Parisian Excursions of Brownian Motion and their Applications in Mathematical Finance

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

In this thesis, we study Parisian excursions, which are defined as excursions of Brownian motion above or below a pre-determined barrier, exceeding a certain time length. Employing a new method, a recursion formula for the densities of single barrier and double barrier Parisian stopping times are computed. This new approach allows us to obtain a semi-closed form solution for the density of the one-sided stopping times, and does not require any numerical inversions of Laplace transforms. Further, it is backed by an intuitive argument which is premised on the recursive nature of the excursions and the strong Markov property of the Brownian motion. The same method is also employed in our computation of the two-sided and the double barrier Parisian stopping times. In turn, the resultant densities are used to price Parisian options. In particular, we provide numerical expressions for down-and-in Parisian calls. Additionally, we study the tail of the distribution of the two-sided Parisian stopping time. Based on the asymptotic properties of its distribution, we propose an approximation for the option prices, alleviating the heavy computational load arising from the recursions. Finally, we use the infinitesimal generator to obtain several results on other variations of Parisian excursions. Specifically, apart from the length, we are interested in the number of excursions and the maximum height achieved during an excursion. Using the same generator, we derive the joint Laplace transform of the occupation times of the Brownian motion above and below zero, but only starting the clock each time after a certain length.

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Chapter 1

Introduction

Parisian options were first introduced by Chesney, Jeanblanc and Yor [14]. They are path dependent options whose payoff depends not only on the final value of the underlying asset, but also on the path trajectory of the underlying above or below a predetermined barrier L. For example, the owner of a Parisian down-and-out call loses the option when the underlying asset price S reaches the level L and remains constantly below this level for a time interval longer than D, while for a Parisian down-and-in call, the same event gives the owner the right to exercise the option. Parisian options are a kind of barrier option. However, it has the advantage of not being as easily manipulated by an influential agent as a simple barrier option, and thus is a guarantee against easy arbitrage.

No explicit pricing formula is known for this type of option. Previous literature has largely focused on using Laplace transforms to price Parisian options. In Chesney et al. [14], Dassios and Wu [21], and Shröder [43], the problem is reduced to finding the Laplace transform of the Parisian stopping time, which is the first time the length of the excursion reaches level D. In [14], the Laplace transform of the stopping time was obtained using the Brownian meander and Azema martingale while Dassios and Wu [21] introduced a perturbed Brownian motion and a semi-Markov model to obtain the Laplace transform. In both of these, an explicit form of the Laplace transform of the distribution of the Parisian stopping time and consequently that of the option price is found. Other methods of pricing Parisian options include the PDE method, studied by Haber, Schönbucher and Wilmott [30], pricing by simulation, as in Anderluh [5] and Bernard and Boyle [8], and a combinatorial approach in Costabile [16]. Zhu and Chen [45] provided an analytic solution that involves a double integral, using a coordinate transform.

There exist also other types of Parisian options. Cumulative Parisian options, which are related to the total excursion time above (or below) a barrier, are studied in [14], while double-sided Parisian options are introduced in Dassios and Wu [19] and Anderluh and Weide [6]. Parisian option pricing under a jump diffusion model has been studied by Albrecher [3], and Chesney and Gauthier [13] looked at American Parisian options. Edokko options, which are generalisations of Parisian options, are introduced in Fujita and Miura [27]. Further, other types of path-dependent options such as α-percentile options have been explored in Miura [38], Akahori [2] and Dassios [17].

Several papers have also studied techniques to numerically invert the Laplace transforms of the option prices. Labart and Lelong [34, 35] used an inversion formula based on the Abate and Whitt [1] method. Bernard, Courtois and Quittard-Pinon [9] obtained numerical prices by approximating the Laplace transforms using a linear combination of fractional functions. This resulted in an approximate solution rather than an exact one, albeit to a high degree of accuracy. In this thesis, we propose a different method to obtain the option price without numerically inverting its Laplace transform. Instead, we work directly with the Laplace transform of the stopping time and simply use it to obtain a recursive formula for the density. We always know that a recursive formula for the density function exists and is discontinuous in D because if t is the first time the length of the excursion reaches D, and $kD < t < (k+1)D$, the excursion must start at $t-D$ which is between $(k-1)D < t-D < kD$, and there cannot be any excursions greater than length D before this. Hence, the density for the stopping time where t is between $kD < t < (k+1)D$ can be computed from the density of the previous step. Furthermore, to find the density for $kD < t < (k+1)D$, we will see later that we only need to compute a finite sum of k terms, allowing for a simple and fast procedure. For small time intervals, we give a direct and intuitive probabilistic proof of the formula for the density function. For larger time steps, we write the density function as a recursive equation which can be solved numerically. Furthermore, we also show how the prices of Parisian options can be computed from the density of the Parisian stopping time.

Two-sided Parisian options are options which are knocked in or out when the underlying asset spends D amount of time consecutively either above or below a single barrier. While the same intuitive argument does not work for the two-sided case, we can obtain a recursion for the density of the two-sided Parisian stopping time. The formula is very similar to that of the one-sided case. However, when we study the tails of the two distributions, we find that the two-sided stopping time has an exponential tail, while the one-sided stopping time has a heavier tail. This fact allows us to present an alternative method for pricing the option which is faster than computing the recursions. Moreover, we extend the method to also price double barrier Parisian options. Double barrier Parisian options are introduced in Dassios and Wu [19] and Anderluh and Weide [6], and the Laplace transforms for the price of these options are obtained. We derive the prices of double barrier Parisian options without numerically inverting its Laplace transform.

Besides the lengths of excursions, we also look at the heights of Parisian excursions. This has been studied in Gauthier [28] and Pitman and Yor [41], and is also related to Brownian excursion areas studied in Louchard [36, 37] and Perman and Wellner [39]. In particular, we obtain the Laplace transform of the stopping time which is the first time the Brownian motion makes an excursion above the barrier of a certain length, and hits a second barrier during the excursion. In the context of options, this will ensure that the stock price does not stay around the barrier during the excursion of interest and is thus less easily manipulated. In the Parisian default framework, as studied in Broeders and Chen [11] and Chen and Suchanecki [12], this ensures that companies are given not just a grace period but also some leeway on capital shortfall. Furthermore, Albrecher and Lautscham [4] generalised the classical ruin concept to a concept of bankruptcy under which the probability of bankruptcy increases the more negative the surplus becomes.

A generalisation of this framework leads us to consider the counting process of Parisian excursions. This has not been studied in the literature, but it is closely related to the Brownian local time, as seen in Karatzas and Shreve [32], Louchard [36] and Pitman and Yor [40]. Although not done in this thesis, this can have applications in mathematical finance, for example it can be used to price a bond that pays off a continuous payment whenever the price of a share is below a certain level for a certain period of time. This kind of bond can be used as insurance for the firm. We present two methods, the first one more rudimentary where we obtain the Laplace transform of the number of Parisian excursions. The second method uses the perturbed Brownian motion introduced in Dassios and Wu [21] and the piecewise deterministic semi-Markov model as detailed in Davis [22]. We extend further to derive the Laplace transform of the Parisian stopping time for the Brownian meander and also the joint Laplace transform of the Parisian occupation time above and below a barrier, which is the occupation times, but with a qualifying period for each excursion. The Brownian meander has been studied extensively, for example in Durrett and Inglehart [23, 24] and Hooghiemstra [29], the maximum, first entrance times and occupation time distributions of the Brownian meander are derived, while in Imhof [31], some joint densities involving the value and time of the maximum over a fixed time interval for the Brownian motion and Brownian meander are obtained.

This thesis is organised as follows:

Chapter 2 states an important result for the density of the Parisian stopping time. A recursive formula is derived for the density and we provide both an intuitive argument as well as a formal proof of this result. Furthermore, we propose a new procedure for pricing Parisian options and an algorithm for pricing one-sided Parisian options is also given.

Chapter 3 extends the results of the above to the two-sided case. We compare the tails of the one and two-sided Parisian stopping time distributions and the exact formula for the asymptotic behaviour of the two-sided case is derived.

Chapter 4 generalises to the case of double barrier Parisian stopping times. The procedure for pricing double barrier Parisian options is given in the chapter.

Chapter 5 derives the Laplace transform of a new stopping time, which is the first time the Brownian motion makes an excursion of a certain length and also achieves a minimum height during the excursion. We use a semi-Markov model to prove this result, and the same model will also be used in the next few chapters.

Chapter 6 provides some results on the counting process of Parisian excursions up to an exponential time. In particular, we look at the Laplace transform of the number of excursions of a certain length above or below a barrier, the joint distribution of the number of excursions above and below the barrier, and the joint distribution of the number of excursions above the barrier of different lengths.

Chapter 7 further explores the counting process using the piecewise deterministic semi-Markov model. We obtain the Laplace transforms of some stopping times related to more than one Parisian type excursions.

Chapter 8 looks at the Parisian stopping time for the Brownian meander and its Laplace transform is obtained.

Chapter 9 extends the framework to explore Parisian occupation times with a qualifying period. We obtain the joint Laplace transform of the cumulative occupation time of the Brownian motion above and below the barrier but we only start the clock each time after a qualifying period.

Chapter 10 concludes this thesis.

Chapter 2

One-sided Parisian Options

Parisian options are options whose payoff depends on the path trajectory of the underlying asset above or below a barrier. For instance, a Parisian down-and-in call is a call option that gets knocked in when the underlying stays below a barrier for D amount of time consecutively. This is illustrated in the following picture, where the option is knocked in at $\tau^-_{L,D}$. The mathematical definition of $\tau^-_{L,D}$ will be given in the next section but we use it here for illustrative purposes. It is the stopping time which is the first time the underlying process goes below barrier L, for $L > 0$ and stays below for a period longer than D.

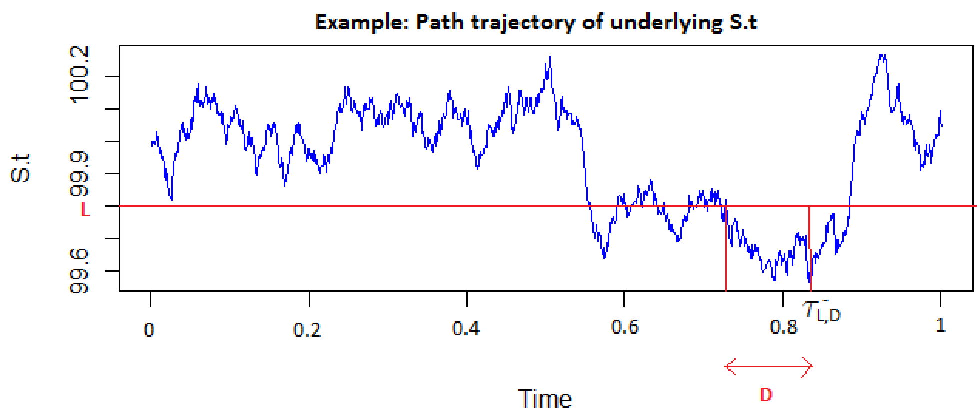

Figure 2.1: Illustration of a Parisian Down-and-in Call

In order to price Parisian options, we need to find the distribution of the Parisian stopping time. Here, we propose a new method to numerically obtain the density of the Parisian stopping time. This comes in the form of a recursion formula, and thus we note that the procedure is more efficient for long window length relative to the time to maturity. In this chapter, we look at the one-sided case, where we are only interested in excursions either above or below the barrier. The Laplace transform of the density is obtained in Chesney, Jeanblanc and Yor [14] using the Azema martingale and in Dassios and Wu [21] using a semi-Markov model. We show how this Laplace transform can be analytically inverted into a recursion. The advantage of this method is that the recursions are easy to program as the resulting formula only involves a finite sum and does not require a numerical inversion of the Laplace transform. We then propose a new algorithm for pricing Parisian options. This chapter is mostly based on Dassios and Lim [18], which was published earlier this year.

2.1 Definitions

We will use the same definitions for the excursions as in Chesney et al [14]. Let S be the underlying asset following a geometric Brownian motion, and $\mathcal{Q}$ denote the risk neutral probability measure. The dynamics for S under $\mathcal{Q}$ is

$$dS_t = S_t(rdt + \sigma dW_t), \qquad S_0 = x \tag{2.1}$$

where W_t is a standard Brownian motion under $\mathcal{Q}$, and r and σ positive constants. We also introduce the notations

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right)$$

so that the asset price $S_t = xe^{\sigma(mt+W_t)}$. For $L > 0$, we define

$$g^S_{L,t} = \sup\{s \le t | S_s = L\}, \qquad d^S_{L,t} = \inf\{s \ge t | S_s = L\}$$

with the usual convention that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. The trajectory of S between $g^S_{L,t}$ and $d^S_{L,t}$ is the excursion which straddles time t. We are interested here in $t - g^S_{L,t}$, which is the age of the excursion at time t. For $D > 0$, we now define

$$\begin{aligned}
\tau^+_{L,D}(S) &= \inf\{t \ge 0 | \mathbf{1}_{S_t > L}(t - g^S_{L,t}) \ge D\} \\
\tau^-_{L,D}(S) &= \inf\{t \ge 0 | \mathbf{1}_{S_t < L}(t - g^S_{L,t}) \ge D\}.
\end{aligned}$$

Hence, $\tau^+_{L,D}(S)$ is the first time that the length of the excursion of process S above the barrier L reaches level D, while $\tau^-_{L,D}(S)$ corresponds to the excursion below level L. We also introduce the following notation for the stopping times where we refer to the standard Brownian motion W instead of S. Furthermore, without loss of generality since any time t of interest can be expressed in units of the window length D, we let $D = 1$ from now on and drop its notation.

$$\begin{aligned}\tau_b^+ &= \inf\{t \geq 0 | \mathbf{1}_{W_t > b}(t - g^W_{b,t}) \geq 1\} \\ \tau_b^- &= \inf\{t \geq 0 | \mathbf{1}_{W_t < b}(t - g^W_{b,t}) \geq 1\}.\end{aligned}$$

We denote by $C_i^d(x, T)$ the price of a Parisian down-and-in call with initial underlying price x, maturity T, and parameters K, L, D, r fixed. The owner of a Parisian down-and-in option receives the payoff only if there is an excursion below the level L which is of length greater than D. This will be the case if $\tau_L^-(S) \leq T$. We have the price formula

$$C_i^d(x, T) = E_{\mathcal{Q}}\left[e^{-rT}\mathbf{1}_{\{\tau_L^-(S)\leq T\}}(xe^{\sigma(mT+W_T)} - K)^+\right].$$

We introduce a new probability measure $\mathcal{P}$, which makes $Z_t = W_t + mt$ a standard Brownian motion under $\mathcal{P}$. Applying Girsanov's Theorem, we have

$$C_i^d(x, T) = E_{\mathcal{P}}\left[e^{-(r+\frac{1}{2}m^2)T}\mathbf{1}_{\{\tau_b^-\leq T\}}e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+\right].$$

To simplify things, we also let

$$^*C_i^d(x, T) = e^{(r+\frac{1}{2}m^2)T}C_i^d(x, T).$$

We denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of the Brownian motion $(Z_t, t \geq 0)$. Then τ_b^- is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion

$$\begin{aligned}^*C_i^d(x, T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^-\leq T\}}E_{\mathcal{P}}\left[e^{mZ_T}\left(xe^{\sigma Z_T} - K\right)^+ | \mathcal{F}_{\tau_b^-}\right]\right] \\ &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^-\leq T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}}e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}}dy\right].\end{aligned}$$

We will first look at the stopping times τ_b^- and τ_b^+. We want to obtain the density functions for these two random variables. We denote by $f_b^-(t)$ the density function of τ_b^- and $f_b^+(t)$ the density function of τ_b^+.

2.2 Density of the one-sided Parisian stopping time

In this section, we present the recursive formula for the density function of τ_b^-. First, we give the intuitive proof for the first two steps of the recursion. This results in explicit formulas for when the time frame we are interested in is only at most twice that of the window length. We then provide a more formal proof which will give a recursive equation for all values of t. We present the proof for the excursions below the barrier, τ_b^-, and the result for τ_b^+ follows due to the symmetry of Brownian motion.

Theorem 2.1 *For $b \leq 0$, the density function of τ_b^- can be written as a recursion as follows:*

$$f_b^-(t) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ... \tag{2.2}$$

for $t > 1$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \frac{1}{2\pi\sqrt{t}} e^{-\frac{b^2}{2t}}, \qquad \text{for } t > 0 \tag{2.3}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s) \frac{\sqrt{s-1}}{2\pi s} ds, \qquad \text{for } t > k+1 \tag{2.4}$$

and

$$f_b^+(t) = f_{-b}^-(t). \tag{2.5}$$

2.2.1 Intuitive Proof for $1 < t < 3$

We look at the case $b = 0$, where we start at the barrier, $S_0 = L$. We denote by T_x the first hitting time of level x of a standard Brownian motion, and recall the notation g_t as the last time the Brownian motion is at 0 before time t. We want to find the density of τ_0^-, which is the first time the excursion reaches length 1. The density of τ_0^- vanishes for $t < 1$. For $1 < t < 2$, the excursion must start at $0 < t - 1 < 1$. Now, we modify the problem slightly and find instead $P(\tau_0^- - 1 \in dt)$, the probability density for t being the start of the excursion greater than length 1. For $0 < t < 1$, we condition the value of the Brownian motion at time 1. At time 1, the probability that the start of an excursion of length 1 occurred at time t is equal to the probability that t is the time of the last exit time g_1, that the Brownian motion travelled to x between time t to time 1, and that the Brownian motion does not hit 0 before a further time period t, such that the total time spent above 0 is 1. The required probability is obtained by integrating over x.

$$\begin{aligned} P(\tau_0^- - 1 \in dt) &= \int_0^\infty P(g_1 \in dt, W_1 \in dx, T_x \geq t)dx \\ &= P(g_1 \in dt) \int_0^\infty P(W_1 \in dx | g_1 = t) P(T_x \geq t) dx \end{aligned}$$

where we condition on the value of the Brownian motion at time 1. The distribution of g_1 follows the arcsine law (see Chung [15]), and is

$$P(g_1 \in dt) = \frac{1}{\pi\sqrt{t}\sqrt{1-t}}dt.$$

We note that $W_1 | g_1 = t$ has the same distribution as a Brownian meander of excursion length $1-t$ and has density (see [15])

$$P(W_1 \in dx | g_1 = t) = \frac{x}{2(1-t)} e^{-\frac{x^2}{2(1-t)}} dx.$$

Thus, we have

$$\begin{aligned} P(\tau_0^- - 1 \in dt) &= \frac{1}{\pi\sqrt{t}\sqrt{1-t}}dt \frac{1}{2} \int_0^\infty \frac{x}{1-t} e^{-\frac{x^2}{2(1-t)}} \int_t^\infty \frac{x}{\sqrt{2\pi u^3}} e^{-\frac{x^2}{2u}} du dx \\ &= \frac{1}{\pi\sqrt{t}\sqrt{1-t}} \frac{1}{2} \sqrt{1-t} dt \\ &= \frac{1}{2\pi\sqrt{t}} dt. \end{aligned}$$

We denote this by $L_0(t)$. For $0 < t < 1$, $L_0(t)$ is the probability that t is the start of one excursion greater than length 1. For $1 < t < 2$, however, there can be up to 2 excursions, and since we are only interested in the first excursion greater than length 1, we subtract the probability that there are indeed 2 excursions. We denote by $L_1(t)$ the probability density of t being the start of two excursions greater than length 1, for $1 < t < 2$. We break this probability up into 3 parts, the probability that the Brownian motion makes a first excursion of length 1, $L_0(s-1)$, that it travelled to x at time s, hits 0 again at time u, $s < u < t$, and that starting at 0 at time u, it will make a second excursion of length 1 at time t, $L_0(t-u)$.

The required probability is then obtained by integrating over all s, x and u.

$$\begin{aligned}
L_1(t) &= \int_1^t L_0(s-1)\int_0^\infty P(W_s\in dx|g_s=s-1)\int_s^t P(T_x\in du)L_0(t-u)\\
&= \int_1^t L_0(s-1)\int_0^\infty xe^{-\frac{x^2}{2}}\int_s^t \frac{x}{\sqrt{2\pi(u-s)^3}}e^{-\frac{x^2}{2(u-s)}}\frac{1}{2\pi\sqrt{t-u}}dudsdx\\
&= \int_1^t L_0(s-1)\frac{1}{2\pi}\frac{\sqrt{t-s}}{t-s+1}ds\\
&= \int_1^t L_0(t-s)\frac{1}{2\pi}\frac{\sqrt{s-1}}{s}ds
\end{aligned}$$

where we condition on the start of the first excursion greater than length 1 $s-1$, the value of the Brownian motion at the end of this excursion W_s, and the first time the Brownian motion comes back to zero again after that u. Moreover, $L_0(t-u)$ is the probability that t is the start of an excursion with length larger than 1, given that we start from 0 at u. For $2<t<3$, the density of τ_0^- is $L_0(t-1)-L_1(t-1)$. The same argument follows by induction for $t>3$ and we obtain the recursion.

2.2.2 General case ($b \le 0$)

Below we give the formal proof for the recursive formula of the theorem for time $t \ge 1$.
Proof. For simplicity, we define the following function.

$$\Psi(x) = 1 + x\sqrt{2\pi}e^{\frac{x^2}{2}}\mathcal{N}(x)$$

where $\mathcal{N}(x)$ denotes the standard normal distribution function. The Laplace transform $\hat{h}(\beta)$ of a function $h(t)$ on the positive real line is defined by

$$\mathcal{L}(h(t)) = \hat{h}(\beta) = \int_0^\infty e^{-\beta t}h(t)dt.$$

For $b \le 0$, the Laplace transform of the density $f_b^-(t)$ of the stopping time (with $D=1$) is

$$\hat{f}_b^-(\beta) = \frac{e^{\sqrt{2\beta}b}}{\Psi(\sqrt{2\beta})}.$$

Chesney et al. [14] obtained this using the Azema martingale, while Dassios and Wu [21] derived the same result using a semi-Markov model which we will be using later. Instead of inverting this numerically, we find a direct formula for $f_b^-(t)$ by writing the above equation as a renewal equation, which can then be solved recursively. First, we rewrite $\Psi(\sqrt{2\beta})$ as

$$\begin{aligned}
\frac{1}{\beta}e^{-\beta}\Psi\left(\sqrt{2\beta}\right) &= \frac{e^{-\beta}}{\beta}+2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
&= \frac{e^{-\beta}}{\beta}+\sqrt{\frac{\pi}{\beta}}\left(1+2\int_{0}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right) \\
&= \sqrt{\frac{\pi}{\beta}}+\frac{e^{-\beta}}{\beta}+\int_0^1\frac{e^{-\beta s}}{\sqrt{s}}ds \\
&= \sqrt{\frac{\pi}{\beta}}+\int_1^\infty e^{-\beta s}ds+\left(\int_0^\infty\frac{e^{-\beta s}}{\sqrt{s}}ds-\int_1^\infty\frac{e^{-\beta s}}{\sqrt{s}}ds\right) \\
&= 2\sqrt{\frac{\pi}{\beta}}+\frac{1}{\beta}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds \\
&= 2\sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right).
\end{aligned}$$

So we have

$$\hat{f}_b^-(\beta) = \frac{e^{-\beta}e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)} \tag{2.6}$$

$$= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k. \tag{2.7}$$

We denote

$$\hat{L}_k(\beta)=\frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k. \tag{2.8}$$

Since $\hat{L}_1(\beta)\to 0$ as $\beta\to\infty$, and $\hat{L}_k(\beta)$ is continuous and decreasing in β, there exists $\beta^*>0$ such that the above expansion from line (2.6) to (2.7) is valid for all $\beta>\beta^*$. Furthermore, we have the following Laplace inversions

$$\mathcal{L}^{-1}\left(\frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}\right) = \frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}} \tag{2.9}$$

$$\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right) = \frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}}. \tag{2.10}$$

Equation (2.9) can be checked by integrating

$$
\begin{aligned}
& \int_0^\infty e^{-\beta t}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b^2}{2t}}dt \\
= \quad & \frac{e^{\sqrt{2\beta}b}}{2\sqrt{2\beta}}\int_0^\infty \frac{\sqrt{2\beta}-\frac{b}{t}}{2\pi\sqrt{t}}e^{-\frac{(b+\sqrt{2\beta}t)^2}{2t}}dt+\frac{e^{-\sqrt{2\beta}b}}{2\sqrt{2\beta}}\int_0^\infty \frac{\sqrt{2\beta}+\frac{b}{t}}{2\pi\sqrt{t}}e^{-\frac{(b-\sqrt{2\beta}t)^2}{2t}}dt \\
= \quad & \frac{e^{\sqrt{2\beta}b}}{2\sqrt{\pi\beta}}
\end{aligned}
$$

where both integrals are evaluated using a change of variable $x=\frac{b\pm\sqrt{2\beta}t}{\sqrt{t}}$and the second term turns out to be zero. The LHS of (2.10) is the product of two functions whose inversion is known, so by taking their convolution we get

$$
\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \int_0^t \frac{1}{2\pi\sqrt{t-s}}\frac{1}{2s^{3/2}}\mathbf{1}_{\{s>1\}}ds \\
&= \left[-\frac{\sqrt{t-s}}{2\pi t\sqrt{s}}\right]_1^t=\frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}}.
\end{aligned}
$$

Hence, taking the Laplace inversion of equation (2.8), we obtain that L_k is the k^{th} convolution of (2.10), and L_0 is the expression obtained in (2.9). Finally, we note that for $n<t<n+1$, $L_k(t)$ is zero for $k>n$, so we only need a finite sum up to n, where the series expansion is valid for $\beta>\beta^*$. ■

2.2.3 General case $(b>0)$

We let T_b be the first hitting time of a standard Brownian motion of level b. For $b>0$, we are only concerned with the case where $T_b<D$. Without loss of generality, we take $D=1$. If $T_b\geq 1$, the Parisian stopping time $\tau_b^-=1$ since we are already below the barrier, and the problem simplifies.

Theorem 2.2 *For $f_b^-(t,T_b<1)$ the probability density function of τ_b^- on the set $\{T_b<1\}$,*

$$
f_b^-(t,T_b<1)=\sum_{k=0}^{n-1}(-1)^k L_k(t-1), \qquad \textit{for } n<t\leq n+1,\ n=1,2,... \tag{2.11}
$$

for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \mathbf{1}_{\{0<t\leq 1\}} \frac{1}{2\pi\sqrt{t}} e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}} \frac{1}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right) \tag{2.12}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s) \frac{\sqrt{s-1}}{2\pi s} ds, \qquad \text{for } t > k+1. \tag{2.13}$$

Furthermore,

$$f_b^+(t, T_b < 1) = f_{-b}^-(t, T_{-b} < 1). \tag{2.14}$$

Proof. In this case, we have

$$\begin{aligned}
E\left[e^{-\beta\tau_b^-(t)}\mathbf{1}_{\{T_b<1\}}\right] &= E\left[e^{-\beta(T_b+\tau_0^-)}\mathbf{1}_{\{T_b<1\}}\right] \\
&= E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right] \frac{1}{\Psi\left(\sqrt{2\beta}\right)} \\
&= e^{-\beta} \sum_{k=0}^{\infty} (-1)^k \frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{2\sqrt{\pi\beta}} \left(\frac{1}{2\sqrt{\pi\beta}} \int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k.
\end{aligned}$$

As in the previous case, there exists some β^* such that the series expansion is valid for $\beta > \beta^*$ since $\hat{L}_1(\beta) \to 0$ as $\beta \to \infty$ and $\hat{L}_k(\beta)$ is continuous and decreasing in β. We have

$$\begin{aligned}
L_0(t) = \mathcal{L}^{-1}\left(\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{2\sqrt{\pi\beta}}\right) &= \mathbf{1}_{\{0<t\leq 1\}} \int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} \frac{1}{2\pi\sqrt{t-s}} ds \\
&\quad + \mathbf{1}_{\{t>1\}} \int_0^1 \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} \frac{1}{2\pi\sqrt{t-s}} ds \\
&= \mathbf{1}_{\{0<t\leq 1\}} \frac{1}{2\pi\sqrt{t}} e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}} \frac{1}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right)
\end{aligned}$$

and L_k for $k = 1, 2, \ldots$ is the same as the previous case. ■

2.3 Pricing one-sided Parisian Options

2.3.1 Down-and-in Parisian call

We look in particular at the case of a down-and-in call option. Let S be the underlying asset price as in (2.1), L the barrier level, and m, b, l defined as in section 2.1.

$$m = \frac{1}{\sigma}\left(r - \frac{\sigma^2}{2}\right), \qquad b = \frac{1}{\sigma}\ln\left(\frac{L}{x}\right), \qquad k = \frac{1}{\sigma}\ln\left(\frac{K}{x}\right).$$

We denote by $Z(.)$ the probability density function of a standard normal random variable, and $\mathcal{N}_\rho(.,.)$ the joint cumulative function for a pair of bivariate standard normal random variables with correlation coefficient ρ. We present the following pricing formula for ${}^*C_i^d(x,T)$.

Theorem 2.3 *The price of a down-and-in Parisian call option on the underlying S with barrier $L < x$ (ie. $b < 0$) and maturity time $T > 1$, is given by*

$$ {}^*C_i^d(x,T) = \sqrt{2\pi}\int_0^T f_b^-(t)\left(x\psi(\sigma+m, h_b, b, \rho, t) - K\psi(m, h_b', b, \rho, t)\right) dt \tag{2.15}$$

where $f_b^-(t)$ is the density function of the Parisian stopping time with barrier b as in Theorem 2.1, and we define the function

$$\begin{aligned}\psi(x,y,b,\rho,t) \quad &= \quad e^{\frac{x^2(1+T-t)+2bx}{2}}\left(Z(-x)\mathcal{N}\left(\frac{-x\rho - y}{\sqrt{1-\rho^2}}\right) - \rho Z(y)\mathcal{N}\left(\frac{-x-\rho y}{\sqrt{1-\rho^2}}\right)\right.\\ &\qquad \left. -x\left(\mathcal{N}(-x) - \mathcal{N}_\rho(-x,y)\right)\right) \end{aligned}\tag{2.16}$$

and

$$h_b = \frac{1}{\sqrt{1+T-t}}\left(k - b - (\sigma+m)(1+T-t)\right) \tag{2.17}$$

$$h_b' = \frac{1}{\sqrt{1+T-t}}\left(k - b - m(1+T-t)\right) \tag{2.18}$$

$$\rho = \frac{1}{\sqrt{1+T-t}}. \tag{2.19}$$

Proof. As in the previous section, we change to a measure $\mathcal{P}$ under which Z_t is a standard Brownian motion. Furthermore, since τ_b^- is an $\mathcal{F}_t$ stopping time, by the strong Markov property of Brownian motion, we have

$$\begin{aligned}
{}^{*}C_i^d(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^- \le T\}} E\left[e^{mZ_T}\left(xe^{\sigma Z_T}-K\right)^+ |\mathcal{F}_{\tau_b^-}\right]\right] \\
&= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b^- \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}} e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}} dy\right].
\end{aligned}$$

It is easy to see that τ_b^- and $Z_{\tau_b^-}$ are independent (see Chesney et al. [14] for more detail). We denote the density functions of τ_b^- and $Z_{\tau_b^-}$ by $f_b^-(t)$ and $v(z)$ respectively. The density of $Z_{\tau_b^-}$ is associated to the Brownian meander and for window length $D = 1$ is (see Yor [44] for more detail)

$$v(dz) = P(Z_{\tau_b^-} \in dz) = (b-z)e^{-\frac{(z-b)^2}{2}}\mathbf{1}_{\{z<b\}}dz.$$

So we have

$$\begin{aligned}
{}^{*}C_i^d(x,T) &= \int_0^T \int_{-\infty}^{\infty} f_b^-(t)v(dz) \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-z)^2}{2(T-t)}} dydt \\
&= \sqrt{2\pi}\int_0^T f_b^-(t) \int_{-\infty}^{b}\int_k^{\infty} (b-z)e^{-\frac{(z-b)^2}{2}} e^{my}\left(xe^{\sigma y}-K\right) \frac{1}{2\pi\sqrt{T-t}} e^{-\frac{(y-z)^2}{2(T-t)}} dydzdt.
\end{aligned}$$

We are interested to evaluate the double integral with respect to y and z.

$$\begin{aligned}
& \frac{1}{2\pi\sqrt{T-t}} \int_{-\infty}^{b}\int_k^{\infty} e^{my}(xe^{\sigma y}-K)(b-z)e^{-\frac{(z-b)^2}{2}} e^{-\frac{(y-z)^2}{2(T-t)}} dzdy \\
= & \frac{1}{2\pi\sqrt{T-t}} \int_{-\infty}^{b}\int_k^{\infty} xe^{(\sigma+m)y}(b-z)e^{-\frac{(z-b)^2}{2}} e^{-\frac{(y-z)^2}{2(T-t)}} dzdy \\
& -\frac{1}{2\pi\sqrt{T-t}} \int_{-\infty}^{b}\int_k^{\infty} Ke^{my}(b-z)e^{-\frac{(z-b)^2}{2}} e^{-\frac{(y-z)^2}{2(T-t)}} dzdy.
\end{aligned}$$

We look at the first integral on the RHS. The integrand can be written as the joint density function of a bivariate normal distribution.

$$\begin{aligned}
& x\frac{1}{2\pi\sqrt{T-t}} \int_{-\infty}^{b}\int_k^{\infty} e^{(\sigma+m)y}(b-z)e^{-\frac{(z-b)^2}{2}} e^{-\frac{(y-z)^2}{2(T-t)}} dzdy \\
= \; & x\exp\left\{\frac{(\sigma+m)^2(1+T-t)+2b(\sigma+m)}{2}\right\}\frac{1}{2\pi\sqrt{T-t}} \\
& \int_{-\infty}^{b}\int_k^{\infty} (b-z)\exp\left\{-\frac{(y-(b+(\sigma+m)(1+T-t)))^2}{2(T-t)}\right\}\exp\left\{-\frac{(z-(b+(\sigma+m)))^2}{2(T-t)/(1+T-t)}\right\}
\end{aligned}$$

$$\exp\left\{\frac{2(y-(b+(\sigma+m)(1+T-t)))(z-(b+(\sigma+m)))}{2(T-t)}\right\}dzdy$$

$$= x\exp\left\{\frac{(\sigma+m)^2(1+T-t)+2b(\sigma+m)}{2}\right\}$$

$$\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{-(\sigma+m)}\int_{h_b}^{\infty}(-v-(\sigma+m))\exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\}dudv \tag{2.20}$$

where we have used the transformation $u=\frac{y-(b+(\sigma+m)(1+T-t)}{\sqrt{1+T-t}}$ and $v=z-(b+(\sigma+m))$, h_b, h_b' and ρ as defined in (2.3) - (2.5). Now, we have the following result for (U,V) bivariate normal with mean 0, variance 1, and correlation coefficient ρ.

$$\begin{aligned}
&\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{h_b}^{\infty}\int_{-\infty}^{-(\sigma+m)} ve^{-\frac{(u^2-2\rho uv+v^2)}{2(1-\rho^2)}}dudv\\
=\;&\frac{1}{2\pi}\int_{h_b}^{\infty}\int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}}\left(\sqrt{1-\rho^2}w+\rho u\right)e^{-\frac{1}{2}(u^2+w^2)}dudw
\end{aligned}$$

where we used the transformation $v-\rho u=w\sqrt{1-\rho^2}$. Now applying integration by parts, we obtain

$$\begin{aligned}
&\frac{\sqrt{1-\rho^2}}{2\pi}\int_{h_b}^{\infty}\left[-e^{-\frac{1}{2}(u^2+w^2)}\right]_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}}+\frac{\rho}{2\pi}\int_{h_b}^{\infty}ue^{-\frac{1}{2}u^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}}e^{-\frac{1}{2}w^2}dwdu\\
=\;&\frac{\sqrt{1-\rho^2}}{2\pi}\int_{h_b}^{\infty}-e^{-\frac{((\sigma+m)^2+2\rho(\sigma+m)u+u^2)}{2(1-\rho^2)}}du\\
&+\frac{\rho}{2\pi}\left\{\left[-e^{-\frac{1}{2}u^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}}e^{-\frac{1}{2}w^2}dw\right]_{h_b}^{\infty}+\int_{h_b}^{\infty}e^{-\frac{1}{2}u^2}e^{-\frac{1}{2}\left(\frac{-(\sigma+m)-\rho u}{\sqrt{1-\rho^2}}\right)^2}\left(\frac{-\rho}{\sqrt{1-\rho^2}}\right)du\right\}\\
=\;&-\frac{\sqrt{1-\rho^2}}{2\pi}\int_{h_b}^{\infty}e^{-\frac{1}{2}((\sigma+m)^2+2\rho(\sigma+m)u+u^2)}du\left(1+\frac{\rho^2}{1-\rho^2}\right)+\frac{\rho}{2\pi}e^{-\frac{1}{2}h_b^2}\int_{-\infty}^{\frac{-(\sigma+m)-\rho h_b}{\sqrt{1-\rho^2}}}e^{-\frac{1}{2}w^2}dw.
\end{aligned}$$

Here, we apply another transformation $v=\frac{u+(\sigma+m)\rho}{\sqrt{1-\rho^2}}$ to the first integral above. This gives us

$$\begin{aligned}
&-\frac{1}{2\pi}\int_{\frac{h_b+(\sigma+m)\rho}{\sqrt{1-\rho^2}}}^{\infty}e^{-\frac{1}{2}((\sigma+m)^2+v^2)}dv+\rho Z(h_b)\mathcal{N}\left(\frac{-(\sigma+m)-\rho h_b}{\sqrt{1-\rho^2}}\right)\\
=\;&-Z(-(\sigma+m))\mathcal{N}\left(\frac{-(\sigma+m)\rho-h_b}{\sqrt{1-\rho^2}}\right)+\rho Z(h_b)\mathcal{N}\left(\frac{-(\sigma+m)-\rho h_b}{\sqrt{1-\rho^2}}\right).
\end{aligned}$$

So we have

$$
\begin{aligned}
& \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{-(\sigma+m)} \int_{h_b}^{\infty} (-v-(\sigma+m)) \exp\left\{-\frac{u^2-2\rho uv+v^2}{2(1-\rho^2)}\right\} du dv \\
= \; & Z(-(\sigma+m))\mathcal{N}\left(\frac{-(\sigma+m)\rho-h_b}{\sqrt{1-\rho^2}}\right) - \rho Z(h_b)\mathcal{N}\left(\frac{-(\sigma+m)-\rho h_b}{\sqrt{1-\rho^2}}\right) \\
& -(\sigma+m)\left(\mathcal{N}(-(\sigma+m)) - \mathcal{N}_\rho(-(\sigma+m), h_b)\right).
\end{aligned}
$$

Substituting this back into (2.18), we obtain $x\psi(\sigma+m, h_b, b, \rho, t)$. Doing the same for the second integral, we get $x\psi(\sigma+m, h_b, b, \rho, t) - K\psi(m, h'_b, b, \rho, t)$. ■

Theorem 2.4 *The price of a down-and-in Parisian call option on the underlying S with barrier $L > x$ (ie. $b > 0$) and maturity time $T > 1$, is given by*

$$
\begin{aligned}
{}^*C_i^d(x,T) &= x\phi(\sigma+m) - K\phi(m) \\
& \quad +\sqrt{2\pi}\int_0^T f_b^-(t; T_b < 1)\left(x\psi(\sigma+m, h_b, b, \rho, t)\right) - K\psi(m, h'_b, b, \rho, t))\, dt
\end{aligned} \tag{2.21}
$$

where $f_b^-(t, T_b < 1)$ is the density function of the Parisian stopping time with barrier b in Theorem 2.2, and ψ, h_b, h'_b, ρ defined as in Theorem 2.3, and we also used the function

$$
\begin{aligned}
\phi(x) &= e^{\frac{x^2 T}{2}}\left(\mathcal{N}(b-x) - \mathcal{N}_{\frac{1}{\sqrt{T}}}\left(b-x, \frac{k-xT}{\sqrt{T}}\right)\right) \\
& \quad -e^{\frac{x^2T+4bx}{2}}\left(\mathcal{N}(-b-x) - \mathcal{N}_{\frac{1}{\sqrt{T}}}\left(-b-x, \frac{k-2b-xT}{\sqrt{T}}\right)\right).
\end{aligned} \tag{2.22}
$$

Proof. For $b > 0$, we split into the case when $T_b > 1$ and $T_b < 1$.

$$
\begin{aligned}
{}^*C_i^d(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b^- \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}} e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}} dy\right] \\
& \quad +E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b^- \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}} e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}} dy\right].
\end{aligned}
$$

For $z < b$, the law of Z_1 on the set $\{T_b > 1\}$ is

$$
P(Z_1 \in dz, T_b > 1) = P(Z_1 \in dz) - P(Z_1 \in dz, T_b < 1)
$$

$$= \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}} - e^{-\frac{(z-2b)^2}{2}}\right) dz$$

where the second term is due to the reflection about b. Since we start below the barrier, $\tau_b^- = 1$ if $T_b > 1$. So we have

$$\begin{aligned}
& E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b^- \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}} e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}} dy\right] \\
= \; & E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{1 \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-1)}} e^{-\frac{(y-Z_1)^2}{2(T-1)}} dy\right] \\
= \; & \frac{1}{\sqrt{2\pi(T-1)}} \int_{-\infty}^{b}\int_{k}^{\infty} e^{my}\left(xe^{\sigma y} - K\right) e^{-\frac{(y-z)^2}{2(T-1)}} \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}} - e^{-\frac{(z-2b)^2}{2}}\right) dzdy \\
= \; & x\phi(\sigma+m) - K\phi(m)
\end{aligned}$$

where the last step involves writing the integrand as the density function of a pair of bivariate normal random variables as before to obtain a joint cumulative distribution function. On the set $\{T_b < 1\}$, $Z_{\tau_b^-}$ is again independent of τ_b^-, so we have

$$\begin{aligned}
& E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b^- \le T\}} \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b^-)}} e^{-\frac{(y-Z_{\tau_b^-})^2}{2(T-\tau_b^-)}} dy\right] \\
= \; & \int_{0}^{T}\int_{-\infty}^{\infty} f_b^-(t;T_b<1)v(dz) \int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y} - K\right)^+ \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-z)^2}{2(T-t)}} dydt \\
= \; & \sqrt{2\pi}\int_{0}^{T} f_b^-(t;T_b<1)\left(x\psi(\sigma+m, h_b, b, \rho, t)\right) - K\psi(m, h_b', b, \rho, t)\right) dt
\end{aligned}$$

where the proof is as before. ■

2.3.2 Down-and-out Parisian call

A Parisian down-and-out call can be priced using the prices for the down-and-in calls with the same barrier, strike price and initial asset price. We let $C_{BS}(x, T)$ denote the price of a vanilla call option with initial asset price x, maturity T, and strike price K. Then

$$C_{BS}(x, T) = E_{\mathcal{Q}}\left[e^{-rT}(S_T - K)^+\right].$$

We also denote $C_o^d(x,T)$ as the price of a down-and-out call with the same parameters. We have for $L \leq x$ ($b \leq 0$),

$$\begin{aligned} C_o^d(x,T) &= E_{\mathcal{Q}}\left[e^{-rT}(S_T-K)^+\mathbf{1}_{\{\tau_b^->T\}}\right] \\ &= E_{\mathcal{Q}}\left[e^{-rT}(S_T-K)^+\right] - E_{\mathcal{Q}}\left[e^{-rT}(S_T-K)^+\mathbf{1}_{\{\tau_b^-\leq T\}}\right] \\ &= C_{BS}(x,T) - e^{-(r+\frac{1}{2}m^2)T}\left({}^*C_i^d(x,T) - x\phi(\sigma+m) - K\phi(m)\right). \end{aligned}$$

This parity relationship allows us to price the down-and-out Parisian calls.

For $L > x$ ($b > 0$), we only need to consider the case when $T_b < 1$, since when $T_b > 1$, $\tau_b^- = 1$ and the option is knocked out. Hence, the price of the option is

$$\begin{aligned} C_o^d(x,T) &= E_{\mathcal{Q}}\left[e^{-rT}(S_T-K)^+\mathbf{1}_{\{\tau_b^->T\}}\mathbf{1}_{\{T_b<1\}}\right] \\ &= E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}(S_T-K)^+e^{-rT}\right] - E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b^-<T\}}(S_T-K)^+e^{-rT}\right] \\ &= E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}E_{\mathcal{Q}}\left[(S_T-K)^+e^{-rT}|\mathcal{F}_{T_b}\right]\right] - e^{-(r+\frac{1}{2}m^2)T}\left({}^*C_i^d(x,T) - x\phi(\sigma+m) - K\phi(m)\right) \\ &= E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}C_{BS}(L,T-T_L)\right] - e^{-(r+\frac{1}{2}m^2)T}\left({}^*C_i^d(x,T) - x\phi(\sigma+m) - K\phi(m)\right) \\ &= \int_0^1 \frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}C_{BS}(L,T-t)dt - e^{-(r+\frac{1}{2}m^2)T}\left({}^*C_i^d(x,T) - x\phi(\sigma+m) - K\phi(m)\right). \end{aligned}$$

2.3.3 Up-and-in Parisian call

For an up-and-in Parisian call, we have the following pricing formulae:

Theorem 2.5 *The price of an up-and-in Parisian call on the underlying S with barrier $L > x$ (ie. $b > 0$) and maturity time $T > 1$, is given by*

$$ {}^*C_i^u(x,T) = \sqrt{2\pi}\int_0^T f_b^+(t)(x\psi(-(\sigma+m),h_b,b,-\rho,t) - K\psi(-m,h_b',b,-\rho,t))dt \qquad (2.23)$$

where $f_b^+(t)$ is the density function of the Parisian stopping time with barrier b as in Theorem 2.1, h_b, h_b' and ρ are defined as in Theorem 2.3, and $\bar{\mathcal{N}}_\rho(x,y) = P(X > x, Y > y)$ is the survival function of the bivariate normal random variables X and Y with correlation coefficient ρ.

Theorem 2.6 *The price of an up-and-in Parisian call on the underlying S with barrier $L < x$*

(ie. $b < 0$) and maturity time $T > 1$, is given by

$$
\begin{aligned}
{}^*C_i^u(x,T) &= x\phi'(\sigma+m) - K\phi'(m) \\
&\quad + \sqrt{2\pi}\int_0^T f_b^+(t;T_b<1)(x\psi(-(\sigma+m),h_b,b,-\rho,t) - K\psi(-m,h_b',b,-\rho,t))dt
\end{aligned} \tag{2.24}
$$

where $f_b^+(t;T_b<1)$ is the density function of τ_b^+ conditioned on the set $T_b < 1$ as in Theorem 2.2, and ψ, h_b, h_b', and ρ are as in the previous theorem. Furthermore, the function $\phi'(x)$ is defined as

$$
\phi'(x) = e^{\frac{x^2T}{2}}\left(\bar{\mathcal{N}}_\rho\left(b-x,\frac{k-xT}{\sqrt{T}}\right) - e^{\frac{x^2T+4bx}{2}}\bar{\mathcal{N}}_\rho\left(-b-x,\frac{k-(2b+xT)}{\sqrt{T}}\right)\right). \tag{2.25}
$$

2.3.4 Up-and-out Parisian call

We denote by C_o^u the price of a up-and-out Parisian call. As in above for down-and-out call options, the Parisian up-and-out calls can be priced using parity relationships. For $b > 0$, we have

$$C_o^u(x,T) = C_{BS}(x,T) - C_o^d(x,T)$$

and for $b \le 0$,

$$C_o^u(x,T) = \int_0^1 \frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}C_{BS}(L,T-t)dt - e^{-(r+\frac{1}{2}m^2)T}\left({}^*C_i^u(x,T) - x\phi'(\sigma+m) - K\phi'(m)\right).$$

2.3.5 Put-call parities

Furthermore, we can obtain the prices of the one-sided Parisian put options by using some put-call parity relations. We denote by $P_o^d(x,T,K,L)$ the price of a down-and-out Parisian put with initial asset price $S_0 = x$, maturity T, strike price K and barrier L and so on. Quoting the results obtained in Labart and Lelong [34] Section 5, we have

$$
\begin{aligned}
P_o^d(x,T,K,L) &= xKC_o^u(\frac{1}{x},T,\frac{1}{K},\frac{1}{L}) \\
P_o^u(x,T,K,L) &= xKC_o^d(\frac{1}{x},T,\frac{1}{K},\frac{1}{L}) \\
P_i^u(x,T,K,L) &= xKC_i^d(\frac{1}{x},T,\frac{1}{K},\frac{1}{L}) \\
P_i^d(x,T,K,L) &= xKC_i^u(\frac{1}{x},T,\frac{1}{K},\frac{1}{L}).
\end{aligned}
$$

2.4 Numerical Results

The following table shows the density and cumulative function for $b = 0$ at intervals of 0.5, computed using a time step of $h = 0.001$.

Table 2.1: Density $f_0^-(t)$ for $0 < t \leq 10$

t	$f_0^-(t)$	$F_0^-(t)$	t	$f_0^-(t)$	$F_0^-(t)$
1.5	0.225192	0.224967	6.0	0.032951	0.596578
2.0	0.159195	0.318230	6.5	0.029312	0.612044
2.5	0.115597	0.385764	7.0	0.026296	0.625858
3.0	0.089488	0.436448	7.5	0.023763	0.638296
3.5	0.071858	0.476398	8.0	0.021613	0.649571
4.0	0.059334	0.508918	8.5	0.019768	0.659854
4.5	0.050062	0.536056	9.0	0.018171	0.669282
5.0	0.042972	0.559146	9.5	0.016778	0.677967
5.5	0.037410	0.579104	10.0	0.015554	0.686003

The following graph shows the distribution function $F_0(t)$ plotted against t, for $0 \leq t \leq 50$.

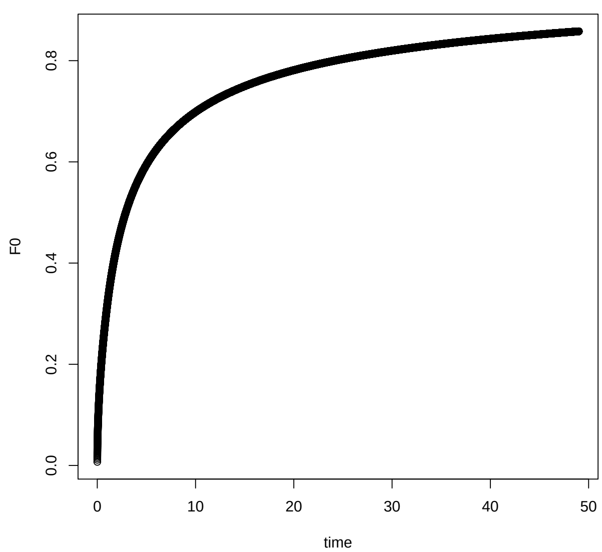

Figure 2.2: Graph of $F_0(t)$ vs t for $0 < t \leq 50$

Below is a table of the prices of Parisian down-and-in calls. Using the same parameters as in Bernard, Courtois and Quittard-Pinon [9], $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 95$, and

$L = 90$, we obtain the same results up to 2 decimal places at different window lengths D and initial stock price S_0.

Table 2.2: Price of Parisian Down-and-in call

S_0	$D = 1$ month	$D = 2$ months	$D = 3$ months	$D = 4$ months
80	2.599144	1.917126	1.325256	0.894224
82	2.915856	1.951244	1.278959	0.833757
84	3.024509	1.850371	1.158805	0.732982
86	2.862540	1.630234	0.983769	0.605966
88	2.466282	1.337957	0.783991	0.471268
90	1.969965	1.034182	0.589757	0.345171
92	1.558517	0.798889	0.445765	0.255106
94	1.223695	0.610794	0.332570	0.185555
96	0.957872	0.465487	0.247251	0.134427
98	0.747613	0.353669	0.183210	0.097016
100	0.581894	0.267937	0.135329	0.069763

The table below gives a comparison of the CPU times for our algorithm and that using the Laplace inversion technique in [34], computed using the above parameters and $S_0 = 90$. Due to the increasing number of recursions required, the computation times increase rapidly as the window length decreases. As we can see in the table below, our algorithm is very efficient for long window lengths relative to the time to maturity. For window lengths of 2 months and above, the CPU time required for this algorithm is less than a second. However, for window length of 1 month, our algorithm is slower because of the large number of recursions.

Table 2.3: Computation times (s)

D	Recursion formula	Laplace inversion
1 month	2.56	1.06
2 months	0.98	1.24
3 months	0.70	1.48
4 months	0.58	1.66

2.A Appendix to Chapter 2

The code is written in R. First we compute the density $f_b^-(t)$ for different values of t, using n number of steps and h for the size of each time step. Using the numerical values of $f_b^-(t)$, we can do a numerical integration and use Theorem 2.3 to obtain the price of the down-and-in Parisian call. We note that since we have chosen the window length D as the unit of time, all parameters (r, σ) are correspondingly normalised depending on the window length. Below is the code for pricing a down-and-in Parisian call option using the parameters $\sigma = 0.2$, $r = 0.05$, $T = 1$ year, $K = 95$, and $L = 90$, number of time steps $n = 1000$, $D = 3$ months, and initial price $S_0 = 92$.

```
# load package
library(mnormt)

#parameters
n <- 1000
t <- 4
r <- 0.05
sigma <- 0.2
S0 <- 92
L <- 90
K <- 95

t<-t-1
h<-1/n
r<-r/(t+1)
sigma<-sigma/sqrt(t+1)
b<- 1/sigma*log(L/S0)
m<- 1/sigma*(r-sigma^2/2)

f<-mat.or.vec(t*n,1) #vector of densities for tau

L1<-1/sqrt(pi*(1:(t*n)-0.5)*h)*exp(-b^2/(2*(1:(t*n)-0.5)*h))
L2<-mat.or.vec(t*n,1) #vector of Lk's
x<-sqrt(((1:(t*n)-0.5)*h))/(pi*(1+((1:(t*n)-0.5)*h)))
```

```
f<-L1

for(i in 1:(t-1)) {
y<-convolve(L1[((i-1)*n+1):((t-1)*n)],rev(x[1:((t-i)*n)]),type = "open")
L2[(i*n+1):(t*n)]<-y[1:((t-i)*n)]*h
f<-f+L2*(-1/2)^i
L1<-L2
L2<-mat.or.vec(t*n,1)
}

f<-f/(2*sqrt(pi)) #we obtain the density for \tau_b^-.

rho<-1/sqrt(1+((t*n):1-0.5)*h)
c1<-S0*exp(((sigma+m)^2*(1+((t*n):1-0.5)*h)+2*b*(sigma+m))/2)
c2<-K*exp((m^2*(1+((t*n):1-0.5)*h)+2*b*m)/2)
k1<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-(sigma+m)*(1+((t*n):1-0.5)*h))
l1<-rep(-(sigma+m),times=(t*n))
k2<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-m*(1+((t*n):1-0.5)*h))
l2<-rep(-m,times=(t*n))

mnorm1<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l,k)
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm1[i]<-pmnorm(c(l1[i],k1[i]),c(0,0),varcov)
}

mnorm1<-pnorm(l1)-mnorm1

mnorm2<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l',k')
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm2[i]<-pmnorm(c(l2[i],k2[i]),c(0,0),varcov)
}
```

```
mnorm2<-pnorm(l2)-mnorm2

q<-c1*(dnorm(l1)*pnorm((l1*rho-k1)/sqrt(1-rho^2))
   -rho*dnorm(k1)*pnorm((l1-rho*k1)/sqrt(1-rho^2))
   -(sigma+m)*mnorm1)-c2*(dnorm(l2)*pnorm((l2*rho-k2)/sqrt(1-rho^2))
   -rho*dnorm(k2)*pnorm((l2-rho*k2)/sqrt(1-rho^2))-m*mnorm2)
q<-sqrt(2*pi)*q

price<-f%*%q*h*exp(-(r+0.5*m^2)*(t+1))
```

For $L > x$ ($b > 0$), there is an extra term for $T_b > 1$. The code for $b > 0$ is

```
# load package
library(mnormt)

#parameters
n <- 1000
t <- 4
r <- 0.05
sigma <- 0.2
S0 <- 80
L <- 90
K <- 95

t<-t-1
h<-1/n
r<-r/(t+1)
sigma<-sigma/sqrt(t+1)
b<- 1/sigma*log(L/S0)
m<- 1/sigma*(r-sigma^2/2)
k<- 1/sigma*log(K/S0)

f<-mat.or.vec(t*n,1) #vector of densities for tau
```

```
L1<-mat.or.vec(t*n,1) #vector of Lk starting at L0
L1[1:n]<-1/sqrt(pi*(1:n-0.5)*h)*exp(-b^2/(2*(1:n-0.5)*h))
L1[(n+1):(t*n)]<-2/sqrt((pi*((n+1):(t*n)-0.5)*h))*
                 pnorm(-b*sqrt(1-1/(((n+1):(t*n)-0.5)*h)))*
                 exp(-b^2/(2*(((n+1):(t*n)-0.5)*h)))
L2<-mat.or.vec(t*n,1) #vector of Lk's
x<-sqrt(((1:(t*n)-0.5)*h))/(pi*(1+((1:(t*n)-0.5)*h)))

f<-L1

for(i in 1:(t-1)) {
y<-convolve(L1[((i-1)*n+1):((t-1)*n)],rev(x[1:((t-i)*n)]),type = "open")
L2[(i*n+1):(t*n)]<-y[1:((t-i)*n)]*h
f<-f+L2*(-1/2)^i
L1<-L2
L2<-mat.or.vec(t*n,1)
}

f<-f/(2*sqrt(pi))

rho<-1/sqrt(1+((t*n):1-0.5)*h)
c1<-S0*exp(((sigma+m)^2*(1+((t*n):1-0.5)*h)+2*b*(sigma+m))/2)
c2<-K*exp((m^2*(1+((t*n):1-0.5)*h)+2*b*m)/2)
k1<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-(sigma+m)*(1+((t*n):1-0.5)*h))
l1<-rep(-(sigma+m),times=(t*n))
k2<-1/sqrt(1+((t*n):1-0.5)*h)*(1/sigma*log(K/S0)-b-m*(1+((t*n):1-0.5)*h))
l2<-rep(-m,times=(t*n))

mnorm1<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l,k)
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm1[i]<-pmnorm(c(l1[i],k1[i]),c(0,0),varcov)
}
```

```
mnorm1<-pnorm(l1)-mnorm1

mnorm2<-mat.or.vec(t*n,1) #cdf of bivariate normal computed at (l',k')
for(i in 1:(t*n)) {
varcov<-matrix(c(1,rho[i],rho[i],1),2,2)
mnorm2[i]<-pmnorm(c(l2[i],k2[i]),c(0,0),varcov)
}

mnorm2<-pnorm(l2)-mnorm2

q<-c1*(dnorm(l1)*pnorm((l1*rho-k1)/sqrt(1-rho^2))
   -rho*dnorm(k1)*pnorm((l1-rho*k1)/sqrt(1-rho^2))
   -(sigma+m)*mnorm1)-c2*(dnorm(l2)*pnorm((l2*rho-k2)/sqrt(1-rho^2))
   -rho*dnorm(k2)*pnorm((l2-rho*k2)/sqrt(1-rho^2))-m*mnorm2)
q<-sqrt(2*pi)*q

price<-f%*%q*h

rhot<-1/sqrt(t+1)
varcov<-matrix(c(1,rhot,rhot,1),2,2)
phi1<-exp((sigma+m)^2*(t+1)/2)*(pnorm(b-(sigma+m))
       -pmnorm(c(b-(sigma+m),(k-(sigma+m)*(t+1))/sqrt(t+1)),c(0,0),varcov))-
       exp((sigma+m)^2*(t+1)/2+4*b*(sigma+m)/2)*((pnorm(-b-(sigma+m))
       -pmnorm(c(-b-(sigma+m),(k-2*b-(sigma+m)*(t+1))/sqrt(t+1)),c(0,0),varcov)))
phi2<-exp(m^2*(t+1)/2)*(pnorm(b-m)
       -pmnorm(c(b-m,(k-m*(t+1))/sqrt(t+1)),c(0,0),varcov))-
       exp(m^2*(t+1)/2+4*b*m/2)*((pnorm(-b-m)
       -pmnorm(c(-b-m,(k-2*b-m*(t+1))/sqrt(t+1)),c(0,0),varcov)))

price<-price+S0*phi1-K*phi2

price<-price*exp(-(r+0.5*m^2)*(t+1))
```

Chapter 3

Two-sided Parisian Options

Two-sided Parisian options get knocked in or out when the underlying either stays D amount of time above or below the barrier, whichever comes first. The stopping time $\tau_{L,D}$ is the first time the underlying process either stays above or below the barrier L for a period longer than length D, and is defined using the two one-sided Parisian stopping times,

$$\tau_{L,D} = \tau^{+}_{L,D} \wedge \tau^{-}_{L,D}.$$

The knock in mechanism is illustrated in the following graph. The min-in Parisian call is knocked in at time $\tau_{L,D}$, where the underlying has in this case spent D amount of time above the barrier.

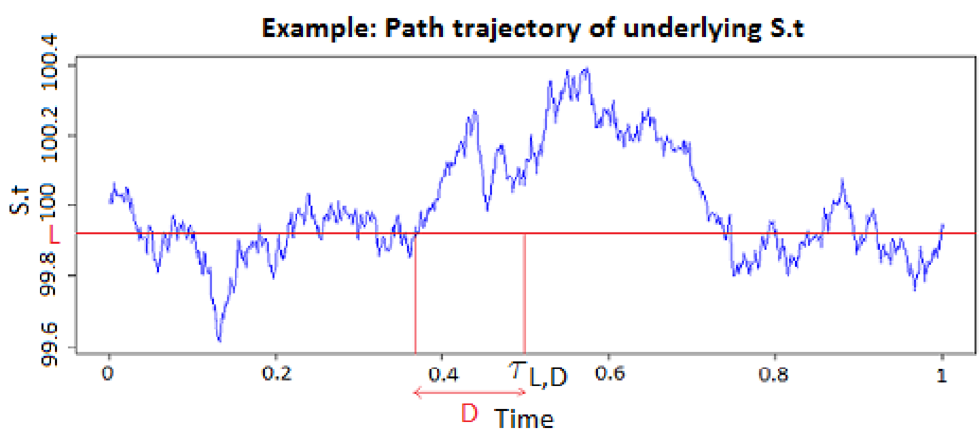

Figure 3.1: Illustration of a Parisian min-in call

To simplify notation, we take both the window lengths to be equal to 1, so for barrier b,

we have

$$\tau_b = \tau_b^+ \wedge \tau_b^- .$$

The two-sided Parisian options were first introduced in Dassios and Wu [21]. The Laplace transform of its pricing formula was given in their paper. We extend our results from the previous chapter to the case of two-sided Parisian options. Here, we discover some interesting results about the tails of the distributions of the one and two-sided Parisian stopping times. We assume that the underlying asset follows a geometric Brownian motion, with dynamics as in (2.1). Denoting by $C_i^{min}(x,T)$ the price of a Parisian min-in call option with initial underlying price x, maturity T and parameters K, L, D and r fixed, we have the pricing formula

$$^*C_i^{min}(x,T) = E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b \le T\}} e^{mZ_T}(xe^{\sigma Z_T} - K)^+\right].$$

3.1 Density of the two-sided Parisian stopping time

In this section, we give an analytical formula for the density of the two-sided Parisian stopping time. The formula is very similar to that for the one-sided stopping time.

Theorem 3.1 *For $f_0(t)$ the probability density function of τ_0,*

$$f_0(t) = \sum_{k=0}^{n-1} (-1)^k L_k(t-1), \qquad \text{for } n < t \le n+1,\ n = 1, 2, ... \tag{3.1}$$

for $t > 1$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \frac{1}{\pi\sqrt{t}}, \qquad \text{for } t > 0 \tag{3.2}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s)\frac{\sqrt{s-1}}{\pi s} ds, \qquad \text{for } t > k+1. \tag{3.3}$$

Proof. The Laplace transform of the density of τ_0 is (see Dassios and Wu [21])

$$\hat{f}_0(\beta) = \frac{1}{\Psi(\sqrt{2\beta}) - e^{\beta}\sqrt{\pi\beta}}.$$

We have

$$
\begin{aligned}
\frac{1}{\beta}e^{-\beta}\left(\Psi\left(\sqrt{2\beta}\right)-e^{\beta}\sqrt{\pi\beta}\right) &= \frac{e^{-\beta}}{\beta}+2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx-\sqrt{\frac{\pi}{\beta}} \\
&= \frac{e^{-\beta}}{\beta}+\sqrt{\frac{\pi}{\beta}}\left(1+2\int_{0}^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right)-\sqrt{\frac{\pi}{\beta}} \\
&= \frac{e^{-\beta}}{\beta}+\int_0^1\frac{e^{-\beta s}}{\sqrt{s}}ds \qquad (3.4)\\
&= \int_1^{\infty}e^{-\beta s}ds+\left(\int_0^{\infty}\frac{e^{-\beta s}}{\sqrt{s}}ds-\int_1^{\infty}\frac{e^{-\beta s}}{\sqrt{s}}ds\right) \\
&= \sqrt{\frac{\pi}{\beta}}+\frac{1}{\beta}\int_1^{\infty}\frac{e^{-\beta s}}{2s^{3/2}}ds \\
&= \sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{\sqrt{\pi\beta}}\int_1^{\infty}\frac{e^{-\beta s}}{2s^{3/2}}ds\right).
\end{aligned}
$$

So

$$
\begin{aligned}
\hat{f}_0(\beta) &= \frac{e^{-\beta}}{\sqrt{\pi\beta}\left(1+\frac{1}{\sqrt{\pi\beta}}\int_1^{\infty}\frac{e^{-\beta s}}{2s^{3/2}}ds\right)} \\
&= e^{-\beta}\sum_{k=0}^{\infty}(-1)^k\frac{1}{\sqrt{\pi\beta}}\left(\frac{1}{\sqrt{\pi\beta}}\int_1^{\infty}\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k
\end{aligned}
$$

and as above, there exists a β^* such that this is valid for all $\beta > \beta^*$. Hence we have the recursive solution. ■

Remark 3.2 *The only difference between the one-sided and two-sided Parisian stopping time densities is that there is a factor of 2 in the formula for the one-sided stopping time.*

Similarly for $b > 0$, we have the following recursive solution for the density of τ_b on the set $\{T_b < 1\}$.

Theorem 3.3 *For $b > 0$, we denote $f_b(t, T_b < 1)$ the probability density function of the two-sided stopping time τ_b on the set $\{T_b < 1\}$. We have*

$$
f_b(t, T_b < 1) = \sum_{k=0}^{n-1}(-1)^k L_k(t-1), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ... \qquad (3.5)
$$

for $t > 0$, where $L_k(t)$ is defined recursively as follows:

$$L_0(t) = \mathbf{1}_{\{0<t\le 1\}} \frac{1}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}} \frac{2}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right) \tag{3.6}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s) \frac{\sqrt{s-1}}{\pi s} ds, \qquad \textit{for } t > k+1. \tag{3.7}$$

Proof. We have

$$\begin{aligned}
E\left[e^{-\beta\tau_b}\mathbf{1}_{\{T_b<1\}}\right] &= E\left[e^{-\beta(T_b+\tau_0)}\mathbf{1}_{\{T_b<1\}}\right] \\
&= E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right] \frac{1}{\Psi(\sqrt{2\beta}) - e^{\beta}\sqrt{\pi\beta}} \\
&= e^{-\beta} \sum_{k=0}^{\infty} (-1)^k \frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{\sqrt{\pi\beta}} \left(\frac{1}{\sqrt{\pi\beta}} \int_1^{\infty} \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k .
\end{aligned}$$

We have

$$\begin{aligned}
L_0(t) &= \mathcal{L}^{-1}\left(\frac{E\left[e^{-\beta T_b}\mathbf{1}_{\{T_b<1\}}\right]}{\sqrt{\pi\beta}}\right) \\
&= \mathbf{1}_{\{0<t\le 1\}} \int_0^t \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} \frac{1}{\pi\sqrt{t-s}} ds + \mathbf{1}_{\{t>1\}} \int_0^1 \frac{b}{\sqrt{2\pi s^3}} e^{-\frac{b^2}{2s}} \frac{1}{\pi\sqrt{t-s}} ds \\
&= \mathbf{1}_{\{0<t\le 1\}} \int_{\frac{b}{\sqrt{t}}}^{\infty} \frac{2}{\pi\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sqrt{\frac{x^2}{tx^2-b^2}} dx + \mathbf{1}_{\{t>1\}} \int_b^{\infty} \frac{2}{\pi\sqrt{2\pi}} e^{-\frac{x^2}{2}} \sqrt{\frac{x^2}{tx^2-b^2}} dx \\
&= \mathbf{1}_{\{0<t\le 1\}} \int_{\frac{b^2}{t}}^{\infty} \frac{1}{\pi\sqrt{2\pi}} e^{-\frac{y}{2}} \sqrt{\frac{1}{ty-b^2}} dy + \mathbf{1}_{\{t>1\}} \int_{b^2}^{\infty} \frac{1}{\pi\sqrt{2\pi}} e^{-\frac{y}{2}} \frac{1}{\sqrt{ty-b^2}} dy \\
&= \mathbf{1}_{\{0<t\le 1\}} \frac{2}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \int_0^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx + \mathbf{1}_{\{t>1\}} \frac{2}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \int_{b\sqrt{t-1}}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} dx \\
&= \mathbf{1}_{\{0<t\le 1\}} \frac{1}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} + \mathbf{1}_{\{t>1\}} \frac{2}{\pi\sqrt{t}} e^{-\frac{b^2}{2t}} \mathcal{N}\left(-b\sqrt{\frac{t-1}{t}}\right)
\end{aligned}$$

and we have that L_k for $k = 1, 2, ...$ is the same as the previous case.

For $b < 0$, we have

$$f_b(t, T_b < 1) = f_{-b}(t, T_{-b} < 1)$$

due to the symmetry of the standard Brownian motion. ■

3.2 Tail distributions of the two-sided Parisian stopping time

In this section, we prove that the two-sided stopping time τ_0 has an exponential tail, unlike that for the one-sided stopping time τ_0^-. We present some numerical results and graphs to see what happens when $t \to \infty$. Furthermore, we compare this to the one-sided stopping time τ_0^-, which has a heavier tail as we will see.

Theorem 3.4 *We denote $\bar{F}_0(t)$ as the tail of the distribution of the two-sided Parisian stopping time with barrier 0, τ_0. It has an exponential tail. As $t \to \infty$, we have*

$$\bar{F}_0(t) \sim C_{\beta^*} e^{-\beta^* t} \tag{3.8}$$

for some constant C_{β^*} and $\beta^* > 0$ such that $-\beta^*$ is the unique solution of the equation

$$\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta} = 0 \tag{3.9}$$

and

$$C_{\beta^*} = 2e^{-\beta^*}. \tag{3.10}$$

Proof. First, we have

$$\begin{aligned}
\hat{f}_0(\beta) &= \frac{1}{\Psi(\sqrt{2\beta}) - e^{\beta}\sqrt{\pi\beta}} \\
&= \frac{1}{\beta\left(\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta}\right)} \\
&= \frac{1}{1 + \int_0^1 (1 - e^{-\beta v}) \frac{1}{2v^{3/2}} dv} \\
&= \int_0^\infty e^{-u} e^{-u \int_0^1 (1-e^{-\beta v}) \frac{1}{2v^{3/2}} dv} du \\
&= E\left(e^{-\beta X_T}\right)
\end{aligned}$$

where X_T is a subordinator (Lévy process) with Lévy measure $\frac{1}{2v^{3/2}}$ for $v < 1$ at an independent exponential time $T \sim Exp(1)$. Hence, we observe an interesting connection between the distributions of the Parisian stopping time and that of the Lévy process X_T. This suggests possibilities for further study. The first step above follows from (3.4) and the second step can be derived as below:

$$
\begin{aligned}
\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv &= \int_0^1 \int_0^v \beta e^{-\beta u} du \frac{1}{2v^{3/2}} dv \\
&= \int_0^1 \beta e^{-\beta u} \int_u^1 \frac{1}{2v^{3/2}} dv du \\
&= \int_0^1 \beta e^{-\beta u} \left(\frac{1}{\sqrt{u}} - 1 \right) du \\
&= \beta \left(\int_0^1 \frac{e^{-\beta s}}{\sqrt{s}} ds + \frac{e^{-\beta}}{\beta} \right) - 1.
\end{aligned}
$$

Next, we define two new discrete random variables $\overline{T}$ and $\underline{T}$:

$$
\begin{aligned}
P(\overline{T} = kh) &= e^{-kh}(1-e^{-h}) \qquad k = 0, 1, ... \\
P(\underline{T} = kh) &= e^{-(k-1)h}(1-e^{-h}) \qquad k = 1, 2, ...
\end{aligned}
$$

We note that $\overline{T}$ is the upper bound for T and $\underline{T}$ is its lower bound. Hence, we have that $P(\underline{T} \leq t) \leq P(T \leq t) \leq P(\overline{T} \leq t)$, and thus

$$
P(X_{\overline{T}} > x) \leq P(X_T > x) \leq P(X_{\underline{T}} > x)
$$

since X_t is a subordinator and hence increasing. Our aim is to show that as $h \to 0$, both $P(X_{\overline{T}} > x)$ and $P(X_{\underline{T}} > x)$ converges to the same limit which is then equal to $P(X_T > x)$.

Now we have

$$
E\left(e^{-\beta X_{\overline{T}}}\right) = \sum_{k=0}^{\infty} e^{-kh}(1-e^{-h}) e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv}
$$

and

$$
E\left(e^{-\beta X_{\underline{T}}}\right) = \sum_{k=1}^{\infty} e^{-(k-1)h}(1-e^{-h}) e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv}.
$$

We look first at $E\left(e^{-\beta X_{\overline{T}}}\right)$. We define the function $\hat{g}_h(\beta)$ as

$$
\hat{g}_h(\beta) = e^{-h\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv}.
$$

Then

$$\begin{aligned}
\sum_{k=0}^{\infty} e^{-kh}(1-e^{-h}) e^{-kh\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv} &= \sum_{k=0}^{\infty} e^{-kh}(1-e^{-h})\left(\hat{g}_h(\beta)\right)^k \\
&= \frac{1-e^{-h}}{1-\hat{g}_h(\beta)e^{-h}}.
\end{aligned}$$

Now, we denote by $\overline{L}(x)$ the tail distribution $P(X_{\overline{T}} > x)$, and $\hat{\overline{L}}(\beta)$ its Laplace transform. So we have

$$\begin{aligned}
\hat{\overline{L}}(\beta) &= \frac{1-\frac{1-e^{-h}}{1-\hat{g}_h(\beta)e^{-h}}}{\beta} \\
&= \frac{e^{-h}\frac{1-\hat{g}_h(\beta)}{\beta}}{1-\hat{g}_h(\beta)e^{-h}}
\end{aligned}$$

$$\hat{\overline{L}}(\beta)\left(1-\hat{g}_h(\beta)e^{-h}\right) = e^{-h}\frac{1-\hat{g}_h(\beta)}{\beta}.$$

Inverting the Laplace transform on both sides, we have

$$\overline{L}(x) - \int_0^x \overline{L}(x-y)dG_h(y)e^{-h} = e^{-h}\overline{G}_h(x).$$

Let β^* be such that $-\beta^*$ is the solution to the equation

$$1+\int_0^1 (1-e^{-\beta v})\frac{1}{2v^{3/2}}dv = 0.$$

We note that this equation has a unique negative solution, because the expression on the left hand side of this equation is decreasing for negative β. Furthermore, as $\beta \to 0$, the expression approaches 1, and as $\beta \to -\infty$, the expression approaches $-\infty$. Next, we define $\overline{L}^*(x)$ as

$$\overline{L}(x)e^{\beta^* x} = \overline{L}^*(x).$$

Then we have

$$\begin{aligned}
\overline{L}^*(x)e^{-\beta^* x} - \int_0^x \overline{L}^*(x-y)e^{-\beta^*(x-y)}dG_h(y) &= e^{-h}\overline{G}_h(x) \\
\overline{L}^*(x) - \int_0^x \overline{L}^*(x-y)e^{\beta^* y}dG_h(y) &= e^{-h}e^{\beta^* x}\overline{G}_h(x).
\end{aligned}$$

By the key renewal theorem (see Feller [26]), we have that as $x \to \infty$,

$$\begin{aligned}
\overline{L}^*(x) &\to \frac{\int_0^\infty e^{-h} e^{\beta^* y}\overline{G}_h(y)dy}{\int_0^\infty y e^{\beta^* y} dG_h(y)} \\
&= \frac{e^{-h}\left(1 - e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}\right)}{-\beta^* \frac{d}{d\beta^*}\hat{g}_h(\beta^*)} \\
&= \frac{e^{-h}\left(1 - e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}\right)}{-\beta^* \left(h\int_0^1 e^{\beta^* v}\frac{1}{2\sqrt{v}}dv\right) e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}}.
\end{aligned}$$

We denote this by $\overline{C}_h$. When $h \to 0$, we get

$$\begin{aligned}
\overline{C}_h &= \frac{\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}{-\beta^* \int_0^1 e^{\beta^* v}\frac{1}{2\sqrt{v}}dv} \\
&= \frac{-1}{-\frac{\beta^*}{2}\left(\int_0^1 e^{\beta^* v}\frac{1}{\sqrt{v}}dv - \frac{e^{\beta^*}}{\beta^*}\right) - \frac{e^{\beta^*}}{2}} \\
&= 2e^{-\beta^*}.
\end{aligned}$$

Likewise, we denote by $\bar{l}(x)$ the tail distribution $P(X_{\underline{T}} > x)$, and $\hat{\bar{l}}(\beta)$ its Laplace transform. Similarly, we can compute

$$\begin{aligned}
\hat{\bar{l}}(\beta) &= \frac{1 - \frac{1-e^{-h}}{1-\hat{g}_h(\beta)e^{-h}}\hat{g}_h(\beta)}{\beta} \\
&= \frac{\frac{1-\hat{g}_h(\beta)}{\beta}}{1-\hat{g}_h(\beta)e^{-h}}
\end{aligned}$$

$$\hat{\bar{l}}(\beta)\left(1 - \hat{g}_h(\beta)e^{-h}\right) = \frac{1-\hat{g}_h(\beta)}{\beta}.$$

Inverting the Laplace transform, we have

$$\bar{l}(x) - \int_0^x \bar{l}(x-y)dG_h(y)e^{-h} = \bar{G}_h(x)$$

and we define $\bar{l}^*(x)$ as

$$\bar{l}(x)e^{\beta^* x} = \bar{l}^*(x).$$

Then we have

$$\bar{l}^*(x) - \int_0^x \bar{l}^*(x-y)e^{\beta^* y}dG_h(y) = e^{\beta^* x}\overline{G}_h(x).$$

By the key renewal theorem,

$$\begin{aligned}
\bar{l}^*(x) &\to \frac{\int_0^\infty e^{\beta^* y}\overline{G}_h(y)dy}{\int_0^\infty ye^{\beta^* y}dG_h(y)} \\
&= \frac{1 - e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}}{-\beta^*\left(h\int_0^1 e^{\beta^* v}\frac{1}{2\sqrt{v}}dv\right)e^{-h\int_0^1 (1-e^{\beta^* v})\frac{1}{2v^{3/2}}dv}}.
\end{aligned}$$

We denote this by $\underline{C}_h$. When $h \to 0$, we get

$$\underline{C}_h = 2e^{-\beta^*}.$$

Finally, we note that since we have

$$\begin{array}{rcl}
e^{\beta^* x}P(X_{\overline{T}} > x) \leq & e^{\beta^* x}P(X_T > x) & \leq e^{\beta^* x}P(X_{\underline{T}} > x) \\
\overline{L}^*(x) \leq & e^{\beta^* x}\bar{F}_0(x) & \leq \bar{l}^*(x)
\end{array}$$

and as $h \to 0$, $\overline{L}^*(x)$ and $\bar{l}^*(x)$ converges to the same limit as $x \to \infty$, we have that $e^{\beta^* x}\bar{F}_0(x)$ also converges to this limit as $x \to \infty$. We thus have the result

$$\bar{F}_0(t) \to 2e^{-\beta^*}e^{-\beta^* t}$$

as $t \to \infty$. ■

Remark 3.5 *We can compute* β^* *numerically to be 0.854 and* $C_{\beta^*} = 2e^{-\beta^*}$ *to be 0.851.*

3.3 Numerical Results

The table below presents the survival function for both τ_0 and τ_0^-, computed using a time step of $h = 0.001$ with R.

Table 3.1: One and two-sided survival functions for $0 < t \leq 10$

t	$\bar{F}_0(t)$	$\bar{F}_0^-(t)$	t	$\bar{F}_0(t)$	$\bar{F}_0^-(t)$
1.5	0.555931	0.775033	6.0	0.015114	0.403422
2.0	0.369469	0.681770	6.5	0.010779	0.387956
2.5	0.242144	0.614236	7.0	0.007910	0.374142
3.0	0.159600	0.563552	7.5	0.006003	0.361704
3.5	0.105503	0.523602	8.0	0.004726	0.350429
4.0	0.070093	0.491082	8.5	0.003866	0.340146
4.5	0.046893	0.463944	9.0	0.003278	0.330718
5.0	0.031679	0.440854	9.5	0.002872	0.322033
5.5	0.021687	0.420896	10.0	0.002586	0.313997

We can see that the two-sided survival function goes to 0 much faster than the one-sided case.

The following graph compares the density functions of the one and two-sided case. The red line represents $f_0(t)$ while the black line $f_0^-(t)$, plotted against time. This graph suggests that $f_0^-(t)$ has a heavier tail.

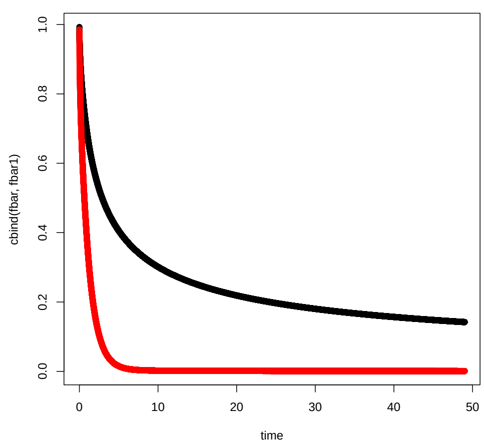

Figure 3.2: Graph of $f_0(t)$ and $f_0^-(t)$ vs t for $0 < t \leq 50$

The following graph depicts the tails $\bar{F}_0(t)$ (black) and the approximation $C_{\beta^*}e^{-\beta^* t}$ (red).

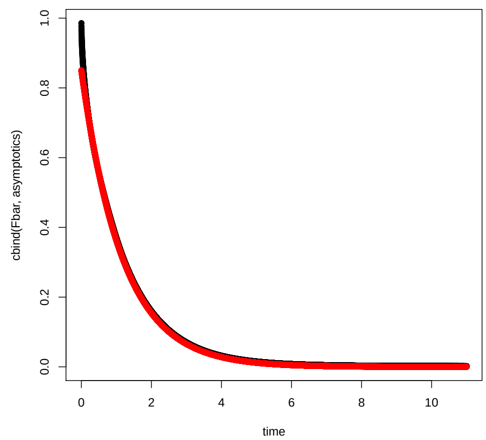

Figure 3.3: Graph of $\bar{F}_0(t)$ and $C_{\beta^*}e^{-\beta^* t}$ vs t for $0 < t \leq 20$

From Figure 3.2, we can see that the tail of the one-sided case is heavier than that of the two-sided case, and Figure 3.3 suggests that the approximation is rather good for the two-sided case. The following graph plots $\bar{F}_0^-(t)$ against $\ln(t)$.

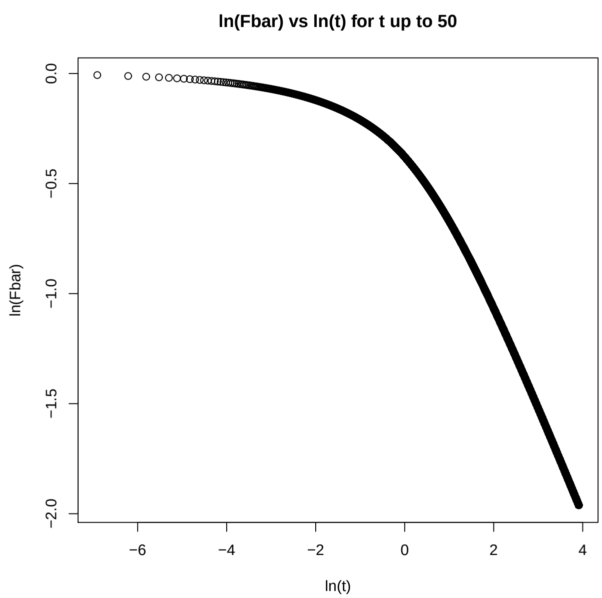

Figure 3.4: Graph of $\ln(\bar{F}_0^-(t))$ vs $\ln(t)$ for $0 < t \leq 50$

Remark 3.6 *The above graph has a slope of 0.5, thus suggesting that the one-sided stopping* *time has a power tail with exponent 0.5. This has, however, not been proved mathematically and thus provides another possibility for future research.*

Remark 3.7 *Based on the asymptotics in Theorem 3.4, we propose a new method of obtaining the density of the two-sided stopping time when $b = 0$. From the recursions in Theorem 3.1, we can compute the closed form formulas for the density $f_0(t)$ for $1 < t \leq 4$. For $t > 4$, we approximate the density with the asymptotics. We have from Theorem 3.1*

$$L_0(t) = \frac{1}{\pi\sqrt{t}}, \quad \textit{for } t > 0.$$

$$\begin{aligned}
L_1(t) &= \int_1^t L_1(t-s)\frac{s-1}{\pi s}ds \\
&= \int_0^{t-1} \frac{\sqrt{t-s-1}}{\pi^2(t-s)\sqrt{s}}ds \\
&= \left[2\arctan\left(\sqrt{\frac{s}{t-s-1}}\right) - \frac{2}{\sqrt{t}}\arctan\left(\sqrt{\frac{s}{t(t-s-1)}}\right)\right]_0^{t-1} \\
&= \frac{1}{\pi} - \frac{1}{\pi\sqrt{t}}.
\end{aligned}$$

$$\begin{aligned}
L_2(t) &= \int_1^{t-1} L_2(t-s)\frac{\sqrt{s-1}}{\pi s}ds \\
&= \frac{1}{\pi^2}\int_1^{t-1}\left(\frac{\sqrt{t-s-1}}{t-s}ds - \frac{\sqrt{t-s-1}}{\sqrt{s}(t-s)}\right)ds \\
&= \frac{1}{\pi^2}\left[2\arctan(\sqrt{t-s-1}) - 2\sqrt{t-s-1}\right]_1^{t-1} \\
&\quad -\frac{1}{\pi^2}\left[2\arctan\sqrt{\frac{s}{t-s-1}} - \frac{2\arctan\left(\sqrt{\frac{s}{t(t-s-1)}}\right)}{\sqrt{t}}\right]_1^{t-1} \\
&= \frac{2\sqrt{t-2}}{\pi^2} - \frac{4\arctan\sqrt{\frac{t-2}{t}}}{\pi^2\sqrt{t}}.
\end{aligned}$$

Hence, we have an approximation for the density:

$$f_0(t) = \begin{cases} \frac{1}{\pi\sqrt{t}} & \text{for } 1 < t \leq 2 \\ \frac{2}{\pi\sqrt{t}} - \frac{1}{\pi} & \text{for } 2 < t \leq 3 \\ \frac{2}{\pi\sqrt{t}} - \frac{1}{\pi} - \frac{4}{\pi^2}\tan^{-1}\sqrt{t-2} + \frac{4}{\pi^2\sqrt{t}}\tan^{-1}\sqrt{\frac{t-2}{t}} + \frac{2}{\pi^2}\sqrt{t-2} & \text{for } 3 < t \leq 4 \\ 2\beta^* e^{-\beta^*(t+1)} & \text{for } t > 4 \end{cases},$$

where $\beta^* = 0.854$.

3.4 Pricing two-sided Parisian Options

3.4.1 Min-call-in Parisian call

A min-call-in Parisian call is a call option that gets knocked in, as the name suggests, when either the underlying makes an excursion above the barrier or an excursion below the barrier of a certain length. Here, we price a min-call-in Parisian call with the same window length $D = 1$ above and below the barrier.

Theorem 3.8 *The price of a two-sided Parisian-in option on the underlying* S *with barrier* L *and maturity time* $T > 1$*, is*

$$\begin{aligned} {}^*C_i^{min}(x,T) &= x\phi(\sigma+m) - K\phi(m) \\ &+ \sqrt{\frac{\pi}{2}}\int_0^T f_b(t;T_b<1)\left(x\psi(\sigma+m,h_b,b,\rho,t) + \psi(-(\sigma+m),h_b,-b,-\rho,t)\right. \\ &\left.-K\left(\psi(-m,h_b',-b,-\rho,t) + \psi(m,h_b',b,\rho,t)\right)\right)dt \end{aligned} \tag{3.11}$$

where $f_b(t;T_b<1)$ *is the density function of the two-sided Parisian stopping time with barrier* b *as in Theorem 3.3, and* $\phi(x)$*,* $\psi(x,y,b,\rho,t)$*,* h_b*,* h_b' *and* ρ *are defined as before.*

Proof. We denote by $\mathcal{F}_t = \sigma(Z_s, s \leq t)$ the natural filtration of the Brownian motion $(Z_t, t \geq 0)$. Then τ_b is an $\mathcal{F}_t$-stopping time, and by the strong Markov property of Brownian motion

$$\begin{aligned} {}^*C_i^{min}(x,T) &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\leq T\}} E\left[e^{mZ_T}\left(xe^{\sigma Z_T}-K\right)^+ | \mathcal{F}_{\tau_b}\right]\right] \\ &= E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau_b\leq T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^+ \frac{1}{\sqrt{2\pi(T-\tau_b)}} e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right] \end{aligned}$$

$$
\begin{aligned}
&= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\
&\quad +E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b\le 1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right].
\end{aligned}
$$

If $T_b>1$, $\tau_b=1$, so we have

$$
\begin{aligned}
& E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\
&= E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b>1\}}\mathbf{1}_{\{1\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-1)}}e^{-\frac{(y-Z_1)^2}{2(T-1)}}dy\right]\\
&= \frac{1}{\sqrt{2\pi(T-1)}}\int_{-\infty}^{b}\int_{k}^{\infty} e^{my}(xe^{\sigma y}-K)e^{-\frac{(y-z)^2}{2(T-1)}}\frac{1}{\sqrt{2\pi}}\left(e^{-\frac{z^2}{2}}-e^{-\frac{(z-2b)^2}{2}}\right)dzdy\\
&= x\phi(\sigma+m)-K\phi(m).
\end{aligned}
$$

For $T_b\le 1$, Z_{τ_b} is independent of τ_b, so we have

$$
\begin{aligned}
& E_{\mathcal{P}}\left[\mathbf{1}_{\{T_b\le 1\}}\mathbf{1}_{\{\tau_b\le T\}}\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-\tau_b)}}e^{-\frac{(y-Z_{\tau_b})^2}{2(T-\tau_b)}}dy\right]\\
&= \int_0^T\int_{-\infty}^{\infty} f_b(t;T_b<1)v(dz)\int_{-\infty}^{\infty} e^{my}\left(xe^{\sigma y}-K\right)^{+}\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydt\\
&= \int_0^T\int_{-\infty}^{b}\int_{k}^{\infty} f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt\\
&\quad +\int_0^T\int_{b}^{\infty}\int_{k}^{\infty} f_b(t;T_b<1)\frac{z-b}{2}e^{\frac{(z-b)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt\\
&= \sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^{b}\int_{k}^{\infty} f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt\\
&\quad +\sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^{b}\int_{k}^{\infty} f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt\\
&= \sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^{b}\int_{k}^{\infty} f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-z)^2}{2(T-t)}}dydzdt\\
&\quad +\sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^{b}\int_{k}^{\infty} f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt\\
&= \sqrt{\frac{\pi}{2}}\int_0^T f_b(t;T_b<1)(x\psi(\sigma+m,h,b,\rho,t)-K\psi(m,h',b,\rho,t))dt
\end{aligned}
$$

$$+\sqrt{\frac{\pi}{2}}\int_0^T f_b(t;T_b<1)(x\psi(-(\sigma+m),h_b,b,-\rho,t)-K\psi(-m,h_b',b,-\rho,t))dt$$

since we have

$$\begin{aligned}
&\sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_k^\infty f_b(t;T_b<1)\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}e^{my}(xe^{\sigma y}-K)\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt\\
=&\sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_k^\infty f_b(t;T_b<1)e^{2b(\sigma+m)}\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}xe^{(\sigma+m)(y-2b)}\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y-2b+z)^2}{2(T-t)}}dydzdt\\
=&\sqrt{\frac{\pi}{2}}\int_0^T\int_{-\infty}^b\int_{k-2b}^\infty f_b(t;T_b<1)e^{2b(\sigma+m)}\frac{b-z}{2}e^{\frac{(b-z)^2}{2}}xe^{(\sigma+m)y}\frac{1}{2\pi\sqrt{T-t}}e^{-\frac{(y+z)^2}{2(T-t)}}dydzdt
\end{aligned}$$

and

$$\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^b\int_k^\infty xe^{(\sigma+m)y}e^{-\frac{(y+z)^2}{2(T-t)}}(b-z)e^{-\frac{(z-b)^2}{2}}dzdy \tag{3.12}$$

$$= e^{\frac{(\sigma+m)^2(1+T-t)-2b(\sigma+m)}{2}}\frac{x}{2\pi\sqrt{T-t}}\int_{-\infty}^b\int_{k-2b}^\infty(b-z)\exp\{-\frac{(y+(b-(\sigma+m)(1+T-t)))^2}{2(T-t)}\} \tag{3.13}$$

$$\exp\{-\frac{(z-(b-(\sigma+m)))^2}{2(T-t)/(1+T-t)}\}\exp\{-\frac{2(y+(b-(\sigma+m)(1+T-t)))(z-(b-(\sigma+m)))}{2(T-t)}\}dydzdt$$

$$= xe^{\frac{(\sigma+m)^2(1+T-t)-2b(\sigma+m)}{2}}\frac{1}{2\pi\sqrt{1-\rho^2}}\int_{-\infty}^{\sigma+m}\int_{h_b}^\infty(-v+(\sigma+m))e^{-\frac{u^2+2\rho uv+v^2}{2}}dudv \tag{3.14}$$

$$= xe^{-2b(\sigma+m)}\psi(-(\sigma+m),h_b,b,-\rho,t)$$

where expression (3.13) is obtained from (3.12) by a manipulation of the exponents, and from expression (3.13) to (3.14) we have used the transformation $u=\frac{y+(b-(\sigma+m)(1+T-t))}{\sqrt{1+T-t}}$ and $v=z-(b-(\sigma+m))$. ■

3.4.2 Min-call-out Parisian Call

For the knock-out call with the same parameters, we have

$$\begin{aligned}
C_i^{min}(x,T) &= E_{\mathcal{Q}}\left[\mathbf{1}_{\{\tau_b>t\}}(S_T-K)^+e^{-rT}\right]\\
&= E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}(S_T-K)^+e^{-rT}\right]-E_{\mathcal{Q}}\left[\mathbf{1}_{\{T_b<1\}}\mathbf{1}_{\{\tau_b\le T\}}(S_T-K)^+e^{-rT}\right]\\
&= \int_0^1\frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}C_{BS}(L,T-t)dt-\left(C_i^{min}(x,T)-(x\phi(\sigma+m)-K\phi(m))\right).
\end{aligned}$$

3.4.3 Numerical results

The following table gives the prices of the two-sided Parisian option for different values of initial asset price S_0 and window length D, for parameters $K = 95$, $L = 90$, $T = 1$ year, $r = 0.05$ and $\sigma = 0.2$.

Table 3.2: Price of Parisian min-in call

S_0	$D = 1$ week	$D = 2$ weeks	$D = 1$ month	$D = 2$ months
80	2.817708	2.809610	2.660829	2.123282
82	3.471103	3.430688	3.145066	2.482966
84	4.203278	4.101558	3.737759	3.096815
86	5.050461	4.978642	4.724678	4.261088
88	6.535228	6.639547	6.589191	6.342500
90	6.897115	6.895460	6.891562	6.872088

Chapter 4

Double-barrier Parisian options

In this chapter, we look at double-barrier Parisian options, where the option gets knocked in or knocked out when the stock price goes either above the upper barrier or below the lower barrier for a certain period of time. This is illustrated below, where the option gets knocked in at $\tau^{L_2}_{L_1,D}$. We define the new stopping time $\tau^{L_2}_{L_1,D}$, which is the first time the underlying process either spends D amount of time consecutively above the barrier L_2 or D amount of time consecutively below the barrier L_1, and we have for $L_1 < L_2$,

$$\tau^{L_2}_{L_1,D} = \tau^{-}_{L_1,D} \wedge \tau^{+}_{L_2,D}.$$

The knock in mechanism is illustrated below, where the option gets knocked in at $\tau^{L_2}_{L_1,D}$, in this case having spent D amount of time below the barrier L_1.

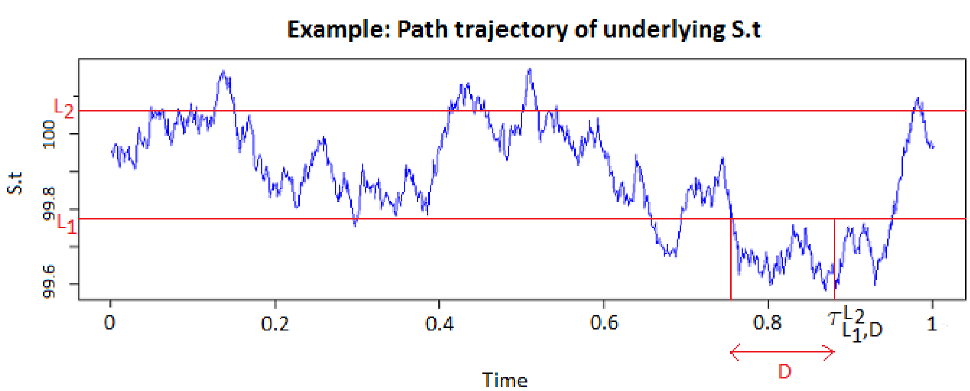

Figure 4.1: Illustration of a Parisian Double-barrier min-in call

Taking the window lengths to be equal to 1, we have for $b_1 < b_2$,

$$\tau_{b_1}^{b_2} = \tau_{b_1}^- \wedge \tau_{b_2}^+.$$

4.1 Density of the double-barrier Parisian stopping time

We can also price double barrier Parisian options using the same method as before. We have the following definition for the double barrier Parisian stopping time. In order to price double-barrier Parisian options, we first find the density of the double-barrier Parisian stopping time $\tau_{b_1}^{b_2}$. We split into the case when the long excursion occurs above the upper barrier and when it occurs below the lower barrier. For the first case, we have the following recursion.

Theorem 4.1 *Let $f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-)$ denote the probability density function of τ on the set $\tau_{b_2}^+ < \tau_{b_1}^-$. Then for $b_1 \leq 0 \leq b_2$, we have*

$$f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-) = \sum_{k=0}^{n-1} (-1)^k \left(L_k(t-1) - \tilde{L}_k(t-1) \right), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ... \tag{4.1}$$

for $t > 1$, where $L_k(t)$ and $\tilde{L}_k(t)$ are defined recursively as follows:

$$L_0(t) = \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}} + \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}}, \ \text{for } t > 0 \tag{4.2}$$

$$L_{k+1}(t) = \int_1^{t-k} L_k(t-s) \left(\frac{\sqrt{s-1}}{2\pi s} \left(1 + e^{-\frac{(b_2-b_1)^2}{2(s-1)}} \right) - \frac{b_2 - b_2}{\sqrt{2\pi} s^{3/2}} e^{-\frac{(b_1-b_2)^2}{2s}} \mathcal{N}\left(-\frac{b_2 - b_1}{\sqrt{s(s-1)}} \right) \right) ds \tag{4.3}$$

$$\tilde{L}_0(t) = \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}} - \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}}, \ \text{for } t > 0 \tag{4.4}$$

$$\tilde{L}_{k+1}(t) = \int_1^{t-k} \tilde{L}_k(t-s) \left(\frac{\sqrt{s-1}}{2\pi s} \left(1 - e^{-\frac{(b_2-b_1)^2}{2(s-1)}} \right) + \frac{b_2 - b_2}{\sqrt{2\pi} s^{3/2}} e^{-\frac{(b_1-b_2)^2}{2s}} \mathcal{N}\left(-\frac{b_2 - b_1}{\sqrt{s(s-1)}} \right) \right) ds \tag{4.5}$$

for $t > k+1$.

Proof. We have the following Laplace transform for $\tau_{b_1}^{b_2}$ (see Anderluh and Weide [6] Theorem 3.2):

$$
\begin{aligned}
E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^{+}<\tau_{b_1}^{-}\}}\right) &= \frac{e^{-\sqrt{2\beta}b_1}\Psi\left(\sqrt{2\beta}\right)-e^{\sqrt{2\beta}b_1}\Psi\left(-\sqrt{2\beta}\right)}{e^{\sqrt{2\beta}(b_2-b_1)}\Psi\left(\sqrt{2\beta}\right)^2-e^{\sqrt{2\beta}(b_1-b_2)}\Psi\left(-\sqrt{2\beta}\right)^2} \\
&= \frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2}+e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)+e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \\
&\quad -\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2}-e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)-e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)}. \qquad (4.6)
\end{aligned}
$$

From the one-sided case, we have the following equalities

$$
\begin{aligned}
e^{-\beta}\frac{1}{\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right) &= e^{\sqrt{2\beta}(b_2-b_1)/2}\left(\frac{e^{-\beta}}{\beta}+2\sqrt{\frac{\pi}{\beta}}\mathcal{N}\left(\sqrt{2\beta}\right)\right) \\
&= e^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
e^{-\beta}\frac{1}{\beta}e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right) &= e^{\sqrt{2\beta}(b_1-b_2)/2}\left(\frac{e^{-\beta}}{\beta}-2\sqrt{\frac{\pi}{\beta}}\int_{-\infty}^{-\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right) \\
&= e^{\sqrt{2\beta}(b_1-b_2)/2}\left(\frac{e^{-\beta}}{\beta}-\sqrt{\frac{\pi}{\beta}}\left(1-2\int_0^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right)\right) \\
&= e^{\sqrt{2\beta}(b_1-b_2)/2}\left(-2\sqrt{\frac{\pi}{\beta}}+\sqrt{\frac{\pi}{\beta}}+\frac{e^{-\beta}}{\beta}+2\sqrt{\frac{\pi}{\beta}}\int_0^{\sqrt{2\beta}}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right) \\
&= e^{\sqrt{2\beta}(b_1-b_2)/2}\left(-2\sqrt{\frac{\pi}{\beta}}+2\sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)\right) \\
&= e^{\sqrt{2\beta}(b_1-b_2)/2}2\sqrt{\frac{\pi}{\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right).
\end{aligned}
$$

So combining the two, we have for the denominator in the first term of (4.6)

$$
\begin{aligned}
& e^{-\beta}\frac{1}{\beta}\left(e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right)+e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)\right) \\
= \ & e^{\sqrt{2\beta}(b_2-b_1)/2}2\sqrt{\frac{\pi}{\beta}}\left(1+\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)+e^{\sqrt{2\beta}(b_1-b_2)/2}2\sqrt{\frac{\pi}{\beta}}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)
\end{aligned}
$$

$$= e^{\sqrt{2\beta}(b_2-b_1)/2} 2\sqrt{\frac{\pi}{\beta}} \left(1 + \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds + e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds\right).$$

The first term can thus be written as an infinite series summation as below:

$$\begin{aligned}
&\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} + e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right) + e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \\
= \; & \frac{e^{-\beta}\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} + e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{2\sqrt{\pi\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}} \\
& \times \sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds + e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k \\
= \; & e^{-\beta}\frac{e^{\sqrt{2\beta}b_1} + e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}} \sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds + e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k.
\end{aligned}$$

We note again that the series summation is valid because the term in the brackets is a continuous and decreasing function of β, hence there exists a β^* such that the series is convergent for all $\beta > \beta^*$. Similarly, the second term in equation (4.6) can be written as an infinite series summation

$$\begin{aligned}
&\frac{\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} - e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{e^{\sqrt{2\beta}(b_2-b_1)/2}\Psi\left(\sqrt{2\beta}\right) - e^{\sqrt{2\beta}(b_1-b_2)/2}\Psi\left(-\sqrt{2\beta}\right)} \\
= \; & \frac{e^{-\beta}\frac{1}{2}\left(e^{\sqrt{2\beta}(b_1+b_2)/2} - e^{-\sqrt{2\beta}(b_1+b_2)/2}\right)}{2\sqrt{\pi\beta}e^{\sqrt{2\beta}(b_2-b_1)/2}} \\
& \times \sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds - e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k \\
= \; & e^{-\beta}\frac{e^{\sqrt{2\beta}b_1} - e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}} \sum_{k=0}^{\infty}(-1)^k \left(\frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds - e^{\sqrt{2\beta}(b_1-b_2)} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds\right)^k.
\end{aligned}$$

To show that the series expansion is valid, we have that

$$\frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds - e^{\sqrt{2\beta(b_1-b_2)}} \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds \le \frac{1}{2\sqrt{\pi\beta}} \int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}} ds$$

and the right hand side is continuous and decreasing in β, so there exists a β^* such that the expansion is valid for all $\beta > \beta^*$. We also have the following explicit Laplace inversions:

$$\begin{aligned}
\mathcal{L}^{-1}\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}\right) &= \frac{\sqrt{t-1}}{2\pi t}\mathbf{1}_{\{t>1\}} && (4.7)\\
\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \left(\frac{\sqrt{t-1}}{2\pi t}e^{-\frac{(b_2-b_1)^2}{2(t-1)}}\right. \\
&\quad \left.+\frac{b_2-b_1}{\sqrt{2\pi t}}\mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t-1}}\right)\right)\mathbf{1}_{\{t>1\}} && (4.8)\\
\mathcal{L}^{-1}\left(\frac{e^{\sqrt{2\beta}b_1}}{4\sqrt{\pi\beta}}\right) &= \frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}} && (4.9)\\
\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{2\beta}b_2}}{4\sqrt{\pi\beta}}\right) &= \frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}} && (4.10)
\end{aligned}$$

where (4.7) has been proved before and (4.8) can be derived as follows. It is the convolution of the following two functions

$$\begin{aligned}
\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\right) &= \frac{1}{2\pi\sqrt{t}}e^{-\frac{(b_1-b_2)^2}{2t}}\\
\mathcal{L}^{-1}\left(\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right) &= \frac{1}{2t^{3/2}}.
\end{aligned}$$

So we can work out

$$\begin{aligned}
&\mathcal{L}^{-1}\left(e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty \frac{e^{-\beta s}}{2s^{3/2}}ds\right)\\
= \;& \mathbf{1}_{\{t>1\}}\int_1^t \frac{1}{2\pi\sqrt{t-s}}e^{-\frac{(b_1-b_2)^2}{2(t-s)}}\frac{1}{2s^{3/2}}ds\\
= \;& \mathbf{1}_{\{t>1\}}\int_0^{t-1} \frac{1}{2\pi\sqrt{s}}\frac{1}{2(t-s)^{3/2}}e^{-\frac{(b_1-b_2)^2}{2s}}ds\\
= \;& \mathbf{1}_{\{t>1\}}\int_{-\infty}^{\frac{b_1-b_2}{\sqrt{t-1}}} \frac{-x(b_2-b_1)}{2\pi\left(tx^2-(b_1-b_2)^2\right)^{3/2}}e^{-\frac{x^2}{2}}dx\\
= \;& \mathbf{1}_{\{t>1\}}\left(\left[\frac{b_2-b_1}{2\pi t\left(tx^2-(b_1-b_2)^2\right)^{1/2}e^{-\frac{x^2}{2}}}\right]_{-\infty}^{\frac{b_1-b_2}{\sqrt{t-1}}} - \int_{-\infty}^{\frac{b_1-b_2}{\sqrt{t-1}}}\frac{-x(b_2-b_1)}{2\pi t\sqrt{tx^2-(b_1-b_2)^2}}dx\right)\\
= \;& \mathbf{1}_{\{t>1\}}\left(\frac{\sqrt{t-1}}{2\pi t}e^{-\frac{(b_1-b_2)^2}{2(t-1)}} - \int_{\frac{(b_1-b_2)^2}{t-1}}^\infty \frac{b_2-b_1}{2\pi t}\frac{1}{2\sqrt{ty-(b_1-b_2)^2}}e^{-\frac{y}{2}}dy\right)
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{1}_{\{t>1\}} \left(\frac{\sqrt{t-1}}{2\pi t} e^{-\frac{(b_1-b_2)^2}{2(t-1)}} - \frac{b_2-b_1}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(b_1-b_2)^2}{2t}} \int_{\frac{b_2-b_1}{\sqrt{t(t-1)}}}^{\infty} e^{-\frac{y^2}{2}} dy \right) \\
&= \mathbf{1}_{\{t>1\}} \left(\frac{\sqrt{t-1}}{2\pi t} e^{-\frac{(b_1-b_2)^2}{2(t-1)}} - \frac{b_2-b_1}{\sqrt{2\pi} t^{3/2}} e^{-\frac{(b_1-b_2)^2}{2t}} \mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{t(t-1)}} \right) \right).
\end{aligned}
$$

Inverting the two terms in (4.6) and writing them as convolutions, we get the result. ■

Remark 4.2 *We can check that when $b_1 = b_2$, we get the same result as for the two-sided case for $b = b_1 = b_2$.*

We have a similar result for when the long excursion below the lower barrier is reached first, and we state here without proof.

Theorem 4.3 *Let $f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+)$ denote the probability density function of $\tau_{b_1}^{b_2}$ on the set $\tau_{b_1}^- < \tau_{b_2}^+$. Then*

$$
f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+) = \sum_{k=0}^{n-1} (-1)^k \left(L_k(t-1) + \tilde{L}_k(t-1) \right), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ... \tag{4.11}
$$

for $t > 1$, where $L_k(t)$ and $\tilde{L}_k(t)$ are defined as above.

Theorem 4.4 *For $b_1 < b_2 < 0$, we have for $T_{b_2} < 1$,*

$$
f_{b_1}^{b_2}(t, \tau_{b_2}^+ < \tau_{b_1}^-, T_{b_2} < 1) = \sum_{k=0}^{n-1} (-1)^k \left(L_k(t-1) - \tilde{L}_k(t-1) \right), \qquad \text{for } n < t \leq n+1,\ n = 1, 2, ... \tag{4.12}
$$

for $t > 1$, where $L_k(t)$ and $\tilde{L}_k(t)$ are defined recursively as before, but $L_0(t)$ and $\tilde{L}_0(t)$ are now different:

$$
\begin{aligned}
L_0(t) &= \mathbf{1}_{\{0 \leq t \leq 1\}} \left(\frac{1}{4\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}} + \frac{1}{4\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}} \right) \\
&\quad + \mathbf{1}_{\{t>1\}} \left(\frac{1}{4\pi\sqrt{t}} e^{-\frac{b_1^2}{2t}} \mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} + \frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}} \right) \right. \\
&\quad + \frac{1}{4\pi\sqrt{t}} e^{-\frac{(2b_2-b_1)^2}{2t}} \mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}} - \frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}} \right) \\
&\quad \left. + \frac{1}{2\pi\sqrt{t}} e^{-\frac{b_2^2}{2t}} \mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}} \right) \right)
\end{aligned} \tag{4.13}
$$

$$\begin{aligned}
L_{k+1}(t) &= \int_1^{t-k} L_k(t-s)\left(\frac{\sqrt{s-1}}{2\pi s}\left(1+e^{-\frac{(b_2-b_1)^2}{2(s-1)}}\right)\right. \\
&\quad \left.-\frac{b_2-b_2}{\sqrt{2\pi}s^{3/2}}e^{-\frac{(b_1-b_2)^2}{2s}}\mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{s(s-1)}}\right)\right)ds \qquad (4.14) \\
\tilde{L}_0(t) &= \mathbf{1}_{\{0\le t\le 1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}-\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\right) \\
&\quad +\mathbf{1}_{\{t>1\}}\left(\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right. \\
&\quad +\frac{1}{4\pi\sqrt{t}}e^{-\frac{(2b_2-b_1)^2}{2t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right) \\
&\quad \left.-\frac{1}{2\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right)\right) \qquad (4.15) \\
\tilde{L}_{k+1}(t) &= \int_1^{t-k} \tilde{L}_k(t-s)\left(\frac{\sqrt{s-1}}{2\pi s}\left(1-e^{-\frac{(b_2-b_1)^2}{2(s-1)}}\right)\right. \\
&\quad \left.+\frac{b_2-b_2}{\sqrt{2\pi}s^{3/2}}e^{-\frac{(b_1-b_2)^2}{2s}}\mathcal{N}\left(-\frac{b_2-b_1}{\sqrt{s(s-1)}}\right)\right)ds \qquad (4.16)
\end{aligned}$$

for $t > k+1$. And for $\tau_{b_1}^- < \tau_{b_2}^+$, we have

$$f_{b_1}^{b_2}(t, \tau_{b_1}^- < \tau_{b_2}^+, T_{b_2} < 1) = \sum_{k=0}^{n-1}(-1)^k\left(L_k(t-1)+\tilde{L}_k(t-1)\right), \qquad \textit{for } n < t \le n+1, n = 1, 2, ... \qquad (4.17)$$

Proof. For $b_1 < b_2 < 0$, the Laplace transform of the stopping time on the set $\tau_{b_2}^+ < \tau_{b_1}^-$ and $T_{b_2} < 1$ is

$$\begin{aligned}
& E\left(e^{-\beta\tau_{b_1}^{b_2}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}\mathbf{1}_{\{T_{b_2}<1\}}\right) \\
= \; & E\left(e^{-\beta(T_{b_2}+\tau_{b_1-b_2}^0)}\mathbf{1}_{\{\tau_0^+<\tau_{b_1-b_2}^-\}}\mathbf{1}_{\{T_{b_2}<1\}}\right) \\
= \; & E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)E\left(e^{-\beta\tau_{b_1-b_2}^0}\mathbf{1}_{\{\tau_0^+<\tau_{b_1-b_2}^-\}}\right)
\end{aligned}$$

and

$$
\begin{aligned}
& E\left(e^{-\beta\tau^0_{b_1-b_2}}\mathbf{1}_{\{\tau_0^+<\tau^-_{b_1-b_2}\}}\right) \\
= \; & e^{-\beta}\frac{e^{\sqrt{2\beta}(b_1-b_2)}+1}{4\sqrt{\pi\beta}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds+e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k \\
& +e^{-\beta}\frac{e^{\sqrt{2\beta}(b_1-b_2)}-1}{4\sqrt{\pi\beta}}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds-e^{\sqrt{2\beta}(b_1-b_2)}\frac{1}{2\sqrt{\pi\beta}}\int_1^\infty\frac{e^{-\beta s}}{2s^{3/2}}ds\right)^k.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
L_0(t) &= \mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{e^{\sqrt{2\beta}(b_1-b_2)}+1}{4\sqrt{\pi\beta}}\right) \\
\tilde{L}_0(t) &= \mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{e^{\sqrt{2\beta}(b_1-b_2)}-1}{4\sqrt{\pi\beta}}\right).
\end{aligned}
$$

Now, we have the following Laplace inversions.

$$
\begin{aligned}
& \mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{e^{\sqrt{2\beta}(b_1-b_2)}}{4\sqrt{\pi\beta}}\right) \\
= \; & \mathbf{1}_{\{0\le t<1\}}\int_0^t\frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds+\mathbf{1}_{\{t>1\}}\int_0^1\frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds
\end{aligned}
$$

where the integrals can be evaluated as below.

$$
\begin{aligned}
& \mathbf{1}_{\{0\le t<1\}}\int_0^t\frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds \\
= \; & \mathbf{1}_{\{0\le t<1\}}\int_{-\infty}^{\frac{b_2}{\sqrt{t}}}\frac{1}{2\pi\sqrt{2\pi}}\frac{-x}{\sqrt{x^2t-b_2^2}}e^{-\frac{x^2}{2}}e^{-\frac{(b_2-b_1)^2x^2}{2(tx^2-b_2^2)}}dx \\
= \; & \mathbf{1}_{\{0\le t<1\}}\int_{\frac{b_2^2}{t}}^{\infty}\frac{1}{4\pi\sqrt{2\pi}\sqrt{yt-b_2^2}}e^{-\frac{y}{2}}e^{-\frac{(b_2-b_1)^2y}{2(ty-b_2^2)}}dy \\
= \; & \mathbf{1}_{\{0\le t<1\}}\int_0^\infty\frac{1}{2\pi\sqrt{2\pi t}}e^{-\frac{1}{2t}(x^2+b_2^2)}e^{-\frac{(b_2-b_1)^2}{2t}\left(\frac{x^2+b_2^2}{x^2}\right)}dx \\
= \; & \mathbf{1}_{\{0\le t<1\}}e^{-\frac{b_2^2}{2t}}e^{-\frac{(b_2-b_1)^2}{2t}}\int_0^\infty\frac{1}{2\pi\sqrt{2\pi t}}e^{-\frac{1}{2t}\left(x^2+\frac{(b_2-b_1)^2b_2^2}{x^2}\right)}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \mathbf{1}_{\{0\le t<1\}}\frac{1}{2}e^{-\frac{b_2^2}{2t}}e^{-\frac{(b_2-b_1)^2}{2t}}\left(\int_0^\infty \frac{1+\frac{(b_2-b_1)b_2}{x^2}}{2\pi\sqrt{2\pi}t}e^{-\frac{1}{2t}\left(x-\frac{(b_2-b_1)b_2}{x}\right)^2}e^{-\frac{1}{2t}(2(b_2-b_1)b_2)}dx\right.\\
&\quad\left.+\int_0^\infty \frac{1-\frac{(b_2-b_1)b_2}{x^2}}{2\pi\sqrt{2\pi}t}e^{-\frac{1}{2t}\left(x-\frac{(b_2-b_1)b_2}{x}\right)^2}e^{\frac{1}{2t}(2(b_2-b_1)b_2)}dx\right)\\
&= \mathbf{1}_{\{0\le t<1\}}\frac{1}{2}e^{-\frac{b_2^2}{2t}}e^{-\frac{(b_2-b_1)^2}{2t}}\left(0+\int_{-\infty}^\infty \frac{1}{2\pi\sqrt{2\pi}t}e^{-\frac{y^2}{2}}e^{\frac{1}{2t}(2(b_2-b_1)b_2)}dy\right)\\
&= \mathbf{1}_{\{0\le t<1\}}\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_1^2}{2t}}.
\end{aligned}
$$

Similarly, we compute the integral for $t>1$

$$
\begin{aligned}
&\mathbf{1}_{\{t>1\}}\int_0^1 \frac{-b_2}{\sqrt{2\pi s^3}}e^{-\frac{b_2^2}{2s}}\frac{1}{4\pi\sqrt{t-s}}e^{-\frac{(b_2-b_1)^2}{2(t-s)}}ds\\
= \ &\mathbf{1}_{\{t>1\}}\left(e^{-\frac{b_1^2}{2t}}\frac{1}{4\pi\sqrt{t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}+\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)+e^{-\frac{(2b_2-b_1)^2}{2t}}\frac{1}{4\pi\sqrt{t}}\mathcal{N}\left(\frac{b_2\sqrt{t-1}}{\sqrt{t}}-\frac{b_2-b_1}{\sqrt{t}\sqrt{t-1}}\right)\right).
\end{aligned}
$$

From the previous section, we also have that

$$
\mathcal{L}^{-1}\left(E\left(e^{-\beta T_{b_2}}\mathbf{1}_{\{T_{b_2}<1\}}\right)\frac{1}{4\sqrt{\pi\beta}}\right)=\mathbf{1}_{\{0\le t\le 1\}}\frac{1}{4\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}+\mathbf{1}_{\{t>1\}}\frac{1}{2\pi\sqrt{t}}e^{-\frac{b_2^2}{2t}}\mathcal{N}\left(-b_2\sqrt{\frac{t-1}{t}}\right).
$$

Combining all the above, we get the result. ■

Theorem 4.5 *Using the symmetry of Brownian motion, we have for* $0<b_1<b_2$,

$$
\begin{aligned}
f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-,T_{b_1}<1) &= f_{-b_2}^{-b_1}(t,\tau_{-b_2}^-<\tau_{-b_1}^+,T_{-b_1}<1) &\quad (4.18)\\
f_{b_1}^{b_2}(t,\tau_{b_1}^-<\tau_{b_2}^+,T_{b_1}<1) &= f_{-b_2}^{-b_1}(t,\tau_{-b_1}^+<\tau_{-b_2}^-,T_{-b_1}<1). &\quad (4.19)
\end{aligned}
$$

Proof. The results are due to the symmetry of Brownian motion. The positive barriers can be reflected to give the same result as in the case with negative barriers. ■

4.2 Pricing a double barrier Parisian in call

A double barrier Parisian in call is a call option that gets knocked in at $\tau_{b_1}^{b_2}$ if $\tau_{b_1}^{b_2}\le T$. For such an option with the same parameters as above, lower barrier L_1, upper barrier L_2 with $L_1<S_0<L_2$, i.e. $b_1<0<b_2$, we have the following pricing formula:

Theorem 4.6 *The price of a double barrier Parisian in call with barriers $L_1 < S_0 < L_2$ is*

$$
\begin{aligned}
{}^*C_i^{double}(x,T) &= \int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^- < \tau_{b_2}^+)(x\psi(\sigma+m,h_{b_1},b_1,\rho,t) - K\psi(m,h'_{b_1},b_1,\rho,t))dt \qquad (4.20)\\
&+ \int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+ < \tau_{b_1}^-)(x\psi(-(\sigma+m),h_{b_2},b_2,-\rho,t) - K\psi(-m,h'_{b_2},b_2,-\rho,t))dt,
\end{aligned}
$$

where $\psi(x,y,b,\rho,t)$, h_b, h'_b and ρ are defined as before.

Proof. As above, the price under the measure $\mathcal{P}$ is

$$
\begin{aligned}
& {}^*C_i^{double}(x,T)\\
=\ & E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}e^{mZ_T}(xe^{\sigma Z_T}-K)^+\right]\\
=\ & E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right]\\
=\ & E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_2}^+<\tau_{b_1}^-\}}E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right]\\
& +E_{\mathcal{P}}\left[\mathbf{1}_{\{\tau\le T\}}\mathbf{1}_{\{\tau_{b_1}^-<\tau_{b_2}^+\}}E_{\mathcal{P}}\left[e^{mZ_T}(xe^{\sigma Z_T}-K)^+|\mathcal{F}_\tau\right]\right]\\
=\ & \int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-)\frac{1}{2\pi\sqrt{T-t}}\int_{b_2}^\infty\int_k^\infty \frac{z-b_2}{2}(xe^{\sigma y}-K)e^{my}e^{-\frac{(y-z)^2}{2(T-t)}}e^{-\frac{(z-b)^2}{2}}dydzdt\\
& +\int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^-<\tau_{b_2}^+)\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b_1}\int_k^\infty \frac{b_1-z}{2}(xe^{\sigma y}-K)e^{my}e^{-\frac{(y-z)^2}{2(T-t)}}e^{-\frac{(z-b)^2}{2}}dydzdt\\
=\ & \int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-)\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b_2}\int_k^\infty \frac{b_2-z}{2}(xe^{\sigma y}-K)e^{my}e^{-\frac{(y-2b_2+z)^2}{2(T-t)}}e^{-\frac{(z-b)^2}{2}}dydzdt\\
& +\int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^-<\tau_{b_2}^+)\frac{1}{2\pi\sqrt{T-t}}\int_{-\infty}^{b_1}\int_k^\infty \frac{b_1-z}{2}(xe^{\sigma y}-K)e^{my}e^{-\frac{(y-z)^2}{2(T-t)}}e^{-\frac{(z-b)^2}{2}}dydzdt\\
=\ & \int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-)(x\psi(-(\sigma+m),h_{b_2},b_2,-\rho,t)-K\psi(-m,h'_{b_2},b_2,-\rho,t))dt\\
& +\int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^-<\tau_{b_2}^+)(x\psi(\sigma+m,h_{b_1},b_1,\rho,t)-K\psi(m,h'_{b_1},b_1,\rho,t))dt,
\end{aligned}
$$

where we have made use of the calculations in Theorem 2.3. ■

Theorem 4.7 *The price of a double barrier Parisian in call with barriers $L_1 < L_2 < S_0$ is*

$$
\begin{aligned}
& {}^*C_i^{double}(x,T)\\
=\ & x\phi'(\sigma+m)-K\phi'(m)\\
& +\int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+<\tau_{b_1}^-,T_{b_2}<1)(x\psi(-(\sigma+m),h_{b_2},b_2,-\rho,t)-K\psi(-m,h'_{b_2},b_2,-\rho,t))dt
\end{aligned}
$$

$$+\int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^- < \tau_{b_2}^+, T_{b_2} < 1)(x\psi(\sigma+m,h_{b_1},b_1,\rho,t) - K\psi(m,h'_{b_1},b_1,\rho,t))dt. \quad (4.21)$$

For barriers $S_0 < L_1 < L_2$, the price is

$$\begin{aligned} & {}^*C_i^{double}(x,T) \\ = \; & x\phi(\sigma+m) - K\phi(m) \\ & + \int_0^T f_{b_1}^{b_2}(t,\tau_{b_2}^+ < \tau_{b_1}^-, T_{b_2} < 1)(x\psi(-(\sigma+m),h_{b_2},b_2,-\rho,t) - K\psi(-m,h'_{b_2},b_2,-\rho,t))dt \\ & + \int_0^T f_{b_1}^{b_2}(t,\tau_{b_1}^- < \tau_{b_2}^+, T_{b_2} < 1)(x\psi(\sigma+m,h_{b_1},b_1,\rho,t) - K\psi(m,h'_{b_1},b_1,\rho,t))dt \quad (4.22) \end{aligned}$$

where $\psi(x,y,b,\rho,t)$, $\phi'(x)$, $\phi(x)$, h_b, ρ are all defined as before.

4.3 Double barrier Parisian out call

The price of the double barrier Parisian out call with the underlying price starting in between the two barriers $L_1 < S_0 < L_2$ is

$$C_o^{double}(x,T) = C_{BS}(x,T) - C_i^{double}(x,T).$$

For $L_1 < L_2 < S_0$, the price of the double barrier Parisian out call is

$$C_o^{double}(x,T) = \int_0^1 \frac{-b_2}{\sqrt{2\pi t^3}} e^{-\frac{b_2^2}{2t}} C_{BS}(L_2, T-t)dt - \left(C_i^{double}(x,T) - (x\phi'(\sigma+m) - K\phi'(m))\right)$$

and for $S_0 < L_1 < L_2$,

$$C_o^{double}(x,T) = \int_0^1 \frac{b_1}{\sqrt{2\pi t^3}} e^{-\frac{b_1^2}{2t}} C_{BS}(L_1, T-t)dt - \left(C_i^{double}(x,T) - (x\phi(\sigma+m) - K\phi(m))\right).$$

Chapter 5

Length and height of excursions

In this chapter, we are interested in finding the Laplace transform of the Parisian stopping time conditioned on a certain height. In particular, we define a new one-sided stopping time, which is the first time the Brownian motion makes an excursion of length at least D above zero, and during this excursion also hits a second barrier $L > 0$. This means that the last excursion is of length D if the Brownian motion has already hit L during the excursion, or the excursion is longer than length D and the stopping time is achieved when the Brownian motion hits L. This can be applied to Parisian options where the option gets knocked in or out after the underlying stays above the barrier for a period of time and also hits a second barrier above the first. The motivation for this is to ensure that the underlying does not only stay around the barrier throughout the excursion, hence is less easily manipulated. This is the same as the stopping time which is the first time the Brownian motion makes an excursion of length at least D below zero, and during this excursion also hits a second barrier below the first. This can be applied to calculating default probabilities, and it gives the company some capital allowance on top of a grace period. It is closely related to the concept of bankruptcy introduced in Albrecher and Lautscham [4], where the probability of bankruptcy increases the more negative the surplus becomes. Here, we define a new stopping time, which is the first time the Brownian motion makes an excursion of length at least D below zero, and during this excursion also reaches a second barrier $L < 0$. We use the perturbed Brownian motion in [21] to find the Laplace transform of this stopping time.

5.1 Definitions

Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion starting at 0 and L be the level of the second barrier (the first barrier is set to 0 without loss of generality). The excursions are defined as in Section 2.1:

$$\begin{aligned}
g_{L,t}^W &= \sup\{s \leq t | W_s = L\}, \qquad & d_{L,t}^W &= \inf\{s \geq t | W_s = L\} \\
g_{0,t}^W &= \sup\{s \leq t | W_s = 0\}, \qquad & d_{0,t}^W &= \inf\{s \geq t | W_s = 0\} \\
g_t^W &= \max\left(g_{L,t}^W, g_{0,t}^W\right)
\end{aligned}$$

with the usual convention that $\sup \emptyset = 0$ and $\inf \emptyset = \infty$. The trajectory of W between g_t^W and d_t^W is the excursion which straddles time t. We are interested here in $t - g_t^W$, which is the age of the excursion at time t, as well as $\overline{W}_t$ and $\underline{W}_t$, the maximum and minimum heights achieved during the excursion at time t,

$$\begin{aligned}
\overline{W}_t &= \max_{g_t^W \leq u \leq t} W_u \\
\underline{W}_t &= \min_{g_t^W \leq u \leq t} W_u.
\end{aligned}$$

For $D > 0$ and $L > 0$, we now define the stopping times

$$\begin{aligned}
\overline{\tau}_D^L(W) &= \inf\{t \geq 0 | \mathbf{1}_{\{W_t > 0, \overline{W}_t > L\}}(t - g_t^W) \geq D\} \\
\underline{\tau}_D^{-L}(W) &= \inf\{t \geq 0 | \mathbf{1}_{\{W_t < 0, \underline{W}_t < -L\}}(t - g_t^W) \geq D\}.
\end{aligned}$$

Hence, $\overline{\tau}_D^L(W)$ is the first time that the age of the excursion of the Brownian motion W above zero reaches level D and W hits L in the current excursion, while $\underline{\tau}_D^{-L}(W)$ is the first time that the age of the excursion of W below zero reaches level D and W hits $-L$ in the current excursion. We introduce the perturbed Brownian motion, a new process $X_t^{(\epsilon)}$, $\epsilon > 0$. This process was used in Dassios and Wu [21] to find the Laplace transform of the Parisian stopping time. Define a sequence of stopping times

$$\begin{aligned}
\delta_0 &= 0 \\
\sigma_n &= \inf\{t > \delta_n | W_t = -\epsilon\} \\
\delta_{n+1} &= \inf\{t > \sigma_n | W_t = 0\}
\end{aligned}$$

where $n = 0, 1, ...$ Now define $X_t^{(\epsilon)}$ as

$$\begin{aligned} X_t^{(\epsilon)} &= W_t + \epsilon \quad if \quad \delta_n \leq t < \sigma_n \\ X_t^{(\epsilon)} &= W_t \quad if \quad \sigma_n \leq t < \delta_{n+1}. \end{aligned}$$

We can see that $X_t^{(\epsilon)}$ is a process which starts from ϵ but has the same behaviour as the related Brownian motion and each time it hits the barrier 0 it will have a jump towards the opposite side of the barrier with size ϵ.

5.2 Semi-Markov model

We follow the notation as in [21]. We define the three-state semi-Markov process $Z_t^{(\epsilon)}$ by

$$Z_t^{(\epsilon)} = \begin{cases} 1 & \text{if } X_t^{(\epsilon)} > L \\ 2 & \text{if } 0 \leq X_t^{(\epsilon)} \leq L,\ g_{L,t} > g_{0,t} \\ 3 & \text{if } 0 \leq X_t^{(\epsilon)} \leq L,\ g_{0,t} > g_{L,t} \\ 4 & \text{if } X_t^{(\epsilon)} < 0. \end{cases}$$

We also define $V_t^{(\epsilon)} = t - g_t$, the time spent in the current state. Then $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process, and $Z_t^{(\epsilon)}$ is a semi-Markov process with state space $\{1, 2, 3, 4\}$, where 1 stands for the state when the process $X_t^{(\epsilon)}$ is above the barrier L, 4 corresponds to the state below the barrier 0, and 2 and 3 represent the states when $X_t^{(\epsilon)}$ is in between 0 and L, given it comes in through L or 0 respectively. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$\begin{aligned} P(Z_{t+\Delta t}^{(\epsilon)} = j, i \neq j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t) \\ P(Z_{t+\Delta t}^{(\epsilon)} = i | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t) \end{aligned}$$

for $i = 1, 2, 3, 4$. Define:

$$\begin{aligned} \bar{P}_{ij}(\mu) &= \exp\left\{ - \int_0^{\mu} \sum_{i \neq j} \lambda_{ij}(v) dv \right\} \\ p_{ij}(\mu) &= \lambda_{ij}(\mu)\bar{P}_i(\mu). \end{aligned}$$

In particular, we have the transition densities (see Karatzas and Shreve [32] pp. 99-100):

$$
\begin{aligned}
p_{12}^{\epsilon}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \\
p_{21}^{\epsilon}(t) &= P_{L-\epsilon}(T_L < T_0; T_L \in dt) = P_{\epsilon}(T_0 < T_L; T_0 \in dt) \\
&= \sum_{n=-\infty}^{\infty} \frac{2nL+\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+\epsilon)^2}{2t}} \\
p_{24}^{\epsilon}(t) &= P_{L-\epsilon}(T_0 < T_L; T_0 \in dt) = P_{\epsilon}(T_L < T_0; T_L \in dt) \\
&= \sum_{n=-\infty}^{\infty} \frac{2nL+L-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+L-\epsilon)^2}{2t}} \\
p_{31}^{\epsilon}(t) &= P_{\epsilon}(T_L < T_0; T_L \in dt) = P_{L-\epsilon}(T_0 < T_L; T_0 \in dt) \\
&= \sum_{n=-\infty}^{\infty} \frac{2nL+L-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+L-\epsilon)^2}{2t}} \\
p_{34}^{\epsilon}(t) &= P_{\epsilon}(T_0 < T_L; T_0 \in dt) = P_{L-\epsilon}(T_L < T_0; T_L \in dt) \\
&= \sum_{n=-\infty}^{\infty} \frac{2nL+\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+\epsilon)^2}{2t}} \\
p_{43}^{\epsilon}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}}
\end{aligned}
$$

where P_{ϵ} denotes the probability measure corresponding to a Brownian motion starting at ϵ. We also define the probabilities $p_{ij}^{*\epsilon}(t)$ as the probability of starting in state i and going to state j at time t for the first time, but regardless of whether it has been to any other states. We have

$$
\begin{aligned}
p_{21}^{*\epsilon}(t) &= p_{34}^{*\epsilon}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \\
p_{31}^{*\epsilon}(t) &= p_{24}^{*\epsilon}(t) = \frac{L-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(L-\epsilon)^2}{2t}}.
\end{aligned}
$$

We denote by $P_{ij}^{\epsilon(k)}(t)$ the probability of starting in state i, and having been to state j but not to state k at time t. We have

$$
\begin{aligned}
P_{31}^{\epsilon(4)}(t) &= P_{\epsilon}(T_0 > t, T_L < t) = P_{\epsilon}(T_0 > t) - P_{\epsilon}(T_0 > t, T_L > t) \\
&= \left(2\mathcal{N}\left(\frac{\epsilon}{\sqrt{t}}\right) - 1\right) - \sum_{n=-\infty}^{\infty} \left(2\mathcal{N}\left(\frac{2nL+\epsilon}{\sqrt{t}}\right) - 1 + 2\mathcal{N}\left(\frac{2nL+L-\epsilon}{\sqrt{t}}\right) - 1\right).
\end{aligned}
$$

5.3 Laplace transform of the Parisian stopping time conditioned on a given height

In this section, we proceed to find the Laplace transform of the Parisian stopping time conditioned on a given height, as described above. We only look at the case above the barrier, but the stopping time for the case below the barrier follows by symmetry. We have $\overline{\tau}_D^L = \underline{\tau}_D^{-L}$.

Theorem 5.1 *For $D > 0$, $L > 0$, we have the following Laplace transform for $\overline{\tau}_D^L$.*

$$E(e^{-\beta\overline{\tau}_D^L}) = \frac{1 + \sum_{n=-\infty}^{\infty}\left(2\sqrt{\pi\beta D}e^{\beta D}e^{-\sqrt{2\beta}(2n+1)L}\mathcal{N}\left(\frac{(2n+1)L}{\sqrt{D}} - \sqrt{2\beta D}\right) - e^{-\frac{((2n+1)L)^2}{2D}}\right)}{\Psi(\sqrt{2\beta D}) + \sum_{n=-\infty}^{\infty}\left(2\sqrt{\pi\beta D}e^{\beta D}e^{-\sqrt{2\beta}2nL}\mathcal{N}\left(\frac{2nL}{\sqrt{D}} - \sqrt{2\beta D}\right) - e^{-\frac{(2nL)^2}{2D}}\right)}. \tag{5.1}$$

Proof. First, we let A_k denote the event that the first time the excursion above 0 reaches length D and achieves a maximum of L during the excursion happens during the k^{th} excursion. Given A_k, $\overline{\tau}_D^L$ is made up of $k-1$ excursions above 0 either with length less than D or with length greater than D but does not hit the barrier L, and k excursions below 0 of any length, plus the last excursion with length at least D, and hits the barrier L. Denoting each of these excursions above 0 by U_i^+ and below 0 by U_i^-, $i = 1, ..., k$, we have

$$\begin{aligned}
E(e^{-\beta\overline{\tau}_D^L(X^{(\epsilon)})}) &= \sum_{k=1}^{\infty} E(e^{-\beta\overline{\tau}_D^L(X^{(\epsilon)})}; A_k) \\
&= \sum_{k=1}^{\infty} E(e^{-\beta(U_1^+ + ... + U_{k-1}^+ + U_k^+ + U_1^- + ... + U_k^-)}; A_k).
\end{aligned}$$

Furthermore, on the event A_k, the U_i^+ for $i = 1, ..., k-1$ have distribution $p_{34}^{*\epsilon}(t)$ for $t < D$ and $p_{34}^{\epsilon}(t)$ for $t \geq D$. U_i^- for $i = 1, ..., k$ have distribution $p_{43}^{\epsilon}(t)$, and they are all independent of each other. So we have

$$\begin{aligned}
E(e^{-\beta\overline{\tau}_D^L(X^{(\epsilon)})}) &= \sum_{k=1}^{\infty}\left(\int_0^D e^{-\beta t}p_{34}^{*\epsilon}(t)dt + \int_D^{\infty} e^{-\beta t}p_{34}^{\epsilon}(t)dt\right)^{k-1}\left(\int_0^{\infty} e^{-\beta t}p_{43}^{\epsilon}(t)dt\right)^k \\
&\quad \left(e^{-\beta D}P_{31}^{\epsilon(4)}(D) + \int_D^{\infty} e^{-\beta t}p_{31}^{\epsilon}(t)dt\right) \\
&= \frac{\int_0^{\infty} e^{-\beta t}p_{43}^{\epsilon}(t)dt\left(e^{-\beta D}P_{31}^{\epsilon(4)}(D) + \int_D^{\infty} e^{-\beta t}p_{31}^{\epsilon}(t)dt\right)}{1 - \left(\int_0^D e^{-\beta t}p_{34}^{*\epsilon}(t)dt + \int_D^{\infty} e^{-\beta t}p_{34}^{\epsilon}(t)dt\right)\int_0^{\infty} e^{-\beta t}p_{43}^{\epsilon}(t)dt},
\end{aligned}$$

and the following calculations

$$
\begin{aligned}
\int_0^\infty e^{-\beta t} p_{43}^\epsilon(t)dt &= e^{-\sqrt{2\beta}\epsilon} && (5.2)\\
\int_0^D e^{-\beta t} p_{34}^{*\epsilon}(t)dt &= \int_0^D e^{-\beta t}\frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}}dt\\
&= \frac{1}{2}e^{\sqrt{2\beta}\epsilon}\int_0^D \frac{\epsilon-\sqrt{2\beta}t}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon+\sqrt{2\beta}t)^2}{2t}}dt + \frac{1}{2}e^{-\sqrt{2\beta}\epsilon}\int_0^D \frac{\epsilon+\sqrt{2\beta t}}{\sqrt{2\pi t^3}} e^{-\frac{(\epsilon-\sqrt{2\beta}t)^2}{2t}}dt\\
&= \frac{1}{2}e^{\sqrt{2\beta}\epsilon}\int_{\frac{\epsilon}{\sqrt{D}}+\sqrt{2\beta D}}^\infty \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}dx + \frac{1}{2}e^{-\sqrt{2\beta}\epsilon}\int_{\frac{\epsilon}{\sqrt{D}}-\sqrt{2\beta D}}^\infty \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}}dx\\
&= e^{-\sqrt{2\beta}\epsilon}\mathcal{N}\left(\sqrt{2\beta D}-\frac{\epsilon}{\sqrt{D}}\right) + e^{\sqrt{2\beta}\epsilon}\mathcal{N}\left(-\sqrt{2\beta D}-\frac{\epsilon}{\sqrt{D}}\right) && (5.3)\\
\int_D^\infty e^{-\beta t} p_{31}^\epsilon(t)dt &= \sum_{n=-\infty}^{\infty}\left(e^{-\sqrt{2\beta}(2nL+L-\epsilon)}\mathcal{N}\left(\frac{2nL+L-\epsilon}{\sqrt{D}}-\sqrt{2\beta D}\right)\right.\\
&\quad \left.-e^{\sqrt{2\beta}(2nL+L-\epsilon)}\mathcal{N}\left(-\frac{2nL+L-\epsilon}{\sqrt{D}}-\sqrt{2\beta D}\right)\right) && (5.4)\\
\int_D^\infty e^{-\beta t} p_{34}^\epsilon(t)dt &= \sum_{n=-\infty}^{\infty}\left(e^{-\sqrt{2\beta}(2nL+\epsilon)}\mathcal{N}\left(\frac{2nL+\epsilon}{\sqrt{D}}-\sqrt{2\beta D}\right)\right.\\
&\quad \left.-e^{\sqrt{2\beta}(2nL+\epsilon)}\mathcal{N}\left(-\frac{2nL+\epsilon}{\sqrt{D}}-\sqrt{2\beta D}\right)\right) && (5.5)\\
e^{-\beta D}P_{31}^{\epsilon(4)}(D) &= e^{-\beta D}\left(2\mathcal{N}\left(\frac{\epsilon}{\sqrt{D}}\right)-1\right.\\
&\quad \left.-\sum_{n=-\infty}^{\infty}\left(2\mathcal{N}\left(\frac{2nL+\epsilon}{\sqrt{D}}\right)-1+2\mathcal{N}\left(\frac{2nL+L-\epsilon}{\sqrt{D}}\right)-1\right)\right). && (5.6)
\end{aligned}
$$

By construction,

$$X_t^{(\epsilon)} \xrightarrow{a.s.} W_t \qquad \text{for all } t.$$

As we saw in [21], since $X_t^{(\epsilon)} \to W_t$ almost surely for all t, by taking the limit $\epsilon \to 0$, the quantities defined based on $X^{(\epsilon)}$ converge to those based on W. Furthermore, $e^{-\beta\mp\tau_D^L} < 1$ almost surely, thus dominated convergence theorem applies to get the result for W,

$$
\begin{aligned}
E\left(e^{-\beta\mp\tau_D^L(W)}\right) &= \lim_{\epsilon\to 0} E\left(e^{-\beta\mp\tau_D^L(X^{(\epsilon)})}\right)\\
&= \lim_{\epsilon\to 0}\frac{\int_0^\infty e^{-\beta t}p_{43}^\epsilon(t)dt\left(e^{-\beta D}P_{31}^{\epsilon(4)}(D)+\int_D^\infty e^{-\beta t}p_{31}^\epsilon(t)dt\right)}{1-\left(\int_0^D e^{-\beta t}p_{34}^{*\epsilon}(t)dt+\int_D^\infty e^{-\beta t}p_{34}^\epsilon(t)dt\right)\int_0^\infty e^{-\beta t}p_{43}^\epsilon(t)dt} && (5.7)
\end{aligned}
$$

Therefore, plugging in the above calculations in equations (5.2) - (5.6) into equation (5.7) and applying L'Hopital's rule, we obtain the result in equation (5.1). ■

Chapter 6

The counting process of Parisian excursions

In this chapter, we study the distribution of the number of excursions away from 0 made by the Brownian motion. As 0 is a regular point of the Brownian motion (see Bertoin [10], Karatzas and Shreve [32]), the process makes infinitely many excursions away from 0. However, it can only make finitely many excursions whose length exceeds D in a finite time. Hence, it is only meaningful to study the number of excursions of duration greater than D. The counting process of excursions is related to the Brownian local time, and this gives us some motivation for studying them.

Here, we study the number of excursions indexed by an exponential time, and obtain some common distributions. We also find some connections with the Brownian local time. We modify slightly the Parisian stopping time. The new stopping time is defined as the first time the Brownian motion returns to the origin after having completed an excursion in excess of length D. This is different from the previously defined Parisian stopping time, which is the first time the length of an excursion reaches length D. We use the strong Markov property of Brownian motion to obtain the distribution of the number of excursions.

6.1 Definition

Let $W = (W_t)_{t \geq 0}$ be a standard Brownian motion starting at 0 as before, and we define g_t^W and d_t^W as in Section 2.1. The intervals (g_t^W, d_t^W) are the excursion intervals of W which straddles time t, each of length $(d_t^W - g_t^W)$. Taking all excursion intervals before time t, they form a countable union of disjoint open sets,

$$I = \bigcup_{\alpha \in A} (g_{t_\alpha}^W, d_{t_\alpha}^W)$$

and each of these intervals contains a number $t_\alpha(w)$. For $D_1, D_2 > 0$, define the stopping times

$$\begin{aligned}
\tilde{\tau}_{D_1}^{+}(W) &= \inf\{d_t^W > 0 \mid \mathbf{1}_{\{W_t>0\}}(d_t^W - g_t^W) \geq D_1\} \\
\tilde{\tau}_{D_2}^{-}(W) &= \inf\{d_t^W > 0 \mid \mathbf{1}_{\{W_t<0\}}(d_t^W - g_t^W) \geq D_2\} \\
\tilde{\tau}_{(D_1,D_2)}^{+}(W) &= \inf\{d_t^W > 0 \mid D_1 \leq \mathbf{1}_{\{W_t>0\}}(d_t^W - g_t^W) \leq D_2\} \\
\tilde{\tau}_{(D_1,D_2)}^{-}(W) &= \inf\{d_t^W > 0 \mid D_1 \leq \mathbf{1}_{\{W_t<0\}}(d_t^W - g_t^W) \leq D_2\}.
\end{aligned}$$

So we have the following interpretations: $\tilde{\tau}_{D_1}^{+}(W)$ is the first time W completes an excursion of duration greater than D_1 above 0, $\tilde{\tau}_{D_2}^{-}(W)$ is the first time W completes an excursion of duration greater than D_2 below 0, for $D_1 < D_2$, $\tilde{\tau}_{(D_1,D_2)}^{+}(W)$ is the first time W completes an excursion of duration between D_1 and D_2 above 0, and $\tilde{\tau}_{(D_1,D_2)}^{-}(W)$ is the first time W completes an excursion of duration between D_1 and D_2 below 0.

We are interested in the number of excursions made by S_t before time t, so we define the following

$$\begin{aligned}
\overline{N}_t^{D_1}(W) &= \#\{(g_u^W, d_u^W) \in I \mid 0 \leq d_u^W \leq t, S_u > 0, d_u^W - g_u^W \geq D_1\} \\
\underline{N}_t^{D_2}(W) &= \#\{(g_u^W, d_u^W) \in I \mid 0 \leq d_u^W \leq t, S_u < 0, d_u^W - g_u^W \geq D_2\} \\
N_t^{D_1,D_2}(W) &= \overline{N}_t^{D_1}(W) + \underline{N}_t^{D_2}(W).
\end{aligned}$$

Hence, $\overline{N}_t^{D_1}(W)$ is the number of excursions made by the process W above 0 of length D_1, $\underline{N}_t^{D_2}(W)$ is the number of excursions made by the process W below 0 of length D_2, and $N_t^{D_1,D_2}(W)$ is the total number of excursions above 0 of length D_1 and below 0 of length D_2.

We also define the stopping times

$$\tilde{\tau}_{D_1}^{n+}(W) = \inf\{t > 0 \mid \overline{N}_t^{D_1}(W) = n\}$$

$$\tilde{\tau}_{D_2}^{n-}(W) \quad = \quad \inf\{t > 0 \mid \underline{N}_t^{D_2}(W) = n\}$$

where $\tilde{\tau}_{D_1}^{0+}(W) = 0$. By the strong Markov property of Brownian motion, $\tilde{\tau}_{D_1}^{n+}(W)$ is the sum of n independent $\tilde{\tau}_{D_1}(W)$'s.

6.2 The modified Parisian stopping time

Here, we look at the stopping time which is the first time the Brownian motion completes an excursion of length D_1 above or below zero. This is different from that in the literature, which looks at the first time the length of the excursion reaches D_1. Here we only look at the excursions above zero but we have the same result for excursions below zero by symmetry.

Theorem 6.1 *For a standard Brownian motion W_t with $W_0 = 0$, we have the following result:*

$$E(e^{-\beta\tilde{\tau}_{D_1}^{+}(W)}) = \frac{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1}}{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)} \tag{6.1}$$

and $\tilde{\tau}_{D_1}^{n+}$ is the sum of n independent $\tilde{\tau}_{D_1}$'s, so

$$E(e^{-\beta\tilde{\tau}_{D_1}^{n+}(W)}) = \left(\frac{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1}}{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}\right)^n. \tag{6.2}$$

We also have

$$\begin{aligned} & E\left(e^{-\beta\tilde{\tau}_{D_1}^{+}(W)}\mathbf{1}_{\{\tilde{\tau}_{D_1}^{+}(W)<\tilde{\tau}_{D_2}^{-}(W)\}}\right) \\ = \; & \frac{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) + \sqrt{D_1}e^{-\beta D_2}\Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}} \end{aligned} \tag{6.3}$$

$$\begin{aligned} & E(e^{-\beta\tilde{\tau}_{D_2}^{-}(W)}\mathbf{1}_{\{\tilde{\tau}_{D_2}^{-}(W)<\tilde{\tau}_{D_1}^{+}(W)\}}) \\ = \; & \frac{\sqrt{D_1}e^{-\beta D_2}\Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) + \sqrt{D_1}e^{-\beta D_2}\Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}}, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} & E(e^{-\beta\tilde{\tau}_{(D_1,D_2)}^{+}(W)}\mathbf{1}_{\{\tilde{\tau}_{(D_1,D_2)}^{+}(W)<\tilde{\tau}_{D_2}^{+}(W)\}}) \\ = \; & \frac{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - \sqrt{D_1}e^{-\beta D_2}\Psi\left(\sqrt{2\beta D_2}\right)}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)} \end{aligned} \tag{6.5}$$

$$
\begin{aligned}
& E(e^{-\beta\tilde{\tau}^{+}_{D_2}(W)}\mathbf{1}_{\{\tilde{\tau}^{+}_{D_2}(W)<\tilde{\tau}^{+}_{(D_1,D_2)}(W)\}}) \\
= \;& \frac{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)-2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}. \qquad (6.6)
\end{aligned}
$$

Proof. These results are obtained using the same method as that in Dassios and Wu [21]. The difference is that in [21], the stopping time is the time when the length of the excursion first reached D_1, but here, it is the time when the Brownian motion first makes an excursion above zero of length D_1, and comes back down to 0. We proceed as in [21]. Here, we denote by $\tilde{\tau}^{+}_{D_1}(X^{(\epsilon)})$ the stopping time as above, but defined based on the perturbed Brownian motion $X^{(\epsilon)}$.

We define the two-state semi-Markov process $Z_t^{(\epsilon)}$ by

$$
Z_t^{(\epsilon)} = \begin{cases} 1 & \text{if } X_t^{(\epsilon)} > 0 \\ 2 & \text{if } X_t^{(\epsilon)} < 0, \end{cases} .
$$

We also define $V_t^{(\epsilon)} = t - g_t$, the time spent in the current state. Then $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process, and $Z_t^{(\epsilon)}$ is a semi-Markov process with state space $1, 2$, where 1 stands for the state when the process $X_t^{(\epsilon)}$ is above 0, and 2 is the state when the process is below 0. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$
\begin{aligned}
P(Z_{t+\Delta t}^{(\epsilon)} = j, i \neq j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t) \\
P(Z_{t+\Delta t}^{(\epsilon)} = i, | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= 1 - \sum_{i\neq j}\lambda_{ij}(u)\Delta t + o(\Delta t)
\end{aligned}
$$

for $i = 1, 2, 3, 4$. Define:

$$
\begin{aligned}
\bar{P}_{ij}(\mu) &= \exp\left\{-\int_0^{\mu}\sum_{i\neq j}\lambda_{ij}(v)dv\right\} \\
p_{ij}(\mu) &= \lambda_{ij}(\mu)\bar{P}_i(\mu).
\end{aligned}
$$

In particular, we have the following transition probabilities:

$$
p_{12}^{\epsilon}(t) = p_{21}^{\epsilon}(t) = \frac{\epsilon}{\sqrt{2\pi t^3}}e^{-\frac{\epsilon^2}{2t}}
$$

$$P_{12}^{\epsilon}(t) \;=\; P_{21}^{\epsilon}(t)=\int_0^t p_{12}^{\epsilon}(s)ds = 1-\bar{P}_{12}^{\epsilon}.$$

Now, we let A_j^i denote the event that the first excursion above 0 longer than D_1 occurs during the i^{th} excursion above 0, and the first excursion below 0 longer than D_2 occurs during the j^{th} excursion below 0. Then we have

$$\begin{aligned}
& E(e^{-\alpha_1\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})-\alpha_2\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})<\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})\}}) \\
= & \sum_{i=1}^{\infty}\sum_{j=i}^{\infty} E(e^{-\alpha_1\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})-\alpha_2\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})<\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})\}}|A_j^i)P(A_j^i).
\end{aligned}$$

Next, we observe that $\tilde{\tau}_{D_1}^{+}$ comprises of $i-1$ full excursions above 0 with length less than D_1, $i-1$ full excursions below 0 with length less than D_2 and one last excursion above 0 with length at least D_1. Similarly, $\tilde{\tau}_{D_2}^{-}$ comprises of and additional $j-i$ full excursions above 0 of any length, $j-i$ full excursions below 0 of length less than D_2, and one last excursion below 0 of length at least D_2. We denote by $U_{1,k}$ as the length of the k^{th} excursion above 0, and $U_{2,k}$ as the length of the k^{th} excursion below 0. $U_{i,k}$ satisfies the conditions: $U_{1,k}<D_1$ for $k=1,...,i-1$, $U_{2,k}<D_2$ for $k=1,...,j-1$, $U_{1,i}\geq D_1$ and $U_{2,j}\geq D_2$. So we have

$$\begin{aligned}
& E(e^{-\alpha_1\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})-\alpha_2\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})<\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})\}}|C) \\
= & E(e^{-\alpha_1\left(\sum_{k=1}^{i-1}(U_{1,k}+U_{2,k})+U_{1,i}\right)-\alpha_2\sum_{k=1}^{j}(U_{1,k}+U_{2,k})}|C) \\
= & \left(\int_0^{D_1} e^{-(\alpha_1+\alpha_2)s}\frac{p_{12}^{\epsilon}(s)}{P_{12}^{\epsilon}(D_1)}\right)^{i-1}\left(\int_{D_1}^{\infty} e^{-(\alpha_1+\alpha_2)s}\frac{p_{12}^{\epsilon}(s)}{\bar{P}_{12}^{\epsilon}(D_1)}ds\right) \\
& \left(\int_0^{D_2} e^{-(\alpha_1+\alpha_2)s}\frac{p_{21}^{\epsilon}(s)}{P_{21}^{\epsilon}(D_2)}\right)^{i-1}\left(\int_0^{\infty} e^{-\alpha_2 s}p_{12}^{\epsilon}(s)ds\right)^{j-i} \\
& \left(\int_0^{D_2} e^{-\alpha_2 s}\frac{p_{21}^{\epsilon}(s)}{P_{21}^{\epsilon}(D_2)}ds\right)^{j-i}\left(\int_{D_2}^{\infty} e^{-\alpha_2 s}\frac{p_{21}^{\epsilon}(s)}{\bar{P}_{21}^{\epsilon}(D_2)}ds\right)
\end{aligned}$$

and

$$P(A_j^i)=P_{12}^{\epsilon}(D_1)^{i-1}P_{21}^{\epsilon}(D_2)^{j-1}\bar{P}_{12}^{\epsilon}(D_1)\bar{P}_{21}^{\epsilon}(D_2).$$

Hence we have

$$E(e^{-\alpha_1\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})-\alpha_2\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}_{D_1}^{+}(X^{(\epsilon)})<\tilde{\tau}_{D_2}^{-}(X^{(\epsilon)})\}})$$

$$
\begin{aligned}
= & \sum_{i=1}^{\infty}\sum_{j=i}^{\infty}\left(\int_0^{D_1} e^{-(\alpha_1+\alpha_2)s}p^{\epsilon}_{12}(s)ds\right)^{i-1}\left(\int_{D_1}^{\infty} e^{-(\alpha_1+\alpha_2)s}p^{\epsilon}_{12}(s)ds\right) \\
& \left(\int_0^{D_2} e^{-(\alpha_1+\alpha_2)s}p^{\epsilon}_{21}(s)ds\right)^{i-1}\left(\int_0^{\infty} e^{-\alpha_2 s}p^{\epsilon}_{12}(s)ds\right)^{j-i} \\
& \left(\int_0^{D_2} e^{-\alpha_2 s}p^{\epsilon}_{21}(s)ds\right)^{j-i}\left(\int_{D_2}^{\infty} e^{-\alpha_2 s}p^{\epsilon}_{21}(s)ds\right).
\end{aligned}
$$

Taking $\alpha_2 = 0$ and $\alpha_1 = \beta$, we have

$$
E(e^{-\beta\tilde{\tau}^+_{D_1}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}^+_{D_1}(X^{(\epsilon)})<\tilde{\tau}^-_{D_2}(X^{(\epsilon)})\}}) = \frac{\int_{D_1}^{\infty} e^{-\beta s}p^{\epsilon}_{12}(s)ds}{1-\int_0^{D_1} e^{-\beta s}p^{\epsilon}_{12}(s)ds\int_0^{D_2} e^{-\beta s}p^{\epsilon}_{21}(s)ds}. \tag{6.7}
$$

We also have

$$
\begin{aligned}
\int_{D_1}^{\infty} e^{-\beta s}p^{\epsilon}_{12}(s)ds &= e^{-\sqrt{2\beta}\epsilon}\mathcal{N}\left(\frac{\epsilon}{\sqrt{D_1}}-\sqrt{2\beta D_1}\right)-e^{\sqrt{2\beta}\epsilon}\mathcal{N}\left(-\frac{\epsilon}{\sqrt{D_1}}-\sqrt{2\beta D_1}\right) && (6.8)\\
\int_0^{D_1} e^{-\beta s}p^{\epsilon}_{12}(s)ds &= e^{-\sqrt{2\beta}\epsilon}\mathcal{N}\left(\sqrt{2\beta D_1}-\frac{\epsilon}{\sqrt{D_1}}\right)+e^{\sqrt{2\beta}\epsilon}\mathcal{N}\left(-\sqrt{2\beta D_1}-\frac{\epsilon}{\sqrt{D_1}}\right) && (6.9)\\
\int_0^{D_2} e^{-\beta s}p^{\epsilon}_{21}(s)ds &= e^{-\sqrt{2\beta}\epsilon}\mathcal{N}\left(\sqrt{2\beta D_2}-\frac{\epsilon}{\sqrt{D_2}}\right)+e^{\sqrt{2\beta}\epsilon}\mathcal{N}\left(-\sqrt{2\beta D_2}-\frac{\epsilon}{\sqrt{D_2}}\right) && (6.10)
\end{aligned}
$$

We let $\epsilon \to 0$. By construction,

$$
X^{(\epsilon)}_t \xrightarrow{a.s.} W_t \quad \text{for all } t.
$$

The stopping time defined based on $X^{(\epsilon)}$ converge to those based on W (see [21]). Furthermore, since $e^{-\beta\tilde{\tau}^+_{D_1}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}^+_{D_1}(X^{(\epsilon)})<\tilde{\tau}^-_{D_2}(X^{(\epsilon)})\}} < 1$ almost surely, dominated convergence theorem applies and we have

$$
\begin{aligned}
E\left(e^{-\beta\tilde{\tau}^+_{D_1}(W)}\mathbf{1}_{\{\tilde{\tau}^+_{D_1}(W)<\tilde{\tau}^-_{D_2}(W)\}}\right) &= \lim_{\epsilon\to 0} E\left(e^{-\beta\tilde{\tau}^+_{D_1}(X^{(\epsilon)})}\mathbf{1}_{\{\tilde{\tau}^+_{D_1}(X^{(\epsilon)})<\tilde{\tau}^-_{D_2}(X^{(\epsilon)})\}}\right)\\
&= \lim_{\epsilon\to 0}\frac{\int_{D_1}^{\infty} e^{-\beta s}p^{\epsilon}_{12}(s)ds}{1-\int_0^{D_1} e^{-\beta s}p^{\epsilon}_{12}(s)ds\int_0^{D_2} e^{-\beta s}p^{\epsilon}_{21}(s)ds}.
\end{aligned}
$$

Now, substituting the above calculations (6.8) - (6.10) into equation (6.7) and using L'Hopital's rule, we obtain the result (6.3). (6.1) can be obtained by taking $D_2 \to \infty$, and the other three results can be derived in the same way. ■

6.3 Laplace transform of the number of Parisian excursions

In this section, we present some probability results on the number of Parisian excursions above and below zero.

Theorem 6.2 *Let $\tilde{T}$ be exponentially distributed, with parameter β. For a standard Brownian motion W_t with $W_0 = 0$, we have the following:*

$$\overline{N}_{\tilde{T}}^{D_1}(W) \sim Geometric\left(\frac{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1}}{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}\right) \tag{6.11}$$

$$P(\overline{N}_{\tilde{T}}^{D_1}(W) = n) = \left(\frac{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1}}{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}\right)^n \left(1 - \frac{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1}}{e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}\right). \tag{6.12}$$

Proof. For $n = 0, 1, 2, ...$, we have:

$$\begin{aligned}
P(\overline{N}_{\tilde{T}}^{D_1}(W) = n) &= \int_0^\infty \beta e^{-\beta t} P(\overline{N}_t^{D_1}(W) = n)dt \\
&= \int_0^\infty \beta e^{-\beta t}\left(P(\tilde{\tau}_{D_1}^{n+}(W) \le t) - P(\tilde{\tau}_{D_1}^{(n+1)+}(W) \le t)\right) dt \\
&= E(e^{-\beta \tilde{\tau}_{D_1}^{n+}(W)}) - E(e^{-\beta \tilde{\tau}_{D_1}^{(n+1)+}(W)}) \\
&= \left(E(e^{-\beta \tilde{\tau}_{D_1}^{+}(W)})\right)^n - \left(E(e^{-\beta \tilde{\tau}_{D_1}^{+}(W)})\right)^{n+1}
\end{aligned}$$

and the result follows from (6.1). ■

As $D \to 0$, we have the following representation for the Brownian local time.

Theorem 6.3 (P. Lévy (1948)) *The local time at the origin of the Brownian motion W satisfies*

$$\lim_{D\to 0}\sqrt{\frac{\pi D}{2}}\overline{N}_t^D(W) \to L_t(0) \qquad a.s.. \tag{6.13}$$

Proof. See Karatzas and Shreve [32] (Theorem 2.21). ■

We also have the distribution for $L_{\tilde{T}}(0)$.

$$L_{\tilde{T}}(0) \sim Exp(2\sqrt{2\beta}). \tag{6.14}$$

It is easy to check that if we take $D_1 \to 0$ in equation (6.9), $\overline{N}^D_{\tilde{T}}(W)$ converges in distribution to $L_{\tilde{T}}(0)$.

Theorem 6.4 *We have the following joint probability for the number of excursions above and below zero. For $D_1, D_2 > 0$, we have*

$$P(\overline{N}^{D_1}_{\tilde{T}}(W) = n_1, \underline{N}^{D_2}_{\tilde{T}}(W) = n_2) = \binom{n_1+n_2}{n_2} g_1(\beta)^{n_1} g_2(\beta)^{n_2} \left(1 - g_1(\beta) - g_2(\beta)\right) \tag{6.15}$$

for $n_1 = 0, 1, 2, ...,$ $n_2 = 0, 1, 2, ...,$ where

$$\begin{aligned} g_1(\beta) &= E\left(e^{-\beta\tilde{\tau}^+_{D_1}(W)} \mathbf{1}_{\{\tilde{\tau}^+_{D_1}(W) < \tilde{\tau}^-_{D_2}(W)\}}\right) \\ &= \frac{\sqrt{D_2} e^{-\beta D_1} \Psi\left(\sqrt{2\beta D_1}\right) - 2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2} e^{-\beta D_1} \Psi\left(\sqrt{2\beta D_1}\right) + \sqrt{D_1} e^{-\beta D_2} \Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}} \qquad (6.16) \\ g_2(\beta) &= E\left(e^{-\beta\tilde{\tau}^-_{D_2}(W)} \mathbf{1}_{\{\tilde{\tau}^-_{D_2}(W) < \tilde{\tau}^+_{D_1}(W)\}}\right) \\ &= \frac{\sqrt{D_1} e^{-\beta D_2} \Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2} e^{-\beta D_1} \Psi\left(\sqrt{2\beta D_1}\right) + \sqrt{D_1} e^{-\beta D_2} \Psi\left(\sqrt{2\beta D_2}\right) - 2\sqrt{\pi\beta D_1 D_2}}. \qquad (6.17) \end{aligned}$$

Proof. We denote $\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W)$ as the first time the process completes n_1 excursions above 0 of length greater than D_1 and n_2 excursions below 0 of length greater than D_2, in any order. Then $\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W)$ is the sum of n_1 independent copies of $\tilde{\tau}^+_{D_1}(W)\mathbf{1}_{\{\tilde{\tau}^+_{D_1}(W) < \tilde{\tau}^-_{D_2}(W)\}}$ and n_2 independent copies of $\tilde{\tau}^-_{D_2}(W)\mathbf{1}_{\{\tilde{\tau}^-_{D_2}(W) < \tilde{\tau}^+_{D_1}(W)\}}$.

Since there are $\binom{n_1+n_2}{n_2}$ number of ways to order the excursions, we have that the probability that there are exactly n_1 excursions above 0 of length at least D_1 and n_2 excursions below 0 of length at least D_2 is the probability that $\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W)$ happens before time t, minus the probability that the next time there is an excursion above 0 of length D_1 or below 0 of length D_2 happens before time t. Also, because of the strong Markov property, the stopping times are all independent of each other. So we have

$$\begin{aligned} & P(\overline{N}^{D_1}_{\tilde{T}}(W) = n_1, \underline{N}^{D_2}_{\tilde{T}}(W) = n_2) \\ = & \int_0^\infty \beta e^{-\beta t} P(\overline{N}^{D_1}_t(W) = n_1, \underline{N}^{D_2}_t(W) = n_2) dt \\ = & \int_0^\infty \beta e^{-\beta t} \left(\binom{n_1+n_2}{n_2} P(\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W) \leq t) \right. \end{aligned}$$

$$
\begin{aligned}
&-\binom{n_1+n_2}{n_2}P(\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W)+\tilde{\tau}^{+}_{D_1}(W)\mathbf{1}_{\{\tilde{\tau}^{+}_{D_1}(W)<\tilde{\tau}^{-}_{D_2}(W)\}}\le t) \\
&-\binom{n_1+n_2}{n_2}P(\tilde{\tau}^{n_1,n_2}_{D_1,D_2}(W)+\tilde{\tau}^{-}_{D_2}(W)\mathbf{1}_{\{\tilde{\tau}^{-}_{D_2}(W)<\tilde{\tau}^{+}_{D_1}(W)\}}\le t)\Big)\,dt \\
= &\binom{n_1+n_2}{n_2}\left(g_1(\beta)^{n_1}g_2(\beta)^{n_2}-g_1(\beta)^{n_1+1}g_2(\beta)^{n_2}-g_1(\beta)^{n_1}g_2(\beta)^{n_2+1}\right) \\
= &\binom{n_1+n_2}{n_2}g_1(\beta)^{n_1}g_2(\beta)^{n_2}(1-g_1(\beta)-g_2(\beta))
\end{aligned}
$$

for $n_1 = 0, 1, 2, ...$ and $n_2 = 0, 1, 2, ...$ ■

Next, we look at the joint probability of excursions above 0, but of different lengths.

Theorem 6.5 *For $D_1 < D_2$, the joint probability of the number of excursions above 0 longer than D_1 and the number of excursions above 0 longer than D_2 is*

$$
P(\overline{N}^{D_1}_{\tilde{T}}(W)=n_1,\overline{N}^{D_2}_{\tilde{T}}(W)=n_2)=\binom{n_1}{n_2}h_1(\beta)^{n_1-n_2}h_2(\beta)^{n_2}\left(1-h_1(\beta)-h_2(\beta)\right) \tag{6.18}
$$

for $n_1 = 0, 1, 2, ...,$ $n_2 = 0, 1, 2, ..., n_1,$ where

$$
\begin{aligned}
h_1(\beta) &= E(e^{-\beta\tilde{\tau}^{+}_{(D_1,D_2)}(W)}\mathbf{1}_{\{\tilde{\tau}^{+}_{(D_1,D_2)}(W)<\tilde{\tau}^{+}_{D_2}(W)\}}) \\
&= \frac{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)-\sqrt{D_1}e^{-\beta D_2}\Psi\left(\sqrt{2\beta D_2}\right)}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)} \qquad (6.19) \\
h_2(\beta) &= E(e^{-\beta\tilde{\tau}^{+}_{D_2}(W)}\mathbf{1}_{\{\tilde{\tau}^{+}_{D_2}(W)<\tilde{\tau}^{+}_{(D_1,D_2)}(W)\}}) \\
&= \frac{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)-2\sqrt{\pi\beta D_1 D_2}}{\sqrt{D_2}e^{-\beta D_1}\Psi\left(\sqrt{2\beta D_1}\right)}. \qquad (6.20)
\end{aligned}
$$

Proof. We first note that

$$
P(\overline{N}^{D_1}_{\tilde{T}}(W)=n_1,\overline{N}^{D_2}_{\tilde{T}}(W)=n_2)=P(\overline{N}^{(D_1,D_2)}_{\tilde{T}}(W)=n_1-n_2,\overline{N}^{D_2}_{\tilde{T}}(W)=n_2).
$$

Then we proceed in the same way as above, with the two stopping times of interest now being $\tilde{\tau}^{+}_{(D_1,D_2)}(W)$ and $\tilde{\tau}^{+}_{D_2}(W)$. We want the probability of having n_1 excursions above 0 of length at least D_1, and out of these n_1 excursions, n_2 of them being longer than D_2. Hence, this is the probability of having n_2 of $\tilde{\tau}^{+}_{D_2}(W)\mathbf{1}_{\{\tilde{\tau}^{+}_{D_2}(W)<\tilde{\tau}^{+}_{(D_1,D_2)}(W)\}}$ and n_1-n_2 of $\tilde{\tau}^{+}_{(D_1,D_2)}(W)\mathbf{1}_{\{\tilde{\tau}^{+}_{(D_1,D_2)}(W)<\tilde{\tau}^{+}_{D_2}(W)\}}$ occurring before time t, minus off the probability that the next time an excursion above 0 of length at least D_2 or between D_1 and D_2 will happen before time t. Furthermore, there are $\binom{n_1}{n_2}$ number of ways to order the excursions, and by the strong Markov property, the stopping times are all independent. Hence, we derive the result. ■

Chapter 7

Counting the excursions using a piecewise deterministic model

In this chapter, we look again at the counting process introduced in the previous chapter, this time from another point of view. Here, we define a piecewise deterministic semi-Markov process with a special state once the excursion reaches a certain length, and use the infinitesimal generator to find the Laplace transform of the first time the process makes n excursions above or below zero. We also obtain the joint distribution of T_L and $\overline{N}^{*D}_{T_L}(W)$, which we will define later, is the number of excursions made by the Brownian motion above 0 longer than length D before it hits level L.

7.1 Laplace transform of the first time there are n excursions above 0

Here, we find the Laplace transform of the first time the Brownian motion makes n excursions above 0 of length D. This is however different from the previous section as we now stop when we reach length D in the n^{th} excursion.

7.1.1 Definitions

We denote $\overline{N}^{*D}_t(W)$ and $\underline{N}^{*D}_t(W)$ as the number of excursions above and below 0 up to time t respectively, similar to above, but each excursion is counted once it reaches length D. We have the following definitions:

$$\begin{aligned}
\overline{N}_t^{*D_1}(W) &= \#\{(g_s^W, d_s^W) \in I \mid 0 \le g_s^W \le t - D_1, W_s > 0, d_s^W - g_s^W \ge D_1\} \\
\underline{N}_t^{*D_2}(W) &= \#\{(g_s^W, d_s^W) \in I \mid 0 \le g_s^W \le t - D_2, W_s < 0, d_s^W - g_s^W \ge D_2\} \\
N_t^{*D_1,D_2}(W) &= \overline{N}_t^{D_1}(W) + \underline{N}_t^{D_2}(W)
\end{aligned}$$

where I is the union of open sets of excursion intervals as defined in Section 6.1. We also define the stopping times

$$\begin{aligned}
\tau_{D_1}^{n+}(W) &= \inf\{t > 0 \mid \overline{N}_t^{*D_1}(W) = n\} \\
\tau_{D_2}^{n-}(W) &= \inf\{t > 0 \mid \underline{N}_t^{*D_2}(W) = n\}
\end{aligned}$$

for $n = 1, 2, ...$ and $\tau_{D_1}^{0+} = 0$. We are interested in this section to compute the Laplace transform of $\tau_D^{n+}(W)$.

7.1.2 Semi-Markov model

We again use the perturbed Brownian motion defined in Chapter 5, which is a Brownian motion which makes a jump of size ϵ towards the opposite side of the barrier every time it hits 0, and denote this new process by $X^{(\epsilon)} = (X_t^{(\epsilon)})_{t \ge 0}$. We now introduce the piecewise deterministic semi-Markov process Z_t, this time with an additional state 1^* for the case when the process has spent more than D amount of time above 0,

$$Z_t^{(\epsilon)} = \begin{cases} 1 & \text{if } X_t^{(\epsilon)} > 0,\ t - g_t < D \\ 1^* & \text{if } X_t^{(\epsilon)} > 0,\ t - g_t \ge D \\ 2 & \quad\quad \text{if } X_t^{(\epsilon)} < 0 \end{cases}.$$

Whenever the Brownian motion is below 0, $Z_t^{(\epsilon)}$ is in state 2, and when the Brownian motion is above 0, $Z_t^{(\epsilon)}$ is either in state 1 or 1^* when the Brownian motion is above 0, depending on whether the current excursion has exceeded length D. We also define $V_t^{(\epsilon)} = t - g_t$, the time spent by $X_t^{(\epsilon)}$ in the current state. Then $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process and $Z_t^{(\epsilon)}$ is thus a 3-state semi-Markov process. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$P(Z_{t+\Delta t}^{(\epsilon)} = j, i \ne j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) \quad = \quad \lambda_{ij}(u)\Delta t + o(\Delta t)$$

$$P(Z_{t+\Delta t}^{(\epsilon)} = i | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) = 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t)$$

for $i = 1, 2, 3, 4$. Define:

$$\begin{aligned}
\bar{P}_{ij}(\mu) &= \exp\left\{ -\int_0^{\mu} \sum_{i \neq j} \lambda_{ij}(v) dv \right\} \\
p_{ij}(\mu) &= \lambda_{ij}(\mu)\bar{P}_i(\mu).
\end{aligned}$$

In particular, transition from state 1 to 2 is the first hitting time of ϵ for the Brownian motion and occurs up to time D. If the Brownian motion stays in state 1 up to time D, then transition occurs from state 1 to 1^* at time D. The transition probability from state 1^* to 2 is the first hitting time of ϵ given that it does not occur before time D. Hence, we have the following expressions for the transition probabilities:

$$\begin{aligned}
p_{12}^{\epsilon}(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} & \text{if } 0 < t < D \\ 0 & t > D \end{cases} \\
p_{11^*}^{\epsilon}(t) &= \begin{cases} \bar{P}_{12}^{\epsilon}(D) = 2\mathcal{N}\left(\frac{\epsilon}{\sqrt{D}}\right) - 1 & \text{if } t = D \\ 0 & \text{otherwise} \end{cases} \\
p_{1^*2}^{\epsilon}(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \frac{1}{\bar{P}_{12}^{\epsilon}(D)} = \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \frac{1}{2\mathcal{N}\left(\frac{\epsilon}{\sqrt{D}}\right) - 1} & \text{if } t > D \\ 0 & \text{otherwise} \end{cases} \\
p_{21}^{\epsilon}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}}.
\end{aligned}$$

Furthermore, $p_{1^*1}^{\epsilon} = p_{21^*}^{\epsilon} = 0$. The transition intensities λ_{ij} are

$$\begin{aligned}
\bar{P}_i^{\epsilon}(t) &= e^{-\int_0^t \sum_{j \neq i} \lambda_{ij}(v) dv} \\
\lambda_{ij}^{\epsilon}(t) &= \frac{p_{ij}^{\epsilon}(t)}{\bar{P}_i^{\epsilon}(t)}.
\end{aligned}$$

We also define $\hat{P}_{ij}^{\epsilon}(\beta)$ and $\tilde{P}_{ij}^{\epsilon}(\beta)$ in order to simplify notation.

$$\begin{aligned}
\tilde{P}_{ij}^{\epsilon}(\beta) &= \int_0^{\infty} e^{-\beta s} p_{ij}^{\epsilon}(s) ds \\
\hat{P}_{ij}^{\epsilon}(\beta) &= \int_0^{d} e^{-\beta s} p_{ij}^{\epsilon}(s) ds.
\end{aligned}$$

7.1.3 Results

Theorem 7.1 *We have the following Laplace transform for the stopping time $\tau_{D_1}^{n+}$:*

$$E(e^{-\beta\tau_{D_1}^{n+}}) = \frac{1}{\Psi(\sqrt{2\beta D})}\left(\frac{\Psi(\sqrt{2\beta D}) - 2\sqrt{\pi\beta D}}{\Psi(\sqrt{2\beta D})}\right)^{n-1}. \tag{7.1}$$

Proof. To simplify notation, we denote $\overline{N}_t^\epsilon = \overline{N}_t^{*D_1}(X^{(\epsilon)})$ the counting process as defined above but this time based on the process $X^{(\epsilon)}$ rather than W. We consider the infinitesimal generator for the process $(Z_t^{(\epsilon)}, \overline{N}_t^\epsilon, V_t^{(\epsilon)})$. Consider a function which is smooth enough and of the form

$$f(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) = f_{Z_t^{(\epsilon)}}(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, t)$$

where f_i, $i = 1, 2, 3, 4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathcal{A}$ is defined as the operator such that

$$f(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) - \int_0^t \mathcal{A}f(\overline{N}_s^\epsilon, V_s^{(\epsilon)}, Z_s^{(\epsilon)}, s)ds$$

is a martingale. Hence we solve $\mathcal{A}f = 0$ subject to certain conditions to obtain martingales of the form $f(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t)$ to which we can apply the optional stopping theorem to obtain the Laplace transform we are interested in. More precisely, we have

$$\begin{aligned}
\mathcal{A}f_1(n,u,t) &= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{12}(u)(f_2(n,0,t) - f_1(n,u,t)) \\
&\quad + \lambda_{11^*}(u)(f_{1*}(n+1,d,t) - f_1(n,d,t)) = 0 \qquad (7.2)\\
\mathcal{A}f_2(n,u,t) &= \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial u} + \lambda_{21}(u)(f_1(n,0,t) - f_2(n,u,t)) = 0 \qquad (7.3)\\
\mathcal{A}f_{1^*}(n,u,t) &= \frac{\partial f_{1^*}}{\partial t} + \frac{\partial f_{1^*}}{\partial u} + \lambda_{1^*2}(u)(f_2(n,0,t) - f_{1^*}(n,u,t)) = 0. \qquad (7.4)
\end{aligned}$$

We assume f_i has the form

$$f_i(n,u,t) = \theta^n g_i(u)e^{-\beta t}$$

where $0 < \theta \leq 1$, $\beta > 0$ are constants. Since we are only interested in the excursion in state 1 for $0 \leq V_t^{(\epsilon)} < D$, and in state 1^* for $V_t^{(\epsilon)} > D$, so we solve (7.2) for $0 < u < D$ and (7.4) for $u > D$ subject to the conditions $g_1(D) = g_{1^*}(D)$ and $\lim_{u\to\infty} g_i(u) = 0$ for $i = 1^*, 2$. We get

$$\begin{aligned}
g_1(u) &= \int_u^D e^{-\int_u^s(\beta+\lambda_{12}(v))dv}(\lambda_{12}(s)g_2(0) + \theta\lambda_{11^*}(s)g_{1^*}(D) - \lambda_{11^*}(s)g_1(D))ds \\
&\quad + e^{-\int_u^D(\beta+\lambda_{12}(s))ds}g_1(D), \quad 0 < u < D \qquad (7.5)
\end{aligned}$$

$$g_2(u) = \int_u^\infty e^{-\int_u^s(\beta+\lambda_{21}(v))dv}\lambda_{21}(s)dsg_1(0) \tag{7.6}$$

$$g_{1^*}(u) = \int_u^\infty e^{-\int_u^s(\beta+\lambda_{1^*2}(v))dv}\lambda_{1^*2}(s)dsg_2(0), \quad u > D. \tag{7.7}$$

Taking $u = D$ in (7.7) and $u = 0$ in (7.5) and (7.6), we have

$$\begin{aligned}
g_1(0) &= e^{-\beta D}\bar{P}^\epsilon_{12}(D)g_1(D) + \hat{P}^\epsilon_{12}g_2(0) + \theta e^{-\beta D}\bar{P}^\epsilon_{12}(D)g_{1^*}(D) - e^{-\beta D}\bar{P}^\epsilon_{12}(D)g_1(D) \\
g_2(0) &= \tilde{P}^\epsilon_{21}g_1(0) \\
g_1(D) &= g_{1^*}(D) = \tilde{P}^\epsilon_{1^*2}g_2(0).
\end{aligned}$$

Solving for θ, we have

$$\theta = \frac{1 - \hat{P}^\epsilon_{21}\tilde{P}^\epsilon_{21}}{e^{-\beta D}\bar{P}^\epsilon_{12}(D)\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}}.$$

Hence, we obtain the martingale

$$M_t = \theta^{\overline{N}^\epsilon_t}g_i(V_t^{(\epsilon)})e^{-\beta t}$$

for $i = 1, 1^*, 2$. We apply the optional stopping theorem to M_t with the stopping time $\tau^{n+}_{D_1}(X^{(\epsilon)}) \wedge t$ to get

$$E(M_{\tau^{n+}_{D_1}(X^{(\epsilon)})\wedge t}) = E(M_0). \tag{7.8}$$

Since the function $g_1(u)$ is continuous, it is bounded on $[0, D]$, and so $g_1(V_t^{(\epsilon)})$ is bounded by a constant K on $[0, D]$ for all ω. Furthermore, $g_2(u)$ and $g_{1^*}(u)$ are both continuous and decreasing in u, so $g_2(V_t^{(\epsilon)})$ is bounded by some constant for all u and $g_{1^*}(V_t^{(\epsilon)})$ is bounded by some constant for $u > D$, for all ω. Hence dominated convergence applies and we have

$$\begin{aligned}
\lim_{t\to\infty} E(M_{\tau^{n+}_D(X^\epsilon)\wedge t}) &= E(M_{\tau^{n+}_D(X^\epsilon)}) \\
&= E(\theta^n e^{-\beta\tau^{n+}_D(X^\epsilon)}g_{1^*}(D)) \\
&= \left(\frac{1 - \hat{P}^\epsilon_{21}\tilde{P}^\epsilon_{21}}{e^{-\beta D}\bar{P}^\epsilon_{12}(D)\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}}\right)^n \tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}g_1(0)E(e^{-\beta\tau^{n+}_D(X^{(\epsilon)})}).
\end{aligned}$$

Since

$$E(M_0) = g_1(0),$$

we have

$$E(e^{-\beta\tau_D^{n+}(X^\epsilon)}) = \frac{1}{\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}} \left(\frac{e^{-\beta D}\bar{P}^\epsilon_{12}(D)\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}}{1-\hat{P}^\epsilon_{21}\tilde{P}^\epsilon_{21}} \right)^n .$$

We let $\epsilon \to 0$. By construction,

$$X_t^{(\epsilon)} \xrightarrow{a.s.} W_t \quad \text{for all } t.$$

The stopping time defined based on $X_t^{(\epsilon)}$ also converge to those of the Brownian motion W_t almost surely (see [21]). Furthermore, $e^{-\beta\tau} < 1$ almost surely, and thus dominated convergence theorem applies to get the result for W_t

$$\begin{aligned} E(e^{-\beta\tau_D^{n+}(W)}) &= \lim_{\epsilon\to 0} E(e^{-\beta\tau_D^{n+}(X_t^{(\epsilon)})}) \\ &= \lim_{\epsilon\to 0} \frac{1}{\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}} \left(\frac{e^{-\beta D}\bar{P}^\epsilon_{12}(D)\tilde{P}^\epsilon_{1^*2}\tilde{P}^\epsilon_{21}}{1-\hat{P}^\epsilon_{21}\tilde{P}^\epsilon_{21}} \right)^n \\ &= \frac{1}{\Psi(\sqrt{2\beta D})} \left(\frac{\Psi(\sqrt{2\beta D}) - 2\sqrt{\pi\beta D}}{\Psi(\sqrt{2\beta D})} \right)^{n-1}, \end{aligned}$$

where we have used the calculations in (6.8) - (6.10), and this completes the proof. ■

Remark 7.2 *We can check that when $n = 1$, we get the result for the one-sided Parisian stopping time as obtained in [14] and [21].*

7.2 Number of excursions before hitting L

In this section, we are interested to count the number of excursions above zero that are of length at least D, before the process hits level L. We denote by $T_L(W)$ the first hitting time of the Brownian motion of level L

$$T_L(W) = \inf\{t > 0 | W_t = L\}.$$

7.2.1 Semi-Markov model

We use the same perturbed Brownian motion $X_t^{(\epsilon)}$ and for the piecewise deterministic Markov process Z_t, we have one more state for when the process is above L.

$$Z_t^\epsilon = \begin{cases} 1 & \text{if } X_t^{(\epsilon)} > L \\ 2 & \text{if } 0 < X_t^{(\epsilon)} < L,\, t - g_t < D \\ 2^* & \text{if } 0 < X_t^{(\epsilon)} < L,\, t - g_t \geq D \\ 3 & \text{if } X_t^{(\epsilon)} < 0 \end{cases}.$$

Then $Z_t^{(\epsilon)}$ is in state 3 whenever the Brownian motion is below 0, and in state 1 when it is above L. It is either in state 2 or 2^* depending on whether the current excursion has exceeded length D. We define $V_t^{(\epsilon)} = t - g_t$ to be the time spent in the current state. $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process, and $Z_t^{(\epsilon)}$ is thus a 4-state semi-Markov process. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$\begin{aligned} P(Z_{t+\Delta t}^{(\epsilon)} = j, i \neq j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t) \\ P(Z_t^{(\epsilon)} = i | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t) \end{aligned}$$

for $i = 1, 2, 2^*, 3$. We define:

$$\begin{aligned} \bar{P}_{ij}(\mu) &= \exp\left\{ -\int_0^\mu \sum_{i \neq j} \lambda_{ij}(v) dv \right\} \\ p_{ij}(\mu) &= \lambda_{ij}(\mu)\bar{P}_i(\mu). \end{aligned}$$

In particular, we have that $p_{21}^\epsilon(t)$ is the probability of hitting L at time $t < D$ before hitting 0, $p_{23}^\epsilon(t)$ is the probability of hitting 0 at time $t < D$ before hitting L, and $p_{22^*}^\epsilon$ is the probability of the process staying between 0 and L up to time D. $p_{2^*1}^\epsilon(t)$ is the probability of hitting L at time $t > D$ and $p_{2^*3}^\epsilon(t)$ is the probability of hitting 0 at time $t > D$, given the process stays between 0 and L up to time D. These probabilities are (see Karatzas and Shreve [32] (Chapter 2)):

$$p_{21}^\epsilon(t) = \begin{cases} P_\epsilon(T_L \in dt, T_L < T_0) = \sum_{n=-\infty}^{\infty} \frac{2nL+L-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+L-\epsilon)^2}{2t}} & \text{if } 0 < t < D \\ 0 & \text{otherwise} \end{cases}$$

$$
\begin{aligned}
p^{\epsilon}_{23}(t) &= \begin{cases} P_{\epsilon}(T_0 \in dt, T_0 < T_L) = \sum_{n=-\infty}^{\infty} \frac{2nL+\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+\epsilon)^2}{2t}} & \text{if } 0 < t < D \\ 0 & \text{otherwise} \end{cases} \\
p^{\epsilon}_{22^*}(t) &= \begin{cases} P_{\epsilon}(T_0 > t, T_L > t) = \sum_{n=-\infty}^{\infty} \left(2\mathcal{N}\left(\frac{2nL+\epsilon}{\sqrt{t}}\right) - 1 + 2\mathcal{N}\left(\frac{2nL+L-\epsilon}{\sqrt{t}}\right) - 1\right) & \text{if } t > D \\ 0 & \text{otherwise} \end{cases} \\
p^{\epsilon}_{2^*3}(t) &= \begin{cases} \frac{P_{\epsilon}(T_0 \in dt, T_0 < T_L)}{\bar{P}^{\epsilon}_2(D)} = \sum_{n=-\infty}^{\infty} \frac{2nL+\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+\epsilon)^2}{2t}} \frac{1}{\bar{P}_2(D)} & \text{if } t > D \\ 0 & \text{otherwise} \end{cases} \\
p^{\epsilon}_{2^*1}(t) &= \begin{cases} \frac{P_{\epsilon}(T_L \in dt, T_L < T_0)}{\bar{P}^{\epsilon}_2(D)} = \sum_{n=-\infty}^{\infty} \frac{2nL+L-\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+L-\epsilon)^2}{2t}} \frac{1}{\bar{P}_2(D)} & \text{if } t > D \\ 0 & \text{otherwise} \end{cases} \\
p^{\epsilon}_{32}(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}}.
\end{aligned}
$$

The transition intensities λ_{ij} are

$$
\begin{aligned}
\bar{P}^{\epsilon}_i(t) &= e^{-\int_0^t \sum_{j \neq i} \lambda_{ij}(v) dv} \\
\lambda_{ij}(t) &= \frac{p^{\epsilon}_{ij}(t)}{\bar{P}^{\epsilon}_i(t)}.
\end{aligned}
$$

We also define $\hat{P}^{\epsilon}_{ij}(\beta)$ and $\tilde{P}^{\epsilon}_{ij}(\beta)$ in order to simplify notation.

$$
\begin{aligned}
\tilde{P}^{\epsilon}_{ij}(\beta) &= \int_0^{\infty} e^{-\beta s} p^{\epsilon}_{ij}(s) ds \\
\hat{P}^{\epsilon}_{ij}(\beta) &= \int_0^{D} e^{-\beta s} p^{\epsilon}_{ij}(s) ds.
\end{aligned}
$$

7.2.2 Results

We have the following theorem for the joint Laplace transform of the number of excursions of length D and the first hitting time of level L.

Theorem 7.3 *For $0 < \theta \leq 1$, $\beta > 0$, we have*

$$
E\left(\theta^{\overline{N}^{*D}_{T_L}(W)} e^{-\beta T_L(W)}\right) = \frac{\sum_{n>0,odd} \left((1-\theta) f(\sqrt{2\beta D}, \frac{nL}{\sqrt{D}}) + 2\theta\sqrt{\pi\beta D} e^{-\sqrt{2\beta} nL}\right)}{\sum_{n>0,even} \left((1-\theta) f(\sqrt{2\beta D}, \frac{nL}{\sqrt{D}}) + 2\theta\sqrt{\pi\beta D} e^{-\sqrt{2\beta} nL}\right) + 2\theta\sqrt{\pi\beta D}} \tag{7.9}
$$

where

$$f(x,y) = e^{-xy}\Phi(x,y) + e^{xy}\Phi(-x,y) \tag{7.10}$$

and $\Phi(x,y)$ *is*

$$\Phi(x,y) = \sqrt{2\pi}x\mathcal{N}(x-y) + e^{-\frac{x^2}{2}}. \tag{7.11}$$

Proof. We consider the infinitesimal generator for the process $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$. Consider a function which is smooth enough and of the form

$$f(\bar{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) = f_{Z_t^{(\epsilon)}}(\bar{N}_t^\epsilon, V_t^{(\epsilon)}, t)$$

where f_i, $i = 1,2,3,4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathcal{A}$ is defined as the operation such that

$$f(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t) - \int_0^t \mathcal{A}f(\overline{N}_s^\epsilon, V_s^{(\epsilon)}, Z_s^{(\epsilon)}, s)ds$$

is a martingale. Hence we solve $\mathcal{A}f = 0$ subject to certain conditions to obtain martingales of the form $f(\overline{N}_t^\epsilon, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, t)$. We have

$$\begin{aligned}
\mathcal{A}f_1(n,u,t) &= \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{12}(u)(f_2(n,0,t) - f_1(n,u,t)) = 0 && (7.12)\\
\mathcal{A}f_2(n,u,t) &= \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial u} + \lambda_{21}(u)(f_1(n,0,t) - f_2(n,u,t))\\
&\quad +\lambda_{23}(u)(f_3(n,0,t) - f_2(n,u,t))\\
&\quad +\lambda_{22^*}(u)(f_{2^*}(n+1,D,t) - f_2(n,D,t)) = 0 && (7.13)\\
\mathcal{A}f_{2^*}(n,u,t) &= \frac{\partial f_{2^*}}{\partial t} + \frac{\partial f_{2^*}}{\partial u} + \lambda_{2^*1}(u)(f_1(n,0,t) - f_{2^*}(n,u,t))\\
&\quad +\lambda_{2^*3}(u)(f_3(n,0,t) - f_{2^*}(n,u,t)) = 0 && (7.14)\\
\mathcal{A}f_3(n,u,t) &= \frac{\partial f_3}{\partial t} + \frac{\partial f_3}{\partial u} + \lambda_{32}(u)(f_2(n,0,t) - f_3(n,u,t)) = 0. && (7.15)
\end{aligned}$$

We assume f_i has the form

$$f_i(n,u,t) = \theta^n g_i(u)e^{-\beta t}$$

where $0 < \theta \leq 1$, $\beta > 0$ are constants. Since we are only interested in the length of the excursion in state 2, we solve (7.13) for $0 < u < D$ and (7.14) for $u > D$ subject to the conditions $g_2(D) = g_{2^*}(D)$ and $\lim_{u\to\infty} g_i(u) = 0$ for $i = 1, 2^*, 3$. We are not interested in what happens in state 1, so we can ignore the first equation. We get

$$
\begin{aligned}
g_2(u) &= g_2(D)e^{-\int_u^D(\beta+\lambda_{21}(v)+\lambda_{23}(v))dv} + \int_u^D e^{-\int_u^s(\beta+\lambda_{21}(v)+\lambda_{23}(v))dv}(\lambda_{21}(s)g_1(0) \\
&\quad +\lambda_{23}(s)g_3(0) + \lambda_{22^*}(s)\theta g_{2^*}(D) - \lambda_{22^*}(s)g_2(D))\,ds, \quad 0 < u < D \qquad (7.16) \\
g_{2^*}(u) &= \int_u^\infty e^{-\int_u^s(\beta+\lambda_{2^*1}(v)+\lambda_{2^*3}(v))dv}(\lambda_{2^*1}(s)g_1(0) + \lambda_{2^*3}g_3(0))ds, \quad u > D \qquad (7.17) \\
g_3(u) &= \int_u^\infty e^{-\int_u^s(\beta+\lambda_{32}(v))dv}\lambda_{32}(s)g_2(0)ds. \qquad (7.18)
\end{aligned}
$$

Taking $u = D$ in (7.17) and $u = 0$ in (7.16) and (7.18), we have

$$
\begin{aligned}
g_2(0) &= \hat{P}^\epsilon_{21}g_1(0) + \hat{P}^\epsilon_{24}g_4(0) + \theta\bar{P}^\epsilon_2(D)g_{2^*}(D) \\
g_{2^*}(D) &= \tilde{P}^\epsilon_{2^*1}g_1(0) + \tilde{P}^\epsilon_{2^*3}g_3(0) \\
g_3(0) &= \tilde{P}^\epsilon_{43}g_2(0).
\end{aligned}
$$

Solving the three equations with $g_2(D) = g_{2^*}(D)$, we get

$$(1 - \tilde{P}^\epsilon_{32}\hat{P}^\epsilon_{23} - \theta\bar{P}^\epsilon_2(D)\tilde{P}^\epsilon_{32}\tilde{P}^\epsilon_{2^*3})g_3(0) = (\tilde{P}^\epsilon_{32}\hat{P}^\epsilon_{21} + \tilde{P}^\epsilon_{32}\bar{P}^\epsilon_2(D)\theta\tilde{P}^\epsilon_{2^*1})g_1(0).$$

We apply the optional stopping theorem to the martingale

$$M_t = \theta^{\overline{N}^\epsilon_t}g_i(V^{(\epsilon)}_t)e^{-\beta t}$$

for $i = 1, 2, 2^*, 3$ and the stopping time $T_L(X^{(\epsilon)})$. Since $T_L(X^{(\epsilon)})$ is finite almost surely, we have

$$E(M_{T_L(X^{(\epsilon)})\wedge t}) = E(M_0). \qquad (7.19)$$

Since $g_2(u)$ is continuous on the closed set $[0, D]$, we have $g_2(V^{(\epsilon)}_t)$ is bounded by a constant on $[0, D]$. Furthermore, $g_{2*}(u)$ and $g_3(u)$ are continuous and decreasing, hence $g_{2^*}(V^{(\epsilon)}_t)$ and $g_3(V^{(\epsilon)}_t)$ are bounded. So dominated convergence applies and on the left hand side we have

$$
\begin{aligned}
\lim_{t\to\infty} E(M_{T_L(X^{(\epsilon)})\wedge t}) &= E(M_{T_L(X^{(\epsilon)})}) \\
&= E(\theta^{\overline{N}^{*D}_{T_L}(X^{(\epsilon)})}e^{-\beta T_L(X^{(\epsilon)})}g_1(0)).
\end{aligned}
$$

Since

$$E(M_0) = g_3(0),$$

we have

$$E(\theta^{\overline{N}^{*D}_{T_L}(X^{(\epsilon)})} e^{-\beta T_L(X^{(\epsilon)})}) = \frac{\tilde{P}^{\epsilon}_{32}(\hat{P}^{\epsilon}_{21} + \theta \bar{P}^{\epsilon}_{2}(D)\tilde{P}^{\epsilon}_{2^*1})}{1 - \tilde{P}^{\epsilon}_{32}\hat{P}^{\epsilon}_{23} - \theta \bar{P}^{\epsilon}_{2}(D)\tilde{P}^{\epsilon}_{32}\tilde{P}^{\epsilon}_{2^*3}}. \tag{7.20}$$

We let $\epsilon \to 0$. By construction,

$$X^{(\epsilon)} \xrightarrow{a.s.} W_t \quad \text{for all } t.$$

The stopping time defined based on $X^{(\epsilon)}_t$ also converge to those of the Brownian motion W_t almost surely. Furthermore, $e^{-\beta T_L(X^{(\epsilon)})} < 1$ almost surely, and thus dominated convergence applies to get the result for W_t. We get

$$\begin{aligned}
& E(\theta^{\overline{N}^{*D}_{T_L}(W)} e^{-\beta T_L(W)}) \\
= \; & \lim_{\epsilon \to 0} E(\theta^{\overline{N}^{*D}_{T_L}(X^{(\epsilon)})} e^{-\beta T_L(X^{(\epsilon)})}) \\
= \; & \lim_{\epsilon \to 0} \frac{\tilde{P}^{\epsilon}_{32}(\hat{P}^{\epsilon}_{21} + \theta \bar{P}^{\epsilon}_{2}(D)\tilde{P}^{\epsilon}_{2^*1})}{1 - \tilde{P}^{\epsilon}_{32}\hat{P}^{\epsilon}_{23} - \theta \bar{P}^{\epsilon}_{2}(D)\tilde{P}^{\epsilon}_{32}\tilde{P}^{\epsilon}_{2^*3}} \\
= \; & \frac{\sum\limits_{n>0,\text{odd}} \Big((1-\theta) f(\sqrt{2\beta D}, \frac{nL}{\sqrt{d}}) + 2\theta\sqrt{\pi\beta D} e^{-\sqrt{2\beta} nL}\Big)}{\sum\limits_{n>0,\text{even}} \Big((1-\theta) f(\sqrt{2\beta D}, \frac{nL}{\sqrt{D}}) + 2\theta\sqrt{\pi\beta D} e^{-\sqrt{2\beta} nL}\Big) + 2\theta\sqrt{\pi\beta D}},
\end{aligned}$$

where we have used the calculations in (5.2) - (5.6) and (6.8) - (6.10), and this completes the proof. ■

Remark 7.4 *It is easy to check that when $\theta = 1$, we obtain the Laplace transform of the first hitting time of the Brownian motion,*

$$E(e^{-\beta T_L(W)}) = e^{-\sqrt{2\beta} L}.$$

Chapter 8

Parisian excursions for the Brownian meander

In this chapter, we are interested to find the Parisian stopping time associated with the Brownian meander, which is Brownian motion conditioned to be positive. In particular we want to find the first time the Brownian meander makes an excursion above the level $L > 0$ and spends at least length D above the level. Some previous studies done on the Brownian meander include Durrett and Inglehart [23, 24], Imhof [31] and Yor [44].

8.1 Parisian stopping time for Brownian meander

We use the same method as the previous chapter to obtain the Laplace transform of the Parisian stopping time for the Brownian meander.

8.1.1 Definitions

The excursions are defined as before.

$$\begin{aligned} g_{L,t}^{W} &= \sup\{s \leq t | W_s = L\}, \qquad d_{L,t}^{W} = \inf\{s \geq t | W_s = L\} \\ g_{0,t}^{W} &= \sup\{s \leq t | W_s = 0\}, \qquad d_{0,t}^{W} = \inf\{s \geq t | W_s = 0\} \\ g_{t}^{W} &= \max(g_{L,t}^{W}, g_{0,t}^{W}) \end{aligned}$$

with the usual convention that $\sup\emptyset = 0$ and $\inf\emptyset = \infty$. The trajectory of W between $g_{L,t}^W$ and $d_{L,t}^W$ is the excursion about L which straddles time t. We are interested in the stopping time $\tau^+_{L,D}$ as defined in Chapter 2, the first time we have an excursion above L of length D. However, now the Brownian motion is conditioned to stay positive throughout the whole excursion. We denote by $W^* = (W^*_t)_{t\geq 0}$ this meander process. Then we define the Parisian stopping time based on this process,

$$\tau^+_{L,D}(W^*) = \inf\{t > 0 \mid W^*_t > L, t - g_t \leq D\}.$$

8.1.2 Semi-Markov model

Now, we introduce the doubly perturbed Brownian motion $Y^{(\epsilon)}_t$, $\epsilon > 0$, defined as follows. Define a sequence of stopping times

$$\begin{aligned} \delta_0 &= 0 \\ \sigma_n &= \inf\{t > \delta_n | W_t = -\epsilon\} \\ \delta_{n+1} &= \inf\{t > \sigma_n | W_t = 0\} \end{aligned}$$

where $n = 0, 1, ...$ Now define the process $X^{(\epsilon)}_t$ as

$$\begin{aligned} X^{(\epsilon)}_t &= W_t + \epsilon \quad if \quad \delta_n \leq t < \sigma_n \\ X^{(\epsilon)}_t &= W_t \quad if \quad \sigma_n \leq t < \delta_{n+1}. \end{aligned}$$

This process is the perturbed Brownian motion used in the previous chapters. Now, we define another sequence of stopping times with respect to the process $X^{(\epsilon)}_t$ and the barrier L:

$$\begin{aligned} \zeta_0 &= 0 \\ \eta_n &= \inf\{t > \zeta_n | X^{(\epsilon)}_t = L\} \\ \zeta_{n+1} &= \inf\{t > \eta_n | X^{(\epsilon)}_t = L - \epsilon\} \end{aligned}$$

where $n = 0, 1, ...$ Then we introduce a new process $Y^{(\epsilon)}_t$ defined as

$$\begin{aligned} Y^{(\epsilon)}_t &= X^{(\epsilon)}_t + \epsilon \quad \zeta_n \leq t < \eta_n \\ Y^{(\epsilon)}_t &= X^{(\epsilon)}_t \quad \eta_n \leq t < \zeta_{n+1}. \end{aligned}$$

This is a process which starts at ϵ and behaves like a Brownian motion except that each time it hits the barrier 0 or L it will jump towards the opposite side of the barrier with size ϵ.

The piecewise deterministic semi-Markov process $Z_t^{(\epsilon)}$ has three states. We do not take into account the state when $W_t < 0$ because we condition W_t to be positive throughout. $Z_t^{(\epsilon)}$ is defined as

$$Z_t^{(\epsilon)} = \begin{cases} 1 & \text{if } Y_t^{(\epsilon)} > L \\ 2 & \text{if } 0 < Y_t^{(\epsilon)} < L,\ g_t^L > g_t^0 \\ 3 & \text{if } 0 < Y_t^{(\epsilon)} < L,\ g_t^L < g_t^0 \end{cases}.$$

We define $V_t^{(\epsilon)} = t - g_t$ to be the time spent in the current state. Then $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process, and $Z_t^{(\epsilon)}$ is a 3-state semi-Markov process. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$\begin{aligned} P(Z_{t+\Delta t}^{(\epsilon)} = j, i \neq j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t) \\ P(Z_t^{(\epsilon)} = i | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t) \end{aligned}$$

for $i = 1, 2, 3$. Define:

$$\begin{aligned} \bar{P}_{ij}(\mu) &= \exp\left\{ - \int_0^\mu \sum_{i \neq j} \lambda_{ij}(v) dv \right\} \\ p_{ij}(\mu) &= \lambda_{ij}(\mu) \bar{P}_i(\mu). \end{aligned}$$

In particular, we have

$$\begin{aligned} p_{12}^\epsilon(t) &= \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \\ p_{21}^\epsilon(t) &= P_{L-\epsilon}(T_L \in dt | T_L < T_0) = \sum_{n=-\infty}^{\infty} \frac{2nL + \epsilon}{\sqrt{2\pi t^3}} e^{-\frac{(2nL+\epsilon)^2}{2t}} \frac{L}{L - \epsilon} \\ p_{31}^\epsilon(t) &= P_\epsilon(T_L \in dt | T_L < T_0) = \sum_{n=-\infty}^{\infty} \frac{(2n+1)L - \epsilon}{\sqrt{2\pi t^3}} e^{-\frac{((2n+1)L-\epsilon)^2}{2t}} \frac{L}{\epsilon}. \end{aligned}$$

These transition probabilities are obtained by noting that $p_{12}^\epsilon(t)$ is the probability of hitting L at time t from $L + \epsilon$, $p_{21}^\epsilon(t)$ is the probability of hitting L at time t starting at ϵ, without hitting 0 before, and $p_{31}^\epsilon(t)$ is the probability of hitting L at time t starting at $L - \epsilon$, without hitting 0 before. The transition intensities λ_{ij} are

$$\bar{P}_i^\epsilon(t) = e^{-\int_0^t \sum_{j \neq i} \lambda_{ij}(v) dv}$$

$$\lambda_{ij}(t) \;=\; \frac{p^{\epsilon}_{ij}(t)}{\bar{P}^{\epsilon}_{i}(t)}.$$

As before, we also define $\hat{P}^{\epsilon}_{ij}(\beta)$ and $\tilde{P}^{\epsilon}_{ij}(\beta)$ in order to simplify notation:

$$\begin{aligned}\tilde{P}^{\epsilon}_{ij}(\beta) &= \int_0^{\infty} e^{-\beta s} p^{\epsilon}_{ij}(s) ds \\ \hat{P}^{\epsilon}_{ij}(\beta) &= \int_0^{D} e^{-\beta s} p^{\epsilon}_{ij}(s) ds.\end{aligned}$$

8.1.3 Results

We have the following Laplace transform for the Parisian stopping time of the Brownian meander.

Theorem 8.1 *The Laplace transform of the Parisian stopping time for the Brownian meander is*

$$E\left(e^{-\beta\tau^{+}_{L,D}(W^*)}\right) = \frac{2\sqrt{2\beta}Le^{-\sqrt{2\beta}L}\frac{2}{\sqrt{2\pi D}}e^{-\beta D}}{2\sqrt{2\beta}e^{-2\sqrt{2\beta}L} + (1-e^{-2\sqrt{2\beta}L})(2\sqrt{2\beta}\mathcal{N}(\sqrt{2\beta D}) + \frac{2}{\sqrt{2\pi D}}e^{-\beta D} - \frac{1}{L})}. \quad (8.1)$$

Proof. We consider a function that is smooth enough and of the form

$$f(V^{(\epsilon)}_t, Z^{(\epsilon)}_t, t) = f_{Z^{(\epsilon)}_t}(V^{(\epsilon)}_t, t),$$

where f_i, $i = 1, 2, 3, 4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathcal{A}$ is defined as the operator such that

$$f(V^{(\epsilon)}_t, Z^{(\epsilon)}_t, t) - \int_0^t \mathcal{A}f(V^{(\epsilon)}_s, Z^{(\epsilon)}_s, s)ds$$

is a martingale. Hence we solve $\mathcal{A}f = 0$ subject to certain conditions to obtain martingales of the form $f(V^{(\epsilon)}_t, Z^{(\epsilon)}_t, t)$. More precisely, we have

$$\mathcal{A}f_1(u,t) = \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial u} + \lambda_{12}(u)(f_2(0,t) - f_1(u,t)) = 0 \quad (8.2)$$

$$\mathcal{A}f_2(u,t) = \frac{\partial f_2}{\partial t} + \frac{\partial f_2}{\partial u} + \lambda_{21}(u)(f_1(0,t) - f_2(u,t)) = 0 \quad (8.3)$$

$$\mathcal{A}f_3(u,t) = \frac{\partial f_3}{\partial t} + \frac{\partial f_3}{\partial u} + \lambda_{31}(u)(f_1(0,t) - f_3(u,t)) = 0. \quad (8.4)$$

We assume $f_i(u,t)$ has the form

$$f_i(u,t) = g_i(u)e^{-\beta t}$$

where $\beta > 0$ is a constant. Since we are only interested in the length of the excursion in state 1, we solve (8.2) for $0 < u < D$ and (8.3), (8.4) subject to the conditions $\lim_{u\to\infty} g_i(u) = 0$ for $i = 2, 3$. We get

$$\begin{aligned} g_1(u) &= g_1(D)e^{-\int_u^D (\beta+\lambda_{12}(s))ds} + \int_u^D e^{-\int_u^s (\beta+\lambda_{12}(v))dv}\lambda_{12}(s)dsg_2(0) && (8.5)\\ g_2(u) &= \int_u^\infty e^{-\int_u^s (\beta+\lambda_{21}(v))dv}\lambda_{21}(s)dsg_1(0) && (8.6)\\ g_3(u) &= \int_u^\infty e^{-\int_u^s (\beta+\lambda_{31}(v))dv}\lambda_{31}(s)dsg_1(0). && (8.7)\end{aligned}$$

Taking $u = 0$, we have

$$\begin{aligned} g_1(0) &= g_1(D)\bar{P}_1^\epsilon(D)e^{-\beta D} + \hat{P}_{12}^\epsilon \tilde{P}_{21}^\epsilon g_2(0)\\ g_2(0) &= \tilde{P}_{21}^\epsilon g_1(0)\\ g_3(0) &= \tilde{P}_{31}^\epsilon g_1(0).\end{aligned}$$

Solving the three equations, we get

$$g_1(D) = \frac{1 - \hat{P}_{12}^\epsilon \tilde{P}_{21}^\epsilon}{\bar{P}_1^\epsilon(D)e^{-\beta D}} g_1(0).$$

We apply the optional stopping theorem to the martingale

$$M_t = g_i(V_t^{(\epsilon)})e^{-\beta t}$$

for $i = 1, 2, 3$ and the stopping time $\tau_{L,D}^+(Y^{(\epsilon)})$):

$$E(M_{\tau_{L,D}^+(Y^{(\epsilon)})\wedge t}) = E(M_0). \qquad (8.8)$$

As before, $g_1(u)$ is continuous on the closed set $[0, D]$, hence $g_1(V_t^{(\epsilon)})$ is bounded by some constant for $V_t^{(\epsilon)} \in [0, D]$. Also, we have that $g_2(u)$ and $g_3(u)$ are continuous and decreasing, so $g_2(V_t^{(\epsilon)})$ and $g_3(V_t^{(\epsilon)})$ are bounded by some constant for all $V_t^{(\epsilon)}$. So dominated convergence applies and on the left hand side we have

$$\begin{aligned}
\lim_{t\to\infty} E(M_{\tau^+_{L,D}(Y^{(\epsilon)})\wedge t}) &= E(M_{\tau^+_{L,D}(Y^{(\epsilon)})}) \\
&= E(e^{-\beta\tau^+_{L,D}(Y^{(\epsilon)})} g_1(D))
\end{aligned}$$

and since

$$E(M_0) = g_3(0),$$

we have

$$E(e^{-\beta\tau^+_{L,D}(Y^{(\epsilon)})}) = \frac{\tilde{P}^\epsilon_{31}\bar{P}^\epsilon_1(D)e^{-\beta D}}{1-\hat{P}^\epsilon_{12}\tilde{P}^\epsilon_{21}}.$$

We let $\epsilon \to 0$. By construction,

$$Y^{(\epsilon)} \xrightarrow{a.s.} W_t \quad \text{for all } t.$$

The stopping time defined based on $Y^{(\epsilon)}_t$ also converge to that of the Brownian motion W_t almost surely (see [21]). Furthermore, $e^{-\beta\tau^+_{L,D}(Y^{(\epsilon)})} < 1$ almost surely, and thus dominated convergence applies to get the result for W^*. We get

$$\begin{aligned}
E(e^{-\beta\tau^+_{L,D}(W^*)}) &= \lim_{\epsilon\to 0} E(e^{-\beta\tau^+_{L,D}(Y^{(\epsilon)})}) \\
&= \lim_{\epsilon\to 0} \frac{\tilde{P}^\epsilon_{31}\bar{P}^\epsilon_1(D)e^{-\beta D}}{1-\hat{P}^\epsilon_{12}\tilde{P}^\epsilon_{21}} \\
&= \frac{2\sqrt{2\beta}Le^{-\sqrt{2\beta}L}\frac{2}{\sqrt{2\pi D}}e^{-\beta D}}{2\sqrt{2\beta}e^{-2\sqrt{2\beta}L} + (1-e^{-2\sqrt{2\beta}L})(2\sqrt{2\beta}\mathcal{N}(\sqrt{2\beta D}) + \frac{2}{\sqrt{2\pi D}}e^{-\beta D} - \frac{1}{L})},
\end{aligned}$$

where we have used the calculations in equations (5.2) - (5.6) and (6.8) - (6.10), and this completes the proof. ■

Chapter 9

Parisian occupation time - Occupation time with a qualifying period

An extension of the framework used in the previous two chapters results in formulas for the joint Laplace transform of the occupation time above and below 0 of a Brownian motion, but with a qualifying period for each excursion. Here, we compute the occupation time above and below 0, but we only start the clock after an excursion has reached a certain length. This result can be used to price bonds where continuous payments are made whenever the price of the underlying has stayed below a certain level for a period of time.

9.1 The semi-Markov model

As before, we use the semi-perturbed Brownian motion $X_t^{(\epsilon)}$ and define the piecewise deterministic semi-Markov process $Z_t^{(\epsilon)}$ by

$$Z_t^{(\epsilon)} = \begin{cases} 1 & \text{if } X_t^{(\epsilon)} > 0,\, t - g_t < D_1 \\ 1^* & \text{if } X_t^{(\epsilon)} > 0,\, t - g_t \geq D_1 \\ 2 & \text{if } X_t^{(\epsilon)} < 0,\, t - g_t < D_2 \\ 2^* & \text{if } X_t^{(\epsilon)} < 0,\, t - g_t \geq D_2 \end{cases}.$$

We also define $V_t^{(\epsilon)} = t - g_t$, the time spent by $X_t^{(\epsilon)}$ in the current state. Then $(Z_t^{(\epsilon)}, V_t^{(\epsilon)})$ is a Markov process. $Z_t^{(\epsilon)}$ is thus a semi-Markov process. The process transitions to state 1^* when it has spent D_1 amount of time in state 1, and transitions to state 2^* when it has spent

D_2 amount of time in state 2. The transition probabilities $\lambda_{ij}(u)$ for $Z_t^{(\epsilon)}$ satisfy:

$$
\begin{aligned}
P(Z_{t+\Delta t}^{(\epsilon)} = j, i \neq j | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= \lambda_{ij}(u)\Delta t + o(\Delta t) \\
P(Z_{t+\Delta t}^{(\epsilon)} = i | Z_t^{(\epsilon)} = i, V_t^{(\epsilon)} = u) &= 1 - \sum_{i \neq j} \lambda_{ij}(u)\Delta t + o(\Delta t)
\end{aligned}
$$

for $i = 1, 1^*, 2, 2^*$. Define:

$$
\begin{aligned}
\bar{P}_{ij}(\mu) &= \exp\left\{ - \int_0^\mu \sum_{i \neq j} \lambda_{ij}(v) dv \right\} \\
p_{ij}(\mu) &= \lambda_{ij}(\mu)\bar{P}_i(\mu).
\end{aligned}
$$

Now for $i = 1, 2$, we define the occupation times Z_t^i as

$$
Z_t^i(X^{(\epsilon)}) = \int_0^t \mathbf{1}_{\{Z_t^{(\epsilon)} \in i^*\}} ds.
$$

To simplify notation, we will refer to this as Z_t^i and use the notation $Z_t^i(W)$ for the corresponding occupation time of the Brownian motion. First, we find the joint Laplace transform of Z_t^1 and Z_t^2. As before, the transition probabilities satisfy:

$$
\begin{aligned}
p_{12}^\epsilon(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} & \text{if } 0 < t < D_1 \\ 0 & t > D_1 \end{cases} \\
p_{11^*}^\epsilon(t) &= \begin{cases} \bar{P}_{12}(D_1) = 2\mathcal{N}\left(\frac{\epsilon}{\sqrt{D_1}}\right) - 1 & \text{if } t = D_1 \\ 0 & \text{otherwise} \end{cases} \\
p_{1^*2}^\epsilon(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \frac{1}{\bar{P}_{12}(D_1)} & \text{if } t > D_1 \\ 0 & \text{otherwise} \end{cases} \\
p_{21}^\epsilon(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} & \text{if } 0 < t < D_2 \\ 0 & t > D_2 \end{cases} \\
p_{22^*}^\epsilon(t) &= \begin{cases} \bar{P}_{21}(D_2) = 2\mathcal{N}\left(\frac{\epsilon}{\sqrt{D_2}}\right) - 1 & \text{if } t = D_2 \\ 0 & \text{otherwise} \end{cases} \\
p_{2^*1}^\epsilon(t) &= \begin{cases} \frac{\epsilon}{\sqrt{2\pi t^3}} e^{-\frac{\epsilon^2}{2t}} \frac{1}{\bar{P}_{21}(D_2)} & \text{if } t > D_2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

and we have the transition intensities λ_{ij}

$$\begin{aligned}
\bar{P}^{\epsilon}_{ij}(t) &= e^{-\int_0^t \lambda_{ij}(v)dv} \\
\bar{P}^{\epsilon}_{i}(t) &= e^{-\int_0^t \sum_{j\neq i} \lambda_{ij}(v)dv} \\
\lambda_{ij}(t) &= \frac{p^{\epsilon}_{ij}(t)}{\bar{P}^{\epsilon}_{i}(t)}.
\end{aligned}$$

We also define $\tilde{P}^{\epsilon}_{ij}(\beta)$ and $\hat{P}^{\epsilon}_{ij}(\beta)$ in order to simplify notation:

$$\begin{aligned}
\tilde{P}^{\epsilon}_{ij}(\beta) &= \int_0^{\infty} e^{-\beta s} p^{\epsilon}_{ij}(s)ds \\
\hat{P}^{\epsilon}_{ij}(\beta) &= \int_0^{D_i} e^{-\beta s} p^{\epsilon}_{ij}(s)ds.
\end{aligned}$$

9.2 Laplace transform for the joint Parisian occupation times

Theorem 9.1 *We have the following representation for the Laplace transform of the occupation times Z^1_t and Z^2_t*

$$E\left(\int_0^{\infty} e^{-\beta t} e^{-\alpha_1 Z^1_t(W)} e^{-\alpha_2 Z^2_t(W)} dt\right) = \frac{\varphi(\alpha_1,\alpha_2,\beta)}{\varphi'(\alpha_1,\alpha_2,\beta)}, \tag{9.1}$$

where φ and φ' are defined as below.

$$\begin{aligned}
\varphi(x,y,z) &= \frac{1}{\sqrt{z}}\Big(\mathcal{N}(\sqrt{2zD_2}) - \mathcal{N}(-\sqrt{2zD_1})\Big) + \frac{1}{\sqrt{y+z}} e^{yD_2}\mathcal{N}(-\sqrt{2(y+z)D_2}) \\
&\quad + \frac{1}{\sqrt{x+z}} e^{xD_1}\mathcal{N}(-\sqrt{2(x+z)D_1}) \qquad (9.2) \\
\varphi'(x,y,z) &= \sqrt{z}\Big(\mathcal{N}(\sqrt{2zD_2}) - \mathcal{N}(-\sqrt{2zD_1})\Big) + \sqrt{y+z}\, e^{yD_2}\mathcal{N}(-\sqrt{2(y+z)D_2}) \\
&\quad + \sqrt{x+z}\, e^{xD_1}\mathcal{N}(-\sqrt{2(x+z)D_1}). \qquad (9.3)
\end{aligned}$$

Proof. We consider a function of the form

$$f(t, V^{(\epsilon)}_t, Z^{(\epsilon)}_t, Z^1_t, Z^2_t) = f_{Z^{(\epsilon)}_t}(t, V^{(\epsilon)}_t, Z^1_t, Z^2_t)$$

where f_i, $i = 1, 2, 3, 4$ are functions from $\mathbb{R}^2$ to $\mathbb{R}$. The generator $\mathcal{A}$ is defined as the operator such that

$$f(t, V_t^{(\epsilon)}, Z_t^{(\epsilon)}, Z_t^1, Z_t^2) - \int_0^t \mathcal{A}f(s, V_s^{(\epsilon)}, Z_s^{(\epsilon)}, Z_s^1, Z_s^2)ds$$

is a martingale. In this case, we have

$$\begin{aligned} \mathcal{A}f_i &= \frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial u} + \sum_{j \neq i} \lambda_{ij}(u)(f_j(t, 0, z^1, z^2) - f_i(t, u, z^1, z^2)) \qquad (9.4) \\ &\quad + \lambda_{ii^*}(u)(f_{i^*}(t, D_i, z^1, z^2) - f_i(t, D_i, z^1, z^2)) = 0 \\ \mathcal{A}f_{i^*} &= \frac{\partial f_{i^*}}{\partial z^i} + \frac{\partial f_{i^*}}{\partial t} + \frac{\partial f_{i^*}}{\partial u} + \sum_{j \neq i} \lambda_{ij}(u)(f_j(t, 0, z^1, z^2) - f_i(t, u, z^1, z^2)) = 0. \qquad (9.5) \end{aligned}$$

To obtain martingales $f_{Z_t}(t, V_t^{(\epsilon)}, Z_t^1, Z_t^2)$, we solve $\mathcal{A}f = 0$ subject to certain conditions. We assume $f_i(t, V_t^{(\epsilon)}, Z_t^1, Z_t^2)$ takes the form

$$f_i(t, V_t^{(\epsilon)}, Z_t^1, Z_t^2) = \int_0^t e^{-\beta s} e^{-\alpha_1 Z_s^1} e^{-\alpha_2 Z_s^2} ds + e^{-\beta t} e^{-\alpha_1 Z_t^1} e^{-\alpha_2 Z_t^2} g_i(V_t^{(\epsilon)})$$

where β, α_1 and α_2 are positive constants. Substituting this in equations (9.4) and (9.5) and then equating to 0, we have

$$g_i'(u) - (\beta + \sum_{j \neq i} \lambda_{ij}(u))g_i(u) = 1 + \sum_{j \neq i} \lambda_{ij}(u)g_j(0) + \lambda_{ii^*}(u)(g_{i^*}(D_i) - g_i(D_i)) \qquad (9.6)$$

and

$$g_{i^*}'(u) - (\beta + \alpha_i + \sum_{j \neq i^*} \lambda_{i^*j}(u))g_{i^*}(u) = 1 + \sum_{j \neq i^*} \lambda_{i^*j}(u)g_j(0). \qquad (9.7)$$

We solve (9.6) for $0 < u < D_i$ and (9.7) for $u > D_i$ subject to the conditions $g_i(D_i) = g_{i^*}(D_i)$ and $\lim_{u \to \infty} g_{i^*}(u) = 0$ for $i = 1, 2$. We get

$$\begin{aligned} g_i(u) &= \int_u^{D_i} e^{-\int_u^s (\beta + \sum_{j \neq i} \lambda_{ij}(v))dv} \left(\sum_{j \neq i} \lambda_{ij}(s) g_j(0) + 1 \right) ds \\ &\quad + e^{-\int_u^{D_i} (\beta + \sum_{j \neq i} \lambda_{ij}(v))dv} g_i(D_i), \quad 0 < u < D_i \qquad (9.8) \\ g_{i^*}(u) &= \int_u^{\infty} e^{-\int_u^s (\beta + \alpha_i + \sum_{j \neq i^*} \lambda_{i^*j}(v))dv} \left(\sum_{j \neq i^*} \lambda_{i^*j}(s) g_j(0) + 1 \right) ds, \quad u > D_i. \qquad (9.9) \end{aligned}$$

Taking $u = 0$ in (9.8) and $u = D_i$ in (9.9), we get

$$\begin{aligned} g_i(0) &= \sum_{j\neq i} g_j(0)\hat{P}^{\epsilon}_{ij}(\beta) + \frac{1}{\beta}\left(1 - e^{-\beta D_i}\bar{P}^{\epsilon}_i(D_i) - \sum_{j\neq i}\hat{P}^{\epsilon}_{ij}(\beta)\right) + e^{-\beta D_i}\bar{P}^{\epsilon}_i(D_i) g_i(D_i) \\ g_{i^*}(D_i) &= \sum_{j\neq i} e^{(\beta+\alpha_i)D_i}\tilde{P}^{\epsilon}_{i^*j}(\beta+\alpha_i) g_j(0) + \frac{1}{\beta+\alpha_i}\left(1 - \sum_{j\neq i} e^{(\beta+\alpha_i)D_i}\tilde{P}^{\epsilon}_{i^*j}(\beta+\alpha_i)\right). \end{aligned}$$

Solving the two equations, we have

$$\begin{aligned} g_i(0) &= \sum_{j\neq i} g_j(0)\left(\hat{P}^{\epsilon}_{ij}(\beta) + e^{\alpha_i D_i}\bar{P}^{\epsilon}_i(D_i)\tilde{P}^{\epsilon}_{i^*j}(\beta+\alpha_i)\right) \\ &\quad + \frac{1}{\beta}\left(1 - e^{-\beta D_i}\bar{P}^{\epsilon}_i(D_i) - \sum_{j\neq i}\hat{P}^{\epsilon}_{ij}(\beta)\right) \\ &\quad + \frac{1}{\beta+\alpha_i} e^{-\beta D_i}\bar{P}^{\epsilon}_i(D_i)\left(1 - \sum_{j\neq i} e^{(\beta+\alpha_i)D_i}\tilde{P}^{\epsilon}_{i^*j}(\beta+\alpha_i)\right). \end{aligned}$$

We then can solve for $g_i(0)$. In the two-state case,

$$g_1(0) = \frac{h(\beta,\alpha_1,\alpha_2)}{1 - \left(\hat{P}^{\epsilon}_{12}(\beta) + e^{\alpha_1 D_1}\bar{P}^{\epsilon}_1(D_1)\tilde{P}^{\epsilon}_{1^*2}(\beta+\alpha_1)\right)\left(\hat{P}^{\epsilon}_{21}(\beta) + e^{\alpha_2 D_2}\bar{P}^{\epsilon}_2(D_2)\tilde{P}^{\epsilon}_{2^*1}(\beta+\alpha_2)\right)}$$

where

$$\begin{aligned} h(\beta,\alpha_1,\alpha_2) &= \left(\hat{P}^{\epsilon}_{12}(\beta) + e^{\alpha_1 D_1}\bar{P}^{\epsilon}_1(D_1)\tilde{P}^{\epsilon}_{1^*2}(\beta+\alpha_1)\right)\left(\frac{1}{\beta}\left(1 - e^{-\beta D_2}\bar{P}^{\epsilon}_2(D_2) - \hat{P}^{\epsilon}_{21}(\beta)\right)\right. \\ &\quad \left. + \frac{1}{\beta+\alpha_2} e^{-\beta D_2}\bar{P}^{\epsilon}_2(D_2)(1 - e^{(\beta+\alpha_2)D_2}\tilde{P}^{\epsilon}_{2^*1}(\beta+\alpha_2))\right) \\ &\quad + \frac{1}{\beta}\left(1 - e^{-\beta D_1}\bar{P}^{\epsilon}_1(D_1) - \hat{P}^{\epsilon}_{12}(\beta)\right) \\ &\quad + \frac{1}{\beta+\alpha_1} e^{-\beta D_1}\bar{P}^{\epsilon}_1(D_1)(1 - e^{(\beta+\alpha_1)D_1}\tilde{P}^{\epsilon}_{1^*2}(\beta+\alpha_1)). \end{aligned}$$

Now, we use the martingale

$$f_i(t, V^{(\epsilon)}_t, Z^1_t, Z^2_t) = \int_0^t e^{-\beta s} e^{-\alpha_1 Z^1_s} e^{-\alpha_2 Z^2_s} ds + e^{-\beta t} e^{-\alpha_1 Z^1_t} e^{-\alpha_2 Z^2_t} g_i(V^{(\epsilon)}_t) \tag{9.10}$$

for $i = 1, 2$. We also have that

$$E(f_i(t, V_t^{(\epsilon)}, Z_t^1, Z_t^2)) = E(f(0,0,0,0)) = g_1(0)$$

holds for all $t > 0$. When $t \to \infty$, the second term in (9.10) becomes 0 and we have

$$E\left(\int_0^\infty e^{-\beta t} e^{-\alpha_1 Z_t^1} e^{-\alpha_2 Z_t^2} dt\right) = g_1(0).$$

Now we let $\epsilon \to 0$. By construction,

$$X_t^{(\epsilon)} \xrightarrow{a.s.} W_t \quad \text{for all } t.$$

As in [21], since $X_t^{(\epsilon)}$ converges to W_t almost surely for all t, the quantities defined based on $X^{(\epsilon)}$ also converge to those of the Brownian motion W almost surely. Furthermore, $\int_0^\infty e^{-\beta t} e^{-\alpha_1 Z_t^1} e^{-\alpha_2 Z_t^2} dt < \frac{1}{\beta}$ almost surely, and thus dominated convergence theorem applies to get the result for W_t,

$$\begin{aligned}
& E\left(\int_0^\infty e^{-\beta t} e^{-\alpha_1 Z_t^1(W)} e^{-\alpha_2 Z_t^2(W)} dt\right) \\
= & \lim_{\epsilon \to 0} E\left(\int_0^\infty e^{-\beta t} e^{-\alpha_1 Z_t^1} e^{-\alpha_2 Z_t^2} dt\right) \\
= & \lim_{\epsilon \to 0} \frac{h(\beta, \alpha_1, \alpha_2)}{1 - \left(\hat{P}_{12}^\epsilon(\beta) + e^{\alpha_1 D_1} \bar{P}_1^\epsilon(D_1) \tilde{P}_{1^*2}^\epsilon(\beta + \alpha_1)\right)\left(\hat{P}_{21}^\epsilon(\beta) + e^{\alpha_2 D_2} \bar{P}_2^\epsilon(D_2) \tilde{P}_{2^*1}^\epsilon(\beta + \alpha_2)\right)} \\
= & \frac{\varphi(\alpha_1, \alpha_2, \beta)}{\varphi'(\alpha_1, \alpha_2, \beta)},
\end{aligned}$$

where ϕ and ϕ' are as defined in equations (9.2) and (9.3). ■

Remark 9.2 *When $\alpha_2 = 0$ and $D_1 = 0$, we can simplify to get*

$$E\left(\int_0^\infty e^{-\beta t} e^{-\alpha_1 Z_t^1(W)} dt\right) = \frac{1}{\sqrt{\beta(\beta + \alpha_1)}},$$

same as the result for the occupation time above 0 without a qualifying period.

Chapter 10

Conclusion

In conclusion, there are several main results in this thesis. Firstly, we derive an analytical formula for the density of the one-sided Parisian stopping time. This is in the form of a recursion and it provides an alternative way to price Parisian options which is easy to program and fast to compute for long window lengths relative to the maturity time. We extend this result to obtain the density of the two-sided Parisian stopping time and also the double barrier Parisian stopping time. For the two-sided stopping time, we give an asymptotic result for the tail of the distribution.

Furthermore, we study the counting process of Parisian excursions, which are excursions that exceed a certain length. We obtain the Laplace transforms of some distributions related to the number of excursions. These results can be applied to mathematical finance, for example we can price an option which pays off an amount proportional to the number of times the stock price stays above a certain level for a period of time. Finally, we use the perturbed Brownian motion and the piecewise deterministic semi-Markov framework to derive several other interesting results. In particular, we obtain the Laplace transform of the Parisian stopping time conditioned on a given height. This can be used to price options which get knocked in not only when the share price stays above a barrier for a certain period of time, but also it must hit another barrier above the first one. We also obtain the Laplace transform of the Parisian stopping time for a Brownian meander. Finally, we derive the joint Laplace transform of the occupation times above and below zero, but with a qualifying period.

Further research can be done to find the Parisian stopping time for other kinds of processes, such as the Bessel process or the Ornstein Uhlenbeck process. One can also explore further into stopping times involving both the height and length of the excursion, as some of these cannot be derived using the current framework. It will also be good to look into applying some of these results to mathematical finance, such as option pricing or calculating the probability of default. An example would be a bond which pays off a continuous amount whenever the price of an underlying asset falls below a certain barrier for a certain period of time, or the probability of default for a firm that is liable to default only if the price of its share has dropped below a certain level and stayed below the level for a period of time, during which it also goes below a second barrier.

Bibliography

[1] ABATE, J. and WHITT, G. *Numerical Inversion of Laplace transforms of Probability Distributions.* ORSA J. Comput., Vol. 7, No. 1, (1995), pp. 36-43.

[2] AKAHORI, J. *Some Formulae for a New Type of Path-Dependent Option.* Ann. Appl. Probab., Vol. 5, No. 2, (1995), pp. 383-388.

[3] ALBRECHER, H., KORTSCHAK, D., ZHOU, X. *Pricing of Parisian options for a jump-diffusion model with two-sided jumps.* Appl. Math. Finance, Vol. 19, No. 2, (2012), pp. 97-129.

[4] ALBRECHER, H. and LAUTSCHAM, V. *From Ruin to Bankruptcy for Compound Poisson Surplus Processes.* Astin Bull., Vol. 43, No. 2, (2013), pp. 213-243.

[5] ANDERLUH, J.H.M. *Pricing Parisians and barriers by hitting time simulation.* Eur. J. Finance, Vol. 14, No. 2, (2008), pp. 137-156.

[6] ANDERLUH, J.H.M. and VAN DER WEIDE, J.A.M. *Double-sided Parisian option pricing.* Finance Stoch. Vol. 13, (2009), pp. 205-238.

[7] AVELLANEDA, M. and WU, L. *Pricing Parisian-style Options with a Lattice Method.* Int. J. Theor. Appl. Finance, Vol. 2, No. 1, (1999), pp. 1-16.

[8] BERNARD, C., BOYLE, P. *Monte Carlo methods for pricing discrete Parisian options.* Eur. J. Finance, Vol. 17, No. 3, (2011), pp. 169-196.

[9] BERNARD, C., COURTOIS, O. and QUITTARD-PINON, F. *A New Procedure for Pricing Parisian Options.* J. Derivatives, Vol. 12, No. 4 (2005), pp. 45-53.

[10] BERTOIN, J. *Lévy Process.* Cambridge university press, 1996.

[11] BROEDERS, D. and CHEN, A. *Pension regulation and the market value of pension liabilities: A contingent claims analysis using Parisian options.* J. Banking Finance, Vol. 34, No. 6, (2010), pp. 1201-1214.

[12] CHEN, A. and SUCHANECKI, M. *Default Risk, Bankruptcy Procedure and the Market Value of Life Insurance Liabilities.* Insurance Math. Econom., Vol. 40, (2007), pp. 231-255.

[13] CHESNEY, M., GAUTHIER, L. *American Parisian options.* Finance Stoch., Vol. 10, No. 4, (2006), pp. 475-506.

[14] CHESNEY, M., JEANBLANC-PICQUE, M. and YOR, M. *Brownian excursions and Parisian barrier options.* Adv. in Appl. Probab., Vol. 29, No. 1, (1997), pp. 165-184.

[15] CHUNG, K.L. *Excursions in Brownian motion.* Ark. Mat., Vol. 14, No. 2, (1976), pp. 155-177.

[16] COSTABILE, M. *A combinatorial approach for pricing Parisian options.* Decis. Econ. Finance (2002), pp. 111-125.

[17] DASSIOS, A. *The Distribution of the Quantile of a Brownian motion with drift and the pricing of related path-dependent options.* Ann. Appl. Probab. 1995, Vol. 5, No. 2, pp. 389-398.

[18] DASSIOS, A. and LIM, J.W. *Parisian Option Pricing: A Recursive Solution for the Density of the Parisian Stopping Time.* SIAM J. Financial Math., Vol. 4, No. 1, (2013), pp. 599-615.

[19] DASSIOS, A. and WU, S. *Double-barrier Parisian options.* J. Appl. Probab., Vol. 48, (2011), pp. 1-20.

[20] DASSIOS, A. and WU, S. *On barrier strategy dividends with Parisian implementation delay for classical surplus process.* Insurance Math. Econom., Vol. 45, No. 2, (2009), pp. 195-202.

[21] DASSIOS, A. and WU, S. *Perturbed Brownian motion and its application to Parisian option pricing.* Finance Stoch., Vol. 14, No. 3, (2009), pp. 473-494.

[22] DAVIS, M.H.A. *Piecewise-deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models.* J. R. Stat. Soc. Ser. B Stat. Methodol., Vol. 46, No. 3, (1984), pp. 353-388.

[23] DURRETT, R.T. and INGLEHART, D.L. *Functionals of Brownian Meander and Brownian Excursion.* Ann. Probab., Vol. 5, No. 1, (1977), pp. 130-135.

[24] DURRETT, R.T., INGLEHART, D.L. and MILLER, D.R. *Weak Convergence to Brownian Meander and Brownian Excursion.* Ann. Probab., Vol. 5, No. 1, (1977), pp. 117-129.

[25] FELLER, W., *An Introduction to Probability Theory and Its Applications, Vol. I.* Third edition. 1968.

[26] FELLER, W., *An Introduction to Probability Theory and Its Applications, Vol. II.* Third edition. 1968.

[27] FUJITA, T., MIURA, R. *Edokko Options: A New Framework of Barrier Options.* Asia Pac. Financ. Markets, Vol. 9, No. 2, (2003), pp. 141-151.

[28] GAUTHIER, L. *Excursions Height- and Length-Related Stopping Times, and Application to Finance.* Adv. in Appl. Probab., Vol. 34, No. 4, (2002), pp. 846-868.

[29] HOOGHIEMSTRA, G. *On explicit occupation time distributions for Brownian processes.* Stat. Probab. Lett., Vol. 56, (2002), pp. 405-417.

[30] HABER, R.J., SCHÖNBUCHER, P.J., and WILMOTT, P. *Pricing Parisian Options.* J. Derivatives, Vol. 6, No. 3, (1999), pp. 71-79.

[31] IMHOF, J.P. *Density Factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications.* J. Appl. Probab., Vol. 21, No. 3, (1984), pp. 500-510.

[32] KARATZAS, I., SHREVE, S.E., *Brownian Motion and Stochastic Calculus.* Second Edition, Springer-Verlag, 1991.

[33] KNIGHT, F.B. *On the Excursion Process of Brownian Motion.* Trans. Amer. Math. Soc., Vol. 258, No. 1, (1980), pp. 77-86.

[34] LABART, C. and LELONG, J. *Pricing double barrier Parisian Options using Laplace transforms.* Int. J. Theor. Appl. Finance, Vol. 12, (2009), pp. 1944.

[35] LABART, C. and LELONG, J. *Pricing Parisian options using Laplace transforms.* Bankers Markets and Investors, Vol. 99, (2009).

[36] LOUCHARD, G. *Kac's Formula, Lévy's Local Time and Brownian Excursion.* J. Appl. Probab., Vol. 21, No. 3, (1984), pp. 479-499.

[37] LOUCHARD, G. *The Brownian Excursion Area: A Numerical Analysis.* Comput. Math. Appl., Vol. 10, No. 6, (1984), pp. 413-417.

[38] MIURA, R. *A Note on Look-back Options based on order statistics.* Hitotsubashi J. Commerce Management, Vol. 27, (1992), pp. 15-28.

[39] PERMAN, M. and WELLNER, J.A. *On the Distribution of Brownian Areas.* Ann. Appl. Probab., Vol. 6, No. 4, (1996), pp. 1091-1111.

[40] PITMAN, J. and YOR, M. *Hitting, Occupation and Inverse Local Times of One-Dimensional Diffusions: Martingale and Excursion Approaches.* Bernoulli, Vol. 9, No. 1 (2003), pp. 1-24.

[41] PITMAN, J. and YOR, M. *On the distribution of Ranked Heights of Excursions of a Brownian Bridge.* Ann. Probab., Vol. 29, No. 1, (2001), pp. 361-384.

[42] REVUZ, D. and YOR, M., *Continuous Martingales and Brownian Motion.* Third edition, Springer, 1999.

[43] SCHRÖDER, M. *Brownian excursions and Parisian barrier options: a note.* J. Appl. Prob., Vol. 40, No. 4, (2003), pp. 855-864.

[44] YOR, M. *Some Aspects of Brownian Motion Part II.* Birkhäuser Verlag (Lectures in Mathematics), ETH Zurich, 1991.

[45] ZHU, S.P., CHEN, W.T. *Pricing Parisian and Parasian options analytically.* J. Econom. Dynam. Control, Vol. 37, No. 4, (2013), pp. 875-896.