue connected-matching of unspecified parity can be used to provide a blue odd connected-matching.

This situation is similar to that found in Case A (see page 52). In that case, $G[W]$ was the union of odd green-components. We were able to apply a (parity-free) technical result (Theorem 2.6.11) to $G[W]$ to give a connected-matching of appropriate size, colour and parity to complete the proof in that case. In Case F, if $c = 4\alpha_1$, in the same way, we apply the main (even-even-odd) result from [FŁ07b] to $G[W]$, giving a connected-matching of appropriate size, colour and parity to complete the proof. Alternatively, if $c = \alpha_1 + 2\alpha_2$ or $c = \alpha_1 + 2\alpha_3$, we apply Theorem B to $G[W]$, giving either a connected-matching of appropriate size, colour and parity or one of a list of particular coloured structures, completing Case F.

Moving on to Case G, recall that in Case B (starting on page 52) we had a similar bound for w and that, considering the decomposition, we were able to apply Lemma 2.6.12 to show that, in fact, W must be either trivially small or contain roughly $\alpha_1 k$ vertices. In Case G, we may do the same thing but must utilise an alternative technical lemma (specifically [FŁ07b, Lemma 13]).

In Case B, for w small, we applied a two-coloured (even) connected-matching stability result (Lemma 2.6.15) to each of $G[X]$ and $G[Y]$, giving a specified coloured structure. Analogous two-coloured connected-matching stability results for odd connected-matchings and for mixed parity connected-matchings will appear in [KSS07b]. In Case G, depending on the relative sizes of α_1, α_2 and α_3, we use one or more of these applied to each of $G[X]$ and $G[Y]$ to specify the coloured structure of G.

The remaining case is when W contains almost $\alpha_1 k$ vertices. In Case B, we were able to show that X and Y each included close to half the remaining vertices (and were, thus, of size close to $\alpha_1 k$). Then, since $G[X]$ and $G[Y]$ contained only red and blue edges, Corollary 2.6.15 gave either a red or blue connected-matching in each of X and Y on roughly $\frac{2}{3}\alpha_1 k$ vertices. Considering these matchings together with the graphs $G[X, W]$ and $G[Y, W]$ (each of which contained only red and blue edges) and making use of Lemma 2.6.18, we were able to find either a red connected-matching on at least $\alpha_1 k$ vertices or a blue connected-matching on at least $\alpha_2 k$ vertices. In Case G, $G[X]$, $G[Y]$, $G[X, W]$ and $G[Y, W]$ each contain only red and green edges. This time, we use the same approach, making use of corollaries to the additional two-coloured stability results discussed in the above paragraph to find either a red connected-matching on at least

$\alpha_1 k$ vertices or a green odd connected-matching on at least $\alpha_3 k$ vertices, completing the proof of the connected-matching stability result. Note that the argument needed is similar to that given in Cases B.i–B.iii (see pages 54–59) but is slightly more involved since one of the possible connected-matchings we seek is odd.

Note that, alternatively, when c takes the first form, that is, when the graph contains slightly fewer than $4\alpha_1 k$ vertices, a result of Kohayakawa, Simonovits and Skokan [KSS09a, Theorem 6] gives either a red odd connected-matching on at least $\alpha_1 k$ vertices, a blue odd connected-matching on at least $\alpha_2 k$ vertices, a green odd connected-matching on at least $\alpha_3 k$ vertices or one of two particular structures. The first structure is the same structure obtained by following the approach above and the second immediately implies that the graph contains a red connected-matching on at least $\alpha_1 k$ vertices. Thus, this result can be used to slightly shorten the proof of the connected-matching stability result.

Having proved the connected-matching stability result, we apply it to the reduced graph, as in Section 2.9, the result being that $\mathcal{G}$ contains either a connected-matching or a particular coloured structure. A version of Theorem 2.3.4 for one even and two odd cycles can be used to blow up the connected-matchings to cycles. Thus, it suffices to deal with the coloured structures. Most of the potential coloured structures have already been seen in Section 2.5. The only one that has not already been seen resembles the graph shown in Figure 2.105 but with greater freedom of colouring:

Definition 2.13.1. *For x, c positive, let $\mathcal{L}(x,c)$ be the class of edge-multicoloured graphs defined as follows:*

A given two-multicoloured graph $H = (V,E)$ belongs to $\mathcal{L}$ if the vertex set of V can be partitioned into $X_1 \cup X_2 \cup X_3 \cup X_4$ such that

- *(i)* $|X_1|, |X_2|, |X_3|, |X_4| \geq x$;
- *(ii) H is c-almost-complete; and*
- *(iii) (a) all edges present in $H[X_1], H[X_2], H[X_3], L[X_4]$ are coloured red,*
  - *(b) all edges present in $H[X_1, X_2]$, $H[X_3, X_4]$ are coloured green.*

For each coloured structure, the proof follows the pattern established in Sections 2.10–2.12, that is, we lift the structure found in the reduced graph back to the original graph where we exploit it to force either a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices, a blue cycle on $\langle \alpha_2 n \rangle$ vertices or a green cycle on $\langle \alpha_3 n \rangle$ vertices. For the structures already dealt with, the proofs in Sections 2.10–2.12 must be adapted slightly since we now require the blue cycle to have odd length.

For the structure not already seen, we outline the key steps in the proof that the existence of such a structure in the reduced graph implies the existence, in the original graph, of either a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices, a blue cycle on $\langle \alpha_2 n \rangle$ vertices or a green cycle on $\langle \alpha_3 n \rangle$ vertices:

Given a three-coloured graph on at least $4\langle\langle \alpha_1 n \rangle\rangle - 3$ vertices, suppose that its reduced graph $\mathcal{G}$ contains a structure from $\mathcal{L}((\alpha_1 - \eta^{1/4})k, 4\eta^4 k)$ and that $\alpha_1 \geq \frac{2}{3}\alpha_2 - \eta^{1/4} \geq \frac{2}{3}\alpha_3 - \eta^{1/4}$. Then, the vertex set $\mathcal{V}$ of $\mathcal{G}$ has partition $\mathcal{V} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \mathcal{X}_3 \cup \mathcal{X}_4 \cup \mathcal{W}$ with

$$|\mathcal{X}_1| = |\mathcal{X}_2| = |\mathcal{X}_3| = |\mathcal{X}_4| \geq (\alpha_1 - \eta^{1/4})k \tag{2.93}$$

such that all edges present in $\mathcal{G}[\mathcal{X}_1], \mathcal{G}[\mathcal{X}_2], \mathcal{G}[\mathcal{X}_3], \mathcal{G}[\mathcal{X}_4]$ are coloured exclusively red and all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2], \mathcal{G}[\mathcal{X}_3, \mathcal{X}_4]$ are coloured exclusively green. We also know that $\mathcal{G}$ is $4\eta^4 k$-almost-complete.

Considering the colouring of $\mathcal{G}$ and the sizes of $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3$ and $\mathcal{X}_4$, we can show that all edges present in $G[\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{X}_3 \cup \mathcal{X}_4]$ must be coloured either blue or green since the presence of a red edge in $G[\mathcal{X}_1 \cup \mathcal{X}_2, \mathcal{X}_3 \cup \mathcal{X}_4]$ would give a red connected-matching on at least $\alpha_1 k$ vertices.

Similarly, since $|\mathcal{X}_1| + |\mathcal{X}_2| \geq (2\alpha_1 - 2\eta^{1/4})k > (\alpha_3 + 2\eta^{1/4})k$, for instance, $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_3]$ and $\mathcal{G}[\mathcal{X}_2, \mathcal{X}_3]$ cannot both contain a green edge, as such a pair of edges could be used along with $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_2]$ to give a green odd connected-matching on at least $\alpha_3 k$ vertices. Thus, without loss of generality, we may assume that all edges present in $\mathcal{G}[\mathcal{X}_1, \mathcal{X}_3] \cup \mathcal{G}[\mathcal{X}_2, \mathcal{X}_4]$ are coloured exclusively blue.

We then show that the original graph must have a similar coloured-structure which can be exploited to force a cycle of appropriate length, colour and parity. We partition the vertices of G into sets X_1, X_2, X_3, X_4 and W corresponding to the partition of the vertices of $\mathcal{G}$ into $\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4$ and $\mathcal{W}$. Having done so, each X_i contains roughly $\alpha_1 n$ vertices. We can then prove the following claim:

Claim 2.13.2. *We can remove at most $\eta^{1/4}n$ vertices from each of X_1, X_2, X_3 and X_4 so that, in what remains,*

(i) $G_1[X_1], G_1[X_2], G_1[X_3]$ *and* $G_1[X_4]$ *are each* $\eta^{1/4}n$*-almost-complete;*

(ii) $G_2[X_1, X_3]$ *and* $G_2[X_2, X_4]$ *are each* $\eta^{1/4}n$*-almost-complete;*

(iii) $G_3[X_1, X_2]$ *and* $G_3[X_3, X_4]$ *are each* $\eta^{1/4}n$*-almost-complete.*

Given this colouring, we see that it is possible to construct a red cycle in each of $G[X_1], G[X_2], G[X_3]$ and $G[X_4]$ of any (non-trivial) length up to the size of that part of the graph, a blue cycle in each of $G[X_1, X_3]$ and $G[X_2, X_4]$ of any (non-trivial) even length up to twice the size of the smaller part and a green cycle in each of $G[X_1, X_2]$ and $G[X_3, X_4]$ of any (non-trivial) length up to twice the size of the smaller part. Given the relative sizes of α_1, α_2 and α_3, the blue and green cycles found exceed the length required to complete the proof but, as they arise from bipartite graphs, they are not odd.

We then consider the vertices of W, showing that they can be partitioned into $W_1 \cup W_2 \cup W_3 \cup W_4$ such that, for each i, all vertices in W_i have large red degree to X_i. Then, since we consider a graph on $4\langle\langle \alpha_1 n \rangle\rangle - 3$ vertices, we have $|X_i \cup W_i| \geq \langle\langle \alpha_1 n \rangle\rangle$ for some i and can, thus, obtain a red cycle on $\langle\langle \alpha_1 n \rangle\rangle$ vertices, completing the proof.

2.14 Conclusions

Together, [KSS09a], [BS09] and this chapter give exact values for the Ramsey number of any triple of sufficiently long cycles (except when all three cycles are even but of different lengths). We now discuss briefly what is known for four or more colours beginning with the case when all the cycles in question are of odd length.

Recall, from Section 1.4, that Bondy and Erdős gave the following bounds for the r-coloured Ramsey number of odd cycles

$$2^{r-1}(n-1) + 1 \leq R(C_n, C_n, \dots, C_n) \leq (r+2)!n$$

and conjectured that the lower bound gives the true value of the Ramsey number.

In 2012, Łuczak, Simonovits and Skokan [ŁSS12] gave an improved asymptotic upper bound. For n odd and $r \geq 4$, they proved that

$$R(C_n, C_n, \ldots, C_n) \leq r2^r n + o(n)$$

as $n \to \infty$.

Note that the conjecture still stands and has been confirmed for three colours by Kohayakawa, Simonovits, and Skokan [KSS09a]. The structures providing the lower bound are well known and easily constructed. For two colours, the structure is simply two classes of $n-1$ vertices coloured such that all edges within each class are coloured red and all edges between classes are coloured blue (see Figure 2.107).

Figure 2.107: Coloured structure giving the lower bound for two colours.

The relevant r-coloured structure is obtained by taking two copies of the $(r-1)$-coloured structure and colouring all edges between the copies with colour r (see Figure 2.108).

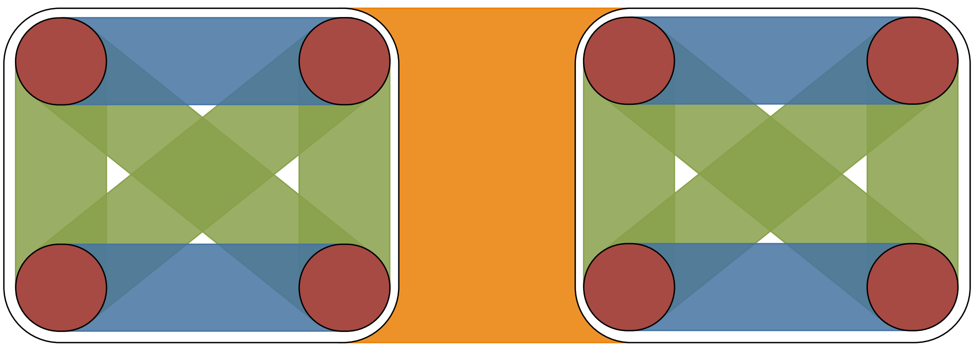


Figure 2.108: Coloured structure giving the lower bound for four colours.

Notice that these structures also give a lower bound for the r-coloured Ramsey number when the cycles have different lengths. Thus, for $n_1 \geq n_2 \geq \cdots \geq n_r$ all odd, we have

$$R(C_{n_1}, C_{n_2}, \ldots, C_{n_r}) \geq 2^{r-1}(n_1 - 1) + 1.$$

In 1976, Erdős, Faudree, Rousseau and Schelp [EFRS76] considered the case when one cycle is much longer than the others, proving in the case of odd cycles that, if n is much larger than m, ℓ, k all odd, then

$$R(C_n, C_m, C_\ell, C_k) = 8n - 7,$$

thus showing that the above bound is tight in that case.

This 'doubling-up' process can also be used to provide structures giving sensible lower bounds for mixed parity multicolour Ramsey numbers. For example, consider the case of two even and two odd cycles. The three-coloured graph shown in Figure 2.109 below, was used earlier to provide a lower bound for $R(C_n, C_m, C_\ell)$ in the case that n, m are even and ℓ is odd. Taking two copies of the graph and colouring all the edges between the copies with a fourth colour gives a four-coloured graph providing a lower bound for $R(C_n, C_m, C_\ell, C_k)$, in the case that n, m are even and ℓ, k are odd.

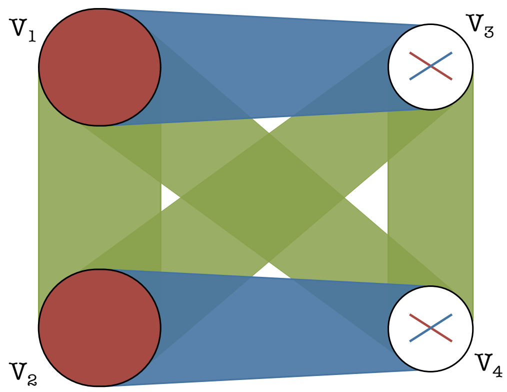


Figure 2.109: Structure providing a lower bound for even-even-odd case.

As the number of colours increases, there are simply too many mixed parity cases to discuss each one here or to give conjectures for the exact or asymptotic Ramsey numbers. However, looking at the structures already seen for three colours and 'doubling-up' would seem to be a good place to start.

For even cycles, this 'doubling-up' is not an option and the Ramsey numbers grow more slowly as the number of colours increases. Indeed, Łuczak, Simonovits and Skokan [ŁSS12], proved that the r-coloured Ramsey number for even cycles essentially grows no faster than linearly in r, proving that, for n even,

$$R(C_n, C_n, \ldots, C_n) \leq rn + o(n)$$

as $n \to \infty$.

Recall that Bondy and Erdős [BE73] proved that, for $n \geq 3$ even,

$$R(C_n, C_n) = \tfrac{3}{2}n - 1.$$

The simple structure providing the lower bound is shown in Figure 2.110 below. It has two classes of vertices, the first containing $n-1$ vertices and the second $\frac{1}{2}n - 1$ vertices. It is coloured such that all edges within the first class receive one colour and all other edges receive the second.

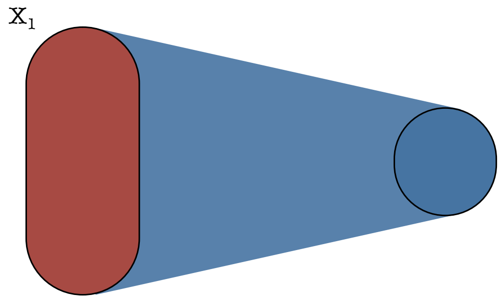


Figure 2.110: Coloured structure giving the lower bound in the two coloured even case.

This structure is easily extended to give a lower bound for the multicoloured odd cycles (see Figure 2.111) showing that for r colours

$$R(C_n, C_n, \ldots, C_n) \geq \tfrac{1}{2}(r+1)n - r + 1.$$

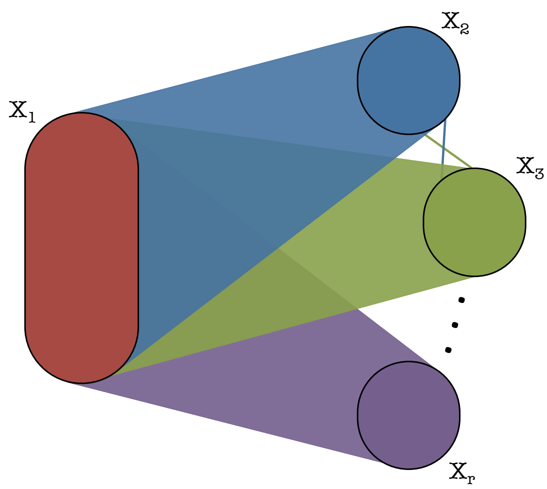


Figure 2.111: Structure providing a lower bound for r-coloured even cycle result.

It can also be adapted to provide a lower bound in the case that the cycles are all even but are of different lengths, showing that, for $n_1 \geq n_2, \ldots, n_r$ all even,

$$R(C_{n_1}, C_{n_2}, \ldots, C_{n_r}) \geq n_1 + \tfrac{1}{2}n_2 + \cdots + \tfrac{1}{2}n_r - r + 1. \tag{2.94}$$

Also in [EFRS76], Erdős, Faudree, Rousseau and Schelp showed that, for n much larger than m, ℓ, k all even,

$$R(C_n, C_m, C_\ell) = n + \tfrac{1}{2}n + \tfrac{1}{2}\ell - 2,$$
$$R(C_n, C_m, C_\ell, C_k) = n + \tfrac{1}{2}n + \tfrac{1}{2}\ell + \tfrac{1}{2}k - 3.$$

Thus, for two or three colours, the bound in (2.94) is tight when one of the cycles is much longer than the others. Notice, also, that this bound agrees with the asymptotic result of Figaj and Łuczak in [FŁ07a].

However, as shown by the exact result of Benevides and Skokan [BS09], this bound can be beaten slightly in the case of three even cycles of equal length. In that case, the less easily extended structure shown in Figure 2.112 gives $R(C_n, C_n, C_n) = 2n$.

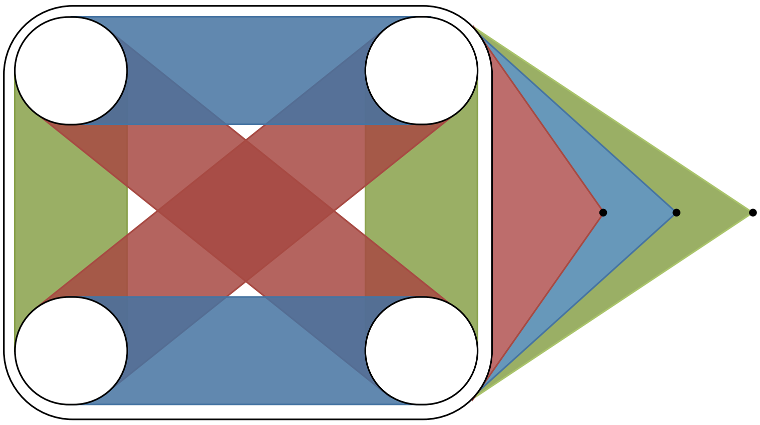


Figure 2.112: Structure providing the lower bound in the paper of Benevides and Skokan.

Based on the results discussed, one might be tempted to conjecture an asymptotic r-colour result for even cycles of the form

$$R(C_{\langle\langle \alpha_1 n\rangle\rangle}, C_{\langle\langle \alpha_2 n\rangle\rangle}, \ldots, C_{\langle\langle \alpha_r n\rangle\rangle}) = \tfrac{1}{2}\left(\alpha_1 + \alpha_2 + \cdots + \alpha_r + \max\{\alpha_1, \alpha_2, \ldots, \alpha_r\}\right) n + o(n).$$

However, in 2006, in the case of r cycles of equal even length n, Sun Yongqi, Yang

Yuansheng, Xu Feng and Li Bingxi [YYXB06] proved, that

$$R(C_n, C_n, \ldots, C_n) \geq (r-1)n - 2r + 4,$$

suggesting that the true form of such a result for even cycles is much more complicated.

There is potential to apply the methods used in this chapter to the case of four or more colours but there are limitations which could make this quite difficult. For instance. two key sets of tools used to prove the stability result (Theorem B) were decompositions (which were used to find large two-coloured subgraphs within three-coloured graphs) and connectivity results (which reduce the difficulty of finding a connected-matching in a two coloured graph). The most basic such connectivity result states that a two-coloured graph is connected in one of its colours. Such results do not apply (or are much more complicated) for three-coloured graphs. Therefore, an alternative approach or (even) more case analysis could well be necessary.

Chapter 3

Removal lemmas for equations over finite fields

In this chapter, we prove a removal lemma for two particular classes of multinomials. Letting F be a finite field of of order q and $X_1, X_2, \dots X_m$ be subsets of that field, we prove that, if a multinomial of the form

$$x_1x_2\dots x_{p_1} + x_{p_1+1}x_{p_1+2}\dots x_{p_2} + \cdots + x_{p_s+1}x_{p_s+2}\dots x_{m-1} + x_m = b$$

$$\text{or} \qquad x_1x_2\dots x_{p_1} + x_{p_1+1}x_{p_1+2}\dots x_{p_2} + \cdots + x_{p_s+1}x_{p_s+2}\dots x_m = 0$$

has $o(q^{m-1})$ solutions $(x_1, x_2, x_3, \dots, x_m)$ with $x_i \in X_i$, then we can delete $o(q)$ elements from each X_i so that no solutions remain.

Analogous results are known for linear equations over finite groups and for systems of equations over finite fields. However, probably the most natural place to begin an exploration of this topic is the famous Triangle Removal Lemma of Ruzsa and Szemerédi which states that a graph on n vertices containing $o(n^3)$ triangles can be made triangle-free by the removal of $o(n^2)$ edges (where we say that a graph G contains k copies of a graph H if G has, as subgraphs, k graphs isomorphic to H). This result can be stated more precisely as follows:

Theorem 3.1. (The Triangle Removal Lemma [RS78]). *For every $\epsilon > 0$, there exists $N_{3.1} = N_{3.1}(\epsilon)$ and $\delta = \delta_{3.1}(\epsilon)$ such that, if H is a graph on $n \geq N_{3.1}$ vertices with at most δn^3 triangles, then one can remove from H at most ϵn^2 edges to obtain a graph that contains no triangles.*

The standard proof of this result, as seen in Section 1.5, utilises Szemerédi's Regularity Lemma, meaning that $N_{3.1}$ is taken to be a tower of twos of height proportional to ϵ^{-5}. In 1997, Gowers [Gow97] demonstrated that $N_{1.5.2}$ must be at least as large as a tower of twos of height proportional to $\epsilon^{-1/16}$, meaning that significant improvement in the size of $N_{3.1}$ would require a proof not utilising regularity. Such a proof was provided by Fox [Fox11] who provided a new proof not using regularity which gives a bound for $N_{3.1}$ that is a tower of height proportional to $\log(\epsilon^{-1})$.

The Triangle Removal Lemma can be generalised in a number of ways, for instance, from triangles to general graphs:

Theorem 3.2. (The Graph Removal Lemma [EFR86]). *Let K be a fixed (directed) graph on k vertices. Given $\epsilon > 0$, there exists $N_{3.2} = N_{3.2}(\epsilon, k)$ and $\delta = \delta_{3.2}(\epsilon, k)$ such that, if H is a (directed) graph on $n \geq N_{3.2}$ vertices with at most δn^k copies of K, then one can remove from H at most ϵn^2 edges to obtain a graph that contains no (directed) copies of K.*

In 2005, Green proved a version of Szemerédi's Regularity Lemma for Abelian groups and derived as a consequence the below removal lemma for linear equations over Abelian groups:

Theorem 3.3. (Removal Lemma for equations over Abelian groups [Gre05]). *Given $\epsilon > 0$ and an integer m, there exists $N_{3.3} = N_{3.3}(\epsilon, m)$ and $\delta = \delta_{3.3}(\epsilon, m)$ such that the following holds:*

Letting G be a finite Abelian group of order $N \geq N_{3.3}$ and g a fixed element of G, if $X_1, X_2, \ldots, X_m$ are subsets of G such that there are at most δN^{m-1} solutions to the equation $x_1 + x_2 + \cdots + x_m = g$ with $x_i \in X_i$, then it is possible to remove at most ϵN elements from each set X_i so that no solutions remain.

More recently, Král', Serra and Vena [KSV09] were able to use the Graph Removal Lemma to give a proof of Green's result which also works for non-Abelian groups:

Theorem 3.4. (Removal Lemma for equations over groups [KSV09]). *Given $\epsilon > 0$ and an integer m, there exists $N_{3.4} = N_{3.4}(\epsilon, m)$ and $\delta = \delta_{3.4}(\epsilon, m)$ such that the following holds:*

Letting G be a finite group of order $N \geq N_{3.4}$ and g a fixed element of G, if $X_1, X_2, \ldots, X_m$ are subsets of G such that there are at most δN^{m-1} solutions to the equation $x_1 x_2 \ldots x_m = g$

with $x_i \in X_i$, then it is possible to remove at most ϵN elements from each set X_i so that no solutions remain.

Later, the same authors and, independently, Shapira proved a version of the above for systems of linear equations over finite fields:

Theorem 3.5. (Removal Lemma for systems of linear equations [KSV12], [Sha10]). Given $\epsilon > 0$ and an integer m, there exists $q_{3.5} = q_{3.5}(\epsilon, m)$ and $\delta = \delta_{3.5}(\epsilon, m)$ such that the following holds:

Letting $F = \mathbb{F}_q$ be the finite field of order $q \geq q_{3.5}$ and $k \geq 1$ an integer, if A is a $(k \times m)$ matrix of rank k with coefficients in F, b is a k-dimensional vector over F and $X_1, X_2, \ldots, X_m$ are subsets of F such that there are at most δq^{m-k} solutions to the system $Ax = b$ with $x_i \in X_i$, then it is possible to remove at most ϵq elements from each X_i so that no solutions remain.

Both proofs made use of the following coloured hypergraph analogue of the graph removal lemma which we will also use later in this chapter (note that the colourings referred to in the result are not required to be proper):

Theorem 3.6. (The coloured Hypergraph Removal Lemma [Ish09], [AT10]). *Let K be a fixed r-uniform c-coloured hypergraph on k vertices. Given $\epsilon > 0$, there exists $N_{3.6} = N_{3.6}(\epsilon, k)$ and $\delta = \delta_{3.6}(\epsilon, k)$ such that, if H is an r-uniform c-coloured hypergraph on $n \geq N_{3.6}$ vertices with fewer than δn^k copies of K, then one can remove from H at most ϵn^r edges to obtain a hypergraph containing no copies of K.*

3.1 Results

We begin by asking a general question:

'Can the theorems of the first section be extended
to general multinomials and polynomials?'

Notice that a simple extension to multinomials can be obtained from Theorem 3.5 by setting k equal to one, $x_i = y_i^p$ and $X_i = \{y^p : y \in Y_i\}$:

Corollary 3.1.1. *Given $\epsilon > 0$ and an integer m, there exists $q_{3.1.1} = q_{3.1.1}(\epsilon, m)$ and $\delta = \delta_{3.1.1}(\epsilon, m)$ such that the following holds:*

Letting $F = \mathbb{F}_q$ be a finite field of order $q \geq q_{3.1.1}$ and $p \geq 1$ an integer, if $b \in F$ and $Y_1, Y_2, \dots, Y_m$ are subsets of F such that there are at most δq^{m-1} solutions to the equation $y_1^p + y_2^p + \dots + y_m^p = b$ with $y_i \in Y_i$ for all i, then it is possible to remove at most ϵq elements from each set Y_i so that no solutions remain.

Extending these results to general polynomials and multinomials appears to be surprisingly difficult. However, we have been able to extend them to two particular classes of multinomials and the remainder of this chapter is dedicated to proving these extensions:

Theorem 3.1.2. *Given $\epsilon > 0$ and an integer m, there exists $q_{3.1.2} = q_{3.1.2}(\epsilon, m)$ and $\delta = \delta_{3.1.2}(\epsilon, m)$ such that the following holds:*

Letting $F = \mathbb{F}_q$ be the finite field of order $q \geq q_{3.1.2}$, b be an element of F and $s, p_1, \dots, p_s$ integers, if $X_1, X_2, \dots, X_m$ are subsets of F such that the equation

$$x_1 x_2 \dots x_{p_1} + x_{p_1+1} x_{p_1+2} \dots x_{p_2} + \dots + x_{p_s+1} x_{p_s+2} \dots x_{m-1} + x_m = b$$

has at most δq^{m-1} solutions with $x_i \in X_i$ for all i, then it is possible to remove at most ϵq elements from each X_i so that no solutions remain.

The last term, consisting of a single variable x_m, seems somewhat unnatural but, so far, we have only been able to remove it when $b = 0$:

Theorem 3.1.3. *Given $\epsilon > 0$ and an integer m, there exists $q_{3.1.3} = q_{3.1.3}(\epsilon, m)$ and $\delta = \delta_{3.1.3}(\epsilon, m)$ such that the following holds:*

Letting $F = \mathbb{F}_q$ be the finite field of order $q \geq q_{3.1.3}$ and $s, p_1, \dots, p_s$ integers, if $X_1, \dots, X_m$ are subsets of F such that the equation

$$x_1 x_2 \dots x_{p_1} + x_{p_1+1} x_{p_1+2} \dots x_{p_2} + \dots + x_{p_s+1} x_{p_s+2} \dots x_m = 0$$

has at most δq^{m-1} solutions with $x_i \in X_i$ for all i, then it is possible to remove at most ϵq elements from each X_i so that no solutions remain.

3.2 Illustrations

In order to better illustrate the method of proof we will use, we repeat the proof of Theorem 3.4 found in [KSV09] and then prove a specific case of Theorem 3.1.2.

Proof of Theorem 3.4. Fix $\epsilon_0 > 0$, $m \geq 3$. Let G be a finite group of order N, let g be an element of G and let $X_1, X_2, \ldots, X_m$ be sets of elements of G.

We define the directed graph H on vertex set $G \times \{1, 2, \ldots, m\}$ with the following edges:

(i) for $1 \leq i \leq m-1$, we include a directed-edge from (u, i) to $(v, i+1)$ if there exists an element $x_i \in X_i$ such that $ux_i = v$ and label this edge $[x_i, i]$;

(ii) we include a directed-edge from (u, m) to $(v, 0)$ if there exists an element $x_m \in X_m$ such that $ux_m g^{-1} = v$ and label this edge $[x_m, m]$.

Notice that, for each $x_i \in X_i$, H contains N edges labelled with $[x_i, i]$. Observe, also, that any directed cycle in H of length m gives a solution to the equation $x_1 x_2 \ldots x_m = g$. Indeed, if $[x_1, 1], [x_2, 2], \ldots, [x_m, m]$ are the labels of the edges of the cycle and it contains the vertex $(u, 1)$, then $ux_1 x_2 \ldots x_m g^{-1} = v$ by definition of H. Conversely, a solution $x_1, x_2, \ldots, x_m$ of the equation corresponds to N disjoint directed cycles of length m in H:

$$(u, 1), (ux_1, 2), (ux_1x_2, 3), \ldots, (ux_1x_2 \ldots x_{m-1}, m), (ux_1x_2 \ldots x_m g^{-1}, 1) = (u, 1),$$

one for each of the N possible choices of $u \in G$.

Suppose, now, that there are at most $\epsilon_0 N^{m-1}$ solutions to the equation

$$x_1 x_2 \ldots x_m = g \text{ with } x_i \in X_i.$$

Then, by correspondence of the cycles of H and the solutions of the above, the directed graph contains no more than $\epsilon_0 N^m$ distinct directed cycles of length m.

We then apply the Graph Removal Lemma (Theorem 3.2) to H with $\epsilon = \epsilon_0 / m^m$: Since H has at most $\epsilon_0 N^m = \epsilon(mN)^m$ copies of the directed cycle of length m, provided $N \geq N_{3.2}(\epsilon, m)$, there is a set E of at most $\delta(mN)^2$ edges such that $H - E$ contains no directed cycle on m vertices (for some δ depending only on ϵ and m).

Let Y_i be the set of those elements $x \in X_i$ such that E contains at least N/m directed-edges labelled with $[x, i]$. Since $|E| \leq \delta(mN)^2$, the size of each Y_i is at most $m|E|/N \leq$

$\delta m^3 N$. Set $X_i' = X_i \backslash Y_i$. Then, since the size of Y_i is bounded by $\delta m^3 N$, where δ depends on ϵ and m only, and $\delta \to 0$ as $\epsilon_0 \to 0$, all that remains is to prove that there are no solutions to

$$x_1 x_2 \dots x_m = g \text{ with } x_i \in X_i'.$$

Assume that there is a solution to the above and consider the N edge disjoint directed cycles corresponding to $x_1, x_2, \dots, x_m$. Each of these cycles contains at least one of the edges of E and every edge contained within these cycles has one of the labels $[x_1, 1], [x_2, 2], \dots, [x_m, m]$. Since the cycles are disjoint, E will contain at least N/m edges with the same label, $[x_i, i]$. Consequently, $x_i \in Y_i$ and, thus, $x_i \notin X_i'$, a contradiction. $\square$

The below is a specific case of Theorem 3.1.2 intended to illustrate the method of proof used later to prove Theorems 3.1.2 and 3.1.3, whose proofs follow in Sections 3.3 and 3.4.

Claim 3.2.1. *Given $\epsilon > 0$, there exists $q_{3.2.1} = q_{3.2.1}(\epsilon)$ and $\delta = \delta_{3.2.1}(\epsilon)$ such that the following holds: Letting $F = \mathbb{F}_q$ be the finite field of order $q \geq q_{3.2.1}$ and $X_1, X_2, \dots, X_7$ subsets of F, if the equation*

$$x_1 x_2 + x_3 x_4 + x_5 x_6 + x_7 = b$$

has at most δq^6 solutions with $x_i \in X_i$, then it is possible to remove at most ϵq elements from each X_i so that no solution remains.

Proof. We consider the above equation and, motivated by re-writing it as

$$(c_0 x_1 x_2 + c_1) + (c_0 x_3 x_4 + c_2) + (c_0 x_5 x_6 - c_1 - c_2) + c_0 x_7 = c_0 b, \tag{3.1}$$

define a large hypergraph H and smaller fixed hypergraph K so that copies of K in H will correspond to solutions of (3.1).

Let H have vertex set $V = F \times \{0, 1, 2, \dots 8\}$ and write V_i for the vertices arising from $F \times i$ for $i \in \{0, 1, 2\}$ (each to V_i corresponding to a c_i) and W_i for the vertices arising from $F \times (i + 2)$ for $i \in \{1, 2, 3, 4, 5, 6\}$.

For $c_i \in V_i$ $(i = 0, 1, 2)$, $q_j \in W_j$ $(j = 1, 2, \dots, 6)$, we include the following edges:

Include edge	labelled	coloured	if there exists
(c_0, c_1, c_2, q_1)	$[x_1, 1]$	1	$x_1 \in X_1$ s.t. $c_0 x_1 = q_1$
(c_1, c_2, q_1, q_2)	$[x_2, 2]$	2	$x_2 \in X_2$ s.t. $q_1 x_2 + c_1 = q_2$
(c_0, c_1, c_2, q_3)	$[x_3, 3]$	3	$x_3 \in X_3$ s.t. $c_0 x_3 = q_3$
(c_1, c_2, q_3, q_4)	$[x_4, 4]$	4	$x_4 \in X_4$ s.t. $q_3 x_4 + c_2 = q_4$
(c_0, c_1, c_2, q_5)	$[x_5, 5]$	5	$x_5 \in X_5$ s.t. $c_0 x_5 = q_5$
(c_1, c_2, q_5, q_6)	$[x_6, 6]$	6	$x_6 \in X_6$ s.t. $q_5 x_6 - c_1 - c_2 = q_6$
(c_0, q_2, q_4, q_6)	$[x_7, 7]$	7	$x_7 \in X_7$ s.t. $q_2 + q_4 + q_6 + c_0 x_7 = c_0 b$

We define the hypergraph K as follows: K has nine vertices, v_0, v_1, v_2, w_1, w_2, w_3, w_4, w_5, w_6, and seven edges, $v_0v_1v_2w_1$ coloured 1, $v_1v_2w_1w_2$ coloured 2, $v_0v_1v_2w_3$ coloured 3, $v_1v_2w_3w_4$ coloured 4, $v_0v_1v_2w_5$ coloured 5, $v_1v_2w_5w_6$ coloured 6 and $v_0w_2w_4w_6$ coloured 7.

Notice that, for each $x_i \in X_i$, H contains q^3 edges labelled with $[x_i, i]$. Observe, also, that any coloured copy of K contained within H gives a solution to the equation. Conversely, a solution $x_1, x_2, \ldots, x_7$ of the equation corresponds to q^3 edge disjoint coloured copies of K.

Suppose, now, that there are at most $\epsilon_0 q^6$ solutions to the equation

$$x_1x_2 + x_3x_4 + x_5x_6 + x_7 = b.$$

Then, by correspondence of the coloured copies of K in H and the solutions of the above, the directed graph contains no more than $\epsilon_0 q^9$ distinct coloured copies of K.

Then, recalling that H is a 4-uniform hypergraph on $N_0 = 9q$ vertices, we apply the coloured Hypergraph Removal Lemma (Theorem 3.6) to H with $\epsilon = \epsilon_0/9^9$. Since H has at most $\epsilon_0 q^9 = \epsilon(9q)^9$ coloured copies of K, provided $q \geq N_{3.6}$, there is a set E of at most δN_0^4 edges such that $H - E$ contains no coloured copy of K (for some δ depending only on ϵ).

Let Y_i be the set of those elements $x \in X_i$ such that E contains at least $q^3/7$ edges labelled with $[x_i, i]$. Since $|E| \leq \delta(N_0)^4$, the size of each Y_i is at most $7|E|/(q^3) \leq 9^5\delta q$. Set $X_i' = X_i \backslash Y_i$. Since the size of Y_i is bounded in terms of δ and $\delta \to 0$ as $\epsilon_0 \to 0$, all that remains is to prove that there are no solutions to

$$x_1x_2 + x_3x_4 + x_5x_6 + x_7 = b \text{ with } x_i \in X_i'.$$

Assume that there is a solution to the above and consider the q^3 edge disjoint coloured copies of K corresponding to $x_1, x_2, \ldots, x_m$. Each contains at least one of the edges of E and every edge contained within these has one of the labels $[x_1, 1], [x_2, 2], \ldots, [x_7, 7]$. Since the cycles are disjoint, E will contain at least $q^3/7$ edges with the same label, $[x_i, i]$. Consequently, $x_i \in Y_i$ and, thus, $x_i \notin X_i'$, a contradiction. □

3.3 Proof of Theorem 3.1.2

Consider the equation

$$x_1 x_2 \ldots x_{p_1} + x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + \cdots + x_{p_s+1} x_{p_s+2} \ldots x_{m-1} + x_m = b,$$

which can be re-written as

$$(c_0 x_1 x_2 \ldots . x_{p_1} + c_1) + (c_0 x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + c_2) + \ldots$$
$$+ (c_0 x_{p_{s-1}+1} x_{p_{s-1}+2} x_{p_s} + c_s) + (c_0 x_{p_s+1} \ldots x_{m-1} - c_1 - c_2 - \ldots - c_s) + c_0 x_m = c_0 b.$$

Let $H = (V, E)$ have vertex set $V = F \times \{0, 1, 2, \ldots s, s+1, \ldots, s+m-1\}$.

We write V_i for the vertices arising from $F \times i$ for $i \in \{0, \ldots, s\}$ (each V_i corresponding to a c_i) and W_i for the vertices arising from $F \times (i+s)$ for $i \in \{1, \ldots, m-1\}$.

(Setting $p_0 = 0$ and $p_{s+1} = m$) for $c_i \in V_i$, $q_j \in W_j$, we include the following edges:

(i) For $1 \leq k \leq s+1$,
we include $(c_0, c_1, c_2, \ldots c_s, q_{p_{k-1}+1})$ labelled $[x_{p_{k-1}+1}, p_{k-1}+1]$, coloured $p_{k-1}+1$ if there exists $x_{p_{k-1}+1} \in X_{p_{k-1}+1}$ s.t. $c_0 x_{p_{k-1}+1} = q_{p_{k-1}+1}$;

(ii) for $1 \leq k \leq s+1$, for $p_{k-1}+1 < j < p_k$,
we include $(c_1, c_2, \ldots c_s, q_{j-1}, q_j)$ labelled $[x_j, j]$, coloured j
if there exists $x_j \in X_j$ s.t. $q_{j-1} x_j = q_j$;

(iii) for $1 \leq k \leq s$,
we include $(c_1, c_2, \ldots c_s, q_{p_k-1}, q_{p_k})$ labelled $[x_{p_k}, p_k]$, coloured p_k
if there exists $x_{p_k} \in X_{p_k}$ s.t. $q_{p_k-1} x_{p_k} + c_k = q_{p_k}$;

(iv) for $k = s+1$,
we include $(c_1, c_2, \dots c_s, q_{m-2}, q_{m-1})$ labelled $[x_{m-1}, m-1]$, coloured $m-1$ if there exists $x_{m-1} \in X_{m-1}$ s.t. $q_{m-2}x_{m-1} - c_1 - c_2 - \dots - c_s = q_{m-1}$;

(v) finally, we include $(c_0, q_{p_1}, q_{p_2}, \dots, q_{p_s}, q_{m-1})$ labelled $[x_m, m]$, coloured m if there exists $x_m \in X_m$ s.t. $q_{p_1} + q_{p_2} + \dots + q_{p_s} + q_{m-1} + c_0 x_m = c_0 b$.

We define the hypergraph K as follows: K has $s+m$ vertices $v_0, v_1, \dots, v_s, w_1, w_2, \dots, w_{m-1}$ and m edges each of size $(s+2)$:

(i) For $1 \le k \le s+1$, $(v_0, v_1, v_2, \dots v_s, w_{p_{k-1}+1})$ coloured $p_{k-1}+1$;

(ii) for $1 \le k \le s+1$, for $p_{k-1}+1 < j < p_k$, $(v_1, v_2, \dots v_s, w_{j-1}, w_j)$ coloured j;

(iii) for $1 \le k \le s$, $(v_1, v_2, \dots v_s, w_{p_k-1}, w_{p_k})$ coloured p_k;

(iv) for $k = s+1$, $(v_1, v_2, \dots v_s, w_{m-2}, w_{m-1})$ coloured $m-1$;

(v) and, finally, $(v_0, w_{p_1}, w_{p_2}, \dots, w_{p_s}, w_{m-1})$ coloured m.

Notice that, for each $x_i \in X_i$, H contains q^{s+1} edges labelled with $[x_i, i]$. Observe, also, that any coloured copy of K contained within H gives a solution to the equation by definition. Conversely, solution $x_1, x_2, \dots, x_m$ of the equation corresponds to q^{s+1} disjoint coloured copies of K.

Suppose, now, that there are at most $\epsilon_0 q^{m-1}$ solutions to the equation

$$x_1 x_2 \dots x_{p_1} + x_{p_1+1} x_{p_1+2} \dots x_{p_2} + \dots + x_{p_s+1} x_{p_s+2} \dots x_{m-1} + x_m = b.$$

Then, by correspondence of the coloured copies of K in H and the solutions of the above, the directed graph contains no more than $\epsilon_0 q^{s+m}$ distinct coloured copies of K.

Then, recalling that H is an $(s+2)$-uniform hypergraph on $N_0 = (s+m)q$ vertices, we apply the coloured Hypergraph Removal Lemma (Theorem 3.6) to H with $\epsilon = \epsilon_0/(s+m)^{s+m}$: Since H has at most $\epsilon_0 q^{s+m} = \epsilon((s+m)q)^{s+m}$ coloured copies of K, provided $q \ge N_{3.6}$, there is a set E of at most δN_0^{s+2} edges such that $H - E$ contains no coloured copy of K (for some δ depending only on ϵ and m).

Let Y_i be the set of those elements $x \in X_i$ such that E contains at least q^{s+1}/m edges labelled with $[x_i, i]$. Since $|E| \le \delta(N_0)^{s+2}$, the size of each Y_i is at most $m|E|/(q^{s+1}) \le$

$\delta(s+m)^{s+3}q$. Set $X_i' = X_i \backslash Y_i$. Since the size of Y_i is bounded in terms of δ (depending only on ϵ and m) and $\delta \to 0$ as $\epsilon_0 \to 0$, all that remains is to prove that there are no solutions to

$$x_1 x_2 \ldots x_{p_1} + x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + \cdots + x_{p_s+1} x_{p_s+2} \ldots x_{m-1} + x_m = b \text{ with } x_i \in X_i'.$$

Assume that there is a solution to the above and consider the q^{s+1} edge disjoint coloured copies of K corresponding to $x_1, x_2, \ldots, x_m$. Each contains at least one of the edges of E and every edge contained within these has one of the labels $[x_1, 1], [x_2, 2], \ldots, [x_m, m]$. Since the cycles are disjoint, E will contain at least q^{s+1}/m edges with the same label, $[x_i, i]$. Consequently, $x_i \in Y_i$ and, thus, $x_i \notin X_i'$, a contradiction. □

3.4 Proof of Theorem 3.1.3

Consider the equation

$$x_1 x_2 \ldots x_{p_1} + x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + \cdots + x_{p_s+1} x_{p_s+2} \ldots x_m = 0,$$

which can be re-written as

$$\begin{aligned}(c_0 x_1 x_2 \ldots . x_{p_1} + c_1) + (c_0 x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + c_2) + \ldots \\ + (c_0 x_{p_{s-2}+1} x_{p_{s-2}+2} \ldots x_{p_{s-1}} + c_{s-1}) + (c_0 x_{p_{s-1}+1} \ldots x_{p_s} - c_1 - c_2 \ldots - c_{s-1}) \\ + (c_0 x_{p_s+1} x_{p_s+2} \ldots x_m) = 0.\end{aligned}$$

Let $H = (V, E)$ have vertex set $V = F \times \{0, 1, 2, \ldots s-1, s, \ldots, s+m-2\}$.

We write V_i for the vertices arising from $F \times i$ for $i \in \{0, \ldots, s-1\}$ (each V_i corresponding to c_i) and W_i for the vertices arising from $F \times (i+s-1)$ (for $i \in \{1, \ldots, m-1\}$).

(Setting $p_0 = 0$ and $p_{s+1} = m$), for $c_i \in V_i$, $q_j \in W_j$, we include the following edges:

(i) For $1 \leq k \leq s+1$,
include $(c_0, c_1, c_2, \ldots c_{s-1}, q_{p_{k-1}+1})$ labelled $[x_{p_{k-1}+1}, p_{k-1}+1]$, coloured $p_{k-1}+1$ if there exists $x_{p_{k-1}+1} \in X_{p_{k-1}+1}$ s.t. $c_0 x_{p_{k-1}+1} = q_{p_{k-1}+1}$;

(ii) for $1 \leq k \leq s+1$, for $p_{k-1}+1 < j < p_k$,
include $(c_1, c_2, \ldots c_{s-1}, q_{j-1}, q_j)$ labelled $[x_j, j]$, coloured j
if there exists $x_j \in X_j$ s.t. $q_{j-1}x_j = q_j$;

(iii) for $1 \leq k \leq s$,
include $(c_1, c_2, \ldots c_{s-1}, q_{p_k-1}, q_{p_k})$ labelled $[x_{p_k}, p_k]$, coloured p_k,
if there exists $x_{p_k} \in X_{p_k}$ s.t. $q_{p_k-1}x_{p_k} + c_k = q_{p_k}$;

(iv) for $k = s$,
include $(c_1, c_2, \ldots c_{s-1}, q_{p_s-1}, q_{p_s})$ labelled $[x_{p_s}, p_s]$, coloured p_s
if there exists $x_{p_s} \in X_{p_s}$ s.t. $q_{p_s}x_{p_s-1} - c_1 - c_2 - \ldots - c_{s-1} = q_{p_s}$;

(v) finally, include $(q_{p_1}, q_{p_2}, \ldots, q_{p_s}, q_{m-1})$ labelled $[x_m, m]$, coloured m
if there exists $x_m \in X_m$ s.t. $q_{p_1} + q_{p_2} + \cdots + q_{p_s} + q_{m-1}x_m = 0$.

We define the hypergraph K as follows: K has $s+m-1$ vertices v_0, v_1,…, v_{s-1}, w_1, w_2,…, w_{m-1} and m edges, each of size $(s+1)$:

(i) For $1 \leq k \leq s+1$, $(v_0, v_1, v_2, \ldots, v_{s-1}, w_{p_{k-1}+1})$ coloured $p_{k-1}+1$;

(ii) for $1 \leq k \leq s+1$, for $p_{k-1}+1 < j < p_k$, $(v_1, v_2, \ldots v_{s-1}, w_{j-1}, w_j)$ coloured j;

(iii) for $1 \leq k \leq s$, $(v_1, v_2, \ldots, v_{s-1}, w_{p_k-1}, w_{p_k})$ coloured p_k;

(iv) for $k = s$, $(v_1, v_2, \ldots, v_{s-1}, w_{p_s-1}, w_{p_s})$ coloured p_s;

(v) and, finally, $(w_{p_1}, w_{p_2}, \ldots, w_{p_s}, w_{m-1})$ coloured m.

Notice that, for each $x_i \in X_i$, H contains q^s edges labelled with $[x_i, i]$. Observe, also, that any coloured copy of K contained within H gives a solution to the equation by definition. Conversely, solution $x_1, x_2, \ldots, x_m$ of the equation corresponds to q^s disjoint coloured copies of K.

Suppose, now, that there are at most $\epsilon_0 q^{m-1}$ solutions to the equation

$$x_1x_2\ldots x_{p_1} + x_{p_1+1}x_{p_1+2}\ldots x_{p_2} + \cdots + x_{p_s+1}x_{p_s+2}\ldots x_m = 0.$$

Then, by correspondence of the coloured copies of K in H and the solutions of the above, the directed graph contains no more than $\epsilon_0 q^{s+m-1}$ distinct coloured copies of K.

Then, recalling that H is an $(s+1)$-uniform hypergraph on $N_0 = (s+m-1)q$ vertices, we apply the coloured graph removal lemma to H with $\epsilon = \epsilon_0/(s+m-1)^{s+m-1}$: Since H has at most $\epsilon_0 q^{s+m-1} = \epsilon((s+m-1)q)^{s+m-1}$ coloured copies of K, provided $q \geq N_{3.6}$, there is a set E of at most δN_0^{s+1} edges such that $H - E$ contains no coloured copy of K (for some δ depending only on ϵ and m).

Let Y_i be the set of those elements $x \in X_i$ such that E contains at least q^s/m edges labelled with $[x_i, i]$. Since $|E| \leq \delta(N_0)^{s+1}$, the size of each Y_i is at most $m|E|/(q^s) \leq \delta(s+m-1)^{s+2}q$. Set $X_i' = X_i \backslash Y_i$. Since the size of Y_i is bounded in terms of δ (depending only on ϵ and m) and $\delta \to 0$ as $\epsilon_0 \to 0$, all that remains is to prove that there are no solutions to

$$x_1 x_2 \dots x_{p_1} + x_{p_1+1} x_{p_1+2} \dots x_{p_2} + \dots + x_{p_s+1} x_{p_s+2} \dots x_m = 0 \text{ with } x_i \in X_i'.$$

Assume that there is a solution to the above and consider the q^s edge disjoint coloured copies of K corresponding to $x_1, x_2, \dots, x_m$. Each contains at least one of the edges of E and every edge contained within these has one of the labels $[x_1, 1], [x_2, 2], \dots, [x_m, m]$. Since the cycles are disjoint, E will contain at least q^s/m edges with the same label, $[x_i, i]$. Consequently, $x_i \in Y_i$ and, thus, $x_i \notin X_i'$, a contradiction. □

3.5 Conclusions and open problems

The final singleton term x_m in the multinomial

$$x_1 x_2 \dots x_{p_1} + x_{p_1+1} x_{p_1+2} \dots x_{p_2} + \dots + x_{p_s+1} x_{p_s+2} \dots x_{m-1} + x_m = b$$

in Theorem 3.1.2 seems somewhat unnatural. However, our proof, as given in Section 3.3, relies upon this 'extra' term, in that, without it, the required correspondence between solutions of the above and copies of K in H cannot be established. This is because, in order for each solution, $x_1, x_2, \dots, x_m$ to correspond to q^{s+1} disjoint copies of K, it is necessary that, given a copy of K, one can re-construct from the definitions of H and K the values of $c_0, c_1 \dots, c_s$ and thus $x_0, x_1 \dots, x_s$. In the proof in Section 3.3, this is possible and is based upon K having edges corresponding to each product and an additional edge corresponding to the sum. However, dropping the singleton term x_m would reduce the number of edges available by one, causing difficulties which we were unable to overcome except in the case where $b = 0$.

However, we offer the following conjecture:

Conjecture 3.5.1. *Given $\epsilon > 0$ and an integer m, there exists $q_{3.1.3} = q_{3.5.1}(\epsilon, m)$ and $\delta = \delta_{3.5.1}(\epsilon, m)$ such that the following holds: Letting $F = \mathbb{F}_q$ be the finite field of order $q \geq q_{3.5.1}$ and $s, p_1, \ldots, p_s$ integers, if $X_1, X_2, \ldots, X_m$ are subsets of F such that the equation*

$$x_1 x_2 \ldots x_{p_1} + x_{p_1+1} x_{p_1+2} \ldots x_{p_2} + \cdots + x_{p_s+1} x_{p_s+2} \ldots x_m = b$$

has at most δq^{m-1} solutions with $x_i \in X_i$ for all i, then it is possible to remove at most ϵq elements from each X_i so that no solutions remain.

Chapter 4

Fractional colouring

In this chapter, we consider the problem of bounding the fractional chromatic number of a triangle-free graph with maximum degree at most three.

Recall, from Section 1.3, that, for a graph $G = (V, E)$, a function w which assigns to each independent set of vertices I a real number $w(I) \in [0, 1]$ is called a *weighting*; that the *weight* $w[v]$ of a vertex $v \in V$ with respect to a weighting w is defined to be the sum of $w(I)$ over all independent sets containing v; and that a *fractional colouring* is a weighting w such that, for each $v \in V$, $w[v] \geq 1$. Recall also that the *size* $|w|$ of a fractional colouring w is the sum of $w(I)$ over all independent sets I and that the *fractional chromatic number* $\chi_f(G)$ is defined to be the infimum of $|w|$ over all possible fractional colourings of G.

By Lemma 1.3.1, for a graph G and a positive rational q, the following are equivalent:

(i) $\chi_f(G) \leq q$;

(ii) there exists an integer N and a multi-set $\mathcal{W}$ of at most qN independent sets in G such that each vertex is contained in exactly N sets from $\mathcal{W}$;

(iii) there exists a probability distribution π on the independent sets of G such that, for each vertex v, the probability that v is contained in a random independent set (with respect to π) is at least $1/q$.

Finally, recall that the problem of finding $\chi_f(G)$ for a given a graph G can be viewed as the LP-relaxation of the integer programming problem of finding $\chi(G)$. Thus, for any graph G, we have $\chi_f(G) \leq \chi(G)$.

Brooks' Theorem asserts that any triangle-free subcubic graph has chromatic number at most three and, thus, has fractional chromatic number at most three. On the other hand, Fajtlowicz [Faj78] observed that the independence number of the generalised Petersen Graph $P(7,2)$ is at most five, which implies that $\chi_f(P(7,2)) \geq 14/5 = 2.8$.

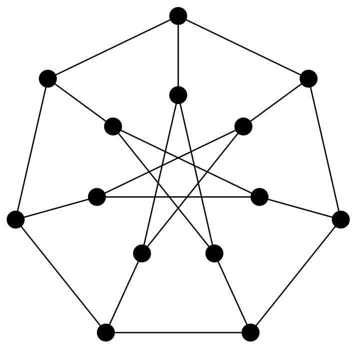


Figure 4.1: The generalised Petersen graph $P(7,2)$.

In 2001, Heckman and Thomas [HT01] conjectured that, if G is a triangle-free subcubic graph, then $\chi_f(G) \leq 2.8$. This conjecture is based on a result of Staton [Sta79] (see also [Jon90, HT01]) that any triangle-free subcubic graph contains an independent set of size at least $5n/14$, where n is the number of vertices of G. As shown by the graph $P(7,2)$, this result is optimal.

Hatami and Zhu [HZ10] proved that under the same assumptions, $\chi_f(G) \leq 3 - 3/64 \approx 2.953$. More recently, Lu and Peng [LP12] were able to improve this bound to $\chi_f(G) \leq 3 - 3/43 \approx 2.930$. We offer a new probabilistic proof which improves this bound to $3 - 1/11 \approx 2.909$:

Theorem 4.1. *If G is a triangle-free subcubic graph, then*

$$\chi_f(G) \leq 32/11.$$

Subsequently to the work done in this thesis Liu [Liu12] was able to improve this bound to $3 - 3/43 \approx 2.867$. More recently still, Dvořák, Sereni and Volec [DSV13] proved the conjectured bound of 2.8. Both proofs are quite long and use different methods to our proof.

In the remainder of this chapter, we set up the machinery necessary to prove Theorem 4.1 before illustrating our method by giving a short proof of a slightly weaker result. We then discuss how to complete the proof of Theorem A via a long case analysis found in Appendix A.

Note that, recently,

4.1 Definitions and notation

Recall that the length of a path P, denoted by $|P|$, is defined to be the number of edges in the path. If P is a path and $x, y \in V(P)$, then we write xPy for the subpath of P between x and y. The same notation is used when P is a cycle with a specified orientation, in which case xPy is the subpath of P from x to y with respect to the orientation. We will refer to the edges in directed graphs as *arcs* where, if xy is an arc, then x is its *tail* and y its *head*.

Given a graph G, we call a collection of m edges whose removal from $E(G)$ renders G disconnected an *edge-cut* of size m. We call an edge cut of size 1 a *bridge* and say that a graph which does not contain a bridge is *bridgeless*.

4.2 An algorithm

For now, let us assume that G is a triangle-free bridgeless cubic graph (we will extend the arguments which follow to general triangle-free subcubic graphs later).

By a well-known theorem of Petersen (see, for instance, [Die05, Corollary 2.2.2]), G has a 2-factor. It will be helpful to choose a 2-factor satisfying the conditions in the following result of Kaiser and Škrekovski [KŠ08]:

Theorem 4.2.1 ([KŠ08, Corollary 4.5])**.** *Every cubic bridgeless graph contains a 2-factor whose edge set intersects each inclusion-wise minimal edge-cut in G of size 3 or 4.*

Among all 2-factors of G satisfying the condition of Theorem 4.2.1, we choose a 2-factor F with as many components as possible.

In what follows, we will refer to the perfect matching complementary to F as M. If $u \in V(G)$, then u' denotes the opposite end of the edge of M containing u, that is, the M-mate of u. For the sake of brevity, we refer to the M-mate of u simply as the *mate* of u. We fix a reference orientation of each cycle of F and let u_{+k} (where k is a positive integer) denote the vertex reached from u by following k consecutive edges of F in accordance with the fixed orientation. The symbol u_{-k} is defined symmetrically.

We write u_+ and u_- for u_{+1} and u_{-1} respectively. These vertices are referred to as the *F-neighbours* of u. To simplify the notation for mates we write, for example, u'_{+2} instead of $(u_{+2})'$.

We now describe **Algorithm 1**, an algorithm to construct a random independent set I in G. An independent set is said to be *maximum* if no other independent set has larger cardinality. Given a set $X \subseteq V(G)$, we define $\Phi(X) \subseteq X$ as follows:

(a) if $F[X]$ is a path, then $\Phi(X)$ is either a maximum independent set of $F[X]$ or its complement in X, each with probability $1/2$;

(b) if $F[X]$ is a cycle, then $\Phi(X)$ is a maximum independent set in $F[X]$, chosen uniformly at random;

(c) if $F[X]$ is disconnected, then $\Phi(X)$ is the union of the sets $\Phi(X \cap V(K))$, where K ranges over all components of $F[X]$.

In **Phase 1** of the algorithm, we choose an orientation $\vec{\sigma}$ of M by directing each edge of M independently at random, choosing each direction with probability $1/2$. A vertex u is *active* (with respect to $\vec{\sigma}$) if u is a head of $\vec{\sigma}$; otherwise, it is *inactive.*

An *active run* of $\vec{\sigma}$ is a maximal set R of vertices such that the induced subgraph $F[R]$ is connected and each vertex in R is active. Thus, $F[R]$ is either a path or a cycle. We let

$$\sigma^1 = \bigcup_R \Phi(R),$$

where R ranges over all active runs of $\vec{\sigma}$. Thus, σ^1 is an independent set I (which will be modified by subsequent phases of the algorithm and eventually become its output). The vertices of σ^1 are referred to as those *added in Phase 1.*

In **Phase 2**, we add to I all the active vertices u such that each neighbour of u is inactive. Observe that, if an active run consists of a single vertex u, then u will be added to I either in Phase 1 or in Phase 2.

In **Phase 3**, we consider the set of all vertices of G which are not contained in I and have no neighbour in I. We call such vertices *feasible.* Note that each feasible vertex must be inactive. A *feasible run* R is defined analogously to an active run, except that each vertex in R is required to be feasible.

We define

$$\sigma^3 = \bigcup_R \Phi(R),$$

where R now ranges over all feasible runs. All of the vertices of σ^3 are added to I.

In **Phase 4**, we add to I all the feasible vertices with no feasible neighbours. As with Phase 2, a vertex which forms a feasible run by itself is certain to be added to I either in Phase 3 or in Phase 4.

When referring to the random independent set I in Sections 4.3–4.5, we mean the set output from Phase 4 of Algorithm 1. We represent the random choices made during an execution of Algorithm 1 by the triple $\sigma = (\vec{\sigma}, \sigma^1, \sigma^3)$ which we call a *situation*. Thus, the set Ω of all situations is the sample space in our probabilistic scenario. An *event* is any subset of Ω. Note that, if we know the situation σ associated with a particular run of Algorithm 1, we can determine the resulting independent set $I = I(\sigma)$. We will say that an event $\Gamma \subseteq \Omega$ *forces* a vertex $u \in V(G)$ if u is included in $I(\sigma)$ for any situation $\sigma \in \Gamma$.

4.3 Templates and diagrams

Throughout this and the subsequent sections, let u be a fixed vertex of G and let $v = u'$. Furthermore, let Z be the cycle of F containing u. All cycles of F are taken to have a preferred orientation which enables us to use notation such as uZv for subpaths of these cycles.

We will analyse the probability of the event $u \in I(\sigma)$, where σ is a random situation produced by Algorithm 1. To this end, we classify situations based on what they look like in the vicinity of u.

A *template* in G is a 5-tuple $\Delta = (\Delta, \Delta^1, \Delta^{\bar{1}}, \Delta^3, \Delta^{\bar{3}})$, for which the following hold:

(i) Δ is an orientation of a subgraph of M,

(ii) Δ^1 and $\Delta^{\bar{1}}$ are disjoint sets of heads of Δ,

(iii) Δ^3 and $\Delta^{\bar{3}}$ are disjoint sets of tails of Δ.

We set $\Delta^* = \Delta^1 \cup \Delta^{\bar{1}} \cup \Delta^3 \cup \Delta^{\bar{3}}$. The *weight* of Δ, denoted by $w(\Delta)$, is defined as

$$w(\Delta) = |E(\Delta)| + |\Delta^*|\,.$$

A situation $\sigma = (\vec{\sigma}, \sigma^1, \sigma^3)$ *weakly conforms to* Δ if the following hold:

(i) $\Delta \subseteq \vec{\sigma}$,

(ii) $\Delta^1 \subseteq \sigma^1$, and

(iii) $\Delta^{\bar{1}} \cap \sigma^1 = \emptyset$.

If, in addition,

(iv) $\Delta^3 \subseteq \sigma^3$ and $\Delta^{\bar{3}} \cap \sigma^3 = \emptyset$,

then we say that σ *conforms to* Δ. The *event defined by* Δ, denoted by $\Gamma(\Delta)$, consists of all situations conforming to Δ.

In the above definition, we can think of Δ^1 and $\Delta^{\bar{1}}$ as specifying which vertices must or must not be added to I in Phase 1. Note, however, that a vertex u in an active run of length one will be added to I in Phase 2 even if $u \in \Delta^{\bar{1}}$. Similarly, Δ^3 and $\Delta^{\bar{3}}$ specify which vertices will or will not be added to I in Phase 3, with an analogous provision for feasible runs of length one.

To facilitate the discussion, we represent templates by pictorial *diagrams*. These usually show only the neighbourhood of the distinguished vertex u, and the following conventions apply for a diagram representing a template Δ:

- the vertex u is circled;
- solid and dotted lines represent edges and non-edges of G, respectively;
- u_- is shown to the left of u, while v_- is shown to the right of v;
- the arcs of Δ are shown with arrows;
- the vertices in Δ^1 ($\Delta^{\bar{1}}$, Δ^3, $\Delta^{\bar{3}}$, respectively) are shown with a star (crossed star, triangle, crossed triangle, respectively);
- sometimes, only one end-vertex of an arc will be shown (so an edge of G may actually be represented by one or two arcs of the diagram) but the other end-vertex can still be assigned one of the above symbols.

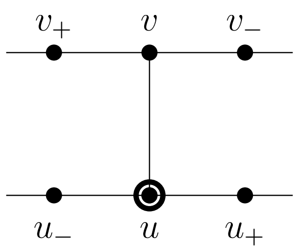


Figure 4.2: The location of neighbours of u and v.

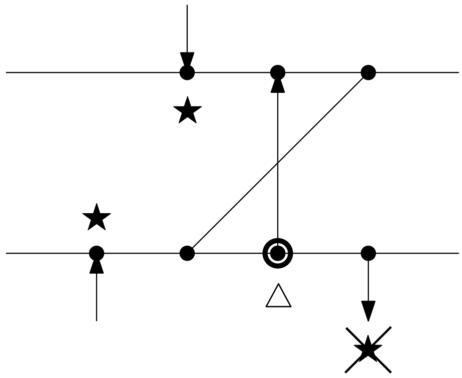


Figure 4.3: A diagram.

An arc with only one endvertex in a diagram is called an *outgoing* or an *incoming* arc, depending on its direction. A diagram is *valid* in a graph G if all of its edges are present in G and each edge of G is given at most one orientation in the diagram. Thus, a diagram is valid in G if and only if it determines a template in G. An event defined by a diagram is *valid* in G if the diagram is valid in G.

A sample diagram is shown in Figure 4.3. The corresponding event consists of all situations $(\vec{\sigma}, \sigma^1, \sigma^3)$ such that v, v_+, u_{-2} and u'_+ are heads of $\vec{\sigma}$, σ^1 includes v_+ and u_{-2} but does not include u'_+, and σ^3 includes u.

We call a template Δ *admissible* if $\Delta^3 \cup \Delta^{\bar{3}}$ is either empty or contains only u and, in the latter case, u is feasible in any situation weakly conforming to Δ. All the templates we consider will be admissible. Therefore, we state the subsequent definitions and results in a form restricted to this case.

We will need to estimate the probability of an event defined by a given template. If it were not for the sets Δ^1, $\Delta^{\bar{3}}$, etc., this would be simple as the orientations of distinct edges represent independent events. However, the events, say, $u_1 \in \Delta^1$ and $u_2 \in \Delta^1$ (where u_1 and u_2 are vertices) are in general not independent, and the amount of their dependence is influenced by the orientations of certain edges of F. To keep the dependence under control, we introduce the following concept:

A *sensitive pair* of a template Δ is an ordered pair (x, y) of vertices in $\Delta^1 \cup \Delta^{\bar{1}}$ such that x and y are contained in the same cycle W of F, the path xWy has no internal vertex in Δ^* and one of the following conditions holds:

(a) $x, y \in \Delta^1$ or $x, y \in \Delta^{\bar{1}}$, $x \neq y$, the path xWy has odd length and contains no tail of Δ;

(b) $x \in \Delta^1$ and $y \in \Delta^{\bar{1}}$ or vice versa, the path xWy has even length and contains no tail of Δ;

(c) $x = y \in \Delta^1$, W is odd and W contains no tail of Δ.

Sensitive pairs of the form (x, x) are referred to as *circular*; the other ones are *linear.*

A sensitive pair (x, y) is *k-free* (where k is a positive integer) if xWy contains at least k vertices which are not heads of Δ. Furthermore, any pair of vertices which is not sensitive is considered k-free for any integer k.

We define a number $q(\Delta)$ in the following way: If $u \in \Delta^3$ and Z is an odd cycle, then $q(\Delta)$ is the probability that all vertices of Z are feasible with respect to a random situation from $\Gamma(\Delta)$; otherwise, $q(\Delta)$ is defined as 0.

Observation 4.3.1. *Let Δ be a template in G. Then,*

(i) $q(\Delta) = 0$ if $u \notin \Delta^3$ or Z contains a head of Δ or Z is even, and

(ii) $q(\Delta) \leq 1/2^t$ if Z contains at least t vertices which are not tails of Δ.

The following lemma is a basic tool for estimating the probability of an event given by a template.

Lemma 4.3.2. *Let G be a graph and Δ an admissible template in G such that*

(i) Δ has ℓ linear sensitive pairs, the i-th of which is x_i-free ($i = 1, \dots, \ell$), and

(ii) Δ has c circular sensitive pairs, the i-th of which is y_i-free ($i = 1, \dots, c$).

Then,

$$\mathbf{P}(\Gamma(\Delta)) \geq \left(1 - \sum_{i=1}^{\ell} \frac{1}{2^{x_i}} - \sum_{i=1}^{c} \frac{1}{5 \cdot 2^{y_i}} - \frac{q(\Delta)}{5}\right) \frac{1}{2^{w(\Delta)}}. \tag{4.1}$$

Proof. Consider a random situation σ. We need to estimate the probability that σ conforms to Δ. We begin by investigating the probability P_W that σ weakly conforms to Δ.

In Phase 1, the orientation σ is chosen by directing each edge of M independently at random, each direction being chosen with probability 1/2. Therefore, the probability that the orientation of each edge in the subgraph specified by Δ agrees with the orientation chosen at random is $(1/2)^{|E(\Delta)|}$.

As noted above, the sets $\Delta^1, \Delta^{\bar{1}}, \Delta^3$ and $\Delta^{\bar{3}}$ prescribe vertices to be added or not added in Phases 1 and 3 of the algorithm.

Suppose, for now, that $\Delta^3 \cup \Delta^{\bar{3}}$ is empty and that every active run R satisfies $|R \cap (\Delta^1 \cup \Delta^{\bar{1}})| = 1$ and is either a path or an even cycle. Then, a given vertex in Δ^1 is added in Phase 1 with probability 1/2. Likewise, a given vertex in $\Delta^{\bar{1}}$ is not added in Phase 1 with probability 1/2. Indeed, R has either one or two maximum independent sets and $\Phi(R)$ either chooses between the maximum independent set and its complement or between the two maximum independent sets.

There are $|\Delta^1|$ vertices required to be added in Phase 1 and $|\Delta^{\bar{1}}|$ vertices required to not be added in Phase 1. These events are independent each with probability 1/2, giving the resultant probability

$$\mathbf{P}(\Delta^1 \subseteq \sigma_1, \Delta^{\bar{1}} \cap \sigma_1 = \emptyset) = \left(\frac{1}{2}\right)^{|\Delta^1|+|\Delta^{\bar{1}}|}. \tag{4.2}$$

In this case, the probability P_W is then obtained by multiplying (4.2) by $(1/2)^{|E(\Delta)|}$ and agrees with (4.1), completing the proof in that case.

Thus, we now assume that $\Delta^3 \cup \Delta^{\bar{3}}$ contains only u and assess the probability that σ conforms to Δ under the assumption that it conforms weakly and that $|R \cap (\Delta^1 \cup \Delta^{\bar{1}})| = 1$ for every active run R. Then, since Δ is admissible, u is feasible with respect to σ. Let R be the feasible run containing u. Suppose that R is a path or an even cycle. Then, if $u \in \Delta^3$, it is added in Phase 3 with probability 1/2 and, if $u \in \Delta^{\bar{3}}$, it is not added in Phase 3 with probability 1/2. Since u is the only vertex allowed in $\Delta^3 \cup \Delta^{\bar{3}}$, we obtain

$$\mathbf{P}(\Delta^3 \subseteq \sigma^3, \Delta^{\bar{3}} \cap \sigma^3 = \emptyset) = \begin{cases} \frac{1}{2} & \text{if } u \in \Delta^3 \cup \Delta^{\bar{3}}, \\ 1 & \text{otherwise.} \end{cases}$$

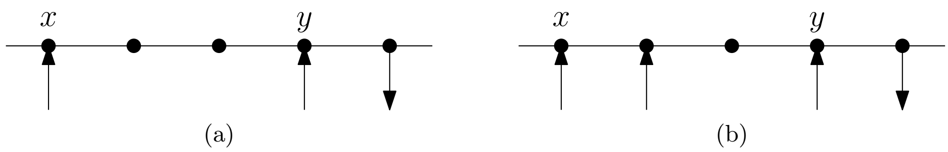


Figure 4.4: The probability that x and y are in distinct active runs in a conforming random situation is: (a) 3/4, (b) 1/2.

The assumption that u is feasible whenever σ weakly conforms to Δ implies that the addition of u to σ^3 is independent of the preceding random choices.

Note that we can relax the assumption that $|R \cap (\Delta^1 \cup \Delta^{\bar{1}})| = 1$ to allow, for instance, $|R \cap \Delta^1| \geq 1$, provided that the vertices of Δ^1 are appropriately spaced. Suppose that x and y are in the same component W of F and that all vertices of xWy are active after the choice of orientations in Phase 1. Let R be the active run R containing xWy:

If $|xWy|$ is even, Φ will choose both x and y with probability 1/2 for addition to I in Phase 1, an increase compared to the probability 1/4 if they are in different active runs. On the other hand, if $|xWy|$ is odd, then the probability of adding both x and y is zero as x and y cannot both be contained in $\Phi(R)$. Thus, if $x, y \in \Delta^1$ and $|xWy|$ is odd, then x and y must be in distinct active runs with respect to any situation conforming to Δ. As a result, we will, in general, get a lower value for the probability in (4.2), which will depend on the sensitive pairs involved in Δ.

Let (x, y) be a k-free sensitive pair contained in a cycle W of F, and let the internal vertices of xWy which are not heads of Δ be denoted by $x_1, \ldots, x_k$. Suppose that (x, y) is of type (a), say, $x, y \in \Delta^1$. The active runs of x and y with respect to σ will be separated if we require that at least one of $x_1, x_2, \ldots, x_k$ is the tail of an arc of σ, which happens with probability $1 - (1/2)^k$. The same computation applies to a sensitive pair of type (b).

Now suppose that (x, x) is sensitive of type (c), that is, x is the only member of Δ^1 belonging to an odd cycle W of length ℓ. If some vertex of W is the tail of an arc of $\vec{\sigma}$, then x will be added in Phase 1 with probability 1/2 as usual. It can happen, however (with probability $(1/2)^{\ell-1}$) that all the vertices of W are heads in $\vec{\sigma}$, in which case $\Phi(V(W))$ is one of ℓ maximum independent sets in W. If this happens, x will be added to I with probability $(1/2)(\ell-1)/\ell \geq 2/5$ rather than 1/2; this results in a reduction in $\mathbf{P}(\Gamma(\Delta))$ of at most $(1/5)(1/2)^{\ell-1}(1/2)^{w(\Delta)}$.

Finally, let us consider the situation where $u \in \Delta^3$ and the feasible run containing u is cyclic, that is, the case where every vertex in C_u is feasible. If Z is even, then this has no effect as u is still added in Phase 3 with probability $1/2$. If Z is odd, then u is added in Phase 3 with probability at least $2/5$ instead. Thus, if the probability of all vertices in C_u being feasible is $q(\Delta)$, then the resultant loss of probability from $\mathbf{P}(\Gamma(\Delta))$ is at most $q(\Delta)/(5 \cdot 2^{w(\Delta)})$.

Putting all this together gives:

$$\mathbf{P}(\Gamma(\Delta)) = \left(1 - \sum_{i=1}^{\ell} \frac{1}{2^{x_i}} - \sum_{i=1}^{c} \frac{1}{5 \cdot 2^{y_i}} - \frac{q(\Delta)}{5}\right) \frac{1}{2^{w(\Delta)}},$$

as required. □

Note that, by a careful analysis of the template in question, it is sometimes possible to obtain a bound better than that given by Lemma 4.3.2; however, the latter bound will usually be sufficient for our purposes.

A template without any sensitive pairs is called *weakly regular*. If a weakly regular template Δ has $q(\Delta) = 0$, then it is *regular*. By Lemma 4.3.2, if Δ is a regular template, then $\mathbf{P}(\Gamma(\Delta)) \geq (1/2)^{w(\Delta)}$.

The analysis is often much more involved if sensitive pairs are present. To allow for a brief description of a template Δ, we say that Δ is *covered* (in G) by ordered pairs of vertices (x_i, y_i), where $i = 1, \ldots, k$, if every sensitive pair of Δ is of the form (x_i, y_i) for some i. In most cases, our information on the edge set of G will only be partial; although we will not be able to tell for sure whether any given pair of vertices is sensitive, we will be able to restrict the set of possibly sensitive pairs.

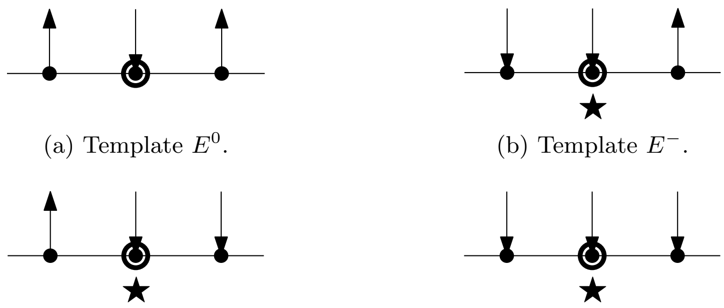
For brevity, we also use $(x, y)^\ell$ to denote an ℓ-free pair of vertices (x, y). Thus, we may write, for instance, that a template Δ is covered by pairs $(x, y)^2$ and $(z, z)^4$. By Lemma 4.3.2, we then have $\mathbf{P}(\Gamma(\Delta)) \geq (1/2)^{w(\Delta)} \cdot (1 - 1/4 - 1/80)$.

We extend the terminology used for templates to events defined by templates. Suppose that Δ is a template in G. The properties of $\Gamma(\Delta)$ simply reflect those of Δ. Thus, we say that the event $\Gamma(\Delta)$ is *regular* (*weakly regular*) if Δ is regular (weakly regular), and we set $q(\Gamma(\Delta)) = q(\Delta)$. A pair of vertices is said to be *k-free* for $\Gamma(\Delta)$ if it is k-free for Δ. $\Gamma(\Delta)$ is *covered* by a set of pairs of vertices if Δ is.

4.4 Events forcing a vertex

In this section, we build up a repertoire of events forcing the distinguished vertex u. (Recall that u is forced by an event Γ if u is contained in $I(\sigma)$ for every situation $\sigma \in \Gamma$.) In our analysis, we will distinguish various cases based on the local structure of G and show that, in each case, the total probability of these events (and, thus, the probability that $u \in I$) is large enough.

Suppose, first, that σ is a situation for which u is active. By the description of the Algorithm, we will have $u \in I$ if either both u_+ and u_- are inactive, or $u \in \sigma^1$. Thus, each of the templates E^0, E^-, E^+ and $E^\pm$, represented by the diagrams in Figure 4.5, defines an event which forces u. These events are pairwise disjoint. Observe that, by the assumption that G is triangle-free, each of the diagrams is valid in G.



(a) Template E^0. (b) Template E^-. (c) Template E^+. (d) Template $E^\pm$.

Figure 4.5: Some templates defining events which force u.

It is not difficult to estimate the probabilities of these events. The event E^0 is regular of weight 3, so $\mathbf{P}(E^0) \geq 1/8$ by Lemma 4.3.2. Similarly, E^+ and E^- are regular of weight 4 and have probability at least 1/16 each. The weakly regular event $E^\pm$ has weight 4 and the only potentially sensitive pair is (u, u). If the pair is sensitive, the length of Z must be odd and hence at least 5; thus, the pair is 2-free. By Lemma 4.3.2,

$$\mathbf{P}(E^\pm) \geq \frac{1}{16} \cdot \frac{19}{20} = \frac{1.9}{32}.$$

Note that, if Z has a chord (for instance, uv), then $E^\pm$ is actually regular, which improves the above estimate to 1/16.

By the above,

$$\mathbf{P}(E^0 \cup E^+ \cup E^- \cup E^\pm) \geq \frac{4+2+2+1.9}{32} = \frac{9.9}{32}.$$

These events cover most of the situations where $u \in I$. To prove Theorem 4.1, we will need to find other situations forcing u whose total probability is at least around one tenth of the above. Although this number is much smaller, finding the required events turns out to be a more difficult task.

4.5 Illustration

Recall that, in order to prove Theorem A, we need to present disjoint events forcing the fixed vertex u whose probabilities sum to at least 11/32. Thus far, we have found that

$$\mathbf{P}(E^0 \cup E^+ \cup E^- \cup E^\pm) \geq \begin{cases} \frac{9.9}{32} & \text{if } uv \text{ is not a chord} \\ \frac{10}{32} & \text{if } uv \text{ is a chord.} \end{cases}$$

Therefore, we need to find further valid events whose total probability is sufficient to give $\mathbf{P}(u \in I) \geq 11/32$.

It turns out that this is not always possible, making it necessary to use a compensation step in which those vertices with surplus probability donate that probability to those vertices which are deficient. The complete proof is quite involved and includes a long case analysis. For that reason, we postpone it to Appendix A. Instead, we will prove the following as an illustration of the method of proof:

Theorem 4.1′. *If G is a bridgeless cubic graph with girth at least six, then*

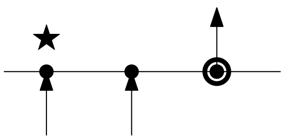
$$\chi_f(G) \leq 32/11.$$

That is, in addition to the assumptions that G is triangle free, cubic and bridgeless, we will assume that G contains no four-cycles and no five-cycles.

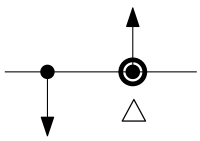
Since Figure 4.5 exhausts all the possibilities where u is active, we now turn to the situations where u is inactive. Suppose that an event forces u although u is inactive. We find that, if u_- is active, then u_{-2} must be added in Phase 1. If u_- is inactive, then there are several configurations which allow u to be forced since u can be added in Phase 3. However, the result also depends on the configurations around u_+ and v. We will express the events forcing u as combinations of certain 'primitive' events.

Let us begin by defining templates A, B (so called *left templates*). Recall that the vertex v is the mate of u. Diagrams corresponding to the templates are given in Figure 4.6:

template	heads of $\vec{\sigma}$	other conditions
A	v, u_-, u_{-2}	$u_{-2} \in \sigma^1$
B	v, u'_-	$u \in \sigma^3$



(a) Template A.



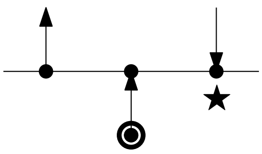
(b) Template B.

Figure 4.6: Left templates.

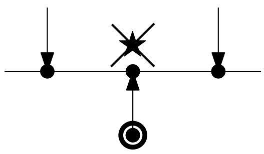
In addition, for $P \in \{A, B\}$, the template P^* is obtained by exchanging all '$-$' signs for '$+$' signs in this description. These are called *right templates*. In our diagrams, templates such as A restrict the situation to the left of u, while templates such as A^* restrict the situation to the right.

We also need primitive templates related to v and its neighbourhood (*upper templates*), since the configuration here is also relevant (see Figure 4.7):

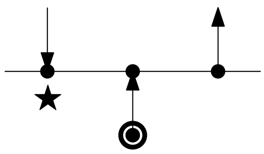
template	heads of $\vec{\sigma}$	other conditions
D^-	v, v_-, v'_+	$v_- \in \sigma^1$
D^0	v, v_-, v_+	$v \notin \sigma^1$
D^+	v, v'_-, v_+	$v_+ \in \sigma^1$



(a) Template D^-.



(b) Template D^0.



(c) Template D^+.

Figure 4.7: Upper templates.

We define the templates obtained from the left, right and upper events as their combinations. More precisely, for $P, Q \in \{A, B\}$ and $R \in \{D^-, D^0, D^+\}$, we define PQR to be the template Δ such that

$$\Delta = P \cup Q^* \cup R,$$
$$\Delta^1 = P^1 \cup (Q^*)^1 \cup R^1,$$

and so on for the other constituents of the template. The same symbol PQR will be used for the event defined by the template. If the result is not a legitimate template (for instance, because an edge is assigned both directions or because u is required to be both in Δ^3 and $\Delta^{\bar{3}}$), then the event is an empty one and is said to be *invalid*, just as if it were defined by an invalid diagram.

Let Σ be the set of all valid events PQR given by the above templates. Thus, Σ includes, for example, the events AAD^0 or ABD^+. However, some of them may be invalid and the probability of others will, in general, depend on the structure of G. We will examine this dependence in the following section. It is not hard to check (using the description of Algorithm 1) that each of the valid events in Σ forces u and also that each of them is given by an admissible template, as defined in Section 4.3.

We now show that the total probability of the valid events in Σ is sufficient to give $\mathbf{P}(u \in I) \geq 11/32$, dividing the remainder of the proof into two cases:

Case A: uv is not a chord

Recall that v denotes the vertex u' and Z denotes the cycle of F containing u. Define $C_v \neq Z$ to be the cycle of F containing v. Note that, since G has girth at least six, the set $\{u_{-2}, u_-, u_+, u_{+2}, v_-, v_+\}$ is independent.

As observed above, $\mathbf{P}(E^0 \cup E^- \cup E^+ \cup E^\pm) \geq 9.9/32 = 79.2/256$.

Thus, we need to find events in Σ whose total probability is at least $8.8/256$.

Then, consider the weakly regular event BBD^+ (shown in Figure 4.8). Since $|V(Z)| \geq 6$, Observation 4.3.1(ii) implies that $q(BBD^+) \leq 1/8$. Then, since the event has weight 7, by Lemma 4.3.2, we have $\mathbf{P}(BBD^+) \geq 1.95/256$. We also get the same estimate for BBD^- and BBD^0.

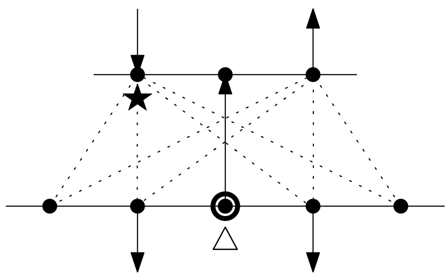


Figure 4.8: The event BBD^+ used in Case A.

Since u_- is not adjacent to either v_- or v_+, the event ABD^+ (shown in Figure 4.9) is valid. The event is regular and has weight 9, so, by Lemma 4.3.2, $\mathbf{P}(ABD^+) \geq 0.5/256$.

The same applies to the events ABD^-, ABD^0, BAD^+, BAD^- and BAD^0. Thus, the probability of the union of these six events is at least $3/256$. Together with the other events described so far, the probability exceeds $88/256$, completing the proof in this case.

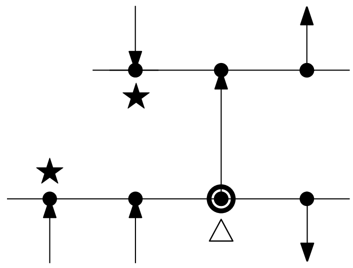


Figure 4.9: The event ABD^+ used in Case A.

Case B: uv is a chord

Observe that, in this case, one useful simplification is that, by Observation 4.3.1(i), we now have $q(\Delta) = 0$ for any template Δ. Thus, in particular, we have $\mathbf{P}(E^\pm) \geq 1/16$, which implies

$$\mathbf{P}(E^0 \cup E^- \cup E^+ \cup E^\pm) \geq \frac{10}{32}.$$

Thus, we need to find events in Σ whose total probability is at least $1/32$.

Recall that we assume that G has girth at least six and hence, again, we may assume that the set $\{u_{-2}, u_-, u_+, u_{+2}, v_-, v_+\}$ is independent. Again, we begin by considering the event BBD^+. The event is regular so, by Lemma 4.3.2, we have $\mathbf{P}(BBD^+) \geq 0.25/32$. The same applies to the events BBD^0 and BBD^-. Thus, it suffices to find additional events of total probability at least $0.25/32 = 2/256$.

Next, we consider the event ABD^- of weight 9 (shown in Figure 4.10). Since neither (u_{-2}, v_-) nor its reverse is a sensitive pair, the event is regular. Thus, $\mathbf{P}(ABD^-) \geq 0.5/256$. By symmetry, we also have $\mathbf{P}(BAD^+) \geq 0.5/256$.

The argument for ABD^- can also be applied to the event ABD^+ (shown in Figure 4.11), whose diagram has two outgoing arcs, to give a contribution of at least $0.25/256$, unless the vertex set of the path vZu is $\{v, v_+, u'_+, v'_-, u_-, u\}$, which cannot occur since G has girth at least six. Next, consider the event ABD^0 (shown in Figure 4.12). It is valid and covered by the pair (v, u_{-2}). Since the girth of G is at least six and the diagram of ABD^0 contains only one outgoing arc (namely $u_+u'_+$), the pair is 1-free. Thus, by Lemma 4.3.2, we have $\mathbf{P}(ABD^0) \geq 0.25/256$ and, by symmetry, have $\mathbf{P}(BAD^0) \geq 0.25/256$.

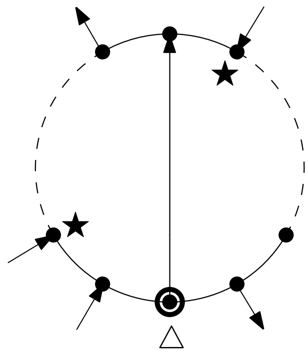


Figure 4.10: The event ABD^- used in Case B.

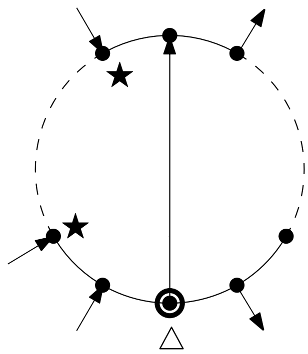


Figure 4.11: The event ABD^+ used in Case B.

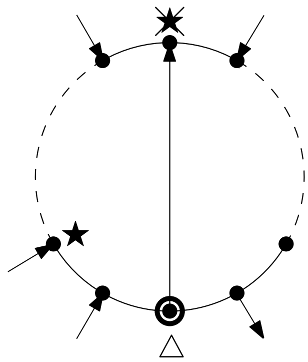


Figure 4.12: The event ABD^0 used in Case B.

Thus, we have found a sufficient probability to complete the proof of Theorem 4.1′. □

4.6 Subcubic graphs

Appendix A expands on Section 4.5 to give a proof of Theorem A for triangle-free bridgeless cubic graphs. Thus, it remains to generalise from triangle-free bridgeless cubic graphs to triangle-free subcubic graphs. This generalisation is perhaps most clear when phrased in terms of the second equivalent definition of the fractional chromatic number as given in Lemma 1.3.1:

Having shown that, for a triangle-free bridgeless cubic graph, G', $\chi_f(G') \leq k = 32/11$, by Lemma 1.3.1, there exists an integer N such that kN is an integer and we can colour the vertices of G' using N-tuples from $\{1, 2, \ldots, kN\}$ in such a way that adjacent vertices receive disjoint lists of colours.

Now, suppose that G is an arbitrary triangle-free subcubic graph. We show, by induction on the number of vertices of G, that $\chi_f(G) \leq k$. The base cases where $|V(G)| \leq 3$ are trivial. Suppose that G has a bridge and choose a block B_1 incident with only one bridge e. (A *block* of G is a maximal connected subgraph of G without cut-vertices.) Let B_2 be the other component of $G - e$. For $i = 1, 2$, the induction hypothesis implies that B_i ($i = 1, 2$) admits a colouring by N_i-tuples from a list of $\lfloor kN_i \rfloor$ colours, for a suitable integer N_i. Setting N to be a common multiple of N_1 and N_2 such that kN is an integer, we see that each B_i has an N-tuple colouring by colours $\{1, \ldots, kN\}$. Furthermore, since $k > 2$, we may permute the colours used for B_1 so as to make the end-vertices of e coloured by disjoint N-tuples. The result is a valid N-tuple colouring of G by kN colours, showing $\chi_f(G) \leq k$.

We may, thus, assume that G is bridgeless; in particular, it has minimum degree 2 or 3. In fact, we may assume that it contains a vertex of degree 2 for, otherwise, we are done by Sections 4.2–4.4 and Appendix A. If G contains at least two vertices of degree 2, we can form a graph G'' by taking two copies of G and joining the two copies of each vertex of degree 2 by an edge. Since G'' is a cubic bridgeless graph and contains G as a subgraph, we find that $\chi_f(G) \leq k$.

It remains to consider the case where G is bridgeless and contains exactly one vertex v_0 of degree 2. Let G_0 be the bridgeless cubic graph obtained by suppressing v_0, and let e_0 denote the edge corresponding to the pair of edges incident at v_0 in G. By Theorem 4.2.1, G_0 has a 2-factor F_0 containing e_0 such that $E(F_0)$ intersects every inclusion-wise minimal edge-cut of size 3 or 4 in G_0.

Let G_1 be obtained from two copies of G by joining the copies of v_0 by an edge. Thus, G_1 is a cubic graph with precisely one bridge. The 2-factor F_0 of G_0 yields a 2-factor F_1 of G_1 in the obvious way. Moreover, every inclusion-wise minimal edge-cut of size 3 or 4 in G_1 is intersected by $E(F_1)$. In that case, the arguments of Sections 4.2–4.4 and Appendix A still work even though G_1 is not bridgeless. Consequently, $\chi_f(G_1) \leq k$ and, since G is a subgraph of G_1, we infer that $\chi_f(G) \leq k$ as well, which completes the proof of Theorem 4.1. □

Chapter 5

An analogue of Vizing's Theorem for intersecting hypergraphs

Recall that, when considering (proper) edge-colourings of a graph G, the Theorems of Shannon [Sha49] and Vizing [Viz64] give the following bounds for the chromatic index of a multigraph G:

$$\chi'(G) \leq \tfrac{3}{2}\Delta,$$
$$\chi'(G) \leq \Delta + \mu,$$

where Δ is the maximum degree of the vertices of G and μ is the maximum multiplicity of the edges of G.

A natural question to ask would be:

'Can these results be generalised to hypergraphs?'

In this chapter, we consider a possible first step towards answering that question, namely:

'How many edges can an intersecting hypergraph have?'

where a hypergraph $H = (V, E)$ is intersecting if, for any edges $e_1, e_2 \in E$, $e_1 \cap e_2 \neq \emptyset$.

In order to illustrate the connection, we first define the notion of proper colouring that we will use for hypergraphs. We define a *proper k-edge-colouring* of a hypergraph H to be an assignment of a colour to each edge such that any two edges sharing at least one vertex receive different colours and at most k colours are used.

We can then define the *chromatic-index* $\chi'(H)$ to be the minimum k such that there exists a proper k-edge-colouring of H. Under these definitions, if H is an intersecting hypergraph, then

$$|e(H)| = \chi'(H).$$

In what follows, we allow multiple copies of edges. Therefore, all references to hypergraphs should be understood as referring to multi-hypergraphs.

Given a vertex $v \in V(H)$, we define the *degree* of that vertex $d(v)$ to be the number of edges (including multiple copies of the same edge) incident at v and define the *maximum degree* $\Delta(H)$ to be the maximum of $d(v)$ over $v \in V(H)$. Similarly, we define the *multiplicity* $\mu(e)$ of an edge to be the number of copies of that edge present in the hypergraph and define the *maximum multiplicity* $\mu(H)$ to be the maximum of $\mu(e)$ over $e \in E(H)$.

In 1981, Füredi [Für81] proved that, for a intersecting r-uniform simple hypergraph H with maximum degree Δ,

$$\chi'(H) = |e(H)| \leq (r - 1 + 1/r)\Delta.$$

This bound takes the same form as Shannon's bound and, indeed, agrees with it for $r = 2$. Furthermore, Füredi proved that this maximum is uniquely obtained by a particular class of hypergraphs and that, for any other intersecting r-uniform simple hypergraph,

$$\chi'(H) = |e(H)| \leq (r - 1)\Delta.$$

The class of hypergraphs in question are the finite projective planes, where the finite projective plane of order $r-1$ can be defined to be an r-uniform hypergraph on r^2-r+1 vertices satisfying the following properties:

(i) any pair of vertices belong to some edge;

(ii) any two edges share exactly one vertex;

(iii) every vertex has degree r.

Such hypergraphs are known to exist when r is a prime power and are known to not exist for $r = 6$ [BR49] and $r = 10$ [Lam91].

For intersecting 3-uniform simple hypergraphs H, Füredi's result gives

$$\chi'(H) \leq |e(H)| \leq \tfrac{7}{3}\Delta,$$

with equality only if H is the projective plane of order two. This well known hypergraph, known as the Fano Plane, is shown below.

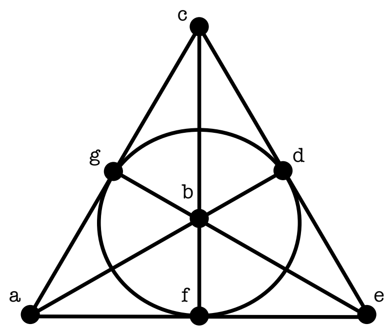


Figure 5.1: The Fano Plane.

In the remainder of this chapter, we consider intersecting 3-uniform hypergraphs (and allow multiple copies of edges). We prove an upper bound for $|e(H)|$ more similar in form to Vizing's bound and that this upper bound is obtained only by (multiple copies of) the Fano plane:

Theorem 5.1. *Let H be an intersecting 3-uniform hypergraph with maximum degree Δ and maximum multiplicity μ. Then,*

$$\chi'(H) = |e(H)| \leq 2\Delta + \mu.$$

Furthermore, the unique structure achieving this maximum is μ copies of the Fano Plane.

Given the form of this bound, it is tempting to conjecture that Vizing's Theorem and Theorem 5.1 are special cases of a result that would say that, for r-uniform hypergraphs,

$$\chi'(H) \leq (r-1)\Delta + \mu.$$

Alternatively, defining $\mu_r(H)$ to be the maximum number of edges having r vertices in common, we can re-write Vizing's bound as

$$\chi'(G) \leq \mu_1 + \mu_2$$

and the bound for 3-uniform hypergraphs suggested by Theorem 5.1 as

$$\chi'(H) \leq 2\mu_1 + \mu_3.$$

In this notation, the form of the latter bound is less pleasing and we notice that it takes no specific account of edges sharing two vertices, raising the possibility that the true form of an analogue of Vizing's Theorem for r-uniform hypergraphs may need to be more complicated.

5.1 Proof of Theorem 5.1

Let us begin by defining a property, which we will make much use of:

Definition 5.1.1. *We say that an intersecting hypergraph H is loosely-intersecting if, given any edge e_1 and any vertex $v \in e_1$, there exists an edge e_2 such that $e_1 \cap e_2 = v$.*

For a hypergraph H, we define

$$W(H) = \{v \in V(H) \text{ s.t. } v = e_1 \cap e_2 \text{ for some } e_1, e_2 \in E\}.$$

Then, by definition, if H is loosely-intersecting, $W(H) = V(H)$ and every vertex v belongs to at least two distinct edges e_1, e_2, which intersect each other only at v.

Observe that it suffices to prove Theorem 5.1 for loosely-intersecting hypergraphs. Indeed, it can be shown that an intersecting 3-uniform hypergraph which is not loosely-intersecting has fewer than $2\Delta + \mu$ edges as follows: Suppose H is such a hypergraph, then there exists some (multi-)edge e_1 and $x \in e_1$ such that, whenever $x \in e_1 \cap e_2$, $|e_1 \cap e_2| \geq 2$. In that case, we can delete x from (every copy of) e_1, obtaining an intersecting hypergraph H' in which e_1 spans only two vertices. Notice then that, since every edge must intersect e_1, we have $|e(H)| = |e(H')| \leq 2\Delta(H') + \mu(H') \leq 2\Delta(H) + \mu(H)$.

Notice, also, that loosely connected 3-graphs with at most six vertices have at most 2Δ edges. Indeed, suppose such a hypergraph has $n \leq 6$ vertices and degree sequence

$d_1, d_2, \ldots, d_n$. Then, $3e(H) = \Sigma d_i \leq 6\Delta$ so $e(H) \leq 2\Delta$. Thus, we may restrict our attention to loosely-intersecting 3-graphs with at least seven vertices.

It is known (see, for instance, [Tal05]) that, if H is an intersecting 3-graph, then $W(H)$ has cardinality at most 7. We could use this result to restrict our attention to loosely-intersecting 3-graphs with exactly seven vertices. However, for the sake of completeness, in the proof that follows, we prove directly that the 3-graphs we consider can have at most seven vertices.

The following observation allows us, when considering 3-graphs on exactly seven vertices, to assume that every vertex belongs to at least three distinct edges:

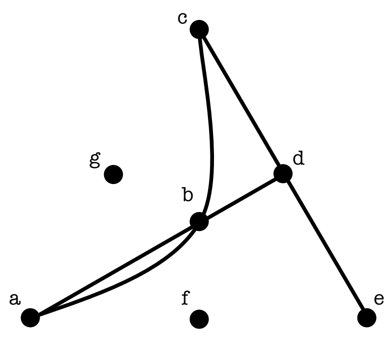
Observation 5.1.2. *Let H be a 3-uniform hypergraph on seven vertices with at least $2\Delta + \mu$ edges. Then, every vertex has degree at least 3μ.*

Proof. Suppose the vertices of H have degree sequence $\Delta = d_1 \geq d_2 \geq \cdots \geq d_7$ with $d_i < 3\mu$ for some i. Then, $3e(H) = \sum d_i < 6\Delta + 3\mu$, so $e(H) < 2\Delta + \mu$. □

We now proceed to complete the proof of Theorem 5.1 by case analysis. In what follows, we suppose that H is a 3-uniform hypergraph with maximum degree Δ, maximum multiplicity μ and at least $2\Delta + \mu$ edges. We will refer to the vertices of H as $a, b, c, \ldots$.

Case A: H has at least two edges sharing some pair of vertices.

Without loss of generality, suppose that H includes the edges abc and abd. Then, since H is loosely-intersecting, there must exist an edge which intersects abc only at c. Since H is intersecting, this edge must be, without loss of generality, the edge cde.



Now, consider the vertex e. Since H is loosely-intersecting, there must be an edge intersecting cde only at e. Since H is intersecting, this edge must be, without loss of generality, either abe or aef.

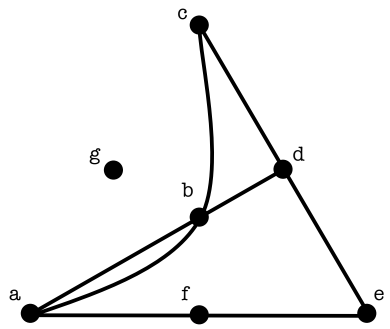
Suppose that H includes the edge abe. Then, since H is loosely-intersecting, considering $a \in abd$ forces H to include an edge intersecting abd only at a. Thus, since H is intersecting, it must include, without loss of generality, either aef or ace.

Thus, we may, in fact, consider the following two cases:

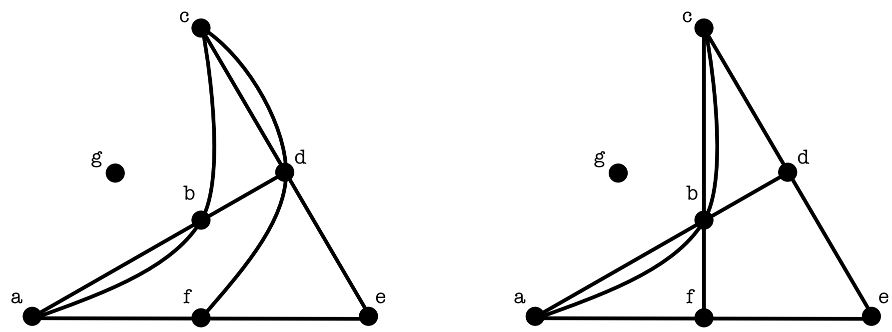
(i) H includes the edges abc, abd, cde and aef;

(ii) H includes the edges abc, abd, cde, abe and ace but not aef.

Case A.i: H includes the edges abc, abd, cde and aef.

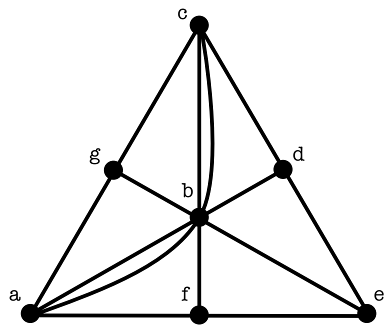


Since H is loosely-intersecting, considering vertex f forces H to include, as an edge, without loss of generality, either cdf or bcf.



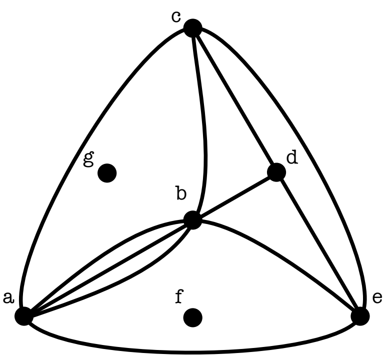
In either case, we consider the seventh vertex g. Since H is loosely-intersecting, there must exist two distinct edges intersecting each other only at g. Therefore, we seek pairs of vertices from $\{a, b, c, d, e, f\}$ which cover all the edges identified so far.

In the first case, only two such pairs exist, namely ac and ad, but these contain a common vertex, giving a contradiction. In the second case, exactly two such pairs cover all the edges, namely ac and be. Thus, H must include, as edges, acg and beg. Since these are the only such pairs, g has degree at most 2μ.

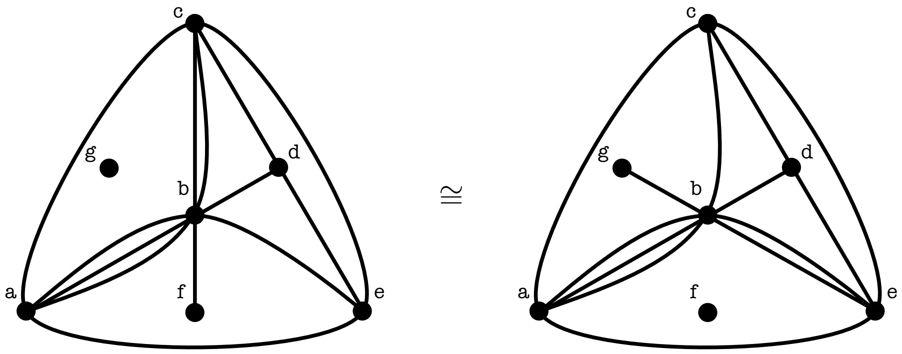


Consider a potential eighth vertex h. In order that h may belong to an edge, recalling that H is intersecting, we seek pairs (or singletons) from $\{a, b, c, d, e, f, g\}$ covering all the edges identified thus far. There are no such pairs (or singletons), so H cannot have an eighth vertex. Thus, H has exactly seven vertices and, by Observation 5.1.2, each of those vertices has degree at least 3μ, giving a contradiction and completing the proof in Case A.i.

Case A.ii: H includes the edges abc, abd, cde, abe and ace but not aef.

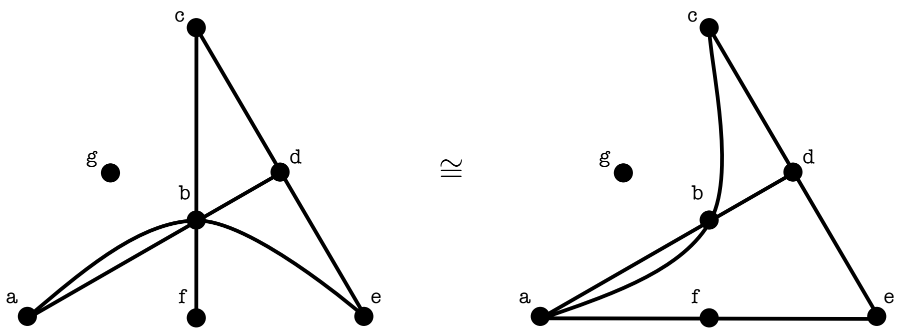


Since H is loosely-intersecting, considering b in the edge abc forces H to include an edge intersecting abc only at b. Since H is intersecting, such an edge must intersect all other known edges. Without loss of generality, the only two possibilities for such an additional edge are bde and beg. Similarly, considering b in abe forces H to include at least one of bcd and bcf. Observe, though, that the hypergraphs obtained by adding either bcf or beg to those edges already found are isomorphic to each other.



Thus, without loss of generality, H includes either both of the edges bde and bcd or the edge bcf.

Suppose that H includes the edge bcf. Then, consider the edges abd, abe, cde and bcf. Exchanging the roles of a and b and of c and e gives a situation identical to that considered in case A.i.



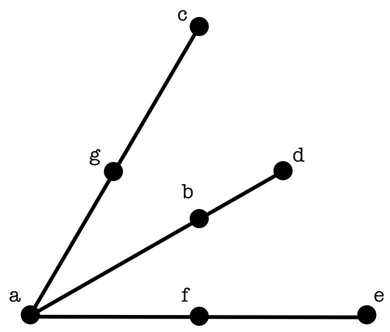
Thus, we may suppose that H includes the edges bde and bcd. In that case, notice that the edges found thus far span only five vertices but recall that H has at least seven vertices. Three pairs, namely ad, be and bc, cover every edge found so far. Thus, considering the sixth vertex f, H must include the edge adf and also at least one of the edges bcf or bef. In either case, consider the seventh vertex g. Since H is loosely-intersecting, g belongs to two distinct edges. However, none of the pairs from $\{a, b, c, d, e, f\}$ cover all of the edges already found, giving a contradiction and completing the proof in Case A.

Case B: Every two edges of H intersect in exactly one vertex.

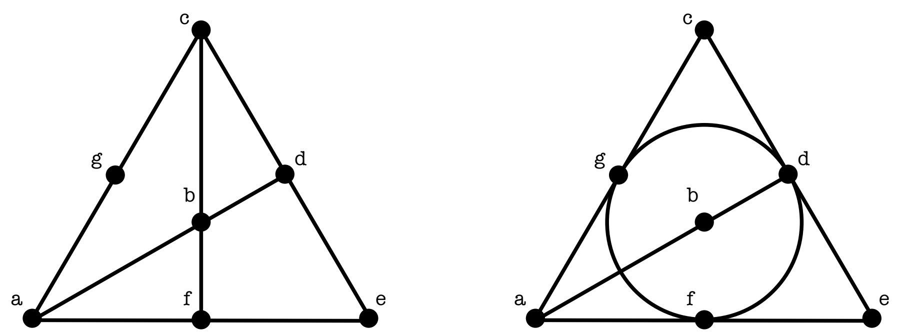
Recall that, since H is loosely-intersecting, every vertex in H belongs to at least two distinct edges. Suppose, in fact, that every vertex in H belongs to exactly two distinct edges. Suppose the vertex a belongings to abd and aef. Then, considering vertex e

forces H to include, without loss of generality, the edge cde. Then, the only other edge that can exist in H is bcf, resulting in a 3-graph with $|e(H)| \leq 2\Delta$.

Thus, we may assume that H has a vertex, say a, belonging to three distinct edges. Thus, H must include, without loss of generality, the edges abd, acg and aef.



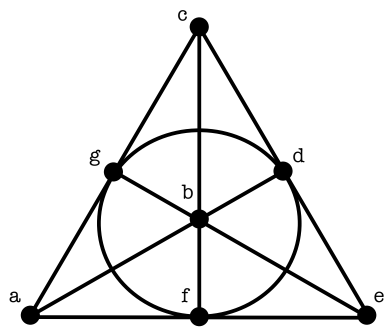
Then, consider vertex c. Recalling that H is loosely-intersecting, we see that H must include, without loss of generality, the edge cde. Similarly, considering vertex f, we see that there must exist an edge intersecting aef only at f. The only possibilities for such edges are bcf or dgf.



The two situations are, in fact, identical save for the naming of the vertices. In either case, there is only one pair spanning all the edges found so far. Thus, recalling that every vertex must belong to at least two distinct edges, we may assume that H has no more than the seven vertices identified so far and that, by Observation 5.1.2, every vertex has degree at least 3μ. Thus, in fact, H includes both bcf and dgf. There remain three vertices (b, e and g) which belong to only two of the edges found thus far. Considering spanning pairs, we see that the only possible additional edge is beg.

At this point all vertices belong to exactly three distinct edges and, recalling that any pair of edges must intersect in exactly one vertex, no more distinct edges can be present.

The distinct edges found form the Fano Plane. Since the hypergraph spans exactly seven vertices and each vertex belongs to exactly three distinct edges, by Observation 5.1.2, every edge must have multiplicity μ, thus, completing the proof.



$\square$

Bibliography

[AH77] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[AHK77] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[AT10] Tim Austin and Terence Tao. Testability and repair of hereditary hypergraph properties. *Random Structures Algorithms*, 36(4):373–463, 2010.

[BS09] F. S. Benevides and J. Skokan. The 3-colored Ramsey Number of even cycles. *J. Combinatorial Theory Ser. B*, 99(4):690–708, 2009.

[Bol98] Béla Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[BE73] J. A. Bondy and P. Erdős. Ramsey numbers for cycles in graphs. *J. Combinatorial Theory Ser. B*, 14:46–54, 1973.

[BM08] J. A. Bondy and U. S. R. Murty. *Graph theory*, volume 244 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2008.

[Bro41] R. L. Brooks. On colouring the nodes of a network. *Proc. Cambridge Philos. Soc.*, 37:194–197, 1941.

[BR49] R. H. Bruck and H. J. Ryser. The nonexistence of certain finite projective planes. *Canadian J. Math.*, 1:88–93, 1949.

[Chv72] V. Chvátal. On Hamilton's ideals. *J. Combinatorial Theory Ser. B*, 12:163–168, 1972.

[Con09] David Conlon. A new upper bound for diagonal Ramsey numbers. *Ann. of Math. (2)*, 170(2):941–960, 2009.

[Die05] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[Dir52] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc. (3)*, 2:69–81, 1952.

[DSV13] Zdeněk Dvořák, Jean-Sébastien Sereni, and Jan Volec. Subcubic triangle-free graphs have fractional chromatic number at most 14/5. Preprint at `arXiv:1301.5296v1 [math.CO]`, 2013.

[Erd47] P. Erdős. Some remarks on the theory of graphs. *Bull. Amer. Math. Soc.*, 53:292–294, 1947.

[Erd81] P. Erdős. On the combinatorial problems which I would most like to see solved. *Combinatorica*, 1(1):25–42, 1981.

[EFRS76] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp. Generalized Ramsey theory for multiple colors. *J. Combinatorial Theory Ser. B*, 20(3):250–264, 1976.

[EFR86] P. Erdős, P. Frankl, and V. Rödl. The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent. *Graphs Combin.*, 2(2):113–121, 1986.

[EG59] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar*, 10:337–356 (unbound insert), 1959.

[ES35] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[Faj78] S. Fajtlowicz. On the size of independent sets in graphs. In *Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1978)*, Congress. Numer., XXI, pages 269–274, Winnipeg, Man., 1978. Utilitas Math.

[FS74] R. J. Faudree and R. H. Schelp. All Ramsey numbers for cycles in graphs. *Discrete Math.*, 8:313–329, 1974.

[Fer13] D. G. Ferguson. The Ramsey number of one even and two odd cycles. In preparation, 2013.

[FŁ07a] A. Figaj and T. Łuczak. The Ramsey number for a triple of long even cycles. *J. Combin. Theory Ser. B*, 97(4):584–596, 2007.

[FŁ07b] A. Figaj and T. Łuczak. The Ramsey number for a triple of large cycles. Preprint at `arXiv:0709.0048 [math.CO]`, 2007.

[Fox11] Jacob Fox. A new proof of the graph removal lemma. *Ann. of Math. (2)*, 174(1):561–579, 2011.

[Für81] Zoltán Füredi. Maximum degree and fractional matchings in uniform hypergraphs. *Combinatorica*, 1(2):155–162, 1981.

[GG67] L. Gerencsér and A. Gyárfás. On Ramsey-type problems. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, 10:167–170, 1967.

[Gow97] W. T. Gowers. Lower bounds of tower type for Szemerédi's uniformity lemma. *Geom. Funct. Anal.*, 7(2):322–337, 1997.

[Gre05] B. Green. A Szemerédi-type regularity lemma in abelian groups, with applications. *Geom. Funct. Anal.*, 15(2):340–376, 2005.

[GRSS07] A. Gyárfás, M. Ruszinkó, G. N. Sárközy, and E. Szemerédi. Three-color Ramsey numbers for paths. *Combinatorica*, 27(1):35–69, 2007.

[HZ10] Hamed Hatami and Xuding Zhu. The fractional chromatic number of graphs of maximum degree at most three. *SIAM J. Discrete Math.*, 23(4):1762–1775, 2009/10.

[HT01] Christopher Carl Heckman and Robin Thomas. A new proof of the independence ratio of triangle-free cubic graphs. *Discrete Math.*, 233(1-3):233–237, 2001. Graph theory (Prague, 1998).

[Ish09] Yoshiyasu Ishigami. A simple regularization of hypergraphs. Preprint at `arXiv:math/0612838 [math.CO]`, 2009.

[Jon90] Kathryn Fraughnaugh Jones. Size and independence in triangle-free graphs with maximum degree three. *J. Graph Theory*, 14(5):525–535, 1990.

[KŠ08] Tomáš Kaiser and Riste Škrekovski. Cycles intersecting edge-cuts of prescribed sizes. *SIAM J. Discrete Math.*, 22(3):861–874, 2008.

[KR01] G. Károlyi and V. Rosta. Generalized and geometric Ramsey numbers for cycles. *Theoret. Comput. Sci.*, 263(1-2):87–98, 2001. Combinatorics and computer science (Palaiseau, 1997).

[KSS09a] Y. Kohayakawa, M. Simonovits, and J. Skokan. The 3-colored Ramsey number of odd cycles. *J. Combinatorial Theory Ser. B*, 2009. Accepted.

[KSS09b] Y. Kohayakawa, M. Simonovits, and J. Skokan. Stability of the Ramsey number of cycles. Manuscript, 2009.

[KS96] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 295–352. János Bolyai Math. Soc., Budapest, 1996.

[KSV09] Daniel Král', Oriol Serra, and Lluís Vena. A combinatorial proof of the removal lemma for groups. *J. Combin. Theory Ser. A*, 116(4):971–978, 2009.

[KSV12] Daniel Král', Oriol Serra, and Lluís Vena. A removal lemma for systems of linear equations over finite fields. *Israel J. Math.*, 187:193–207, 2012.

[Lam91] C. W. H. Lam. The search for a finite projective plane of order 10. *Amer. Math. Monthly*, 98(4):305–318, 1991.

[Liu12] Chun-Hung Liu. An upper bound on the fractional chromatic number of triangle-free subcubic graphs. Preprint at `arXiv:1211.4229v1 [math.CO]`, 2012.

[LP12] Linyuan Lu and Xing Peng. The fractional chromatic number of triangle-free graphs with $\Delta \leq 3$. *Discrete Math.*, 312(24):3502–3516, 2012.

[Łuc99] T. Łuczak. $R(C_n, C_n, C_n) \leq (4 + o(1))n$. *J. Combin. Theory Ser. B*, 75(2):174–187, 1999.

[ŁSS12] Tomasz Łuczak, Miklós Simonovits, and Jozef Skokan. On the multi-colored Ramsey numbers of cycles. *J. Graph Theory*, 69(2):169–175, 2012.

[MM63] J. Moon and L. Moser. On Hamiltonian bipartite graphs. *Israel J. Math.*, 1:163–165, 1963.

[NS08] V. Nikiforov and R. H. Schelp. Cycles and stability. *J. Combin. Theory Ser. B*, 98(1):69–84, 2008.

[Rad94] Stanisław P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994.

[Ram30] F. P. Ramsey. On a problem of formal logic. *Proc. London Math. Soc. Series 2*, 30:264–286, 1930.

[Ros73] Vera Rosta. On a Ramsey-type problem of J. A. Bondy and P. Erdős. I, II. *J. Combinatorial Theory Ser. B*, 15:94–104; ibid. 15 (1973), 105–120, 1973.

[RS78] I. Z. Ruzsa and E. Szemerédi. Triple systems with no six points carrying three triangles. In *Combinatorics (Proc. Fifth Hungarian Colloq., Keszthely, 1976), Vol. II*, volume 18 of *Colloq. Math. Soc. János Bolyai*, pages 939–945. North-Holland, Amsterdam, 1978.

[SU97] Edward R. Scheinerman and Daniel H. Ullman. *Fractional graph theory.* Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1997. A rational approach to the theory of graphs, With a foreword by Claude Berge, A Wiley-Interscience Publication.

[Sha49] Claude E. Shannon. A theorem on coloring the lines of a network. *J. Math. Physics*, 28:148–151, 1949.

[Sha10] Asaf Shapira. A proof of Green's conjecture regarding the removal properties of sets of linear equations. *J. Lond. Math. Soc. (2)*, 81(2):355–373, 2010.

[Spe75] Joel Spencer. Ramsey's theorem—a new lower bound. *J. Combinatorial Theory Ser. A*, 18:108–115, 1975.

[Spe94] Joel Spencer. *Ten lectures on the probabilistic method.* Siam, second edition, 1994.

[Sta79] William Staton. Some Ramsey-type numbers and the independence ratio. *Trans. Amer. Math. Soc.*, 256:353–370, 1979.

[Sze75] E. Szemerédi. On sets of integers containing no k elements in arithmetic progression. *Polska Akademia Nauk. Instytut Matematyczny. Acta Arithmetica*, 27:199–245, 1975.

[Sze78] E. Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978.

[Tal05] John Talbot. The intersection structure of t-intersecting families. *Electron. J. Combin.*, 12:Note 18, 4 pp. (electronic), 2005.

[Viz64] V. G. Vizing. On an estimate of the chromatic class of a p-graph. *Diskret. Analiz No.*, 3:25–30, 1964.

[Wil03] Robin Wilson. *Four colours suffice: How the map problem was solved.* Penguin Books Ltd, 2003.

[YYXB06] Sun Yongqi, Yang Yuansheng, Feng Xu, and Li Bingxi. New lower bounds on the multicolor Ramsey numbers $R_r(C_{2m})$. *Graphs Combin.*, 22(2):283–288, 2006.

Appendix A

Fractional colouring

This appendix includes the remaining parts of the proof of Theorem 4.1.

Recall that we consider a cubic bridgeless graph G and that, in Section 4.4, we showed that

$$\mathbf{P}(E^0 \cup E^+ \cup E^- \cup E^{\pm}) \geq \begin{cases} \frac{79.2}{256} & \text{if } uv \text{ is not a chord} \\ \frac{80}{256} & \text{if } uv \text{ is a chord,} \end{cases}$$

and defined a set of events Σ, each forcing u. Recall also that, if we could show that $\mathbf{P}(u \in I) \geq 11/32$, then Lemma 1.3.1 would give us $\chi_f(G) \leq 32/11$ as required.

Recall that, in Section 4.5, we proved the required result in the case that G had girth at least six. The proof in this appendix follows essentially the same approach but requires a long case analysis since short cycles introduce additional dependancies.

A.1 Outline

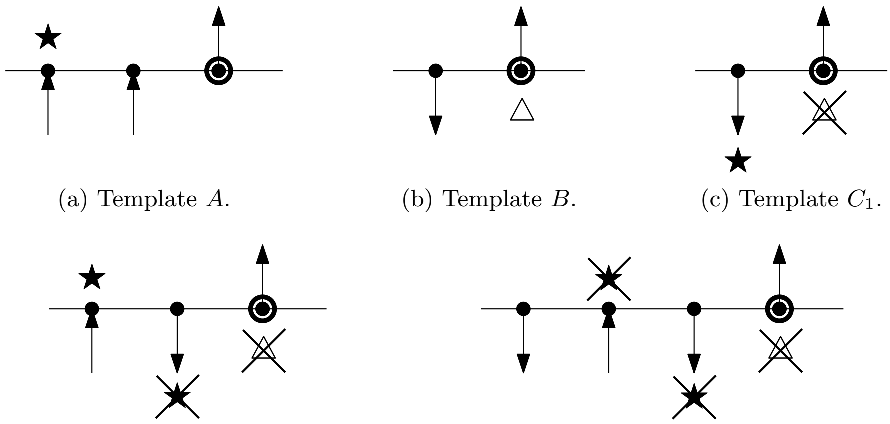
Before beginning the case analysis, we expand the definition of Σ to include more events, define a small amount of additional terminology and prove a lemma that will be useful later.

In Section A.4, we will consider the case when uv is not a chord of Z and in Section A.5 we will consider the case when uv is a chord of Z. In both cases, it will turn out that the contribution from the events in Σ is not always sufficient, which will make it necessary to modify the algorithm in order to allow vertices with surplus probability to donate that probability to those vertices which are deficient. This augmentation step will be referred to as Phase 5 of the Algorithm and will be discussed in Section A.6.

A.2 Additional templates

Let us recall the definition of the left-templates A and B and define three further left-templates C_1, C_2, C_3. Diagrams corresponding to the templates are given in Figure A.1:

template	heads of $\vec{\sigma}$	other conditions
A	v, u_-, u_{-2}	$u_{-2} \in \sigma^1$
B	v, u'_-	$u \in \sigma^3$
C_1	v, u'_-	$u \notin \sigma^3, u'_- \in \sigma^1$
C_2	v, u'_-, u_{-2}	$u \notin \sigma^3, u'_- \notin \sigma^1, u_{-2} \in \sigma^1$
C_3	v, u'_-, u_{-2}, u'_{-3}	$u \notin \sigma^3, u'_- \notin \sigma^1, u_{-2} \notin \sigma^1$



(a) Template A. (b) Template B. (c) Template C_1.

(d) Template C_2. (e) Template C_3.

Figure A.1: Left-templates.

As before, given a left-template P, the right-template P^* is obtained by exchanging all '$-$' signs for '$+$' signs in this description.

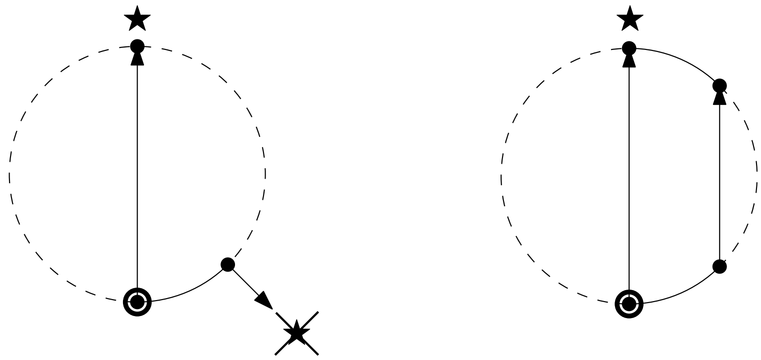
For $P, Q \in \{A, B, C_1, C_2, C_3\}$ and $R \in \left\{D^-, D^0, D^+\right\}$, we define PQR to be the template Δ such that

$$\Delta = P \cup Q^* \cup R,$$
$$\Delta^1 = P^1 \cup (Q^*)^1 \cup R^1,$$

and define Σ be be the set of all valid events PQR given by the above templates.

It is not hard to check (using the description of Algorithm 1) that each of the valid events in Σ forces u and also that each of them is given by an admissible template.

Note that, in the analysis which follows, in some cases, the structure of G may make some of the symbols in a diagram redundant. For instance, consider the diagram in Figure A.2(a) and let R_1 be the event corresponding to the associated template Δ_1. Since the weight of Δ_1 is 4, Lemma 4.3.2 implies a lower bound for $\mathbf{P}(R_1)$ which is slightly below 1/16. However, if we happen to know that the mate of u_+ is v_-, then we can remove the symbol at u'_+; the resulting diagram encodes the same event and comes with a better bound of 1/8. We will describe this situation by saying that the symbol at u'_+ in the diagram for Δ_1 is *removable* (under the assumption that $u'_+ = v_-$).



(a) Δ_1.

(b) Δ_2.

Figure A.2: The symbol at u'_+ in the diagram defining the template Δ_1 becomes removable if we add the assumption that $u'_+ = v_-$.

A.3 Additional terminology

We define some additional terminology and and prove a lemma that will be used to rule out some of the cases in the analysis found in Section A.5:

If G is a graph and $X, Y \subseteq V(G)$, then $E(X, Y)$ is the set of edges of $G[X, Y]$. We let $\partial(X)$ denote the set $E(X, V(G) - X)$. For a subgraph $H \subseteq G$, we write $\partial(H)$ for $\partial(V(H))$ and extend the definition of the symbol $E(X, Y)$ to subgraphs in an analogous way. The *neighbourhood* of a vertex u of G is the set $N(u)$ of its neighbours. We define $N[u] = N(u) \cup \{u\}$ and call this set the *closed neighbourhood* of u.

Lemma A.3.1. *Let C be a cycle of F. If there exist vertex-disjoint cycles D_1 and D_2 in G such that $V(C) = V(D_1) \cup V(D_2)$, then the following hold:*

(i) $2 \leq |E(D_1, D_2)| \leq 4$,

(ii) if the length of D_1 or D_2 equals 5, then $|E(D_1, D_2)| \leq 3$.

Proof. Let $d = |E(D_1, D_2)|$. We prove (i). Clearly, $d \geq 2$ since at least two edges of C join D_1 to D_2. Suppose that $d \geq 5$. We claim that the 2-factor F' obtained from F by replacing C with D_1 and D_2 satisfies the condition of Theorem 4.2.1. If not, then there is an inclusionwise minimal edge-cut Y of G of size 3 or 4 disjoint from $E(F')$. Since Y intersects $E(F)$, it must separate D_1 from D_2 and hence contain $E(D_1, D_2)$. But then $|Y| \geq 5$, a contradiction which shows that F' satisfies the condition of Theorem 4.2.1. Having more components than F, it contradicts the choice of F. Thus, $d \leq 4$.

(ii) Assume that $d = 4$ and that the length of, say, D_1 equals 5. Let F' be defined as in part (i). By the same argument, $E(D_1, D_2)$ is the unique inclusionwise minimal edge-cut of G disjoint from $E(F')$. Let K_1 be the component of $G - E(D_1, D_2)$ containing D_1. Since $\partial(D_1)$ contains exactly one edge of K_1, this edge is a bridge in G, contradicting the assumption that G is bridgeless. □

A.4 Analysis: uv is not a chord

Recall that v denotes the vertex u' and Z denotes the cycle of F containing u. In this section, we begin with the case where v is contained in a cycle $C_v \neq Z$ of F (that is, uv is not a chord of Z). We define a number $\epsilon(u)$ as follows:

$$(u) = \begin{cases} 1 & \text{if } uv \text{ is contained in a 4-cycle,} \\ 0 & \text{if } u \text{ has no } F\text{-neighbour contained in a 4-cycle intersecting } C_v, \\ -1 & \text{otherwise.} \end{cases}$$

The vertices with $\epsilon(u) = -1$ will be called *deficient of type 0*.

The aim of this section if to prove the following Proposition. This requires a case analysis in which the end of each case is marked by ▲.

Proposition A.4.1. *If uv is not a chord of Z, then*

$$\mathbf{P}(u \in I) \geq \frac{88 + \epsilon(u)}{256}.$$

Proof. As observed in Section 4.4, the probability of the event $E^0 \cup E^- \cup E^+ \cup E^\pm$ is at least $79.2/256$. Thus, we seek additional probability of $(8.8 + \epsilon(u))/256$.

Case 1. *The edge uv is contained in two 4-cycles.*

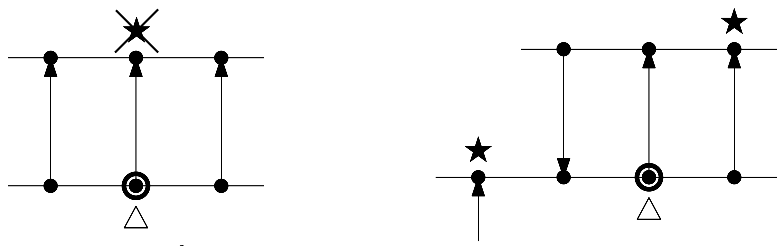
Consider the event BBD^0 of weight 5 (see the diagram in Figure A.3(a)). We claim that $\mathbf{P}(BBD^0) \geq 8/256$. Note that, for any situation $\sigma \in BBD^0$, at least one of the vertices v_-, v_+ is added to I in Phase 1. It follows that, for any such situation, u_- or u_+ is infeasible. Thus, $q(BBD^0) = 0$ and, by Lemma 4.3.2,

$$\mathbf{P}(BBD^0) \geq \frac{1}{2^5} = \frac{8}{256}.$$

Next, we consider the event ABD^- of weight 7 (see Figure A.3(b)). Since $u_{-2} \in \sigma^1$, for any situation $\sigma \in ABD^-$, u_{-2} is infeasible and hence $q(ABD^-) = 0$. Furthermore, ABD^- contains no sensitive pair and, thus, it is regular. Lemma 4.3.2 implies that $\mathbf{P}(ABD^-) \geq 2/256$. This shows that

$$\mathbf{P}(u \in I) \geq 89.2/256.$$

We remark that a further contribution of 2/256 could be obtained from the event AC_1D^-, but it will not be necessary. ▲



(a) BBD^0. (b) ABD^-.

Figure A.3: The events used in Case 1 of the proof of Lemma A.4.1.

Case 2. *uv is contained in one 4-cycle.*

We may assume that u_+ is adjacent to v_-. From Figure A.4(a), we see that the event BBD^0 is weakly regular; we will estimate $q(BBD^0)$. Let σ be a random situation from BBD^0. If C_v is even, then $v_- \in \sigma^1$, which makes u_+ infeasible, so $q(BBD^0) = 0$. Thus, we may assume that C_v is odd; since G is triangle-free, the length of C_v is at least 5.

Thus, it contains at least two vertices other than v, v_-, v_+; consequently, the probability that all the vertices of C_v are active is at most $1/4$. If all the vertices of C_v are active, then $v_- \in \sigma^1$ (and hence u_+ is infeasible) with probability at least $2/5$. It follows that

$$q(BBD^0) \leq \mathbf{P}(v_- \notin \sigma^1 \mid \sigma \in BBD^0) \leq \frac{1}{10}.$$

By Lemma 4.3.2, $\mathbf{P}(BBD^0) \geq 98/100 \cdot 1/64 = 3.92/256$.

Consider the weakly regular event BBD^- (Figure A.4(b)). Observe, first, that the event is valid in G as u_- and v_+ are not neighbours. Since u_+ is infeasible with respect to any situation from BBD^-, we have $q(BBD^-) = 0$ and so BBD^- is regular. Lemma 4.3.2 implies that $\mathbf{P}(BBD^-) \geq 4/256$.

Finally, consider the events ABD^- and AC_1D^- (Figure A.4(c) and (d)); note that the only difference between them is that, for $\sigma \in ABD^-$, $u \in \sigma^3$, whereas, for $\sigma \in AC_1D^-$, $u \notin \sigma^3$. Both events, however, force u. Observe that their validity does not depend on whether u_{-2} and v_+ are neighbours: even if they are, the diagram prescribes consistent orientations at both ends of the edge $u_{-2}v_+$. The events are regular of weight 8, and, thus, $\mathbf{P}(ABD^- \cup AC_1D^-) \geq 2/256$. This proves that $\mathbf{P}(u \in I) > 89.1/256$. ▲

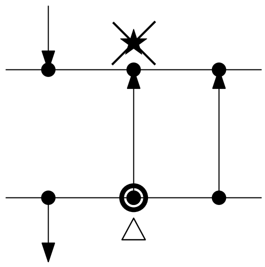
Having dealt with the above cases, we may now assume that the set $\{u_-, u_+, v_-, v_+\}$ is independent.

Case 3. *M includes the edges $u_{-2}v_+$ and $u_{+2}v_-$.*

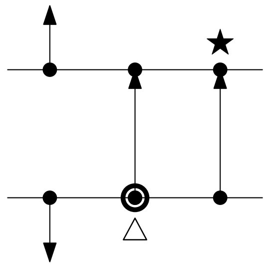
The event BBD^+ (Figure A.5(a)) is regular of weight 7; thus, $\mathbf{P}(BBD^+) \geq 2/256$. Similarly, $\mathbf{P}(BBD^-) \geq 2/256$. We also have $\mathbf{P}(BBD^0) \geq 2/256$ since v_+ and v_- have mates on Z, ensuring that one of the vertices of Z is infeasible and thus $q(BBD^0) = 0$. Furthermore, $\mathbf{P}(ABD^- \cup BAD^+) \geq 2/256$ by Lemma 4.3.2.

We may assume that $u'_- \neq v_{+2}$ and $u'_+ \neq v_{-2}$, since, otherwise, u has a neighbour contained in a 4-cycle and $\epsilon(u) = -1$. In that case, the bound $\mathbf{P}(u \in I) \geq 87.2/256$, proved so far, would be sufficient.

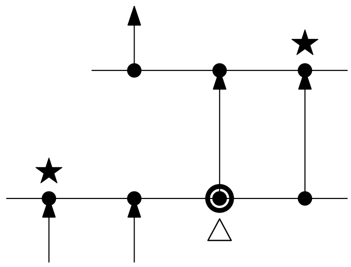
If u_+ or u_- have a mate on Z, then $E^{\pm}$ is regular and hence $\mathbf{P}(E^{\pm}) = 16/256$. This adds further $0.8/256$ to $\mathbf{P}(u \in I)$, making it reach $88/256$, which is sufficient. Thus, we may assume that u'_- and u'_+ are not contained in Z.



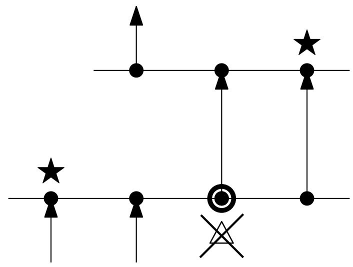
(a) BBD^0.



(b) BBD^-.

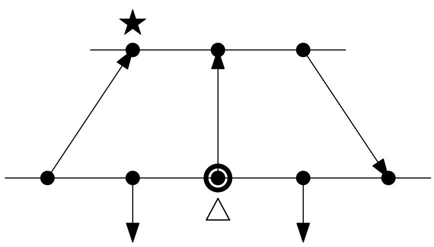


(c) ABD^-.

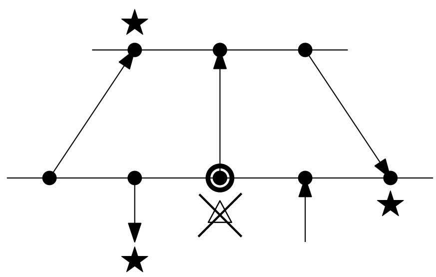


(d) AC_1D^-.

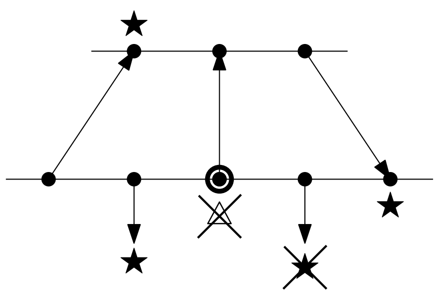
Figure A.4: The events used in Case 2 of the proof of Lemma A.4.1.



(a) BBD^+.



(b) C_1AD^+.



(c) $C_1C_2D^+$.

Figure A.5: Some of the events used in Case 3 of the proof of Lemma A.4.1.

Consider the event C_1AD^+ given by the diagram in Figure A.5(b). The only vertices which can be included in a sensitive pair are u'_-, u_{+2} and v_+. Neither (u_{+2}, u_{+2}) nor (v_+, v_+) are circular sensitive pairs since both Z and C_v contain a tail in C_1AD^+ (u_- and v_-, respectively). Hence, the only possible circular sensitive pair is (u'_-, u'_-). As for linear sensitive pairs, the only possibility is (v_+, u'_-): the vertex u_{+2} is ruled out since none of u'_- and v_+ is contained in Z, and the pair (u'_-, v_+) cannot be sensitive as v_- is a tail in C_1AD^+. (Note that the sensitivity of a pair depends on the order of the vertices in the pair.) Summarizing, the sensitive pair is (u'_-, u'_-) or (v_+, u'_-) and it is clear that both pairs cannot be sensitive at the same time.

If (u'_-, u'_-) is sensitive, then the cycle of F containing u'_- contains at least four vertices which are not heads in C_1AD^+. Consequently, the pair (u'_-, u'_-) is 4-free and Lemma 4.3.2 implies $\mathbf{P}(C_1AD^+) \geq 79/80 \cdot 0.5/256 > 0.49/256$.

On the other hand, if (v_+, u'_-) is sensitive, we know that $|v_+C_vu'_-|$ is odd and our assumption that $u'_- \neq v_{+2}$ implies that the pair (v_+, u'_-) is 2-free. By Lemma 4.3.2, $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$. As this estimate is weaker than the preceding one, C_1AD^+ is guaranteed to have probability at least 0.375/256. Symmetrically, $\mathbf{P}(AC_1D^-) \geq 0.375/256$.

So far, we have accumulated a probability of 87.95/256. The missing bit can be supplied by the event $C_1C_2D^+$ of weight 10 (Figure A.5(c)). Since the mates of u_- and u_+ do not belong to Z, any sensitive pair will involve only the vertices u'_-, u'_+ and v_+, and it is not hard to check that there will be at most two such pairs. Since $u'_- \neq v_{+2}$, each of these pairs is 1-free. If one of them is 2-free, then $\mathbf{P}(C_1C_2D^+) \geq 1/4 \cdot 0.25/256 > 0.06/256$ by Lemma 4.3.2, which is more than the amount missing to 88/256.

We may thus assume that none of these pairs is 2-free. This implies that (v_+, u'_-) is not a sensitive pair, as $|v_+C_vu'_-|$ would have to be odd and strictly between 1 and 3. Thus, there are only two possibilities: (a) $C_1C_2D^+$ is covered by (u'_-, u'_+) and (u'_+, u'_-), or (b) it is covered by (v_+, u'_+) and (u'_+, u'_-). The former case corresponds to u'_+ and u'_- being contained in a cycle W of F of length 4, which is impossible by the choice of F. In the latter case, u'_+ and u'_- are contained in C_v; in fact, $u'_+ = v_{+3}$ and $u'_- = v_{+5}$. Although Lemma 4.3.2 does not give us a non-zero bound for $\mathbf{P}(C_1C_2D^+)$, we can get one by exploiting the fact that G is triangle-free. Since $v_{+2}v_{+4} \notin E(M)$, the probability that both v_{+2} and v_{+4} are tails with respect to the random situation σ is 1/4, and these events are independent of orientations of the other edges of G. Thus, the probability that σ weakly conforms to the template for $C_1C_2D^+$ and v_{+2}, v_{+4} are tails is $1/2^7 = 2/256$.

Under this condition, σ will conform to the template with probability $1/2^5$ (a factor 1/2 for each symbol in the diagram). Consequently, $\mathbf{P}(C_1C_2D^+) > 0.06/256$, again a sufficient amount. ▲

Case 4. *M includes the edge $u_{-2}v_+$ but not $u_{+2}v_-$.*

As in the previous case, $\mathbf{P}(BBD^-) \geq 2/256$. Consider the weakly regular event BBD^+ (Figure A.6(a)). Since $u'_{-2} = v_+ \in \sigma^1$ for any $\sigma \in BBD^+$, we have $q(BBD^+) = 0$. By Lemma 4.3.2, $\mathbf{P}(BBD^+) \geq 2/256$.

The event BBD^0 is also weakly regular, and it is not hard to see that $q(BBD^0) \leq 1/10$ (using the fact that the length of C_v is at least 5). Lemma 4.3.2 implies that $\mathbf{P}(BBD^0) \geq 98/100 \cdot 2/256 = 1.96/256$.

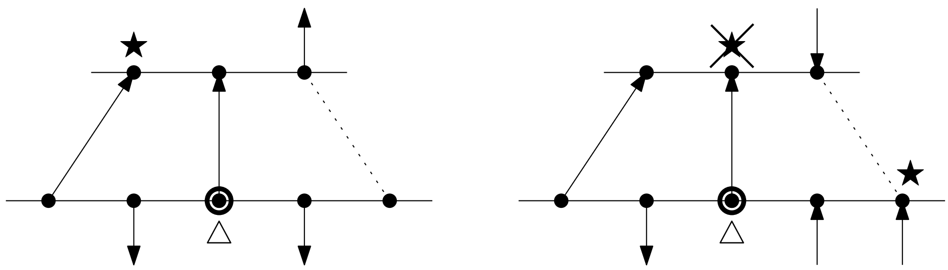
Each of the events BAD^0 (Figure A.6(b)), BAD^+ and BAD^- is regular and has weight 9. By Lemma 4.3.2, it has probability at least 0.5/256. Furthermore, the regular event ABD^- has $\mathbf{P}(ABD^-) \geq 1/256$, also by regularity. So far, we have shown that $\mathbf{P}(u \in I) \geq 87.66/256$. As in the previous case, this enables us to assume that u'_- and u'_+ are not vertices of Z. Furthermore, it may be assumed that $u'_- \neq v_{+2}$, for otherwise $\epsilon(u) = -1$ and the current estimate on $\mathbf{P}(u \in I)$ is sufficient.

If M includes the edge u_-v_{-2}, then AC_1D^- is regular and $\mathbf{P}(AC_1D^-) \geq 0.5/256$, which would make the total probability exceed 88/256. Let us, therefore, assume the contrary.

The event C_1AD^- is covered by $(u'_-, v_-)^2$ and $q(C_1AD^-) = 0$, so the probability of C_1AD^- is at least $3/4 \cdot 0.25/256$. Similarly, C_1AD^+ is covered by (v_+, u'_-). Suppose for a moment that this pair is 2-free; we then get $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.25/256$. The event C_1AD^0 is covered by (v, u'_-) and (u'_-, v). Our assumptions imply for each of the pairs that it is 2-free. By Lemma 4.3.2, $\mathbf{P}(C_1AD^0) \geq 1/2 \cdot 0.25/256$. The contribution we have obtained from $C_1AD^+ \cup C_1AD^- \cup C_1AD^0$ is at least 0.5/256, which is sufficient to complete the proof in this subcase.

It remains to consider the possibility that (v_+, u'_-) is not 2-free in the diagram for C_1AD^+. It must be that the path $v_+C_vu'_-$ includes v'_- and has length 3. The probability bound for C_1AD^+ is now reduced to $1/2 \cdot 0.25/256$. However, now, C_1AD^0 is covered by (u'_-, v), and we find that $\mathbf{P}(C_1AD^0) \geq 3/4 \cdot 0.25/256$. In other words, as before, we have

$$\mathbf{P}(C_1AD^+ \cup C_1AD^- \cup C_1AD^0) \geq 0.5/256.$$



(a) BBD^+. (b) BAD^0.

Figure A.6: Some of the events used in Case 4 of the proof of Lemma A.4.1.

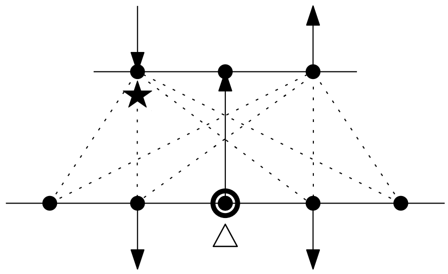


Figure A.7: The event BBD^+ used in the final part of the proof of Lemma A.4.1.

▲

By symmetry, it remains to consider the following case:

Case 5. *The set* $\{u_{-2}, u_-, u_+, u_{+2}, v_-, v_+\}$ *is independent.*

Consider the weakly regular event BBD^+ (Figure A.7). As before, since $|V(Z)| \geq 5$, if Z is odd, Observation 4.3.1(ii) implies that $q(BBD^+) \leq 1/4$. Since the event has weight 7, $\mathbf{P}(BBD^+) \geq 1.9/256$ by Lemma 4.3.2. We get the same estimate for BBD^- and BBD^0.

Since u_- is not adjacent to either of v_- and v_+, the event ABD^+ is valid. It is regular so $\mathbf{P}(ABD^+) \geq 0.5/256$. The same applies to the events ABD^-, ABD^0, BAD^+, BAD^- and BAD^0. Thus, the probability of the union of these six events is at least 3/256. Together with the other events described so far, the probability is at least 87.9/256. As in the previous cases, this means that we may assume that the mate of u_+ is not contained in Z, since, otherwise, we would obtain a further 0.8/256 from the event $E^{\pm}$ and reach the required amount.

Since the length of C_v is at least 5, u'_+ is not adjacent to both v_- and v_+. Suppose that it is not adjacent to v_+ (the other case is symmetric). Then, AC_1D^+ is covered by the pair $(v_+, u'_+)^2$. Hence, $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$. The total probability of $u \in I$ is therefore larger than $88/256$, which concludes the proof. ▲

□

A.5 Analysis: uv is a chord

In this section, we continue the analysis of Section A.4, this time considering the case where uv is a chord of Z. Although this case is more complicated, one useful simplification is that, by Observation 4.3.1(i), we now have $q(\Delta) = 0$ for any template Δ. In particular, $\mathbf{P}(E^{\pm}) \geq 16/256$, which implies

$$\mathbf{P}(E^0 \cup E^- \cup E^+ \cup E^{\pm}) \geq \frac{80}{256}.$$

Roughly speaking, since the probability needed to prove Theorem B is 88/256, we need to find events in Σ whose total probability is at least 8/256. However, like in Section A.4, we may actually require a higher probability or be satisfied with a lower one, depending on the type of the vertex. The surplus probability will be used to compensate for the deficits in Section A.6.

Recall that, at the beginning of Section A.4, we defined deficient vertices of type 0 and associated a number $\epsilon(u)$ with the vertex u provided that uv is not a chord of a cycle of F. We are now going to provide similar definitions for the opposite case, introducing a number of new types of deficient vertices.

Suppose that uv is a chord of Z which is not contained in any 4-cycle of G. The vertex u is *deficient* if it satisfies one of the conditions in Table A.1. (See the illustrations in Figure A.8.) Since the conditions are mutually exclusive, this also determines the *type* of the deficient vertex u.

We now extend the definition to cover the symmetric situations. Suppose that u satisfies the condition of type II when the implicit orientation of Z is replaced by its reverse — which also affects notation such as u_+, uZv etc. In this case, we say that u is deficient of type II*. (As seen in Figure A.9, the picture representing the type is obtained by a flip about the vertical axis.) The same notation is used for all the other types except types 0 and I. A type such as II* is called the *mirror* type of type II.

type of u	condition	$\epsilon(u)$
I	the path v_-vv_+ is contained in a 4-cycle in G, neither the path u_-uu_+ nor the edge uv are contained in a 4-cycle, and u is not of types Ia, Ib, Ia* or Ib*	-0.5
Ia	$\lvert uZv\rvert = 4$ and M includes the edges $u_{+2}v_+$, $u_{-2}v_-$, while $u_+v_{+2} \notin E(M)$	-2
Ib	$\lvert uZv\rvert = 4$ and M includes the edges $u_{+2}v_+$, $u_{-2}v_-, u_+v_{+2}$	-1.5
II	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert \geq 7$ and M includes all of the edges u_-v_{+2}, $u_{-2}v_+$, $u_{-3}u_+$, while $v_{+3}v_- \notin E(M)$	-0.125
IIa	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert = 6$, and M includes all of the edges $u_{-2}v_+$, $u_{-3}u_+$ and u_-u_{-4},	-0.5
III	$\lvert uZv\rvert = 4$, $\lvert vZu\rvert = 8$ and M includes all of the edges $u_{-2}v_+$, $u_{-3}u_+$, $v_{+3}v_-$ and u_-u_{-4}	-0.125

Table A.1: The type of a deficient vertex u provided that uv is a chord of Z, and the associated value $\epsilon(u)$.

Note that even with this extension, the types of a deficient vertex remain mutually exclusive. Furthermore, we have the following observation which will be used repeatedly without explicit mention:

Observation A.5.1. *If u is deficient (of type different from 0), then its mate v is not deficient.*

Proof. Let u be as stated. A careful inspection of Table A.1 and Figure A.8 shows that the path u_-uu_+ is not contained in any 4-cycle. It follows that v is not deficient of type I, Ia, Ib or their mirror variants. Suppose that v is deficient. By symmetry, u also does not belong to the said types and, hence, the types of both u and v are II, IIa, III or the mirror variants. As seen from Figure A.8, when u is of any of these types, the path u_-uu_+ belongs to a 5-cycle in G. By symmetry again, the same holds for v_-vv_+. The only option is that u belongs to type III and v to III*, or vice versa. But this is clearly impossible: if u is of type III or III*, then one of its neighbours on Z is contained in a 4-cycle, and this is not the case for any neighbour of v on Z. Hence, v cannot be of type III or III*. This contradiction shows that v is not deficient. □

We will often need to apply the concept of a type to the vertex v rather than u. This may at first be somewhat tricky; for instance, to obtain the definition of 'v is of type IIa*', one needs to interchange u and v in the definition of type IIa in Table A.1 and then perform the reversal of the orientation of Z. In this case, the resulting condition will be that $\lvert uZv\rvert = 4$, $\lvert vZu\rvert = 6$ (here the two changes cancel each other out) and M includes the edges $v_{+2}u_{-}$, $v_{+3}v_{-}$ and $v_{+}v_{+4}$. To spare the reader from having to turn Figure A.8 around repeatedly, we illustrate the various cases where v is deficient in Figure A.10.

Table A.1 also associates the value $\epsilon(u)$ with each type. By definition, a type with an asterisk (such as II*) has the same value assigned as the corresponding type without an asterisk.

We now extend the function ϵ to all vertices of G. It has been defined for all deficient vertices, as well as for all vertices whose mate is contained in a different cycle of F. Suppose that w is a non-deficient vertex whose mate w' is contained in the same cycle of F. We set

$$\epsilon(w) = \begin{cases} -\epsilon(w') & \text{if } w' \text{ is deficient,} \\ 0 & \text{otherwise.} \end{cases}$$

Our aim in the remainder of this section is to prove the following proposition. As in the proof of Proposition A.4.1, we mark the end of each case by ▲; furthermore, the end of each subcase is marked by △.

Proposition A.5.2. *If uv is a chord of Z, then we have*

$$\mathbf{P}(u \in I) \geq \frac{88 + \epsilon(u)}{256}.$$

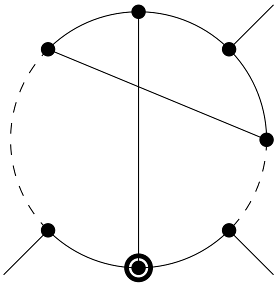
Proof. We distinguish a number of cases based on the structure of the neighbourhood of u in G.

Case 1. *The edge uv is contained in a 4-cycle.*

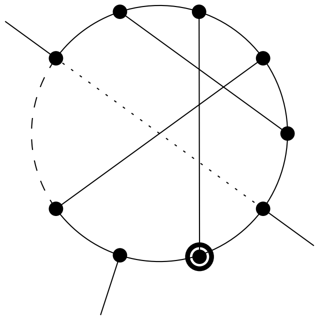
Observe that, in this case, neither u nor v is deficient.

Suppose that $uvv_{-}u_{+}$ is a 4-cycle (the argument in the other cases is the same). Consider, first, the possibility that $v_{-}u_{+}$ is an edge of M. The event BBD^0 is (valid and) regular. By Lemma 4.3.2, $\mathbf{P}(BBD^0) \geq 4/256$. Since this lower bound increases to $8/256$ if $u_{-}v_{+}$ is an edge of M (and since v is not deficient), we may actually assume that this is not the case. Consequently, $\mathbf{P}(BBD^{-}) \geq 4/256$ as BBD^{-} is regular. The total contribution is $8/256$, as desired.

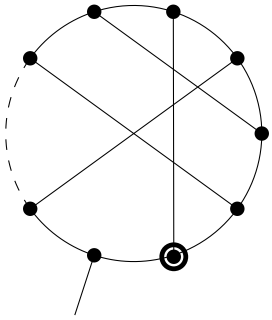
We may, thus, assume that $v_{-}u_{+}$ is an edge of F and no edge of M has both end-vertices in $\{u_{-}, u_{+}, v_{-}, v_{+}\}$. Since the events BBD^0 and BBD^{-} are regular, we have $\mathbf{P}(BBD^0 \cup BBD^{-}) \geq 4/256$.



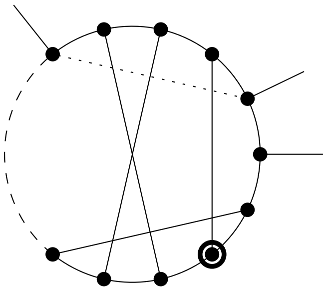
(a) Type I (only one of the two possibilities shown).



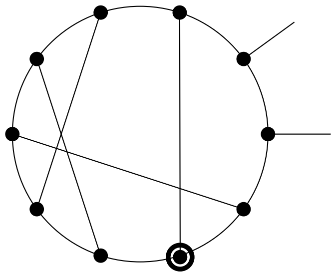
(b) Type Ia.



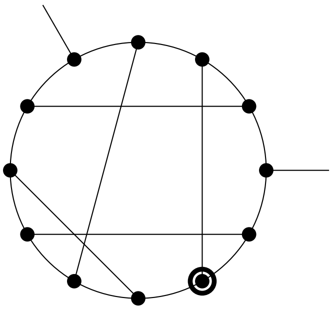
(c) Type Ib.



(d) Type II.



(e) Type IIa.



(f) Type III.

Figure A.8: Deficient vertices.

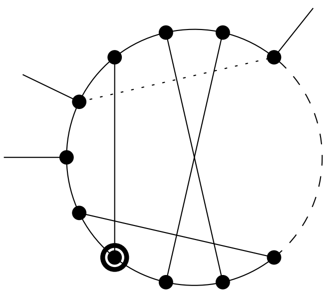
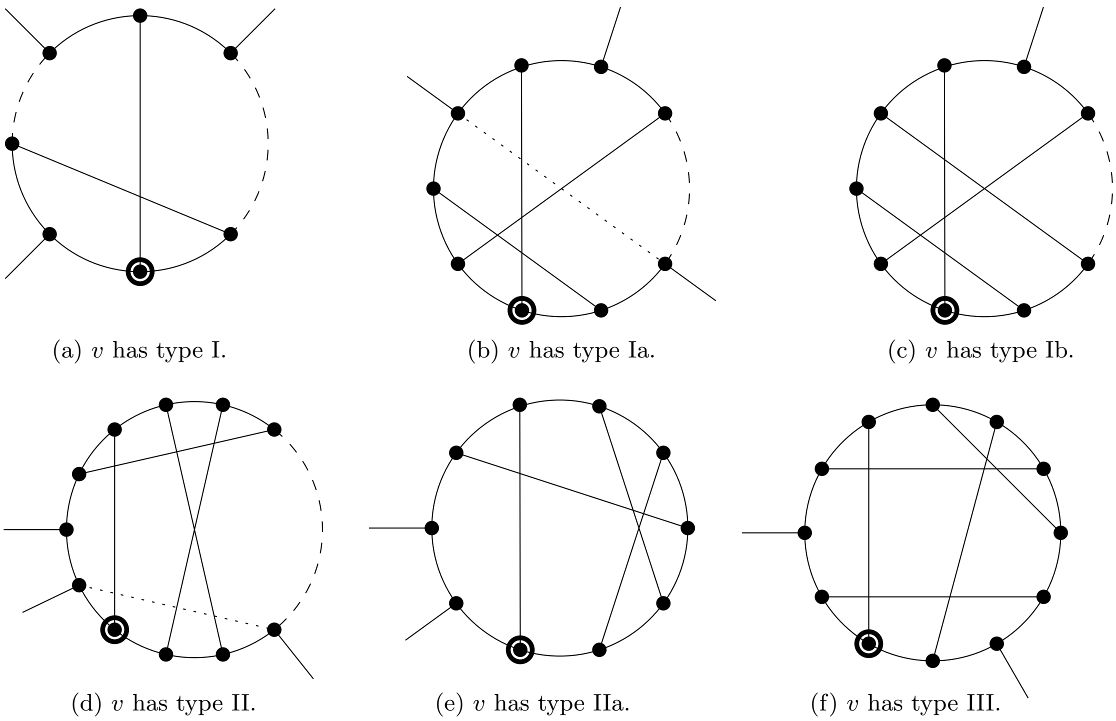
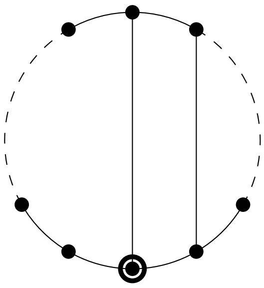


Figure A.9: A deficient vertex u of type II*.

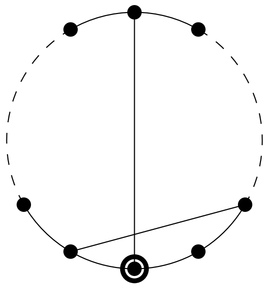


(a) v has type I.
(b) v has type Ia.
(c) v has type Ib.
(d) v has type II.
(e) v has type IIa.
(f) v has type III.

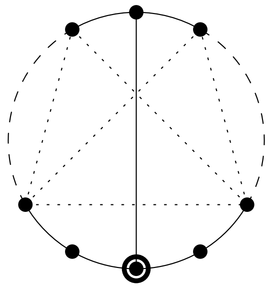
Figure A.10: The situation when the vertex v is deficient. As usual, the vertex u is circled.



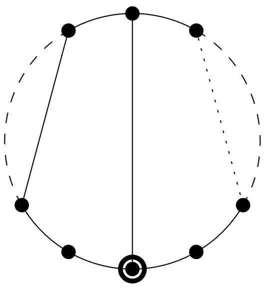
(a) Case 1 (one of the possibilities).



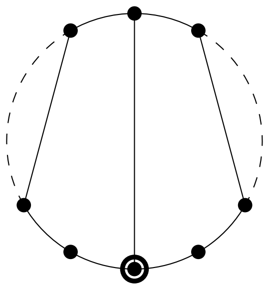
(b) Case 2 (one of the possibilities).



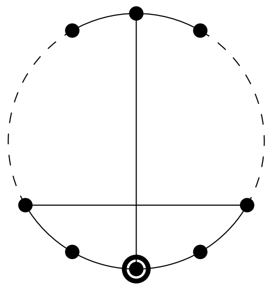
(c) Case 3.



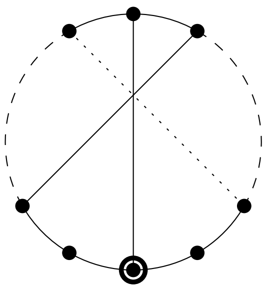
(d) Case 4.



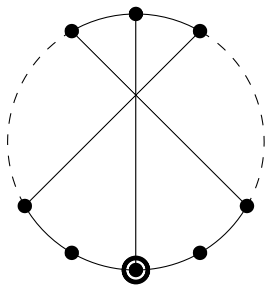
(e) Case 5.



(f) Case 6.



(g) Case 7.



(h) Case 8.

Figure A.11: The main cases in the proof of Proposition A.5.2. Relevant non-edges are represented by dotted lines, paths are shown as dashed lines.

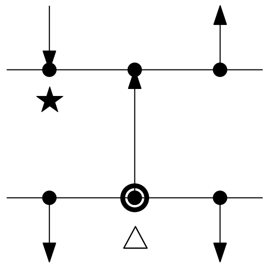


Figure A.12: The event BBD^+.

A further probability of 4/256 is provided by the regular events BAD^0 and BAD^-. Indeed, although the template BAD^0 has weight 8, which would only yield $\mathbf{P}(BAD^0) \geq 1/256$ by Lemma 4.3.2, the estimate is improved to 2/256 by the fact that the associated diagram has a removable symbol at v. The same applies to the event BAD^-. We conclude

$$\mathbf{P}(BBD^0 \cup BBD^- \cup BAD^0 \cup BAD^-) \geq 8/256,$$

as required. ▲

We will henceforth assume that uv is not contained in a 4-cycle. Note that this means that the set $\{u_-, u_+, v_-, v_+\}$ is independent. Consider the regular event BBD^+ (Figure A.12).

By Lemma 4.3.2, we have

$$\mathbf{P}(BBD^+) \geq \frac{2}{256}.$$

The same applies to the events BBD^0 and BBD^-. Thus, in the subsequent cases, it suffices to find additional events of total probability at least $(2 + \epsilon(u))/256$.

Case 2. *The path u_-uu_+ is contained in a 4-cycle.*

Suppose that $u_-uu_+u_{+2}$ is such a 4-cycle. (The other case is symmetric.) Consider the events C_1AD^+ and BAD^+. Since the condition of Case 1 does not hold, and, by the assumption that G is triangle-free, the set $\{u_+, v_-, v_+\}$ is independent in G. Furthermore, each of the events is regular and, by Lemma 4.3.2, each of them has probability at least 1/256. Thus, it remains to find an additional contribution of $\epsilon(u)$.

We distinguish several subcases based on the deficiency and type of the vertex v. Since u_-uu_+ is contained in a 4-cycle, v is either not deficient, or is deficient of type I, Ia, Ib, Ia* or Ib*.

Subcase 2.1. *v is not deficient.*

In this subcase, $\epsilon(u) \leq 0$, so there is nothing to prove. △

Subcase 2.2. *v is deficient of type I.*

By the definition of type I, both of the following conditions hold:

- $u_+v_{+2} \notin E(M)$ or $|uZv| \geq 5$,
- $u_-v_{-2} \notin E(M)$ or $|vZu| \geq 5$.

Moreover, we have $\epsilon(u) = 0.5$.

We may assume that M includes the edge $u_{-2}v_-$ since, otherwise, the event ABD^- is regular (see Figure A.13(a)) and has probability at least $0.5/256$, as required.

The event ABD^+ (Figure A.13(b)) is covered by the pair (v_+, u_{-2}). Consequently, we may assume that $|vZu| = 4$: otherwise the pair is 1-free, and since the event has weight 8, we have $\mathbf{P}(ABD^+) \geq 0.5/256$ by Lemma 4.3.2.

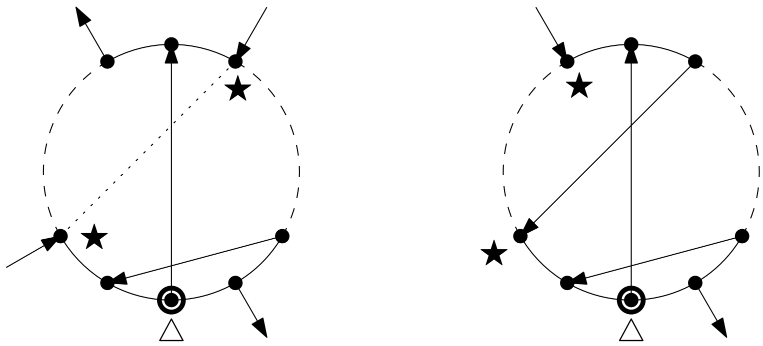
By a similar argument applied to the event C_1AD^-, we infer that $|uZv| = 4$. Thus, the length of Z is 8 and the structure of $G[V(Z)]$ is as shown in Figure A.14(a). The regular event $C_1C_2D^+$ (Figure A.14(b)) has probability at least $0.5/256$, which is sufficient. This concludes the present subcase.

△

Subcase 2.3. *v is deficient of type Ia, Ib, Ia* or Ib*.*

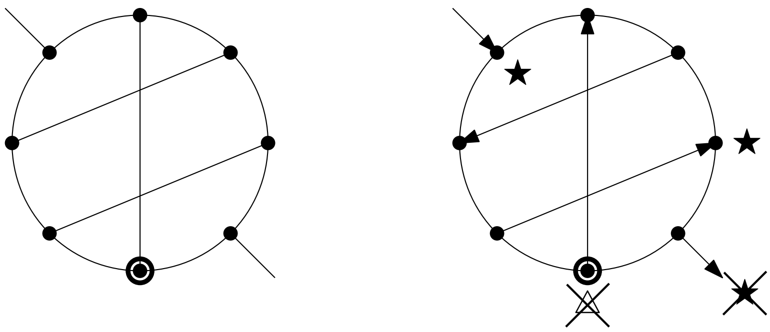
By symmetry, we may assume that v is either of type Ia* (if $u_{-2}v_-$ is not an edge of M) or Ib* (otherwise). Accordingly, we have either $\epsilon(u) = 2$ or $\epsilon(u) = 1.5$.

The regular event $C_1C_2D^+$ provides a contribution of $1/256$. If $u_{-2}v_- \notin E(M)$ (thus, v is of type Ia* and $\epsilon(u) = 2$), then the event ABD^- is also regular (including when $|vZu| = 4$) and $\mathbf{P}(ABD^-) \geq 1/256$, a sufficient amount.



(a) (b)

Figure A.13: Subcase 2.2 of the proof of Proposition A.5.2: (a) The event ABD^{-} if $u_{-2}v_{-} \notin E(M)$. (b) The event ABD^{+}.



(a) (b)

Figure A.14: (a) A configuration in Subcase 2.2 of the proof of Proposition A.5.2. (b) The event $C_1C_2D^{+}$.

It remains to consider the case that $u_{-2}v_- \in E(M)$. The required additional probability of $0.5/256$ is supplied by the event ABD^+, which is covered by the 1-free pair (v_+, u_{-2}). △

Having completed these subcases, the discussion of Case 2 is complete. ▲

From here on, we assume that none of the conditions of Cases 1 or 2 hold. In particular, v is not deficient of type I, Ia, Ib or their mirror types. We distinguish further cases based on the set of edges induced by M on the set

$$U = \{u_{-2}, u_{+2}, v_-, v_+\}.$$

Note that the length of the paths uZv and vZu is now assumed to be at least 4. We call a path *short* if its length equals 4.

Case 3. $E(M[U]) = \emptyset$.

We claim that, if v is deficient, then its type is III or III*. Indeed, for types I, Ia, Ib and their mirror types, u_-uu_+ would be contained in a 4-cycle and this configuration has been covered by Case 2. For types II, IIa and their mirror variants, U would not be an independent set. Since type 0 is ruled out for trivial reasons, types III and III* are the only ones that remain. The only subcase compatible with these types is Subcase 3.2; in the other subcases, v is not deficient and we have $\epsilon(u) \leq 0$. This will simplify the discussion in the present case.

We begin by considering the event ABD^-. By the assumptions, it is valid. Since neither (u_{-2}, v_-) nor its reverse is a sensitive pair, the event is regular. Thus, $\mathbf{P}(ABD^-) \geq 0.5/256$. By symmetry, we have $\mathbf{P}(BAD^+) \geq 0.5/256$.

We distinguish several subcases, in each of which we try to accumulate a further $(1 + \epsilon(u))/256$ worth of probability.

Subcase 3.1. *None of uZv and vZu is short.*

Consider the event ABD^0. By the assumptions, it is valid and covered by (v_+, u_{-2}). Since vZu is not short and the diagram of ABD^0 contains only one outgoing arc (namely

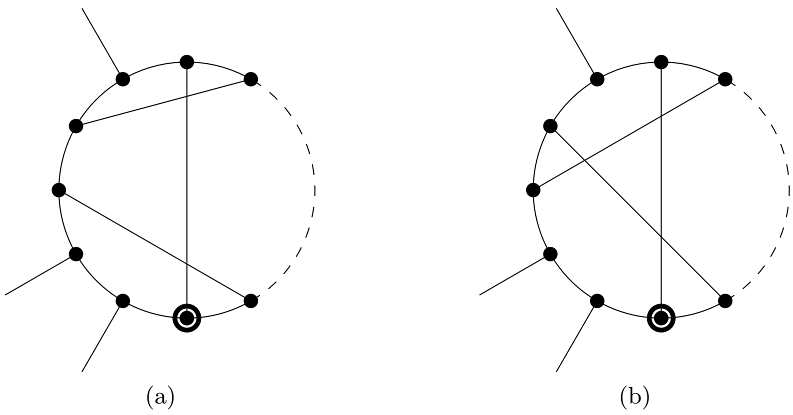


Figure A.15: Two cases where the event ABD^+ cannot be used in Subcase 3.1 of the proof of Proposition A.5.2.

$u_+u'_+$), the pair is 1-free. By Lemma 4.3.2, $\mathbf{P}(ABD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$. By symmetry, $\mathbf{P}(BAD^0) \geq 0.25/256$.

The argument for ABD^0 also applies to the event ABD^+ (whose diagram has two outgoing arcs), unless the vertex set of the path vZu is $\{v, v_+, u'_+, v'_-, u_-, u\}$ (in which case we get the two possibilities in Figure A.15). If this does not happen, then we obtain a contribution of at least 0.25/256 again.

Let us examine the exceptional case in Figure A.15(a) (i.e., $u'_+ = u_{-3}$ and $v'_- = v_{+2}$). The event $C_1C_1D^+$ is covered by $(u'_-, u'_-)^4$. By Lemma 4.3.2, $\mathbf{P}(C_1C_1D^+) \geq 79/80 \cdot 1/256 > 0.98/256$.

Consider now the situation of Figure A.15(b). The event AAD^+ is valid, since $\{u_{-2}, u_{+2}, v_+\}$ is an independent set by assumption and it is regular. We infer that $\mathbf{P}(AAD^+) \geq 0.25/256$.

To summarize the above three paragraphs, we proved

$$\mathbf{P}(ABD^+ \cup C_1C_1D^+ \cup AAD^+) \geq 0.25/256.$$

By symmetry, in the above paragraphs, we proved that

$$\mathbf{P}(BAD^- \cup C_1C_1D^- \cup AAD^-) \geq 0.25/256.$$

Together with the events ABD^0 and BAD^0 considered earlier, this makes for a total contribution of at least 1/256. As noted at the beginning of Case 3, $\epsilon(u) \leq 0$ so this is sufficient. △

Subcase 3.2. *The path vZu is short but uZv is not.*

In this subcase, v may be deficient of type III*, in which case $\epsilon(u) = 0.125$; otherwise, $\epsilon(u) \leq 0$.

The event BAD^- is covered by the pair (u_{+2}, v_-), which is 1-free unless v'_+ and u'_- are the only internal vertices of the path $u_{+2}Zv_-$. However, this situation would be inconsistent with our choice of F since $\partial(Z)$ would have size 4. (Recall that $\partial(Z)$ is the set of edges of G with one end in $V(Z)$.) Consequently, $\mathbf{P}(BAD^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Moreover, if u'_+ (which is a tail in BAD^-) is contained in $u_{+2}Zv_-$, then $\mathbf{P}(BAD^-) \geq 0.5/256$.

The same discussion applies to the event BAD^0. In particular, if $u'_+ \in V(u_{+2}Zv_-)$, then the probability of the union of these two types is at least $1/256$. This is a sufficient amount unless v is deficient of type III*, in which case a further $0.25/256$ is obtained from the regular event AAD^-.

We may, thus, assume that $u'_+ \notin V(u_{+2}Zv_-)$ (so v is not deficient). The event AC_1D^- is then covered by $(u'_+, u'_+)^3$ (we are taking into account the arc incident with v_+) and hence $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$ by Lemma 4.3.2.

The event AC_2D^- is covered by the pair $(u_{+2}, v_-)^1$ and has probability at least $1/2 \cdot 0.0625/256 > 0.03/256$. We claim that $\mathbf{P}(BAD^- \cup BAD^0 \cup AAD^-) \geq 0.75/256$. Since the total amount will exceed $1/256$, this will complete the present subcase:

Suppose first that $v'_+ \in V(uZv)$. Then the event BAD^0 is regular and $\mathbf{P}(BAD^0) \geq 0.5/256$. In addition, BAD^- has only one sensitive pair (u_{+2}, v_-). This pair is 1-free since, otherwise, v'_+ and u'_- would be the only internal vertices of the path $u_{+2}Zv_-$ and Z would be incident with exactly four non-chord edges of M, a contradiction with the choice of F. Thus, $\mathbf{P}(BAD^-) \geq 0.25/256$ and the claim is proved.

Let us therefore assume that $v'_+ \notin V(uZv)$. We again distinguish two possibilities according to whether u'_- is contained in uZv or not. If $u'_- \in V(uZv)$, then $\mathbf{P}(BAD^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$ as BAD^- is covered by $(u_{+2}, v_-)^1$. Similarly, $\mathbf{P}(BAD^0) \geq$

0.25/256. The event AAD^- is regular of weight 10, whence $\mathbf{P}(AAD^-) \geq 0.25/256$. The total probability of these three events is at least 0.75/256, as claimed.

To complete the proof of the claim, we may assume that $u'_- \notin V(uZv)$. The only possibly sensitive pair of BAD^- and BAD^0 is now 2-free, implying a probability bound of $3/4 \cdot 0.5/256$ for each event. Thus, $\mathbf{P}(BAD^- \cup BAD^0) \geq 0.75/256$, finishing the proof of the claim and the whole subcase. △

Subcase 3.3. *Both vZu and uZv are short.*

In this subcase, Z is an 8-cycle; by our assumptions, it has only one chord uv. Recall also that in this subcase, $\epsilon(u) \leq 0$.

Consider the event AC_1D^-. Since it is covered by $(u'_+, u'_+)^3$, we have $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$ by Lemma 4.3.2. By symmetry, $\mathbf{P}(C_1AD^+) \geq 0.24/256$ so the total probability so far is 0.48/256.

Suppose now that the vertices u'_+ and u'_- are located on different cycles of F. By Lemma 4.3.2, $\mathbf{P}(C_1C_1D^0) \geq 39/40 \cdot 0.5/256 > 0.48/256$. Similarly, $\mathbf{P}(C_1C_1D^+) \geq 77/80 \cdot 0.5/256 > 0.48/256$, which makes for a sufficient contribution.

We may, thus, assume that u'_+ and u'_- are on the same cycle, say Z', of F. Suppose that they are non-adjacent, in which case $C_1C_1D^0$ is covered by $(u'_+, u'_-)^2$ and $(u'_-, u'_+)^2$, and its probability is at least $1/2 \cdot 0.5/256 = 0.25/256$. If neither v'_- nor v'_+ are on Z', then the same computation applies to $C_1C_1D^+$ and $C_1C_1D^-$ so the total probability accumulated so far is $(0.48 + 0.25 + 0.25 + 0.25)/256 > 1/256$ by Lemma 4.3.2. We may, thus, assume, without loss of generality, that $v'_+ \in V(u'_+Z'u'_-)$. Under this assumption, $C_1C_1D^-$ is covered by $(u'_-, u'_+)^1$ and, thus, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. At the same time, $\mathbf{P}(C_1C_1D^+)$ is similarly seen to be at least 0.125/256, which makes the total probability at least $(0.48 + 0.25 + 0.25 + 0.125)/256 > 1/256$.

It remains to consider the possibility that u'_+ and u'_- are adjacent. In this case, $\mathbf{P}(AC_1D^- \cup C_1AD^+) \geq 0.5/256$, so we need to find additional 0.5/256. The event $C_1C_2D^+$ has a template covered by $(u'_-, u'_-)^2$, and hence its probability is at least $19/20 \cdot 0.25/256 > 0.23/256$. Similarly, $\mathbf{P}(C_2C_1D^-) \geq 0.23/256$. The same argument applies to the events $C_1C_3D^+$ and $C_3C_1D^-$, resulting in a total probability of $(0.5 + 4 \cdot 0.23)/256 > 1/256$. This finishes Case 3. △

▲

Case 4. $E(M[U]) = \{u_{-2}v_+\}$.

In this case, two significant contributions are from the regular events ABD^- and BAD^+:

$$\mathbf{P}(ABD^-) \geq \frac{1}{256},$$
$$\mathbf{P}(BAD^+) \geq \frac{0.5}{256}.$$

We distinguish several subcases; in each of them, we try to accumulate a contribution of $(0.5 + \epsilon(u))/256$ from other events. In particular, if u is deficient of type I, IIa or IIa* (and $\epsilon(u) = -0.5$), we are done.

Let us consider the vertex v. We claim that, if v is deficient, then it must be of type II* or IIa*. Indeed, the assumption that u_-uu_+ is not contained in a 4-cycle excludes types I, Ia, Ib and their mirror variants. An inspection of the type definitions shows that, if v is of type II or IIa, then M includes the edge $u_{+2}v_-$, which we assume not to be the case. Finally, if v is of type III or III*, then $u_{-2}v_+$ is not an edge of M, another contradiction with our assumption.

The only types that remain for v are II* and IIa*. Observe that, if v is of one of these types, then uZv is short.

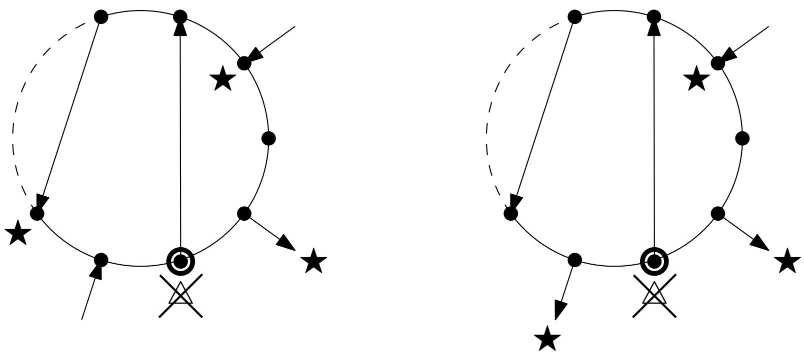
Subcase 4.1. *The path uZv is not short.*

By the above discussion, v is not deficient of either type, whence $\epsilon(u) \leq 0$. The event BAD^- is covered by $(u_{+2}, v_-)^1$ (consider the outgoing arc incident with u_-). It follows that $\mathbf{P}(BAD^-) \geq 0.25/256$. The same argument applies to BAD^0, and thus

$$\mathbf{P}(ABD^- \cup BAD^+ \cup BAD^- \cup BAD^0) \geq \frac{1 + 0.5 + 0.25 + 0.25}{256} = \frac{2}{256}.$$

△

We have observed that, if v is deficient, then it must be of type II* or IIa*. Since this requires that the F-neighbours of u'_- are v_+ and v'_-, it can only happen in the following subcase.



(a) (b)

Figure A.16: Two events used in Subcase 4.2 of the proof of Proposition A.5.2: (a) AC_1D^-, (b) $C_1C_1D^-$.

Subcase 4.2. *The vertices u'_+ and u'_- are non-adjacent.*

Consider the events AC_1D^- and $C_1C_1D^-$ (Figure A.16). If the event AC_1D^- has a sensitive pair, it is either (u'_+, u'_+) or (u'_+, u_{-2}).

Suppose first that u'_+ is distinct from u_{-3}. In this case, Lemma 4.3.2 implies that $\mathbf{P}(AC_1D^-) \geq 3/4 \cdot 0.5/256$ no matter whether $u'_+ \in V(Z)$ or not. Secondly, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$ (by Lemma 4.3.2 again) so the total contribution is at least $0.625/256$, which is sufficient if v is either not deficient or is deficient of type II*. It remains to consider the possibility that v is deficient of type IIa*. In this case, AC_1D^- is covered by $(u'_+, u'_+)^4$; by Lemma 4.3.2, $\mathbf{P}(AC_1D^-) \geq 79/80 \cdot 0.5/256 > 0.49/256$. Similarly, we obtain $\mathbf{P}(C_1C_1D^-) > 0.49/256$ and $\mathbf{P}(C_2C_1D^-) \geq 0.24/256$. The total contribution is $1.22/256 > (0.5 + \epsilon(u))/256$.

We may thus suppose that $u'_+ = u_{-3}$; since this is incompatible with v being of type II* as well as IIa*, we find that v is not deficient and $\epsilon(u) \leq 0$. We have $\mathbf{P}(C_1C_1D^-) \geq 3/4 \cdot 0.5/256$ (whether u'_- is contained in vZu or outside Z) since the event $C_1C_1D^-$ is covered by a single 2-free pair (either (u'_-, u'_-) or (u'_-, u_{-3})) and the weight of the event is 9. It remains to find a further contribution of $0.125 + \epsilon(u)$ to reach the target amount. In particular, we may assume that u is not deficient of type II.

If $u'_- \neq v_{+2}$, the event $C_1C_1D^0$ is covered by (v_+, u'_-) and (u'_-, u_{-3}). Using Lemma 4.3.2, we find that $\mathbf{P}(C_1C_1D^0) \geq 1/2 \cdot 0.5/256$, which is sufficient.

Thus, the present subcase boils down to the situation where u'_- is adjacent to v_+ (that is, $u'_- = v_{+2}$) and $u'_+ = u_{-3}$. Since u is not deficient of type II, it must be that v_-v_{+3} is an edge of M. In this case, the only events of non-zero probability in Σ are the events ABD^-, BAD^+ and $C_1C_1D^-$ considered above. Fortunately, the condition that $v_-v_{+3} \in E(M)$ increases the probability bound for $C_1C_1D^+$ from $3/4 \cdot 0.5/256$ to $0.5/256$, as required. △

As all the subcases where v is deficient have been covered in Subcase 4.2, we may henceforth assume that $\epsilon(u) \leq 0$ and seek find a further contribution of $(0.5 + \epsilon(u))/256$.

Subcase 4.3. *The vertices u'_+ and u'_- are adjacent, uZv is short and $u'_+ \neq u_{-3}$.*

Suppose first that u'_- (and u'_+) is contained in Z. The event $C_2C_1D^-$ is then covered by the 1-free pair (u'_-, u_{-2}) or (u'_+, u_{-2}). Since its weight is 9, we have $\mathbf{P}(C_2C_1D^-) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Note that the event $C_3C_1D^-$ is valid; it is also regular so $\mathbf{P}(C_3C_1D^-) \geq 0.25/256$. Together, this yields $0.5/256$, which is sufficient.

We may therefore assume that u'_- (and u'_+) are not contained in Z. The event $C_2C_1D^-$ is covered by (u'_+, u'_-) or its reverse, each of which is 3-free. By Lemma 4.3.2, $\mathbf{P}(C_2C_1D^-) \geq 39/40 \cdot 0.5/256 > 0.48/256$. The event $C_3C_1D^-$, if irregular, has the same sensitive pair and it is now 2-free. Since the weight of its diagram is 10, $\mathbf{P}(C_3C_1D^-) \geq 19/20 \cdot 0.25/256 > 0.23/256$. The total contribution exceeds the desired $0.5/256$. △

Subcase 4.4. *The vertices u'_+ and u'_- are adjacent, uZv is short and $u'_+ = u_{-3}$.*

Suppose first that the path v_+Zu_{-4} contains at least two vertices distinct from v'_-. Then the event C_1AD^+ (see Figure A.17) is covered by $(v_+, u_{-4})^2$. Since the weight of C_1AD^+ is 10, we have $\mathbf{P}(C_1AD^+) \geq 3/4 \cdot 0.25/256$. The events $C_1C_2D^+$ and $C_1C_3D^+$ have weight 11 but the diagram of each of them has a removable symbol at u_{-3} so we get the same bound of $3/4 \cdot 0.25/256$ for each of $C_1C_2D^+$ and $C_1C_3D^+$, since each of the diagrams is covered by one 2-free pair. The total contribution is at least $0.56/256$.

If v'_- is the only internal vertex of v_+Zu_{-4}, then the above events are in fact regular and we obtain an even higher contribution. Thus, we may assume that either v_+ and u_{-4} are neighbours on Z, or v_+Zu_{-4} contains two internal vertices and one of them is v'_-.

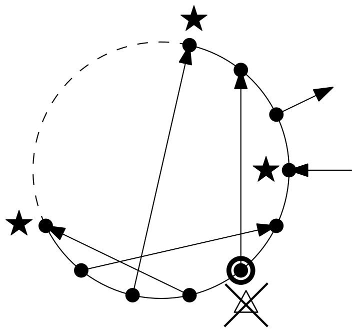


Figure A.17: The event C_1AD^+ used in Subcase 4.4 of the proof of Proposition A.5.2.

The former case is ruled out since we are assuming (from the beginning of Case 4) that u is not deficient of type IIa. It remains to consider the latter possibility. Here, v'_- is either v_{+2} or v_{+3}. In fact, it must be v_{+3}, since otherwise u would be deficient of type I, which has also been excluded at the beginning of Case 4. But, then, u is deficient of type III, so $\epsilon(u) = -0.125$. At the same time, the unique sensitive pair for each of the events C_1AD^+, $C_1C_2D^+$ and $C_1C_3D^+$, considered above, is now 1-free; the probability of the union of these events is, thus, at least $3 \cdot 1/2 \cdot 0.25/256 = 0.375/256 = (0.5 + \epsilon(u))/256$ as necessary. △

▲

Case 5. $E(M[U]) = \{u_{-2}v_+, u_{+2}v_-\}$.

As in Case 4, the probability of the event ABD^- is at least 1/256; by symmetry, $\mathbf{P}(BAD^+) \geq 1/256$. We claim that the resulting contribution of 2/256 is sufficient because $\epsilon(u) \leq 0$. Clearly, v is not of type 0. Applying the definitions of the remaining types to v, we find that none of them is compatible with the presence of the edges $u_{-2}v_+$ and $u_{+2}v_-$ in M. This shows that $\epsilon(u) \leq 0$. ▲

Case 6. $E(M[U]) = \{u_{-2}u_{+2}\}$.

Recall our assumption that the set $J = \{u_-, u_+, v_-, v_+\}$ is independent. If we suppose that, moreover, both the paths uZv and vZu were short, then the mate of each vertex in J must be outside Z. This means that $|\partial(Z)| = 4$, a contradiction with F satisfying the condition in Theorem 4.2.1. Thus, we may assume, by symmetry, that the path vZu is not short.

The event ABD^- is regular of weight 9 so $\mathbf{P}(ABD^-) \geq 0.5/256$. Similarly, $\mathbf{P}(BAD^+) \geq 0.5/256$. We need to find additional $(1+\epsilon(u))/256$ to add to the probabilities of ABD^- and BAD^+ above. Note also that, if v is deficient, then it must be of type III* and this only happens in Subcase 6.3.

Subcase 6.1. *uZv is not short.*

Assume that u'_+ is not contained in vZu and consider the events ABD^+ and ABD^0. If v'_- is not contained in vZu, then ABD^+ is covered by the pair $(v_+, u_{-2})^2$, and it follows that $\mathbf{P}(ABD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$. Similarly, $\mathbf{P}(ABD^0) \geq 0.375/256$. On the other hand, if v'_- is contained in vZu, then the pair (v_+, u_{-2}) may only be 1-free for ABD^+, whence $\mathbf{P}(ABD^+) \geq 1/2 \cdot 0.5/256 = 0.25/256$, but this decrease is compensated for by the fact that $\mathbf{P}(ABD^0) \geq 0.5/256$ as ABD^0 is now regular. Summarizing, if u'_+ is not contained in vZu, then the probability of $ABD^+ \cup ABD^0$ is at least $0.75/256$.

The event BAD^0 of weight 9 is covered by the pair (u_{+2}, v), which is 1-free since uZv is not short. Hence, $\mathbf{P}(BAD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$. Putting this together, for $u'_+ \notin V(vZu)$, we have

$$\mathbf{P}(ABD^- \cup BAD^+ \cup ABD^+ \cup ABD^0 \cup BAD^0) \geq \frac{0.5 + 0.5 + 0.375 + 0.375 + 0.25}{256} = \frac{2}{256}.$$

Since this is the required amount, we may assume by symmetry that $u'_+ \in V(vZu)$ and $u'_- \in V(uZv)$ (Figure A.18).

If v'_+ is not contained in uZv then in addition to $\mathbf{P}(BAD^0) \geq 0.25/256$ as noted above, we have $\mathbf{P}(BAD^-) \geq 0.25/256$ for the same reasons. On the other hand, $v'_+ \in V(uZv)$ increases the probability bound for BAD^0 to $\mathbf{P}(BAD^0) \geq 0.5/256$ as the event is regular in this case. All in all, the contribution of $BAD^- \cup BAD^0$ is at least $0.5/256$.

By symmetry, $ABD^+ \cup ABD^0$ also contributes at least $0.5/256$. Together with the events ABD^- and BAD^+, which each have a probability of at least $0.5/256$ as discussed above, we have found the required $2/256$. △

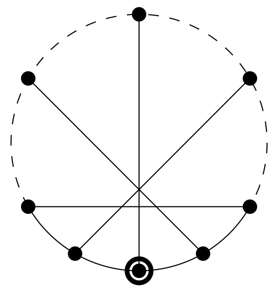


Figure A.18: A configuration in Subcase 6.1 of the proof of Proposition A.5.2.

Thus, the path uZv may be assumed to be short.

Subcase 6.2. $u'_+ \notin V(Z)$.

As in the previous subcase, $\mathbf{P}(ABD^+ \cup ABD^0) \geq 0.75/256$.

The event AC_1D^- has weight 10 (see Figure A.19). If the cycle of F containing u'_+ is odd, it contains at least 3 vertices different from u'_+ and v'_+. Thus, AC_1D^- is covered by $(u'_+, u'_+)^3$. By Lemma 4.3.2, $\mathbf{P}(AC_1D^-) \geq 39/40 \cdot 0.25/256 > 0.24/256$.

Similarly, AC_1D^0 has a diagram of weight 10 and is covered by $(u'_+, u'_+)^4$ and $(v_-, u_{-2})^2$. By Lemma 4.3.2, $\mathbf{P}(AC_1D^0) \geq 59/80 \cdot 0.25/256 > 0.18/256$. The probability of $AC_1D^- \cup AC_1D^0$ is, thus, at least $(0.24 + 0.18)/256 = 0.42/256$, more than the missing $0.25/256$.

△

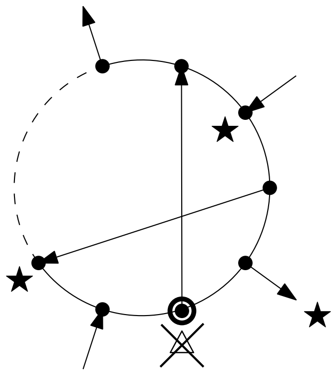


Figure A.19: The event AC_1D^- used in Subcase 6.2 of the proof of Proposition A.5.2.

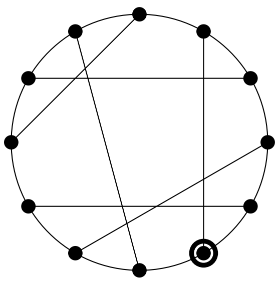


Figure A.20: The situation where v is deficient of type III* in Subcase 6.3 of the proof of Proposition A.5.2.

Subcase 6.3. *$u'_+ \in V(Z)$ and the length of vZu is at least 7.*

We will show that the assumption about vZu increases the contribution of $ABD^+ \cup ABD^0$. Suppose that $v'_- \in V(Z)$. Then, ABD^+ is covered by $(v_+, u_{-2})^2$ and ABD^0 is regular so $\mathbf{P}(ABD^+ \cup ABD^0) \geq (3/4 + 1) \cdot 0.5/256 = 0.875/256$. On the other hand, if $v'_- \notin V(Z)$, then the pair (v_+, u_{-2}) is 3-free for both ABD^+ and ABD^0, and we get the same result:

$$\mathbf{P}(ABD^+ \cup ABD^0) \geq 2 \cdot \frac{7}{8} \cdot \frac{0.5}{256} = 0.875/256.$$

We need to find the additional $(0.125 + \epsilon(u))/256$.

Suppose, first, that v is deficient (necessarily of type III*) so $\epsilon(u) = 0.125$. The induced subgraph of G on $V(Z)$ is then as shown in Figure A.20; in this case, the event $C_1C_1D^-$ is regular and $\mathbf{P}(C_1C_1D^-) \geq 0.5/256$, a sufficient amount.

We may, thus, assume that $\epsilon(u) \leq 0$. Suppose that u'_+ is not adjacent to either u_{-2} or v_+. Then, the event AC_1D^0 is covered by $(v_+, u'_+)^2$ and $(u'_+, u_{-2})^2$. By Lemma 4.3.2, $\mathbf{P}(AC_1D^0) \geq 1/2 \cdot 0.25/256 = 0.125/256$ as required.

The vertex u'_+ can therefore be assumed to be adjacent to u_{-2} or v_+. The event $C_1C_1D^-$ has only one sensitive pair, namely (u'_-, u'_+) or its reverse (if $u'_- \in V(Z)$) or (u'_-, u'_-) (if u'_- is outside Z). If this is a 1-free pair, then, by Lemma 4.3.2, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256 > 0.125/256$, as required. In the opposite case, it must be that u'_- is a neighbour of u'_+. Then, however, we observe that $\mathbf{P}(ABD^+)$ and $\mathbf{P}(ABD^0)$ are both at least 0.5/256 (as the events are regular) and this increase provides the missing 0.125/256.

△

Thus, we may assume that the length of vZu is 5 or 6. If the length of vZu is 5, then

u_-uu_+ belongs to a cycle of length 4, which has already been considered in Case 2. Thus, to complete the discussion of Case 6, it remains to consider the following subcase.

Subcase 6.4. *uZv is short, $u'_+ \in V(Z)$, and the length of vZu is 6.*

The vertex u'_+ equals either v_{+2} or v_{+3}. Suppose, first, that $u'_+ = v_{+2}$. Then, each edge in $\partial(Z)$ is incident with a vertex in $\{v_{+3}, u_-, v_-, v_+\}$. By the choice of F, M must contain an edge with both ends in the latter set. For trivial reasons, the only candidate is $v_{+3}v_-$ (Figure A.21(a)). However, this is also not an edge of M since the 5-cycles $u_{-2}Zu_{+2}$ and v_-Zv_{+3} would contradict Lemma A.3.1(ii). (See Figure A.21(b) for illustration.)

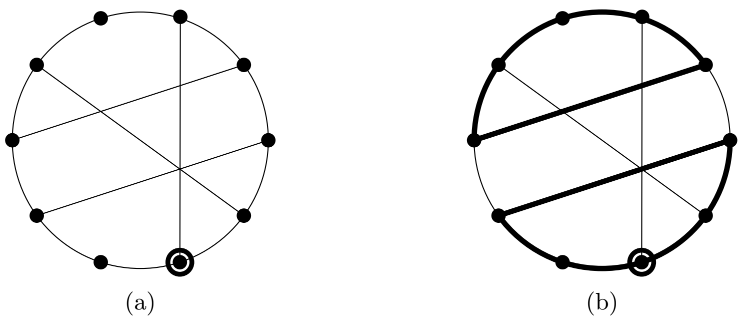


Figure A.21: The use of Lemma A.3.1 in Subcase 6.4 of the proof of Proposition A.5.2. (a) The cycle Z and its chords assuming that $u'_+ = v_{+2}$. (b) The two 5-cycles (bold) contradicting Lemma A.3.1(ii).

Thus, $u'_+ = v_{+3}$. Here, each edge of $\partial(Z)$ is incident with a vertex in $\{u_-, v_-, v_+, v_{+2}\}$, and it is easy to see that one of these edges must be incident with v_+. There are two possibilities for an edge with both ends in $\{u_-, v_-, v_+, v_{+2}\}$, namely v_-v_{+2} or u_-v_{+2}. In either case, the event ABD^0 is easily seen to be regular and, thus, $\mathbf{P}(ABD^0) \geq 0.5/256$. In fact, this concludes the discussion if $v_-v_{+2} \in E(M)$, since then u is deficient of type I and $\epsilon(u) = -0.5$, thus, the contribution of 0.5/256 is sufficient.

In the remaining case that $u_-v_{+2} \in E(M)$, we need a further 0.5/256, which is provided by the regular event ABD^+. △

▲

Case 7. $E(M[U]) = \{u_{-2}v_-\}$.

If both the paths uZv and vZu are short, then each edge of $\partial(Z)$ is incident with a vertex in $\{u_-, u_+, u_{+2}, v_+\}$. Our assumptions imply that no edge of M joins two of these vertices, so $|\partial(Z)| = 4$, contradicting our choice of F. We may therefore assume that at least one of vZu and uZv is not short.

In all the subcases, we can use the regular event BAD^+, for which we have $\mathbf{P}(BAD^+) \geq 0.5/256$. Hence, we need to find an additional probability of $(1.5 + \epsilon(u))/256$.

Subcase 7.1. *vZu is short.*

In this subcase, the path v_-vv_+ is contained in a 4-cycle and it is not hard to see that u must be deficient of type I (neither uv nor u_-uu_+ is contained in a 4-cycle, and the missing edge $u_{+2}v_+$ rules out cases Ia* and Ib*). Thus, $\epsilon(u) = -0.5$ and we need to find further 1/256 worth of probability.

Observe first that, by our assumptions, the set $\{u_-, u_+, u_{+2}, v_+\}$ is independent. We will distinguish several cases based on whether u'_-, u'_+ and v'_+ are contained in Z (and hence in $u_{+3}Zv_{-2}$) or not.

If $u'_+ \in V(Z)$, then the events BAD^0 and BAD^- are regular, and each of them has probability 0.5/256, which provides the necessary 1/256.

Suppose thus that $u'_+ \notin V(Z)$ and consider, first, the case that $u'_- \notin V(Z)$. The event C_1AD^+ is covered by the pair $(u'_-, u'_-)^4$, so, by Lemma 4.3.2, its probability is $\mathbf{P}(C_1AD^+) \geq 79/80 \cdot 0.25/256 > 0.24/256$. The event C_1AD^0 has up to two sensitive pairs: it is covered by $(u'_-, u'_-)^4$ and $(u_{+2}, v_-)^2$, where the latter pair is 2-free because uZv is not short. We obtain $\mathbf{P}(C_1AD^0) \geq 59/80 \cdot 0.25/256 > 0.18/256$.

To find the remaining 0.58/256 (still for $u'_- \notin V(Z)$), we use the events BAD^0 and BAD^-. We claim that their probabilities add up to at least 0.75/256. Indeed, if $v'_+ \notin V(Z)$, then both BAD^0 and BAD^- are covered by the pair $(u_{+2}, v_-)^2$ (which is 2-free because uZv is not short and $u'_- \notin V(Z)$). By Lemma 4.3.2, they have probability at least 0.375/256 each. On the other hand, if $v'_+ \in V(Z)$, then BAD^0 is regular and BAD^- is covered by $(u_{+2}, v_-)^1$ so $\mathbf{P}(BAD^0) \geq 0.5/256$ and $\mathbf{P}(BAD^-) \geq 0.25/256$. For both of the possibilities, $\mathbf{P}(BAD^0 \cup BAD^-) \geq 0.75/256$ as claimed.

We can, therefore, assume that $u'_- \in V(Z)$ (and $u'_+ \notin V(Z)$, of course). A large part of the required 1/256 is provided by the event $C_1C_1D^+$, which is covered by the pair $(u'_+, u'_+)^4$, so $\mathbf{P}(C_1C_1D^+) \geq 79/80 \cdot 0.5/256 > 0.49/256$.

A final case distinction will be based on the location of v'_+. Suppose, first, that $v'_+ \notin V(Z)$. We claim that the length of uZv is at least 7. If not, then, since uZv is not short, the length of Z is 9 or 10. At the same time, Z has at least 3 chords (incident with u, u_- and u_{-2}) and, therefore, $|\partial(Z)| \leq 4$. By the choice of F and the assumption that the mates of u_+ and v_+ are outside Z, Z has length 10 and $\partial(Z)$ is of size 2. In addition, u_{+2} is incident with a chord of Z whose other endvertex w is contained in $u_{+3}Zv_{-2}$. However, $|uZv| = 6$ implies that $w \in \{u_{+3}, u_{+4}\}$, contradicting the assumption that G is simple and triangle-free. We conclude that $|uZv| \geq 7$, as claimed.

This observation implies that, for the event BAD^0, the only possibly sensitive pair, namely (u_{+2}, v_-), is 2-free. Hence, $\mathbf{P}(BAD^0) \geq 3/4 \cdot 0.5/256 = 0.375/256$. Hence, $\mathbf{P}(BAD^-) \geq 0.375/256$ and this amount is sufficient.

It remains to consider the case that $v'_+ \in V(Z)$. Being regular, the event BAD^0 has probability at least 0.5/256. Thus, it is sufficient to find further events forcing u of total probability at least 0.01/256. It is easiest to consider the mutual position of u'_- and v'_+ on $u_{+3}Zv_{-2}$. If $u'_- \in V(v'_+Zv_{-2})$, then the event C_1AD^+ is regular and has probability at least 0.25/256. In the opposite case, $C_1C_1D^0$ is covered by the pair $(u'_+, u'_+)^4$, which means that $\mathbf{P}(C_1C_1D^0) \geq 79/80 \cdot 0.5/256 > 0.49/256$. In both cases, the probability is sufficiently high. △

Having dealt with Subcase 7.1, we can use the event AAD^+, which is covered by $(v_+, u_{-2})^2$. By Lemma 4.3.2, $\mathbf{P}(AAD^+) \geq 3/4 \cdot 0.5/256 = 0.375/256$ and, hence, $\mathbf{P}(BAD^+ \cup AAD^+) \geq 0.875/256$. Since v is not deficient, we seek a further contribution of at least 1.125/256.

Subcase 7.2. *Neither vZu nor uZv is short.*

Consider the event ABD^+ of weight 8 (Figure A.22) which is covered by the pair (v_+, u_{-2}). Since vZu is not short, the vertices in the pair are not neighbours. Furthermore, if the pair is sensitive, then the path v_+Zu_{-2} contains at least two internal vertices, one of which is different from u'_+. Thus, the pair is 1-free and, by Lemma 4.3.2,

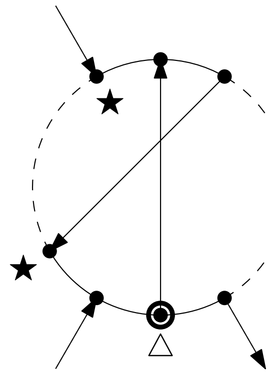


Figure A.22: The event ABD^+ used in Subcase 7.2 of the proof of Proposition A.5.2.

$\mathbf{P}(ABD^+) \geq 1/2 \cdot 1/256$. If the pair (v_+, u_{-2}) is actually 2-free in ABD^+, then the estimate increases to $3/4 \cdot 1/256$.

The event BAD^0 is covered by the pair (u_{+2}, v_-), which is 1-free as uZv is not short; moreover, if $u'_- \notin V(uZv)$, then the pair is 2-free. Thus, $\mathbf{P}(BAD^0) \geq 1/2 \cdot 0.5/256 = 0.25/256$ or $3/4 \cdot 0.5/256 = 0.375/256$ in the respective cases.

If the higher estimates hold for both the events ABD^+ and BAD^0 considered above, then the contributions of these events total

$$\frac{0.75 + 0.375}{256} = \frac{1.125}{256},$$

which is sufficient.

Suppose, first, that we get the higher estimate for $\mathbf{P}(ABD^+)$, that is, that (v_+, u_{-2}) is 2-free in ABD^+. By the above, it may be assumed that $u'_- \in V(uZv)$ and the pair (u_{+2}, v_-) is not 2-free in BAD^0. We need to find an additional $0.125/256$. To this end, we use the event BAD^- of weight 9. The probability of BAD^- is at least $1/2 \cdot 0.5/256$ (which is sufficient) if (u_{+2}, v_-) is 1-free in BAD^-. This could be false only if $\left\{u'_-, v'_+\right\} = \{u_{+3}, v_{-2}\}$; for each of the corresponding two possibilities, the event BAD^0 is a regular one, contradicting the assumption that (u_{+2}, v_-) is not 2-free in BAD^0.

It remains to discuss the possibility that (v_+, u_{-2}) is not 2-free in ABD^+. In that case, the length of vZu is 6 and $u'_+ \in \{v_{+2}, v_{+3}\}$. Since the lower bound to $\mathbf{P}(AAD^+)$ increases to $0.5/256$ in this case, the total probability of BAD^+, AAD^+ and ABD^+ is at least $1.5/256$. In addition, we have a contribution of $1/2 \cdot 0.5/256$ from BAD^0 so we need to add a further $0.25/256$.

Assume, first, that $u'_- \neq v_{-2}$ and consider $C_1C_1D^-$. We claim that $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$. This is certainly true if $u'_- \notin V(Z)$ since $C_1C_1D^-$ has weight 9 and it is covered by $(u'_-, u'_-)^4$. Suppose, thus, that $u'_- \in V(Z)$. There is at most one sensitive pair for $C_1C_1D^-$ ((u'_-, u'_+) or (u'_-, v_-) or none). If the event is regular or the sensitive pair is 1-free, then $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$, as required. Otherwise, since there is only one outgoing arc in the diagram for $C_1C_1D^-$, u'_- must be adjacent to v_- or u'_+. The former case is ruled out by the assumption $u'_- \neq v_{-2}$. In the latter case, the 5-cycle $uvZv_{+2}u_-$ and the cycle $u'_+u_+Zv_-u_{-2}$ provide a contradiction with Lemma A.3.1(ii).

We may, therefore, assume that $u'_- = v_{-2}$. Consider the cycles v_-Zu_{-2} and u_-Zv_{-2}. Since each of the edges $v_{-2}v_-$, $u_{-2}u_-$, uv and $u_+u'_+$ has one endvertex in each of the cycles, Lemma A.3.1(i) implies that neither u_{-3} nor v_+ have their mate in u_-Zv_{-2}. We claim that $P(BAD^-) \geq 7/8 \cdot 0.5/256$. The event is covered by the pair (u_{+2}, v_-) so, by Lemma 4.3.2, it suffices to show that the pair is 3-free. If not, then $|u_{+2}Zv_-| = 3$ and u_{+3} is the only vertex of $u_{+2}Zv_-$ which is not a head of BAD^-. In that case, however, $\partial(Z)$ consists of the four edges of M incident with a vertex from $\{u_{+2}, u_{+3}, v_+, v_{+2}, v_{+3}\} - \{u'_+\}$, contradicting the choice of F. We conclude that $P(BAD^-) \geq 7/8 \cdot 0.5/256$, as claimed. Since this contribution exceeds the required 0.25/256, the discussion of Subcase 7.2 is complete. △

Subcase 7.3. *uZv is short and either the length of vZu is at least 7, or $u'_+ \notin V(vZu)$.*

The event ABD^+ is covered by $(v_+, u_{-2})^2$ by assumption. Thus, $\mathbf{P}(ABD^+) \geq 3/4 \cdot 1/256$. In view of the events BAD^+ (probability at least 0.5/256) and AAD^+ (probability at least $3/4 \cdot 0.5/256$), we need to collect further 0.375/256.

Suppose, first, that $u'_+ \notin V(vZu)$. The event AC_1D^+ of weight 9 is covered by $(u'_+, u'_+)^4$ and $(v_+, u_{-2})^2$. By Lemma 4.3.2, $\mathbf{P}(AC_1D^+) \geq 59/80 \cdot 0.5/256 > 0.36/256$. The event AC_2D^+ of weight 11 is covered by $(v_+, u_{-2})^2$; thus, $\mathbf{P}(AC_2D^+) \geq 3/4 \cdot 0.125/256$, which together with $\mathbf{P}(AC_1D^+)$ yields more than the required 0.375/256.

We may, therefore, assume that $u'_+ \in V(vZu)$, which increases $\mathbf{P}(AAD^+)$ to at least 0.5/256 (so the missing probability is now 0.25/256).

Suppose that u'_- and u'_+ are non-adjacent. If $u'_- \notin V(Z)$, then $C_1C_1D^-$ is covered by $(u'_-, u'_-)^3$. Otherwise, it is covered by $(u'_-, u'_+)^1$ (we have to consider v'_+ here). In either case, $\mathbf{P}(C_1C_1D^-) \geq 1/2 \cdot 0.5/256$, as required.

We may, thus, assume that u'_- and u'_+ are adjacent. The event AC_1D^+ has weight 9 and at most one possibly sensitive pair; this pair is (u'_+, u_{-2}) if $(u'_-)_+ = u'_+$, or (u'_+, v_+) otherwise. If the sensitive pair is 2-free, we are done since $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. In the opposite case, we get two possibilities.

The first possibility is that u'_+ is adjacent to u_{-2}, so $u'_+ = u_{-3}$. In this case, the 5-cycle $u_{-3}u_+Zv_-u_{-2}$ and the cycle $uvZu_{-4}u_-$ provide a contradiction with Lemma A.3.1(ii).

The second possibility is that u'_+ is adjacent to v_+, that is, $u'_+ = v_{+2}$. Here, the event AC_2D^+ is regular, and $\mathbf{P}(AC_2D^+) \geq 0.25/256$, as desired. △

Subcase 7.4. *uZv is short, the length of vZu is 6 and $u'_+ \in V(vZu)$.*

The vertex u'_+ equals either v_{+2} or v_{+3}. Each of the events ABD^+, BAD^+, AAD^+ (considered earlier) now have probability at least 0.5/256. We need to find an additional 0.5/256.

If $u'_+ = v_{+2}$, then each edge of $\partial(Z)$ is incident with a vertex from the set $\{u_-, u_{+2}, v_+, v_{+3}\}$. By the choice of F, some edge of M must join two of these vertices; our assumptions imply that the only candidate is the edge $u_{-3}u_{+2}$. The events AC_2D^+, C_2AD^+ and $C_2C_2D^+$ are regular with AC_2D^+ having a removable symbol and their probabilities are easily computed to be at least 0.25/256, 0.125/256 and 0.125/256, respectively. This adds up to the required 0.5/256.

On the other hand, if $u'_+ = v_{+3}$, then each edge of $\partial(Z)$ is incident with $\{u_-, u_{+2}, v_+, v_{+2}\}$. In two of the cases, there is a pair of 5-cycles which yields a contradiction with Lemma A.3.1(ii): if $u_{+2}v_{+2} \in E(M)$, then the cycles are $u_{-3}Zu_+$ and $u_{+2}Zv_{+2}$, while, if $u_-v_{+2} \in E(M)$, then the cycles are u_-uvZv_{+2} and $u_+2Zv_-u_{-2}u_{-3}$. All the other cases are ruled out by the assumptions (notably, the assumption that $u_{+2}v_+ \notin E(M)$). △

The only possibility in Case 7 not covered by the above subcases is that uZv is short, vZu has length 5 and $u'_+ \in V(vZu)$. This is, however, excluded by our choice of Z: the cycle Z of length 9 would have at least three chords, implying $|\partial(Z)| \in \{1, 3\}$, which is impossible. ▲

Case 8. $E(M[U]) = \{u_{-2}v_-, u_{+2}v_+\}$.

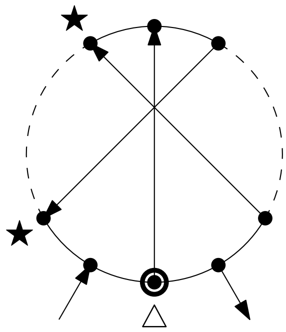


Figure A.23: The event ABD^+ used in Subcase 8.1 of the proof of Proposition A.5.2.

We will call a chord f of Z *bad* if $f \in \{u_-u_{+3}, u_+u_{-3}, u_-v_{-2}, u_+v_{+2}\}$.

Subcase 8.1. *Neither uZv nor vZu is short and Z has no bad chord.*

The event ABD^+ has one sensitive pair, namely (v_+, u_{-2}) (see Figure A.23). We claim that this pair is 2-free. Suppose not; then it must be that u'_+ is an internal vertex of v_+Zu_{-2} and there is exactly one other internal vertex in the path. This would mean that the edge of M incident with u_+ is a bad chord, contrary to the assumption. Hence, (v_+, u_{-2}) is 2-free in ABD^+ and $\mathbf{P}(ABD^+) \geq 3/4 \cdot 1/256$ as the weight of ABD^+ is 8.

For a similar reason (using the symmetry in the definition of a bad chord), $\mathbf{P}(BAD^-) \geq 3/4 \cdot 1/256$. Since v is not deficient in this subcase, it suffices to find a further $0.5/256$ to reach the desired bound.

Suppose, first, that u'_- and u'_+ are not neighbours.

If u'_- and u'_+ are contained in two distinct cycles of F, both different from Z, then, by Lemma 4.3.2, we have $\mathbf{P}(C_1C_1D^+) \geq 39/40 \cdot 0.5/256$ and the same estimate holds for $C_1C_1D^0$ and $C_1C_1D^-$. Thus,

$$\mathbf{P}(C_1C_1D^+ \cup C_1C_1D^0 \cup C_1C_1D^-) \geq \frac{1.46}{256},$$

much more than the required amount.

If u'_+ and u'_- are contained in the same cycle $Z' \neq Z$ of F, then the event $C_1C_1D^+$ is covered by $(u'_+, u'_-)^2$ and $(u'_-, u'_+)^2$. By Lemma 4.3.2, $\mathbf{P}(C_1C_1D^+) \geq 1/2 \cdot 0.5/256$. Since the same holds for $C_1C_1D^0$ and $C_1C_1D^-$, we find a sufficient contribution of $0.75/256$.

If, say, u'_+ is contained in Z and u'_- is not, then $C_1C_1D^+$ is covered by the pairs

$(v_+, u'_+)^2$ and $(u'_-, u'_-)^4$ (note that the first pair is 2-free since $u'_+ \neq v_{+2}$ by the absence of bad chords). Using Lemma 4.3.2, we find that $\mathbf{P}(C_1C_1D^+) \geq 59/80 \cdot 0.5/256 > 0.36/256$. Similarly, C_1AD^- is covered by $(u_{+2}, v_-)^2$ and $(u'_-, u'_-)^4$ so, by Lemma 4.3.2, $\mathbf{P}(C_1AD^-) \geq 59/80 \cdot 0.5/256 > 0.36/256$. Thus,

$$\mathbf{P}(C_1C_1D^+ \cup C_1AD^-) \geq \frac{0.36 + 0.36}{256} = \frac{0.72}{256}$$

and we are done.

Thus, still in the case that u'_+ and u'_- are not neighbours, we may assume that they are both contained in Z. Consider the event AC_1D^+. If $u'_- \in V(vZu)$, then the event is covered by a single 2-free pair, namely (v_+, u'_+) or (u'_+, u_{-2}), so $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. On the other hand, if $u'_- \in V(uZv)$, then AC_1D^+ is regular if $u'_+ \in V(uZv)$, or is covered by $(v_+, u'_+)^2$ and $(u'_+, u_{-2})^2$ otherwise. Summing up, $\mathbf{P}(AC_1D^+) \geq 1/2 \cdot 0.5/256$. Symmetrically, $\mathbf{P}(C_1AD^-) \geq 1/2 \cdot 0.5/256$ and we have found the necessary $0.5/256$.

We may, thus, assume that u'_- and u'_+ are neighbours.

If they are contained in a cycle of F different from Z, then the event C_1AD^- is covered by the 2-free pair (u_{+2}, v_-) so $\mathbf{P}(C_1AD^-) \geq 3/4 \cdot 0.5/256$. By symmetry, $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$, making for a sufficient contribution of $1.5/256$.

We may, thus, suppose that u'_- and u'_+ are both contained in vZu. By the absence of bad chords, u'_+ is not a neighbour of v_+ nor u_{-2}. Thus, the event AC_1D^+ is covered by a single 2-free pair, namely (v_+, u'_+) or (u'_+, u_{-2}), and $\mathbf{P}(AC_1D^+) \geq 3/4 \cdot 0.5/256$. Moreover, $\mathbf{P}(C_1AD^-) \geq 3/4 \cdot 0.5/256$ since the event is covered by $(u_{+2}, v_-)^1$ so

$$\mathbf{P}(AC_1D^+ \cup C_1AD^-) \geq \frac{0.375 + 0.375}{256} = \frac{0.75}{256},$$

as required. This finishes Subcase 8.1. △

Subcase 8.2. *Neither uZv nor vZu is short but Z has a bad chord.*

By symmetry, we may assume that at least one of u_+v_{+2}, u_+u_{-3} is a bad chord of Z (see Figure A.24).

Consider, first, the possibility, that $u_+v_{+2} \in E(M)$. By Lemma A.3.1(i), uv and $u_{-2}v_+$

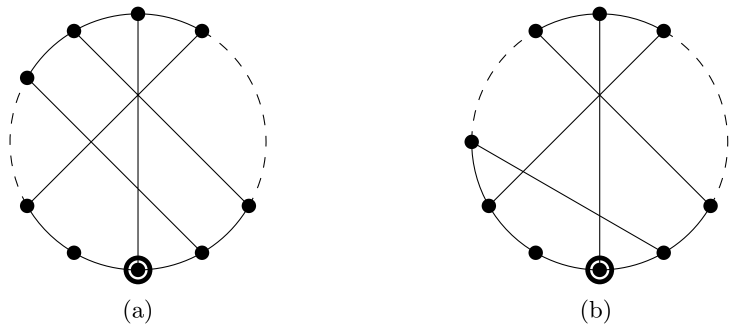


Figure A.24: The possibilities in Subcase 8.2 of the proof of Proposition A.5.2.

are the only two chords of Z with one endvertex in $v_{+3}Zu$ and the other one in $u_{+3}Zv$. In particular, $u'_{-} \notin V(uZv)$.

We will use the events BAD^{-}, ABD^{+}, $C_1C_1D^{-}$ and C_1AD^{-}. Let us estimate their probabilities. The event BAD^{-} of weight 8 is covered by the pair (u_{+2}, v_{-}), which is 2-free as uZv is not short. Thus, $\mathbf{P}(BAD^{-}) \geq 3/4 \cdot 1/256$ by Lemma 4.3.2. Similarly, ABD^{+} is covered by the pair $(v^{+}, u_{-2})^{1}$ and, therefore, $\mathbf{P}(ABD^{+}) \geq 1/2 \cdot 1/256$. The event $C_1C_1D^{-}$ of weight 9 is covered by $(u'_{-}, u'_{-})^{4}$, implying $\mathbf{P}(C_1C_1D^{-}) \geq 79/80 \cdot 0.5/256$. Finally, the event C_1AD^{-} of weight 9 is covered by the pairs $(u'_{-}, u'_{-})^{4}$ and $(u_{+2}, v_{-})^{2}$ (the latter of which is, again, 2-free since uZv is not short). By Lemma 4.3.2, $\mathbf{P}(C_1AD^{-}) \geq 59/80 \cdot 0.5/256$. Summarizing,

$$\mathbf{P}(BAD^{-} \cup ABD^{+} \cup C_1C_1D^{-} \cup C_1AD^{-}) > \frac{0.75 + 0.5 + 0.49 + 0.36}{256} = \frac{2.1}{256},$$

which is sufficient.

We may, therefore, assume that $u_{+}u_{-3}$ is a bad chord (Figure A.24(b)). The length of vZu is at least 6, as can be seen by considering the cycles $u_{-3}Zu_{+}$ and $u_{+2}Zv_{+}$ and using Lemma A.3.1(ii). Furthermore, Lemma A.3.1(i) implies that $u'_{-} \neq u_{-4}$ since, otherwise, the cycles $u_{+}Zv_{-}u_{-2}u_{-3}$ and $u_{-}uvZu_{-4}$ would provide a contradiction.

We distinguish three cases based on the position of u'_{-}. Assume, first, that u'_{-} is contained in vZu. The regular event ABD^{+} has probability at least 1/256. The event BAD^{-} is covered by the pair (u_{+2}, v_{-}), which is 2-free since uZv is not short. Thus, $\mathbf{P}(BAD^{-}) \geq 3/4 \cdot 1/256$. Finally, the event $C_1C_1D^{-}$ is covered by the pair (u'_{-}, u_{-3}), which is 2-free since $u'_{-} \neq u_{-4}$, as noted above. Consequently,

$$\mathbf{P}(ABD^{+} \cup BAD^{-} \cup C_1C_1D^{-}) \geq \frac{1}{256} + \frac{0.75}{256} + \frac{0.75 \cdot 0.5}{256} = \frac{2.125}{256},$$

more than the required 2/256.

Suppose, next, that u'_- is contained in uZv. Note that $u'_- \neq v_{-2}$ by Lemma A.3.1(i). Since the event ABD^+ is covered by $(v_+, u_{-2})^1$, $\mathbf{P}(ABD^+) \geq 1/2 \cdot 1/256$. Similarly, BAD^- is covered by $(u_{+2}, v_-)^1$ and so $\mathbf{P}(BAD^-) \geq 1/2 \cdot 1/256$. The event $C_1C_1D^-$ is covered by the 2-free pair (v_+, u_{-3}) and, thus, $\mathbf{P}(C_1C_1D^-) \geq 3/4 \cdot 0.5/256 = 0.375/256$. The same bound is valid for $C_1C_1D^+$. Finally, $\mathbf{P}(C_1C_1D^0) \geq 1/2 \cdot 0.5/256$ as the event is covered by $(u'_-, v_-)^2$ and $(v_+, u_{-3})^2$. Altogether, we have

$$\mathbf{P}(ABD^+ \cup BAD^- \cup C_1C_1D^- \cup C_1C_1D^+ \cup C_1C_1D^0) \geq \frac{0.5 + 0.5 + 0.375 + 0.375 + 0.25}{256} = \frac{2}{256}.$$

The last remaining possibility is that u'_- is not contained in Z. We have $\mathbf{P}(ABD^+) \geq 0.5/256$ and $\mathbf{P}(BAD^-) \geq 0.75/256$ by standard arguments. The event $C_1C_1D^-$ is covered by the pair $(u'_-, u'_-)^4$ so $\mathbf{P}(C_1C_1D^-) \geq 79/80 \cdot 0.5/256 > 0.49/256$ by Lemma 4.3.2. Similarly, $\mathbf{P}(C_1C_1D^+) \geq 59/80 \cdot 0.5/256 > 0.36/256$ since the event is covered by $(u'_-, u'_-)^4$ and $(v_+, u_{-3})^2$. The total contribution is at least 2.1/256. This concludes Subcase 8.2. $\triangle$

We may now assume, without loss of generality, that the path uZv is short; note that this means that u is deficient of type Ia or Ib. In the former case, there is nothing to prove as $2 + \epsilon(u) = 0$. Therefore, suppose that u is of type Ib (i.e., $u'_+ = v_{+2}$). Since $\epsilon(u) = -1.5$, it remains to find events forcing u with total probability at least 0.5/256. It is sufficient to consider the event ABD^+ of weight 8, which is covered by the 1-free pair (v_+, u_{-2}), and, therefore, $\mathbf{P}(ABD^+) \geq 0.5/256$ by Lemma 4.3.2. This finishes the proof of Case 8 and the whole proposition. ▲

□

A.6 Augmentation

In this section, we show that it is possible to apply the augmentation step mentioned in the preceding sections.

Suppose that u is a deficient vertex of G and $v = u'$. Let us continue to use Z to denote the cycle of the 2-factor F containing u. The *sponsor* $s(u)$ of u is one of its neighbours, defined as follows:

- if u is deficient of type 0 (recall that this type was defined at the beginning of Section A.4), then $s(u)$ is the F-neighbour u with $\epsilon(s(u)) = 1$; if there are two such F-neighbours, we choose $s(u) = u_-$,
- if u is deficient of any other type (in particular, $v \in V(Z)$), then $s(u) = v$.

Observation A.6.1. *Every vertex is the sponsor of at most one other vertex.*

Proof. Clearly, a given vertex can only sponsor its own neighbours, that is, its mate and F-neighbours. Suppose that u is the sponsor of its mate v; thus, $u \in C_v$. Suppose also that u is the sponsor of one of its F-neighbours, say u_+. Then, uv belongs to a 4-cycle intersecting $C_{u'_+}$ but this is not possible since $C_{u'_+} \neq C_v$.

The only remaining possibility is that u is the sponsor of both of its F-neighbours. In that case, both u_+ and u_- are deficient of type 0 and $\epsilon(u) = 1$. Thus, uv is contained in a 4-cycle but neither u_+ nor u_- is, giving rise to a contradiction. □

Recall that $N[u]$ denotes the closed neighbourhood of u, that is, $N[u] = N(u) \cup \{u\}$. An independent set J in G is said to be *favourable* for u if $N[u] \cap J = \{s(u)\}$. The *receptivity* of u, denoted $\rho(u)$, is the probability that a random independent set (with respect to the distribution given by Algorithm 1) is favourable for u. We say that u is *k-receptive* ($k \geq 0$) if the receptivity of u is at least $k/256$.

For an independent set J, we let $p(J)$ denote the probability that the random independent set produced by Algorithm 1 is equal to J. We fix an ordering $J_1, \ldots, J_s$ of all independent sets J in G such that $p(J) > 0$. Furthermore, an ordering $u_1, \ldots, u_r$ of all deficient vertices is chosen in such a way that $|\epsilon(u_i)| \leq |\epsilon(u_j)|$ if $1 \leq i < j \leq s$ (which we refer to as the *monotonicity* of the ordering).

Let u_i be a deficient vertex. We let $\tilde{N}(u_i)$ be the set of all deficient neighbours u_j of u_i such that $j < i$; furthermore, we put $\tilde{N}[u_i] = \tilde{N}(u_i) \cup \{u_i\}$. We define $\eta(u_i)$ as

$$\eta(u_i) = \sum_{u_j \in \tilde{N}[u_i]} |\epsilon(u_j)| .$$

We aim to replace $s(u_i)$ with u_i in some of the independent sets that are favourable for u_i, thereby boosting the probability of the inclusion of u_i in the random independent set I. Clearly, this requires that the receptivity of u_i is at least $|\epsilon(u_i)|/256$, since, otherwise, the probability of $u_i \in I$ cannot be increased to the required $88/256$ in this way. We also need to take into account the fact that an independent set may be favourable for u_i and its neighbour at the same time, but the replacement can only take place once. To dispatch the replacements in a consistent way, the following lemma will be useful. We remark that the number $p(u_i, J_j)$ which appears in the statement will turn out to be the probability that u_i is added to the random independent set during Phase 5 of the execution of the algorithm.

Lemma A.6.2. *If the receptivity of each deficient vertex u_i is at least $\eta(u_i)$, then we can choose a non-negative real number $p(u_i, J_j)$ for each deficient vertex u_i and each independent set J_j in such a way that the following holds:*

(i) $p(u_i, J_j) = 0$ whenever J_j is not favourable for u_i;

(ii) for each deficient vertex u_i, $\sum_j p(u_i, J_j) \cdot p(J_j) = |\epsilon(u_i)|/256$;

(iii) for each independent set J_j and deficient vertex u_i, $\sum_{u_t \in \tilde{N}[u_i]} p(u_t, J_j) \leq 1$.

Proof. We may view the numbers $p(u_i, J_j)$ as arranged in a matrix (with rows corresponding to vertices) and choose them in a simple greedy manner as follows. For each $i = 1, \ldots, r$ in this order, we determine $p(u_i, J_1)$, $p(u_i, J_2)$ and so on. Let $\vec{r_i}$ be the i-th row of the matrix, with zeros for the entries that are yet to be determined. Furthermore, let $\vec{p} = (p(J_1), \ldots, p(J_s))$.

For each i, j such that J_j is favourable for u_i, $p(u_i, J_j)$ is chosen as the maximal number such that $\vec{r_i} \cdot \vec{p}^T \leq |\epsilon(u_i)|/256$, and its sum with any number in the j-th column corresponding to a vertex in $\tilde{N}(u_i)$ is at most one. In other words, we set

$$p(u_i, J_j) = \min\Big(\frac{|\epsilon(u_i)|/256 - \sum_{\ell=1}^{j-1} p(u_i, J_\ell) \cdot p(J_\ell)}{p(J_j)}, 1 - \sum_{u_\ell \in \tilde{N}(u_i)} p(u_\ell, J_j)\Big) \qquad \text{(A.1)}$$

if J_j is favourable for u_i and $p(u_i, J_j) = 0$ otherwise. Note that the denominator in the fraction is non-zero since every independent set J_j with $1 \leq j \leq s$ has $p(J_j) > 0$. By the construction, properties (i) and (iii) in the lemma are satisfied and so is the inequality $\vec{r_i} \cdot \vec{p}^T \leq |\epsilon(u_i)|/256$ in property (ii). We need to prove the converse inequality.

Suppose that, for some i, $\vec{r_i} \cdot \vec{p}^T$ is strictly smaller than $|\epsilon(u_i)|/256$. This means that, in (A.1), for each j such that J_j is favourable for u_i, $p(u_i, J_j)$ equals the second term in the outermost pair of brackets. In other words, for each such j, we have

$$\sum_{u_\ell \in \tilde{N}[u_i]} p(u_\ell, J_j) = 1.$$

Thus, we can write

$$\sum_{\substack{j \\ J_j \text{ favourable for } u_i}} \Big(\sum_{u_\ell \in \tilde{N}[u_i]} p(u_\ell, J_j) \Big) \cdot p(J_j) = \sum_{\substack{j \\ J_j \text{ favourable for } u_i}} p(J_j) \tag{A.2}$$
$$= \rho(u_i) \geq \eta(u_i) = \sum_{u_\ell \in \tilde{N}[u_i]} \frac{|\epsilon(u_\ell)|}{256},$$

where the inequality on the second line follows from our assumption on the receptivity of u_i.

On the other hand, the expression on the first line of (A.2) is dominated by the sum of the scalar products of $\vec{p}$ with the rows corresponding to vertices in $\tilde{N}[u_i]$. For each such vertex u_ℓ, we know from the first part of the proof that $\vec{r_\ell} \cdot \vec{p}^T \leq |\epsilon(u_\ell)|/256$. Comparing with (A.2), we find that we must actually have equality both here and in (A.2); in particular,

$$\vec{r_i} \cdot \vec{p}^T = \frac{|\epsilon(u_i)|}{256},$$

a contradiction. □

For brevity, we will say that an event $X \subseteq \Omega$ is *favourable for u* if the independent set $I(\sigma)$ is favourable for u for every situation $\sigma \in X$. We lower-bound the receptivity of deficient vertices as follows:

Proposition A.6.3. *Let u be a deficient vertex. The following holds:*

(i) u is 1.9-receptive;

(ii) if u is of type 0, then it is 3-receptive;

(iii) if u is of type Ia or Ib (or their mirror types), then it is 8-receptive.

Proof. All the event(s) discussed in this proof will be favourable for u, as it is easy to check. To avoid repetition, we shall not state this property in each of the cases.

(i) First, let u be a deficient vertex of type I. We distinguish three cases, in each case presenting an event which is favourable for u and has sufficient probability. If $u_{-2}u_{+2}$ is not an edge of M, then the event Q_1 given by the diagram in Figure A.25(a) is valid. Since it is a regular diagram of weight 7, $\mathbf{P}(Q_1) \geq 2/256$ by Lemma 4.3.2. Thus, $\rho(u) \geq 2/256$ as Q_1 is favourable for u.

We may, thus, assume that $u_{-2}u_{+2} \in E(M)$. Suppose that neither u'_- nor u'_+ is contained in vZu. Consider the event Q_2, given by the diagram in Figure A.25(b). Since the edge uv is not contained in a 4-cycle (u being deficient), neither v_- nor v_+ is the mate of u_+ so the diagram is valid. The event is covered by the pair (u'_+, u'_+). If the pair is sensitive, then the cycle of F containing u'_+ has length at least 5 and, hence, it contains at least two vertices different from u'_+, v'_- and v'_+. Thus, the pair is 2-free and we have $\mathbf{P}(Q_2) \geq 19/20 \cdot 2/256 = 1.9/256$ by Lemma 4.3.2.

By symmetry, we may assume that each of uZv and vZu contain one of u'_- and u'_+. Hence, the event Q_3, defined by Figure A.25(c), is regular and $\mathbf{P}(Q_3) \geq 2/256$. (The event is valid for the same reason as Q_2.)

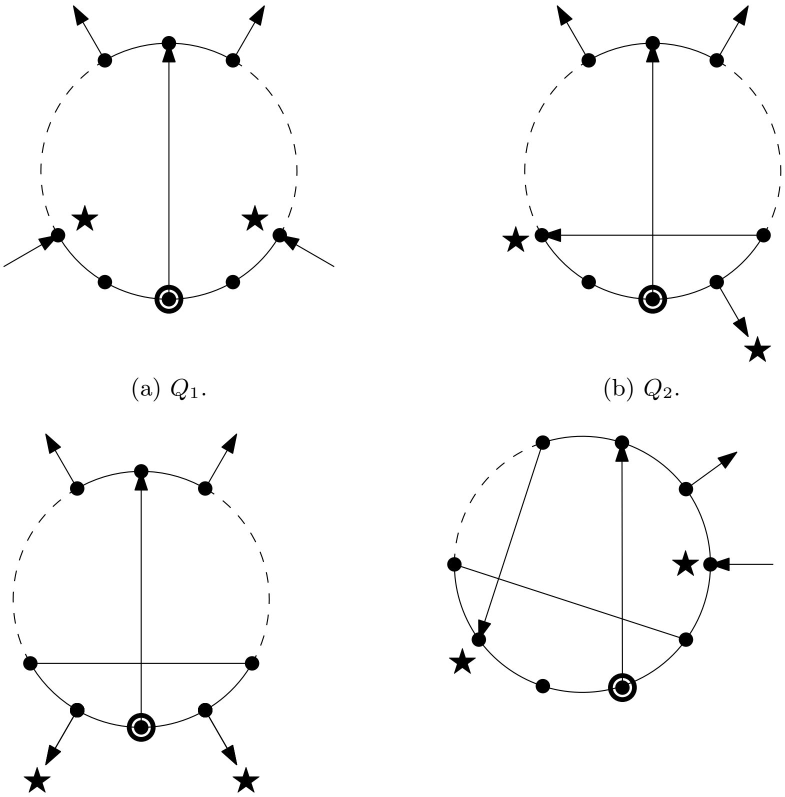
To finish part (i), it remains to discuss deficient vertices of types other than I. In view of parts (ii) and (iii), it suffices to look at types II, IIa, III and their mirror variants. Each of these types is consistent with the diagram in Figure A.25(d) or its symmetric version. The diagram of weight 6 defines a regular event Q_4, whose probability is at least 4/256 by Lemma 4.3.2. This proves part (i).

We prove (ii). Let u be deficient of type 0. We may assume that u_- is contained in a 4-cycle intersecting the cycle C_v; in particular, the mates of u_- and u_{-2} are contained in C_v. By the definition of type 0, we also know that neither u_{-2} nor u_{+2} has a neighbour in $\{v_-, v_+\}$.

Suppose that the set $\{u_{-2}, u_{+2}, v_-, v_+\}$ is independent. Since $u'_- \in V(C_v)$, the event R defined by the diagram in Figure A.26(a) is regular and it is easy to see that it is favourable for u and its probability is at least 1/256. Since the same holds for the events R^+ and R^-, obtained by reversing the arrow at v_- or v_+, respectively, we have shown that u is 3-receptive in this case.

If M includes the edge $u_{-2}v_+$, then both R and R^+ remain valid events and the probability of each of them increases to at least 2/256, showing that u is 4-receptive. An analogous argument applies if M includes $u_{-2}v_-$.

It remains to consider the possibility that $u_{+2}v_-$ or $u_{+2}v_+$ is in M. Suppose that



(a) Q_1.

(b) Q_2.

(c) Q_3 (note that each of uZv, vZu contains one of u'_+, u'_-).

(d) Q_4.

Figure A.25: Events used in the proof of Proposition A.6.3(i).

$u_{+2}v_- \in E(M)$. The event R^- remains valid and regular; its probability increases to at least 2/256. Let S^+ and T^+ be the events given by diagrams in Figure A.26(b) and (c), respectively. It is easy to check that R^+, S^+ and T^+ are pairwise disjoint and favourable for u. The event S^+ is covered by the pair $(u_{+2}, u_-)^1$ and Lemma 4.3.2 implies that $\mathbf{P}(S^+) \geq 0.5/256$. The event T^+ is regular and $\mathbf{P}(T^+) \geq 0.5/256$. Since $\mathbf{P}(R^+ \cup S^+ \cup T^+) \geq 3/256$, u is 3-receptive.

In the last remaining case, namely $u_{+2}v_+ \in E(M)$, we argue similarly. Let S^- and T^- be the events obtained by reversing both arcs incident with v_+ and v_- in the diagram for S^+ or T^+, respectively. It is routine to check that $\mathbf{P}(R^- \cup S^- \cup T^-) \geq 3/256$ and the events are favourable for u. Hence, u is 3-receptive. The proof is finished.

Part (iii) follows by considering the event defined by the diagram in Figure A.27. Note that the event is regular and its probability is at least $1/2^5 = 8/256$. Furthermore, the event is favourable for the vertex u. Thus, u is 8-receptive. □

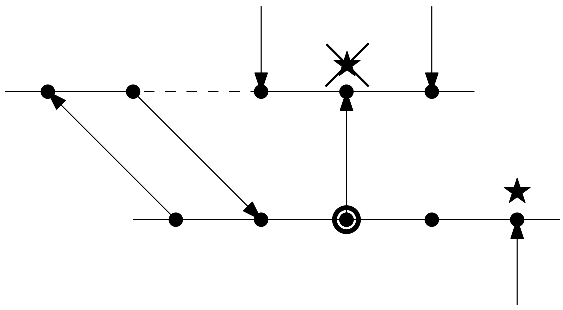
We now argue that Proposition A.6.3 implies the assumption of Lemma A.6.2 that the receptivity of a deficient vertex u_i is at least $\eta(u_i)$. By the monotonicity of the ordering $u_1, \ldots, u_r$ and the fact that $\left|\tilde{N}[u_i]\right| \leq 4$ and each deficient vertex has at least one non-deficient neighbour (namely its sponsor), we have $\eta(u_i) \leq 3\,|\epsilon(u_i)|$. From Proposition A.6.3 and the definition of $\epsilon(u_i)$ (see the beginning of Section A.4 and Table A.1), it is easy to check that u_i is $(3\,|\epsilon(u_i)|)$-receptive, which implies the claim.

Hence, the assumption of Lemma A.6.2 is satisfied. Let $p(u_i, J_j)$ be the numbers whose existence is guaranteed by Lemma A.6.2. We can finally describe Algorithm 2, which consists of the four phases of Algorithm 1, followed by **Phase 5** described below.

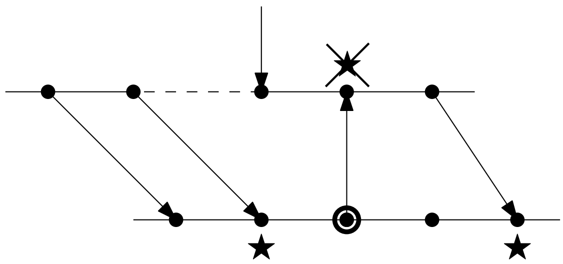
Assume a fixed independent set $I = J_j$ was produced by Phase 4 of the algorithm. We construct a sequence of independent sets $I^{(0)}, \ldots, I^{(r)}$. At the i-th step of the construction, u_i may or may not be added and we will ensure that

$$\mathbf{P}(u_i \text{ is added at } i\text{-th step}) = p(u_i, J_j). \tag{A.3}$$

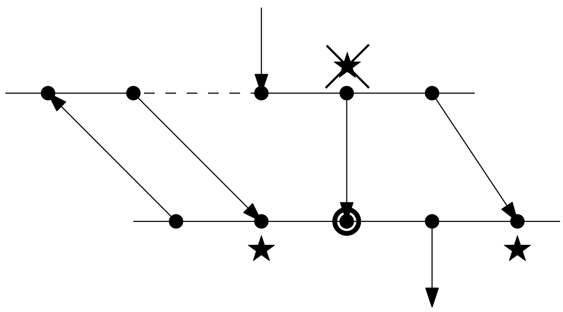
At the beginning, we set $I^{(0)} = I$. For $1 \leq i \leq r$, we define $I^{(i)}$ as follows. If $u_i \in I$ or I is not favourable for u_i, we set $I^{(i)} = I^{(i-1)}$. Otherwise, by (A.3) and property (iii) of Lemma A.6.2, the probability that none of u_i's neighbours has been added before is at



(a) R.



(b) S^+.



(c) T^+.

Figure A.26: Events used in the proof of Proposition A.6.3(ii) for vertices of type 0. Only the possibility that $u'_{-2} = (u'_-)_-$ is shown but the events remain valid if $u'_{-2} = (u'_-)_+$ (that is, if the chords of Z incident with u_- and u_{-2} cross).

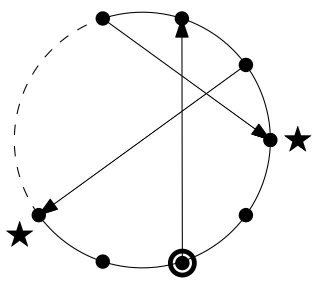


Figure A.27: The event used in the proof of Proposition A.6.3(iii) for vertices of type Ia and Ib and their mirror types.

least

$$1 - \sum_{u_\ell \in \tilde{N}(u_i)} p(u_\ell, J_j) \geq p(u_i, J_j).$$

Thus, by including u_i based on a suitably biased independent coin flip, it is possible to make the probability of inclusion of u_i in Phase 5 (conditioned on $I = J_j$) exactly equal to $p(u_i, J_j)$. The output of Algorithm 2 is the set $I' := I^{(r)}$.

We analyze the probability that a deficient vertex u_i is in I'. By Lemma A.4.1 and Proposition A.5.2,

$$\mathbf{P}(u_i \in I) \geq \frac{88 + \epsilon(u_i)}{256}.$$

By the above and property (ii) of Lemma A.6.2, the probability that u_i is added to I' during Phase 5 equals

$$\begin{aligned}\mathbf{P}(u_i \text{ is added in Phase 5}) &= \sum_{j=1}^{s} \mathbf{P}(u_i \text{ is added in Phase 5} \mid I = J_j) \cdot \mathbf{P}(I = J_j) \\ &= \sum_{j=1}^{s} p(u_i, J_j) \cdot p(J_j) = \frac{|\epsilon(u_i)|}{256}.\end{aligned}$$

Since u_i is deficient, $\epsilon(u_i) < 0$; therefore, we obtain

$$\begin{aligned}\mathbf{P}(u_i \in I') &= \mathbf{P}(u_i \in I) + \mathbf{P}(u_i \text{ is added in Phase 5}) \\ &\geq \frac{88 + \epsilon(u_i)}{256} - \frac{\epsilon(u_i)}{256} = \frac{88}{256}.\end{aligned}$$

If w is a vertex of G which is the sponsor of a (necessarily unique) deficient vertex u_i, then the probability of the removal of w in Phase 5 is equal to the probability of the addition of u_i, namely $|\epsilon(u_i)|/256$. From Lemma A.4.1 and Proposition A.5.2, it follows that $\mathbf{P}(w \in I)$ is high enough for $\mathbf{P}(w \in I')$ to be still greater than or equal to 88/256.

Finally, if a vertex w is neither deficient nor the sponsor of a deficient vertex, it is not affected by Phase 5 and hence $\mathbf{P}(w \in I') \geq 88/256$ as well. Applying Lemma 1.3.1 to Algorithm 2, completes the proof that $\chi_f(G) \leq 256/88 = 32/11$ for bridgeless cubic graphs.