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Graph powers, partitions, and other extremal problems

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is my own work, except where otherwise stated.

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Abstract

Graph theory is the study of networks of objects (called vertices) joined by links (called edges). Since many real world problems can be represented by a graph, graph theory has applications in areas such as sociology, chemistry, and computing. In this thesis, a number of open problems in graph theory are studied.

An old conjecture due to Erdös, Gyárfás, and Pyber says that in any edge-colouring of a complete graph with r colours, it is possible to cover all the vertices with r vertex-disjoint monochromatic cycles. So far, this conjecture has been proved only for $r = 2$. In this thesis, it is shown that in fact this conjecture is false for all $r \geq 3$. In contrast to this, it is shown that in any edge-colouring of a complete graph with three colours, it is possible to cover all the vertices with three vertex-disjoint monochromatic *paths*, proving a particular case of a conjecture due to Gyárfás. In addition, using some results about partitioning coloured graphs the value of certain Ramsey Numbers is determined. In particular the Ramsey number of a path of length n versus the power of a path of length n is calculated, solving a conjecture of Allen, Brightwell, and Skokan.

A recent question posed by Hegarty asks how few edges the power of a regular graph can have. The rth power of a graph G is constructed from G by adding an edge between any two vertices within distance r of each other. Hegarty showed that if G is a regular, connected graph, then G^3 is either complete or satisfies $e(G^3)/e(G) \geq 1 + \epsilon$ where $\epsilon \approx 0.87$. Hegarty asked whether similar results hold for other powers of graphs. In this thesis his question is answered for every $r \geq 4$ by determining how small the ratio $e(G^r)/e(G)$ can be for a regular connected graph.

Finally, progress is made on a conjecture of Manickam, Miklós, and Singhi concerning nonnegative k-sums (sums of k distinct elements) in a set of n numbers. Manickam, Miklós, and Singhi conjectured that if $n \geq 4k$ and we have a set of real numbers $x_1, \ldots, x_n$ satisfying $x_1 + \cdots + x_n \geq 0$, then there are at least $\binom{n-1}{k-1}$ nonnegative k-sums from $\{x_1, \ldots, x_n\}$. It is shown that this conjecture holds whenever $n \geq 10^{46}k$, giving the first linear bound on this conjecture.

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Chapter 1

Introduction and preliminaries

A graph $G = (V, E)$ is given by a set V of vertices, and a set $E \subseteq \binom{V}{2}$ of unordered pairs of vertices. Extremal problems, described by Bollobas as being "at the very heart of graph theory" [9] are questions of the following form:

> "How large does a parameter of a graph G need to be to guarantee that G contains a certain substructure?"

This question can be easy or hard depending on what parameter one considers and what substructure one looks for. Many important theorems and conjectures in graph theory can be phrased as an extremal problem. Indeed, Mantel's Theorem tells us how many edges a graph needs to have to ensure that it contains a triangle. Turan's Theorem tells us how many edges a graph needs to have to guarantee that it contains a clique of a certain order. Dirac's Theorem tells us how large the minimum degree of a graph needs to be to guarantee that it contains a Hamiltonian Cycle. Ramsey's Theorem gives a bound on how many vertices a graph needs to have to guarantee that it contains either a large complete graph or a large independent set.

More generally, extremal questions can be asked about almost any mathematical structure. For example the Erdős-Ko-Rado Theorem tells us how many subsets from $\{1, \ldots, n\}$ of order k we can have such that any two of them intersect. The Cauchy-Davenport theorem tells us how large two sets $A, B \subseteq \mathbb{Z}_p$ (for a prime p) can be so that the order of their sumset $A + B = \{a + b : a \in A, b \in B\}$ is bounded by some constant.

In this thesis we give solutions and partial solutions to a number of extremal problems. Some of these, such as the Erdős-Gyárfás-Pyber Conjecture and the Manickam-Miklós-Singhi Conjecture are old problems which have been open for 20 years and have attracted much attention. Others, such as a question of Hegarty about graph powers, were only asked recently, but have begun to attract attention. Over the remainder of this chapter, we will describe in detail the problems which are studied in this thesis.

1.1 Nonnegative k-sums in a set of numbers with nonnegative sum

Consider the following extremal problem:

"Suppose that we have a set of numbers $x_1, \dots, x_n$ satisfying $x_1 + \dots + x_n \geq 0$. How many subsets $A \subset \{x_1, \dots, x_n\}$ must satisfy $\sum_{a \in A} a \geq 0$?"

By choosing $x_1 = n - 1$ and $x_2 = \dots = x_n = -1$ we see that the answer to this question can be at most 2^{n-1}. In fact, this example has the minimal number of nonnegative sets. Indeed, for any set $A \subset \{x_1, \dots, x_n\}$ either A or $\{x_1, \dots, x_n\} \setminus A$ must have nonnegative sum, so there must always be at least 2^{n-1} nonnegative subsets in any set of numbers $\{x_1, \dots, x_n\}$ with nonnegative sum.

A more difficult extremal problem arises if we count only subsets of fixed order. By again considering the example when $x_1 = n - 1$ and $x_2 = \dots = x_n = -1$ we see that there are sets of n numbers with nonnegative sums which have only $\binom{n-1}{k-1}$ nonnegative k-sums (sums of k distinct numbers). Manickam, Miklós, and Singhi conjectured that for $n \geq 4k$ this assignment gives the least possible number of nonnegative k-sums.

Conjecture 1.1.1 (Manickam, Miklós, Singhi, [41, 42])**.** *Suppose that $n \geq 4k$, and we have n real numbers $x_1, \dots, x_n$ such that $x_1 + \dots + x_n \geq 0$. Then, at least $\binom{n-1}{k-1}$ subsets $A \subset \{x_1, \dots, x_n\}$ of order k satisfy $\sum_{a \in A} a \geq 0$*

Conjecture 1.1.1 appeared in [42] where it was phrased in terms of calculating invariants of an association scheme known as the *Johnson Scheme*. In [41], Conjecture 1.1.1 was phrased in the combinatorial form in which it is stated above. In this thesis we will speak only about the combinatorial version—we refer the reader to [42, 7] for more details about the association scheme version.

A motivation for the bound "$n \geq 4k$" is that for $k \geq 3$ and $n = 3k+1$ there exists an assignment of values to $x_1, \dots, x_{3k+1}$ which results in less than $\binom{n-1}{k-1}$ nonnegative k-sums. Indeed, letting $x_1 = x_2 = x_3 = 2-3k$ and $x_4 = \cdots = x_{3k+1} = 3$ gives an assigment satisfying $x_1 + \cdots + x_{3k+1} = 0$ but having $\binom{3k-2}{k}$ nonnegative k-sums, which is less than $\binom{3k}{k-1}$ for $k \geq 3$. Notice that these examples exist only when $n = 3k+1$. Thus it is possible that the bound "$n \geq 4k$" could be slightly strengthened in Conjecture 1.1.1. For example for $k = 3$, Chowdhury proved that Conjecture 1.1.1 holds with the improved bound of $n \geq 11$, and that this bound is best possible [12].

Despite the apparent simplicity of the statement of Conjecture 1.1.1, it has been open for over two decades. Many partial results have been proven. The conjecture has been proven for $k \leq 3$ by Manickam [40] and independently by Chiaselotti and Marino [21]. It has been proven whenever $n \equiv 0 \pmod{k}$ by Manickam and Singhi [42].

In addition, several results have been proved establishing the conjecture when n is large compared to k. Manickam and Miklós [41] showed that the conjecture holds when $n \geq (k-1)(k^k + k^2) + k$ holds. Tyomkyn [52] improved this bound to $n \geq k(4e \log k)^k \approx e^{ck \log \log k}$. Recently Alon, Huang, and Sudakov [3] showed that the conjecture holds when $n \geq 33k^2$. Subsequently Frankl [20] gave an alternative proof of the conjecture in a range of the form $n \geq 3k^3/2$. To date, Alon, Huang, and Sudakov's bound of $n \geq 33k^2$ stands as the best known bound for Conjecture 1.1.1.

In this thesis we improve these bounds by showing that the conjecture holds in a range when n is linear with respect to k. In Chapter 2 we prove a theorem which shows that Conjecture 1.1.1 holds whenever we have $n \geq 10^{46}k$. The method we use to prove this theorem is inspired by Katona's proof of the Erdős-Ko-Rado Theorem [35].

1.2 Edge growth in graph powers

For two sets of numbers $A, B \subseteq \mathbb{Z}_p$, their sumset is defined to be the set $A + B = \{a + b : a \in A, b \in B\}$. Consider the following question "for two sets A and B of fixed order, how small can the set $A + B$ be?". When p is prime, this question was answered, by Cauchy and Davenport in the following theorem.

Theorem 1.2.1 (Cauchy [11], Davenport [13]). *Let p be a prime, and $A, B \subseteq \mathbb{Z}_p$. Then we have either $A + B = \mathbb{Z}_p$ or*

$$|A + B| \geq |A| + |B| - 1. \tag{1.1}$$

If we take both A and B to be the arithmetic progression $\{a, 2a, 3a, \ldots, ka\}$ for some a and k, we see that it is possible for equality to hold in (1.1). In Chapter 3 we study graph-theoretic analogues of Theorem 1.2.1.

Before we can state the results that we will look at, we will need a few definitions. The distance between two vertices x,y in G is defined as the length of the shortest path between them in G. The rth power of a graph G, denoted G^r, is constructed from G by adding an edge between two vertices x and y when they are within distance r in G. Define the diameter of a connected graph G, $\mathrm{diam}(G)$, as the minimal r such that G^r is complete (alternatively, the maximal distance between two vertices in G). For a group G and a set $A \subseteq G$, the Cayley Graph of A, denoted $Cay(G, A)$, is defined to be the graph with vertex set G with gh an edge whenever gh^{-1} or $hg^{-1} \in A$ holds.

It is easy to see that Theorem 1.2.1 has the following corollary.

Corollary 1.2.2. *Let p be a prime, A a subset of $\mathbb{Z}_p$, and $G = Cay(\mathbb{Z}_p, A)$. Then for any integer $r <$ $\mathrm{diam}(G)$:*

$$e(G^r) \geq r\, e(G).$$

An interesting question to ask is whether analogues of Corollary 1.2.2 hold for more general graphs G. In particular since the Cayley graphs $Cay(\mathbb{Z}_p, A)$ are always regular and (when p is prime and $A \neq \emptyset$) connected, we might focus on regular, connected G. In [34] Hegarty proved the following theorem:

Theorem 1.2.3 (Hegarty, [34]). *Suppose G is a regular, connected graph which satisfies* $\mathrm{diam}(G) \geq 3$. *Then we have*

$$e(G^3) \geq (1 + \epsilon)\, e(G),$$

with $\epsilon \approx 0.087$

In other words, the cube of G retains the original edges of G and gains a positive proportion of new ones. In Chapter 3 we give an alternative proof of this theorem with an improved constant of $\epsilon = \frac{1}{6}$. Since we announced this result, DeVos and Thomassé [15] further improved the constant in Theorem 1.2.3 to $\epsilon = \frac{3}{4}$. They also showed that the constant cannot be improved further by exhibiting a sequence of regular graphs G_n, such that $e(G_n^3)/e(G_n) \to \frac{7}{4}$ as $n \to \infty$.

Theorem 1.2.3 leads to the question of how the growth behaves for other powers of the graph G. Note that Theorem 1.2.3 cannot be used recursively to obtain such a result – since the cube of a regular graph is not necessarily regular. In [34] it was shown that Theorem 1.2.3 does not hold with G^3 replaced by G^2 for any $\epsilon > 0$, and it was asked what happens for higher powers. In Chapter 3 we will address this question for 4th powers and higher. For every $r \geq 4$, we determine how small the ratio $e(G^r)/e(G)$ can be for a regular, connected graph of diameter at least r.

1.3 Ramsey Theory

Ramsey Theory is a branch of mathematics concerned with finding ordered substructures in a mathematical structure which may, in principle, be highly disordered. An early example of a result in Ramsey Theory is a theorem due to Van der Waerden [53], which says that for for any k and $r \geq 1$ there is a number $W(k, r)$, such that any colouring of the numbers $1, 2, \ldots, W(k, r)$ with r colours contains a monochromatic k-term arithmetic progression. A special case of a theorem due to Ramsey [49] says that for every n, there exists a number $R(n)$, such that every 2-edge-coloured complete graph on more than $R(n)$ vertices contains a monochromatic complete graph on n vertices. The number $R(n)$ is called a *Ramsey number*.

A central definition in Ramsey Theory is the *generalized Ramsey number* $R(G)$ of a graph G: the minimum n for which every 2-edge-colouring of K_n contains a monochromatic copy of G. This is the so called *diagonal Ramsey number* of a graph G. For a pair of graphs G and H the Ramsey number of G versus H, $R(G, H)$, is defined to be the minimum n for which every 2-edge-colouring of K_n with the colours red and blue contains either a red copy of G or a blue copy of H. This is the so called *non-diagonal Ramsey number* of G versus H. Although there have been many results which give good bounds on Ramsey numbers of graphs [24], the exact value of the Ramsey number $R(G,H)$ is only known when G and H each belong to one of a few families of graphs.

One of the first Ramsey numbers to be determined exactly was the Ramsey number of the path.

Theorem 1.3.1 (Gerencsér and Gyárfás, [22]). *For $m \le n$ we have that*

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

Recall that the kh power of a path of order n is the graph constructed with vertex set $1, \ldots, n$ and ij an edge whenever $1 \le |i-j| \le k$. Allen, Brightwell, and Skokan conjectured the following generalization of the $n = m$ case of Theorem 1.3.1.

Conjecture 1.3.2 (Allen, Brightwell, Skokan, [2]). *For all k and $n \ge k+1$, we have*

$$R(P_n, P_n^k) = (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor.$$

In Chapter 6 we prove this conjecture. As an intermediate result we find an upper bound on the Ramsey number of a path versus a balanced complete multipartite graph. A *balanced complete k-partite graph* on km vertices, K_m^k, is a graph whose vertices can be partitioned into k sets $A_1, \ldots, A_k$ such that $|A_1| = \cdots = |A_k| = m$ for all i, and there is an edge between $a_i \in A_i$ and $a_j \in A_j$ if, and only if, $i \ne j$. In Chapter 6 we show that for all n, m, and k we have

$$R(P_n, K_m^k) \le (k-1)(n-1) + k(m-1) + 1. \tag{1.2}$$

By considering a union of disjoint red copies of K_{n-1}, it is easy to show that equality holds in (1.2) whenever $m \equiv 1 \pmod{n-1}$. Notice that a special case of (1.2) we obtain that $R(P_n, K_{m,m}) = n + 2m - 2$ whenever $m \equiv 1 \pmod{n-1}$. This is also a corollary of the following earlier theorem due to Häggkvist.

Theorem 1.3.3 (Häggkvist, [32])**.** *If $m, \ell \equiv 1 \pmod{n-1}$, then we have*

$$R(P_n, K_{m,\ell}) = n + m + \ell - 2.$$

The special case of (1.2) when $m = 1$ gives the following theorem which was essentially proved by Erdős [17], as observed by Parsons [44].

Theorem 1.3.4 (Erdős, [17])**.** *For all n and m we have*

$$R(P_n, K_m) = (m-1)(n-1) + 1.$$

1.4 Partitioning graphs into monochromatic subgraphs

Recall that in [22], Gerencsér and Gyárfás proved Theorem 1.3.1 and so determined the Ramsey Number of a path. In the same paper, they proved the following.

Theorem 1.4.1 (Gerencsér and Gyárfás, [22])**.** *The vertices of every 2-edge-coloured complete graph can be covered by two vertex-disjoint monochromatic paths of different colours.*

The proof of Theorem 1.4.1 is so short that it was originally published in a footnote of [22]. Indeed to see that the theorem holds, simply find a red path R in K_n and a vertex-disjoint blue path B in K_n such that $|R| + |B|$ is as large as possible. Let r and b be endpoints of R and B respectively. If there is a vertex $x \notin R \cup B$, then it is easy to see that the triangle $\{x, r, b\}$ contains either a red path between x and r or a blue path between x and b. This path can be joined to R or B contradicting maximality of $|R| + |B|$.

The relation between Theorems 1.3.1 and 1.4.1 is that it is possible to determine the weaker bound $R(P_n, P_m) \leq n + m - 1$ on the Ramsey Number of a path using Theorem 1.4.1. Indeed Theorem 1.4.1 implies that every 2-edge-coloured K_{n+m-1} can be covered by a red path R and a disjoint blue path B. Clearly these paths cannot cover all the vertices unless $|R| \geq n$ or $|B| \geq m$.

Theorem 1.4.1 has led to many results and conjectures about covering coloured graphs by monochromatic subgraphs. One of these is a conjecture due to Gyárfás which generalises Theorem 1.4.1.

Conjecture 1.4.2 (Gyárfás, [27]). *The vertices of every r-edge-coloured complete graph can be covered with r vertex-disjoint monochromatic paths.*

According to [28], Erdős offered 25 – 50 US Dollars for a solution of the $r = 3$ case of this conjecture. Erdős, Gyárfás, and Pyber made the following stronger conjecture.

Conjecture 1.4.3 (Erdős, Gyárfás & Pyber, [18]). *The vertices of every r-edge-coloured complete graph can be covered with r vertex-disjoint monochromatic cycles.*

When dealing with these conjectures, the empty set, a single vertex, and a single edge between two vertices are considered to be paths and cycles. It is worth noting that neither of the above conjectures require the monochromatic paths covering K_n to have distinct colours. Indeed as we shall see, there are examples of r-edge-coloured complete graphs which cannot be covered by r vertex-disjoint monochromatic paths without repeating colours. Whenever the vertices of a graph G are covered by vertex-disjoint subgraphs $H_1, H_2, \ldots, H_k$, we say that $H_1, H_2, \ldots, H_k$ *partition* G.

Most effort has focused on Conjecture 1.4.3. It was shown in [18] that there is a function $f(r)$ such that, for all n, any r-edge-coloured K_n can be partitioned into $f(r)$ monochromatic cycles. The best known upper bound for $f(r)$ is due to Gyárfás, Ruszinkó, Sárközy, and Szemerédi [30] who show that, for large n, $100r \log_2 r$ monochromatic cycles are sufficient to partition the vertices of an r-edge-coloured K_n.

For small r, there has been more progress. The case $r = 2$ of Conjecture 1.4.3 is closely related to Lehel's Conjecture, which says that any 2-edge-coloured complete graph can be partitioned into two monochromatic cycles *with different colours*. This conjecture first appeared in [4] where it was proved for some special types of colourings of K_n. Gyárfás [26] showed that the vertices of a 2-edge-coloured complete graph can be covered by two monochromatic cycles with different colours intersecting in at most one vertex. Łuczak, Rödl, and Szemerédi [39] showed, using the Regularity Lemma, that Lehel's Conjecture holds for $r = 2$ for large n. Later, Allen [1] gave an alternative proof that works for smaller (but still large) n, and which avoids the use of the Regularity Lemma. Lehel's Conjecture was finally shown to be true for all n by Bessy and Thomassé [6], using a short, elegant argument.

For $r = 3$, Gyárfás, Ruszinkó, Sárközy, and Szemerédi proved the following theorem.

Theorem 1.4.4 (Gyárfás, Ruszinkó, Sárközy & Szemerédi, [31])**.** *Suppose that the edges of K_n are coloured with three colours. There are three vertex-disjoint monochromatic cycles covering all but $o(n)$ vertices in K_n.*

In [31], it is also shown that, for large n, 17 monochromatic cycles are sufficient to partition *all* the vertices of every 3-edge-coloured K_n.

Despite Theorem 1.4.4 being an approximate version of the case $r = 3$ of Conjecture 1.4.3, in Chapter 4 we show that the conjecture is false for all $r \geq 3$

Interestingly, in all the counterexamples to Conjecture 1.4.3 that we construct it is possible to cover all except one of the vertices with r disjoint monochromatic cycles. Therefore the counterexamples we construct are quite "mild" and leave room for further work to either find better counterexamples, or to prove better approximate versions of the conjecture similar to Theorem 1.4.4.

The disproof of Conjecture 1.4.3 also raises the question of whether Conjecture 1.4.2 holds for $r \geq 3$ or not. In Chapter 5, we will prove the $r = 3$ case of Conjecture 1.4.2.

All the partitioning results mentioned so far have been about partitioning coloured complete graphs. There have also been a number of interesting results and conjectures about partitioning coloured graphs which are not complete into monochromatic subgraphs. For example Sárközy [50] considered r-edge-coloured graphs of fixed independence number and bounded the number of monochromatic cycles needed to partition such graphs. Balogh, Barát, Gerbner, Gyárfás, and Sárközy [5] considered 2-edge-coloured graphs G with minimum degree $|G|/2$ and showed that $|G| - o(|G|)$ vertices in such a graph can be covered by two disjoint monochromatic cycles.

A complete bipartite graph is *balanced* if both its parts have the same size. Gyárfás and Lehel proved the following theorem about partitioning a 2-edge coloured balanced complete bipartite graph. The proof of this theorem appears implicitly in [29], and the statement appears in [26].

Theorem 1.4.5 (Gyárfás & Lehel, [26, 29])**.** *Suppose that the edges of $K_{n,n}$ are coloured with two colours such that one of the parts of $K_{n,n}$ is contained in a monochromatic connected component. Then there exist two disjoint monochromatic paths with different colours which cover all, except possibly one, of the vertices of $K_{n,n}$.*

In Chapter 5 we sharpen Theorem 1.4.5 by showing that the two disjoint monochromatic paths of different colours can actually cover *all* the vertices of $K_{n,n}$. This theorem is used in Chapter 5 in the proof of the $r = 3$ case of Conjecture 1.4.2.

1.5 Non-diagonal partitioning results

Recall that although Ramsey Theory initially focused on finding bounds for just the quantity $R(K_n, K_n)$, it quickly developed into looking for bounds on the more general quantities $R(G, H)$ for a pair of graphs G and H. This happened partly because calculating $R(G, H)$ for certain pairs of graphs G and H could shed light on the original problem, and partly because calculating $R(G, H)$ for any pairs of graphs is an interesting problem in its own right.

So far most results about partitioning coloured graphs have partitioned a coloured complete graph into a small number of monochromatic graphs $G_1, \dots, G_k$ such that the graphs $G_1, \dots, G_k$ all have the same structure. These could be seen as "diagonal partitioning results". During the proofs of some results in this thesis we found it useful to partition a 2-edge-coloured complete graph into two monochromatic graphs G and H which have very different structure. These can be seen as "non-diagonal partitioning results". The most important of these results is a strengthening of the original Gerencsér-Gyárfás Theorem about partitioning a 2-edge-coloured graph into two monochromatic paths. It turns out that Theorem 1.4.1 can be strengthened to give the following.

Lemma 1.5.1. *Suppose that the edges of K_n are coloured with the colours red and blue. Then there is a vertex-partition of K_n into a red path and a blue balanced complete bipartite graph.*

Lemma 1.5.1 plays an important role in this thesis. It is used in Chapter 5 in the proof of the $r = 3$ case of Conjecture 1.4.2. A generalisation of Lemma 1.5.1 is used in Chapter 6 in the proof of (1.2) and Conjecture 1.3.2. Lemma 1.5.1 easily implies that $R(P_n, K_{m,m}) = n + 2m - 2$ holds, which is a special case of Häggkvist's result (Theorem 1.3.3).

In view of the importance of Lemma 1.5.1, we give a proof of it here.

Proof of Lemma 1.5.1. Notice that a graph with no edges is a complete bipartite graph (with one of the parts empty). Therefore, any 2-edge-coloured K_n certainly has a partition into a red path and a blue complete bipartite graph (by assigning all of K_n to be one of the parts of the complete bipartite graph). Partition K_n into a red path P and a complete bipartite graph $B(X, Y)$ with parts X and Y such that the following hold.

(i) $\max(|X|, |Y|)$ is as small as possible.

(ii) $|P|$ is as small as possible (whilst keeping (i) true).

We are done if $|X| = |Y|$ holds. Therefore, without loss of generality, suppose that we have $|X| < |Y|$.

Suppose that $P = \emptyset$. Then let y be any vertex in Y, $P' = \{y\}$, $Y' = Y - y$, and $X' = X$. This new partition of K_n satisfies $\max(|Y'|, |X'|) < |Y| = \max(|X|, |Y|)$, contradicting minimality of the original partition in (i).

Now, suppose that P is nonempty. Let p be an end vertex of P.

If there is a red edge py for $y \in Y$, then note that letting $P' = P + y$ and $Y' = Y - y$ gives a partition of K_n into a red path and a complete bipartite graph $B(X, Y')$ with parts X and Y'. However we have $\max(|Y'|, |X|) < |Y| = \max(|X|, |Y|)$, contradicting minimality of the original partition in (i).

If all the edges between p and Y are blue, then note that letting $P' = P - p$ and $X' = X + p$ gives a partition of K_n into a red path and a complete bipartite graph

$B(X', Y)$ with parts X' and Y. We have that $\max(|X'|, |Y|) = |Y| = \max(|X|, |Y|)$ and $|P'| < |P|$, contradicting minimality of the original partition in (ii). $\square$

A few other "non-diagonal partitioning results" are proved in this thesis. In Chapter 6 we give a generalisation of Lemma 1.5.1 which says that for any k, every 2-edge-coloured complete graph can be partitioned into k red paths and a blue balanced complete $(k+1)$-partite graph. If the red edges of the complete graph form a connected graph, then this result can be improved—in this case it is possible to partition the graph into k red paths and a blue balanced complete $(k+2)$-partite graph.

The ideas in the above proof of Lemma 1.5.1 are also important in this thesis. The proof could be summarised as "first we find a partition of our graph which is in some way extremal and then we show that it possesses the properties that we want". A number of other proofs we present in this thesis also have the same basic idea.

1.6 Notation

Notation that we will use is standard, apart from several exceptions which are explicitly mentioned below. Notation for graphs which we will use can be found in [16]. Notation for hypergraphs which we will use can be found in [8].

For a graph G, $V(G)$ denotes the set of vertices of G, and $E(G)$ the set of edges. For a set of vertices S in a graph G, the induced subgraph of G with vertex set S is denoted by $G[S]$. For two sets of vertices S and T in a graph G, let $G[S, T]$ be the subgraph of G with vertex set $S \cup T$ with st an edge of $G[S, T]$ whenever $s \in S$ and $t \in T$. A graph G is called bipartite if its vertices can be partitioned into two sets X and Y such that there are no edges within X or Y. In this case we say that X and Y give a *bipartition* of G and the sets X and Y are called the *parts* or *classes* of the bipartition.

We will often identify a graph G with its vertex set $V(G)$. Whenever we say that two subgraphs of a graph are "disjoint" we will always mean vertex-disjoint. Whenever a graph G is covered by vertex-disjoint subgraphs $H_1, H_2, \ldots, H_k$, we say that $H_1, H_2, \ldots, H_k$ *partition* G.

For two vertices $u, v \in V(G)$, the *distance* between u and v, denoted $d(u, v)$, is defined to be the length of the shortest path between them. The rth power of G, denoted G^r, is the graph with vertex set $V(G)$, and xy an edge whenever x and y are within distance r of each other. The *diameter* of a connected graph is the smallest r for which G^r is complete (or, alternatively, the maximum possible distance between a pair of vertices in G).

For $v \in V(G)$, define its *neighbourhood* as $N(v) = \{u \in V(G) : uv \in E(G)\}$. For a vertex $v \in G$, its degree is defined as the number of edges containing v. The maximum and minimum degrees of G are defined as $\Delta(G) = \max_{v\in G} d(v)$ and $\delta(G) = \min_{v\in G} d(v)$ respectively. Similarly for a set $S \subseteq V(G)$ we let $\Delta(S) = \max_{v\in S} d(v)$ and $\delta(S) = \min_{v\in S} d(v)$.

For a hypergraph H, the vertex-degree of a vertex $v \in H$ is the number of edges containing v. We say that H is d-regular if every vertex has vertex-degree d. The complete k-uniform hypergraph with n vertices is denoted by $\mathcal{K}_n^{(k)}$.

A *linear forest* is a disjoint union of paths. A balanced complete k-partite graph, denoted K_m^k, is a graph whose vertices can be partitioned into k sets $A_1, \ \dots, A_k$ such that $|A_1| = \cdots = |A_k| = m$ for all i, and there is an edge between $a_i \in A_i$ and $a_j \in A_j$ if, and only if, $i \neq j$. A fact that we use about complete k-partite graphs is that K_m^k always contains a copy of the power of a path P_{km}^{k-1}.

For a group G and a set $A \subseteq G$, the Cayley Graph of A, denoted $Cay(G, A)$, is defined to be the graph with vertex set G with gh an edge whenever hg^{-1} or $gh^{-1} \in A$ holds.

Throughout Chapters 5 and 6 it will be convenient to have special notation for dealing with paths in graphs. Often we will define paths in a graph G by giving its sequence of vertices $p_1, p_2, \dots, p_k \in G$ such that p_ip_{i+1} is an edge in G. For a nonempty path P, we will distinguish between the two endpoints of P saying that one endpoint is the "start" of P and the other is the "end" of P. Thus we will often say things like "Let P be a path from u to v". Let P be a path from a to b in G and Q a path from c to d in G. If P and Q are vertex-disjoint and bc is an edge in G, then we define $P + Q$ to be the unique path from a to d formed by joining P and Q with the edge bc. If P is a path and Q is a subpath of P sharing an endpoint with P, then $P - Q$ will denote the subpath of P with vertex set $V(P) \setminus V(Q)$.

Throughout Chapters 4 – 6 we will deal with edge-coloured graphs. Whenever a graph is coloured with two colours, the colours will be called "red" and "blue". When there are three colours, they will be called "red", "blue", and "green". If a graph G is coloured with some number of colours we define the *red colour class* of G to be the subgraph of G with vertex set $V(G)$ and edge set consisting of all the red edges of G. We say that G is *connected in* red, if the red colour class is a connected graph. A red component of a graph G is a connected component of the red colour class of G.

For any function f defined on uncoloured graphs we define f_r to be that function evaluated on the red colour class of a coloured graph. For example $d_r(v)$ denotes the number of red edges containing a vertex v, $\Delta_r(G)$ denotes the maximum red degree of G, etc.

Similar definitions are made for the colours blue and green as well (using subscripts "b" and "g" instead of "r").

We will need the following special 3-colourings of the complete graph.

Definition 1.6.1. *Suppose that the edges of K_n are coloured with three colours. We say that the colouring is **4-partite** if there exists a partition of the vertex set into four nonempty sets A_1, A_2, A_3, and A_4 such that the following hold.*

- *The edges between A_1 and A_4, and the edges between A_2 and A_3 are red.*
- *The edges between A_2 and A_4, and the edges between A_1 and A_3 are blue.*
- *The edges between A_3 and A_4, and the edges between A_1 and A_2 are green.*

The edges within the sets A_1, A_2, A_3, and A_4 can be coloured arbitrarily. The sets A_1, A_2, A_3, and A_4 will be called the "classes" of the 4-partition.

When dealing with 4-partite colourings of K_n, the classes will always be labelled "A_1", "A_2", "A_3", and "A_4", with colours between the classes as in the above definition. See Figure 1.1 for an illustration of a 4-partite colouring of K_n. The following lemma gives a useful alternative characterization of 4-partite colourings of K_n.

Lemma 1.6.2. *Suppose that the edges of K_n are coloured with three colours. The colouring is 4-partite if and only it is disconnected in each colour and there is a red*

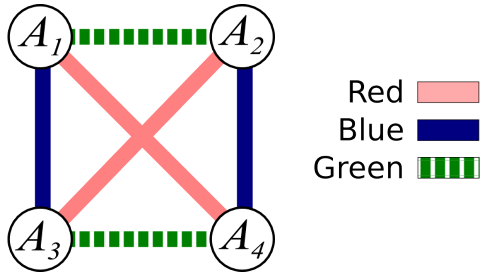

Figure 1.1: A 4-partite colouring of K_n.

connected component C_1 and a blue connected component C_2 such that all of the sets $C_1 \cap C_2$, $(V(K_n) \setminus C_1) \cap C_2$, $C_1 \cap (V(K_n) \setminus C_2)$, and $(V(K_n) \setminus C_1) \cap (V(K_n) \setminus C_2)$ are nonempty.

Proof. Suppose that we have a red component C_1 and a blue component C_2 as in the statement of the lemma. Let $A_1 = C_1 \cap (V(K_n) \setminus C_2)$, $A_2 = (V(K_n) \setminus C_1) \cap C_2$, $A_3 = (V(K_n) \setminus C_1) \cap (V(K_n) \setminus C_2)$, and $A_4 = C_1 \cap C_2$.

Since C_1 and C_2 are red and blue components respectively, all the edges between A_1 and A_2 and between A_3 and A_4 are green. Since K_n is not connected in green, there cannot be any green edges between A_1 and A_3. Therefore, since $A_1 \subseteq C_1$ and $A_3 \cap C_1 = \emptyset$, all the edges between A_1 and A_3 are blue. Similarly, the edges between A_1 and A_4 are all red. Since K_n is not connected in red or green, the edges between A_2 and A_4 are all blue. Since K_n is not connected in blue or green, the edges between A_2 and A_3 are all red. This ensures that the sets A_1, A_2, A_3, and A_4 form the classes of a 4-partite colouring of K_n.

For the converse, suppose that A_1, A_2, A_3, and A_4 form the classes of a 4-partite colouring. Choose $C_1 = A_1 \cup A_4$ and $C_2 = A_2 \cup A_4$ to obtain components as in the statement of the lemma. $\square$

Just like 4-partite colourings are special colourings of K_n, we will need special colourings of $K_{n,n}$ in the proof of Theorem 5.1.1.

Definition 1.6.3. *Let $K_{n,n}$ be a 2-edge-coloured balanced complete bipartite graph with partition classes X and Y. We say that the colouring on $K_{n,n}$ is **split** if it is possible*

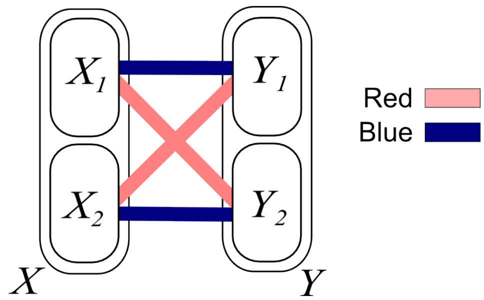

Figure 1.2: A split colouring of $K_{n,n}$.

to partition X into two nonempty sets X_1 and X_2, and Y into two nonempty sets Y_1 and Y_2, such that the following hold.

- *The edges between X_1 and Y_2, and the edges between X_2 and Y_1 are red.*
- *The edges between X_1 and Y_1, and the edges between X_2 and Y_2 are blue.*

The sets X_1, X_2, Y_1, and Y_2 will be called the "classes" of the split colouring.

When dealing with split colourings of $K_{n,n}$ the classes will always be labelled "X_1", "X_2", "Y_1", and "Y_2" with colours between the classes as in the above definition. See Figure 1.2 for an illustration of a split colouring of K_n. The following lemma gives an alternative characterization of split colourings of $K_{n,n}$.

Lemma 1.6.4. *Let $K_{n,n}$ be a 2-edge-coloured balanced complete bipartite graph. The colouring on $K_{n,n}$ is split if and only if none of the following hold.*

(i) *$K_{n,n}$ is connected in some colour.*

(ii) *There is a vertex u such that all the edges containing u have the same colour.*

Proof. Suppose that $K_{n,n}$ is not split and (i) fails to hold. We will show that (ii) holds. Let X and Y be the classes of the bipartition of $K_{n,n}$. Let C be any red component of $K_{n,n}$, $X_1 = X \cap C$, $X_2 = X \setminus C$, $Y_1 = Y \cap C$, and $Y_2 = Y \setminus C$. If all these sets are nonempty, then G is split with classes X_1, X_2, Y_1, and Y_2. To see this note that there cannot be any red edges between X_1 and Y_2, or between X_2 and Y_1 since C is a red component. There cannot be any blue edges between X_1 and Y_1, or between X_2 and Y_2 since $K_{n,n}$ is disconnected in blue.

Assume that one of the sets X_1, X_2, Y_1, or Y_2 is empty. If X_1 is empty, then C is entirely contained in Y and hence consists of a single vertex u, giving rise to case (ii) of the lemma. If X_2 is empty, then note that Y_2 must be nonempty. Indeed, otherwise we would have $C = K_{n,n}$, contradicting our assumption that (i) fails to hold. Let u be any vertex in Y_2. For any v, the edge uv must be blue, since $X \subseteq C$ holds. Thus again (ii) holds. The cases when Y_1 or Y_2 are empty are proved in the same way by symmetry.

For the converse, note that if $K_{n,n}$ is split, then the red components are $X_1 \cup Y_1$ and $X_2 \cup Y_2$, and that the blue components are $X_1 \cup Y_2$ and $X_2 \cup Y_1$. It is clear that neither (i) nor (ii) can hold. $\square$

A simple corollary of Lemma 1.6.4 is that a 2-edge-colouring of $K_{n,n}$ is split if, and only if, neither of the parts of $K_{n,n}$ is contained in a monochromatic connected component.

Chapter 2

Nonnegative k-sums in a set of numbers with nonnegative sum

2.1 Introduction

Suppose that we have a set of numbers $x_1, \ldots, x_n$ satisfying $x_1 + \cdots + x_n \geq 0$. How many subsets $A \subset \{x_1, \ldots, x_n\}$ of order k must satisfy $\sum_{a \in A} a \geq 0$? By choosing $x_1 = n-1$ and $x_2 = \cdots = x_n = -1$ we see that the answer can be at most $\binom{n-1}{k-1}$. Manickam, Miklós, and Singhi conjectured that for $n \geq 4k$ this assignment gives the least possible number of nonnegative k-sums.

Conjecture 1.1.1 (Manickam, Miklós, Singhi, [41, 42])**.** *Suppose that $n \geq 4k$, and we have n real numbers $x_1, \ldots, x_n$ such that $x_1 + \cdots + x_n \geq 0$. At least $\binom{n-1}{k-1}$ subsets $A \subset \{x_1, \ldots, x_n\}$ or order k satisfy $\sum_{a \in A} a \geq 0$*

As mentioned in the introduction, there have been many results establishing the conjecture when n is large compared to k. Manickam and Miklós [41] showed that the conjecture holds when $n \geq (k-1)(k^k + k^2) + k$ holds. Tyomkyn [52] improved this bound to $n \geq k(4e \log k)^k \approx e^{ck \log \log k}$. Recently Alon, Huang, and Sudakov [3] showed that the conjecture holds when $n \geq 33k^2$. Subsequently Frankl [20] gave an alternative proof of the conjecture in a range of the form $n \geq 3k^3/2$. To date, Alon, Huang, and Sudakov's bound of $n \geq 33k^2$ stands as the best known bound for Conjecture 1.1.1.

The aim of this chapter is to improve these bounds by showing that the conjecture holds in a range when n is linear with respect to k.

Theorem 2.1.1. *Suppose that* $n \geq 10^{46}k$*, and we have* n *real numbers* $x_1, \ldots, x_n$ *such that* $x_1 + \cdots + x_n \geq 0$*. At least* $\binom{n-1}{k-1}$ *subsets* $A \subset \{x_1, \ldots, x_n\}$ *of order* k *satisfy* $\sum_{a \in A} a \geq 0$

It is worth noticing at this point that there seem to be connections between the problem and results mentioned so far in this chapter, and the Erdős-Ko-Rado Theorem about intersecting families of sets. A family $\mathcal{A}$ of sets is said to be *intersecting* if any two members of $\mathcal{A}$ intersect. The Erdős-Ko-Rado Theorem [19] says that for $n \geq 2k$, any intersecting family $\mathcal{A}$ of subsets of $[n]$ of order k, must satisfy $|\mathcal{A}| \leq \binom{n-1}{k-1}$. The extremal family of sets in the Erdős-Ko-Rado Theorem is formed by considering the family of all k-sets which contain a particular element of $[n]$. This is exactly the family $\mathcal{A}$ that we obtain from the extremal case of the Manickam-Miklós-Singhi Conjecture if we let the members of $\mathcal{A}$ be the nonnegative k-sums from $x_1, \ldots, x_n$. In addition, many of the methods used to approach Conjecture 1.1.1 are similar to proofs of the Erdős-Ko-Rado Theorem. The method we use to prove Theorem 2.1.1 in this chapter is inspired by Katona's proof of the Erdős-Ko-Rado Theorem in [35].

Suppose that we have a hypergraph $\mathcal{H}$ together with an assignment of real numbers to the vertices of $\mathcal{H}$ given by $f : V(\mathcal{H}) \to \mathbb{R}$. We can extend f to the powerset of $V(\mathcal{H})$ by letting $f(A) = \sum_{v \in A} f(v)$ for every $A \subseteq V(\mathcal{H})$. We say that an edge $e \in E(\mathcal{H})$ is *negative* if $f(e) < 0$, and e is *nonnegative* otherwise. We let $e_f^+(\mathcal{H})$ be the number of nonnegative edges of $\mathcal{H}$. Recall that the degree $d(v)$ of a vertex v in a hypergraph $\mathcal{H}$ is the number of edges containing v. A hypergraph $\mathcal{H}$ is d-regular if every vertex has degree d. The minimum degree of a hypergraph $\mathcal{H}$ is $\delta(\mathcal{H}) = \min_{v \in V(\mathcal{H})} d(v)$.

The following observation is key to our proof of Theorem 2.1.1.

Lemma 2.1.2. *Let* $\mathcal{H}$ *be a* d*-regular* k*-uniform hypergraph on* n *vertices. Suppose that for every* $f : V(\mathcal{H}) \to \mathbb{R}$ *satisfying* $\sum_{x \in V(\mathcal{H})} f(x) \geq 0$ *we have* $e_f^+(\mathcal{H}) \geq d$*. Then for every* $f : V(\mathcal{K}_n^{(k)}) \to \mathbb{R}$ *satisfying* $\sum_{x \in V(\mathcal{K}_n^{(k)})} f(x) \geq 0$ *we have* $e_f^+(\mathcal{K}_n^{(k)}) \geq \binom{n-1}{k-1}$ *(and so Conjecture 1.1.1 holds for this particular* n *and* k*).*

Lemma 2.1.2 is proved by an averaging technique similar to Katona's proof of the Erdős-Ko-Rado Theorem (see Section 2.2). This technique has already appeared in the context of the Manickam-Miklós-Singhi Conjecture in [41] where it was used to prove the conjecture when $n \geq (k-1)(k^k + k^2) + k$. See [36] for a survey of other uses of this method in extremal combinatorics.

Lemma 2.1.2 shows that instead of proving the conjecture about the complete graph $\mathcal{K}_n^{(k)}$, it may be possible to find regular hypergraphs which satisfy the condition in Lemma 2.1.2 and hence deduce the conjecture. This motivates us to make the following definition.

Definition 2.1.3. *A k-uniform hypergraph $\mathcal{H}$ has the **MMS-property** if for every $f : V(\mathcal{H}) \to \mathbb{R}$ satisfying $\sum_{x \in V(\mathcal{H})} f(x) \geq 0$ we have $e^+(\mathcal{H}) \geq \delta(\mathcal{H})$.*

Conjecture 1.1.1 is equivalent to the statement that for $n \geq 4k$ the complete hypergraph on n vertices has the MMS-property. Lemma 2.1.2 shows that in order to prove Conjecture 1.1.1 for particular n and k, it is sufficient to find one regular n-vertex k-uniform hypergraph $\mathcal{H}$ with the MMS-property. This hypergraph $\mathcal{H}$ may be much sparser than the complete hypergraph—allowing for very different proof techniques.

Perhaps the first two candidates one chooses for hypergraphs that may have the MMS-property are matchings and tight cycles. The matching $\mathcal{M}_{t,k}$ is defined as the k-uniform hypergraph consisting of tk vertices and t vertex disjoint edges. Notice that $\mathcal{M}_{t,k}$ is 1-regular. The matching $\mathcal{M}_{t,k}$ always has the MMS-property—indeed we have that $\sum_{e \in E(\mathcal{M}_{t,k})} f(e) = \sum_{x \in \mathcal{M}_{t,k}} f(x) \geq 0$, and so one of the edges of $\mathcal{M}_{t,k}$ is nonnegative. This observation was used in [42] to prove Conjecture 1.1.1 whenever k divides n.

The tight cycle $\mathcal{C}_{n,k}$ is defined as the hypergraph with vertex set $\mathbb{Z}_n$ and edges formed by the intervals $\{i \pmod n, i+1 \pmod n, \ldots, i+k \pmod n\}$ for $i \in \mathbb{Z}_n$. It turns out that the tight cycles do not have the MMS-property when $n \not\equiv 0 \pmod k$. To see this for example when $k = 3$ and $n \equiv 1 \pmod k$, let $f(x) = 50$, 50, 50, -101, 50, 50, -101, 50, 50, $-101 \ldots$ for $x = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$.

An interesting question, which we will return to in Section 2.6 is "which hypergraphs have the MMS-property?"

The main result of this chapter is showing that there exist $k(k-1)^2$-regular k-uniform hypergraphs on n vertices which have the MMS-property, for all $n \geq 10^{46}k$.

Theorem 2.1.4. *For $n \geq 10^{46}k$, there are $k(k-1)^2$-regular k-uniform hypergraphs on n vertices, $\mathcal{H}_{n,k}$, with the property that for every $f : V(\mathcal{H}_{n,k}) \to \mathbb{R}$ satisfying $\sum_{x \in V(\mathcal{H}_{n,k})} f(x) \geq 0$ we have $e^+(\mathcal{H}_{n,k}) \geq k(k-1)^2$.*

Combining Theorem 2.1.4 and Lemma 2.1.2 immediately implies Theorem 2.1.1.

Throughout this chapter, we will use notation from Additive Combinatorics for sumsets $A + B = \{a + b : a \in A, b \in B\}$ and translates $A + x = \{a + x : a \in A\}$.

The structure of this chapter is as follows. In Section 2.2 we prove Lemma 2.1.2. In Section 2.3, we define the graphs $\mathcal{H}_{n,k}$ used in Theorem 2.1.4 and prove some of their basic properties. In Section 2.4, we prove Theorem 2.1.4 with the weaker bound of $n \geq 14k^4$ in order to illustrate the main ideas in the proof of Theorem 2.1.4. In Section 2.5 we prove Theorem 2.1.4. In Section 2.6, we conclude by discussing the techniques used in this chapter and whether they could be used to prove Conjecture 1.1.1 in general.

2.2 Proof of the averaging lemma

Here we prove Lemma 2.1.2.

Proof. Suppose that we have a function $f : \{1, \ldots, n\} \to \mathbb{R}$ satisfying $\sum_{x \in \{1,\ldots,n\}} f(x) \geq 0$. Consider a random permutation σ of $\{1, \ldots, n\}$, chosen uniformly out of all permutations of $\{1, \ldots, n\}$. We define a function $f_\sigma : \{1, \ldots, n\} \to \mathbb{R}$ given by $f_\sigma : x \to f(\sigma(x))$. Clearly $\sum_{x \in \{1,\ldots,n\}} f_\sigma(x) \geq 0$. We will count $\mathbb{E}(e^+_{f_\sigma}(\mathcal{H}))$ in two different ways. For an edge $e \in \mathcal{K}_n^{(k)}$, we have

$$\mathbb{P}(\sigma(e) \in \mathcal{H}) = \frac{e(\mathcal{H})}{\binom{n}{k}} = \frac{d}{\binom{n-1}{k-1}}$$

Therefore we have

$$\mathbb{E}(e^+_{f_\sigma}(\mathcal{H})) = \sum_{\substack{e \in \mathcal{K}_n^{(k)}, \\ f(e) \geq 0}} \mathbb{P}(\sigma(e) \in \mathcal{H}) = e^+(\mathcal{K}_n^{(k)}) \frac{d}{\binom{n-1}{k-1}}$$

However, by the assumption of the lemma, $\mathbb{E}(e^+_{f_\sigma}(\mathcal{H}))$ is at least d. This gives us

$$e^+(\mathcal{K}_n^{(k)}) \geq \binom{n-1}{k-1}.$$

□

2.3 Construction of the hypergraphs $H_{n,k}$

In this section we construct graphs $\mathcal{H}_{n,k}$ which satisfy Theorem 2.1.4. We also prove some basic properties which the graphs $\mathcal{H}_{n,k}$ have.

Define the clockwise interval between a and $b \in \mathbb{Z}_n$ to be $[a, b] = \{a, a+1, \ldots, b\}$. The graph $\mathcal{H}_{n,k}$ has vertex set $\mathbb{Z}_n$. We define k-edges $e(v, i, j)$ as follows:

$$e(v, i, j) = [v, v+i-1] \cup [v+i+j, v+j+k-1]$$

The edges of $\mathcal{H}_{n,k}$ are given by $e(v, i, j)$ for $v \in \mathbb{Z}_n$ and $i, j \in \{1, \ldots, k-1\}$. In other words $\mathcal{H}_{n,k}$ consists of all the double intervals of order k, where the distance between the two intervals is at most $k-1$.

Notice that the graph $\mathcal{H}_{n,k}$ is indeed $k(k-1)^2$ regular.

In order to deal with the graphs $\mathcal{H}_{n,k}$ it will be convenient to assign a particular set $E(v)$ of $O(k^2)$ edges to each vertex v. First, for each vertex v in $\mathcal{H}_{n,k}$ and $i, j \in [1, k-1]$, we will define a set of edges, $E(v, i, j)$. Then $E(v)$ will be a union of the sets $E(v, i, j)$.

The definition of the sets $E(v, i, j)$ is quite tedious. However the sets $E(v, i, j)$ are constructed to satisfy only a few properties. One property that we will need is that for fixed, v, i, j certain intervals can be formed as disjoint unions of edges in $E(v, i, j)$. See Figures 2.1 – 2.4 for illustrations of the precise configurations that we will use. Another property that we will need is that no edge $e \in \mathcal{H}_{n,k}$ is contained in too many of the sets $E(v, i, j)$. See Lemmas 2.3.1 and 2.3.2 for precise statements of these two properties.

Over the next four pages we define the sets $E(v, i, j)$.

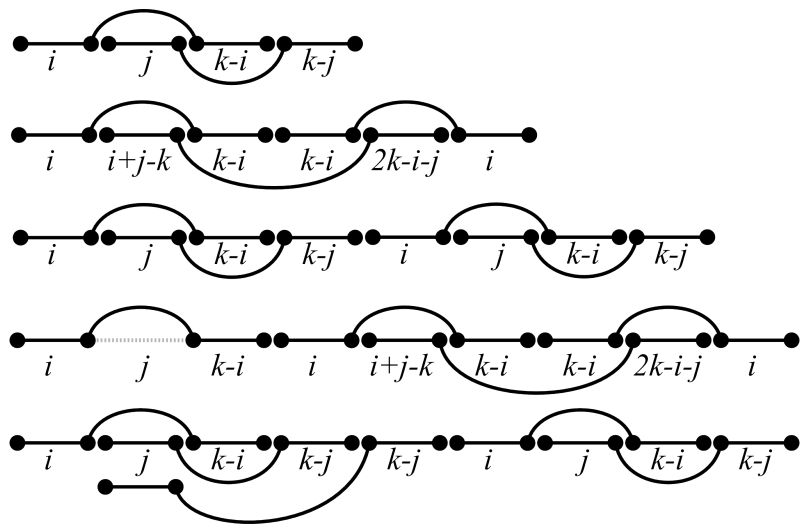

Figure 2.1: The edges in $E(v, i, j)$ when we have $i + j \geq k$ and $i \geq j$.

If $i + j \geq k$ and $i \geq j$, then we let

$$\begin{aligned} E(v,i,j) = \{&e(v,i,j), e(v+k+j, i, i+j-k),\\ &e(v+k+i+j, i+j-k, 2k-2i), e(v+i, j, k-i),\\ &e(v+k+i+2j, k-i, 2k-i-j), e(v+i, j, 2k-i-j),\\ &e(v+3k-j, i, j), e(v+3k-j+i, j, k-i),\\ &e(v+i, i+j-k, 2k-2i), e(v+i+j, k-i, 2k-i-j),\\ &e(v+2k, i, j), e(v+2k+i, j, k-i)\}. \end{aligned}$$

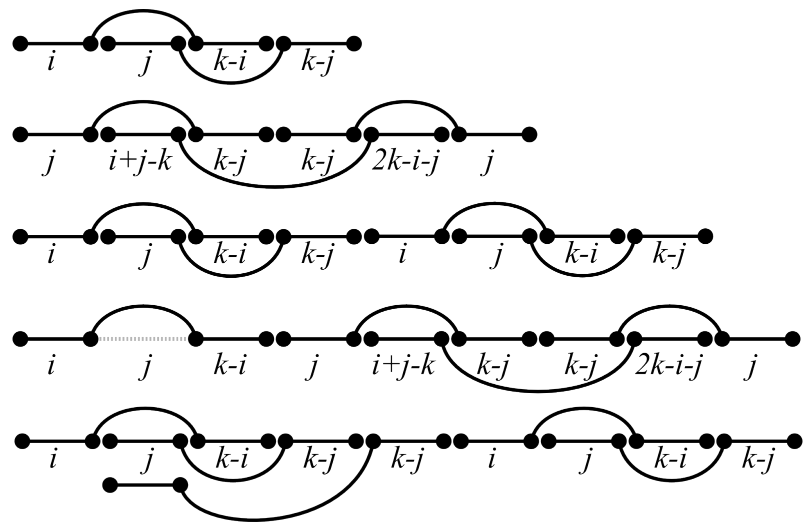

Figure 2.2: The edges in $E(v,i,j)$ when we have $i+j \geq k$ and $j < i$.

If $i+j \geq k$ and $j < i$, then we let

$$\begin{aligned}E(v,i,j) = \{&e(v,i,j), e(v+k+j, j, i+j-k),\\ &e(v+k+2j, i+j-k, 2k-2j), e(v+i, j, k-i),\\ &e(v+k+i+2j, k-j, 2k-i-j),\\ &e(v+i, j, 2k-i-j), e(v+3k-j, i, j),\\ &e(v+3k-j+i, j, k-i), e(v, j, i+j-k),\\ &e(v+j, i+j-k, 2k-2j), e(v+i+j, k-j, 2k-i-j),\\ &e(v+2k, i, j), e(v+2k+i, j, k-i)\}.\end{aligned}$$

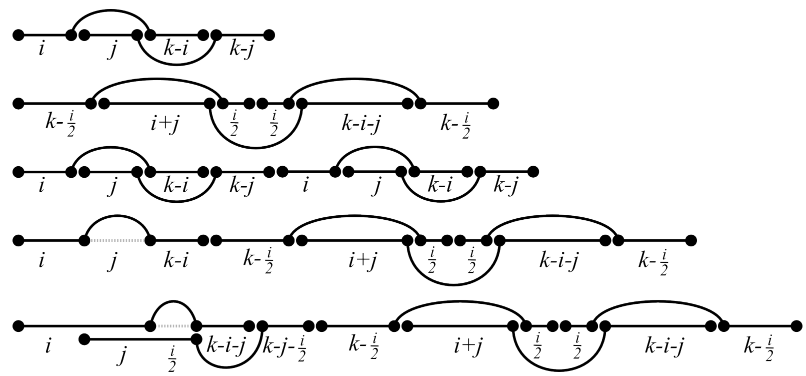

Figure 2.3: The edges in $E(v, i, j)$ when we have $i + j < k$ and i is even.

If $i + j < k$ and i is even, then we let

$$
\begin{aligned}
E(v, i, j) = \{&e(v, i, j), e(v + k + j, k - \frac{i}{2}, i + j), \\
&e(v + 2k + j - \frac{i}{2}, i + j, i), e(v, i + j, \frac{i}{2}), \\
&e(v + 2k + i + 2j, \frac{i}{2}, k - \frac{i}{2}), e(v + i, j + \frac{i}{2}, k - i - j), \\
&e(v + 2k - j, k - \frac{i}{2}, i + j), e(v + 3k - j - \frac{i}{2}, i + j, i), \\
&e(v + 3k + i, \frac{i}{2}, k - \frac{i}{2}), e(v, k - \frac{i}{2}, i + j), \\
&e(v + k - \frac{i}{2}, i + j, i), e(v + k + i + j, \frac{i}{2}, k - i - j), \\
&e(v + i, j, k - i), e(v + 2k, i, j), e(v + 2k + i, j, k - i)\}.
\end{aligned}
$$

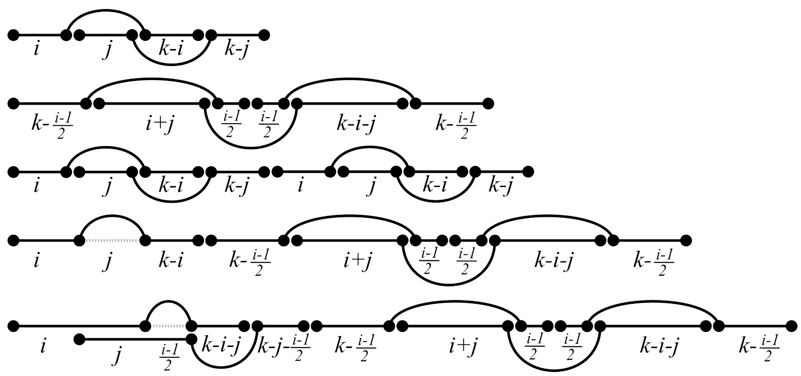

Figure 2.4: The edges in $E(v, i, j)$ when we have $i + j < k$ and i is odd.

If $i + j < k$ and i is odd, then we let

$$\begin{aligned}E(v, i, j) = \{&e(v, i, j), e(v + k + j, k - \frac{i-1}{2}, i + j),\\ &e(v + 2k + j - \frac{i-1}{2}, i + j, i), e(v, i + j, \frac{i-1}{2}),\\ &e(v + 2k + i + 2j, \frac{i-1}{2}, k - \frac{i-1}{2}), e(v + i, j + \frac{i-1}{2}, k - i - j),\\ &e(v + 2k - j, k - \frac{i-1}{2}, i + j), e(v + 3k - j - \frac{i-1}{2}, i + j, i),\\ &e(v + 3k + i, \frac{i-1}{2}, k - \frac{i-1}{2}), e(v, k - \frac{i-1}{2}, i + j),\\ &e(v + k - \frac{i-1}{2}, i + j, i), e(v + k + i + j, \frac{i-1}{2}, k - i - j),\\ &e(v + i, j, k - i), e(v + 2k, i, j), e(v + 2k + i, j, k - i)\}.\end{aligned}$$

We define $E^-(v, i, j)$ to be the set of edges corresponding to edges in $E(v, i, j)$, but going anticlockwise (i.e. $E^-(v, i, j) = \{\{x_1, \ldots, x_k\} : \{v - (x_1 - v), \ldots, v - (x_k - v)\} \in E(v, i, j)\}$). For each vertex v, we let

$$E(v) = \bigcup_{i,j \in [1,k-1]} E(v, i, j) \cup E^-(v, i, j).$$

Notice that from the definition of $E(v, i, j)$, we certainly have $E(v, i, j) \leq 15$ for every $i, j \in [1, k - 1]$, which implies that $|E(v)| \leq 15(k - 1)^2$. Also, since $e(v, i, j) \in$

$E(v)$ for every $i, j \in [1, k-1]$, we have that $E(v) \geq (k-1)^2$. Therefore, we have $|E(v)| = \Theta(k^2)$.

There are only two features of the sets $E(v, i, j)$ that will be needed in the proof of Theorem 2.1.4. One is that sequences of edges similar to the ones in Figures 2.1 – 2.4 exist in $E(v, i, j)$. This allows us to prove the following lemma.

Lemma 2.3.1. *Suppose that $i, j \in [1, k-1]$ and all the edges in $E(v, i, j)$ are negative. The following hold.*

(i) $f([v, v + 2k - 1]) < 0$.

(ii) $f([v, v + 3k - 1]) < 0$.

(iii) $f([v, v + 4k - 1]) < 0$.

(iv) $f([v + i, v + i + j - 1]) < 0 \implies f([v, v + 4k + j - 1]) < 0$.

(v) $f([v + i, v + i + j - 1]) \geq 0 \implies f([v, v + 5k - j - 1]) < 0$.

Proof. Figures 2.1 – 2.4 illustrates the constructions that are used in the proof of this lemma.

(i) This follows from the fact that $e(v, i, j), e(v+i, j, k-i) \in E(v, i, j)$ and $e(v, i, j) \cup e(v + i, j, k - i) = [v, v + 2k - 1]$.

(ii) For $i + j \geq k$ and $i \geq j$, this follows from the fact that $e(v, i, i+j-k), e(v+i, i+j-k, 2k-2i), e(v+i+j, k-i, 2k-i-j) \in E(v, i, j)$ and $e(v, i, i+j-k) \cup e(v+i, i+j-k, 2k-2i) \cup e(v+i+j, k-i, 2k-i-j) = [v, v+3k-1]$. The other cases are similar.

(iii) This follows from the fact that $e(v, i, j), e(v+i, j, k-i), e(v+2k, i, j), e(v+2k+i, j, k-i) \in E(v, i, j)$ and $e(v, i, j) \cup e(v+i, j, k-i) \cup e(v+2k, i, j) \cup e(v+2k+i, j, k-i) = [v, v+4k-1]$.

(iv) For $i + j \geq k$ and $i \geq j$, this follows from the fact that $e(v, i, j), e(v+k+j, i, i+j-k), e(v+k+i+j, i+j-k, 2k-2i), e(v+k+i+2j, k-i, 2k-i-j) \in E(v, i, j)$

and $e(v,i,j) \cup e(v+k+j,i,i+j-k) \cup e(v+k+i+j,i+j-k,2k-2i) \cup e(v+k+i+2j,k-i,2k-i-j) \cup [v+i,v+i+j-1] = [v,v+4k+j-1]$. The other cases are similar.

(v) For $i+j \geq k$ and $i \geq j$, this follows from the fact that $e(v,i,j), e(v+i,j,k-i), e(v+i,j,2k-i-j), e(v+3k-j,i,j), e(v+3k-j+i,j,k-i) \in E(v,i,j)$ and also $e(v,i,j) \cup e(v+i,j,k-i) \cup e(v+i,j,2k-i-j) \cup e(v+3k-j,i,j) \cup e(v+3k-j+i,j,k-i) = [v,v+5k-j-1]$ and $e(v+i,j,k-i) \cap e(v+i,j,2k-i-j) = [v+i,v+i+j-1]$. The other cases are similar.

□

The other feature of the sets $E(v,i,j)$ that we need is that no edge is contained in too many of the sets $E(v,i,j)$. This is quantified in the following lemma. For the duration of this chapter, we fix the constant $C_1 = 110$.

Lemma 2.3.2. *Let e be an edge in $\mathcal{H}_{n,k}$. The edge e is contained in at most C_1 of the sets $E(v,i,j) \cup E^-(v,i,j)$ for $v \in V(\mathcal{H}_{n,k})$, and $i,j \in [1,k-1]$.*

Proof. Notice that there are 55 edges mentioned in the definition of $E(v,i,j)$. For $t = 1,\ldots,55$, let $F^t(v,i,j)$ be the singleton containing the tth edge in the definition of $E(v,i,j)$, i.e. $F^1(v,i,j) = \{e(v,i,j)\}$, $F^2(v,i,j) = \{e(v+k+j,i,i+j-k)\}, \ldots, F^{55}(v,i,j) = \{e(v+2k+i,j,k-i)\}$. This definition is purely formal—for certain i and j, it is possible that an edge in $F^t(v,i,j)$ is not an edge of $\mathcal{H}_{n,k}$ (for example $F^3(v,i,j)$ contains the edge $e(v+k+i+j,i+j-k,2k-2i)$ which is not an edge of $\mathcal{H}_{n,k}$ if $2k-2i \geq k$). Similarly it is possible for $F^t(v,i,j)$ to be empty for certain i and j—for example $F^{52}(v,i,j)$ should contain $e(v+k+i+j,\frac{i-1}{2},k-i-j)$ which is not defined when i is even.

Clearly $E(v,i,j) \subseteq \bigcup_{t=1}^{55} F^t(v,i,j)$ holds. Also, it is straightforward to check that for fixed t, the sets $F^t(v,i,j)$ are all disjoint for $v \in V(\mathcal{H}_{n,k})$, and $i,j \in [1,k-1]$. Indeed for fixed t, if we have $e(u,a,b) \in F^t(v,i,j)$, then it is always possible to work out v, i, and j uniquely in terms of u, a, and b. These two facts, together with the Pigeonhole Principle imply that the edge e can be contained in at most 55 of the sets $E(v,i,j)$ for $v \in V(\mathcal{H}_{n,k})$, and $i,j \in [1,k]$. The lemma follows, since $C_1 \geq 2 \cdot 55 = 110$. □

A useful corollary of Lemma 2.3.2 is that an edge e can be contained in at most 110 of the sets $E(v)$ for $v \in V(\mathcal{H}_{n,k})$.

2.4 Hypergraphs of order $O(k^4)$ which have the MMS-property.

In this section we prove Theorem 2.1.4, with a weaker bound of $n \geq 14k^4$. This proof has many of the same ideas as the proof of Theorem 2.1.4, but is much shorter. We therefore present it in order to illustrate the techniques that we will use in proving Theorem 2.1.4, and hopefully aid the reader to understand that theorem.

Theorem 2.4.1. *For $n \geq 14k^4$, and every function $f : V(\mathcal{H}_{n,k}) \to \mathbb{R}$ which satisfies $\sum_{x \in V(\mathcal{H}_{n,k})} f(x) \geq 0$ we have $e_f^+(\mathcal{H}_{n,k}) \geq k(k-1)^2$.*

Proof. Suppose for the sake of contradiction that we have a function $f : V(\mathcal{H}_{n,k}) \to \mathbb{R}$ satisfying $\sum_{x \in V(\mathcal{H}_{n,k})} f(x) \geq 0$ such that we have $e_f^+(\mathcal{H}_{n,k}) < k(k-1)^2$.

The proof of the theorem rests on two claims. The first of these says that any sufficiently small interval I in $\mathbb{Z}_n$ is contained in a negative interval of almost the same order as I.

Claim 2.4.2. *Let I be an interval in $\mathbb{Z}_n$ such that $|I| \leq n - 2k$. Then there is an interval $J = [j_1, j_t]$ which satisfies the following:*

(i) $|J| \leq |I| + 2k$.

(ii) $I \subseteq J$.

(iii) $f(J) < 0$.

Proof. Without loss of generality, we may assume that I is the interval $[2k, 2km + l]$ for some $l \in [0, 2k-1]$ and $m \leq \frac{n}{2k} - 1$. First we will exhibit $2k(k-1)^2$ sets of vertex-disjoint edges covering I.

For $v \in \{0 \dots 2k-1\}$, $i, j \in \{1, \dots, k-1\}$ we let

$$\mathcal{D}(v,i,j) = \bigcup_{t=0}^{m} \Big(e(v+2tk, i, j) \cup e(v+2tk+i, j, k-i) \Big)$$

Notice that an edge $e(u,a,b)$ is contained only in the sets $\mathcal{D}(u \pmod{2k}, a, b)$ and $\mathcal{D}(u-k+b \pmod{2k}, k-b, a)$. Therefore, since there are at less than $k(k-1)^2$ nonnegative edges in $\mathcal{H}_{n,k}$, there are some v_0,i_0 and j_0 for which the set $\mathcal{D}(v_0,i_0,j_0)$ contains only negative edges. Letting $J = \bigcup \mathcal{D}(v_0,i_0,j_0) = [v_0, v_0+2k(m+1)]$ implies the claim. $\square$

The second claim that we need shows that any sufficiently large interval which does not contain nonnegative edges in $\mathcal{H}_{n,d}$ must be negative.

Claim 2.4.3. *Let $I = [i_1, i_m]$ be an interval in $\mathbb{Z}_n$ which satisfies the following:*

(i) $|I| \geq 12k$.

(ii) *There are no nonnegative edges of $\mathcal{H}_{n,k}$ contained in I.*

We have that $f(I) < 0$.

Proof. Let $R_0 = \{v \in I : f([0, v-1]) < 0\}$ and $R_m = \{v \in I : f([v,m]) < 0\}$. Let $Q^- = \{i \in [1,k-1] : f([1,i]) < 0\}$ and $Q^+ = \{k-i \in [1,k-1] : f([1,i]) \geq 0\}$.

Since I contains only negative edges, parts (iv) and (v) of Lemma 2.3.1 imply that we have that $(Q^- \cup Q^+) + 4k \subseteq R_0$. Part (iii) of Lemma 2.3.1 implies that $4k \in R_0$. Then, parts (i) and (ii) of Lemma 2.3.1 imply that $(Q^- \cup Q^+ \cup \{0\}) + tk \subseteq R_0$ for any $t \in \left\{6, 7, \dots, \left\lfloor \frac{m}{k} \right\rfloor - 1\right\}$. This implies that we have $R_0 \cap [u, u+k-1] \geq |Q^- \cup Q^+ \cup \{0\}|$ for any $u \in [6k, m-k-1]$.

Notice that $Q^- \cup Q^+$ contains at least one element from each of the sets $\{1, k-1\}, \dots, \left\{ \left\lfloor \frac{k}{2} \right\rfloor, \left\lceil \frac{k}{2} \right\rceil \right\}$. This implies that for every $u \in \{6k, \dots, m-k-1\}$ we have

$$|R_0 \cap [u, u+k-1]| \geq |Q^- \cup Q^+ \cup \{0\}| \geq \left\lfloor \frac{k}{2} \right\rfloor + 1 > \frac{k}{2}.$$

Similarly we obtain $|R_m \cap [u, u+k-1]| > \frac{k}{2}$ for every $u \in \{k, \ldots, m-7k\}$. By choosing $u = 6k$, we have that $|R_0 \cap [6k, 7k-1]|, |R_m \cap [6k, 7k-1]| > \frac{k}{2}$, and hence there exists some $i \in [6k, 7k-1]$ such that $i \in R_0, R_m$ hold. This gives us $f([0,m]) = f([0,i]) + f([i+1,m]) < 0$, proving the claim. $\square$

We now prove the theorem. Suppose that every interval of order $14k$ in $\mathcal{H}_{n,k}$ contains a nonnegative edge. Since there are at least $\frac{n}{14k} \geq k^3$ such disjoint intervals in $\mathcal{H}_{n,k}$, we have at least k^3 nonnegative edges in $\mathcal{H}_{n,k}$, contradicting our initial assumption that $e_f^+(\mathcal{H}_{n,k}) < k(k-1)^2$.

Suppose that there is an interval I of order $14k$ in $\mathcal{H}_{n,k}$ which contains only negative edges. Applying Claim 2.4.2 to $V(\mathcal{H}_{n,k} \setminus I)$ we obtain an interval $J \subseteq I$ such that $f(V(\mathcal{H}_{n,k}) \setminus J) < 0$ and $|J| \geq 12k$. Applying Claim 2.4.3 to J we obtain that $f(J) < 0$. Therefore, we have $f(V(\mathcal{H}_{n,k})) = f(J) + f(V(\mathcal{H}_{n,k}) \setminus J) < 0$ contradicting the assumption that $f(V(\mathcal{H}_{n,k})) \geq 0$ in the theorem $\square$

It is not hard to see that Claim 2.4.3 would still be true if we allowed I to contain a small number of nonnegative edges. The proof of Theorem 2.1.4 is similar to the proof of Theorem 2.4.1 since it also consists of two main claims which are analogues of Claims 2.4.2 and 2.4.3. However the analogue of Claim 2.4.3 is much stronger since it allows for $O(k^3)$ nonnegative edges to be contained in I. This is the main improvement in the proof of Theorem 2.4.1 which is needed to obtain the linear bound which we have in Theorem 2.1.4.

2.5 Proof of Theorem 2.1.4

In this section we use ideas from Sections 2.3 and 2.4 in order to Theorem 2.1.4.

Proof of Theorem 2.1.4. For convenience, we fix the following constants for the duration of the proof.

$$\begin{array}{ll} C_0 = 10^{46} & \epsilon_0 = 10^{-9} \\ C_1 = 110 & \epsilon_1 = 10^{-18} \\ C_2 = 10^{16} & \epsilon_2 = 10^{-6} \\ C_3 = 28 & \epsilon_3 = 10^{-2} \\ & \epsilon_4 = 0.1 \\ & \epsilon_5 = 0.25 \end{array}$$

Let $n \geq C_0 k$, and let $\mathcal{H}_{n,k}$ be the hypergraph defined in Section 2.3. Recall that for any vertex $v \in V(\mathcal{H}_{n,k})$, we have $|E(v)| = \Theta(k^2)$.

Definition 2.5.1. *We say that a vertex v in $\mathcal{H}_{n,d}$ is* ***bad*** *if at least $\epsilon_0 k^2$ of the edges in $E(v)$ are nonnegative and* ***good*** *otherwise.*

Let $G_{\mathcal{H}}$ be the set of good vertices in $\mathcal{H}_{n,k}$.

Suppose that we have a function $f : V(\mathcal{H}_{n,k}) \to \mathbb{R}$ such that we have $e_f^+(\mathcal{H}_{n,k}) < k(k-1)^2$. We will show that $f(V(\mathcal{H}_{n,k})) < 0$ holds. The proof of the theorem consists of the following two claims.

Claim 2.5.2. *Let I be an interval in $\mathbb{Z}_n$ such that $|I| \leq n - 4C_2 k$. There is an interval $J = [j_1, j_t]$ which satisfies the following:*

(i) *$|J| \leq |I| + 4C_2 k$.*

(ii) *$I \subseteq J$.*

(iii) *Both $j_1 - 1$ and $j_t + 1$ are good.*

(iv) *$f(J) < 0$.*

Claim 2.5.3. *Let $I = [i_1, i_m]$ be an interval in $\mathbb{Z}_n$ which satisfies the following:*

(i) *$C_3 k \leq |I| \leq (C_3 + 4C_2)k$.*

(ii) *Both i_1 and i_m are good.*

(iii) *Every subinterval of I of order k, contains at most $\epsilon_1 k$ bad vertices.*

We have that $f(I) < 0$.

Once we have these two claims, the theorem follows easily:

First suppose that no intervals in $\mathbb{Z}_n$ of order $(C_3 + 4C_2)k$ satisfies condition (iii) of Claim 2.5.3. This implies that there are at least $\epsilon_1 C_0 k/(C_3 + 4C_2)$ bad vertices in $\mathcal{H}_{n,k}$. Then Claim 2.3.2 together with the definition of "bad" implies that there are at least $\epsilon_0\epsilon_1 C_0 k^3/C_1(C_3+4C_2)$ nonnegative edges in $\mathcal{H}_{n,k}$. However, since $\epsilon_0\epsilon_1 C_0/C_1(C_3+4C_2) \geq 1$, this contradicts our assumption that $e^+_f(\mathcal{H}_{n,k}) < k(k-1)^2$.

Now, suppose that there is an interval I of order $(C_3 + 4C_2)k$ which satisfies condition (iii) of Claim 2.5.3. Notice that all subintervals of I will also satisfy condition (iii) of Claim 2.5.3. Applying Claim 2.5.2 to $V(\mathcal{H}_{n,k})\setminus I$ gives an interval $J \subseteq I$ which satisfies all the conditions of Claim 2.5.3 and also $f(V(\mathcal{H}_{n,k})\setminus J) < 0$. Applying Claim 2.5.3 to J implies that we also have $f(J) < 0$. We have $\sum_{v\in\mathcal{H}_{n,k}} f(v) = f(V(\mathcal{H}_{n,k})\setminus J)+f(J) < 0$, contradicting our initial assumption and proving the theorem.

It remains to prove Claims 2.5.2 and 2.5.3.

Proof of Claim 2.5.2. Without loss of generality, we may assume that I is the interval $[0, 2km + l]$ for some $l \in [0, 2k - 1]$ and $m < \frac{n}{2k} - 2C_2$. We partition $[1, 2k]$ into two sets as follows.

Definition 2.5.4. *For $r \in [1, 2k]$ we say that r is **unblocked** if for every $t \in [-C_2, m+C_2]$, there are some $i, j \in [1, k-1]$ such that both of the edges $e(2tk + r, i, j)$ and $e(2tk + r + i, j, k-i)$ are negative. We say that r is **blocked** otherwise.*

Notice that if r is unblocked, then for every $t_1 \in [-C_2, 0]$ and $t_2 \in [m, m + C_2]$ we have that $f([2t_1k + r, 2t_2k + r - 1]) < 0$. Therefore the claim holds unless either $2t_1k + r - 1$ or $2t_2k + r$ is bad. Therefore, for each r which is unblocked, we can assume that all the vertices in either $\{r - 1 - 2kC_2, r - 1 - 2k(C_2 - 1), \ldots, r - 1\}$ or $\{r + 2km, r + 2k(m + 1), \ldots, r + 2k(m + C_2)\}$ are bad.

To each $r \in [1, 2k]$, we assign a set of nonnegative edges, $P(r)$, as follows:

- If r is blocked, then there is some $t_r \in [-C_2, m + C_2]$, such that for every $i, j \in [1, k-1]$ one of the edges $e(2t_rk+r, i, j)$ or $e(2t_rk+r+i, j, k-i)$ is nonnegative. We let $P(r)$ be the set of these edges. Notice that this ensures that $|P(r)| \geq (k-1)^2$.

Also, note that for fixed a,b,c the $P(r)$ can contain at most one edge of the form $e(a+2tk,b,c)$ for any $t \in [-C_2, m+C_2]$.

- If r is unblocked we know that all the vertices in either $\{r-1-2kC_2, r-1-2k(C_2-1), \ldots, r-1\}$ or $\{r+2km, r+2k(m+1), \ldots, r+2k(m+C_2)\}$ are bad. Let $P(r)$ be the set of nonnegative edges in $E(r-1-2kC_2) \cup E(r-1-2k(C_2-1)) \cup \cdots \cup E(r-1) \cup E(r+2km) \cup E(r+2k(m+1)) \cup \cdots \cup E(r+2k(m+C_2))$. Since at least C_2 of these vertices are bad, Lemma 2.3.2 together with the Pigeonhole Principle implies that $|P(r)| \geq \frac{C_2\epsilon_0}{C_1}k^2$.

Notice that an edge e can be in at most 2 of the sets $P(r)$ for r blocked. This is because it can be in at most one such set as an edge of the form "$e(tk+r,i,j)$" and in at most one such set and as an edge of the form "$e(tk+r+i,j,k-i)$". Therefore we have:

$$\left|\bigcup_{r \text{ blocked}} P(r)\right| \geq \sum_{r \text{ blocked}} \frac{1}{2}(k-1)^2 \tag{2.1}$$

Lemma 2.3.2 implies that an edge e can be in at most C_1 of the sets $P(r)$ for r unblocked. Therefore we have:

$$\left|\bigcup_{r \text{ unblocked}} P(r)\right| \geq \sum_{r \text{ unblocked}} \frac{C_2\epsilon_0}{(C_1)^2}k^2 \tag{2.2}$$

We claim that for any $s \in [1, 2k]$, we have

$$\left|\left(\bigcup_{t\in[-C_2,m+C_2]} E(s+2tk)\right) \cap \left(\bigcup_{r \text{ blocked}} P(r)\right)\right| \leq 2|E(s)|. \tag{2.3}$$

Indeed, otherwise the Pigeonhole Principle implies that for some $r \in [1, 2k]$, t_1, t_2, $t_3 \in [-C_2, m+C_2]$, and $i, j \in [1, k-1]$ we have three distinct edges $e(r+2t_1k, i, j)$, $e(r+2t_2k, i, j)$, and $e(r+2t_3k, i, j)$ which are are all contained in $\left(\bigcup_{t\in[-C_2,m+C_2]} E(s+2tk)\right) \cap \left(\bigcup_{r \text{ blocked}} P(r)\right)$. This means that there are some r_1, r_2, and $r_3 \in [1, 2k]$ which are blocked, such that $e(r+2t_lk, i, j) \in P(r_l)$ holds for $l = 1$, 2 and 3. Since each r_l is blocked, all the edges in $P(r_l)$ are of the form $e(2t'k+r_l, i', j')$ or $e(2t'k+r_l+i', j', k-i')$

for some $t' \in [-C_2, m+C_2]$ and $i', j' \in [1, k-1]$. This, together with $e(r+2t_l k, i, j) \in P(r_l)$, implies that we have $r_1, r_2, r_3 \in \{r, r-k+j\}$. This means that for some distinct $l, l' \in \{1,2,3\}$, we have $r_l = r_{l'}$, which means that both $e(r+2t_l k, i, j)$ and $e(r+2t_{l'} k, i, j)$ are contained in $P(r_l)$. However, this contradicts our definition of $P(r_l)$ for r_l blocked which allowed only one edge of the form $e(r+2tk, i, j)$ to be in $P(r_l)$ for fixed r, i and j. This shows that (2.3) holds for all $s \in [1, 2k]$.

Recall that for all vertices s we have $|E(s)| \le C_1 k^2$. This, together with (2.3) implies that we have

$$\begin{aligned}
\left|\left(\bigcup_{s \text{ unblocked}} P(s)\right) \cap \left(\bigcup_{r \text{ blocked}} P(r)\right)\right| &\le \left|\left(\bigcup_{\substack{s \text{ unblocked,} \\ t \in [-C_2, m+C_2]}} E(s+2tk)\right) \cap \left(\bigcup_{r \text{ blocked}} P(r)\right)\right| \\
&\le \sum_{s \text{ unblocked}} 2|E(s)| \\
&\le \sum_{s \text{ unblocked}} 2C_1 k^2. \qquad (2.4)
\end{aligned}$$

Putting (2.1), (2.2), and (2.4) together, we obtain:

$$\begin{aligned}
e_f^+(\mathcal{H}_{n,k}) &\ge \left|\bigcup_{r \text{ blocked}} P(r)\right| + \left|\bigcup_{r \text{ unblocked}} P(r)\right| - \left|\left(\bigcup_{s \text{ unblocked}} P(s)\right) \cap \left(\bigcup_{r \text{ blocked}} P(r)\right)\right| \\
&\ge \sum_{r \text{ blocked}} \frac{1}{2}(k-1)^2 + \sum_{r \text{ unblocked}} \frac{C_2 \epsilon_0}{(C_1)^2} k^2 - \sum_{s \text{ unblocked}} 2C_1 k^2 \\
&\ge \sum_{r \text{ blocked}} \frac{1}{2}(k-1)^2 + \sum_{r \text{ unblocked}} \frac{1}{2} k^2 \\
&\ge k(k-1)^2. \qquad (2.5)
\end{aligned}$$

The second last inequality follows from $\frac{C_2 \epsilon_0}{(C_1)^2} - 2C_1 \ge \frac{1}{2}$. The last inequality follows from the fact that "the number of blocked vertices" + "the number of unblocked vertices" = $2k$. However (2.5) contradicts the assumption that there are less than $k(k-1)^2$ nonnegative edges in $\mathcal{H}_{n,k}$, proving the claim. $\square$

It remains to prove Claim 2.5.3.

Proof of Claim 2.5.3. Without loss of generality, we can assume that $I = [0, m]$ for some $m \leq (C_3 + 4C_2)k$.

Recall that we are using notation from additive combinatorics for sumsets and translates. Except where otherwise stated, sumsets will lie in $\mathbb{Z}$. For a set $A \subseteq \mathbb{Z}$, define

$$A \bmod (k) = \{b \in [0, k-1] : b \equiv a \bmod (k) \text{ for some } a \in A\}.$$

For each vertex v, we define a set of vertices $R(v)$ contained in I.

$$R(v) = \{u \in [v+1, m] : f([v, u-1]) < 0 \text{ and } u \text{ is good.}\}$$

$R(v)$ has the following basic properties.

Claim 2.5.5. *The following hold.*

(i) *If $u > v$ and $u \in R(v)$, we have $R(u) \subseteq R(v)$.*

(ii) *Suppose that $t \geq 2$ and we have a set $X \subseteq R(v) \cap [w, w+2k-1]$, for some vertex w. There is a subset $X' \subseteq X$, such that we have $|X'| \geq |X| - 2\epsilon_1 kt$ and $X' + t'k \subseteq R(v)$ for every $t' \in \{2, \ldots, t\}$.*

(iii) *Suppose that we have $X \subseteq [0, 2k-1]$ such that $X + t_0 k \subseteq R(0)$ for some t_0. There is a subset $X' \subseteq X \bmod (k)$, such that $X' + (t_0+3)k \subseteq R(0)$ and $|X'| \geq |X| - 6\epsilon_1 k$.*

(iv) *Suppose that we have $X \subseteq [w, w+k-1] \cap R(0)$ for some w. Then for any $v \geq w + 2k$, we have we have $|R(0) \cap [v, v+k-1]| \geq |X| - 2\epsilon_1 (v - w + 1)k$.*

Proof. (i) This part is immediate from the definition of $R(v)$.

(ii) First, we deal with the case when $t = 2$ or 3. The general case will follow by induction.

Suppose that we have $x \in X$. Since x is good, Lemma 2.3.2 implies that there are at most $\epsilon_0 C_1 k^2$ pairs i, j for which $E(x, i, j)$ contains a nonnegative edge. Therefore, since $\epsilon_0 C_1 < 1$, there must be at least one pair i_0, j_0 for which all the edges in $E(x, i_0, j_0)$ are nonnegative. Combining this with parts (i) and (ii) of Lemma 2.3.1 implies that we have

$$f([v, x+2k-1]), f([v, x+3k-1]) < 0. \tag{2.6}$$

If $t = 2$ we let $X' = X \cap (G_{\mathcal{H}} - 2k)$. The identity 2.6 implies that $X' + 2k \subseteq R(v)$. By condition (iii) of Claim 2.5.3, we know that there are at most $2\epsilon_1 k$ bad vertices in $[w+2k, w+4k-1]$, which implies that $|X'| \geq |X| - 2\epsilon_1 k$.

Similarly, if $t = 3$ we let $X' = X \cap (G_{\mathcal{H}} - 2k) \cap (G_{\mathcal{H}} - 3k)$. The identity 2.6 implies that $X' + 2k, X' + 3k \subseteq R(v)$. By condition (iii) of Claim 2.5.3, we know that there are at most $3\epsilon_1 k$ bad vertices in $[w+2k, w+5k-1]$, which implies that $|X'| \geq |X| - 3\epsilon_1 k$.

Suppose that the claim holds for $t = t_0$ for some $t_0 \geq 3$. We will show that it holds for $t = t_0 + 1$. We know that there is a set $X' \subseteq X + t_0 k$, such that we have $|X'| \geq |X| - \epsilon_1 k t_0$ and $X' + t'k \subseteq R(v)$ for $t' = 2, \ldots, t_0$. Applying the $t = 2$ part of this claim to $X' + t_0 k$ we obtain a set $X'' \subseteq X'$ such that $|X''| \geq |X'| - \epsilon_1 k \geq |X| - \epsilon_1 k(t_0 + 1)$ and also $X'' + (t_0 + 1)k \subseteq R(v)$. This proves the claim by induction.

(iii) Apply part (i) to $X + t_0$ with $t = 3$ to obtain a set X' with $|X'| \geq |X| - 3\epsilon_1 k$ and $X' + t_0 k + \{2k, 3k\} \subseteq R(0)$. Let $X'' = X' \bmod (k)$ to obtain a set satisfying $X'' \subseteq X \bmod (k)$ and $|X''| \geq |X \bmod (k)| - 3\epsilon_1 k$. We have that $X'' + t_0 + 3k = (X' \cap [0, k-1] + t_0 + 3k) \cup (X' \cap [k, 2k-1] + t_0 + 2k) \subseteq X' + t_0 + \{2k, 3k\} \subseteq R(0)$.

(iv) Apply part (i) to X with $t = \left\lfloor \frac{v-w}{k} \right\rfloor + 1$ to obtain a set X' with $|X'| \geq |X| - \epsilon_1 \left(\left\lfloor \frac{v-w}{k} \right\rfloor + 1 \right) k$ and $X' + t'k \subseteq R(0)$ for any $t' = 2, \ldots, \left(\left\lfloor \frac{v-w}{k} \right\rfloor + 1 \right) k$. For any $x \in X'$, either $x + \left\lfloor \frac{v-w}{k} \right\rfloor k$ or $x + \left(\left\lfloor \frac{v-w}{k} \right\rfloor + 1 \right) k$ is in $[v, v+k-1] \cap R_0$, which implies that $|R(0) \cap [v, v+k-1]| \geq |X'| \geq |X| - \epsilon_1 (v - w + 1)k$.

□

To every vertex $v \in I$ and $\epsilon > 0$, we assign sets $Q_\epsilon^+(v)$, $Q_\epsilon^-(v)$, $Q_\epsilon(v) \subseteq [1, k-1]$ as follows.

$$
\begin{aligned}
Q^-_\epsilon(v) &= \{j \in [1, k-1] : f([v+i, v+i+j-1]) < 0 \\
&\qquad\qquad\qquad\qquad \text{for at least } \epsilon k \text{ numbers } i \in [1, k-1]\} \\
Q^+_\epsilon(v) &= \{k-j \in [1, k-1] : f([v+i, v+i+j-1]) \geq 0 \\
&\qquad\qquad\qquad\qquad \text{for at least } \epsilon k \text{ numbers } i \in [1, k-1]\} \\
Q_\epsilon(v) &= Q^-_\epsilon(v) \cup Q^+_\epsilon(v) \cup \{0\}.
\end{aligned}
$$

$Q_\epsilon(v)$ has the following basic properties.

Claim 2.5.6. *The following hold.*

(i) *For any $r \in [0, k]$, we have $Q_{2\epsilon}(v) \subseteq Q_\epsilon(v-r) \cup Q_\epsilon(v-r+k)$.*

(ii) *For $\epsilon \leq \frac{1}{2}$, $x \in [1, k-1]$, and $v \in I$ either x or $k-x$ is in $Q_\epsilon(v)$.*

(iii) *For $\epsilon \leq \frac{1}{2}$ and $v \in I$, we have $|Q_\epsilon(v)| \geq \frac{1}{2}k$.*

Proof. If $j \in Q^-_{2\epsilon}(v)$, then there are at least $2\epsilon k$ numbers $i \in [1, k-1]$ for which $f([v+i, v+i+j-1]) < 0$. For every $r \in [0, k]$ the Pigeonhole Principle implies that there must either be at least ϵk numbers $i \in [1, k-1]$ for which $f([v-r+i, v-r+i+j-1]) < 0$ or at least ϵk numbers $i \in [1, k-1]$ for which $f([v-r+k+i, v-r+k+i+j-1]) < 0$. Therefore we have $Q^-_{2\epsilon}(v) \subseteq Q^-_\epsilon(v-r) \cup Q^-_\epsilon(v-r+k)$. Similarly we obtain $Q^+_{2\epsilon}(v) \subseteq Q^+_\epsilon(v-r) \cup Q^+_\epsilon(v-r+k)$ which implies part (i).

Part (ii) is immediate from the definition of $Q_\epsilon(v)$. Part (iii) follows from (ii). $\square$

The following claim shows that for a good vertex v, there is a certain translate of $Q_{\epsilon_5}(v)$ which will nearly be contained in $R(v)$.

Claim 2.5.7. *For any good vertex v satisfying $0 \leq v \leq m-5k$, there is a $Q' \subseteq Q_{\epsilon_5}(v)$ such that $|Q'| \geq |Q_{\epsilon_5}(v)| - \epsilon_2 k$ and we have*

$$Q' + 4k + v \subseteq R(v).$$

Proof. Let $T \subseteq [1, k-1]$ be the set of $j \in [1, k-1]$ for which there are at least $\epsilon_5 k$ numbers $i \in [1, k-1]$ such that $E(v, i, j)$ contains a nonnegative edge. We have at least $|T|\epsilon_5 k$ pairs $i, j \in [1, k-1]$ for which $E(v, i, j)$ contains a nonnegative edge. Since v is good, Lemma 2.3.2 implies that at most $\epsilon_0 C_1 k^2$ of the sets $E(v, i, j)$ contain nonnegative edges for $i, j \in [1, k-1]$. Therefore, we have $|T|\epsilon_5 k \leq \epsilon_0 C_1 k^2$. We define the set Q' as

$$Q' = \big((Q^-_{\epsilon_5}(v) \setminus T) \cup (Q^+_{\epsilon_5}(v) \setminus T) \cup \{0\}\big) \cap (G_{\mathcal{H}} - 4k).$$

First we prove $Q' + 4k + v \subseteq R(v)$. Suppose that we have $j \in Q^-_{\epsilon_5}(v) \setminus T$. From the definition of T, there are at at more than $k - 1 - \epsilon_5 k$ numbers $i \in [1, k-1]$ such that all the edges in $E(v, i, j)$ are negative. From the definition of $Q^-_{\epsilon_5}(v)$, there are at least $\epsilon_5 k$ numbers $i \in [1, k-1]$ such that $[v+i, v+i+j-1]$ is negative. Therefore, there is some $i \in [1, k-1]$ such that all the edges in $E(v, i, j)$ are negative and also $[v+i, v+i+j-1]$ is negative. Part (iv) of Lemma 2.3.1 implies that we have $f(v, v+4k+j-1) < 0$ and so $(Q^-_{\epsilon_5}(v) \setminus T + 4k + v) \cap G_{\mathcal{H}} \subseteq R(v)$. Similarly, using part (v) of Lemma 2.3.1, it is possible to show that $(Q^+_{\epsilon_5}(v) \setminus T + 4k + v) \cap G_{\mathcal{H}} \subseteq R(v)$. Finally, part (iii) of Lemma 2.3.1 implies that we have $(\{0\} + 4k + v) \cap G_{\mathcal{H}} \subseteq R(v)$, and hence $Q' + 4k + v \subseteq R(v)$.

Now we prove $|Q_{\epsilon_5}(v)| - \epsilon_2 k$. Since $|T| \leq \epsilon_0 C_1 k / \epsilon_5$, we must have

$$|Q_{\epsilon_5}(v) \setminus T| \geq |Q_{\epsilon_5}(v)| - \frac{\epsilon_0 C_1}{\epsilon_5} k. \tag{2.7}$$

Condition (iii) of Claim 2.5.3 implies that

$$|Q'| \geq |Q_{\epsilon_5}(v) \setminus T| - \epsilon_1 k. \tag{2.8}$$

Now, (2.7), (2.8) and $\epsilon_2 \geq \epsilon_0 C_1/\epsilon_5 + \epsilon_1$ imply $|Q'| \geq |Q_{\epsilon_5}(v)| - \epsilon_2 k$, proving the claim. □

Definition 2.5.8. *For $S \subseteq A \times B$ we define*

$$A +_S B = \{a + b : (a, b) \in S\}.$$

The following claim shows that for a certain large set S, a translate of $Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)$ is contained in $R(0)$.

Claim 2.5.9. *There is a set* $S \subseteq Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)$ *such that* $|S| \geq |Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)| - \epsilon_3^2 k^2$ *and we have*

$$\big(Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)\big) + 13k \subseteq R(0).$$

Proof. For every good vertex $v \in I$, Claim 2.5.7 combined with part (ii) of Claim 2.5.5 implies that there is a set $Q_v \subseteq Q_{\epsilon_5}(v)$ such that we have $Q_v + v + \{6k, 7k\} \subseteq R(v)$ and also

$$|Q_v| \geq |Q_{\epsilon_5}(v)| - (7\epsilon_1 + \epsilon_2)k. \tag{2.9}$$

Now, part (i) of Claim 2.5.5 implies that we have

$$\bigcup_{v \in R(0) \cap [6k, 8k-1]} R(v) \subseteq R(0). \tag{2.10}$$

Combining $Q_v + v + \{6k, 7k\} \subseteq R(v)$ with (2.10) implies that we have

$$\bigcup_{v \in (Q_0 + \{6k, 7k\})} (Q_v + v + \{6k, 7k\}) \subseteq R(0). \tag{2.11}$$

We let

$$S = \{(a, b) \in Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k) : a \in Q_0 \text{ and } b \in Q_{a+6k} \cup Q_{a+7k}\}.$$

The identity (2.11) implies that we have

$$
\begin{aligned}
Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k) + 13k &= \{a + b : a \in Q_0 \text{ and} \\
&\qquad b \in (Q_{a+6k} \cup Q_{a+7k}) \cap Q_{2\epsilon_5}(7k)\} + 13k \\
&\subseteq \{a + b : a \in Q_0 \text{ and } b \in Q_{a+6k} \cup Q_{a+7k}\} + 13k \\
&= \left(\bigcup_{a \in Q_0 + 6k} Q_a + a + 7k \right) \cup \left(\bigcup_{a \in Q_0 + 7k} Q_a + a + 6k \right) \\
&\subseteq \bigcup_{a \in (Q_0 + \{6k, 7k\})} (Q_a + a + \{6k, 7k\}) \\
&\subseteq R(0).
\end{aligned}
$$

Now we prove $|S| \geq |Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)| - \epsilon_3^2 k^2$. Notice that for each $a \in [0, k-1]$, part (i) of Claim 2.5.6 implies

$$
Q_{2\epsilon_5}(7k) \subseteq Q_{\epsilon_5}(a + 6k) \cup Q_{\epsilon_5}(a + 7k) \text{ for all } a \in Q_{\epsilon_5}(0). \tag{2.12}
$$

The identity (2.12) combined with (2.9) and $Q_v \subseteq Q_{\epsilon_5}(v)$ implies that for all $a \in [1, k-1]$ we have

$$
\begin{aligned}
|(Q_{a+6k} \cup Q_{a+7k}) \cap Q_{2\epsilon_5}(7k)| &\geq |(Q_{\epsilon_5}(a + 6k) \cup Q_{\epsilon_5}(a + 7k)) \cap Q_{2\epsilon_5}(7k)| \\
&\qquad - 14\epsilon_1 + 2\epsilon_2)k \\
&= |Q_{2\epsilon_5}(7k)| - (14\epsilon_1 + 2\epsilon_2)k.
\end{aligned}
$$

This gives us

$$\begin{aligned}
|S| &= \sum_{a\in Q_0} |(Q_{a+6k} \cup Q_{a+7k}) \cap Q_{2\epsilon_5}(v)| \\
&\geq \sum_{a\in Q_0} \Big(|Q_{2\epsilon_5}(7k)| - (14\epsilon_1 + 2\epsilon_2)k\Big) \\
&\geq \Big(|Q_{\epsilon_5}(0)| - (7\epsilon_1 + \epsilon_2)k\Big)\Big(|Q_{2\epsilon_5}(7k)| - (14\epsilon_1 + 2\epsilon_2)k\Big) \\
&\geq |Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)| - (21\epsilon_1 + 3\epsilon_2)k^2 \\
&\geq |Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)| - \epsilon_3^2 k^2.
\end{aligned}$$

The second last inequality follows from $|Q_{\epsilon_5}(0)|$, $|Q_{2\epsilon_5}(7k)| \leq k$. The last inequality follows from $\epsilon_3^2 \geq 21\epsilon_1 + 3\epsilon_2$. □

Claim 2.5.9 is combined with the following.

Claim 2.5.10. *Suppose that A and $B \subseteq \mathbb{Z}_k$, and satisfy that for any $x \in \mathbb{Z}_k$, either x or $-x \in A$ and either x or $-x \in B$. Let $S \subseteq A \times B$ be a set satisfying $|S| \geq |A \times B| - \epsilon_3^2 k^2$. We have*

$$|A +_S B| \geq \left(\frac{1}{2} + \epsilon_4\right) k.$$

When k is prime, Claim 2.5.10 follows from a theorem due to Lev [38], which itself is closely related to a theorem due to Pollard [47]. In order to prove Claim 2.5.10, we will need some results from additive combinatorics. We define

$$(A + B)_i = \{x \in \mathbb{Z}_k : x = a + b \text{ for at least } i \text{ distinct pairs } (a, b) \in A \times B\}.$$

Notice that we have $(A + B)_{i+1} \subseteq (A + B)_i$.

The proof of Claim 2.5.10 will use the following theorem due to Grynkiewicz.

Theorem 2.5.11 (Grynkiewicz, [25])**.** *Let A and $B \subseteq \mathbb{Z}_k$ and $t \leq k$. We have one of the following.*

(i) *The following holds.*

$$\sum_{i=1}^{t} |(A+B)_i| \geq t|A| + t|B| - 2t^2 + 1. \tag{2.13}$$

(ii) *There are sets $A' \subseteq A$ and $B' \subseteq B$ such that $|A \setminus A'| + |B \setminus B'| \leq t-1$ and we have $A' + B' = (A+B)_t$.*

We define the *stabiliser* of a set $X \in \mathbb{Z}_k$ to be $Stab(X) = \{y \in \mathbb{Z}_k : y + X = X\}$. We use the following theorem due to Kneser.

Theorem 2.5.12 (Kneser, [37])**.** *Let A and $B \subseteq \mathbb{Z}_k$ and H the stabiliser of $A+B$ in $\mathbb{Z}_k$. We have*

$$|A+B| \geq |A+H| + |B+H| - |H|. \tag{2.14}$$

Sumsets in Claim 2.5.10, Theorem 2.5.11 and Theorem 2.5.12 are all in $\mathbb{Z}_k$.

Proof of Claim 2.5.10. Notice that since x or $-x \in A, B$, we must have $|A|, |B| \geq \frac{1}{2}k$. Our initial goal will be to show that we have

$$|(A+B)_{\epsilon_3 k}| \geq \left(\frac{1}{2} + \epsilon_4 + \epsilon_3\right) k. \tag{2.15}$$

Apply Theorem 2.5.11 to A and B with $t = 2\epsilon_3 k$. We split into two cases, depending on which part of Theorem 2.5.11 holds.

(i) Suppose that (2.13) holds. Since we are working over $\mathbb{Z}_k$ in this claim, we have $|(A+B)_i| \leq k$. Combining this with (2.13) implies

$$\begin{aligned}\sum_{i=\epsilon_3 k}^{2\epsilon_3 k} |(A+B)_i| &\geq 2\epsilon_3 k\Big(|A| + |B| - 4\epsilon_3 k\Big) + 1 - \sum_{i=1}^{\epsilon_3 k - 1} |(A+B)_i| \\ &\geq \epsilon_3 k\Big(2|A| + 2|B| - (1+8\epsilon_3)k\Big).\end{aligned}$$

This, together with $(A+B)_{i+1} \subseteq (A+B)_i$ implies that we have

$$|(A+B)_{\epsilon_3 k}| \geq 2|A| + 2|B| - (1+8\epsilon_3)k.$$

The identity (2.15) follows since we have $|A|, |B| \geq \frac{1}{2}k$ and $1 - 8\epsilon_3 \geq 1/2 + \epsilon_4 + \epsilon_3$.

(ii) Suppose that we have two sets A' and B' as in part (ii) of Theorem 2.5.11. Apply Theorem 2.5.12 to the sets A' and B'.

Note that $|A \setminus A'| + |B \setminus B'| \leq t - 1$ together with (2.14) and $|A|, |B| \geq \frac{1}{2}k$ implies that we have

$$\begin{aligned} |(A+B)_{\epsilon_3 k}| &\geq |(A+B)_{2\epsilon_3 k}| \\ &= |A' + B'| \\ &\geq |A' + Stab(A'+B')| + |B' + Stab(A'+B')| - |Stab(A'+B')| \quad (2.16) \\ &\geq |A| + |B| - |Stab(A'+B')| - 2\epsilon_3 k \\ &\geq (1 - 2\epsilon_3)k - |Stab(A'+B')|. \quad (2.17) \end{aligned}$$

If $|Stab(A'+B')| \leq \frac{1}{3}k$, then (2.15) follows (2.17) combined with $1 - 2\epsilon_3 - 1/3 \geq 1/2 + \epsilon_4 + \epsilon_3$.

Otherwise, Lagrange's Theorem implies that $Stab(A'+B')$ is either all of $\mathbb{Z}_k$ or that k is even and $Stab(A'+B')$ is the set of even elements of $\mathbb{Z}_k$. If $Stab(A'+B') = \mathbb{Z}_k$ holds, then we have $A' + Stab(A'+B') = B' + Stab(A'+B') = \mathbb{Z}_k$. Substituting this into (2.16) implies that we have $|(A+B)_{\epsilon_3 k}| = k$ and so (2.15) holds.

Suppose that $Stab(A'+B')$ consists of all the even elements of $\mathbb{Z}_k$. Since for every x, either x or $-x \in A$, there are at least $\frac{1}{4}k$ even elements in A, and at least $\frac{1}{4}k$ odd elements in A. Therefore, since $|A'| \geq |A| - 2\epsilon_3 k$, A' must contain an even element and an odd element. This implies that $A' + Stab(A'+B') = \mathbb{Z}_k$. Similarly $B' + Stab(A'+B') = \mathbb{Z}_k$. Thus (2.16) implies that we have $|(A+B)_{\epsilon_3 k}| = k$ and so (2.15) holds.

Now, we use (2.15) to deduce the claim. Let $T = (A+B)_{\epsilon_3 k} \setminus (A +_S B)$. We have $|A +_S B| + |T| \geq |(A+B)_{\epsilon_3 k}|$. Notice that from the definition of $(A+B)_{\epsilon_3 k}$ we have $\epsilon_3 k|T| + |S| \leq |A \times B|$. This, combined with (2.15) and $|S| \geq |A \times B| - \epsilon_3^2 k^2$ implies that we have

$$\begin{aligned}
|A +_S B| &\geq |(A+B)_{\epsilon_3 k}| - |T| \\
&\geq |(A+B)_{\epsilon_3 k}| - \frac{1}{\epsilon_3 k}(|A \times B| - |S|) \\
&\geq |(A+B)_{\epsilon_3 k}| - \epsilon_3 k \\
&\geq \left(\frac{1}{2} + \epsilon_4\right) k.
\end{aligned}$$

□

Claims 2.5.9 and 2.5.10 cannot be directly combined since sumsets in Claim 2.5.9 are in $\mathbb{Z}$ whereas sumsets in Claim 2.5.10 are in $\mathbb{Z}_k$. However, Claim 2.5.9 gives us a set S such that $|S| \geq |Q_{\epsilon_5}(0) \times Q_{2\epsilon_5}(7k)| - \epsilon_3^2 k^2$ and we have $\big(Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)\big) + 13k \subseteq R(0)$. Part (iii) of Claim 2.5.5 implies that there is a subset $Q' \subseteq (Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)) \bmod (k)$ such that $Q' + 16k \subseteq R(0)$ and we have

$$|Q'| \geq |(Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)) \bmod (k)| - 3\epsilon_1 k. \tag{2.18}$$

By Claim 2.5.10 and part (ii) of Claim 2.5.6, we have

$$|(Q_{\epsilon_5}(0) +_S Q_{2\epsilon_5}(7k)) \bmod (k)| \geq \left(\frac{1}{2} + \epsilon_4\right) k. \tag{2.19}$$

Combining (2.18) and (2.19) implies that $|R(0) \cap [16k, 17k-1]| \geq (1/2 + \epsilon_4 - 3)\, \epsilon_1 k$. Applying part (iv) of Claim 2.5.5 implies that for any $w \in I$, we have

$$|R(0) \cap [w, w+k-1]| \geq \left(\frac{1}{2} + \epsilon_4 - \epsilon_1 \left(\frac{w}{k} + 4\right)\right) k.$$

Combining this with $m \leq (4C_2 + C_3)k$ gives

$$|R(0) \cap [m-17k, m-16k-1]| \geq \left(\frac{1}{2} + \epsilon_4 - \epsilon_1(4C_2 + C_3 + 4)\right). \tag{2.20}$$

We can define $R^{-}(v) = \{u \in I \cap G_{\mathcal{H}} : f([u+1, v]) < 0\}$. By symmetry, we obtain

$$|R^{-}(m) \cap [m - 17k, m - 16k - 1]| \geq \left(\frac{1}{2} + \epsilon_4 - 3\epsilon_1\right) k. \tag{2.21}$$

Now, (2.20), (2.21), and $\epsilon_4 > \epsilon_1(4C_2 + C_3 + 4)$ imply that we have

$$|R(m) \cap [m - 17k, m - 16k - 1]| > \frac{1}{2}k,$$
$$|R^{-}(m) \cap [m - 17k, m - 16k - 1]| > \frac{1}{2}k.$$

Therefore, there is some $v \in [m-17k, m-16k-1]$ such that $v \in R(0)$ and $v-1 \in R^{-}(m)$. By definition of $R(0)$ and $R(m)$ we obtain $f(I) < 0$. □

As mentioned before, Claims 2.5.2 and 2.5.3 imply the theorem. □

2.6 Discussion

In this section we discuss some further directions one might take with our approach to Conjecture 1.1.1.

- The constant 10^{46} in Theorem 2.1.4 can certainly be improved by being more careful in the proof. The main question is whether a better choice of hypergraphs $\mathcal{H}_{n,k}$ can lead to a solution to Conjecture 1.1.1. It is not clear what kind of hypergraphs one should look for. Although in the above theorem, the hypergraphs $\mathcal{H}_{n,k}$ are quite sparse, this does not seem to be crucial in the proof.

- The constant "10^{46}" cannot be reduced to "4" in Theorem 2.1.4 without changing the graphs $\mathcal{H}_{n,k}$. Indeed for large k, the graphs $\mathcal{H}_{5(k-1),k}$ do not have the MMS-property. To see this, consider the following function $f : V(G) \to \mathbb{R}$.

$$f(i) = k - 2 \text{ if } i \equiv 0 \pmod{k-1},$$
$$f(i) = -1 \text{ if } i \not\equiv 0 \pmod{k-1}.$$

It is easy to see that we have $\sum_{x\in V(G)} f(x) = 0$. For two vertices i and j let

$$p(i,j) = \begin{cases} \text{The number of edges of } \mathcal{H}_{5(k-1),k} \text{ containing } i \text{ and } j & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

The graph $\mathcal{H}_{5(k-1),k}$ has five nonnegative vertices $0, k-1, 2(k-1), 3(k-1), 4(k-1)$. An edge $e \in \mathcal{H}_{5(k-1),k}$ is nonnegative if and only if e contains at least two of these vertices. Therefore the number of nonnegative edges in $\mathcal{H}_{5(k-1),k}$ is at most

$$\frac{1}{2} \sum_{\substack{i,j\in\{0,k-1,2(k-1),\\ 3(k-1),4(k-1)\}}} p(i,j) = 5p(0,k-1) + 5p(0,2(k-1)). \tag{2.22}$$

Notice that an edge $e(-v,i,j)$ contains both 0 and $k-1$ if and only if we have

$$i \geq v+1, \tag{2.23}$$
$$j \geq v, \tag{2.24}$$
$$i+j \geq v+k-1. \tag{2.25}$$

It's easy to check that the number of triples (v,i,j) which satisfy (2.23) – (2.25) is less than $\frac{1}{6}k^3 + o(k^3)$, which implies that $p(0,k-1) = \frac{1}{6}k^3 + o(k^3)$.

The only edges $\mathcal{H}_{5(k-1),k}$ which contain 0 and $2(k-1)$ are of the form $e(0,i,k-1)$ for some i, so we have that $p(0,2(k-1)) = k-1$. Therefore, there are less than $\frac{5}{6}k^3 + o(k^3)$ nonnegative edges in $\mathcal{H}_{5(k-1),k}$ which is smaller than $k(k-1)^2$ for large enough k.

The above argument shows that the constant "10^{46}" in Theorem 2.1.4 cannot be reduced to less than 5. This shows that Conjecture 1.1.1 cannot be solved by the argument we used in this chapter without changing the graphs $\mathcal{H}_{n,k}$ to some other construction.

- We conclude with the following general problem.

Problem 2.6.1. *Which hypergraphs have the MMS-property?*

This problem is probably quite hard, since a solution to it would mean a generalization of Conjecture 1.1.1. However, perhaps looking for hypergraphs which have the MMS-property would lead to improved bounds on Conjecture 1.1.1.

Chapter 3

Edge growth in graph powers

3.1 Introduction

Recall that the rth power of G, denoted G^r, is the graph with vertex set $V(G)$, and xy an edge whenever x and y are within distance r of each other. One would expect that when $r \leq \operatorname{diam}(G)$, then G^r has substantially more edges than G. In this chapter we study how small the ratio $e(G^r)/e(G)$ can be for regular graphs G. The motivation for studying this comes from the following consequence of the Cauchy-Davenport Theorem mentioned in the introduction.

Corollary 1.2.1 (Cauchy, Davenport, [11, 13]). *Let p be a prime, G the Cayley graph of a set $A \subseteq Z_p$, and r an integer such that $r \leq \operatorname{diam}(G)$. Then we have*

$$\frac{e(G^r)}{e(G)} \geq r. \tag{3.1}$$

One could ask whether inequalities similar to (3.1) hold for more general families of graphs. Motivated by the fact that Cayley graphs are regular, Hegarty asked this question for regular graphs and proved the following theorem.

Theorem 1.2.3 (Hegarty, [34]). *Let G be a regular, connected graph, which satisfies* diam$(G) \geq 3$. *Then we have*

$$\frac{e(G^3)}{e(G)} \geq 1 + \epsilon. \tag{3.2}$$

Where $\epsilon \approx 0.087$.

In Section 3.2 we give an alternative proof of this result with an improved constant of $\epsilon = \frac{1}{6}$. DeVos and Thomassé subsequently improved the constant further to $\epsilon = \frac{3}{4}$ [6]. The value $\epsilon = \frac{3}{4}$ is optimal in a sense that there exists a sequence of regular graphs of diameter greater than 3, G_m, satisfying $e(G_m^3)/e(G_m) \to \frac{7}{4}$ as $m \to \infty$ [6].

It is natural to ask what happens for other powers of G. For G^2, Hegarty showed that no inequality similar to (3.2) can hold for regular graphs in general, by exhibiting a sequence of regular, connected graphs of diameter greater than 2, G_m, satisfying $e(G_m^2)/e(G_m) \to 1$ as $m \to \infty$ [34]. See Figure 3.1 for examples of sequences of graphs which have this property. Goff [23] studied the 2nd power of regular graphs further. He showed that for any d-regular graph connected graph G such that $\mathrm{diam}(G) > 2$, we have $e(G^2)/e(G) \geq 1 + \frac{3}{2d} - o\left(\frac{1}{d}\right)$. For general d-regular connected graphs G with $\mathrm{diam}(G) > 2$, he showed that the $\frac{3}{2d}$ term in this result cannot be replaced with λ/d for any $\lambda > \frac{3}{2}$. However he showed that with the exception of two families of exceptional graphs, we have $e(G^2)/e(G) \geq 1 + \frac{2}{d} - o\left(\frac{1}{d}\right)$ for all d-regular connected graphs with $\mathrm{diam}(G) > 2$.

In Section 3.3 we consider the case when $r \geq 4$ and determine how small the ratio $e(G^r)/e(G)$ can be in this case for a regular, connected graph G

Theorem 3.1.1. *Let G be a connected, regular graph, and r a positive integer such that* $\mathrm{diam}(G) \geq r$.

- *If $r \equiv 0 \pmod 3$, then we have*

$$\frac{e(G^r)}{e(G)} \geq \frac{r+3}{3} - \frac{3}{2(r+3)}.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$\frac{e(G^r)}{e(G)} \geq \left\lceil \frac{r}{3} \right\rceil.$$

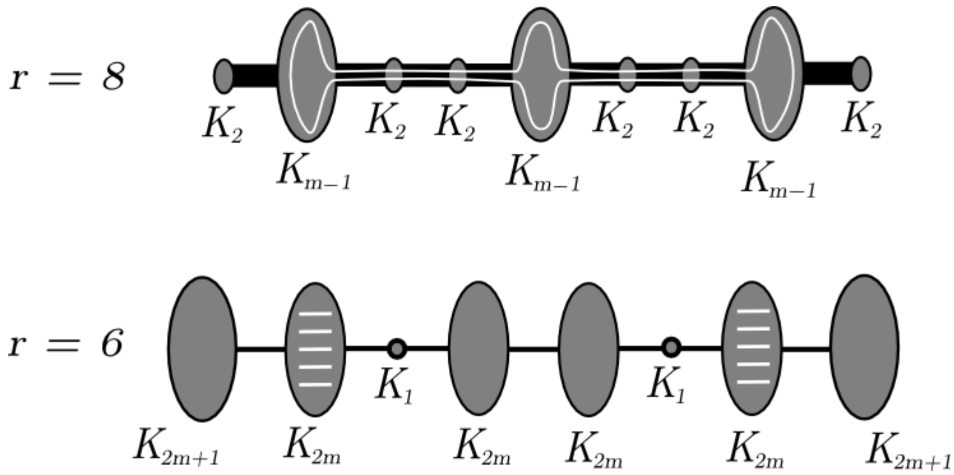

Figure 3.1: Graphs showing the optimality of the cases "$r = 8$" and "$r = 6$" of Theorem 3.1.1. The grey circles represent complete graphs of specified order. The black lines between the sets represent all the edges being present between them. The white cycle in the "$r = 8$" case represents a single cycle passing through all the vertices in the specified sets being removed. The white matchings in the "$r = 6$" case represent a perfect mathing being removed from the specified sets. The "$r = 8$" example also shows that no identity of the form $e(G^2)/e(G) \geq 1 + \epsilon$ for $\epsilon > 0$ holds in general for all regular, connected graphs of diameter greater than 2.

The case $r = 3$ of Theorem 3.1.1 is due to DeVos and Thomassé [6], and will not be proved here.

Theorem 3.1.1 gives a lower bound on the ratio $e(G^r)/e(G)$ for regular graphs. The bounds on $e(G^r)/e(G)$ in Theorem 3.1.1 are optimal in the following sense. For each r, there exists a sequence of regular, connected graphs of diameter at least r, G_m, such that $e(G_m^r)/e(G_m)$ tends to the bound given by Theorem 3.1.1 as m tends to infinity. See Figure 3.1 for a diagram of the sequences that we will construct.

For $r \not\equiv 0 \pmod 3$, we construct the following sequence of graphs G_m. Take disjoint sets of vertices $N_0, ..., N_r$, with $|N_i| = m - 1$ if $i \equiv 1 \pmod 3$ and $|N_i| = 2$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \ldots, r - 1$. Add all the edges within N_i for all i. Remove a cycle passing through all the vertices in $N_1 \cup ... \cup N_{r-1}$. It is easy to see that G_m is m-regular and of diameter r. If $r \equiv 1 \pmod 3$ then $|G_m| = \frac{1}{3}(rm - m + 3r - 3)$ will hold. Since G_m is m-regular, we have $e(G_m) = \frac{1}{6}(rm - m + 3r - 3)m$.

Since G_m^r is complete, we have $e(G_m^r) = \frac{1}{18}(rm - m + 3r - 3)(rm - m + 3r - 4)$. This implies that $e(G_m^r)/e(G_m) \to \left\lceil \frac{r}{3} \right\rceil$ as $m \to \infty$. A similar calculation can be used to show that the same limit holds when $r \equiv 2 \pmod 3$.

For $r \equiv 0 \pmod 3$, we construct the following sequence of graphs G_m to show that Theorem 3.1.1 is optimal. Take disjoint sets of vertices $N_0, ..., N_{r+1}$. Let $|N_0| = |N_{r+1}| = 2m+1$, $|N_i| = 1$ if $i \equiv 2 \pmod 3$, and $|N_i| = 2m$ otherwise. Add all the edges between N_i and N_{i+1} for $i = 0, 1, \ldots, r$. Add all the edges within N_i for all i. Delete a perfect matching from each of the sets N_1 and N_r. This will ensure that G_m is $4m$-regular and has diameter $r+1$. Note that $|G_m| = \frac{1}{3}(4rm + r + 12m + 6)$, and so we have $e(G_m) = \frac{1}{6}(4rm + r + 12m + 6)4m$. The only edges missing from G_m^r will be between N_0 and N_{r+1}, so we have $e(G_m^r) = \frac{1}{18}(4rm + r + 12m + 6)(4rm + r + 12m + 5) - (2m + 1)^2$. This implies that $e(G_m^r)/e(G_m) \to \frac{r+3}{3} - \frac{3}{2(r+3)}$ as $m \to \infty$. This construction is a generalization of one from [6].

The requirement of G being regular in the above theorems is quite restrictive. Following [6], we will instead assume that G has minimum degree $\delta(G)$, and give the following bound on $e(G^r)$ in terms of $|G|$ and $\delta(G)$.

Theorem 3.1.2. *Let G be a connected graph, and r a integer such that* $\operatorname{diam}(G) \geq r$.

- *If $r \equiv 0 \pmod 3$, then we have*

$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)} \right) \delta(G)|G|.$$

- *If $r \not\equiv 0 \pmod 3$, then we have*

$$e(G^r) \geq \frac{1}{2} \left\lceil \frac{r}{3} \right\rceil \delta(G)|G|.$$

The case $r = 3$ of Theorem 3.1.2 is due to DeVos and Thomassé [6], and will not be proved here. Theorem 3.1.2 immediately implies Theorem 3.1.1.

The structure of this chapter is as follows. In Section 3.2 we give an alternative proof of Theorem 1.2.3 with an improved constant of $\epsilon = \frac{1}{6}$. In Section 3.3 we prove

Theorems 3.1.1 and 3.1.2. In Section 3.4 we discuss some related results and conjectures in the area of edge growth in graph powers.

3.2 Cubes

In this section we prove the following theorem.

Theorem 3.2.1. *Suppose G is a regular, connected graph with* $\mathrm{diam}(G) \geq 3$. *Then we have*

$$e(G^3) \geq \left(1 + \frac{1}{6}\right) e(G).$$

Proof. Let the degree of each vertex be d. Note that as G is regular, and not complete, every $v \in V(G)$ will have a non-neighbour in G. Together with connectedness this implies that each $v \in V(G)$ has at least one new neighbour in G^2. This implies the theorem for $d \leq 6$. For the remainder of the proof, we assume that $d > 6$.

The proof rests on the following colouring of the edges of G: For an edge uv in G, colour

$$uv \textbf{ red} \text{ if } |N(u) \cap N(v)| > \frac{2}{3}d,$$
$$uv \textbf{ blue} \text{ if } |N(u) \cap N(v)| \leq \frac{2}{3}d.$$

Notice that if uv is a blue edge, then there are at least $\frac{4}{3}d - 1$ neighbours of u in G^2. This is because u will be connected to everything in $N(u) \cup N(v)$ except itself, and $|N(u) \cup N(v)| \geq \frac{4}{3}d$ for uv blue. If, in addition, we have some x connected to u by an edge (of any colour), then x will be at distance at most 3 from everything in $N(u) \cup N(v) \setminus \{x\}$. Hence x will have at least $\frac{4}{3}d - 1$ neighbours in G^3.

Partition the vertices of G as follows:

$$B = \{v \in V(G) : v \text{ has a blue edge coming out of it}\}, \tag{3.3}$$
$$R = \{v \in V(G) : v \notin B \text{ and there is a } u \in B \text{ such that } uv \text{ is an edge}\}, \tag{3.4}$$
$$S = V(G) \setminus (B \cup R). \tag{3.5}$$

By the above argument, if v is in $B \cup R$, then v has at least $\frac{4}{3}d - 1$ neighbours in G^3. Recall that since G is d-regular, connected and non-complete each $u \in S$ will have at least $d+1$ neighbours in G^3. Summing these two bounds over all vertices in G, gives

$$\begin{aligned}
2e(G^3) &\geq \left(\frac{4}{3}d - 1\right)|B \cup R| + (d+1)|S| \\
&= \left(\frac{4}{3}d - 1\right)|B \cup R| + (d+1)\left(|V(G)| - |B \cup R|\right) \\
&= \frac{7}{6}d|V(G)| + \frac{1}{3}\left(|B \cup R| - \frac{1}{2}|V(G)|\right)(d-6) \\
&= \frac{7}{3}e(G) + \frac{1}{3}\left(|B \cup R| - \frac{1}{2}|V(G)|\right)(d-6).
\end{aligned}$$

Recall that we are considering the case when $d > 6$. Thus to prove that $e(G^3) \geq \frac{7}{6}e(G)$, it suffices to show that $|B \cup R| \geq \frac{1}{2}|V(G)|$. To show this we shall demonstrate that $|S| \leq |R|$. First we need a proposition helping us to find blue edges in G.

Proposition 3.2.2. *For any $v \in V(G)$ there is some $b \in B$ such that $d(v,b) \leq 2$.*

Proof. Suppose $d(v,u) = 3$. Then there are vertices x and y such that $\{v,x,y,u\}$ forms a path between u and v. We will show that one of the edges vx, xy or yu is blue. This will prove the proposition assuming that there are *any* blue edges to begin with. However, it also shows the existence of blue edges because $\operatorname{diam}(G) \geq 3$.

So, suppose that the edges vx and uy are red. Then we have $|N(v) \cap N(x)| > \frac{2}{3}d$, and $|N(u) \cap N(y)| > \frac{2}{3}d$. Using this and $N(u) \cap N(v) = \emptyset$ gives

$$\begin{aligned}
|N(x) \cup N(y)| &\geq |(N(x) \cup N(y)) \cap N(v)| + |(N(x) \cup N(y)) \cap N(u)| \\
&\geq |N(x) \cap N(v)| + |N(y) \cap N(u)| \\
&> \frac{4}{3}d.
\end{aligned}$$

Therefore $|N(x) \cap N(y)| = 2d - |N(x) \cup N(y)| \leq \frac{2}{3}d$. Hence xy is blue, proving the proposition. $\square$

Now we will show that $|S| \leq |R|$. Suppose $r \in R$. By the definition of R, there is a $b \in B$ such that rb is an edge. This edge is necessarily red as $r \notin B$. Using $N(b) \subseteq B \cup R$,we have $|N(r) \cap (B \cup R)| \geq |N(r) \cap N(b)| > \frac{2}{3}d$. Hence

$$|N(r) \cap S| \leq \frac{1}{3}d. \tag{3.6}$$

Suppose $s \in S$. Proposition 3.2.2 implies that there is some $r \in R$ such that sr is an edge. Since sr is red, we have $|N(s) \cap N(r)| > \frac{2}{3}d$. Using this, the fact that $N(s) \subseteq R \cup S$, and (3.6), gives

$$\begin{aligned}
|N(s) \cap R| &\geq |N(s) \cap N(r) \cap R| \\
&= |N(s) \cap N(r)| - |N(s) \cap N(r) \cap S| \\
&\geq |N(s) \cap N(r)| - |N(r) \cap S| \\
&> \frac{1}{3}d. \qquad (3.7)
\end{aligned}$$

Double-counting the edges between S and R using the bounds (3.6) and (3.7) gives a contradiction unless $|S| \leq |R|$. Therefore $|B \cup R| \geq \frac{1}{2}|V(G)|$ as required. □

3.3 Higher powers

In this section we prove Theorems 3.1.1 and 3.1.2.

In this section we will consider graphs which may contain loops. This is because the proof of the results in this section is more natural in this setting.

We will denote graphs which may contain loops by bold letters such as $\mathbf{G}$. For two vertices x and y (possibly $x = y$) we only ever allow one edge between x and y. The neighbourhood of a vertex x, $N(x)$, is defined as the set of vertices adjacent to x. (If there is a loop at x, then $N(x)$ will contain x itself.) The degree of x is $|N(x)|$. For graphs with loops allowed, $\mathbf{G}^r$ is defined identically to how it was defined for loopless graphs. Note that if $\mathbf{G}$ is a graph with loops allowed, then $\mathbf{G}^r$ always has a loop at each vertex. For two sets of vertices X and Y, let $d(X, Y)$ denote the length of a shortest path between a vertex in X and a vertex in Y. If X is a set of vertices, let $N^r(X)$ be the set of vertices distance at most r from X. We abbreviate $N^r(\{x\})$ as $N^r(x)$.

We prove the following theorem, and then deduce Theorem 3.1.2 as a corollary. Many ideas in the proof of Theorem 3.3.1 are taken from [6]. In particular, Claims 3.3.7 and 3.3.8 are analogues of claims proved in [6].

Theorem 3.3.1. *Let* $\mathbf{G}$ *be a connected graph, and* r *a positive integer such that* $r \geq 4$ *and* $\operatorname{diam}(\mathbf{G}) \geq r$.

- *If* $r \equiv 0 \pmod 3$, *then we have*

$$e(\mathbf{G}^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right) \delta(\mathbf{G})|\mathbf{G}| + \frac{1}{2}|\mathbf{G}|.$$

- *If* $r \not\equiv 0 \pmod 3$, *then we have*

$$e(\mathbf{G}^r) \geq \frac{1}{2}\left\lceil \frac{r}{3} \right\rceil \delta(\mathbf{G})|\mathbf{G}| + \frac{1}{2}|\mathbf{G}|.$$

Proof. For convenience, we will set $\delta = \delta(\mathbf{G})$. If P is a path between two vertices x and y, we say that P is a *geodesic* if the length of P is $d(x, y)$. The notion of a geodesic is useful because the neighbourhood of a geodesic must be quite large. This is quantified in the following claim.

Claim 3.3.2. *Let* P *be a length* k *geodesic. Then* $|N(P)| \geq \left(\left\lfloor \frac{k}{3} \right\rfloor + 1\right) \delta$ *holds.*

Proof. Let $x_0,\ x_1, \ldots, x_k$ be the vertices of P (in the order in which they occur along the path). Notice that $N(x_0),\ N(x_3), \ldots, N(x_{3\lfloor \frac{k}{3} \rfloor})$ must all disjoint, since otherwise we could find a shorter path between x_0 and x_k. The sets $N(x_0),\ N(x_3), \ldots, N(x_{3\lfloor \frac{k}{3} \rfloor})$ must also be contained in $N(P)$, and each have order at least δ. This implies the result. $\square$

We now prove the theorem in the case when $r \not\equiv 0 \pmod 3$.

The diameter of $\mathbf{G}$ is at least r, so $\mathbf{G}$ must contain a length r geodesic, P. Claim 3.3.2 implies that the following holds:

$$|\mathbf{G}| \geq |N(P)| \geq \left(\left\lfloor \frac{r}{3} \right\rfloor + 1\right)\delta = \left\lceil \frac{r}{3} \right\rceil \delta. \tag{3.8}$$

Since $\mathbf{G}^r$ contains a loop at every vertex, we have $e(\mathbf{G}^r) = \sum_{v \in V(\mathbf{G})} \left(\frac{1}{2}|N^r(v)| + \frac{1}{2}\right)$. Thus to prove Theorem 3.3.1 it is sufficent to exhibit $\left\lceil \frac{r}{3} \right\rceil \delta$ elements of $N^r(v)$ for each vertex $v \in V(G)$.

Suppose that there exists a length $r-1$ geodesic P_v starting from a vertex v. Then $N(P_v)$ is contained in $N^r(v)$, giving

$$|N^r(v)| \geq |N(P_v)| \geq \left(\left\lfloor \frac{r-1}{3} \right\rfloor + 1\right)\delta = \left\lceil \frac{r}{3} \right\rceil \delta.$$

The second inequality is an application of Claim 3.3.2.

Suppose that all the vertices in $\mathbf{G}$ are within distance $r-1$ of v. In this case we have $N^r(v) = V(\mathbf{G})$, which is of order at least $\left\lceil \frac{r}{3} \right\rceil \delta$ by (3.8). This completes the proof of the case "$r \not\equiv 0 \pmod 3$" of the theorem.

For the rest of the proof fix r such that $r \equiv 0 \pmod 3$. Note that this implies that $r \geq 6$.

If v is a vertex of $\mathbf{G}$, we say that v is *sufficient* if $|N^r(v)| \geq \left(\frac{r}{3} + 1\right)\delta$. Otherwise we say that v is *insufficient*.

The following is a useful property of insufficient vertices.

Claim 3.3.3. *Let v be an insufficient vertex. Then there is some vertex at distance $r+1$ from v.*

Proof. Since $\operatorname{diam}(\mathbf{G}) \geq r$, Claim 3.3.2 implies that $|\mathbf{G}| \geq \left(\frac{r}{3} + 1\right)\delta$. Since v is insufficient, we have $|N^r(v)| < \left(\frac{r}{3} + 1\right)\delta$, and so v cannot be within distance r from all the vertices in the graph. $\square$

The following three claims will allow us to bound the number of insufficient vertices in $\mathbf{G}$.

Claim 3.3.4. *If $2 < d(x,y) < r$ holds for $x, y \in V(\mathbf{G})$, then either x or y is sufficient.*

Proof. Suppose that x is insufficient. By Claim 3.3.3, we can find a length r geodesic starting from x with vertex sequence x, x_1, $x_2, \ldots, x_r$.

Suppose that $N(y) \cap N(x_i) \neq \emptyset$ for some i with $3 \leq i \leq r-3$. In this case $N(x)$, $N(x_3)$, $N(x_6), \ldots, N(x_r)$ are all contained in $N^r(y)$. There are $\frac{r}{3}+1$ of these, they are all disjoint (since x, x_1, $x_2, \ldots, x_r$ form a geodesic), and are of order at least δ. Hence y is sufficient.

Otherwise $N(y) \cap N(x_i) = \emptyset$ for all $3 \leq i \leq r-3$. In this case $N(x)$, $N(y)$, $N(x_3)$, $N(x_6), \ldots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. This contradicts our initial assumption that x is insufficient. □

Claim 3.3.5. *Let x and y be two vertices in $\mathbf{G}$ such that $d(x,y) = r$ or $d(x,y) = r+1$. If there exists a vertex $z \in \mathbf{G}$ such that $d(z,x),\ d(z,y) \geq r-1$, then either x or y is sufficient.*

Proof. Choose any z in $N^{r-1}(\{x,y\}) \setminus N^{r-2}(\{x,y\})$. This set is nonempty by the second assumption of the claim. We will have $d(z,x), d(z,y) \geq r-1$ and either $d(z,x)$ or $d(z,y) = r-1$. Without loss of generality assume that $d(z,x) = r-1$ and $d(z,y) \geq r-1$.

We will show that x is sufficient. Let x, $x_1, \ldots, x_{d(x,y)-1}$, y be a geodesic between x and y. For $i = 1, \ldots, d(x,y)-1$, the triangle inequality implies that

$$d(x,z) - i = d(x,z) - d(x,x_i) \leq d(x_i,z), \tag{3.9}$$

$$d(y,z) - d(x,y) + i = d(y,z) - d(y,x_i) \leq d(x_i,z). \tag{3.10}$$

Averaging (3.9) and (3.10), and using the inequalities $d(z,x), d(z,y) \geq r-1$ and $d(x,y) \leq r+1$ gives

$$\frac{r-3}{2} \leq d(x_i,z). \tag{3.11}$$

If $r \geq 9$, then (3.11) implies that $d(z,x_i) \geq 3$ for all i. Hence $N(x)$, $N(z)$, $N(x_3)$, $N(x_6), \ldots, N(x_{r-3})$ are all disjoint and contained in $N^r(x)$. Hence x is sufficient.

If $r = 6$, then (3.9) and (3.10) imply that $d(z,x_i) \geq 3$ for all x_i except possibly x_3 or x_4. In this case $N(z)$, $N(x_2)$ and $N(x_5)$ are all disjoint and contained in $N^6(x)$.

Hence x is sufficient. $\square$

Claim 3.3.6. *If $d(x, y) = r$ holds for $x, y \in V(\mathbf{G})$, then either x or y is sufficient.*

Proof. Suppose that x and y are insufficient. By Claim 3.3.3 there exists $z \in V(\mathbf{G})$ such that $d(x, z) = r+1$. Let $x,\ x_1, \ldots, x_{r-1}, y$ be a geodesic between x and y. Since x and y are insufficient, Claim 3.3.5 implies that we have $d(z, y) < r-1$. Note that $d(x, z) = r+1$ implies that $N(z) \cap N(x_i) = \emptyset$ for all $i \leq r-2$. Thus $N(z),\ N(x_1),\ N(x_4), \ldots, N(x_{r-2})$ are all disjoint and contained in $N^r(y)$. This contradicts our assumption that y is insufficient. $\square$

Let X be the set of insufficient vertices in $\mathbf{G}$. We define an equivalence relation "$\sim$" on X by letting $x \sim y$ if $d(x, y) \leq 2$. For $r \geq 6$, Claim 3.3.4 implies that this is an equivalence relation. Let $X_1, \ldots, X_l$ be the equivalence classes of "$\sim$".

The following claim gives a lower bound on the order of $\mathbf{G}$.

Claim 3.3.7. $|\mathbf{G}| \geq \left(\frac{r+3}{6}\right) \delta l$

Proof. Claims 3.3.4 and 3.3.6 imply that $d(X_i, X_j) \geq r+1$ for all $i \neq j$. If $d(X_i, X_j) = r+1$ for some i and j, then Claim 3.3.5 implies that we have $d(X_i, z) < r-1$ or $d(X_j, z) < r-1$ for all $z \in V(\mathbf{G})$. Then, Claim 3.3.4 implies that all the vertices outside of X_i and X_j are sufficient. This gives us two cases to consider:

(i) $d(X_i, X_j) \geq r+2$ for all $i \neq j$.

(ii) $d(X_1, X_2) = r+1$ and $l = 2$.

Suppose that (i) holds (this includes the case "$l = 1$"). For each i, choose x_i to be any vertex in X_i. Note that $N^{\left\lfloor \frac{r}{2} \right\rfloor}(x_i)$ contains a length $\left\lfloor \frac{r}{2} \right\rfloor$ geodesic, P. Using Claim 3.3.2 gives

$$\left| N^{\left\lfloor \frac{r}{2} \right\rfloor + 1}(X_i) \right| \geq |N(P)| \geq \left(\left\lfloor \frac{1}{3} \left\lfloor \frac{r}{2} \right\rfloor \right\rfloor + 1 \right) \delta \geq \left(\frac{r+3}{6} \right) \delta.$$

For the last inequality we are using the fact that $r \equiv 0 \pmod 3$. Note that (i) implies that $N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \cap N^{\lfloor \frac{r}{2} \rfloor + 1}(X_j) = \emptyset$ for all i, j. This implies that the following holds:

$$|V(\mathbf{G})| \geq \sum_{i=1}^{l} \left| N^{\lfloor \frac{r}{2} \rfloor + 1}(X_i) \right| \geq \left(\frac{r+3}{6} \right) \delta l.$$

Suppose that (ii) holds. Using Claim 3.3.2 we obtain

$$|V(\mathbf{G})| \geq \left(\frac{r}{3} + 1 \right) \delta = \left(\frac{r+3}{6} \right) \delta l.$$

□

When x is insufficient, the following claim gives a lower bound on the order of $N^r(x)$.

Claim 3.3.8. *Suppose that x is an insufficient vertex in the equivalence class X_i. Then, $|N^r(x)| \geq |X_i| + \frac{r}{3}\delta$ holds.*

Proof. By Claim 3.3.3, we can choose a length r geodesic from x. Let $x, x_1, \ldots, x_r$ be the vertices of this geodesic. Suppose that $X_i \cap N(x_j)$ is nonempty for some x_j. Choose $y \in X_i \cap N(x_j)$. Clearly $j \leq 1$ must hold, since otherwise $N(x),\ N(x_3), N(x_6), \ldots, N(x_r)$ would all be contained in $N^r(y)$, contradicting that y is insufficient (since $y \in X_i$).

Hence $X_i,\ N(x_2),\ N(x_5), \ldots, N(x_{r-1})$ are all disjoint and contained in $N^r(x)$ proving the claim. □

Combining Claims 3.3.7 and 3.3.8 we prove the theorem.

$$
\begin{aligned}
2e(\mathbf{G}^r) - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathbf{G}| - |\mathbf{G}| &= \sum_{x \in V(\mathbf{G})} |N^r(x)| - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathbf{G}| \\
&\geq \sum_{x \text{ sufficient}} \left(\frac{r}{3}+1\right) d + \sum_{i=1}^{l} \left(|X_i| + \frac{r}{3}\delta\right)|X_i| \\
&\qquad - \left(\frac{r+3}{3} - \frac{3}{2(r+3)}\right)\delta|\mathbf{G}| \\
&= \frac{3}{2(r+3)}\delta|\mathbf{G}| + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&\geq \frac{1}{4}\delta^2 l + \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta\right) \\
&= \sum_{i=1}^{l}\left(|X_i|^2 - |X_i|\delta + \frac{1}{4}\delta^2\right) \\
&= \sum_{i=1}^{l}\left(|X_i| - \frac{1}{2}\delta\right)^2 \\
&\geq 0.
\end{aligned}
$$

The first equality uses the fact that $\mathbf{G}^r$ contains a loop at every vertex, hence $2e(\mathbf{G}^r) = \sum_{x \in V(\mathbf{G})} |N^r(x)| + |\mathbf{G}|$. The first inequality follows from the definition of "sufficient vertex" and Claim 3.3.8. The second equality follows from the fact that there are $|\mathbf{G}| - \sum_{i=1}^{l} |X_i|$ sufficient vertices in $\mathbf{G}$. The second inequality follows from Claim 3.3.7. This completes the proof. $\square$

Proof of Theorem 3.1.2. Let $\mathbf{G}$ be a copy of G with a loop added at every vertex. Then $\mathbf{G}^r$ will be isomorphic to G^r with a loop added at every vertex. Note that we have $e(\mathbf{G}^r) = e(G^r) + |G|$, and $\delta(\mathbf{G}) = \delta(G) + 1$. Substitute these into Theorem 3.3.1 obtain the following.

- If $r \equiv 0 \pmod 3$, then we have

$$e(G^r) \geq \left(\frac{r+3}{6} - \frac{3}{4(r+3)}\right)\delta(G)|G| + \left(\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}\right)|G|.$$

- If $r \not\equiv 0 \pmod 3$, then we have

$$e(G^r) \geq \frac{1}{2}\left\lceil\frac{r}{3}\right\rceil\delta(G)|G| + \left(\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}\right)|G|.$$

Note that for $r \geq 3$, both $\frac{r+3}{6} - \frac{3}{4(r+3)} - \frac{1}{2}$ and $\frac{1}{2}\left\lceil\frac{r}{3}\right\rceil - \frac{1}{2}$ are non-negative, so Theorem 3.1.2 follows. □

3.4 Discussion

All the examples constructed above have their diameter close to r. If a graph G has diameter larger than r, it seems that the bounds of Theorem 3.1.1 can be improved. Some results in this direction have been obtained by DeVos, McDonald and Scheide [14].

All the questions from this chapter could be asked for directed graphs as well. In particular one can define directed Cayley graphs for a set $A \subseteq \mathbb{Z}_p$ by letting xy be a directed edge whenever $x - y \in A$. Then the Cauchy-Davenport Theorem implies an identical version of Theorem 1.2.2 for directed Cayley graphs. In this setting it is easy to show that there is growth even for the square of an out-regular oriented graph D (a directed graph is oriented when for a pair of vertices u and v, uv and vu are not both edges). In particular, we have

$$e(D^2) \geq \frac{3}{2}\, e(D).$$

This occurs because every vertex v has $|N_2^{out}(v)| \geq \frac{1}{2}|N_1^{out}(v)| + 1$ in an out-regular oriented graph (here $N_d^{out}(v)$ denotes the dth out-neighbourhood of a vertex—the set of vertices to which there is a *directed* path of length at most d from v). It's easy to see that this is best possible for such graphs. One can construct out-regular oriented graphs where the proportion of vertices v satisfying $|N_2^{out}(v)| = \frac{1}{2}|N_1^{out}(v)| + 1$ is arbitrarily close to 1. We sketch one such construction here. Consider three sets of vertices A_1, A_2, and A_3 of orders satisfying $|A_1| = n$ and $|A_2| = 2m$, $|A_3| = m + 1$ where $n > m$. We add all the edges from A_1 to A_2 and from A_2 to A_3. Each vertex in A_3 has edges going to some set of $2m$ vertices in A_1. Finally, we add edges inside A_2, so that it forms a regular tournament (i.e. there is an edge between any pair of vertices). The resulting oriented graph is $2m$-out-regular. However all vertices v in A_1 satisfy $|N_2^{out}(v)| = m + 1 = \frac{1}{2}|N_1^{out}(v)| + 1$. Therefore by setting n to be sufficiently large compared to m, the ratio $e(D^2)/e(D)$ can be made arbitrarily close to $3/2$ for this oriented graph.

However if we insist on *both* in and out-degrees to be constant, (8) no longer seems tight. Such graphs are always Eulerian. In [51] there is a conjecture attributed to Jackson and Seymour that if an oriented graph D is Eulerian, then $e(D^2) \geq 2\, e(D)$ holds. If this conjecture were proved, it would be an actual generalization of the directed version of Theorem 1.2.2, as opposed to the mere analogues proved above.

Chapter 4

Counterexamples to the Erdős-Gyárfás-Pyber Conjecture

4.1 Introduction

In this chapter we study the following conjecture due to Erdős, Gyárfás, and Pyber.

Conjecture 1.4.3 (Erdős, Gyárfás, Pyber, [18])**.** *The vertices of every r-edge-coloured complete graph can be covered by r vertex-disjoint monochromatic cycles.*

This conjecture has only been proved for $r = 2$. In this case it was first proved for large n by Łuczak, Rödl, and Szemerédi [39] using the regularity lemma. Subsequently Allen [1] proved it for smaller (but still large) n by an argument avoiding regularity. The $r = 2$ case of Conjecture 1.4.3 was finally proved for all n by Bessy and Thomassé [6], using a short, elegant argument.

The goal of this chapter is to show that in fact this conjecture is false for all $r \geq 3$.

Theorem 4.1.1. *Suppose that $r \geq 3$. There exist infinitely many r-edge-coloured complete graphs which cannot be vertex-partitioned into r monochromatic cycles.*

Theorem 4.1.1 is proved in Section 4.2. For a particular counterexample of low order to the case $r = 3$ of Conjecture 1.4.3, see Figure 4.1.

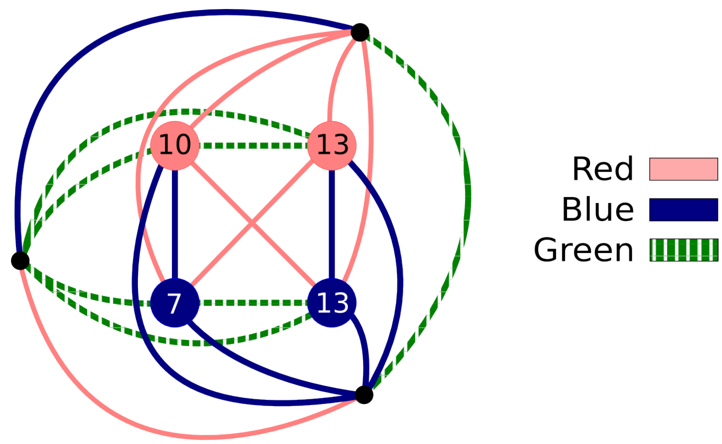

Figure 4.1: A 3-edge colouring of K_{46} which cannot be partitioned into three monochromatic cycles. The small black dots represent single vertices. The large red and blue circles represent red and blue complete graphs of order specified by the numbers inside. The coloured lines between the sets represent all the edges between them being of that colour. This particular colouring is called J_3^1 in this chapter. In Section 3 we prove that this colouring does not allow a partition into three monochromatic cycles.

Theorem 4.1.1 raises a number of open question about partitioning coloured complete graphs. In particular it is not clear whether some modification of Conjecture 1.4.3 holds or not. The counterexamples that we construct in this chapter are very mild—in all the r-coloured graphs that we construct it is possible to cover all *except one* of the vertices by r disjoint monochromatic cycles. Therefore it is still possible that slight refinements of the conjecture are true. In Section 4.3 we discuss a number of such refinements.

4.2 Construction

In this section, we will prove Theorem 4.1.1, by constructing a sequence of r-edge-coloured complete graphs, J_r^m, which cannot be partitioned into r monochromatic cycles for all $r \geq 3$. In order to construct J_r^m, we will first need a sequence of auxiliary r-edge-coloured complete graphs, which cannot be partitioned into r monochromatic paths *with different colours*. According to [28] such colourings were first found by Heinrich.

The following lemma shows that such colourings exist.

Lemma 4.2.1. *For each $r \geq 3$, there exists a sequence of r-edge-coloured complete graphs, H_r^m, which satisfy the following.*

(i) *H_r^m cannot be vertex-partitioned into $r-1$ monochromatic paths.*

(ii) *H_r^m cannot be vertex-partitioned into r monochromatic paths with different colours.*

The proof of Lemma 4.2.1 is somewhat technical and will be performed at the end of this section. First we will show how to use Lemma 4.2.1 to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. For fixed r, let H_r^m be a sequence of graphs satisfying (i) and (ii) of Lemma 4.2.1. We construct a sequence of r-edge-coloured complete graphs, J_r^m, on $|H_r^m| + r$ vertices as follows.

Construction 4.2.2. *We partition the vertices of $|J_r^m|$ into a set H of order $|H_r^m|$ and a set of r vertices $\{v_1, \ldots, v_r\}$. The edges in H are coloured to produce a copy of H_r^m. For each $i \in \{1, \ldots, r\}$, we colour all the edges between v_i and H with colour i. The edge v_1v_2 has colour 3. For $j \geq 3$ the edge v_1v_j has colour 2 and the edge v_2v_j has colour 1. For $3 \leq i < j$, the edge v_iv_j has colour 1.*

We now prove that for every m, J_r^m cannot be partitioned into r disjoint monochromatic cycles.

Suppose that $C_1, \ldots, C_r$ are r disjoint monochromatic cycles in J_r^m. We need to show that $C_1 \cup \cdots \cup C_r \neq J_r^m$. Note that, for any $i \neq j$, the edge v_iv_j has a different colour to the edges between v_i and H. This means that a monochromatic cycle in J_r^m cannot simultaneously pass through edges in $\{v_1, \ldots, v_r\}$ and vertices in H.

Let $P_i = C_i \setminus \{v_1, \ldots, v_r\}$. We claim that, for each i, P_i is a monochromatic path in H. If $C_i \cap \{v_1, \ldots, v_r\} \leq 1$, then this is clear. So, suppose that for $j \neq k$ we have $v_j, v_k \in C_i$. In this case C_i cannot contain vertices in H, since otherwise the edges of C_i which pass through v_j and v_k would have different colours, contradicting the fact that C_i is monochromatic. This means that $P_i = \emptyset$, which is trivially a path.

Therefore $P_1, \ldots, P_r$ partition H into r monochromatic paths. By Lemma 4.2.1, they are all nonempty and not all of different colours. This means that there is a colour, say colour i, which is not present in any of the cycles $C_1, \dots, C_r$. For each j, the fact that P_j is nonempty implies that C_j does not contain edges in $\{v_1, \dots, v_r\}$. But then, the vertex v_i cannot be contained in any of the cycles $C_1, \dots, C_r$ since all the edges between v_i and H have colour i. □

It remains to prove Lemma 4.2.1. The following simple fact will be convenient to state.

Lemma 4.2.3. *Let G be a graph, X an independent set in G, and P a path in G. Then we have*

$$|P \cap X| \leq |P \cap (G \setminus X)| + 1.$$

Proof. Let $x_1, \dots, x_k$ be the vertex sequence of P. For $i \leq k-1$, if x_i is in X, then x_{i+1} must be in $G \setminus X$, implying the result. □

We now prove Lemma 4.2.1.

Proof of Lemma 4.2.1. For $r = 3$, the graphs H_3^m are 3-colourings of K_{43m} constructed as follows.

Construction 4.2.4. *Partition the vertex set of K_{43m} into four classes A_1, A_2, A_3, and A_4 such that $|A_1| = 10m$, $|A_2| = 13m$, $|A_3| = 7m$, and $|A_4| = 13m$. The edges between A_1 and A_2 and between A_3 and A_4 are colour 1. The edges between A_1 and A_3 and between A_2 and A_4 are colour 2. The edges between A_1 and A_4 and between A_2 and A_3 are colour 3. The edges within A_1 and A_2 are colour 3. The edges within A_3 and A_4 are colour 2.*

For $r \geq 4$, the graphs H_r^m are r-coloured complete graphs with $|H_{r-1}^{5m}| + 2m$ vertices constructed as follows.

Construction 4.2.5. *Partition the vertices of H_r^m into two sets H and K such that $|H| = |H_{r-1}^{5m}|$ and $|K| = 2m$. We colour H with colours $1, \dots, r-1$ to produce a copy of H_{r-1}^{5m}. All other edges are coloured with colour r.*

It will be convenient to prove a slight strengthening of the lemma. We will prove that for any $T \subseteq V(H_r^m)$ satisfying $|T| \leq m$, the graph $H_r^m \setminus T$ satisfies parts (i) and (ii) of the lemma.

The proof is by induction on r. First we shall prove the lemma for the initial case, $r = 3$.

Recall that H_3^m is partitioned into four sets A_1, A_2, A_3, and A_4. Let $B_i = A_i \setminus T$. Since $|T| \leq m$, the sets B_1, B_2, B_3, and B_4 are all nonempty. We will need the following claim.

Claim 4.2.6. *The following hold.*

(a) B_2 cannot be covered by a colour 1 path.

(b) B_1 cannot be covered by a colour 2 path.

(c) B_4 cannot be covered by a colour 3 path.

(d) B_4 cannot be covered by a colour 1 path.

(e) $B_1 \cup B_3$ cannot be covered by a colour 1 path contained in $B_1 \cup B_2$ and a disjoint colour 3 path contained in $B_2 \cup B_3$.

(f) $B_2 \cup B_3$ cannot be covered by a colour 1 path contained in $B_3 \cup B_4$ and a disjoint colour 2 path contained in $B_2 \cup B_4$.

Proof.

(a) Let P be any colour 1 path in $H_3^m \setminus T$ which intersects B_2. The path P must then be contained in the colour 1 component $B_1 \cup B_2$. The set B_2 does not contain any colour 1 edges, so Lemma 4.2.3 implies that $|P \cap B_2| \leq |P \cap B_1| + 1$ holds. This, combined with the fact that $|T| \leq m$ holds, implies that we have

$$|P \cap B_2| \leq |P \cap B_1| + 1 \leq |A_1| + 1 = 10m + 1 < 12m \leq |B_2|.$$

This implies that P cannot cover all of B_2.

(b) This part is proved similarly to (a), using the fact that B_1 does not contain any colour 2 edges and that we have $|A_3| + 1 = 7m + 1 < 9m \leq |B_1|$.

(c) This part is proved similarly to (a), using the fact that B_4 does not contain any colour 3 edges and that we have $|A_1| + 1 = 10m + 1 < 12m \leq |B_4|$.

(d) This part is proved similarly to (a), using the fact that B_4 does not contain any colour 1 edges and that we have $|A_3| + 1 = 7m + 1 < 12m \leq |B_4|$.

(e) Let P be a colour 1 path contained in $B_1 \cup B_2$ and let Q be a disjoint colour 3 path contained in $B_2 \cup B_3$. The set B_1 does not contain any colour 1 edges and B_3 does not contain any colour 3 edges, so Lemma 4.2.3 implies that $|(P \cup Q) \cap (B_1 \cup B_3)| \leq |(P \cup Q) \cap B_2| + 2$ holds. This, combined with the fact that $|T| \leq m$ holds, implies that we have

$$|(P \cup Q) \cap (B_1 \cup B_3)| \leq |(P \cup Q) \cap B_2| + 2 \leq |A_2| + 2 = 13m + 2 < 16m \leq (B_1 \cup B_3)$$

This implies that P and Q cannot cover all of $B_1 \cup B_3$.

(f) This part is proved similarly to (e), using the fact that B_2 does not contain any colour 2 edges, B_3 does not contain any colour 1 edges, and that we have $|A_4| + 2 = 13m + 2 < 19m \leq B_2 \cup B_3$.

$\square$

We now prove the lemma for $r = 3$. We deal with parts (i) and (ii) separately.

(i) Suppose, for the sake of contradiction, that P and Q are two monochromatic paths which partition $H_3^m \setminus T$. Note that P and Q cannot have different colours since any two monochromatic paths with different colours in H_3^m can intersect at most three of the four sets B_1, B_2, B_3, and B_4. The colouring $H_3^m \setminus T$ has exactly two components of each colour, so, for each i, the set B_i must be covered by either P or Q. This contradicts case (a), (b), or (c) of Claim 4.2.6 depending on whether P and Q hove colour 1, 2, or 3.

(ii) Suppose, for the sake of contradiction, that P_1, P_2, and P_3 are three monochromatic paths which partition $H_3^m \setminus T$ such that P_i has colour i.

Suppose that $P_2 \subseteq B_1 \cup B_3$. By parts (c) and (d) of Claim 4.2.6, both of the paths P_1 and P_3 must intersect B_4. This leads to a contradiction since none of the paths P_1, P_2, and P_3 intersect B_2.

Suppose that $P_2 \subseteq B_2 \cup B_4$. If $P_1 \subseteq B_1 \cup B_2$ then P_3 must be contained in $B_2 \cup B_3$, contradicting part (e) of Claim 4.2.6. If $P_1 \subseteq B_3 \cup B_4$ then P_3 must be contained in $B_1 \cup B_4$. Therefore $B_2 \cup B_3$ must be covered by P_1 and P_2, contradicting part (f) of Claim 4.2.6. This completes the proof of the lemma for the case $r = 3$.

We now prove the lemma for $r \geq 3$ by induction on r. The initial case $r = 3$ was proved above. Assume that the lemma holds for H_{r-1}^m, for all $m \geq 1$. Let H and K partition H_r^m as in the definition of H_r^m. Suppose that $H_r^m \setminus T$ is partitioned into r monochromatic paths $P_1, \dots, P_r$ (with some of these possibly empty). Without loss of generality we may assume that these are ordered such that each of the paths $P_1, \dots, P_k$ intersects K, and that each of the paths $P_{k+1}, \dots, P_r$ is disjoint from K. Note that we have $k \leq |K| = 2m$. Let $S = H \cap (P_1 \cup \cdots \cup P_k)$. The set $H \setminus T$ does not contain any colour r edges, so Lemma 4.2.3 implies that we have $|S| \leq |K| + k \leq 4m$, and so $|S \cup T| \leq 5m$. We know that $H \setminus (S \cup T)$ is partitioned into $r - k$ monochromatic paths $P_{k+1}, \dots, P_r$, so, by induction, we know that $k = 1$ and that the paths $P_2, \dots, P_r$ are all nonempty and do not all have different colours. This completes the proof since we know that P_1 contains vertices in K, and hence $P_1, \dots, P_r$ are all nonempty, and do not all have different colours.

□

4.3 Discussion

Much of the research on partitioning coloured graphs has focused around Conjecture 1.4.3. Given the disproof of this conjecture, we will spend the remainder of this chapter discussing possible directions for further work.

Although we only constructed counterexamples to Conjecture 1.4.3 for particular n in Section 4.2, it is easy to generalize our construction to work for all $n \geq N_r$, where N_r is a number depending on r. To see this, one only needs to replace the assumption of "m is an integer" with "m is a real number" in Section 4.2, and replace expressions where m appears with suitably chosen integral parts. Doing this and choosing m appropriately will produce r-colourings of K_n which cannot be partitioned into r monochromatic cycles for all sufficiently large n.

Perhaps the most interesting weakening of Conjecture 1.4.3 which may still be true is the following earlier conjecture due to Gyárfás.

Conjecture 1.4.2 (Gyárfás, [27])**.** *The vertices of every r-edge-coloured complete graph can be covered with r vertex-disjoint monochromatic paths.*

It is easy to check that all the r-coloured graphs constructed in Section 4.2 can be partitioned into r monochromatic paths. In addition in Chapter 5 we show that Conjecture 1.4.2 holds for $r = 3$. These two facts together make Conjecture 1.4.2 still seem very plausible to the author.

Another weakening of Conjecture 1.4.3 is the following approximate version.

Conjecture 4.3.1. *For each r there is a constant c_r, such that in every r-edge-coloured complete graph K_n, there are r vertex-disjoint monochromatic cycles covering $n - c_r$ vertices in K_n.*

For $r = 3$, Theorem 1.4.4 shows that a version of Conjecture 4.3.1 is true with c_r replaced with a function $o_r(n)$ satisfying $\frac{o_r(n)}{n} \to 0$ as $n \to \infty$. In a forthcoming paper [46], the author will prove the $r = 3$ case of Conjecture 4.3.1.

Finally we can weaken Conjecture 1.4.3 by removing the constraint that the cycles covering K_n are disjoint.

Conjecture 4.3.2. *Suppose that the edges of K_n are coloured with r colours. There are r (not necessarily disjoint) monochromatic cycles covering all the vertices in K_n.*

A weaker version of this conjecture where "cycles" is replaced with "paths" has appeared in [27]. Our method of finding counterexamples to Conjecture 1.4.3 relied on first finding graphs which cannot be partitioned into r monochromatic paths of different colours. For $r = 3$, using results from Chapter 5, it is easy to show that every 3-edge-coloured complete graph can be covered by three (not necessarily disjoint) paths of different colours. Therefore for $r = 3$, it is unlikely that something similar to our constructions in Section 4.2 can produce counterexamples to Conjecture 4.3.2. It may even possible that, for all r, one can ask for the cycles in Conjecture 4.3.2 to have different colours.

Chapter 5

Partitioning a 3-edge-coloured complete graph into 3 monochromatic paths

5.1 Introduction

Recall that Erdős, Gyárfás, and Pyber conjectured that every r-edge-coloured complete graph can be covered by r vertex-disjoint monochromatic cycles. In Chapter 4 we showed that this conjecture is false for $r \geq 3$. However it is easy to check that the counterexamples we constructed do not disprove the following earlier conjecture due to Gyárfás.

Conjecture 1.4.2 (Gyárfás, [27])**.** *The vertices of every r-edge-coloured complete graph can be covered with r vertex-disjoint monochromatic paths.*

Given the disproof of the Erdős-Gyárfás-Pyber Conjecture, one of the main open questions in the area is whether Conjecture 1.4.2 holds or not. In this chapter we prove the $r = 3$ case of Conjecture 1.4.2.

Theorem 5.1.1. *For $n \geq 1$, suppose that the edges of K_n are coloured with three colours. There is a vertex-partition of K_n into three monochromatic paths.*

The structure of this chapter is as follows. In Section 5.2 we discuss the main idea of the proof of Theorem 5.1.1 and state some of the intermediate results that we will use. In Sections 5.3 and 5.4 we prove Theorem 5.1.1. In Section 5.5 we give some remarks about partitioning coloured graphs into monochromatic paths.

5.2 Preliminaries

Recall that Lemma 1.5.1 shows that every 2-edge-coloured graph can be partitioned into a red path and a blue balanced complete bipartite graph. This Lemma is combined with a result about partitioning 2-edge-coloured balanced complete bipartite graphs into monochromatic paths.

Lemma 5.2.1. *Suppose that the edges of $K_{n,n}$ are coloured with two colours. There is a vertex-partition of $K_{n,n}$ into three monochromatic paths.*

Lemmas 1.5.1 and 5.2.1 together imply that every 3-edge-coloured complete graph can be partitioned into *four* monochromatic paths. Indeed, given a 3-edge-coloured complete graph, first treat blue and green as one colour and apply Lemma 1.5.1 to the graph. this gives a partition of the original graph into a red path and a blue-green balanced complete bipartite graph. Then apply Lemma 5.2.1 to this graph to obtain a partition of the complete graph into 4 monochromatic paths.

The proof of Theorem 5.1.1 is substantially more complicated than the above. The reason for this is that in the above argument we had control over which colours the paths partitioning K_n had. However, as shown in Chapter 4, it is not true that every 3-edge-coloured complete graph can be partitioned into three monochromatic paths *without repeating colours*. Therefore a proof of Theorem 5.1.1 needs to take into account colour repetition somewhere—this is where the main difficulties of the proof come from.

In Chapter 4 we constructed 3-edge-coloured complete graphs which could not be partitioned into three monochromatic paths of different colours (see Lemma 4.2.1). Recall that the colourings that we constructed were 4-partite. It turns out that this is a necessary condition for a 3-edge-coloured K_n to not have a partition into 3 monochromatic paths of different colours. Our proof of Theorem 5.1.1 splits into the following two theorems.

Theorem 5.2.2. *Suppose that the edges of K_n are coloured with three colours such that the colouring is not 4-partite. Then K_n can be vertex-partitioned into three monochromatic paths with different colours.*

Theorem 5.2.3. *Suppose that the edges of K_n are coloured with three colours such that the colouring is 4-partite. Then K_n can be vertex-partitioned into three monochromatic paths, at most two of which have the same colour.*

The proofs of these theorems are quite different. Theorem 5.2.2 is proved by combining more advanced versions of Lemmas 1.5.1 and 5.2.1. Theorem 5.2.3 is proved via a case analysis similar to one performed in [31]. We will use Theorem 5.2.2 in the proof of Theorem 5.2.3.

Recall that a colouring of $K_{n,n}$ is split if, and only if, neither of the parts of $K_{n,n}$ is connected in either colour. The following theorem was stated in the introduction.

Theorem 1.4.5(Gyárfás & Lehel, [26, 29])**.** *Suppose that the edges of $K_{n,n}$ are coloured with two colours. If the colouring is not split, then there exist two disjoint monochromatic paths with different colours which cover all, except possibly one, of the vertices of $K_{n,n}$.*

We will prove the following slight extension of Theorem 1.4.5.

Theorem 5.2.4. *Suppose that the edges of $K_{n,n}$ are coloured with two colours. There is a vertex-partition of $K_{n,n}$ into two monochromatic paths with different colours if and only if the colouring on $K_{n,n}$ is not split.*

Theorem 5.2.4 is the improvement of Lemma 5.2.1 that is needed to prove Theorem 5.2.2.

There exist split colourings of $K_{n,n}$ which cannot be partitioned into two monochromatic paths even when we are allowed to repeat colours. Indeed, any split colouring with classes X_1, X_2, Y_1, and Y_2, satisfying $\big||X_1| - |Y_1|\big| \geq 2$ and $\big||X_1| - |Y_2|\big| \geq 2$ will have this property. Using Theorem 5.2.4, it is not hard to show that any 2-colouring of $K_{n,n}$ which cannot be partitioned into two monochromatic paths must be a split colouring with class sizes as above.

5.3 The case when K_n is not 4-partite

In this section we prove Theorem 5.2.2.

We begin by proving the following strengthening of Lemma 1.5.1

Lemma 5.3.1. *Let G be a graph, and v a vertex in the largest connected component of G. There is a vertex-partition of G into a path P, and two sets A and B, such that there are no edges between A and B, and $|A| = |B|$. In addition P is either empty or starts with v.*

Proof. Let C be the connected component of G containing v. We claim that there is a partition of G into a path P and two sets A and B such that the following hold:

(i) $|A| \leq |B|$.

(ii) There are no edges between A and B.

(iii) P is either empty or starts from v.

(iv) $|A \setminus C|$ is as large as possible (whilst keeping (i) – (iii) true).

(v) $|A|$ is as large as possible (whilst keeping (i) – (iv) true).

(vi) $|P|$ is as large as possible (whilst keeping (i) – (v) true).

To see that such a partition exists, note that letting $P = A = \emptyset$ and $B = V(G)$ gives a partition satisfying (i) – (iii), so there must be a partition having $|A \setminus C|$, $|A|$, and $|P|$ maximum, as required by (iv) – (vi).

Assume that P, A and B satisfy (i) – (vi). We claim that $|A| = |B|$ holds. Suppose, for the sake of contradiction, that we have $|A| < |B|$.

Suppose that P is empty. There are two cases depending on whether $C \subseteq A$ or $C \subseteq B$ holds. Note that, by (ii), we are always in one of these cases.

- Suppose that $C \subseteq A$. By (i) and (ii), there must be some connected component of G, say D, which is contained in B. In this case, let $P' = P$, $A' = (A \setminus C) \cup D$, and $B' = (B \setminus D) \cup C$. Since C was the largest connected component of G, we have $|D| <= |C|$ which implies $|A'| \leq |B'|$. Therefore P', A', and B' partition G, satisfy (i) – (iii), and have $|A' \setminus C| = |A| - |C| + |D| > |A| - |C| = |A \setminus C|$. This contradicts $|A \setminus C|$ being maximal in the original partition.

- Suppose that $C \subseteq B$. In this case we have $v \in B$. Letting $P' = \{v\}$, $A' = A$, and $B' = B - v$ gives a partition satisfying (i) – (v), and having $|P'| > |P|$. This contradicts P being maximal in the original partition.

Suppose that P is not empty. Let u be the end vertex of P. There are two cases depending on whether there are any edges between u and B

- Suppose that for some $w \in B$, uw is an edge. Letting $P' = P + w$, $A' = A$, and $B' = B - w$ gives a partition satisfying (i) – (v), and having $|P'| > |P|$. This contradicts P being maximal in the original partition.
- Suppose that for all $w \in B$, uw is not an edge. Letting $P' = P - u$, $A' = A + u$, and $B' = B$ gives partition satisfying (i) – (iv), and having $|A'| > |A|$. This contradicts A being maximal in the original partition.

$\square$

The following could be seen as a strengthening of Lemma 1.5.1, when one of the colour classes of K_n is connected.

Lemma 5.3.2. *Suppose that G is connected graph. Then at least one of the following holds.*

(i) *There is a path P passing through all but one vertex in G.*

(ii) *There is a vertex-partition of G into a path P, and three nonempty sets A, B_1, and B_2 such that $|A| = |B_1| + |B_2|$ and there are no edges between any two of A, B_1, and B_2.*

Proof. First suppose that for every path P, $G \setminus P$ is connected. Let P be a path in G of maximum length. Let v be an endpoint of P. By maximality, v cannot be connected to anything in $G \setminus P + v$. However, since $P - v$ is a path, $G \setminus P + v$ must be connected, hence it consists of the single vertex v. Thus the path P passes through every vertex in G, proving part (i) of the lemma.

Now, we can assume that there exists a path P_0 such that $G \setminus P_0$ is disconnected. In addition, we assume that P_0 is a shortest such path. The assumption that G is connected implies that P_0 is not empty. Let v_1 be the start of P_0 and v_2 the end of P_0. Let $C_1, \ldots, C_j$ be the connected components of $G \setminus P_0$, ordered such that $|C_1| \geq |C_2| \geq \cdots \geq |C_j|$. The assumption of P_0 being a shortest path, such that $G \setminus P_0$ is disconnected, implies that v_1 and v_2 are both connected to C_t for each $t \in \{1, \ldots, j\}$. Indeed if this were not the case, then either $P_0 - v_1$ or $P_0 - v_2$ would give a shorter path with the required property.

Let u_1 be a neighbour of v_1 in C_1 and u_2 a neighbour of v_2 in C_2. Apply Lemma 5.3.1 to C_1 to obtain a partition of C_1 into a path P_1 and two sets X_1 and Y_1, such that $|X_1| = |Y_1|$ and there are no edges between X_1 and Y_1. Similarly, apply Lemma 5.3.1 to $C_2 \cup \cdots \cup C_j$ to obtain a partition of $C_2 \cup \cdots \cup C_j$ into a path P_2 and two sets X_2 and Y_2, such that $|X_2| = |Y_2|$ and there are no edges between X_2 and Y_2. In addition we can assume that P_1 is either empty or ends at u_1 and that P_2 is either empty or starts at u_2. Since v_1u_1 and v_2u_2 are both edges, we can define the path $Q = P_1 + P_0 + P_2$. Let w_1 be the start of Q, and w_2 the end of Q. We have that either $w_1 \in C_1$ or $w_1 = v_1$ and either $w_2 \in C_2 \cup \cdots \cup C_j$ or $w_2 = v_2$.

If each of the sets X_1, Y_1, X_2, and Y_2 is nonempty, then part (ii) of the lemma holds, using the path Q, $A = X_1 \cup X_2$, $B_1 = Y_1$, and $B_2 = Y_2$.

Suppose that $X_1 = Y_1 = \emptyset$ and $X_2 = Y_2 \neq \emptyset$. In this case w_1 must lie in C_1 since we know that $P_1 \cup X_1 \cup Y_1 = C_1 \neq \emptyset$. Therefore P_1 is nonempty, and so must contain w_1.

Suppose that w_2 has no neighbours in $X_2 \cup Y_2$. Note that in this case $w_2 \neq v_2$ since otherwise $X_2 \cup Y_2 = C_2 \cup \cdots \cup C_j$ would hold, and we know that v_2 has neighbours in $C_2 \ldots C_j$. Therefore, we have $w_2 \in C_2 \cup \cdots \cup C_j$, and so (ii) holds with $P = Q - w_1 - w_2$ as our path, $A = X_2 + w_2$, $B_1 = Y_2$, and $B_2 = \{w_1\}$.

Suppose that w_2 has a neighbour x in $X_2 \cup Y_2$. Without loss of generality, assume that $x \in X_2$. If $|X_2| = |Y_2| = 1$, then part (i) of the lemma holds with $Q + x$ a path covering all the vertices in G except the single vertex in Y_2. If $|X_2| = |Y_2| \geq 2$ then (ii) holds with $P = Q + x - w_1$ as our path, $A = Y_2$, $B_1 = X_2 - x$, and $B_2 = \{w_1\}$.

The case when $X_1 = Y_1 \neq \emptyset$ and $X_2 = Y_2 = \emptyset$ is dealt with similarly. If $X_1 = Y_1 = X_2 = Y_2 = \emptyset$, then Q covers all the vertices in G, so (i) holds. $\square$

We now prove Theorem 5.2.4

Proof of Theorem 5.2.4. Suppose that the colouring of $K_{n,n}$ is split. Two monochromatic paths with different colours can intersect at most three of the sets X_1, X_2, Y_1 and Y_2. This together with the assumption that X_1, X_2, Y_1 and Y_2 are all nonempty implies that $K_{n,n}$ cannot be partitioned into two monochromatic paths with different colours.

It remains to prove that every 2-edge-coloured $K_{n,n}$ which is not split can be partitioned into two monochromatic paths of different colours.

The proof is by induction on n. The case $n = 1$ is trivial. For the remainder of the proof assume that the result holds for $K_{m,m}$ for all $m < n$.

Suppose that $K_{n,n}$ is 2-edge-coloured such that the colouring is not split. Apply Lemma 1.6.4 to $K_{n,n}$. There are two cases to consider, depending on which part of Lemma 1.6.4 occurs.

Case 1: Suppose that the colouring on $K_{n,n}$ satisfies Case (i) of Lemma 1.6.4. Without loss of generality we can assume that $K_{n,n}$ is connected in red.

Apply Lemma 5.3.2 to the red colour class of $K_{n,n}$. If Case (i) of Lemma 5.3.2 occurs, then the theorem follows since we may choose P to be our red path and the single vertex to be our blue path.

So we can assume that we are in case (ii) of Lemma 5.3.2. This gives us a partition of $K_{n,n}$ into a red path P, and three nonempty sets A, B_1, and B_2, such that $|A| = |B_1| + |B_2|$ and all the edges between A, B_1, and B_2 are blue. Let $H = (A \cap X) \cup (B_1 \cap Y) \cup (B_2 \cap Y)$ and $K = (A \cap Y) \cup (B_1 \cap X) \cup (B_2 \cap X)$. Note that $K_{n,n}[H]$ and $K_{n,n}[K]$ are both blue complete bipartite subgraphs of $K_{n,n}$, since all the edges between A and $B_1 \cup B_2$ are blue. Notice that $|A| = |B_1| + |B_2|$ and $|X| = |Y|$ together imply that P contains an even number of vertices. This, together with the fact that the vertices of P must alternate between X and Y, implies that $|X \setminus P| = |Y \setminus P|$.

However $X \setminus P = X \cap (A \cup B_1 \cup B_2)$ and $Y \setminus P = Y \cap (A \cup B_1 \cup B_2)$, so we have that

$$|X \cap A| + |X \cap B_1| + |X \cap B_2| = |Y \cap A| + |Y \cap B_1| + |Y \cap B_2|. \tag{5.1}$$

Equation (5.1), together with $|X \cap A| + |Y \cap A| = |Y \cap B_1| + |Y \cap B_2| + |X \cap B_1| + |X \cap B_2|$ implies that the following both hold:

$$|A \cap X| = |B_1 \cap Y| + |B_2 \cap Y|, \tag{5.2}$$

$$|A \cap Y| = |B_1 \cap X| + |B_2 \cap X|. \tag{5.3}$$

Thus $K_{n,n}[H]$ and $K_{n,n}[K]$ are balanced blue complete bipartite subgraphs of $K_{n,n}$ and so can each be covered by a blue path. If $H = \emptyset$ or $K = \emptyset$ holds, the theorem follows, since $V(K_{n,n}) = V(P) \cup H \cup K$.

So, we can assume that $H \neq \emptyset$ and $K \neq \emptyset$. Equation (5.2), together with $H \neq \emptyset$, implies that $(B_1 \cup B_2) \cap H \neq \emptyset$. Similarly (5.3) together with $K \neq \emptyset$, implies that $(B_1 \cup B_2) \cap K \neq \emptyset$. We also know that B_1 and B_2 are nonempty and contained in $H \cup K$. Combining all of these implies that at least one of the following holds.

(a) $B_1 \cap H \neq \emptyset$ and $B_2 \cap K \neq \emptyset$.

(b) $B_1 \cap K \neq \emptyset$ and $B_2 \cap H \neq \emptyset$.

Suppose that (a) holds. Choose $x \in B_1 \cap H$ and a blue path Q covering H and ending with x. Choose $y \in B_2 \cap K$ and a blue path R covering K and starting with y. Notice that $x \in X$ and $y \in Y$, so there is an edge xy. The edge xy must be blue since it lies between B_1 and B_2. This means that $Q + R$ is a blue path covering $A \cup B_1 \cup B_2 = G \setminus P$, implying the theorem. The case when (b) holds can be treated identically, exchanging the roles of H and K.

Case 2: Suppose that $K_{n,n}$ satisfies Case (ii) of Lemma 1.6.4. Without loss of generality, this gives us a vertex $u \in X$ such that the edge uy is red for every $y \in Y$. Let v be any vertex in Y.

Suppose that the colouring of $K_{n,n} - u - v$ is split with classes X_1, X_2, Y_1, and Y_2. In this case $K_{n,n}[X_1, Y_2]$, $K_{n,n}[X_2, Y_1]$, and $\{v\}$ are all connected in red, and u is connected to each of these by red edges. This means that $K_{n,n}$ is connected in red and we are back to the previous case.

So, suppose that the colouring of $K_{n,n} - u - v$ is not split. We claim that there is a partition of $K_{n,n} - u - v$ into a red path P and a blue path Q such that either P is empty or P ends in Y. To see this, apply the inductive hypothesis to $K_{n,n} - u - v$ to obtain a partition of this graph into a red path P' and a blue path Q'. If P' is empty or P' has an endpoint in Y, then we can let $P = P'$ and $Q = Q'$. Otherwise, the endpoints P' are in X, and so the endpoints of Q' are in Y. Let x be the end of P' and y the end of Q'. If xy is red, let $P = P' + y$ and $Q = Q' - y$. If xy is blue, let $P = P' - x$ and $Q = Q' + x$. In either case, P and Q give a partition of $K_{n,n} - u - v$ into two paths such that either P is empty or P has an endpoint in Y.

Suppose that P is empty. In this case we have a partition of $K_{n,n}$ into a red path $\{u, v\}$ and a blue path Q.

Suppose that P ends in a vertex, w, in Y. The edges uv and uw are both red, so $P + u + v$ is a red path giving the required partition of $K_{n,n}$ into a red path $P + u + v$ and a blue path Q. $\square$

As remarked in the introduction, there are split colourings of $K_{n,n}$ which cannot be partitioned into two monochromatic paths. The following lemma shows that three monochromatic paths always suffice.

Lemma 5.3.3. *Suppose that the edges of $K_{n,n}$ are coloured with two colours. Suppose that the colouring is split with classes X_1, X_2, Y_1, and Y_2. For any two vertices $y_1 \in Y_1$ and $y_2 \in Y_2$, there is a vertex-partition of $K_{n,n}$ into a red path starting at y_1, a red path starting at y_2, and a blue path.*

Proof. Without loss of generality, suppose that $X_1 \leq X_2$ and $Y_1 \leq Y_2$. This, together with $X_1 + X_2 = Y_1 + Y_2$ implies that $X_1 \leq Y_2$ and $Y_1 \leq X_2$ both hold.

$K_{n,n}[X_1, Y_2]$ is a red complete bipartite graph, so we can cover X_1 and $|X_1|$ vertices in Y_2 with a red path starting from y_1. Similarly we can cover Y_2 and $|Y_2|$ vertices in X_1 with a red path starting from y_2. The only uncovered vertices are in Y_2 and X_1. All the edges between these are blue, so we can cover the remaining vertices with a blue path. $\square$

We are now ready to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. The two main cases that we will consider are when K_n is connected in some colour, and when K_n is disconnected in all three colours.

Case 1: Suppose that K_n is connected in some colour. Without loss of generality, we may assume that this colour is red. Apply Lemma 5.3.2 to the red colour class of K_n. If Case (i) of Lemma 5.3.2 occurs, then the theorem follows since we can take P as our red path, the single vertex as our blue path and the empty set as our green path. Therefore, we can assume that Case (ii) of Lemma 5.3.2 occurs, giving us a partition of K_n into a red path P and three sets A, B_1, and B_2 such that $|A| = |B_1| + |B_2|$ and all the edges between A, B_1, and B_2 are blue or green.

If the colouring on $K_n[A, B_1 \cup B_2]$ is not split, we can apply Theorem 5.2.4 to partition $K_n[A, B_1 \cup B_2]$ into a blue path and a green path proving the theorem.

So, assume $K_n[A, B_1 \cup B_2]$ is split with classes X_1, X_2, Y_1, and Y_2, such that $A = X_1 \cup X_2$ and $B_1 \cup B_2 = Y_1 \cup Y_2$. Then, the fact that B_1, B_2, Y_1, and Y_2 are nonempty implies that one of the following holds.

(i) $B_1 \cap Y_1 \neq \emptyset$ and $B_2 \cap Y_2 \neq \emptyset$.

(ii) $B_1 \cap Y_2 \neq \emptyset$ and $B_2 \cap Y_1 \neq \emptyset$.

Assume that (i) holds. Choose $y_1 \in B_1 \cap Y_1$ and $y_2 \in B_2 \cap Y_2$. The edge y_1y_2 must be blue or green since it lies between B_1 and B_2. Assume that y_1y_2 is blue. Apply Lemma 5.3.3 to partition $K_n[A, B_1 \cup B_2]$ into a blue path Q ending with y_1, a blue path R starting from y_2 and a green path S. By joining Q and R with the edge y_1y_2, we obtain a partition of G into three monochromatic paths P, $Q + R$, and S, all of different colours. The cases when (ii) holds or when the edge y_1y_2 is green are dealt with similarly.

Case 2: Suppose that K_n is disconnected in all three colours. Let C be the largest connected component in any colour class. Without loss of generality we may suppose that C is a red connected component. Let D be a blue connected component. Let $C^c = V(K_n) \setminus C$ and $D^c = V(K_n) \setminus D$. One of the sets $C \cap D$, $C^c \cap D$, $C \cap D^c$, or

$C^c \cap D^c$ must be empty. Indeed if all these sets were nonempty, then Lemma 1.6.2 would imply that the colouring is 4-partite, contradicting the assumption of the theorem.

We claim that $D \subseteq C$ or $D^c \subseteq C$ holds. To see this consider four cases depending on which of $C \cap D$, $C^c \cap D$, $C \cap D^c$ or $C^c \cap D^c$ is empty.

- $C \cap D = \emptyset$ implies that all the edges between C and D are green. This contradicts C being the largest component in any colour.
- $C^c \cap D = \emptyset$ implies that $D \subseteq C$.
- $C \cap D^c = \emptyset$ implies that $C \subseteq D$. Since C is the largest component of any colour, this means that $C = D$.
- $C^c \cap D^c = \emptyset$ implies that $D^c \subseteq C$.

We claim that there is a vertex $v \in C$ such that all the edges between v and C^c are green. Indeed if $D \subseteq C$ holds, then choose $v \in D$. If $D^c \subseteq C$ holds, then choose $v \in D^c$. In either case the edges between v and C^c are green.

Apply Lemma 5.3.1 to the red colour class of K_n in order to obtain a partition of K_n into a red path P and two sets A and B such that $|A| = |B|$ and all the edges between A and B are colours 2 or 3. In addition, P is either empty or starts at v. If either of the graphs $K_n[A]$ or $K_n[B]$ is disconnected in red, then we can proceed just as we did after we applied Lemma 5.3.2 in the previous part of the theorem. So assume that both $K_n[A]$ and $K_n[B]$ are connected in red. We claim that one of the sets A or B must be contained in C^c. Indeed otherwise C would intersect each of P, A, and B. Since P, $K_n[A]$, and $K_n[B]$ are connected in red, this would imply that $C = P \cup A \cup B = K_n$ contradicting K_n being disconnected in red. Without loss of generality we may assume that $B \subseteq C^c$. Therefore all the edges between v and B are green.

As before, if the colouring on $K_n[A, B]$ is not split, we can apply Theorem 5.2.4 to partition $K_n[A, B]$ into a blue path and a green path. Therefore assume that the colouring on $K_n[A, B]$ is split.

If the path P is empty, then we must have $v \in A$. Lemma 1.6.4 leads to a contradiction, since we know that all the edges between v and B are green, and $K_n[A, B]$ is split.

Therefore the path P is nonempty. We know that $K_n[A, B]$ is split with classes X_1, X_2, Y_1, and Y_2, such that $A = X_1 \cup X_2$ and $B = Y_1 \cup Y_2$. Choose $y_1 \in Y_1$ and $y_2 \in Y_2$ arbitrarily. Apply Lemma 5.3.3 to $K_n[A, B]$ to partition $K_n[A, B]$ into a green path Q ending with y_1, a green path R starting from y_2, and a blue path S. Notice that the edges y_1v and vy_2 are both green, so $P - v$, S, and $Q + v + R$ give a partition of K_n into three monochromatic paths, all of different colours.

□

5.4 The 4-partite case

In this section, we prove Theorem 5.2.3. Some of the ideas used here are taken from the proof of a similar theorem in [31].

Proof. Let A_1, A_2, A_3, and A_4 be the classes of the 4-partition of K_n, with colours between the classes as in Definition 1.6.1. Our proof will be by induction on n. The initial case of the induction will be $n = 4$, since for smaller n there are no 4-partite colourings of K_n. For $n = 4$, the result is trivial. Suppose that the result holds for K_m for all $m < n$.

For $i = 1$, 2, 3, and 4 we assign three integers r_i, b_i, and g_i to A_i corresponding to the three colours as follows:

(i) Suppose that A_i can be partitioned into three nonempty monochromatic paths R_i, B_i, and G_i of colours red, blue, and green respectively. In this case, let $r_i = |R_i|$, $b_i = |B_i|$, and $g_i = |G_i|$.

(ii) Suppose that A_i can be partitioned into three nonempty monochromatic paths P_1, P_2, and Q such that P_1 and P_2 are coloured the same colour and Q is coloured a different colour. If P_1 and P_2 are red, then we let $r_i = |P_1| + |P_2| - 1$. If Q is red, then we let $r_i = |Q|$. If none of P_1, P_2, or Q are red, then we let $r_i = 1$. We do the same for "blue" and "green" to assign values to b_i and g_i respectively. As a result we have assigned the values $|P_1| + |P_2| - 1$, $|Q|$, and 1 to some permutation of the three numbers r_i, b_i, and g_i.

(iii) Suppose that $|A_i| \leq 2$. In this case, let $r_i = b_i = g_i = 1$.

For each $i \in \{1, 2, 3, 4\}$, A_i will always be in at least one of the above three cases. To see this, depending on whether the colouring on A_i is 4-partite or not, apply either Theorem 5.2.2 or the inductive hypothesis of Theorem 5.2.3 to A_i, in order to partition A_i into three monochromatic paths P_1, P_2, and P_3 at most two of which are the same colour. If $|A_i| \geq 3$ then we can assume that P_1, P_2, and P_3 are nonempty. Indeed if P_1, P_2, or P_3 are empty, then we can remove endpoints from the longest of the three paths and add them to the empty paths to obtain a partition into three nonempty paths, at most two of which are the same colour. Therefore, if $|A_i| \geq 3$, then either Case (i) or (ii) above will hold, whereas if $A_i \leq 2$, then Case (iii) will hold.

For each $i \in \{1, 2, 3, 4\}$, note that the numbers r_i, b_i, and g_i are positive and satisfy $r_i + b_i + g_i \geq |A_i|$. We will need the following definition.

Definition 5.4.1. *A red linear forest F is A_i-**filling** if F is contained in A_i, and either F consists of one path of order r_i, or F consists of two paths F_1 and F_2 such that $|F_1| + |F_2| = r_i + 1$.*

Blue or green A_i-filling linear forests are defined similarly, exchanging the role of r_i for b_i or g_i respectively. We will need the following two claims.

Claim 5.4.2. *Suppose that $i \in \{1, 2, 3, 4\}$, and $|A_i| \geq 2$. There exist two disjoint A_i-filling linear forests with different colours for any choice of two different colours.*

Proof. Claim 5.4.2 holds trivially from the definition of r_i, b_i, and g_i. □

Claim 5.4.3. *Suppose that $i, j \in \{1, 2, 3, 4\}$ such that $i \neq j$ and $K_n[A_i, A_j]$ is red. Let m be an integer such that the following hold.*

$$0 \leq m \leq r_i, \tag{5.4}$$

$$|A_i| - m \leq |A_j|. \tag{5.5}$$

There exists a red path P from A_i to A_i, of order $2|A_i| - m$, covering all of A_i and any set of $A_i - m$ vertices in A_j.

Proof. Note that from the definition of r_i, b_i, and g_i, we can always find an A_i-filling linear forest, F.

Suppose that F consists of one path of order r_i. By (5.4), we can shorten F to obtain a new path F' of order m. By (5.5), we can choose a red path, P, from A_i to A_j consisting of $A_i \setminus F'$ and any $|A_i| - m$ vertices in $A_j \setminus F'$. The path $P + F'$ satisfies the requirements of the claim.

Suppose that F consists of two paths F_1 and F_2 such that $|F_1| + |F_2| = r_i + 1$. By (5.4), we can shorten F_1 and F_2 to obtain two paths F_1' and F_2' such that $|F_1| + |F_2| = m + 1$. By (5.5), we can choose a red path, P, from A_i to A_j consisting of $A_i \setminus F'$ and any $|A_i| - m - 1$ vertices in A_j. By (5.5) there must be at least one vertex, v, in $A_j \setminus P$. The path $P + F_1 + v + F_2$ satisfies the requirements of the claim.

□

We can formulate versions of Claim 5.4.3 for the colours blue or green as well, replacing r_i with b_i or g_i respectively.

To prove Theorem 5.2.3, we will consider different combinations of values of r_i, b_i, and g_i for $i = 1$, 2, 3, and 4 to construct a partition of K_n into three monochromatic paths in each case.

If a partition of K_n into monochromatic paths contains edges in the graph $K_n[A_i, A_j]$ for $i \neq j$, we say that $K_n[A_i, A_j])$ is a *target component* of the partition. Note that a partition of K_n into three monochromatic paths can have at most three target components. This is because a monochromatic path can pass through edges in at most one of graphs $K_n[A_i, A_j]$.

There are two kinds of partitions into monochromatic paths which we shall construct.

- We say that a partition of K_n is *star-like* if the target components are $K_n[A_i, A_j]$, $K_n[A_i, A_k]$, and $K_n[A_i, A_l]$, for (i, j, k, l) some permutation of $(1, 2, 3, 4)$. In this case, all the paths in the partition will have different colours.

- We say that a partition of K_n is *path-like* if the target components are $K_n[A_i, A_j]$, $K_n[A_j, A_k]$, and $K_n[A_k, A_l]$, for (i, j, k, l) some permutation of $(1, 2, 3, 4)$. In this case, two of the paths in the partition will have the same colour.

For $i \in \{1, 2, 3, 4\}$, it is possible to write down sufficient conditions on $|A_i|$, r_i, b_i, and g_i for K_n to have a partition into three monochromatic paths with given target components.

Claim 5.4.4. *Suppose that the following holds:*

$$|A_1| + |A_2| + |A_3| \leq |A_4| + r_1 + b_2 + g_3. \tag{5.6}$$

Then, K_n has a star-like partition with target components $K_n[A_4, A_1]$, $K_n[A_4, A_2]$, and $K_n[A_4, A_3]$ of colours red, blue, and green respectively.

Proof. Using (5.6), we can find three disjoint subsets S_1, S_2, and S_3 of A_4 such that $|S_1| = |A_1| - r_1$, $|S_2| = |A_2| - b_2$, and $|S_3| = |A_3| - g_3$ all hold. By Claim 5.4.3 there is a red path P_1 with vertex set $A_1 \cup S_1$, a blue path P_2 with vertex set $A_2 \cup S_2$, and a green path P_3 with vertex set $A_3 \cup S_3$. The paths P_1, P_2, and P_3 are pairwise disjoint and have endpoints in A_1, A_2, and A_3 respectively.

Depending on whether $A_4 \setminus (P_1 \cup P_2 \cup P_3)$ is 4-partite or not, apply either Theorem 5.2.2 or the inductive hypothesis to find a partition of $A_4 \setminus (P_1 \cup P_2 \cup P_3)$ into three monochromatic paths Q_1, Q_2, and Q_3 at most two of which are the same colour.

We will join the paths P_1, P_2, P_3, Q_1, Q_2, and Q_3 together to obtain three monochromatic paths partitioning all the vertices in K_n.

Suppose that the paths Q_i all have different colours, with Q_1 red, Q_2 blue, and Q_3 green. In this case $P_1 + Q_1$, $P_2 + Q_2$, and $P_3 + Q_3$ are three monochromatic paths forming a star-like partition of K_n.

Suppose that two of the paths Q_i have the same colour. Without loss of generality, we may assume that Q_1 and Q_2 are red and Q_3 is blue. In this case $Q_1 + P_1 + Q_2$, P_2, and $P_3 + Q_3$ are three monochromatic paths forming a star-like partition of K_n. □

Claim 5.4.5. *Suppose that the following all hold:*

$$|A_1| + |A_4| \leq |A_2| + |A_3| + b_4 + g_4 + g_1, \tag{5.7}$$
$$|A_3| + |A_2| \leq |A_1| + |A_4| + b_2 + g_2 + g_3, \tag{5.8}$$
$$|A_1| < |A_2| + g_1, \tag{5.9}$$
$$|A_3| < |A_4| + g_3. \tag{5.10}$$

Then K_n has a path-like partition with target components $K_n[A_1, A_2]$, $K_n[A_3, A_4]$, and $K_n[A_4, A_3]$ of colours green, blue, and green respectively.

Proof. Suppose that we have

$$|A_2| - |A_1| + g_1 \geq |A_4| - |A_3| + g_3. \tag{5.11}$$

The inequality (5.10), together with Claim 5.4.3 ensures that we can find a green path P_1 consisting of all of A_3 and $|A_3| - g_3$ vertices in A_4.

There are two subcases depending on whether the following holds or not:

$$|A_2| - |A_1| \leq |A_4| - |A_3| + g_3. \tag{5.12}$$

Suppose that (5.12) holds. Let $m = |A_1| - |A_2| + |A_4| - |A_3| + g_3$. Note that $|A_1| - m \leq |A_2|$ holds by (5.10), that m is nonnegative by (5.12), and that m is less than g_1 by (5.11). Therefore, we can apply Claim 5.4.3 to find a green path P_2 consisting of A_1 and $|A_1| - m$ vertices in A_2. There remain exactly $|A_4| - |A_3| + g_3$ vertices in each of A_2 and A_4 outside of the paths P_1 and P_2. Cover these with a blue path P_3 giving the required partition.

Suppose that (5.12) fails to hold. Note that (5.10) and the negation of (5.12) imply that $|A_2| > |A_1|$ which, together with the fact that $|A_1| > 0$, implies that $|A_2| \geq 2$. Therefore, we can apply Claim 5.4.2 to A_2 to obtain a blue A_2-filling linear forest, B, and a disjoint green A_2-filling linear forest, G. We construct a blue path P_B and a green path P_G as follows:

Note that $A_4 \setminus P_1$ is nonempty by (5.10), so let u be a vertex in $A_4 \setminus P_1$. If B is the union of two paths B_1 and B_2 such that $|B_1|+|B_2| = b_2+1$, then let $P_B = B_1+\{v\}+B_2$. Otherwise B must be single path of order b_1, and we let $P_B = B$.

Similarly, let v be a vertex in A_1. If G consists of two paths G_1 and G_2, we let $P_G = G_1 + v + G_2$. If G is a single path, we let $P_G = G$.

Note that the above construction and (5.8) imply that the following is true.

$$|A_2 \setminus (P_B \cup P_G)| \leq |A_1 \setminus (P_B \cup P_G)| + |(A_4 \setminus P_1) \setminus (P_B \cup P_G)|. \tag{5.13}$$

The negation of (5.12) is equivalent to the following

$$|A_2| \geq |A_1| + |(A_4 \setminus P_1)|. \tag{5.14}$$

Let P'_B and P'_G be subpaths of P_B and P_G respectively, such that the sum $|P'_B|+|P'_G|$ is as small as possible and we have

$$|A_2 \setminus (P'_B \cup P'_G)| \leq |A_1 \setminus (P'_B \cup P'_G)| + |(A_4 \setminus P'_1) \setminus (P'_B \cup P'_G)|. \tag{5.15}$$

The paths P'_B and P'_G are well defined by (5.13). We claim that we actually have equality in (5.15). Indeed, since A_1, A_2, and $A_4 \setminus P'_1$ are all disjoint, removing a single vertex from P'_B or P'_G can change the inequality (5.15) by at most one. Therefore, if the inequality (5.15) is strict, we know that P'_B and P'_G are not both empty by (5.14), so we can always remove a single vertex from P'_B or P'_G to obtain shorter paths satisfying (5.15), contradicting the minimality of $|P'_B| + |P'_G|$.

Equality in (5.15) implies that $|A_2 \setminus (P_1 \cup P'_B \cup P'_G)| = |(A_1 \cup A_4) \setminus (P_1 \cup P'_B \cup P'_G)|$, so we can choose a green path Q_G from A_1 to A_2 and a disjoint blue path Q_B from $(A_4 \setminus P_1)$ to A_2 such that $Q_B \cup Q_G = (A_1 \cup A_2 \cup A_4) \setminus (P_1 \cup P'_B \cup P'_G)$. The paths P_1, $P'_B + Q_B$, and $P'_G + Q_G$ give us the required partition of K_n.

If the negation of (5.11) holds, we can use the same method, exchanging the roles of A_1 and A_3, and of A_2 and A_4.

□

Clearly, there was nothing particularly special about our choice of target components in Claims 5.4.4 and 5.4.5. Similar sufficient conditions can be written for there to be a star-like partition or a path-like partition for any choice of target components. To prove Theorem 5.2.3, we shall show that either the inequality in Claim 5.4.4 holds or all four inequalities from Claim 5.4.5 hold for some choice of target components.

Without loss of generality we can assume that the following holds:

$$|A_1| \leq |A_2| \leq |A_3| \leq |A_4|. \tag{5.16}$$

Consider the following instances of Claims 5.4.4 and 5.4.5.

There is a star-like partition of K_n with target components $K_n[A_4, A_1]$, $K_n[A_4, A_2]$, and $K_n[A_4, A_3]$ of colours red, blue, and green respectively if the following holds:

$$|A_1| + |A_2| + |A_3| \leq |A_4| + r_1 + b_2 + g_3. \tag{A1}$$

There is a path-like partition of K_n with target components $K_n[A_1, A_2]$, $K_n[A_2, A_4]$, and $K_n[A_4, A_3]$ of colours green, blue, and green respectively if the following holds:

$$\begin{aligned} |A_1| + |A_4| &\leq |A_2| + |A_3| + b_4 + g_4 + g_1, && \text{(B1)} \\ |A_3| + |A_2| &\leq |A_1| + |A_4| + b_2 + g_2 + g_3, && \text{(B2)} \\ |A_1| &< |A_2| + g_1, \\ |A_3| &< |A_4| + g_3. \end{aligned}$$

There is a path-like partition of K_n with target components $K_n[A_1, A_4]$, $K_n[A_4, A_3]$, and $K_n[A_3, A_2]$ of colours red, green, and red respectively if the following holds:

$$\begin{aligned} |A_1| + |A_3| &\leq |A_2| + |A_4| + g_3 + r_3 + r_1, \\ |A_2| + |A_4| &\leq |A_1| + |A_3| + g_4 + r_4 + r_2, && \text{(C2)} \\ |A_1| &< |A_4| + r_1, \\ |A_2| &< |A_3| + r_2. \end{aligned}$$

There is a path-like partition of K_n with target components $K_n[A_1, A_3]$, $K_n[A_3, A_2]$, and $K_n[A_2, A_4]$ of colours blue, red, and blue respectively if the following holds:

$$\begin{aligned}
|A_1| + |A_2| &\leq |A_3| + |A_4| + r_2 + b_2 + b_1, \\
|A_3| + |A_4| &\leq |A_1| + |A_2| + r_3 + b_3 + b_4, && \text{(D2)} \\
|A_1| &< |A_3| + b_1, \\
|A_4| &< |A_2| + b_4. && \text{(D4)}
\end{aligned}$$

There is a path-like partition of K_n with target components $K_n[A_1, A_4]$, $K_n[A_4, A_2]$, and $K_n[A_2, A_3]$ of colours red, blue, and red respectively if the following holds:

$$\begin{aligned}
|A_1| + |A_2| &\leq |A_3| + |A_4| + b_2 + r_2 + r_1, \\
|A_3| + |A_4| &\leq |A_1| + |A_2| + b_4 + r_4 + r_3, && \text{(E2)} \\
|A_1| &< |A_4| + r_1, \\
|A_3| &< |A_2| + r_3. && \text{(E4)}
\end{aligned}$$

There is a path-like partition of K_n with target components $K_n[A_2, A_4]$, $K_n[A_4, A_1]$, and $K_n[A_1, A_3]$ of colours blue, red, and blue respectively if the following holds:

$$\begin{aligned}
|A_1| + |A_2| &\leq |A_3| + |A_4| + r_1 + b_1 + b_2, \\
|A_3| + |A_4| &\leq |A_1| + |A_2| + r_4 + b_4 + b_3, && \text{(F2)} \\
|A_2| &< |A_4| + b_2, \\
|A_3| &< |A_1| + b_3. && \text{(F4)}
\end{aligned}$$

Note that all the unlabelled inequalities hold as a consequence of (5.16) and the positivity of r_i, b_i, and g_i. Thus, to prove the theorem it is sufficient to show that all the labelled inequalities corresponding to some particular letter A, B, C, D, E, or F hold. We split into two cases depending on whether (B1) holds or not.

Case 1: Suppose that (B1) holds.

Note that the following cannot all be true at the same time:

$$|A_3| + r_2 > |A_4| + g_3, \tag{5.17}$$
$$|A_2| + b_4 > |A_3| + r_2, \tag{5.18}$$
$$|A_4| + g_3 > |A_2| + b_4. \tag{5.19}$$

Indeed adding these three inequalities together gives $0 > 0$. Thus the negation of (5.17), (5.18), or (5.19) must hold.

The negation of (5.17) together with $r_2 + b_2 + g_2 \geq |A_2|$ implies that we have (B2). This, together with our assumption that (B1) holds, implies that all the inequalities corresponding to the letter "B" hold.

The negation of (5.18) together with $r_4 + b_4 + g_4 \geq |A_4|$ implies that we have (C2). Therefore, all the inequalities corresponding to the letter "C" hold.

The negation of (5.19), together with $|A_3| \leq r_3 + b_3 + g_3$ implies that (D2) holds. The negation of (5.19), together with $g_3 > 0$ implies that (D4) holds. Therefore, all the inequalities corresponding to the letter "D" hold.

Case 2: Suppose that (B1) does not hold. If (C2) holds, then all the inequalities labelled "C" hold, so we assume that the negation of (C2) holds. We consider three subcases depending on which of (E4) and (F4) hold.

Subcase 1: Suppose that (E4) holds. If (E2) holds, then all the inequalities labelled "E" hold, so we assume that the negation of (E2) holds. Adding the negations of (B1), (C2), and (E2) together, and using $|A_4| \leq r_4 + b_4 + g_4$ gives the following:

$$|A_4| > |A_1| + |A_2| + |A_3| + g_1 + r_2 + r_3.$$

This is stronger than (A1) which implies that all the inequalities corresponding to the letter "A" hold.

Subcase 2: Suppose that (F4) holds. If (F2) holds, then all the inequalities labelled "F" hold, so we assume that the negation of (F2) holds. Adding the negations of (B1), (C2), and (F2) together, and using $|A_4| \leq r_4 + b_4 + g_4$ gives the following:

$$|A_4| > |A_1| + |A_2| + |A_3| + g_1 + r_2 + b_3.$$

This is stronger than (A1) which implies that all the inequalities corresponding to the letter "A" hold.

Subcase 3: Suppose that neither (E4) or (F4) hold. Adding the negations of (B1), (C2), (E4), and (F4) together, and using $|A_4| \leq r_4 + b_4 + g_4$ and $|A_3| \leq r_3 + b_3 + g_3$ gives the following:

$$|A_4| + g_3 > |A_1| + |A_2| + |A_3| + g_1 + r_2 + g_4.$$

This is stronger than (A1) which implies that all the inequalities corresponding to the letter "A" hold. $\square$

5.5 Discussion

It is natural to ask whether the ideas presented in this chapter could make progress with Conjecture 1.4.2 for 4 or more colours. The following lemma is easy to prove using ideas from this chapter.

Lemma 5.5.1. *Suppose that $K_{n,n}$ is coloured with the colours red and blue. Then $K_{n,n}$ can be vertex-partitioned into a red path and two blue balanced complete bipartite graphs.*

This lemma is proved by treating nonedges of $K_{n,n}$ as blue edges and applying Lemma 1.5.1 to the resulting 2-edge-colouring of the complete graph (this is very similar to the proof we gave of Theorem 5.2.4).

Lemma 5.5.1 can be used recursively to show that every r-edge-coloured balanced complete bipartite graph can be partitioned into $2^{r-1} + 1$ monochromatic paths. Combining this with Lemma 1.5.1 shows that every r-edge-coloured complete graph can be partitioned into 2^{r-1} monochromatic paths. Asymptotically this is much worse than the $O(r \log r)$ bound given by Gyárfás, Ruszinkó, Sárközy, and Szemerédi [30]. However, for small r we obtain fairly small bounds on the number of monochromatic paths needed to partition a r-edge-coloured complete graph (for example for $r = 4$ we have that 8 monochromatic paths are sufficient). Therefore it is possible that Lemma 5.5.1 could help with the $r = 4$ or 5 cases of Conjecture 1.4.2. However, new ideas would be needed as well. In particular for $r = 3$ we were able to say a lot about the structure of a 3-edge-coloured complete graph which could not be partitioned into 3 monochromatic paths without repeating colours. For 4 or more colours, we do not even have a guess of what properties such colourings must have.

It is interesting to consider partitions of an edge coloured graph G other than the complete graph. Theorems 1.4.5 and 5.2.4 are results in this direction when G is a balanced complete bipartite graph. We make the following conjecture which would generalise Corollary 5.2.1.

Conjecture 5.5.2. *Suppose that the edges of $K_{n,n}$ are coloured with r colours. There is a vertex-partition of $K_{n,n}$ into $2r-1$ monochromatic paths.*

This conjecture would be optimal, since for all r, there exist r-coloured balanced complete bipartite graphs which cannot be partitioned into $2r-2$ monochromatic paths. We sketch one such construction here. Let X and Y be the classes of the bipartition of a balanced complete bipartite graph. We partition X into $X_1, \dots, X_r$ and Y into $Y_1, \dots, Y_r$ where $|X_i| = 10^i + i$ and $|Y_i| = 10^i + r - i$. The edges between X_i and Y_j are coloured with colour $i + j \pmod r$. It is possible to show that this graph cannot be partitioned into $2r-2$ monochromatic paths. In [33], Haxell showed that every r-edge coloured balanced complete bipartite graph can be partitioned into $O((r \log r)^2)$ monochromatic cycles. Subsequently Peng, Rödl, and Ruciński [45] improved this bound to $O(r^2 \log r)$.

It might be interesting to find out if random graphs can be partitioned into a bounded number of monochromatic paths. In particular, Posa [48] proved that there is a c such that a random graph with $cn \log n$ edges is Hamiltonian with probability tending to 1 as n tends to infinity (by "random graph" we mean the standard Erdős-Renyi model of a random grah—where each edge is chosen uniformly at random with some probablility p). It would be interesting to determine if every 2-edge-colouring random graph with this many edges, has a partition into some fixed number of monochromatic paths.

Chapter 6

Calculating Ramsey Numbers by partitioning coloured graphs

6.1 Introduction

The goal is this chapter is to determine the Ramsey number of a path versus certain graphs.

Our starting point will be the following theorem of Gerencsér and Gyárfás. It is one of the first results about generalized Ramsey numbers (i.e. Ramsey numbers other that $R(K_n, K_m)$).

Theorem 1.3.1 (Gerencsér and Gyárfás, [22]). *For $m \leq n$ we have that*

$$R(P_n, P_m) = n + \left\lfloor \frac{m}{2} \right\rfloor - 1.$$

This theorem is pivotal to this chapter. One of the results that we will prove here is a generalization of the $n = m$ case of the above theorem. The theorem will be applied during the proof of one of our results. In addition the paper in which Theorem 1.3.1 was proved began the theory of partitioning coloured graphs—partitioning coloured graphs will be the main technique we use to prove results in this chapter.

Recall that in the same paper where Gerencsér and Gyárfás proved Theorem 1.3.1, they also proved the following.

Theorem 1.4.1(Gerencsér and Gyárfás, [22])**.** *Every 2-edge-coloured complete graph can be covered by two disjoint monochromatic paths of different colours.*

Any result about partitioning coloured graphs into a small number of monochromatic subgraphs will imply a Ramsey-type result as a corollary. For example Theorem 1.4.1 implies the bound $R(P_n, P_m) \leq n + m - 1$. Indeed Theorem 1.4.1 shows that every 2-edge-coloured K_{n+m-1} can be covered by a red path R and a disjoint blue path B. Clearly these paths cannot cover all the vertices unless $|R| \geq n$ or $|B| \geq m$. This is the main technique we shall use to bound Ramsey numbers throughout this chapter.

Recall that we proved the following lemma in the introduction.

Lemma 1.5.1. *Suppose that the edges of K_n are coloured with two colours. Then K_n can be covered by a red path and a disjoint blue balanced complete bipartite graph.*

Lemma 1.5.1 immediately implies the bound $R(P_n, K_{m,m}) \leq n + 2m - 2$. It turns out that when $m \equiv 1 \pmod{n-1}$, this bound is best possible. The following theorem was proved by Häggkvist.

Theorem 1.3.3 (Häggkvist, [32])**.** *If $m, l \equiv 1 \pmod{n-1}$, then we have*

$$R(P_n, K_{m,l}) = n + m + l - 2.$$

The lower bound on Theorem 1.3.3 comes from considering a colouring of $K_{n+m+l-3}$ consisting of $1+(m+l-2)/(n-1)$ red copies of K_{n-1} and all other edges are coloured blue. The condition $m, l \equiv 1 \pmod{n-1}$ ensures that the number $1 + \frac{(m+l-2)}{(n-1)}$ is an integer.

A theorem we prove in this chapter, is the following generalization of Lemma 1.5.1.

Theorem 6.1.1. *Let $k \geq 1$. Suppose that the edges of K_n are coloured with two colours. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+1)$-partite graph.*

As a corollary of Theorem 6.1.1 we obtain that for $m \equiv 1 \pmod{n-1}$ we have $R(P_n, K_m^t) = (t-1)(n-1) + t(m-1) + 1$. This generalizes a result of Erdős who showed that $R(P_n, K_m) = (t-1)(n-1) + 1$ (see [17, 44]).

If the colouring on K_n is connected in some colour, then the conclusion of Theorem 6.1.1 can be improved.

Theorem 6.1.2. *Let $k \geq 1$. Suppose that the edges of K_n are coloured with the colours red and blue, such that the red spanning subgraph is connected. Then K_n can be covered by k disjoint red paths and a disjoint blue balanced complete $(k+2)$-partite graph.*

In fact we shall prove a slightly stronger result. We will show that under conditions of Theorem 6.1.2, K_n can be covered by a red tree with at most $k+1$ leaves and a disjoint blue balanced $k+2$ partite graph. Theorems 6.1.1 and 6.1.2 will follow from this as corollaries.

A well known remark of Erdős and Rado says that any 2-edge-coloured complete graph is connected in one of the colours. Therefore Theorem 6.1.2 implies that every 2-edge-coloured complete graph can be covered by a monochromatic path and a monochromatic balanced complete tripartite graph (where we have no control over which colour each graph has).

Recall that K_m^t contains a copy of P_{tm}^{t-1}. Therefore, Theorems 6.1.1 and 6.1.2 imply the following.

Corollary 6.1.3. *Let $k \geq 1$. Suppose that K_n is colored with two colours.*

- *K_n can be covered with k disjoint red paths and a disjoint blue kth power of a path.*
- *If K_n is connected in red, then K_n can be covered with k disjoint red paths and a disjoint blue $(k+1)$th power of a path.*

The first part of this corollary may be seen as a generalization of Theorem 1.4.1. We are also able to use Corollary 6.1.3 and Theorem 1.3.1 to determine the Ramsey numbers of a path on n vertices versus a power of a path on n vertices.

Theorem 6.1.4. *For all k and $n \geq k+1$, we have*

$$R(P_n, P_n^k) = (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor.$$

Theorem 6.1.4 solves a conjecture of Allen, Brightwell, and Skokan who asked for the value of $R(P_n, P_n^k)$ in [2].

The structure of this chapter is as follows. In Section 6.2 we prove Theorems 6.1.1 and 6.1.2. In Section 6.3 we determine $R(P_n, K_m^t)$ whenever $m \equiv 1 \pmod{n-1}$ and $R(P_n, P_n^k)$ for all n and k. In Section 6.4 we discuss some further results one might work on using the methods used in this chapter.

6.2 Partitioning coloured complete graphs

The following lemma is an important intermediate step in the proof of Theorems 6.1.1 and 6.1.2.

Lemma 6.2.1. *Suppose that we have a 2-edge-coloured complete graph K_n containing $k+1$ sets $A_0, \dots A_k$, k sets $B_1, \dots B_k$, and k sets $N_1, \dots, N_k$ such that the following hold.*

(i) *The sets $A_0, \dots A_k, B_1, \dots B_k$ partition $V(K_n)$.*

(ii) *For all $1 \leq i < j \leq k$ all the edges between any of the sets A_0, A_i, B_i, A_j, and B_j are blue.*

(iii) *For all i, every red component of B_i intersects N_i.*

(iv) *$|A_0| \geq |A_i|$ for all $i \geq 1$.*

(v) *$|A_i| + |B_i| \geq |A_0|$ for all $i \geq 1$.*

(vi) *For all $i \geq 1$ either $|B_i| \leq 2\min_{t=1}^k |B_t|$ or $|A_i| + |B_i| \leq |A_0| + \min_{t=1}^k |B_t|$ holds.*

Then, there is a partition of K_n into k red paths $P_1, \dots, P_k$ and a blue balanced $k+1$ partite graph. In addition, for each i, the path P_i is either empty or starts in N_i.

Proof. The proof is by induction on the quantity $\sum_{t=1}^{k} |B_t|$.

First we prove the base case of the induction, i.e. we prove the lemma when $\sum_{t=1}^{k} |B_t| = 0$. In this case $B_i = \emptyset$ for all i, and so conditions (iv) and (v) imply that $|A_i| = |A_0|$ for all i. Therefore, by (ii), K_n contains a spanning blue complete $(k+1)$-partite graph with parts $A_0, \ldots, A_k$. We can take $P_1 = \cdots = P_k = \emptyset$ to obtain the required partition.

We now prove the induction step. Suppose that the lemma holds for all 2-edge-coloured complete graphs K'_n containing sets $A'_0, \ldots A'_k$, $B'_1, \ldots B'_k$, and $N'_1, \ldots, N'_k$ as in the statement of the lemma but satisfying $\sum_{t=1}^{k} |B'_t| < \sum_{t=1}^{k} |B_t|$. We will show that the lemma holds for K_n as well.

First we show that if there is a partition of K_n satisfying (i) – (vi), then the sets $A_0, \ldots, A_k$ and $B_1, \ldots, B_k$ can be relabeled to obtain a partition satisfying (i) – (vi) and also the following

$$|A_0| \geq |A_1| \geq \cdots \geq |A_k|, \tag{6.1}$$

$$|B_1| \leq \cdots \leq |B_k|. \tag{6.2}$$

The following claim guarantees this.

Claim 6.2.2. *Let σ be a permutation of $(0, 1, \ldots, k)$ ensuring that $|A_{\sigma(0)}| \geq |A_{\sigma(1)}| \geq \cdots \geq |A_{\sigma(k)}|$ holds. Let τ be a permutation of $(1, \ldots, k)$ ensuring that $|B_{\tau(1)}| \leq \cdots \leq |B_{\tau(k)}|$ holds. Let $A'_i = A_{\sigma(i)}$, $B'_i = B_{\tau(i)}$, and $N'_i = N_{\tau(i)}$. Then the sets A'_i, B'_i, and N'_i satisfy (i) – (vi).*

Proof. Notice that P'_i, A'_i and B'_i satisfy (i) – (iii) trivially.

Since the sets A_i satisfy (iv), we can assume that $\sigma(0) = 0$. This ensures that the sets $A_{\sigma(i)}$ satisfy (iv).

For (v), note that if for some $j \geq 1$, $|A_{\sigma(j)}| + |B_{\tau(j)}| < |A_0|$, then we also have $|A_{\sigma(x)}| + |B_{\tau(y)}| < |A_0|$ for all $x \geq j$ and $y \leq j$. However, the Pigeonhole Principle implies that $\sigma(x) = \tau(y)$ for some $x \geq j$ and $y \leq j$, contradicting the fact that A_i and B_i satisfy (v) for all i.

Suppose that (vi) fails to hold. Then for some j, $|B_{\tau(j)}| > 2 \min_{t=1}^{k} |B_t|$ and $|A_{\sigma(j)}| +$

$|B_{\tau(j)}| > |A_0| + \min_{t=1}^{k} |B_t|$ both hold. If we have $|A_{\tau(i)}| \geq |A_{\sigma(j)}|$ for some $i \geq j$, then $|B_{\tau(i)}| \geq |B_{\tau(j)}| > 2\min_{t=1}^{k} |B_t|$ and $|A_{\tau(i)}|+|B_{\tau(i)}| \geq |A_{\sigma(j)}|+|B_{\tau(j)}| > |A_0|+\min_{t=1}^{k} |B_t|$ both hold, contradicting the fact that A_i and B_i satisfy (vi) for all i. Therefore, we can assume that $|A_{\tau(i)}| < |A_{\sigma(j)}|$ for all $i \geq j$. This, together with $|A_{\sigma(0)}| \geq |A_{\sigma(1)}| \geq \cdots \geq |A_{\sigma(k)}|$ implies that $\{\tau(j), \tau(j+1), \dots, \tau(k)\} \subseteq \{\sigma(j+1), \sigma(j+2), \dots, \sigma(k)\}$, contradicting τ being injective. $\square$

By the above claim, without loss of generality we may assume that the A_is and B_is satisfy (6.1) and (6.2).

Notice that the lemma holds trivially if we have the following.

$$|A_0| = |A_1| + |B_1| = |A_2| + |B_2| = \cdots = |A_k| + |B_k|. \tag{6.3}$$

Indeed, if (6.3) holds, then K_n contains a spanning blue complete $(k+1)$-partite graph with parts $A_0, A_1 \cup B_1 \dots, A_k \cup B_k$, and so taking $P_1 = \cdots = P_k = \emptyset$ gives the required partition.

Therefore, we can assume that (6.3) fails to hold, so there is some j such that $|A_j| + |B_j| > |A_0|$. In addition, we can assume that j is as large as possible, and so $|A_i| + |B_i| = |A_0|$ for all $i > j$.

First we deal with the case when $|B_j| \leq 1$. Notice that in this case (6.2) implies that $|B_i| \leq 1$ for all $i \leq j$. Therefore for each i satisfying $|A_i| + |B_i| > |A_0|$, we have $|B_i| = 1$ and we can let P_i be the single vertex in B_i. For all other i, we let $P_i = \emptyset$. This ensures that $K_n \setminus (P_1, \dots, P_k)$ is a balanced complete k-partite graph with classes $A_1, \dots, A_j, A_{j+1} \cup B_{j+1}, \dots, A_k \cup B_k$, giving the required partition of K_n.

For the remainder of the proof, we assume that $|B_j| \geq 2$. We split into two cases depending on whether B_j is connected in red or not.

Case 1: Suppose that B_j is connected in red. Let v be a vertex in $B_j \cap N_j$. Let $K'_n = K_n - v$, $B'_j = B_j - v$, $N'_j = N_r(v)$ and $A'_i = A_i, B'_i = B_i, N'_i = N_i$ for all other i. We show that the graph K'_n with the sets A'_i, B'_i, and N'_i satisfies (i) – (vi).

Conditions (i), (ii), and (iv) hold trivially for the new sets as a consequence of them holding for the original sets A_j and B_j. Condition (iii) holds trivially whenever $i \neq j$, and holds for $i = j$ as a consequence of B_j being connected in red.

To prove (v), it is sufficient to show that $|A'_j| + |B'_j| \geq |A'_0|$. This is equivalent to $|A_j| + |B_j - v| \geq |A_0|$, which holds as a consequence of $|A_j| + |B_j| > |A_0|$.

We now prove (vi). Note that we have $\min_{t=1}^k |B'_t| = \min(|B_1|, |B'_j|)$. If $\min_{t=1}^k |B'_t| = |B_1|$ holds, then (vi) is satisfied for the new sets $A'_0, \ldots, A'_k, B'_0, \ldots, B'_k$ as a consequence of it being satisfied for the original sets $A_0, \ldots, A_k, B_0, \ldots, B_k$. Now, suppose that we have $\min_{t=1}^k |B'_t| = |B'_j|$. For $i > j$, we have $|A'_i| + |B'_i| = |A'_0|$ which implies that (vi) holds for these i. If $i \leq j$, then we have $|B_i| \leq |B_j|$ which together with $|B_j| \geq 2$ implies that $B'_i \leq 2|B_j| - 2 = 2|B'_j|$ holds.

Therefore, the graph K'_n with the sets A'_i, B'_i,and N'_i satisfies (i) – (vi). We also have $\sum_{t=1}^k |B'_t| = \sum_{t=1}^k |B_t| - 1$, and so, by induction K'_n can be partitioned into k red paths $P'_1, \ldots, P'_k$ starting in $N'_1, \ldots, N'_k$ respectively and a blue balanced $k+1$ partite graph H. Since P'_j starts in $N'_j = N_r(v)$, we have the required partition of K_n into k paths $P'_1, \ldots, v + P'_j, \ldots P'_k$ and a blue balanced $k+1$ partite graph H.

Case 2: Suppose that B_j is disconnected in red. We will find a new partition of K_n into sets $A'_0, \ldots, A'_k$ and $B'_1, \ldots, B'_k$, which together with $N_1, \ldots, N_k$ satisfy (i) – (vi). We will also have $\sum_{t=1}^k |B'_t| < \sum_{t=1}^k |B_t|$ which implies the lemma by induction.

Let B_j^- be the smallest red component of B_j and $B_j^+ = B_j \setminus B_j^-$. There are two subcases, depending on whether we have $|A_j| + |B_j^-| \leq |A_0|$ or not.

Case 2.1: Suppose that we have $|A_j| + |B_j^-| \leq |A_0|$. Let $B'_j = B_j^+$ and $A'_j = A_j \cup B_j^-$, and $A'_i = A_i$, $B'_i = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

To prove (iv), it is sufficient to show that $|A'_0| \geq |A'_j|$ which is true since we are assuming that $|A_j| + |B_j^-| \leq |A_0|$.

To prove (v), it is sufficient to show that $|A'_j| + |B'_j| \geq |A'_0|$ which holds since we have $|A'_j| + |B'_j| = |A_j| + |B_j| \geq |A_0|$.

To prove (vi), note that we have $\min_{t=1}^k |B'_t| = \min(B_1, B'_j)$. If $\min_{t=1}^k |B'_t| = |B_1|$ holds, then (vi) is satisfied for the new sets $A'_0, \ldots, A'_k, B'_0, \ldots, B'_k$ as a consequence of it being satisfied for the original sets $A_0, \ldots, A_k, B_0, \ldots, B_k$. Now, suppose that we have $\min_{t=1}^k |B'_t| = |B'_j|$. For $i > j$, we have $|A'_i| + |B'_i| = |A'_0|$ which implies that (vi) holds for these i. If $i \leq j$, then we have $|B_i| \leq |B_j|$ which together with $|B_j| \leq 2|B_j^+|$ implies that $B'_i \leq 2|B'_j|$ holds.

Notice that we have $\sum_{t=1}^k |B'_t| < \sum_{t=1}^k |B_t|$, and so the lemma holds by induction.

Case 2.2: Suppose that we have $|A_j| + |B_j^-| > |A_0|$.

We claim that in this case $|B_j| \le 2\min_{t=1}^k |B_t|$ holds. Indeed by (vi), we have that either $|B_j| \le 2\min_{t=1}^k |B_t|$ holds, or we have $|A_j| + |B_j| \le |A_0| + \min_{t=1}^k |B_t|$. Adding $|A_j| + |B_j| \le |A_0| + \min_{t=1}^k |B_t|$ to $|A_j| + |B_j^-| > |A_0|$ gives $|B_j^+| < \min_{t=1}^k |B_t|$. This, together with $|B_j| \le 2|B_j^+|$ implies that $|B_j| \le 2\min_{t=1}^k |B_t|$ always holds.

There are two cases, depending on whether we have $j = k$ or not.

Suppose that $j \ne k$. Let $B'_j = B_j^+$, $A'_{j+1} = A_{j+1} \cup B_j^-$, and $A'_i = A_i$, $B'_i = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

To prove (iv), it is sufficient to show that $|A'_0| \ge |A'_{j+1}|$, which holds as a consequence of $|A_{j+1}| + |B_{j+1}| = |A_0|$ and (6.2).

To prove (v), it is sufficient show that $|A'_j|+|B'_j| \ge |A'_0|$, which holds as a consequence of $|B_j^+| \ge |B_j^-|$ and $|A_j| + |B_j^-| > |A_0|$.

We now prove (vi). For $i \ge j+2$, note that we have $|A'_i| + |B'_i| = |A'_0|$ which implies that (vi) holds for these i. For $i \le j$, (vi) holds since we have $|B'_i| \le |B_j| \le 2|B_j^+| = |B'_j|$. For $i = j+1$, we have $|A'_{j+1}| + |B'_{j+1}| \le |A'_0| + \min_{t=1}^k |B'_t|$ as a consequence of $|A'_{j+1}| + |B'_{j+1}| = |A_0| + |B_j^-|$, $|B_j^-| \le \frac{1}{2}|B_j|$, and $|B_j| \le 2\min_{t=1}^k |B_t|$.

Notice that we have $\sum_{t=1}^k |B'_t| < \sum_{t=1}^k |B_t|$, and so the lemma holds by induction.

Suppose that $j = k$. Let $B'_k = B_k^+$, $A'_k = A_0$, $A'_0 = A_k \cup B_k^-$, and $A'_i = A_i, B'_i = B_i$ for all other i. As before, conditions (i) – (iii) hold trivially.

Since $|A_0| \ge |A'_i|$ for all $i \ge 1$, to prove (iv), it is sufficient to show that $|A'_0| \ge |A_0|$. This holds since we assumed that $|A_k| + |B_k^-| > |A_0|$.

To prove (v), we have to show that $|A_i|+|B_i| \ge |A_k|+|B_k^-|$ for all $i < k$ and also that $|A_0| + |B_k^+| \ge |A_k| + |B_k^-|$. We know that for all i we have $|B_k^-| \le \frac{1}{2}|B_k| \le |B_i|$ which, combined with (6.1), implies that we have $|A_i|+|B_i| \ge |A_k|+|B_k^-|$. We also know that $|B_k^+| \ge |B_k^-|$ which, combined with (6.1), implies that we have $|A_0|+|B_k^+| \ge |A_k|+|B_k^-|$.

To prove (vi), note that we have $\min_{t=1}^k |B'_t| = \min(B_1, B'_k)$. If $\min_{t=1}^k |B'_t| = |B'_k|$ holds, then we have $|B'_i| \le 2|B'_k|$ for all i as a consequence of (6.2) and $2|B'_k| \ge |B_k|$. Suppose that $\min_{t=1}^k |B'_t| = |B'_1|$ holds. Then for $i < k$, (vi) is satisfied for the new sets $A'_0, \ldots, A'_k, B'_0, \ldots, B'_k$ as a consequence of it being satisfied for the original sets $A_0, \ldots, A_k, B_0, \ldots, B_k$ and $|A'_0| \ge |A_0|$. For $i = k$, (vi) holds since we have $|B'_k| \le |B_k| \le 2\min_{t=1}^k |B_t|$.

Notice that we have $\sum_{t=1}^{k} |B'_t| < \sum_{t=1}^{k} |B_t|$, and so the lemma holds by induction. $\square$

We now prove the following theorem, which is a strengthening of Theorem 6.1.2

Theorem 6.2.3. *Suppose that K_n is a 2-edge-coloured complete graph which is connected in red. For any $k \geq 2$, the graph K_n can be partitioned into a red tree with at most k leaves and a blue balanced $(k+1)$-partite graph.*

Proof. We will partition K_n into a red tree T, and sets $A_0, A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ with certain properties. For convenience we will define $A = A_0 \cup A_1 \cup \cdots \cup A_k$ and $B = B_1 \cup \cdots \cup B_k$. The tree T will have l leaves which will be called $v_1, v_2, \ldots, v_l$. For a set $S \subseteq K_n$, let $c(S)$ be the order of the largest red component of $K_n[S]$. Define $f(S)$ to be the number of red components contained in S of order $c(A \cup B)$. The tree T, and sets $A_0, A_1, \ldots, A_k$ and $B_1, \ldots, B_k$ are chosen to satisfy the following.

(I) For $1 \leq i < j \leq k$, all the edges between A_0,A_i, A_j, B_i, and B_j are blue.

(II) T has l leaves $v_1, \ldots, v_l$, where $l \leq k$. For $i = 1, \ldots, l$, the leaf v_i, is joined to every red component of B_i by a red edge.

(III) $c(A \cup B)$ is as small as possible, whilst keeping (I) – (II) true.

(IV) $\sum_{t=1}^{k} |f(B_t) - \frac{1}{2}|$ is as small as possible, whilst keeping (I) – (III) true.

(V) $f(A)$ is as small as possible, whilst keeping (I) – (IV) true.

(VI) $|T|$ is as small as possible, whilst keeping (I) – (V) true.

(VII) $|\{i \in \{1, \ldots, k\} : |B_i| \geq c(A \cup B)\}|$ is as large as possible, whilst keeping (I) – (VI) true.

(VIII) $\sum_{\{t:|B_t|<c(A \cup B)\}} |B_t|$ is as large as possible, whilst keeping (I) – (VII) true.

(IX) $\sum_{t=1}^{k} |B_t|$ is as small as possible, whilst keeping (I) – (VIII) true.

(X) $\max_{t=1}^{k} |A_t|$ is as small as possible, whilst keeping (I) – (IX) true.

(XI) $|\{i \in \{1, \ldots, k\} : |A_i| = \max_{t=1}^{k} |A_t|\}|$ is as small as possible, whilst keeping (I) – (X) true.

In order to prove Theorem 6.2.3 we will show that the partition of $A \cup B$ into A_i and B_i satisfies conditions (i), (ii), (iv), (v), and (vi) of Lemma 6.2.1. Then, Lemma 6.2.1 will easily imply the theorem.

Without loss of generality, we may assume that the A_is are labelled such that we have

$$|A_0| \geq |A_1| \geq \cdots \geq |A_k|. \tag{6.4}$$

We begin by proving a sequence of claims.

Claim 6.2.4. *For each i, $f(B_i)$ is either 0 or 1.*

Proof. Suppose that $f(B_i) \geq 2$. Let C be a red component in B_i of order $c(A \cup B)$. Let $B'_i = B_i \setminus C$, $A'_0 = A_0 \cup C$, $T' = T$ and $A'_j = A_j$, $B'_j = B_j$ for other j. It is easy to see that the new partition satisfies (I) – (III). We have that $f(B'_i) = f(B_i) - 1$, which combined with $f(B_i) \geq 2$ implies that $|f(B'_i) - \frac{1}{2}| < |f(B_i) - \frac{1}{2}|$ contradicting minimality of the original partition in (IV). □

Claim 6.2.5. *If we have $f(B_i) = 1$ for some i, then we also have $|B_i| = c(A \cup B)$.*

Proof. Suppose that $f(B_i) = 1$ and $|B_i| > c(A \cup B)$ both hold. Then B_i contains some red connected component C of order strictly less than $c(A \cup B)$. Let $T' = T$, $A'_0 = A \cup C$, $B'_i = B_i \setminus C$, and $A'_t = \emptyset$, $B'_t = B_t$ for all other t.

It is easy to see that the new partition satisfies (I) – (VIII). However $|B'_i| < |B_i|$ and $|B'_t| = |B_t|$ for $t \neq i$ contradicts minimality of the original partition in (IX). □

Claim 6.2.6. *We have that $f(A) \geq 1$.*

Proof. Suppose that we have $f(A) = 0$. Then all the red components of order $c(A \cup B)$ of $A \cup B$ must be contained in B. For each $i \in \{1, \ldots, k\}$, let C_i be a red component of order $c(A \cup B)$ contained in B_i (if one exists). By Claim 6.2.4 any red component of $A \cup B$ or order $c(A \cup B)$ must be one of the C_is. By (II), for $i \in \{1, \ldots, l\}$, if C_i exists, then v_i has a red neighbour u_i in C_i. By red-connectedness of K_n and part (I), every

C_i must be connected to T by a red edge. Therefore, for $i \in \{l+1, \ldots, k\}$, if C_i exists, then there is a red edge $u_i w_i$ between $u_i \in C_i$ and some $w_i \in T$.

Let $A'_0 = A \cup B \setminus \{u_1, \ldots, u_k\}$ and $A'_j = B'_j = \emptyset$ for $j \geq 1$. Let T' be the tree with vertex set $V(T) \cup \{u_1, \ldots, u_k\}$ formed from T by joining u_i to v_i for $i = 1, \ldots, l$ and u_i to w_i for $i = l+1, \ldots, k$.

Clearly the new partition satisfies (I) and (II). However since each of the largest components of $A \cup B$ lost a vertex, we must have $c(K_n \setminus T) < c(A \cup B)$ contradicting minimality of the original partition in (III). $\square$

Claim 6.2.7. *If $i > l$, then $f(B_i) = 1$ holds.*

Proof. Suppose that $f(B_i) = 0$ for some i.

By Claim 6.2.6, there is a red component C of order $c(A \cup B)$ in A. Let $T' = T$, $A'_0 = A \setminus C$, $B'_i = B_i \cup C$, and $A'_t = \emptyset$, $B'_t = B_t$ for all other t.

It is easy to see that the new partition satisfies (I) – (IV). However we have $f(A) = f(A) - 1$ contradicting minimality of the original partition in (V). $\square$

Claim 6.2.8. *For every i, we have $|A_0| \leq |A_i| + c(A \cup B)$.*

Proof. Suppose that for some i we have $|A_0| > |A_i| + c(A \cup B)$. Let C be any red component of A_0. We have $|C| \leq c(A \cup B)$. Let $A'_0 = A_0 \setminus C$, $A'_i = A_i \cup C$, $T' = T$ and $A'_j = A_j$, $B'_j = B_j$ otherwise. It is easy to see that T', A'_j, and B'_j will satisfy (I) – (IX). If the new partition satisfies (X), then we must have $\max_{t=0}^{k} |A'_t| = |A_0|$. However $|A_0| > |A_i| + c(A \cup B)$ ensures that we have $|A'_0|, |A'_i| < |A_0|)$ meaning that the quantity $|\{i \in \{1, \ldots, k\} : |A'_i| = |A'_0|\}|$ must be smaller than it was in the original partition, contradicting (XI). $\square$

Claim 6.2.9. *For every i, we have $|B_i| \geq c(A \cup B)$.*

Proof. Suppose that $|B_i| < c(A \cup B)$ for some i. Notice that this implies that $f(B_i) = 0$. By Claim 6.2.7, we have that $i \leq l$.

First suppose that we have $N_r(v_i) \cap A \neq \emptyset$. Let C be a red component of A which intersects $N_r(v_i)$. Let $T' = T$, $B'_i = B_i \cup C$, and $A'_t = A_t \setminus C$, $B'_t = B_t$, for other t.

The new partition satisfies (I) trivially. By choice of C, new partition satisfies (II). It is easy to see that $c(A'_t), c(B'_t) \leq c(A \cup B)$ for every t which implies that (III) holds for the new partition. Since $f(B_i) = 0$ holds, we have that $f(B'_i) \leq 1$ and hence $|f(B'_i) - \frac{1}{2}| = |f(B_i) - \frac{1}{2}|$ which implies that (IV) holds for the new partition.

It is easy to see that $f(A'_t) \leq f(A_t)$ for all t, which implies that (V) holds for the new partition. Since $T' = T$, (VI) holds for the new partition.

We have that $|B'_t| \geq |B_t|$ for all t. This implies that if the new partition satisfies (VII), then we have $|B'_i| < c(A \cup B)$. However since $|B'_i| > |B_i|$, this contradicts maximality of the original partition in (VIII)

For the remainder of the proof of this claim, we may assume that we have $N_r(v_i) \subseteq B$. There are two cases depending on where the neighbours of v_i lie.

Case 1: Suppose that $N_r(v_i) \subseteq B_i$.

Let $T' = T - v_i$, $B'_i = B_i + v_i$, and $A'_j = A_j$, $B'_j = B_j$ for other j. The resulting partition satisfies (I) since $N_r(v_i) \subseteq B_i$. Condition (II) implies that $B_i + v_i$ is connected in red. This, together with the fact that the neighbour of v_i in T is connected to B'_i by a red edge implies that condition (II) holds for the new partition. The only red component of the new partition which was not a red component of the old partition is $B_i \cup v$, which is of order at most $c(A \cup B)$ because of $|B_i| < c(A \cup B)$. This implies that (III) is satisfied. Since $f(B_i) = 0$, we must have $f(B'_i) = 0$ or 1, which means that $|f(B'_i) - \frac{1}{2}| = |f(B_i) - \frac{1}{2}|$ and hence the new partition satisfies (IV). The new partition satisfies (V) since we have $f(A'_0 \cup \cdots \cup A'_k) = f(A)$. However $|T'| = |T| - 1$, contradicting minimality of the original tree T in (VI).

Case 2: Suppose that $N_r(v_i) \cap B_j \neq \emptyset$ for some $j \neq i$. Let C be a red component of B_j which intersects $N_r(v_i)$. By Claim 6.2.5 we have $c(B_j \setminus C) < c(A \cup B)$.

There are two subcases, depending on whether $j \leq l$ holds.

Case 2.1: Suppose that $j > l$. By Claim 6.2.6 there is a red component $C_A \subseteq A$ of order $c(A \cup B)$. Let $B'_i = B_i \cup C$, $B'_j = (B_j \cup C_A) \setminus C$, $T' = T$ and $A'_t = A_t \setminus C_A$, $B'_t = B_t$ for all other t.

The resulting partition trivially satisfies (I). Condition (II) follows from the fact that B_i is connected to C by a red edge. We have $A'_0 \cup \cdots \cup A'_k \cup B'_1 \cup \cdots \cup B'_k = A \cup B$ which implies that the new partition satisfies (III). Using $|B_i|, |B_j \setminus C| < c(A \cup B)$

we obtain that $f(B_i') = f(B_j') \leq 1$ and $f(B_t') = f(B_t)$ otherwise. This implies that $\sum_{t=1}^{k} |f(B_t') - \frac{1}{2}| = \sum_{t=1}^{k} |f(B_t) - \frac{1}{2}|$, and so the new partition satisfies (IV). However, we have $f(A_0' \cup \cdots \cup A_k') = f(A) - 1$, contradicting minimality of the original partition in (V).

Case 2.2: Suppose that $j \leq l$. Since $i \neq j$, this implies that we have $l \geq 2$.

Let u_i be a red neighbour of v_i in C. By (II), v_j has a red neighbour u_j in C. There must be a red path P between u_i and u_j contained in C.

Notice that joining T and P using the edges $u_i v_i$ and $u_j v_j$ produces a graph T_1 which has $l-2$ leaves and exactly one cycle (which passes thorough P.) By Claim 6.2.6 A contains a red component C_A of order $c(A \cup B)$. By red-connectedness of K_n, there must be some edge xv_j' between $x \in T$ and a vertex $v_j' \in C_A$.

We construct a tree T' and sets A_t' and B_t' as follows.

- Suppose that $x \neq v_t$ for any $t \in \{1, \ldots, l\}$. In this case we let T_2 be the graph with vertices $T_1 + v_j'$, formed from T_1 by adding the edge xv_j'. Notice that T_2 has $l-1$ leaves and exactly one cycle. Therefore, the cycle in T_2 must contain a vertex y of degree at least 3. Let v_i' be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv_i'. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j\}$, v_j' and possibly v_i' (depending on whether the degree of v_i' in T_2 is 2 or not.)

  We also let $A_0' = A \cup B_i \cup B_j \setminus P - v_j'$, $B_i = B_j = \emptyset$, and $A_t = \emptyset$, $B_t' = B_t$, $v_t' = v_t$ for $t \neq i, j$.

- Suppose that $x = v_s$ for some $s \in \{1, \ldots, l\}$ and $f(B_s) = 1$. In this case, Claim 6.2.5 implies that B_s is connected. Let v_s' be a neighbour of x in B_s. Let T_2 be the graph with vertices $T_1 + v_j' + v_s'$, formed from T_1 by adding the edges xv_j' and xv_s'. As before T_2 has $l-1$ leaves and exactly one cycle, which contains a vertex y of degree at least 3. Let v_i' be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv_i'. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j, v_s\}$, v_j', v_s' and possibly v_i' (depending on whether the degree of v_i' in T_2 is 2 or not.)

We also let $A_0' = A \cup B_i \cup B_j \setminus P - v_j'$, $B_i = B_j = \emptyset$, $B_s' = B_s - v_s'$ and $A_t = \emptyset$, $B_t' = B_t$, $v_t' = v_t$ for $t \neq i, j, s$.

- Suppose that $x = v_s$ for some $s \in \{1, \ldots, l\}$ and $f(B_s) = 0$. Let T_2 be the graph with vertices $T_1 + v_j'$, formed from T_1 by adding the edge xv_j'. Then T_2 has $l-2$ leaves and exactly one cycle, which contains a vertex y of degree at least 3. Let v_i' be a neighbour of y on the cycle. We let T' be the tree formed from T_2 by removing the edge yv_i'. The leaves of T' are $\{v_1, \ldots, v_l\} \setminus \{v_i, v_j, v_s\}$, v_j' and possibly v_i' (depending on whether the degree of v_i' in T_2 is 2 or not.)

  We also let $A_0' = A \cup B_i \cup B_j \cup B_s \setminus P - v_j'$, $B_i = B_j = B_s = \emptyset$, and $A_t = \emptyset$, $B_t' = B_t$, $v_t' = v_t$ for $t \neq i, j, s$.

Clearly the new partition satisfies (I). It is easy to see that for all t for which v_t' is defined above, v_t' is connected to all the red components of B_t', so the new partition satisfies (II).

Since $A_0' \cup \cdots \cup A_k' \cup B_1' \cup \cdots \cup B_k' \subseteq A \cup B$, we must have $c(A_0' \cup \cdots \cup A_k' \cup B_1' \cup \cdots \cup B_k') \leq c(A \cup B)$ and hence the new partition satisfies (III). Since for all t, we have $B_t' \subseteq B_t$, the new partition satisfies (IV). Recall that have $c(B_j \setminus C) < c(A \cup B)$, which combined with the fact that P is nonempty and $|C| \leq c(A \cup B)$ implies that $c(B_j \setminus P) < c(A \cup B)$. This, combined with the fact that $c(B_i) < c(A \cup B)$ (and, in the third of the above cases, $c(B_s) < c(A \cup B)$) implies that the red components of $A_1' \cup \cdots \cup A_k'$ are exactly those of A, minus C_A. Therefore we have $f(A_1' \cup \cdots \cup A_k') = f(A) - 1$, contradicting minimality of the original partition in (V). □

Claim 6.2.10. *For every i, we have $|B_i| \leq 2c(A \cup B)$.*

Proof. Suppose that $B_i > 2c(A \cup B)$. Combining this with Claim 6.2.4, means that there is a red component, C in B_i satisfying $|C| < c(A \cup B)$. Let $B_i' = B_i \setminus C$, $A_0' = A_0 \cup C$, and $A_t' = A_t$, $B_t' = B_t$, $T' = T$ otherwise.

The new partition satisfies (I) – (II) trivially. It is easy to see that $c(A_t') = c(A_t)$ and $c(B_t') = c(B_t)$ for every t which implies that (III) holds for the new partition. Also we have $f(A_t') = f(A_t)$ and $f(B_t') = f(B_t)$ for every t which implies that (IV) – (V) hold for the new partition. Since $T' = T$, (VI) holds for the new partition. Since $|B'_t| = |B_t|$ for $t \neq i$ and $|B'_i| \geq c(A \cup B)$, the new partition satisfies (VII) and (VIII).

However, we have that $|B'_i| < |B_i|$ which contradicts minimality of the original partition in (IX). $\square$

We now prove the theorem.

For each $i = 1, \ldots, k$ we define a set $N_i \subseteq A \cup B$. If $i \leq l$, let $N_i = N_r(v_i)$. If $i > l$, let $N_i = \bigcup_{v \in T} N_r(v)$.

We will show that the graph $K_n \setminus T$, together with the sets $A_0, \ldots, A_k$, $B_1, \ldots, B_k$, and $N_1, \ldots, N_k$ satisfies conditions (i) – (vi) of Lemma 6.2.1.

Condition (i) follows from the definition of $A_0, \ldots, A_k$, and $B_1, \ldots, B_k$. Condition (ii) follows immediately from (I). Condition (iii) follows from (II) whenever $i \leq l$ and from red-connectedness of K_n whenever $i \geq k+1$. Condition (iv) follows from the fact that we are assuming (6.4).

Combining Claims 6.2.8 and 6.2.9 implies that we have $|B_i|+|A_i| \geq c(A \cup B)+|A_i| \geq |A_0|$ for all i. This proves condition (v) of Lemma 6.2.1.

Combining Claims 6.2.9 and 6.2.10 implies that we have $2|B_i| \geq 2c(A \cup B) \geq |B_j|$ for all i and j . This proves condition (vi) of Lemma 6.2.1.

Therefore, the graph $K_n \setminus T$, together with the sets $A_0, \ldots, A_k$, $B_1, \ldots, B_k$, and $N_1, \ldots, N_k$ satisfies all the conditions of Lemma 6.2.1. By Lemma 6.2.1, $K_n \setminus T$ can be partitioned into paths $P_1, \ldots, P_k$ starting in $N_1, \ldots, N_k$ and a balanced $(k+1)$-partite graph H. For each i, the path P_i can be joined to T to obtain the required partition of K_n into a tree with at most k leaves $T \cup P_1 \cup \cdots \cup P_k$ and a balanced $(k+1)$-partite graph H. $\square$

We can now deduce Theorems 6.1.1 and 6.1.2.

Proof of Theorem 6.1.2. Every tree with at most k leaves can be partitioned into $k-1$ paths (say by induction on k). Combining this with Theorem 6.2.3 immediately implies Theorem 6.1.2. $\square$

Theorem 6.1.1 follows from Theorem 6.1.2.

Proof of Theorem 6.1.1. Consider a colouring of K_{n+1} formed by adding a vertex, v, to K_n and colouring all the edges containing v red. The resulting colouring is connected in red. By Theorem 6.1.2, it can be partitioned into $k-1$ red paths and a blue balanced $(k+1)$-partite graph. Since all the edges containing v are red, v cannot be contained in a blue balanced $(k+1)$-partite graph. Therefore v is contained in one of the paths, and so removing v from the graph gives the required partition of the original colouring of K_n. □

6.3 Ramsey Numbers

In this section, we use the results of the previous section to determine the the value of the Ramsey number of a path versus certain other graphs.

First we determine $R(P_n, K_m^t)$ whenever $m \equiv 1 \pmod{n-1}$.

Theorem 6.3.1. *If $m \equiv 1 \pmod{n-1}$ then we have*

$$R(P_n, K_m^t) = (t-1)(n-1) + t(m-1) + 1.$$

Proof. For the upper bound, apply Theorem 6.1.1 to the given 2-edge-coloured complete graph on $(t-1)(n-1)+t(m-1)+1$ vertices. This gives us $t-1$ red paths and a blue balanced complete t-partite graph which, cover all the vertices of $K_{(t-1)(n-1)+t(m-1)+1}$. By the Pigeonhole Principle either one of the paths has order at least n or the complete t-partite graph has order at least $t(m-1)+1$. Since the complete t-partite graph is balanced, if it has order more than $t(m-1)+1$, then it must have at least tm vertices.

For the lower bound, consider a colouring of the complete graph on $(t-1)(n-1)+t(m-1)$ vertices consisting of $(t-1)+t(m-1)/(n-1)$ disjoint red copies of K_{n-1} and all other edges coloured blue. The condition $m \equiv 1 \pmod{n-1}$ ensures that we can do this. Since all the red components of the resulting graph have order at most $n-1$, the graph contains no red P_n. The graph contains no a blue K_m^t, since every partition of such a graph would have to intersect at least $(m-1)/(n-1)+1$ of the red copies of K_{n-1} and there are only $(t-1)(n-1)+t(m-1)$ of these. □

In the remainder of this section we will prove Theorem 6.1.4. First we will use Theorem 6.1.1 and 6.1.2 to find upper bounds on $R(P_n, P_m^t)$.

Lemma 6.3.2. *The following statements are true.*

(a) $R(P_n, P_m^t) \leq (n-2)t + m$ *for all* n, m *and* $t \geq 1$.

(b) Suppose that $t \geq 2$ *and* $n, m \geq 1$. *Every* 2*-edge-coloured complete graph on* $(n-1)(t-1) + m$ *vertices which is connected in red contains either a red* P_n *or a blue* P_m^t.

Proof. For part (a), notice that by Theorem 6.1.1, we can partition a 2-edge-coloured $K_{(n-2)t+m}$ into t red paths $P_1, \ldots, P_t$ and a blue tth power of a path P^t. Suppose that there are no red paths of order n in $K_{(n-2)t+m}$. Suppose that i of the paths $P_1, \ldots, P_t$ are of order $n-1$. Without loss of generality we may assume that these are the paths $P_1, \ldots, P_i$. We have $|P^t| + (n-2)(t-i) + (n-1)i \geq |P^t| + |P_1| + \cdots + |P_t| = (n-2)t + m$ which implies $i + |P^t| \geq m$. For each j, let v_j be one of the endpoint of P_j. Notice that since there are no red paths of order n in $K_{(n-2)t+m}$, all the edges in $\{v_1, \ldots, v_i, p\}$ are blue for any $p \in P^t$. This allows us to extend P^t by adding i extra vertices $v_1, \ldots, v_i$ to obtain a tth power of a path of order m.

Part (b) follows immediately from Theorem 6.1.2 and the fact that a balanced t-partite graph contains a spanning $(t-1)$st power of a path. $\square$

The following simple lemma allows us to join powers of paths together.

Lemma 6.3.3. *Let* G *be a graph. Suppose that* G *contains a* $(k-i)$*th power of a path,* P^{k-i}, *and an* $(i-1)$*st power of a path,* Q^{i-1}, *such that the following hold.*

(i) *All the edges between* P^{k-i} *and* Q^{i-1} *are present.*

(ii) $|P^{k-i}| \geq (k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor$.

(iii) $|Q^{i-1}| \geq i\left\lfloor \frac{n}{k+1} \right\rfloor$.

(iv) $|P^{k-i}| + |Q^{i-1}| \geq n$.

Then G contains a kth power of a path on n vertices.

Proof. Without loss of generality, we may assume that P^{k-i} and Q^{i-1} are the shortest such paths contained in G. We claim that this implies that we have $|P^{k-i}|+|Q^{i-1}| = n$. Indeed otherwise (iv) implies that $|P^{k-i}| + |Q^{i-1}| > (k+1)\left\lfloor \frac{n}{k+1} \right\rfloor$, and hence we could remove an endpoint from one of the paths, whilst keeping (ii) and (iii) true.

Let $p_1, \dots, p_{|P^{k-i}|}$ be the vertices of P^{k-i} and $q_1, \dots, q_{|Q^{i-1}|}$ be the vertices of Q^{i-1}. For convenience set $r_P = |P^{k-i}| \pmod{k-i+1}$ and $r_Q = |q^{i-1}| \pmod{i}$. Together with (ii) and (iii), this ensures that we have $|P^{k-i}| = r_P + (k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor$ and $|Q^{i-1}| = r_Q + i\left\lfloor \frac{n}{k+1} \right\rfloor$. It is easy to see that the following sequence of vertices is a kth power of a path on n vertices.

$$
\begin{aligned}
&q_1, \dots q_{r_Q}\\
&p_1, \dots, p_{k-i+1}, q_{r_Q+1}, \dots, q_{r_Q+i}\\
&p_{k-i+2}, \dots, p_{2(k-i+1)}, q_{r_Q+i+1}, \dots, q_{r_Q+2i}\\
&\qquad\qquad\vdots\\
&p_{(k-i+1)\left(\left\lfloor \frac{n}{k+1} \right\rfloor-1\right)+1}, \dots, p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor}, q_{r_Q+(i-1)\left(\left\lfloor \frac{n}{k+1} \right\rfloor-1\right)+1}, \dots, q_{r_Q+i\left(\left\lfloor \frac{n}{k+1} \right\rfloor-1\right)}\\
&p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor+1}, \dots, p_{(k-i+1)\left\lfloor \frac{n}{k+1} \right\rfloor+1+r_P}
\end{aligned}
$$

□

We are now ready to prove Theorem 6.1.4.

Proof of Theorem 6.1.4. For the lower bound $R(P_n, P_n^k) \geq (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$, consider a colouring of $K_{(n-1)k+\left\lfloor \frac{n}{k+1} \right\rfloor-1}$ consisting of k disjoint red copies of K_{n-1} and one disjoint red copy of $K_{\left\lfloor \frac{n}{k+1} \right\rfloor-1}$. All edges outside of these are blue. It is easy to see that when $n \geq k+1$, this colouring contains neither a red path on n vertices nor a blue P_n^k.

It remains to prove the upper bound $R(P_n, P_n^k) \leq (n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$. Let K be a 2-edge-coloured complete graph on $(n-1)k + \left\lfloor \frac{n}{k+1} \right\rfloor$ vertices. Suppose that K does not contain any red paths of order n. We will find a blue copy of P_n^k.

Let C be the largest red component of K. The following claim will give us three cases to consider.

Claim 6.3.4. *One of the following always holds.*

(i) $|C| \geq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor + 1$.

(ii) *There is a set B, such that all the edges between B and $V(K) \setminus B$ are blue and also*
$$n + \left\lfloor \frac{n}{k+1} \right\rfloor \leq |B| \leq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor.$$

(iii) *The vertices of K can be partitioned into k disjoint sets $B_1, \ldots, B_k$ such that for $i \neq j$ all the edges between B_i and B_j are blue and we have*
$$|B_1| \geq |B_2| \geq \cdots \geq |B_k| \geq \left\lceil \frac{n}{k+1} \right\rceil.$$

Proof. Suppose that neither (i) nor (ii) hold.

This implies that all the red components in K have order at most $n + \left\lfloor \frac{n}{k+1} \right\rfloor - 1$. Let B be a subset of $V(K)$ such that the following hold.

(a) All the edges between B and $V(K) \setminus B$ are blue.

(b) $|B| \leq n - 1 + \left\lfloor \frac{n}{k+1} \right\rfloor$.

(c) $|B|$ is as large as possible.

Suppose that there is a red component C' in $V(K) \setminus B$ of order at most $\left\lceil \frac{n}{k+1} \right\rceil - 1$. Let $B' = B \cup C'$. Notice that $n \geq k\left\lfloor \frac{n}{k+1} \right\rfloor + \left\lceil \frac{n}{k+1} \right\rceil$ holds for all integers $n, k \geq 0$. This implies that we have $|B'| = |B| + |C'| \leq 2(n-1) - (k-2)\left\lfloor \frac{n}{k+1} \right\rfloor$ thich implies that either B' is a set satisfying (a) and (b) of larger order than B, or B' satisfies (ii).

Suppose that all the red components in $V(K) \setminus B$ have order at least $\left\lceil \frac{n}{k+1} \right\rceil$. Since $n \geq 2$, we have
$$|V(K) \setminus B| \geq (n-1)(k-1) > (k-2)\left(n - 1 + \left\lfloor \frac{n}{k+1} \right\rfloor\right). \tag{6.5}$$

Using the fact that all red components of K have order at most $n-1+\left\lfloor\frac{n}{k+1}\right\rfloor$, (6.5) implies that $V(K)\setminus B$ must have at least $k-1$ components. Therefore, the components of $V(K)\setminus B$ can be partitioned into $k-1$ sets $B_2,\ldots,B_k$ which, together with $B_1=B$, satisfy (iii). □

We distinguish three cases, depending on which part of Claim 6.3.4 holds.

Case 1: If part (i) of Claim 6.3.4 holds, then there must be some $i\leq k-2$, such that we have

$$(k-i)(n-1)-i\left\lfloor\frac{n}{k+1}\right\rfloor+1\leq|C|\leq(k-i+1)(n-1)-(i-1)\left\lfloor\frac{n}{k+1}\right\rfloor. \quad (6.6)$$

Combining $(k-i)(n-1)-i\left\lfloor\frac{n}{k+1}\right\rfloor+1\leq|C|$ with part (b) of Lemma 6.3.2 shows that C must contain a blue $(k-i)$th power of a path, P^{k-i}, on $n-i\left\lfloor\frac{n}{k+1}\right\rfloor$ vertices. If $i=0$, then P^{k-i} is a copy of P_n^k, and so the theorem holds. Therefore, we can assume that $i\geq 1$.

Notice that (6.6) implies that we have $|V\setminus C|\geq(i-1)(n-1)+i\left\lfloor\frac{n}{k+1}\right\rfloor$. Combining this with part (a) of Lemma 6.3.2 shows that $V\setminus C$ must contain a blue $(i-1)$st power of a path, Q^{i-1}, on $i\left\lfloor\frac{n}{k+1}\right\rfloor+i-1$ vertices.

Since all the edges between C and $V\setminus C$ are blue we can apply Lemma 6.3.3 to P^{k-i} and Q^{i-1} in order to find a blue kth power of a path on n vertices in G.

Case 2: Suppose that there is some set $B\subseteq V(K)$ such that all the edges between B and $V(K)\setminus B$ are blue and also

$$n+\left\lfloor\frac{n}{k+1}\right\rfloor\leq|B|\leq 2(n-1)-(k-2)\left\lfloor\frac{n}{k+1}\right\rfloor.$$

Apply Theorem 1.3.1 to B in order to find a path, P, of order $2\left\lfloor\frac{n}{k+1}\right\rfloor+2$ in B.

Notice that we have $|V(K)\setminus B|\geq(k-2)(n-1)+(k-1)\left\lfloor\frac{n}{k+1}\right\rfloor$. Part (a) of Lemma 6.3.2 shows that $V\setminus B$ must contain a blue $(k-2)$nd power of a path,Q^{k-2} , on $(k-2)\left\lfloor\frac{n}{k+1}\right\rfloor+k-2$ vertices.

Since all the edges between B and $V\setminus B$ are blue we can apply Lemma 6.3.3 with $i=k-1$ in order to find a blue kth power of a path spanning on n vertices in G.

Case 3: Suppose that the vertices of K can be arranged into disjoint sets $B_1, \ldots, B_k$ such that for $i \neq j$ all the edges between B_i and B_j are blue and we have

$$|B_1| \geq |B_2| \geq \cdots \geq |B_k| \geq \left\lceil \frac{n}{k+1} \right\rceil.$$

Let t be the maximum index for which $|B_t| > n-1$. Notice that $|K| \geq k(n-1) + \left\lfloor \frac{n}{k+1} \right\rfloor$ implies that we have $|B_1| + \cdots + |B_t| - t(n-1) \geq \left\lfloor \frac{n}{k+1} \right\rfloor$. Therefore, for $i \leq t$, we can choose numbers x_i satisfying $0 \leq x_i \leq |B_i| - n + 1$ for all i and also $x_1 + \cdots + x_t = \left\lfloor \frac{n}{k+1} \right\rfloor$.

For each $i \leq t$ we have $|B_i| = n - 1 + x_i$, which combined with Theorem 1.3.1, implies that B_i contains a blue path R_i of order $2x_i + 1$. Let $r_{i,0}, r_{i,1}, \ldots, r_{i,2x_i}$ be the vertex sequence of R_i. For each $i \in \{1, \ldots, t\}$ and $j \neq i$ choose a set $A_{i,j}$ of vertices in B_j satisfying $|A_{i,j}| = x_i$. Note that for $j > t$, the identity $|B_j| \geq \left\lceil \frac{n}{k+1} \right\rceil$ implies that we have

$$|A_{1,j}| + \cdots + |A_{t,j}| = \left\lfloor \frac{n}{k+1} \right\rfloor \leq |B_j|. \tag{6.7}$$

For $j \leq t$, the identities $|B_j| \geq n$ and $x_j \leq \left\lfloor \frac{n}{k+1} \right\rfloor$ imply that we have

$$|A_{1,j}| + \cdots + |A_{j-1,j}| + |R_j| + |A_{j+1,j}| + \cdots + |A_{t,j}| = \left\lfloor \frac{n}{k+1} \right\rfloor + x_j + 1 \leq |B_j|. \tag{6.8}$$

Now, (6.7) and (6.8) imply that we can choose the sets $A_{i,j}$, such that $A_{i,j}$ and $A_{i',j}$ are disjoint for $i \neq i'$. In addition, for every $j \leq t$, (6.8) implies that we can choose the sets $A_{i,j}$ to be disjoint from R_j. Let $a_{i,j,1}, \ldots, a_{i,j,x_i}$ be the vertices of $A_{i,j}$. If $n \not\equiv 0 \pmod{k+1}$, then the inequalities in both (6.7) and (6.8) must be strict, and so there must be at least one vertex contained in B_i outside of $R_i \cup A_{i,1} \cup \cdots \cup A_{i,t}$. Let b_i be this vertex.

For $i = 1, \ldots, t$ and $j = 1, \ldots, x_i$, we will define blue paths $P_{i,j}$ of order $k+1$ as follows. If $i = 1$ and $j \in \{1, \ldots, x_1 - 1\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{1,j} = r_{1,2j-1}, r_{1,2j}, a_{1,2,j}, a_{1,3,j}, \ldots, a_{1,k,j}.$$

If $i = 1$ and $j = x_1$, then $P_{i,j}$ has the following vertex sequence.

$$P_{1,x_1} = r_{1,2x_1-1}, r_{1,2x_1}, r_{2,0}, a_{1,3,x_1}, \dots, a_{1,k,x_1}.$$

If $i \in \{2, \dots, t-1\}$ and $j \in \{1, \dots, x_i - 1\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{i,j} = r_{i,2j-1}, a_{i,1,j}, a_{i,2,j}, \dots, a_{i,i-1,j}, r_{i,2j}, a_{i,i+1,j}, a_{i,i+2,j}, \dots, a_{i,k,j}.$$

If $i \in \{2, \dots, t-1\}$ and $j = x_i$, then $P_{i,j}$ has the following vertex sequence.

$$P_{i,x_i} = r_{i,2x_i-1}, a_{i,1,x_i}, a_{i,2,x_i}, \dots, a_{i,i-1,x_i}, r_{i,2x_i}, r_{i+1,0}, a_{i,i+2,x_i}, \dots, a_{i,k,x_i}.$$

If $i = t$ and $j \in \{1, \dots, x_t\}$, then $P_{i,j}$ has the following vertex sequence.

$$P_{t,j} = r_{t,2j-1}, a_{t,1,j}, a_{t,2,j}, \dots, a_{t,t-1,j}, r_{t,2j}.$$

If $n \not\equiv 0 \pmod{k+1}$, we also define a path P_0 of order k with vertex sequence

$$P_0 = r_{1,0}, b_2, b_3, \dots, b_k.$$

If $n \equiv 0 \pmod{k+1}$, let $P_0 = \emptyset$.

Notice that the paths $P_{i,j}$ and $P_{i',j'}$ are disjoint for $(i,j) \neq (i',j')$. Similarly P_0 is disjoint from all the paths $P_{i,j}$. We have the following

$$|P_0| + \sum_{i=1}^{k} \sum_{j=1}^{x_i} |P_{i,j}| = |P_0| + (k+1)(x_1 + \cdots + x_k) = |P_0| + (k+1) \left\lfloor \frac{n}{k+1} \right\rfloor \geq n. \quad (6.9)$$

We claim that the following path is in fact a blue kth power of a path.

$$P = \begin{cases} P_0 + \\ P_{1,1} + P_{1,2} + \cdots + P_{1,x_t} + \\ P_{2,1} + P_{2,2} + \cdots + P_{2,x_2} + \\ \vdots \\ P_{t,1} + P_{t,2} + \cdots + P_{t,x_t}. \end{cases}$$

To see that P is a kth power of a path one needs to check that any pair of vertices a, b at distance at most k along P are connected by a blue edge. It is easy to check that for any such a and b, either $a \in B_i$ and $b \in B_j$ for some $i \neq j$ or a and b are consecutive vertices along P_0 or $P_{i,j}$ for some i, j. In either case ab is blue implying that P is a blue kth power of a path.

The identity (6.9) shows that $|P| \geq n$, completing the proof. □

6.4 Discussion

In this section we dicuss some further directions one might take with the results presented in this chapter.

- It would be interesting to see if there are any other Ramsey numbers which can be determined using the techniques we used in this chapter.

  If G is a graph of (vertex)-chromatic number $\chi(G)$, then $\sigma(G)$ is defined to be the smallest possible order of a colour class in a proper $\chi(G)$-vertex colouring of G. Generalising a construction of Chvatal and Harary, Burr [10] showed that if H is a graph and G is a connected graph and satisfying $|G| \geq \sigma(H)$, then we have

  $$R(G, H) \geq (\chi(H) - 1)(|G| - 1) + \sigma(H) \tag{6.10}$$

  This identity comes from considering a colouring consisting of $\chi(H) - 1$ red copies of $K_{|G|-1}$ and one red copy of $K_{\sigma(H)-1}$. Notice that for a kth power of a path, we have $\chi(P_n^k) = k+1$ and $\sigma(P_n^k) = \left\lfloor \frac{n}{k+1} \right\rfloor$. Therefore, Theorem 6.1.4 shows that (6.10) is best possible when $G = P_n$ and $H = P_n^k$.

It is an interesting question to find other pairs of graphs for which equality holds in (6.10) (see [2, 43]). Allen, Brightwell, and Skokan conjectured that when G is a path, then equality holds in (6.10) for any graph H satisfying $|G| \geq \chi(H)|H|$.

Conjecture 6.4.1 (Allen, Brightwell, and Skokan). *For every graph H, $R(P_n, H) = (\chi(H)-1)(n-1) + \sigma(H)$ whenever $n \geq \chi(H)|H|$.*

It is easy to see that in order to prove Conjecture 6.4.1, it is sufficient to prove it only in the case when H is a (not necessarily balanced) complete multipartite graph.

The techniques used in this chapter look like they may be useful in approaching Conjecture 6.4.1. One reason for this is that several parts of the proof of Theorem 6.1.4 would have worked if we were looking for the Ramsey number of a path versus a balanced complete multipartite graph insead of a power of a path.

- Recall That Lemma 1.5.1 only implies part of Häggkvist result (Theorem 1.3.3). However, it is easy to prove an "unbalanced" version of Lemma 1.5.1 which implies Theorem 1.3.3.

  Lemma 6.4.2. *Suppose that the edges of K_n are coloured with 2 colours and we have an integer t satisfying $0 \leq t \leq n$. Then there is a partition of K_n into a red path and a blue copy of $K_{m,m+t}$ for some integer m.*

  The proof of this lemma is nearly identical to the one we gave of Lemma 1.5.1 in the introduction. Indeed, the only modification that needs to be made is that we need to add the condition "$\big||X| - |Y|\big| \geq t$" on the sets X and Y in the proof of Lemma 1.5.1.

- It would be interesting to see whether Theorems 6.1.1 and 6.1.2 have any applications in the area of partitioning coloured complete graphs. In particular, given that Lemma 1.5.1 played an important role in the proof of the $r = 3$ case of

Conjecture 1.4.2 in Chapter 5, it is possible that Theorems 6.1.1 and 6.1.2 may help with that conjecture.

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