Pricing and Hedging Exotic Options in Stochastic Volatility Models

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The London School of Economics and Political Science

A thesis submitted for the degree of

Doctor of Philosophy

December 2013

Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it). I confirm that Section 3.2 was jointly co-authored with Prof. Elisa Alòs and Prof. Thorsten Rheinländer.

Acknowledgment

First of all, I would like to express my gratitude to Prof. Thorsten Rheinländer for his guidance, support, patience and encouragement throughout my doctoral studies. On one hand, he left me all the freedom I wanted, encouraging and developing me to carry out the independent research. On the other hand, whenever I needed help, he was always there to share his knowledge and technical advices to guide me through the problems, which was essential to the completion of my thesis. I also want to thank for his warm hospitality during my stays at Vienna University of Technology.

My sincere thanks go to Dr. Angelos Dassios. He gave me all the support and advices I needed and readily accepted to supervise me after Thorsten left LSE to teach at Vienna University of Technology.

I am grateful to Jiawei Lim and Yang Yan for all the discussions and help during my study.

And my gratitude goes to my Ph.D. examiners Prof. Peter Mörters and Dr. Hao Xing, for their reading and all the advices on my thesis.

I also want to thank Dr. Umut Cetin, Prof. Mihail Zervos, Dr. Kostas Kardaras, Dr. Erik Baurdoux, Prof. Pauline Barrieu, Ian Marshall and many other staff and colleagues at London School of Economics for helping my research in various ways.

Financial support from the London School of Economics and Political Science is gratefully acknowledged.

The final thanks goes to my beloved family. To Chi Huang, my lovely wife who gave me time, encouragement and support during my doctoral studies, to my parents Ading Hu, Youhong Chen, Peidong Huang and Qunhui Zhang for their support and to Kaokao, my precious daughter.

Abstract

This thesis studies pricing and hedging barrier and other exotic options in continuous stochastic volatility models.

Classical put-call symmetry relates the price of puts and calls under a suitable dual market transform. One well-known application is the semi-static hedging of path-dependent barrier options with European options. This, however, in its classical form requires the price process to observe rather stringent and unrealistic symmetry properties. In this thesis, we provide a general self-duality theorem to develop pricing and hedging schemes for barrier options in stochastic volatility models with correlation.

A decomposition formula for pricing barrier options is then derived by Itō calculus which provides an alternative approach rather than solving a partial differential equation problem. Simulation on the performance is provided.

In the last part of the thesis, via a version of the reflection principle by Désiré André, originally proved for Brownian motion, we study its application to the pricing of exotic options in a stochastic volatility context.

Contents

Chapter 1

Introduction

1.1 Thesis Subject

This thesis studies pricing and hedging barrier and other exotic options in continuous stochastic volatility models.

In the first part of the thesis, we propose an extension of the put-call symmetry theorem (PCS) to cover the case of continuous stochastic volatility models when there is correlation between the price and the volatility process. PCS relates the price of puts and calls under a suitable dual market transform, that is, it allows to infer the price of a call from that of a related put under certain distributional assumptions on the stock price. For example, it implies that if a stock price S follows a Black-Scholes model under a certain pricing martingale measure, with no carrying cost, and $S_0 = 100$, then the price of a 200-strike call option written on S equals to that of two 50-strike put at the same maturity. The PCS relation corresponds to a symmetric distributional reflection property with respect to the bisector of the lift zonoid which is a central object in stochastic convex (Minkowski) geometry, see [27].

Nevertheless, as the correlation brings in some asymmetry, PCS does not hold in this general stochastic environment. Therefore, a general self-duality theorem has been developed to characterize certain relationships between the underlying price process and its dual process involving a changing of numeraire. That is, we show that a generalization of self-duality holds if one replaces in one side of the defining equation the risk-neutral measure $\mathbb{P}$ by a new measure $\mathbb{Q}$, and the process S by a certain modified form D, respectively. An application of this general self-duality allows one to express the no-arbitrage price of the barrier option at the hitting time in terms of a price of a time-dependent put option Γ_τ written on the modified price process. Semi-static hedging is not appropriate in this case as process D is not traded in the market. Nevertheless, we show that one could still perfectly replicate Γ_τ by dynamically trading in stock, realized volatility and bond.

Furthermore, we derive a decomposition formula for pricing vanilla options in general stochastic volatility models. We then adapt this approach to our specific situation, i.e., pricing a time-dependent put option written on the modified price process D under the measure $\mathbb{Q}$. And based on the decomposition formula, we derive an alternative approach of hedging barrier options: the main risk is semi-statically hedged by holding a position in put options written on the stock, and the remaining risk is then dynamically replicated by trading in the realized volatility.

Finally, we derive a closed-form valuation formula for barrier and lookback options in stochastic volatility models via an application of the classical reflection principle approach in the Black-Scholes model.

1.2 Overview of the Literature

1.2.1 Put-call Symmetry and (Quasi) Self-duality

The pioneering works by Bowie and Carr (1994) [7], Bates (1997) [5] and Carr and Chou (1997) [9] introduce the classic put-call symmetry theorem, relating the prices of puts and calls at strikes on opposite of the forward price. Recently, a more complete work by Carr and Lee (2009) [11] illustrates several extensions of the put-call symmetry to more general market conditions. Meanwhile, they introduce the application of PCS in semi-static hedging of path-dependent barrier options with vanilla options, where semi-static hedging refers to trading at most at inception and a finite number of stopping times like hitting times of barriers. Molchanov and Schmutz (2010) [27] extend the PCS theorem to the multivariate case. Tehranchi (2009) [41] studies the symmetry conditions for the price process and the relationship between put-call symmetry and self-duality. Moreover, Detemple (2001) [15] studies an extension of the classical results for models with stochastic volatility, stochastic interest rate and stochastic dividend yield. In particular, Schröder shows in his paper (1999) [37] that PCS holds in rather general models by taking the asset price in a risk-neutral world as a change of numeraire, which comprises a so-called dual market transform, allowing for stochastic coefficients and discontinuities. And much work has been done for exponential Lévy markets, see Farjado (2006) [17] , while Rheinländer and Schmutz (2012) [33] apply a certain power transformation to extend the concept to the notion of quasi self-duality.

1.2.2 Decomposition of option price

Alòs and Ewald (2008) [2] derive a first order decomposition formula for the vanilla option in Heston model by means of Malliavin calculus, in particular the Clark-Ocone formula. Later on, Alòs (2012) [1] extend the decomposition formula to a second order scheme via classical Itō calculus. Meanwhile, in an alternative approach, Antonelli and Scarlatti (2009) [4] study the decomposition problem via Taylor's expansion technique.

1.2.3 Reflection principle and pricing exotic options

This part of work is inspired by Désiré André's reflection principle for Brownian motions [3], [22]. We propose an application of the theorem in stochastic volatility environment based on the Ocone martingale argument. Rheinländer and Sexton (2012) [35] study the reflection property for Ocone martingales in a more general sense.

Regarding pricing exotic options in stochastic volatility models, Lipton (2001) [26] derives a (semi-)analytical solutions for double barrier options in a reduced Heston framework (with zero correlation between the underlying asset's price and variance processes) via the bounded Green's function, while the price of a single barrier option would be implied by setting one of the two barriers to a extreme value. Nevertheless, Faulhaber (2002) [18] shows in his thesis that an extension of these techniques to the general Heston framework fails. More recently, Griebsch and Pilz (2012) [19] develop a (semi-) closed-form valuation formula for continuous barrier options in the reduced Heston framework and approximations for these types of options in the general Heston model.

1.3 Contribution and Organization of this Thesis

In this thesis, we extend the classic self-duality theorem to a general self-duality framework. Based on that, we develop pricing and hedging schemes for barrier options in stochastic volatility models with correlation.

An application of the general self-duality allows one to perfectly replicate the barrier option by dynamically trading in stock, realized volatility and bond. Moreover, a decomposition formula is derived for pricing the barrier options. And we provide an alternative approach of hedging the barrier options: the main risk is semi-statically hedged by holding a position in put options written on the stock, and the remaining risk is then dynamically replicated by trading in the realized volatility.

Furthermore, we prove the reflection principle for the underlying asset price process in continuous stochastic volatility models via a different approach based on the Ocone martingale argument. We then derive the closed-form valuation formula for barrier as well as lookback options.

Let us present the organization of the thesis.

Chapter 2 proposes an extension of the put-call symmetry theorem to cover the case of continuous stochastic volatility models when there is correlation between the price and the volatility process. We start with reviewing some recent work on self-duality, PCS as well as applications in semi-static hedging of barrier options. Then we present our stochastic volatility model and our main theorem on general self-duality. Based on the general self-duality theorem, we derive a replicating portfolio for the barrier options with stock, realized volatility and bond.

In Chapter 3, we extend the decomposition formula for option prices in Heston model by Alòs (2012) [1] to a general stochastic volatility model. We then apply it for pricing and hedging the barrier options. The performance is checked by numerical simulation.

In Chapter 4, We prove the reflection principle for the underlying asset price process in continuous stochastic volatility models and derive the closed-form valuation formula for the barrier and lookback options.

Chapter 2

General self-duality

2.1 Introduction

This chapter studies valuation and hedging of barrier options, and proposes an extension of the known methods to cover the case of continuous stochastic volatility models when there is correlation between the price and the volatility process.

Semi-static refers to trading at most at inception and a finite number of stopping times like hitting times of barriers. The possibility of this hedge, however, requires classically a certain symmetry property of the asset price which has to remain invariant under the duality transformation. This leads naturally to the concept of self-duality which generalises the put-call symmetry, see [9] and more recently [11], [27]. To overcome the symmetry restriction, a certain power transformation has been proposed in the latter two papers which leads to the notion of quasi self-duality. While this works well in the context of exponential Lévy processes, see [33], it does not essentially change the picture for continuous stochastic volatility models. As has been shown in [32], a quasi self-dual price process in this setting is up to the costs of carry the stochastic exponential of a symmetric martingale. In particular, this would exclude any non-zero correlation between the volatility and the price process which is unrealistic.

We propose a different approach to deal with the correlated case: by a multiplicative decomposition, the price process is factorised into a self-dual and a remaining part. This latter part is used as a numeraire for a change of measure. Under this new measure called $\mathbb{Q}$, replacing the risk-neutral measure $\mathbb{P}$, the price process S is no longer a martingale but gets replaced by a modified price process D. We then show that a generalization of self-duality holds if one replaces in one side of the defining equation the measure $\mathbb{P}$ by $\mathbb{Q}$, and the process

S by its modified form D, respectively. In contrast to self-duality which holds only in special circumstances, its general form needs only weak assumptions in the context of continuous stochastic volatility models.

An application of this general self-duality allows one to express the no-arbitrage price of the barrier option at the hitting time τ in terms of a price of a time-dependent put option Γ_τ written on the modified price process. This put option is written on process D which is not a traded asset. This approach, however, leads by the general self-duality to a valuation formula for the barrier option. We moreover show how to perfectly replicate Γ_τ by dynamically trading in stock, realized volatility and bond. It is worth noting that the calculation of this replicating strategy does not involve solving a PDE, in contrast to the method of market completion by trading in stock and vanilla options, see [36].

2.1.1 Overview of the method

Let S be the price process of some risky asset with no carrying cost, modeled as a geometric Brownian motion. Consider a down-and-in call option with strike price K, maturity T and barrier level $B < K$. We denote $\tau := \inf\{t : S_t \leq B\}$ and assume that $S_0 > B$. The payoff of this option is

$$(S_T - K)^+ 1_{\tau \leq T}.$$

Assuming that we are already working under the risk-neutral measure, and that the barrier has been hit before T, the fair price of this option at the barrier is

$$E_\tau \left[(S_T - K)^+ \right],$$

where E_τ denotes the conditional expectation with respect to $\mathcal{F}_\tau$ where $(\mathcal{F}_t)$ is the Brownian filtration. It has been shown in [9] that this conditional expectation is equal to

$$E_\tau \left[\frac{S_T}{B} \left(\frac{B^2}{S_T} - K \right)^+ \right]. \tag{2.1}$$

In fact, it is shown that this observation yields a semi-static hedge for the down-and-in call: at time zero, purchase and hold a European claim $\frac{S_T}{B}\left(\frac{B^2}{S_T} - K\right)^+$. If and when the barrier knocks in, exchange this claim for a $(S_T - K)^+$-claim, at zero cost. By using the concept of self-duality, essentially this argument can be carried over to stochastic volatility models when price process and volatility are uncorrelated.

In the case of stochastic volatility models with correlation, this argument breaks down. In fact, there is no easy way to determine $E_\tau\left[(S_T-K)^+\right]$ in this setting. We propose instead the following steps:

1. By a general self-duality, we show that $E_\tau\left[(S_T-K)^+\right]$ equals the conditional expectation at the barrier τ of some claim Γ_τ. There are two fundamental differences to the uncorrelated case: firstly, Γ_τ is not a European claim but depends on the whole future of stock and volatility from τ to T; secondly, Γ_τ is not written on the stock S, but instead on another hypothetical underlying called D. The name D has been chosen since D replaces S on one side of the equation in the general self-duality, essentially D is a modified price process due to correlation. The dependence on τ is implicit already in the claim (2.1), but hidden since $S_\tau = B$. In contrast, D_τ is a random variable.

2. We then show that D can be explicitly written as product of S and some functional of the volatility. Next, we provide a replicating hedging portfolio for claim Γ_τ, by dynamically trading in stock, realized volatility and bond, which therefore also hedges the down-and-in call.

2.2 Self-duality and semi-static hedging

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where the filtration satisfies the usual conditions with $\mathcal{F}_0$ being trivial up to $\mathbb{P}$-null sets, and fix a finite but arbitrary time horizon $T > 0$. All stochastic processes are RCLL and defined on $[0,T]$. We assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supports at least two independent Brownian motions W and $W^\perp$. Let $E_t^{\mathbb{P}}$ denote the $\mathcal{F}_t$-conditional $\mathbb{P}$-expectation. (In)equalities between stochastic processes are in the sense of indistinguishability, whereas between random variables they are to be understood in the a.s. sense (if the dependency on the measure can be dropped). A martingale measure for a process X is a probability measure $\mathbb{P}$ such that X is a local $\mathbb{P}$-martingale.

Definition 2.1 *Let $S = \exp(X)$ be a martingale with $E^{\mathbb{P}}[S_T] = 1$. We define the probability measure $\widehat{\mathbb{P}}$ by*

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}} = S_T.$$

*The **dual process** $\widehat{S}$ is*

$$\widehat{S} = \frac{1}{S} = \exp(-X).$$

By Bayes' formula, $\widehat{S}$ is a martingale with respect to the ***dual measure*** $\widehat{\mathbb{P}}$.

For a general study of duality transforms we refer to [16]. The following definitions and results are modified from [41]. They differ slightly from the ones used in [41] who uses bounded measurable f instead, and in particular deterministic times. However, all corresponding results in [41] applied in this paper can be adapted to our setting.

Definition 2.2 *Let M be an adapted process. M is* ***symmetric*** *if for any non-negative Borel function f and any stopping time $\tau \in [0,T]$,*

$$E_\tau^{\mathbb{P}}\left[f\left(M_T - M_\tau\right)\right] = E_\tau^{\mathbb{P}}\left[f\left(M_\tau - M_T\right)\right]. \tag{2.2}$$

Here it is permissible that both sides of the equation are infinite. If M is an integrable symmetric process, then condition (2.2) implies that M is a martingale by choosing $f(x) = x$. Note that although f is not non-negative, it can be written as the difference of the two non-negative functions x^+ and x^-, and the result follows then by linearity.

Definition 2.3 *A non-negative adapted process S is* ***self-dual*** *if for any non-negative Borel function g and any stopping time $\tau \in [0,T]$,*

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\mathbb{P}}\left[\left(\frac{S_T}{S_\tau}\right) g\left(\frac{S_\tau}{S_T}\right)\right].$$

Definition 2.4 *Let Y be a semi-martingale with $Y_0 = 0$. Then, there exists a unique semi-martingale Z that satisfies the equation*

$$Z = 1 + \int Z_- dY.$$

The process Z is called the stochastic exponential of Y and is denoted by $\mathcal{E}(Y)$.

Proposition 2.5 *For a continuous semi-martingale Y, the stochastic exponential is given as*

$$\mathcal{E}(Y) = \exp\left(Y - \frac{1}{2}[Y]\right).$$

Theorem 2.6 *[41], Theorem 3.1. The continuous martingale S is self-dual if and only if S is of the form $S = \mathcal{E}(Y)$ for a symmetric continuous local martingale Y.*

Definition 2.7 *Let M be a continuous $(\mathbb{P}, \mathbb{F})$-martingale vanishing at zero and such that $[M]_\infty = \infty$, and consider its Dambis-Dubins-Schwarz (DDS) representation $M = \beta_{[M]}$ where β is some $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$-Brownian motion. The process M is called an **Ocone martingale** if β and $[M]$ are independent.*

The next result is [32], Lemma 18.

Lemma 2.8 *A continuous Ocone martingale M is symmetric.*

The following result has been obtained independently in [11] as well as [27].

Theorem 2.9 *Let S be a continuous self-dual process and τ be the first passage time of the barrier $B \neq S_0$; thus $\tau := \inf\{t : \eta S_t \geq \eta B\}$, where $\eta := sgn(B - S_0)$. Then a barrier option with payoff $G(S_T)\mathbf{1}_{(\tau \leq T)}$ where G is a non-negative Borel function can be replicated by holding a European claim on*

$$\Gamma(S_T) = G(S_T)1_{(\eta S_T \geq \eta B)} + \frac{S_T}{B} G\left(\frac{B^2}{S_T}\right) 1_{(\eta S_T \geq \eta B)}, \tag{2.3}$$

and if and when the barrier knocks in, by exchanging the $\Gamma(S_T)$ claim for the $G(S_T)$ claim with zero cost.

2.3 Generalized self-duality

We consider the following stochastic volatility model on a time interval $[0, T]$ under a risk-neutral measure $\mathbb{P}$:

$$\begin{aligned} dS_t &= rS_t dt + \sigma(V_t)\, S_t\, dZ_t, \qquad & S_0 = s_0 > 0, \\ dV_t &= \mu(V_t)\; dt + \gamma(V_t)\, dW_t, \qquad & V_0 = v_0 > 0. \end{aligned} \tag{2.4}$$

Here $r \geq 0$ denotes the riskless interest rate, and Z, W are two Brownian motions with correlation $\rho \in [-1, 1]$. Let $Z = \rho W + \overline{\rho} W^{\perp}$, where W and $W^{\perp}$ are independent standard Brownian motions and $\overline{\rho} = \sqrt{1 - \rho^2}$. We assume that the functions σ, μ, γ are continuous and of at most linear growth so that there exists a weak solution (S, V), and that $\sigma(V)$ is non-zero on $[0, T]$. The filtration is set to be $\mathbb{F} = \mathbb{F}^{S,V}$, the filtration generated by S and V.

Recall that by the definition of stochastic exponential, we have

$$S = \mathcal{E}\left(\int r dt + \int \sigma(V)\, dZ\right) = \exp\left(\int r dt + \int \sigma(V)\, dZ - \frac{1}{2}\int \sigma^2(V)\, dt\right). \tag{2.5}$$

2.3.1 Standing assumption

Standing assumption. In the sequel we assume that σ is such that all stochastic exponentials of the form $\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)$, with $\lambda \in [-1, 1]$ and ω some Brownian motion adapted to $\mathbb{F}^{S,V}$, are true martingales.

Sufficient conditions for the standing assumption to hold is well-known (see [21] and [34]). We recall here theorems and corollaries related to our model.

Theorem 2.10 *Suppose there exists a positive δ such that*

$$E\left[\exp\left((1+\delta)\int_0^T \sigma^2(V_s)\, ds\right)\right] < \infty$$

Then

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)_T\right] = 1.$$

Proof. See Kallianpur [21], Theorem 7.2.1 and the fact that $\lambda^2 \leq 1$. ■

Theorem 2.11 *Suppose there exists a positive δ and α such that for every $t \in [0, T]$ $(t + \alpha \leq T)$*

$$E\left[\exp\left((1+\delta)\int_t^{t+\alpha} \sigma^2(V_s)\, ds\right)\right] < \infty$$

Then

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)_T\right] = 1.$$

Proof. See Kallianpur [21], Theorem 7.2.2. ■

Theorem 2.12 *Suppose there exists a positive δ such that*

$$\sup_{0 \le t \le T} E\left[\exp\left(\delta\, \sigma^2(V_t)\right)\right] < \infty$$

Then

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)_T\right] = 1.$$

Proof. See Lipster and Shiryaev [25], Section 6.2, Example 3. ■

Corollary 2.13 *Suppose there exists a positive δ and constant C such that*

$$E\left[\exp\left(\delta\, \sigma^2(V_t)\right)\right] < C$$

for each $t \in [0, T]$. Then

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)_T\right] = 1.$$

Proof. See Kallianpur [21], Corollary 7.2.2. ■

Theorem 2.14 *(Novikov) Suppose that*

$$E\left[\exp\left(\frac{\lambda^2}{2} \int_0^T \sigma^2(V_s)\, ds\right)\right] < \infty$$

Then

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)_T\right] = 1.$$

Proof. See Kallianpur [21], Theorem 7.2.3. ■

Definition 2.15 *Let M be a martingale with $M_0 = 0$. M is a* ***BMO-martingale*** *if there exists a constant C such that for all stopping times $\tau \in [0, T]$*

$$E[\,|M_T - M_{\tau-}|\,|\,\mathcal{F}_\tau] \le C.$$

Proposition 2.16 *(John - Nirenberg inequality) Let M be a BMO-martingale. Then there exists $\varepsilon > 0$ such that*

$$E[\exp(\varepsilon\,[M]_T)] < \infty$$

Theorem 2.17 *(BMO-criterion) Suppose $\int \sigma(V)\,d\omega$ is a BMO-martingale, then*

$$E\left[\mathcal{E}\left(\lambda \int \sigma(V)\,d\omega\right)_T\right] = 1.$$

Proof. Proposition 2.16 and Theorem 2.12 yield the result. ■

2.3.2 General self-duality

This model is known to capture empirical results of asset price processes such as volatility clustering and leverage effects. However, the resulting price process is not self-dual when $\rho \neq 0$ (see [11]). It follows by Theorem 2.6 that the driving martingale $\int \sigma\,(V)\;dZ$ cannot be symmetric.

To cope with the asymmetry, we decompose the price process multiplicatively as

$$S = M \times R,$$

where the self-dual part is

$$M = s_0\,\mathcal{E}\left(\overline{\rho}\int \sigma(V)\,dW^{\perp}\right),$$

and the remaining part is

$$R = e^{rt}\mathcal{E}\left(\rho\int \sigma(V)\,dW\right) = e^{rt}R'.$$

Note that under $\mathbb{P}$, by our standing assumption the processes M and R' are martingales with expectation equal to s_0 and 1, respectively.

We take R'_T as Radon-Nikodym derivative to deal with the asymmetry problem via a change of measure:

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\,|_{\mathcal{F}_t} = R'_t, \qquad\qquad t \in [0,T].$$

By Girsanov's theorem,

$$\begin{aligned} W^{\mathbb{Q}} &= W - \int \frac{1}{R'}\, d[R', W] \\ &= W - \rho \int \sigma(V_u)\, du \end{aligned}$$

is a Brownian motion under $\mathbb{Q}$. The **modified price process** D under the measure $\mathbb{Q}$ is defined as

$$D = \frac{S}{R^2} = \frac{M}{R}.$$

We get by integration by parts

$$\begin{aligned} dD &= M dR^{-1} + R^{-1} dM + d[M, R^{-1}] \\ &= D(-rdt - \rho\sigma(V)\, dW + \rho^2\sigma^2(V)\, dt + \overline{\rho}\sigma(V)\, dW^{\perp}) \\ &= D(-rdt - \rho\sigma(V)\, dW^{\mathbb{Q}} + \overline{\rho}\sigma(V)\, dW^{\perp}). \end{aligned}$$

Since $\omega = \overline{\rho}W^{\perp} - \rho W^{\mathbb{Q}}$ is an $\mathbb{Q}$-Brownian motion, we have

$$e^{rt} D = \mathcal{E}\left(\int \sigma\left(V\right)\, d\omega\right),$$

hence by our standing assumption, $e^{rt}D$ is a martingale with expectation equal to 1 under the measure $\mathbb{Q}$.

The process D does not refer to the price process of any traded asset. However, it can be expressed readily in terms of the price and volatility processes. To see this, recall that

$$dV_t = \mu(V_t)\ dt + \gamma(V_t)\, dW_t,$$

hence for $t \in [0,T]$

$$\int_0^t \sigma(V_s)\, dW_s = \int_0^t \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^t \frac{\sigma(V_s)\mu(V_s)}{\gamma(V_s)} ds$$

and

$$D_t = S_t \exp\left(-2rt - 2\rho\left(\int_0^t \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^t \frac{\sigma(V_s)\mu(V_s)}{\gamma(V_s)} ds\right) + \rho^2 \int_0^t \sigma^2(V_s)\, ds\right). \tag{2.6}$$

Furthermore, its stochastic logarithm $\mathcal{L}(D) = \int dD/D$ can be replicated by trading dynamically in the stock, bond, realized as well as cumulative variance:

$$\begin{aligned}
\frac{dD}{D} &= -rdt - \rho\,\sigma(V)\,dW^{\mathbb{Q}} + \overline{\rho}\,\sigma(V)\,dW^{\perp} \\
&= -rdt - \rho\,\sigma(V)\,dW + \rho^2\,\sigma^2(V)\,dt + \overline{\rho}\,\sigma(V)\,dW^{\perp} \\
&= -rdt + (\rho\,\sigma(V)\,dW + \overline{\rho}\,\sigma(V)\,dW^{\perp}) - 2\rho\,\sigma(V)\,dW + \rho^2\,\sigma^2(V)\,dt \\
&= -2rdt + (rdt + \rho\,\sigma(V)\,dW + \overline{\rho}\,\sigma(V)\,dW^{\perp}) - 2\rho\left(\frac{\sigma(V)}{\gamma(V)} dV - \frac{\sigma(V)\mu(V)}{\gamma(V)} dt\right) \\
&+ \rho^2\,\sigma^2(V)\,dt \\
&= -2rdt + \frac{1}{S} dS - 2\rho\frac{\sigma(V)}{\gamma(V)} dV + \left(\rho^2 + 2\rho\frac{\mu(V)}{\sigma(V)\gamma(V)}\right)\sigma^2(V)\,dt.
\end{aligned} \tag{2.7}$$

In this sense we can synthetically create the process $\int dD/D$ as a traded asset, and $\mathbb{Q}$ is a martingale measure for both D and $\mathcal{L}(D)$ (the latter may be a local $\mathbb{Q}$-martingale which is consistent with our definition of a martingale measure).

Definition 2.18 *We denote by $\widehat{\mathbb{Q}}$ the dual measure associated with the process D with respect to $\mathbb{Q}$, where*

$$\frac{d\widehat{\mathbb{Q}}}{d\mathbb{Q}}\,|_{\mathcal{F}_t} = e^{rt} D_t, \qquad t \in [0, T],$$

and by $\widehat{D} = 1/D$ the corresponding dual process.

Theorem 2.19 *For any non-negative Borel function g and any stopping time $\tau \leq T$, we have the **general self-duality***

$$E_\tau^{\mathbb{P}}\left[g\left(\frac{S_T}{S_\tau}\right)\right] = E_\tau^{\widehat{\mathbb{Q}}}\left[g\left(\frac{\widehat{D}_T}{\widehat{D}_\tau}\right)\right].$$

Proof. Notice that in our setting, the process $\overline{\rho}\int \sigma(V)\,dW^{\perp}$ is a continuous Ocone martingale by Ch. 2, Theorem 2.6 of [8]. It follows by Theorem 2.6 that M is self-dual. Let us introduce a new σ-algebra $\mathcal{F}^V$ which contains all the information about the process V, i.e. $\mathcal{F}^V = \mathcal{F}^V_\infty$ where $\left(\mathcal{F}^V_t\right)_{t\geq 0}$ is the augmented filtration generated by V. We have for $p \in (0,1)$ that

$$\begin{aligned}
E^{\mathbb{P}}_\tau\left[\left(\frac{S_T}{S_\tau}\right)^p\right] &= E^{\mathbb{P}}_\tau\left[E^{\mathbb{P}}_\tau\left[\left(\frac{M_T R_T}{M_\tau R_\tau}\right)^p \middle| \mathcal{F}^V\right]\right] \\
&= E^{\mathbb{P}}_\tau\left[\left(\frac{R_T}{R_\tau}\right)^p E^{\mathbb{P}}_\tau\left[\left(\frac{M_T}{M_\tau}\right)^p \middle| \mathcal{F}^V\right]\right].
\end{aligned}$$

By self-duality of M, this equals

$$\begin{aligned}
&E^{\mathbb{P}}_\tau\left[\left(\frac{R_T}{R_\tau}\right)^p E^{\mathbb{P}}_\tau\left[\left(\frac{M_T}{M_\tau}\right)\left(\frac{M_\tau}{M_T}\right)^p \middle| \mathcal{F}^V\right]\right] \\
&= E^{\mathbb{P}}_\tau\left[\left(\frac{M_T}{M_\tau}\right)\left(\frac{M_\tau R_T}{M_T R_\tau}\right)^p\right] \\
&= E^{\mathbb{P}}_\tau\left[\left(\frac{S_T}{S_\tau}\frac{R_\tau}{R_T}\right)\left(\frac{S_\tau R_T^2}{S_T R_\tau^2}\right)^p\right] \\
&= E^{\mathbb{Q}}_\tau\left[e^{r(T-\tau)}\left(\frac{S_T}{S_\tau}\right)^{(1-p)}\left(\frac{R_T^2}{R_\tau^2}\right)^{(p-1)}\right] \\
&= E^{\mathbb{Q}}_\tau\left[e^{r(T-\tau)}\left(\frac{D_T}{D_\tau}\right)^{(1-p)}\right] = E^{\widehat{\mathbb{Q}}}_\tau\left[\left(\frac{D_T}{D_\tau}\right)^{-p}\right]
\end{aligned}$$

Hence the conditional moment-generating functions of $\log\left(\frac{S_T}{S_\tau}\right)$ under $\mathbb{P}$ and $\log\left(\frac{\widehat{D}_T}{\widehat{D}_\tau}\right)$ under $\widehat{\mathbb{Q}}$ are the same in an open interval which implies equality of the conditional distributions. It follows that for any non-negative, bounded Borel function g it holds that

$$\begin{aligned}
E^{\mathbb{P}}_\tau\left[g\left(\frac{S_T}{S_\tau}\right)\right] &= E^{\widehat{\mathbb{Q}}}_\tau\left[g\left(\frac{\widehat{D}_T}{\widehat{D}_\tau}\right)\right] \\
&= E^{\mathbb{Q}}_\tau\left[e^{r(T-\tau)}\frac{D_T}{D_\tau} g\left(\frac{D_\tau}{D_T}\right)\right].
\end{aligned}$$

■

Note that, compared to the classic self-duality, the general self-duality theorem involves replacing the risk-neutral measure $\mathbb{P}$ by $\mathbb{Q}$, and the process S by its modified form D. Therefore, in the next result we calculate the relative entropy, that is, a measure of distance of the measure $\mathbb{Q}$ with respect to $\mathbb{P}$. We show in the next proposition that it is determined by the square of correlation ρ and expectation of the cumulative variance.

Proposition 2.20 *The relative entropy $H(\mathbb{Q},\mathbb{P})$ of the measure $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by*

$$H(\mathbb{Q},\mathbb{P}) = \frac{\rho^2}{2} \cdot E^{\mathbb{Q}}\left[\int_0^T \sigma^2(V_s)\,ds\right]$$

Proof. We recall that

$$\begin{aligned}\left.\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right|_{\mathcal{F}_T} &= R'_T = \mathcal{E}\left(\rho\int\sigma(V)\,dW\right)_T \\ &= \exp\left(\rho\int_0^T\sigma(V_s)\,dW_s - \frac{\rho^2}{2}\int_0^T\sigma^2(V_s)\,ds\right)\end{aligned}$$

hence $H(\mathbb{Q},\mathbb{P})$ is given as

$$\begin{aligned}H(\mathbb{Q},\mathbb{P}) &= E^{\mathbb{P}}\left[\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\log\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right] = E^{\mathbb{Q}}\left[\log\frac{d\,\mathbb{Q}}{d\,\mathbb{P}}\right] \\ &= E^{\mathbb{Q}}\left[\rho\int_0^T\sigma(V_s)\,dW_s - \frac{\rho^2}{2}\int_0^T\sigma^2(V_s)\,ds\right] \\ &= E^{\mathbb{Q}}\left[\rho\int_0^T\sigma(V_s)\,dW_s^{\mathbb{Q}} + \frac{\rho^2}{2}\int_0^T\sigma^2(V_s)\,ds\right] \\ &= \frac{\rho^2}{2}\cdot E^{\mathbb{Q}}\left[\int_0^T\sigma^2(V_s)\,ds\right].\end{aligned}$$

■

2.4 Application to barrier options

2.4.1 General results

The general self-duality allows to derive a pricing formula for the barrier option at the barrier. This pricing formula involves a conditional expectation of a claim written on D. However, as D is not a traded asset this does not serve us to build up a semi-static hedge as in the self-dual case. Instead, an alternative hedging approach is derived in the next chapter. Recall that $\tau := \inf\{t : \eta S_t \geq \eta B\}$, where $\eta := sgn(B - S_0)$. In the sequel, G denotes a non-negative Borel function.

Theorem 2.21 *In the stochastic volatility model (2.4), a barrier option with payoff* $G(S_T)1_{(\tau \le T)}$ *can be priced at time* τ *via no-arbitrage by*

$$E_\tau^{\mathbb{P}}\left[e^{-r(T-\tau)}\Gamma^{\mathbb{P}}(S_T)\right] + E_\tau^{\mathbb{Q}}\left[\Gamma_{\tau\wedge T}^{\mathbb{Q}}\left(\frac{D_T}{D_{\tau\wedge T}}\right)\right], \tag{2.8}$$

where $\Gamma^{\mathbb{P}}(S_T)$ *denotes a European option written on* S_T *with*

$$\Gamma^{\mathbb{P}}(S_T) = G(S_T)1_{(\eta S_T \ge \eta B)},$$

and $\Gamma_t^{\mathbb{Q}}\left(\frac{D_T}{D_t}\right)$ *is written on* $\frac{D_T}{D_t}$ *with*

$$\Gamma_t^{\mathbb{Q}}\left(\frac{D_T}{D_t}\right) = \frac{D_T}{D_t} G\left(\frac{B}{D_T/D_t}\right) 1_{(\eta \frac{D_T}{D_t} > \eta)}. \tag{2.9}$$

Proof. (1) If the barrier is never touched, the contract expires worthless, since at maturity,

$$\Gamma^{\mathbb{P}}(S_T) = 0 \text{ as } 1_{(\eta S_T \ge \eta B)} = 0;$$

and

$$\Gamma_T^{\mathbb{Q}}\left(\frac{D_T}{D_T}\right) = 0 \text{ as } 1_{(\eta \cdot 1 > \eta)} = 0.$$

(2) If the barrier knocks in before maturity, i.e. $\tau \le T$, then $E_\tau^{\mathbb{P}}[e^{-r(T-\tau)}G(S_T)]$ would, according to no-arbitrage arguments, be the fair price of the option at the random time τ, with respect to the chosen martingale measure $\mathbb{P}$. We can write

$$E_\tau^{\mathbb{P}}[e^{-r(T-\tau)}G(S_T)] = E_\tau^{\mathbb{P}}[e^{-r(T-\tau)}G(S_T)1_{(\eta S_T \ge \eta B)}] + E_\tau^{\mathbb{P}}[e^{-r(T-\tau)}G(S_T)1_{(\eta S_T < \eta B)}],$$

where, by the general self-duality, we have

$$E_\tau^{\mathbb{P}}[e^{-r(T-\tau)}G(S_T)1_{(\eta S_T < \eta B)}] = E_\tau^{\mathbb{Q}}\left[\frac{D_T}{D_\tau} G\left(B \cdot \frac{D_\tau}{D_T}\right) 1_{(\eta D_\tau < \eta D_T)}\right].$$

■

Remark 2.22 *Therefore, at the time* $\tau \wedge T$*, the price of the barrier option and the sum of prices of the two options as in the statement are the same. Note also that in the claim (2.8) the two indicator functions differ since the first is in terms of* S *whereas the second is* *in terms of D. This is in contrast to the self-dual case as in (2.3). In the event $[\tau \leq T]$, $S_\tau = B$, whereas in the general self-duality theorem (2.23), D_τ is random which introduces a time-dependency in the claim (2.9).*

In important practical cases, the institution which issues the barrier option can set out the terms of the contingent claim such that both the $\Gamma^{\mathbb{P}}(S_T)$-claim in (2.8) as well as the indicator function in (2.9) are absent. Let us, as an example, rephrase Theorem 2.21 as a corollary in the specific case of a down-and-in call option with strike higher than the barrier level.

Corollary 2.23 *In the stochastic volatility model (2.4), a down-and-in call option with payoff*

$$F(S_T) = \ (S_T - K)^+ 1_{(\inf_{t \leq T} S_t \leq B)} \quad , K \geq B,$$

has the same price as a claim $\Gamma^{\mathbb{Q}}_{\tau \wedge T}\left(\frac{D_T}{D_{\tau \wedge T}}\right)$ with

$$\Gamma^{\mathbb{Q}}_{\tau \wedge T}\left(\frac{D_T}{D_{\tau \wedge T}}\right) = K\left(\frac{B}{K} - \frac{D_T}{D_{\tau \wedge T}}\right)^+$$

where

$$\frac{D_T}{D_{\tau \wedge T}} = \mathcal{E}\left(-r dt + \overline{\rho} \int \sigma(V)\, dW^{\perp} - \rho \int \sigma(V)\, dW^{\mathbb{Q}}\right)_{[\tau \wedge T, T]}.$$

Proof. (1) The value of the $\Gamma^{\mathbb{P}}(S_T)$ option is zero as $K \geq B$ yields

$$(S_T - K)^+ 1_{(S_T \leq B)} = 0;$$

(2) For the $\Gamma^{\mathbb{Q}}_t\left(\frac{D_T}{D_t}\right)$ option,

$$\frac{D_T}{D_t} G\left(\frac{B}{D_T/D_t}\right) 1_{(\eta \frac{D_T}{D_t} > \eta)} = K\left(\frac{B}{K} - \frac{D_T}{D_t}\right)^+ 1_{(\frac{D_T}{D_t} < 1)} = K\left(\frac{B}{K} - \frac{D_T}{D_t}\right)^+;$$

(3) If the barrier is never touched, the contract expires worthless since

$$\frac{D_T}{D_T} = 1 \geq \frac{B}{K};$$

(4) If the barrier knocks in before maturity, i.e. $\tau \leq T$, the statement follows by the generalized self-duality theorem as in the proof of Theorem 2.21. ■

In the next part, we apply the general self-duality in pricing double barrier and sequential barrier options. We assume that the interest rate $r = 0$ for the sake of simplicity.

2.4.2 Double Barrier options

We define new stopping times: $\tau_L := \inf\{t : S_t \leq L\}$, and $\tau_U := \inf\{t : S_t \geq U\}$ for $0 < L < S_0 < U$.

Theorem 2.24 *In the stochastic volatility model (2.4), a double-knocked-out barrier option with a bounded payoff $G(S_T)(1 - 1_{(\tau_U \wedge \tau_L \leq T)})$, can be replicated by holding a claim on*

$$\sum_{n=-\infty}^{\infty} \left[\Gamma^{\mathbb{P}}_{(n)}(S_T) - \Gamma^{\mathbb{Q}}_{(n)\,\tau_U \wedge \tau_L \wedge T} \left(\frac{D_T}{D_{\tau_U \wedge \tau_L \wedge T}} \right) \right] \tag{2.10}$$

where we claim that at most one term in the infinite sum is nonzero, and $\Gamma^{\mathbb{P}}_{(n)}(S_T)$ denotes an option written on S_T with

$$\Gamma^{\mathbb{P}}_{(n)}(S_T) = G\left(\frac{U^n}{L^n} S_T\right) 1_{\left(L < \frac{U^n}{L^n} S_T < U\right)},$$

$\Gamma^{\mathbb{Q}}_{(n)\,t}\left(\frac{D_T}{D_t}\right)$ *is an option written on $\frac{D_T}{D_t}$ with*

$$\Gamma^{\mathbb{Q}}_{(n)\,t}\left(\frac{D_T}{D_t}\right) = \frac{D_T}{D_t} G\left(\frac{U^n}{L^{n-1}} \frac{D_t}{D_T}\right) 1_{\left(L < \frac{U^n}{L^{n-1}} \frac{D_t}{D_T} < U\right)}.$$

Proof. If $L < \frac{U^n}{L^n} S_T < U$, then

$$\frac{U^{n-1}}{L^{n-1}} S_T < L \text{ and } \frac{U^{n+1}}{L^{n+1}} S_T > U.$$

Therefore, for each S_T, at most one term of $\Gamma^{\mathbb{P}}_{(n)}(S_T)$ in the infinite sum is nonzero, because it vanishes outside (L, U). And we claim true for the term of $\Gamma^{\mathbb{Q}}_{(n)\,\tau_U \wedge \tau_L \wedge T}\left(\frac{D_T}{D_{\tau_U \wedge \tau_L \wedge T}}\right)$ with the same argument.

Moreover, note that the absolute value of these two nonzero term is bounded since G is assumed to be a bounded function and D is a positive martingale. Therefore, we may freely interchange expectation and summation by Fubini Theorem.

Now we are ready to show our theorem.

First, if $T < (\tau_U \wedge \tau_L)$, i.e., the barrier never knocks out, then the claim expires worth $G(S_T)\,1_{(L<S_T<U)} = G(S_T)$, as desired. While if the barrier knocks out before maturity, then the claim has the zero value. Note that the probability that the process S hits L and U at the same time equals 0.

If $\tau_U \leq (\tau_L \wedge T)$, by the general self-duality 2.19,

$$
\begin{aligned}
& E^{\mathbb{Q}}_{\tau_U}\left[\Gamma^{\mathbb{Q}}_{(n)\,\tau_U\wedge\tau_L\wedge T}\left(\frac{D_T}{D_{\tau_U\wedge\tau_L\wedge T}}\right)\right] \\
= \; & E^{\mathbb{P}}_{\tau_U}\left[G\left(\frac{U^n}{L^{n-1}}\cdot\frac{S_T}{S_{\tau_U}}\right)1_{\left(L<\frac{U^n}{L^{n-1}}\cdot\frac{S_T}{S_{\tau_U}}<U\right)}\right] \\
= \; & E^{\mathbb{P}}_{\tau_U}\left[G\left(\frac{U^{n-1}}{L^{n-1}}\cdot S_T\right)1_{\left(L<\frac{U^{n-1}}{L^{n-1}}\cdot S_T<U\right)}\right] \\
= \; & E^{\mathbb{P}}_{\tau_U}\left[\Gamma^{\mathbb{P}}_{(n-1)}(S_T)\right]
\end{aligned}
$$

if $\tau_L \leq (\tau_U \wedge T)$, by the general self-duality 2.19,

$$
\begin{aligned}
& E^{\mathbb{Q}}_{\tau_L}\left[\Gamma^{\mathbb{Q}}_{(n)\,\tau_U\wedge\tau_L\wedge T}\left(\frac{D_T}{D_{\tau_U\wedge\tau_L\wedge T}}\right)\right] \\
= \; & E^{\mathbb{P}}_{\tau_L}\left[G\left(\frac{U^n}{L^{n-1}}\cdot\frac{S_T}{S_{\tau_L}}\right)1_{\left(L<\frac{U^n}{L^{n-1}}\cdot\frac{S_T}{S_{\tau_L}}<U\right)}\right] \\
= \; & E^{\mathbb{P}}_{\tau_L}\left[G\left(\frac{U^n}{L^n}\cdot S_T\right)1_{\left(L<\frac{U^n}{L^n}\cdot S_T<U\right)}\right] \\
= \; & E^{\mathbb{P}}_{\tau_L}\left[\Gamma^{\mathbb{P}}_{(n)}(S_T)\right].
\end{aligned}
$$

therefore, in both cases,

$$
\sum_{n=-\infty}^{\infty}\left[\Gamma^{\mathbb{P}}_{(n)}(S_T) - \Gamma^{\mathbb{Q}}_{(n)\,\tau_U\wedge\tau_L\wedge T}\left(\frac{D_T}{D_{\tau_U\wedge\tau_L\wedge T}}\right)\right] = 0
$$

■

2.4.3 Sequential Barrier options

We recall that $\tau_U := \inf\{t : S_t \geq U\}$ and define $\tau_{UL} := \inf\{t \geq \tau_U : S_t \leq L\}$ for $0 < L < S_0 < U$.

Theorem 2.25 *In the stochastic volatility model (2.4), a up-and-in down-and-out sequential barrier option with payoff* $G(S_T)1_{(\tau_U \leq T)}1_{(\tau_{UL} > T)}$*, can be priced via no arbitrage by*

$$\Gamma^{*\mathbb{P}}(S_T) + \Gamma^{*\mathbb{Q}}_{\tau_U \wedge T}\left(\frac{D_T}{D_{\tau_U \wedge T}}\right) \tag{2.11}$$

where $\Gamma^{*\mathbb{P}}(S_T)$ *denotes an option written on* S_T *with*

$$\Gamma^{*\mathbb{P}}(S_T) = G(S_T)1_{(S_T \geq U)},$$

and $\Gamma^{*\mathbb{Q}}_t\left(\frac{D_T}{D_t}\right)$ *is an option written on* $\frac{D_T}{D_t}$ *with*

$$\Gamma^{*\mathbb{Q}}_t\left(\frac{D_T}{D_t}\right) = \frac{D_T}{D_t} G\left(\frac{U}{D_T/D_t}\right) 1_{(D_T > D_t)}.$$

If and when the upper barrier U *knocks in, convert these claims to* $\Gamma^{\mathbb{P}}(S_T) + \Gamma^{\mathbb{Q}}_{\tau_{UL} \wedge T}\left(\frac{D_T}{D_{\tau_{UL} \wedge T}}\right)$ *claims with zero cost, where* $\Gamma^{\mathbb{P}}(S_T)$ *denotes an option written on* S_T *with*

$$\Gamma^{\mathbb{P}}(S_T) = G(S_T) - G(S_T)1_{(S_T \leq L)},$$

and $\Gamma^{\mathbb{Q}}_t\left(\frac{D_T}{D_t}\right)$ *is an option written on* $\frac{D_T}{D_t}$ *with*

$$\Gamma^{\mathbb{Q}}_t\left(\frac{D_T}{D_t}\right) = -\frac{D_T}{D_t} G\left(\frac{L}{D_T/D_t}\right) 1_{(D_T < D_t)}.$$

Then if and when the lower barrier L *knocks in, sell these claims, at zero cost.*

Proof. (1) If $\tau_U > T$, at maturity,

$$\Gamma^{*\mathbb{P}}(S_T) + \Gamma^{*\mathbb{Q}}_T\left(\frac{D_T}{D_T}\right) = 0$$

as desired. Or, at time $\tau_U < T$,

$$
\begin{aligned}
& E^{\mathbb{P}}_{\tau_U}\left[\Gamma^{*\mathbb{P}}(S_T)\right] + E^{\mathbb{Q}}_{\tau_U}\left[\Gamma^{*\mathbb{Q}}_{\tau_U \wedge T}\left(\frac{D_T}{D_{\tau_U \wedge T}}\right)\right] \\
= \;& E^{\mathbb{P}}_{\tau_U}\left[G(S_T) 1_{(\tau_{UL} > T)}\right] \\
= \;& E^{\mathbb{P}}_{\tau_U}\left[\Gamma^{\mathbb{P}}(S_T)\right] + E^{\mathbb{Q}}_{\tau_U}\left[\Gamma^{\mathbb{Q}}_{\tau_{UL} \wedge T}\left(\frac{D_T}{D_{\tau_{UL} \wedge T}}\right)\right]
\end{aligned}
$$

by Theorem 2.21 and the fact that the value of a knock-in option equals the difference between a vanilla and knock-out.

(2) Suppose $\tau_U < T$, and we have converted the claims at time τ_U, then if $\tau_{UL} > T$, at maturity,

$$
\Gamma^{\mathbb{P}}(S_T) + \Gamma^{\mathbb{Q}}_T\left(\frac{D_T}{D_T}\right) = G(S_T)
$$

as desired. If $\tau_{UL} < T$, at time τ_{UL},

$$
\begin{aligned}
& E^{\mathbb{P}}_{\tau_{UL}}\left[\Gamma^{\mathbb{P}}(S_T)\right] + E^{\mathbb{Q}}_{\tau_{UL}}\left[\Gamma^{\mathbb{Q}}_{\tau_{UL}}\left(\frac{D_T}{D_{\tau_{UL}}}\right)\right] \\
= \;& E^{\mathbb{P}}_{\tau_U}\left[G(S_T)\right] - E^{\mathbb{P}}_{\tau_U}\left[G(S_T)\right] \\
= \;& 0
\end{aligned}
$$

by Theorem 2.21. ■

Chapter 3

A decomposition approach of pricing and hedging barrier options

3.1 Introduction

In this chapter, we derive the price of Γ_τ by Malliavin calculus, in particular the Clark-Ocone formula. In a stochastic volatility context, this necessarily involves higher Greeks. Such an approach has been pioneered for European options in the Heston model in [1]. Here we adapt this approach to our specific situation, i.e. hedging of a time-dependent put option written on the modified price process under the measure $\mathbb{Q}$, and generalise it to our general stochastic volatility framework. Moreover, we derive an alternative approach of hedging the barrier options: the main risk is semi-static hedged by holding a position in put options written on the stock, and the remaining risk is then dynamically replicated by trading in the realized volatility.

A related paper is [43] which considers locally risk-minimizing hedging (see [38] for this concept) for general contingent claims. In particular, this is applied to barrier options in a stochastic volatility model with correlation. The main difference to our approach is that in [43] the underlying price process is used as hedging instrument. This leaves some remaining risk as the market is incomplete. In contrast we achieve perfect replication, however have to trade in addition with realized as well as cumulative volatility where it has to be seen how practically feasible this is.

3.2 A decomposition of option prices

This section is a joint work with Prof. Elisa Alòs and Prof. Thorsten Rheinländer.

The goal of this section is to construct a dynamic hedging portfolio for the claim

$$E_t^{\mathbb{Q}}\left[K\left(\frac{B}{K}-\frac{D_T}{D_t}\right)^+\right].$$

In particular, we aim for a decomposition into Black-Scholes, leverage and volatility of volatility terms.

For the sake of simplicity, we firstly work on a plain vanilla put options written on S with payoff

$$G(S_T) = (K - S_T)^+ .$$

Remark 3.1 *Recall that the dynamic of modified price process D under measure $\mathbb{Q}$ is*

$$dD = D\left(-rdt - \rho\,\sigma(V)\,dW^{\mathbb{Q}} + \overline{\rho}\,\sigma(V)\,dW^{\perp}\right)$$

and price process S under $\mathbb{P}$ is

$$dS = S\left(\,rdt + \rho\,\sigma(V)\,dW + \overline{\rho}\,\sigma(V)\,dW^{\perp}\right)$$

where we notice that by changing the sign of the interest rate r, correlation ρ, and the corresponding strike, we arrive at the decomposition and approximation formula for the claim

$$E_t^{\mathbb{Q}}\left[K\left(\frac{B}{K}-\frac{D_T}{D_t}\right)^+\right].$$

We assume that the reader is familiar with the basic results of Malliavin calculus, as given for instance in [28]. Given a standard Brownian motion $W = \{W_t,\, t \in [0,T]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, the set $\mathbb{D}_W^{1,2}$ will denote the domain of the derivative operator D^W. It is well-known that $\mathbb{D}_W^{1,2}$ is a dense subset of $L^2(\Omega)$ and that D^W is a closed and unbounded operator from $L^2(\Omega)$ to $L^2([0,T]\times\Omega)$. We denote $\mathbb{L}_W^{1,2} := L^2([0,T];\mathbb{D}_W^{1,2})$.

Let us moreover fix some notation which we will use in the sequel.

- In the setting of the stochastic volatility model (2.4), we will assume the process $\sigma^2 = \sigma^2(V)$ to be square-integrable and adapted to the filtration generated by the Brownian motion W.

- $P_{BS}(t, x, \sigma)$ denotes the classical vanilla Black-Scholes put option price with initial log-stock price equal to x, strike equal to K at time t.

- $$d_{\pm} := \frac{x - \ln(K) + r(T-t)}{\sigma\sqrt{T-t}} \pm \frac{\sigma}{2}\sqrt{T-t}.$$

- $v_t^2 = \frac{1}{T-t}\int_t^T E^{\mathbb{P}}\left[\left.\sigma_s^2\right|\mathcal{F}_t\right] ds$. That is, v_t^2 denotes the squared time future average volatility.

- $N_t = \int_0^T E^{\mathbb{P}}\left[\left.\sigma_s^2\right|\mathcal{F}_t\right] ds$. Note that $v_t^2 = \frac{1}{T-t}\left(N_t - \int_0^t \sigma_s^2\, ds\right)$. By the martingale representation formula, for every fixed $s > t$, $E^{\mathbb{P}}\left[\left.\sigma_s^2\right|\mathcal{F}_s\right] = E^{\mathbb{P}}\left[\left.\sigma_s^2\right|\mathcal{F}_0\right] + \int_0^s m(s,a)\, dW_a$, for some adapted and square-integrable process $m(s,\cdot)$. In the particular case when $E^{\mathbb{P}}\left[\left.\sigma_s^2\right|\mathcal{F}_s\right] \in \mathbb{D}_W^{1,2}$, for each $s \in [0,T]$, $m(s,a)$ can easily be computed by the Clark-Ocone formula as $m(s,a) = E^{\mathbb{P}}\left[\left.D_a^W \sigma_s^2\right|\mathcal{F}_a\right]$. Then we deduce with stochastic Fubini that $dN_t = \left(\int_t^T m(a,t)da\right) dW_t$.

- $\mathcal{L}_{BS}$ denotes the classical Black-Scholes operator.

- For all $t < T$, V_t denotes the value at time t of a put option with payoff

$$G(S_T) = (K - S_T)^+ .$$

Furthermore, we will use the following result, similar to Lemma 2.1 in [1].

Lemma 3.2 *Let $0 \le t \le s \le T$. Then for every $n \ge 0$, there exists $C = C(n, \rho)$ such that*

$$\left|\partial_x^n G_{BS}(s, X_s, v_s)\right| \le C\left(\int_s^T E^{\mathbb{P}}\left[\left.\sigma_\theta^2\right|\mathcal{F}_s\right] d\theta\right)^{-\frac{1}{2}(n+1)},$$

where $G_{BS}(s, X_s, v_s) := \left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x}\right) P_{BS}(s, X_s, v_s)$.

Now the decomposition can be stated.

Theorem 3.3 *Assume that, for all* $t < T$*,*

$$E^{\mathbb{P}}\left[\int_t^T e^{-r(s-t)}\left|H\left(s,X_s,v_s\right)\sigma_s m(s)\right|\,ds\right|\mathcal{F}_t\right]$$

$$+E^{\mathbb{P}}\left[\int_t^T e^{-r(s-t)}\left|J\left(s,X_s,v_s\right)m^2(s)\right|\,ds\right|\mathcal{F}_t\right]<\infty, \tag{3.1}$$

where

$$X_s := \ln S_s,$$

$$H\left(s,X_s,v_s\right) := \frac{\partial G_{BS}}{\partial x}\left(s,X_s,v_s\right)$$

and

$$J\left(s,X_s,v_s\right) := \left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial x}\right)G_{BS}\left(s,X_s,v_s\right).$$

Then, it follows that

$$\begin{aligned} V_t &= P_{BS}\left(t,X_t,v_t\right) \\ &+\frac{\rho}{2}E^{\mathbb{P}}\left[\int_t^T e^{-r(s-t)}H\left(s,X_s,v_s\right)\sigma_s\,d\left\langle N,W\right\rangle_s\right|\mathcal{F}_t\right] \\ &+\frac{1}{8}E^{\mathbb{P}}\left[\int_t^T e^{-r(s-t)}J\left(s,X_s,v_s\right)\,d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right]. \end{aligned} \tag{3.2}$$

Proof. For fixed $t < T$, we recall

$$V_t = e^{-r(T-t)}E^{\mathbb{P}}\left[G(S_T)\right|\mathcal{F}_t\right].$$

Note that $P_{BS}\left(T,X_T;v_T\right)=V_T$. As V_t is an $\mathbb{P}-$martingale we can then write

$$V_t = e^{-r(T-t)}E^{\mathbb{P}}\left[P_{BS}\left(T,X_T^t;v_T\right)\right|\mathcal{F}_t\right]. \tag{3.3}$$

The remainder of the proof translates verbally from the proof of Theorem 2.1 in [1] to our situation. ■

Remark 3.4 *The proof of the above theorem uses only some integrability and regularity conditions of the volatility process. The volatility process is neither assumed to be Markovian* *nor a diffusion process. Note that, by Lemma 3.2, Condition (3.1) is clearly satisfied if, for example, the volatility process is bounded. Moreover, we can see it is also satisfied by the Heston volatility model when we assume the classical positivity condition (see for example Alòs (2012) [1]).*

3.2.1 An approximation formula for option prices

By freezing the terms $e^{-r(s-t)}H\left(s, X_s, v_s\right)$ and $e^{-r(s-t)}J\left(s, X_s, v_s\right)$ at time t in the expression (3.2), we obtain the following approximation formula for our option price:

$$\begin{aligned} V_t \approx\ & P_{BS}\left(t, X_t, v_t\right) \\ & + \frac{\rho}{2} H\left(t, X_t, v_t\right) E^{\mathbb{P}}\left[\left.\int_t^T \sigma_s \, d\left\langle N, W\right\rangle_s \right| \mathcal{F}_t\right] \\ & + \frac{1}{8} J\left(t, X_t, v_t\right) E^{\mathbb{P}}\left[\left.\int_t^T d\left\langle N, N\right\rangle_s \right| \mathcal{F}_t\right]. \end{aligned} \tag{3.4}$$

Note that, in the above equation, $H\left(t, X_t, v_t\right)$ and $J\left(t, X_t, v_t\right)$ are model-independent and can be written explicitly as:

$$\begin{aligned} H\left(t, X_t, v_t\right) &= \frac{e^{X_t}}{v_t\sqrt{2\pi\left(T-t\right)}} \exp\left(-\frac{d_+^2}{2}\right)\left(1 - \frac{d_+}{v_t\sqrt{T-t}}\right) \\ &= \frac{e^{X_t}}{v_t^2\left(T-t\right)\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right)\left(-d_-\right) \end{aligned}$$

and

$$\begin{aligned} J\left(t, X_t, v_t\right) &= \frac{e^{X_t}}{v_t\sqrt{2\pi\left(T-t\right)}} \exp\left(-\frac{d_+^2}{2}\right)\left[\left(-\frac{d_+}{v_t\sqrt{T-t}} + \frac{d_+^2}{v_t^2\left(T-t\right)}\right)\right. \\ & \left. - \frac{1}{v_t^2\left(T-t\right)}\right] \\ &= \frac{e^{X_t}}{\left(v_t\sqrt{T-t}\right)^3\sqrt{2\pi}} \exp\left(-\frac{d_+^2}{2}\right)\left(d_+d_- - 1\right) \end{aligned}$$

Moreover, the quantities

$$E^{\mathbb{P}}\left[\left.\int_t^T \sigma_s \, d\left\langle N, W\right\rangle_s \right| \mathcal{F}_t\right]$$

and

$$E^{\mathbb{P}}\left[\left.\int_t^T d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right]$$

depend on the chosen stochastic volatility model.

Theorem 3.5 *Under the assumptions of model (2.4), assume that the processes σ and m are bounded. Then, for all $t\in[0,T]$ there exists a constant C such that*

$$\begin{aligned}&\left|V_t-P_{BS}\left(t,X_t;v_t\right)-\frac{\rho}{2}H\left(t,X_t,v_t\right)E^{\mathbb{P}}\left[\left.\int_t^T\sigma_s\,d\left\langle N,W\right\rangle_s\right|\mathcal{F}_t\right]\right.\\&\left.-\frac{1}{8}J\left(t,X_t,v_t\right)E^{\mathbb{P}}\left[\left.\int_t^T d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right]\right|\\&\leq C\left(\left|\rho\right|\left(T-t\right)^{\frac{3}{2}}+\left(T-t\right)^2+\left(T-t\right)^{\frac{5}{2}}\right).\end{aligned}\tag{3.5}$$

Proof. Consider the process $e^{-rt}H\left(t,X_t;v_t\right)U_t+e^{-rt}J\left(t,X_t;v_t\right)I_t$, where

$$U_t:=\frac{1}{8}E^{\mathbb{P}}\left[\left.\int_t^T\sigma_s\,d\left\langle N,W\right\rangle_s\right|\mathcal{F}_t\right]$$

and

$$I_t:=\frac{1}{8}E^{\mathbb{P}}\left[\left.\int_t^T d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right].$$

It is easy to check that

$$H\left(T,X_T;v_T\right)U_T+J\left(T,X_T;v_T\right)I_T=0.$$

Again, the same arguments as in the proof of Theorem 3.7 in [1] allow us to write

$$\begin{aligned}V_t&=P_{BS}\left(t,X_t;v_t\right)+H\left(t,X_t;v_t\right)U_t+J\left(t,X_t;v_t\right)I_t\\&+\frac{\rho}{2}E^{\mathbb{P}}\left[\left.\int_t^T e^{-r(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)H\left(s,X_s,v_s\right)U_s\sigma_s\,d\left\langle N,W\right\rangle_s\right|\mathcal{F}_t\right]\\&+\frac{1}{8}E^{\mathbb{P}}\left[\left.\int_t^T e^{-r(s-t)}\left(\frac{\partial^4}{\partial x^4}-2\frac{\partial^3}{\partial x^3}+\frac{\partial^2}{\partial x^2}\right)H\left(s,X_s,v_s\right)U_s\,d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right]\\&+\frac{\rho}{2}E^{\mathbb{P}}\left[\left.\int_t^T e^{-r(s-t)}\left(\frac{\partial^3}{\partial x^3}-\frac{\partial^2}{\partial x^2}\right)J\left(s,X_s,v_s\right)I_s\sigma_s\,d\left\langle N,W\right\rangle_s\right|\mathcal{F}_t\right]\\&+\frac{1}{8}E^{\mathbb{P}}\left[\left.\int_t^T e^{-r(s-t)}\left(\frac{\partial^4}{\partial x^4}-2\frac{\partial^3}{\partial x^3}+\frac{\partial^2}{\partial x^2}\right)J\left(s,X_s,v_s\right)I_s\,d\left\langle N,N\right\rangle_s\right|\mathcal{F}_t\right]\end{aligned}$$

$$
\begin{aligned}
&=: P_{BS}\left(t, X_t; v_t\right) + e^{-rt} H\left(t, X_t; v_t\right) U_t + e^{-rt} J\left(t, X_t; v_t\right) I_t + \\
&+ T_1 + T_2 + T_3 + T_4.
\end{aligned}
$$

Note that, by Lemma 3.2 and the fact that σ is a bounded process,

$$
\begin{aligned}
T_1 &= \frac{\rho}{2} E^{\mathbb{P}} \left[\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s\right) U_s \sigma_s \, d\left\langle N, W\right\rangle_s \Big| \mathcal{F}_t \right] \\
&\leq C \left|\rho\right| E^{\mathbb{P}} \left[\int_t^T (T-s)^{-5/2} U_s \, d\left\langle N, W\right\rangle_s \Big| \mathcal{F}_t \right].
\end{aligned}
$$

As

$$
U_t := \frac{1}{8} E^{\mathbb{P}} \left[\int_t^T \sigma_s \left(\int_s^T m(r,s) dr \right) ds \Big| \mathcal{F}_t \right],
$$

and

$$
d\left\langle N, W\right\rangle_s = \left(\int_s^T m(a,s) da \right) ds,
$$

it follows that

$$
\begin{aligned}
T_1 \quad \leq \quad & C \left|\rho\right| E^{\mathbb{P}} \left[\int_t^T (T-s)^{-5/2} E^{\mathbb{P}} \left[\int_s^T \left(\int_a^T m(u,a)\, du \right) da \Big| \mathcal{F}_s \right] \right. \\
& \left. \left(\int_s^T m(a,s)\, da \right) ds \Big| \mathcal{F}_t \right],
\end{aligned}
$$

which implies that

$$
T_1 \leq C \left|\rho\right| (T-t)^{\frac{3}{2}}.
$$

In a similar way

$$
\begin{aligned}
T_2 &= \frac{1}{8} E^{\mathbb{P}} \left[\int_t^T e^{-r(s-t)} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2} \right) H\left(s, X_s, v_s\right) U_s \, d\left\langle N, N\right\rangle_s \Big| \mathcal{F}_t \right] \\
&\leq C E^{\mathbb{P}} \left[\int_t^T (T-s)^{-3} E^{\mathbb{P}} \left[\int_s^T \left(\int_a^T m(u,a)\, du \right) da \Big| \mathcal{F}_s \right] \left(\int_s^T m(a,s) da \right)^2 ds \Big| \mathcal{F}_t \right] \\
&\leq C (T-t)^2,
\end{aligned}
$$

$$
T_3 = \frac{\rho}{2} E^{\mathbb{P}} \left[\int_t^T e^{-r(s-t)} \left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) J\left(s, X_s, v_s\right) I_s \sigma_s d\left\langle N, W\right\rangle_s \Big| \mathcal{F}_t \right]
$$

$$\leq C\left|\rho\right| E^{\mathbb{P}}\left[\int_t^T (T-s)^{-3} E^{\mathbb{P}}\left[\int_s^T \left(\int_a^T m(u,a)du\right)^2 dr \middle| \mathcal{F}_s\right] \left(\int_s^T m(a,s)da\right) ds \middle| \mathcal{F}_t\right]$$
$$\leq C\left|\rho\right| (T-t)^2$$

and

$$T_4 = \frac{1}{8} E^{\mathbb{P}}\left[\int_t^T e^{-r(s-t)} \left(\frac{\partial^4}{\partial x^4} - 2\frac{\partial^3}{\partial x^3} + \frac{\partial^2}{\partial x^2}\right) J\left(s, X_s, v_s\right) I_s\, d\left\langle N, N\right\rangle_s \middle| \mathcal{F}_t\right]$$
$$\leq C E^{\mathbb{P}}\left[\int_t^T (T-s)^{-7/2} E^{\mathbb{P}}\left[\int_s^T \left(\int_a^T m(u,a)\,du\right)^2 da \middle| \mathcal{F}_s\right] \left(\int_s^T m(a,s)\,da\right)^2 ds \middle| \mathcal{F}_t\right]$$
$$\leq C(T-t)^{\frac{5}{2}}.$$

■

Remark 3.6 *Once again, the proof of the above theorem uses only some integrability and regularity conditions of the volatility process and similar bounds can be proved under some different hypotheses.*

Theorem 3.7 *Assume the model (2.4), where the process $V \in \mathbb{L}_W^{1,2}$ and the functions $\sigma, \mu, \gamma \in \mathcal{C}_b^1$. Then, for all $t \in [0,T]$ we have that*

$$\begin{aligned}
&d\left\langle N, W\right\rangle_t \\
&= \left(\int_t^T E^{\mathbb{P}}\left[D_t^W \sigma_a^2 \middle| \mathcal{F}_t\right] da\right) dt \\
&= \left(\int_t^T 2\,\gamma(V_t)\, E^{\mathbb{P}}\left[\sigma(V_a)\,\sigma'(V_a) \exp\left[\int_t^a \left(\mu'(V_s) - \frac{1}{2}\ \left(\gamma'(V_s)\right)^2\right) ds \right.\right.\right. \\
&\left.\left.\left. + \int_t^a \gamma'(V_s) dW_s\right] \middle| \mathcal{F}_t\right] da\right) dt
\end{aligned}$$

and

$$\begin{aligned}
&d\left\langle N, N\right\rangle_t \\
&= \left(\int_t^T E^{\mathbb{P}}\left[D_t^W \sigma_a^2 \middle| \mathcal{F}_t\right] da\right)^2 dt
\end{aligned}$$

$$= \left(\int_t^T 2\gamma(V_t) E \left[^{\mathbb{P}} \sigma(V_a)\, \sigma'(V_a) \left[\exp \int_t^a \left(\mu'(V_s) - \frac{1}{2}\ \left(\gamma'(V_s)\right)^2 \right) ds \right.\right.\right.$$
$$\left.\left.\left. + \int_t^a \gamma'(V_s) dW_s \right] \middle| \mathcal{F}_t \right] da \right)^2 dt.$$

Proof. Recall that

$$dN_t = \left(\int_t^T m(a,t) da \right) dW_t, \tag{3.6}$$

where $m(a,t)$ is the process appearing in the martingale representation for the volatility

$$\sigma_s^2 = \sigma_0^2 + \int_0^s m(s,a)\, dW_a.$$

Therefore,

$$d\left\langle N, W \right\rangle_t = \left(\int_t^T m(a,t)\, da \right) dt$$

and

$$d\left\langle N, N \right\rangle_t = \left(\int_t^T m(a,t)\, da \right)^2 dt.$$

Hence, we are interested in the computation of $m(a,t)$. By the Clark-Ocone formula, we know that

$$m(a,t) = E^{\mathbb{P}} \left[D_t^W \sigma_a^2 \middle| \mathcal{F}_t \right].$$

Our goal is to compute

$$D_t^W \sigma_a^2.$$

In the model (2.4), we assume that V follows a SDE of the form

$$dV_a = \mu(V_a)\ dt + \gamma(V_a)\, dW.$$

Then,

$$D_t^W \sigma_a^2 = 2\sigma(V_a)\sigma'(V_a) D_t^W V_a.$$

Next, we compute $D_t^W V_a$. We have

$$V_a = V_0 + \int_0^a \mu(V_s)\, ds + \int_0^a \gamma(V_s)\, dW_s.$$

As we assume that $\mu(V), \gamma(V) \in \mathcal{C}_b^1$, we can see that the Malliavin derivative $D_t^W V_a$ should satisfy

$$D_t^W V_a = \int_t^a \mu'(V_s)\ D_t^W V_s\, ds + \gamma(V_t) + \int_t^a \gamma'(V_s) D_t^W V_s\, dW_s.$$

This equation is linear in $D_t^W V_s$, hence

$$D_t^W V_a = \gamma(V_t) \exp\left[\int_t^a \left(\mu'(V_s) - \frac{1}{2}\ (\gamma'(V_s))^2\right) ds + \int_t^a \gamma'(V_s) dW_s\right],$$

which gives that

$$\begin{aligned}
& m(a,t) \\
& = E^{\mathbb{P}}\left[D_t^W \sigma_a^2 \middle| \mathcal{F}_t\right] \\
& = 2\gamma(V_t) E^{\mathbb{P}}\left[\sigma(V_a)\, \sigma'(V_a) \exp\left(\int_t^a \left(\mu'(V_s) - \frac{1}{2}\ (\gamma'(V_s))^2\right) ds + \right.\right. \\
& \left.\left. \int_t^a \gamma'(V_s) dW_s\right) \middle| \mathcal{F}_t\right],
\end{aligned}$$

and completes the proof. ■

3.3 Decomposition formula

To illustrate the performance of the decomposition formula, we work on the following examples.

3.3.1 Hull and White model

Assume that the volatility process is given by $\sigma(V_t) = \sqrt{V_t}$, where V_t is of the form:

$$dV_t = V_t\, (\mu\ dt + \gamma\, dW_t),$$

and

$$dS_t = S_t \left(r\,dt + \sqrt{V_t}\,dZ_t \right) ,$$

for $t \in [0,T]$, $r \geq 0$, $\mu \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Theorem 3.8 *Given $t \leq T$, the value at time t of a put option with payoff*

$$G(S_T) = \ (K - S_T)^+ \quad ,$$

approximately equals to

$$P_{BS}(t, \ln S_t;\ K, T;\ v_t) + \rho\gamma \cdot \frac{8V_t^{3/2} \cdot S_t \cdot e^{-\frac{d_+^2}{2}} \cdot d_-}{\mu\sqrt{2\pi}v_t^2\,(T-t)} \cdot$$
$$\left[\frac{\exp\left(\frac{3}{2}\mu\,(T-t) + \frac{9}{8}\gamma^2\,(T-t)\right) - 1}{3\,(3\gamma^2 + 4\mu)} + \frac{\exp\left(-\frac{9}{8}\gamma^2 t - \frac{3}{2}\mu t + \mu T\right)}{9\gamma^2 + 4\mu} \right.$$
$$\left. \cdot \left(e^{\frac{9}{8}\gamma^2 t + \frac{1}{2}\mu t} - e^{\frac{9}{8}\gamma^2 T + \frac{1}{2}\mu T} \right) \right] - \frac{\gamma^2}{8} \cdot \frac{S_t \cdot e^{-\frac{d_+^2}{2}} \cdot (d_+ d_- - 1)}{\sqrt{2\pi}(v_t\sqrt{T-t})^{3/2}} \cdot V_t^2$$
$$\cdot \left[\frac{\gamma^2\left(1 - e^{\mu(T-t)}\right)\left(1 - 3e^{\mu(T-t)}\right)}{\mu(3\gamma^2+\mu)(3\gamma^2+2\mu)} + \frac{3\gamma^4\left(1 - e^{\mu(T-t)}\right)^2}{\mu^2(3\gamma^2+\mu)(3\gamma^2+2\mu)} + \right.$$
$$\left. \frac{2\,e^{2\mu(T-t)}\left(1 - e^{3\gamma^2(T-t)}\right)}{3\,(3\gamma^2+\mu)(3\gamma^2+2\mu)} \right]$$

where

$$\begin{aligned} v_t &= \sqrt{\frac{1}{T-t} E^{\mathbb{P}} \left[\int_t^T V_s ds \middle| \mathcal{F}_t \right]}, \\ d_+ &= \frac{\ln \frac{S_t}{K} + (T-t)\left(\frac{1}{2}v_t^2 + r\right)}{v_t\sqrt{T-t}},\ d_- = d_+ - v_t\sqrt{T-t}. \end{aligned}$$

Proof. Theorem 3.7 gives us that

$$m(a,t) \ = \ 2\gamma V_t\, E^{\mathbb{P}} \left(\frac{1}{2} \exp\left[\int_t^a \left(\mu - \frac{1}{2}\,\gamma^2\right) ds + \int_t^a \gamma\, dW_s \right] \middle| \mathcal{F}_t \right)$$

$$= \quad \gamma V_t \, e^{\mu\,(a-t)} \quad ,$$

$$\begin{aligned} d\left\langle N, W\right\rangle_t &= \left(\int_t^T m(a,t)\, da\right) dt \\ &= \left(\frac{\gamma V_t}{\mu}\left(e^{\mu\,(T-t)} - 1\right)\right) dt \end{aligned}$$

and

$$\begin{aligned} d\left\langle N, N\right\rangle_t &= \left(\int_t^T m(a,t)\, dr\right)^2 dt \\ &= \left(\frac{\gamma V_t}{\mu}\left(e^{\mu\,(T-t)} - 1\right)\right)^2 dt. \end{aligned}$$

hence

$$\begin{aligned} & E^{\mathbb{P}}\left(\int_t^T \sigma_s \, d\left\langle N, W\right\rangle_s \bigg| \mathcal{F}_t\right) \\ = \; & E^{\mathbb{P}}\left(\int_t^T \sqrt{V_s}\left(\frac{\gamma V_s}{\mu}\left(e^{\mu\,(T-s)} - 1\right)\right) ds \bigg| \mathcal{F}_t\right) \\ = \; & \frac{\gamma}{\mu}\int_t^T E^{\mathbb{P}}\left[V_s^{3/2}\big| \mathcal{F}_t\right]\,\left(e^{\mu\,(T-s)} - 1\right)\, ds \\ = \; & \frac{\gamma V_t^{3/2}}{\mu}\int_t^T e^{\frac{3}{2}\left(\mu + \frac{3\gamma^2}{4}\right)(s-t)}\,\left(e^{\mu\,(T-s)} - 1\right)\, ds \\ = \; & -\frac{8\gamma V_t^{3/2}}{\mu}\left(\frac{\exp\left(\frac{3}{2}\mu\,(T-t) + \frac{9}{8}\gamma^2\,(T-t)\right) - 1}{3\,(3\gamma^2 + 4\mu)}\right. \\ & \left. + \frac{\exp\left(-\frac{9}{8}\gamma^2 t - \frac{3}{2}\mu t + \mu T\right)\left(e^{\frac{9}{8}\gamma^2 t + \frac{1}{2}\mu t} - e^{\frac{9}{8}\gamma^2 T + \frac{1}{2}\mu T}\right)}{9\gamma^2 + 4\mu}\right) \end{aligned}$$

since by Itō's formula

$$dV_s^{3/2} \;=\; \frac{3}{2}\sqrt{V_s} dV_s + \frac{3}{8}\frac{1}{\sqrt{V_s}} d\left[V\right]_s$$

$$
\begin{aligned}
&= \frac{3}{2} V_s^{3/2} \left(\left(\mu + \frac{\gamma^2}{4} \right) ds + \gamma dW_s \right) \\
&= \frac{3}{2} V_s^{3/2} \left(\left(\gamma dW_s - \frac{\gamma^2}{2} ds \right) + \left(\mu + \frac{3\gamma^2}{4} \right) ds \right)
\end{aligned}
$$

and

$$
\begin{aligned}
& E^{\mathbb{P}} \left(\int_t^T d \langle N, N \rangle_s \bigg| \mathcal{F}_t \right) \\
= \;& E^{\mathbb{P}} \left(\int_t^T \left(\frac{\gamma V_s}{\mu} \left(e^{\mu (T-s)} - 1 \right) \right)^2 ds \bigg| \mathcal{F}_t \right) \\
= \;& \frac{\gamma^2}{\mu^2} \int_t^T E^{\mathbb{P}} \left(V_s^2 \big| \mathcal{F}_t \right) \cdot \left(e^{\mu (T-s)} - 1 \right)^2 ds \\
= \;& \frac{\gamma^2}{\mu^2} V_t^2 \int_t^T e^{\left(2\mu + 3\gamma^2\right)(s-t)} \cdot \left(e^{\mu (T-s)} - 1 \right)^2 ds \\
= \;& -V_t^2 \left(\frac{\gamma^2 \left(1 - e^{\mu(T-t)}\right) \left(1 - 3e^{\mu(T-t)}\right)}{\mu(3\gamma^2 + \mu)(3\gamma^2 + 2\mu)} + \frac{3\gamma^4 \left(1 - e^{\mu(T-t)}\right)^2}{\mu^2(3\gamma^2 + \mu)(3\gamma^2 + 2\mu)} \right. \\
& \left. + \frac{2\, e^{2\mu(T-t)} \left(1 - e^{3\gamma^2(T-t)}\right)}{3\,(3\gamma^2 + \mu)(3\gamma^2 + 2\mu)} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
dV_s^2 &= 2 V_s dV_s + d\,[V]_s \\
&= V_s^2 \left(\left(2\mu + \gamma^2 \right) ds + 2\gamma dW_s \right) \\
&= V_s^2 \left(\left(2\mu + 3\gamma^2 \right) ds + \left(2\gamma dW_s - 2\gamma^2 ds \right) \right).
\end{aligned}
$$

■

3.3.2 Stein & Stein model

Assume that the volatility process is given by $\sigma(V_t) = V_t$, where V_t is a mean-reverting OU process of the form:

$$dV_t = \kappa(\theta - V_t)\ dt + \gamma\, dW_t,$$

and

$$dS_t = S_t\ (r\, dt + V_t\, dZ_t)\ , \tag{3.7}$$

for $t \in [0,T]$, $r \geq 0$, $\kappa \geq 0$, $\theta \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Theorem 3.9 *Given $t \leq T$, the value at time t of a put option with payoff*

$$G(S_T) = \ (K - S_T)^+ \quad ,$$

approximately equals to

$$\begin{aligned}
&P_{BS}(t, \ln S_t;\ K, T;\ v_t) - \rho\gamma \cdot \frac{S_t \cdot \exp\left(-\frac{d_+^2}{2}\right) \cdot d_-}{4\kappa^2\sqrt{2\pi}v_t^2\,(T-t)} \cdot \Bigg([\,\theta^2(4\kappa(T-t) - 9) \\
&+ V_t\,(V_t + 4\theta) + \frac{\gamma^2}{\kappa}(\kappa(T-t) - 1)] + [4\cdot\theta\,\kappa(T-t)(\theta - V_t) + 4\cdot\theta \\
&\cdot(3\theta - 2V_t)]\cdot e^{-\kappa(T-t)} + [\frac{\gamma^2}{\kappa}(\kappa(T-t)+1)\ - (\theta - V_t)(3\theta - V_t) - 2\kappa \\
&\cdot(T-t)(\theta - V_t)^2]\cdot e^{-2\kappa(T-t)}\Bigg) +\ \gamma^2 \cdot \frac{S_t \cdot e^{-\frac{d_+^2}{2}} \cdot (d_+d_- - 1)}{\sqrt{2\pi}(v_t\sqrt{T-t})^{3/2}} \cdot \frac{1}{8\kappa^3} \cdot \\
&\left(\frac{1}{2}\cdot\left[-\frac{5\gamma^2}{4\kappa} + (V_t^2 + 6\cdot\theta V_t - 19\theta^2) + (T-t)\cdot(\,8\cdot\theta^2\kappa + \gamma^2)\right] + \right. \\
&[2\theta\cdot(7\theta - 3V_t) + 4\cdot\theta\cdot(\theta - V_t)\cdot\kappa(T-t)]\cdot e^{-\kappa(T-t)} + [2\theta\cdot(2V_t - 3\theta) \\
&+ (T-t)\cdot(\,\gamma^2 - 2\kappa(\theta - V_t)^2)\ + \frac{\gamma^2}{2\kappa}\ ]\cdot e^{-2\kappa(T-t)} + [2\cdot\theta\cdot(\theta - V_t)] \\
&\left.\cdot\, e^{-3\kappa(T-t)} + \frac{1}{2}\cdot\left[\frac{\gamma^2}{4\kappa} - (\theta - V_t)^2\right]\cdot e^{-4\kappa(T-t)}\right)
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned} v_t &= \sqrt{\frac{1}{T-t}E^{\mathbb{P}}\left[\int_t^T V_s^2 ds \middle| \mathcal{F}_t\right]}, \\ d_+ &= \frac{\ln\frac{S_t}{K}+(T-t)\left(\frac{1}{2}v_t^2+r\right)}{v_t\sqrt{T-t}},\ d_- = d_+ - v_t\sqrt{T-t}. \end{aligned}$$

Proof. Theorem 3.7 gives us that

$$d\langle N,W\rangle_t = 2\gamma\left(\int_t^T E^{\mathbb{P}}\left(V_a|\,\mathcal{F}_t\right)\exp\left[-\kappa(a-t)\right]\,da\right)dt$$

and

$$d\langle N,N\rangle_t = 4\gamma^2\left(\int_t^T E^{\mathbb{P}}\left(V_a|\,\mathcal{F}_t\right)\exp\left[-\kappa(a-t)\right]da\right)^2 dt.$$

Now, as

$$E^{\mathbb{P}}\left(V_a|\,\mathcal{F}_t\right) = \theta + (V_t-\theta)\exp\left[-\kappa(a-t)\right]$$

we can evaluate $d\langle N,W\rangle_t$ and $d\langle N,N\rangle_t$ explicitly as

$$\begin{aligned} & d\langle N,W\rangle_t \\ = \;& 2\gamma\left(\int_t^T \left(\theta+(V_t-\theta)\exp\left[-\kappa(a-t)\right]\right)\exp\left[-\kappa(a-t)\right]\,da\right)dt \\ = \;& 2\gamma\left(\theta\int_t^T \exp\left[-\kappa(a-t)\right]\,da + (V_t-\theta)\int_t^T \exp\left[-2\kappa(a-t)\right]da\right)dt \\ = \;& 2\gamma\left(\theta\left(\frac{1-\exp\left[-\kappa(T-t)\right]}{\kappa}\right)+(V_t-\theta)\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)\right)dt \end{aligned}$$

and

$$\begin{aligned} & d\langle N,N\rangle_t \\ = \;& \gamma^2\left(\theta\int_t^T \exp\left[-\kappa(a-t)\right]da + (V_t-\theta)\int_t^T \exp\left[-2\kappa(a-t)\right]da\right)^2 dt \end{aligned}$$

$$= \quad 4\gamma^2\left(\theta\left(\frac{1-\exp\left[-\kappa(T-t)\right]}{\kappa}\right)+(V_t-\theta)\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)\right)^2 dt.$$

hence

$$E^{\mathbb{P}}\left(\int_t^T \sigma_s\, d\left\langle N,W\right\rangle_s \middle| \mathcal{F}_t\right)$$

$$= \quad E^{\mathbb{P}}\left[\int_s^T V_t\left(2\gamma\theta\left(\frac{1-\exp\left[-\kappa(T-t)\right]}{\kappa}\right)dt+2\gamma\ (V_t-\theta)\right.\right.$$
$$\left.\left.\cdot\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)\right)dt\middle|\mathcal{F}_s\right]$$

$$= \quad 2\gamma\left(\int_s^T\frac{\theta(1-\exp\left[-\kappa(T-t)\right])^2}{2\kappa}E^{\mathbb{P}}\left[V_t\middle|\mathcal{F}_s\right]dt+\int_s^T\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right.$$
$$\left.\cdot E^{\mathbb{P}}\left[V_t^2\middle|\mathcal{F}_s\right]dt\right)$$

$$= \quad \frac{\gamma}{\kappa^2}\left(\frac{1}{2}\cdot\left[(\theta^2(4\kappa(T-s)-9)+V_s(V_s+4\theta)+\frac{\gamma^2}{\kappa}(\kappa(T-s)-1)\right]+[2\theta\right.$$
$$\cdot\kappa(T-s)(\theta-V_s)+2\theta\cdot(3\theta-2V_s)]\cdot e^{-\kappa(T-s)}+\frac{1}{2}\cdot\left[\frac{\gamma^2}{\kappa}(\kappa(T-s)+1)\right.$$
$$\left.\left.-(\theta-V_s)(3\theta-V_s)-2\kappa(T-s)(\theta-V_s)^2\right]\cdot e^{-2\kappa(T-s)}\right)$$

and

$$E^{\mathbb{P}}\left[\int_s^T 4\gamma^2\left(\theta\left(\frac{1-\exp\left[-\kappa(T-t)\right]}{\kappa}\right)+(V_t-\theta)\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)\right)^2\right.$$
$$\left.dt\middle|\mathcal{F}_s\right]$$

$$= \quad 4\gamma^2\left(\int_s^T\left(\frac{\theta(1-\exp\left[-\kappa(T-t)\right])^2}{2\kappa}\right)^2dt+2\int_s^T\left(\frac{\theta(1-\exp\left[-\kappa(T-t)\right])^2}{2\kappa}\right)\right.$$
$$\cdot\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)E^{\mathbb{P}}\left(V_t\middle|\mathcal{F}_s\right)dt+\int_s^T\left(\frac{1-\exp\left[-2\kappa(T-t)\right]}{2\kappa}\right)^2$$
$$\left.E^{\mathbb{P}}\left(V_t^2\middle|\mathcal{F}_s\right)dt\right)$$

$$
\begin{aligned}
= \; & \frac{\gamma^2}{\kappa^3} \cdot \left(\frac{1}{2} \cdot \left[-\frac{5\gamma^2}{4\kappa} + (V_s^2 + 6 \cdot \theta V_s - 19\theta^2) + (T-s) \cdot (\, 8 \cdot \theta^2 \cdot \kappa + \gamma^2) \right] \right. \\
& + [2\theta \cdot (7\theta - 3V_s) + 4 \cdot \theta \cdot (\theta - V_s) \cdot \kappa(T-s)\,] \cdot e^{-\kappa(T-s)} + [\, 2\theta \cdot (2V_s - 3\theta) \\
& \left. + (T-s) \cdot (\, \gamma^2 - 2\kappa(\theta - V_s)^2) \; + \frac{\gamma^2}{2\kappa} \right] \cdot e^{-2\kappa(T-s)} + [\, 2\theta \cdot (\theta - V_s)\,] \cdot e^{-3\kappa(T-s)} \\
& \left. + \frac{1}{2} \cdot \left[\frac{\gamma^2}{4\kappa} - (\theta - V_s)^2 \right] \cdot e^{-4\kappa(T-s)} \right)
\end{aligned}
$$

where

$$
E^{\mathbb{P}}\,(\, V_t \,|\, \mathcal{F}_s) = e^{-\kappa(t-s)} V_s + \theta(1 - e^{-\kappa(t-s)})
$$

$$
\begin{aligned}
E^{\mathbb{P}}\left(\, V_t^2 \,\middle|\, \mathcal{F}_s \right) &= Var\,(\, V_t \,|\, \mathcal{F}_s) + [E^{\mathbb{P}}\,(\, V_t \,|\, \mathcal{F}_s)]^2 \\
&= \frac{\gamma^2}{2\kappa}(1 - e^{-2\kappa(t-s)}) + \left(e^{-\kappa(t-s)} V_s + \theta(1 - e^{-\kappa(t-s)})\right)^2
\end{aligned}
$$

$$
\begin{aligned}
E^{\mathbb{P}}\left[\int_s^T V_t^2 dt \middle| \mathcal{F}_s \right] &= (T-s)\left(\frac{\gamma^2}{2\kappa} + \theta^2\right) + \left(1 - e^{-\kappa(t-s)}\right)(V_t - \theta)\frac{2\theta}{\kappa} \\
&\quad + \left(1 - e^{-2\kappa(t-s)}\right)\left(-\frac{\gamma^2}{4\kappa^2} + \frac{(V_s - \theta)^2}{2\kappa}\right) \; .
\end{aligned}
$$

■

3.3.3 Heston model

Assume that the volatility process is given by $\sigma(V_t) = \sqrt{V_t}$, where V_t is of the form:

$$
dV_t = \kappa(\theta - V_t)\; dt + \gamma \sqrt{V_t} dW_t,
$$

and

$$
dS_t = S_t \left(r\, dt + \sqrt{V_t}\, dZ_t \right) \;, \tag{3.9}
$$

for $t \in [0, T]$, $r \geq 0$, $\kappa \geq 0$, $\theta \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Theorem 3.10 *(Alòs) Given $t \leq T$, the value at time t of a put option with payoff*

$$G(S_T) = \ (K - S_T)^+ \quad ,$$

approximately equals to

$$\begin{aligned}
&P_{BS}(t, \ln S_t;\ K, T;\ v_t) - \rho\gamma \cdot \frac{S_t \cdot \exp\left(-\frac{d_+^2}{2}\right) \cdot d_-}{\kappa\sqrt{2\pi}v_t^2\,(T-t)} \cdot \left[\frac{(V_t - 2\theta)\left(1 - e^{-\kappa(T-t)}\right)}{\kappa}\right. \\
&\left. + (T-t)\left(\theta - (V_t - \theta)e^{-\kappa(T-t)}\right)\right] + \gamma^2 \cdot \frac{S_t \cdot \exp\left(-\frac{d_+^2}{2}\right) \cdot (d_+ d_- - 1)}{8\kappa\sqrt{2\pi}(v_t\sqrt{T-t})^{3/2}} \cdot \left[(T-t)\right. \\
&\left. \cdot \left(\theta - 2(V_t - \theta)e^{-\kappa(T-t)}\right) + \frac{2(V_t - 2\theta)}{\kappa}(1 - e^{-\kappa(T-t)}) + \frac{(2V_t - \theta)}{2\kappa}(1 - e^{-2\kappa(T-t)})\right]
\end{aligned} \tag{3.10}$$

where

$$\begin{aligned}
v_t &= \sqrt{\frac{1}{T-t}E^{\mathbb{P}}\left[\int_t^T V_s ds \middle| \mathcal{F}_t\right]}, \\
d_+ &= \frac{\ln\frac{S_t}{K} + (T-t)\left(\frac{1}{2}v_t^2 + r\right)}{v_t\sqrt{T-t}},\ d_- = d_+ - v_t\sqrt{T-t}.
\end{aligned}$$

Proof. See Alòs (2012) [1] ■

3.4 Numerical simulation

We study the performance of the approximation formula by simulation of the above examples, where we calculate the analytical value of the put option in the Heston model via a closed-form pricing formula introduced by [24] and the put value in the Hull & White model and Stein & Stein model by a finite difference method. Recall that the approximation formula is

$$\begin{aligned}
V_t &\approx P_{BS}\left(t, X_t, v_t\right) \\
&\quad + \frac{\rho}{2}H\left(t, X_t, v_t\right)E^{\mathbb{P}}\left[\int_t^T \sigma_s\, d\left\langle N, W\right\rangle_s \middle| \mathcal{F}_t\right]
\end{aligned}$$

$$+\frac{K}{8}J(t,X_t,v_t)\,E^{\mathbb{P}}\left[\int_t^T d\langle N,N\rangle_s\middle|\mathcal{F}_t\right].$$

and the approximation error is bounded by

$$C\left(|\rho|\,(T-t)^{\frac{3}{2}}+(T-t)^2+(T-t)^{\frac{5}{2}}\right).$$

3.4.1 Hull and White model

Example 3.11 *In table 3.1 and fig. 3.1, we check the goodness of the approximation for a Hull and White model as a function of time to maturity. We can see that the performance remains stable except for very small T.*

Example 3.12 *In table 3.2 and fig. 3.2, we study the goodness of the approximation for a Hull and White model as a function of correlation ρ. Observe that the relative error decreases as the correlation goes to 0, as shown at the boundary.*

Example 3.13 *In fig. 3.3, we plot the goodness of the approximation for a Hull and White model as a function of the correlation ρ and the volatility of volatility γ. The relative error increases with increasing $|\rho|$ and γ, agreeing with our approximation formula.*

Table 3.1. Error of approximation as a function of T in a Hull and White model. We take parameters $S_0 = 100$, $K = 97$, $r = 0.01$, $\mu = 0.2$, $\gamma = 0.1$, $v_0 = 0.04$, $\rho = -0.5$.

T	**approximation**	**put value**	**error**	**error (in %)**
0.125	1.515538766	1.498522772	0.017015993	1.136
0.250	2.574081124	2.556929288	0.017151836	0.671
0.375	3.410587997	3.392529292	0.018058705	0.532
0.500	4.127224642	4.107864613	0.019360029	0.471
0.625	4.766811271	4.745923650	0.020887621	0.440
0.750	5.352036537	5.329470375	0.022566162	0.423
0.875	5.896613379	5.872256662	0.024356717	0.415
1.000	6.409567731	6.383331098	0.026236632	0.411

Table 3.2. Error of approximation as a function of ρ in a Hull and White model. We take $S_0 = 100$, $K = 97$, $r = 0.01$, $\mu = 0.2$, $\gamma = 0.1$, $v_0 = 0.04$, $T = 0.5$.

ρ	**approximation**	**put value**	**error**	**error (in %)**
-1.0	4.144081891	4.115587682	0.028494209	0.692
-0.9	4.140710441	4.114057199	0.026653243	0.648
-0.8	4.137338992	4.112507531	0.024831460	0.604
-0.7	4.133967542	4.110938645	0.023028896	0.560
-0.6	4.130596092	4.109350505	0.021245587	0.517
-0.5	4.127224642	4.107743073	0.019481569	0.474
-0.4	4.123853192	4.106116312	0.017736881	0.432
-0.3	4.120481743	4.104470182	0.016011560	0.390
-0.2	4.117110293	4.102804645	0.014305648	0.349
-0.1	4.113738843	4.101119659	0.012619184	0.308
0	4.110367393	4.099415182	0.010952211	0.267

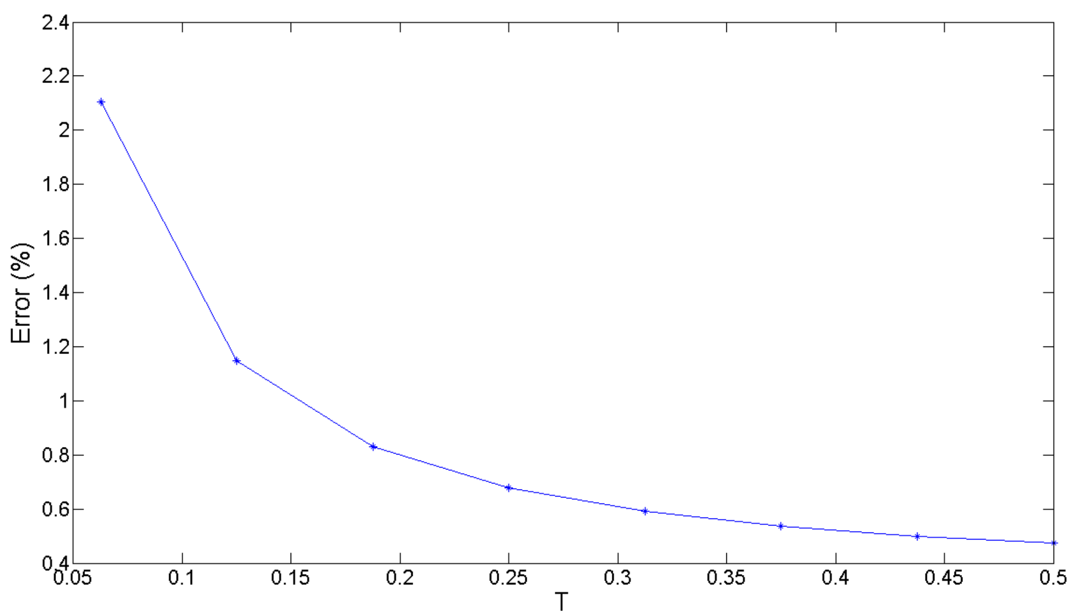

Figure 3.1: Error of approximation as a function of time to maturity in the Hull and White model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\mu = 0.2$, $\gamma = 0.1$, $v_0 = 0.04$, $\rho = -0.5$.

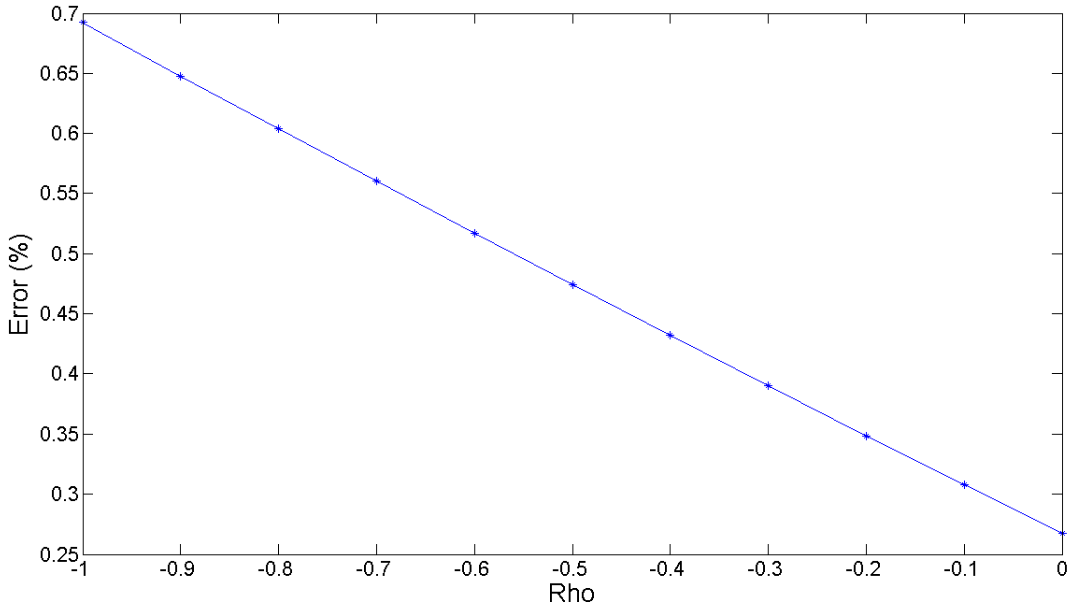

Figure 3.2: Error of approximation as a function of correlation ρ in the Hull and White model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\mu = 0.2$, $\gamma = 0.1$, $v_0 = 0.04$, $T = 0.5$.

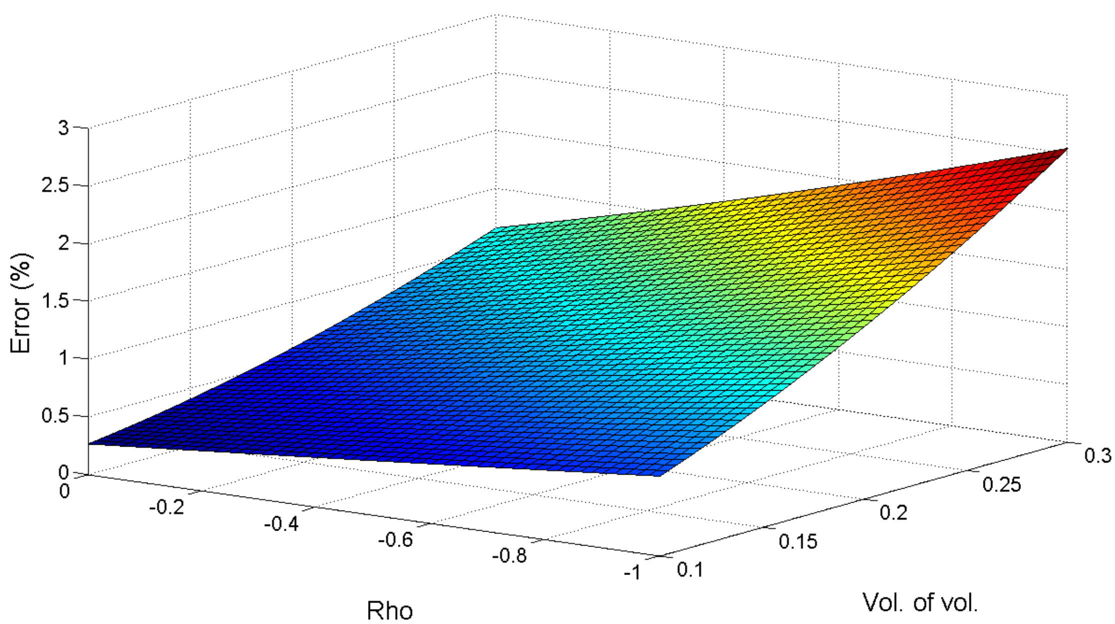

Figure 3.3: Error of approximation as a function of correlation ρ and volatility of volatility γ in the Hull and White model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\mu = 0.2$, $v_0 = 0.04$, $T = 0.5$.

3.4.2 Stein & Stein model

Example 3.14 *In table 3.3 and Fig. 3.4, we check the goodness of the approximation for a Stein & Stein model as a function of time to maturity. We can see that the performance remains stable except for very small* T*.*

Example 3.15 *In table 3.4 and Fig. 3.5, we study the goodness of the approximation for a Stein & Stein model as a function of correlation* ρ*. Observe that the relative error decrease as correlation* ρ *goes to* 0*, as shown in the boundary.*

Example 3.16 *In Fig. 3.6, we plot the goodness of the approximation for a Stein & Stein model as a function of correlation* ρ *and volatility of volatility* γ*.The relative error increases with the increasing of* $|\rho|$ *and* γ*, agreeing with our approximation formula.*

Table 3.3. Error of approximation as a function of T in the Stein & Stein model. We take $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.2$, $\gamma = 0.1$, $v_0 = 0.2$, $\rho = -0.5$.

T	**approximation**	**put value**	**error**	**error (in %)**
0.0625	0.846737210	0.824675451	0.022061759	2.675
0.1250	1.543257615	1.529819401	0.013438213	0.878
0.1875	2.101902173	2.092425078	0.009477096	0.453
0.2500	2.578516375	2.571205558	0.007310817	0.284
0.3125	2.999497654	2.993550522	0.005947132	0.199
0.3750	3.379671188	3.374721890	0.004949298	0.147
0.4375	3.728304336	3.724294965	0.004009371	0.108
0.5000	4.051631314	4.048793836	0.002837478	0.070

Table 3.4. Error of approximation as a function of ρ in the Stein & Stein model. We take $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.2$, $\gamma = 0.1$, $v_0 = 0.2$, $T = 0.5$.

ρ	**approximation**	**put value**	**error**	**error (in %)**
-1.0	4.098634963	4.093649507	0.004985456	0.122
-0.9	4.089234234	4.084699503	0.00453473	0.111
-0.8	4.079833504	4.075740491	0.004093013	0.100
-0.7	4.070432774	4.066771062	0.003661712	0.090
-0.6	4.061032044	4.057789459	0.003242585	0.080
-0.5	4.051631314	4.048793836	0.002837478	0.070
-0.4	4.042230584	4.039782435	0.002448149	0.061
-0.3	4.032829855	4.030753725	0.00207613	0.052
-0.2	4.023429125	4.021706504	0.001722621	0.043
-0.1	4.014028395	4.012639987	0.001388408	0.035
0	4.004627665	4.003553874	0.001073791	0.027

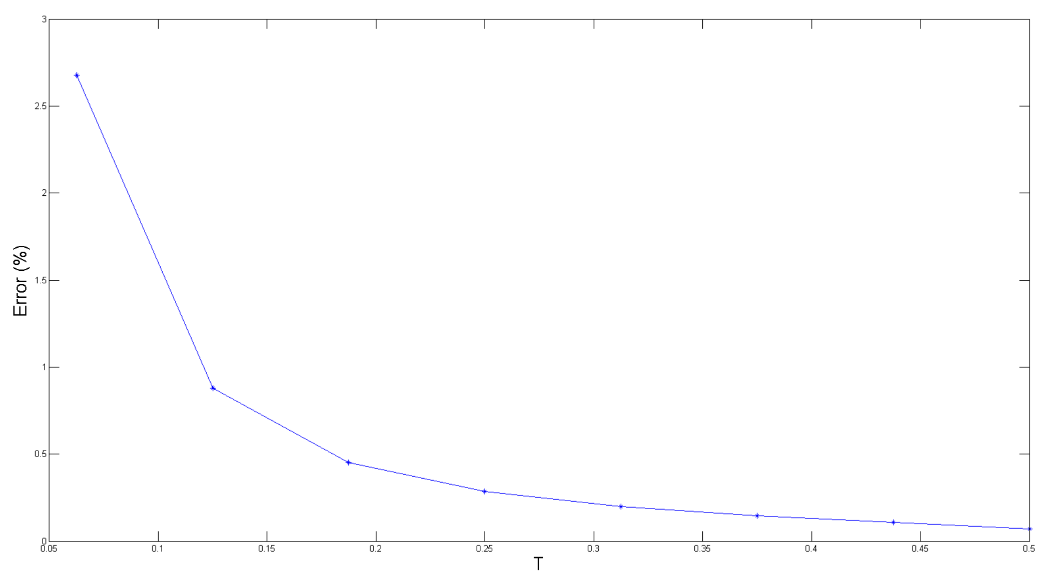

Figure 3.4: Error of approximation as a function of time to maturity in the Stein & Stein model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.2$, $\gamma = 0.1$, $v_0 = 0.2$, $\rho = -0.5$.

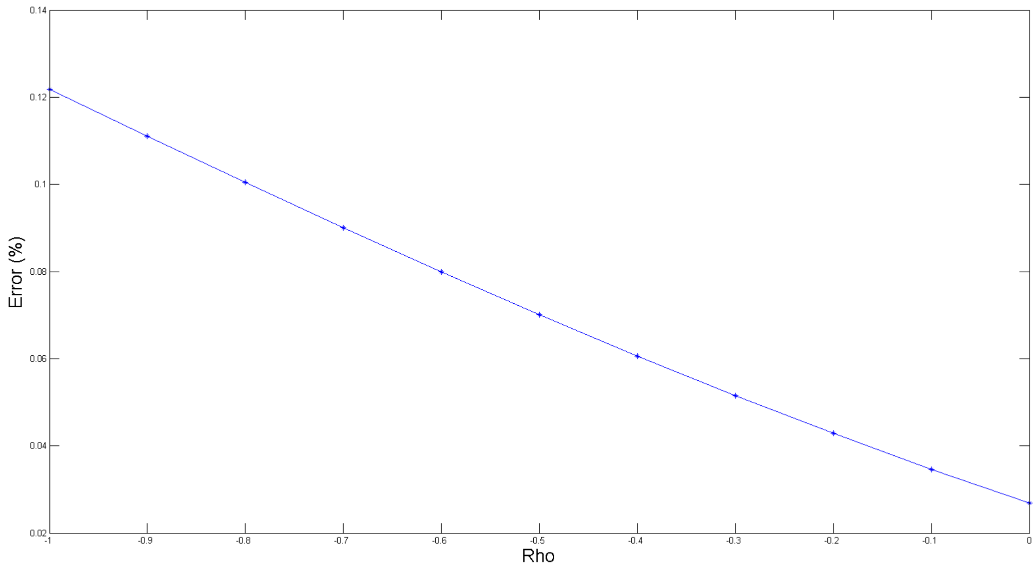

Figure 3.5: Error of approximation as a function of correlation ρ in the Stein & Stein model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.2$, $\gamma = 0.1$, $v_0 = 0.2$, $T = 0.5$.

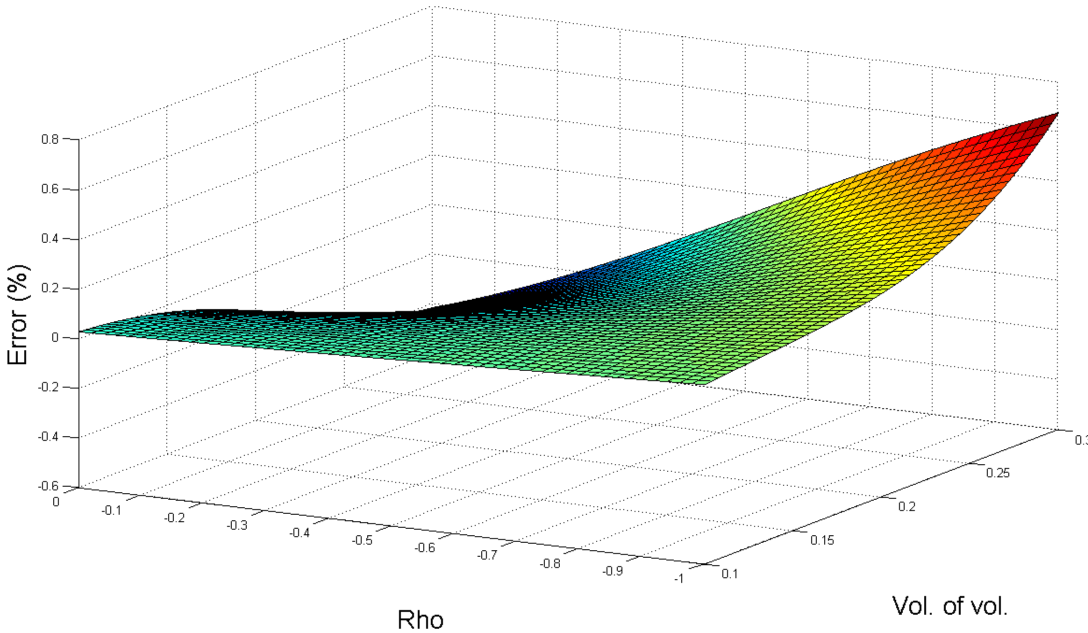

Figure 3.6: Error of approximation as a function of correlation ρ and volatility of volatility γ in the Stein & Stein model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.2$, $v_0 = 0.2$, $T = 0.5$.

3.4.3 Heston model

Example 3.17 *In table 3.5 and fig. 3.7, we check the goodness of the approximation for a Heston model as a function of time to maturity. It is easy to see that our approximation formula performs well, in particular for short time to maturity, and that the relative error increases with* T *as to be expected.*

Example 3.18 *In table 3.6 and fig. 3.8, we study the goodness of the approximation for a Heston model as a function of correlation* ρ*. The relative error decreases as the correlation* ρ *goes to* 0*, as shown at the boundary.*

Example 3.19 *In fig. 3.9, we plot the goodness of the approximation for a Heston model as a function of the correlation* ρ *and the volatility of volatility* γ*. The relative error increases with increasing of* $|\rho|$ *and* γ*, agreeing with our approximation formula.*

Table 3.5. Error of approximation as a function of T in a Heston model. We take $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.04$, $\gamma = 0.2$, $v_0 = 0.04$, $\rho = -0.5$.

T	**approximation**	**analytical value**	**error**	**error (in %)**
0.1	1.266456608	1.265697726	0.000758882	0.060
0.2	2.173526753	2.172501912	0.001024841	0.047
0.3	2.877643169	2.875889221	0.001753948	0.061
0.4	3.469692922	3.467318561	0.002374361	0.068
0.5	3.988368703	3.985075433	0.003293269	0.083
0.6	4.454221182	4.449985742	0.004235440	0.095
0.7	4.879665544	4.874789041	0.004876503	0.100
0.8	5.272871444	5.267372676	0.005498768	0.104
0.9	5.639554004	5.633662731	0.005891273	0.105
1	5.983899168	5.977675950	0.006223219	0.104

Table 3.6. Error of approximation as a function of ρ in a Heston model. We take $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.04$, $\gamma = 0.2$, $v_0 = 0.04$, $T = 0.5$.

ρ	**approximation**	**analytical value**	**error**	**error (in %)**
-1	4.036764420	4.023178421	0.013585999	0.338
-0.9	4.027085277	4.016037327	0.011047949	0.275
-0.8	4.017406133	4.008593041	0.008813092	0.220
-0.7	4.007726990	4.000945628	0.006781362	0.169
-0.6	3.998047846	3.993188030	0.004859816	0.122
-0.5	3.988368703	3.985075433	0.003293269	0.083
-0.4	3.978689559	3.976671426	0.002018133	0.051
-0.3	3.969010416	3.967920622	0.001089793	0.027
-0.2	3.959331272	3.959062429	0.000268843	0.007
-0.1	3.949652129	3.949390104	0.000262025	0.007
0	3.939972985	3.939975652	-0.000002667	0.000

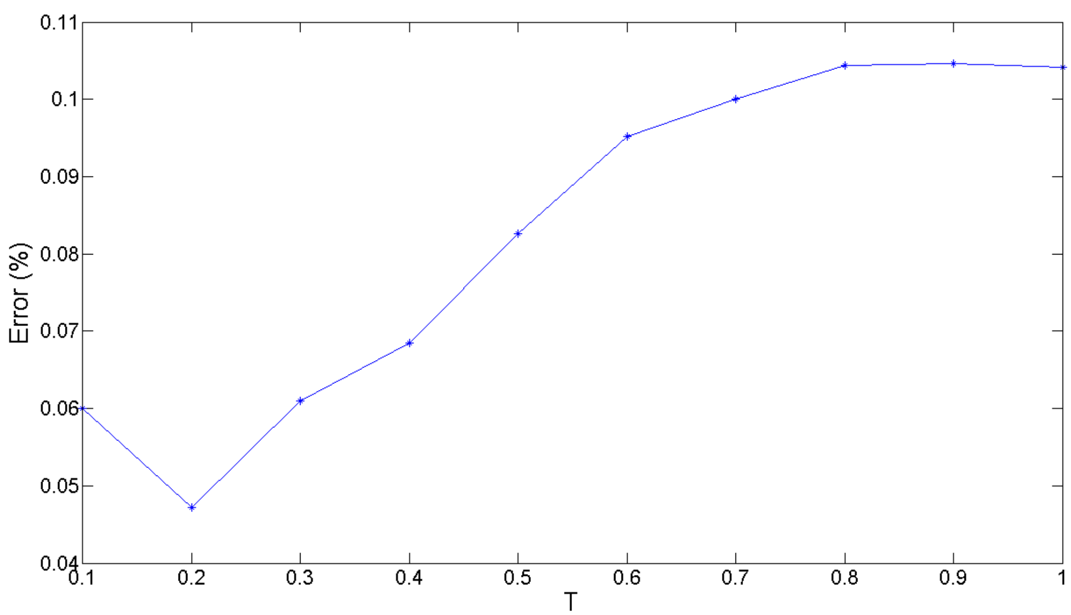

Figure 3.7: Error of approximation as a function of time to maturity in the Heston model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.04$, $\gamma = 0.2$, $v_0 = 0.04$, $\rho = -0.5$.

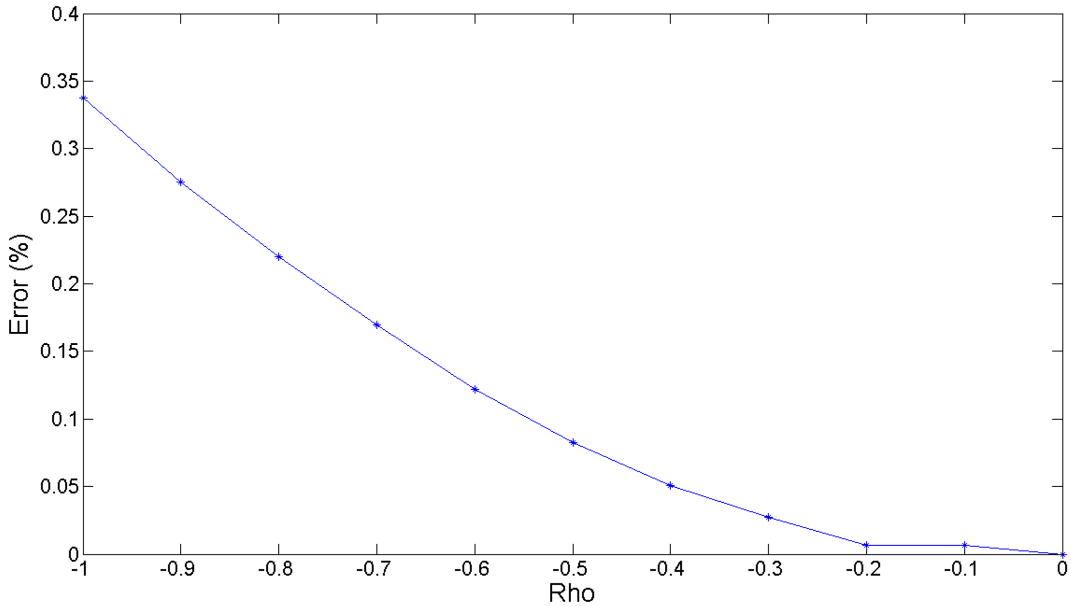

Figure 3.8: Error of approximation as a function of correlation ρ in the Heston model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.04$, $\gamma = 0.2$, $v_0 = 0.04$, $T = 0.5$.

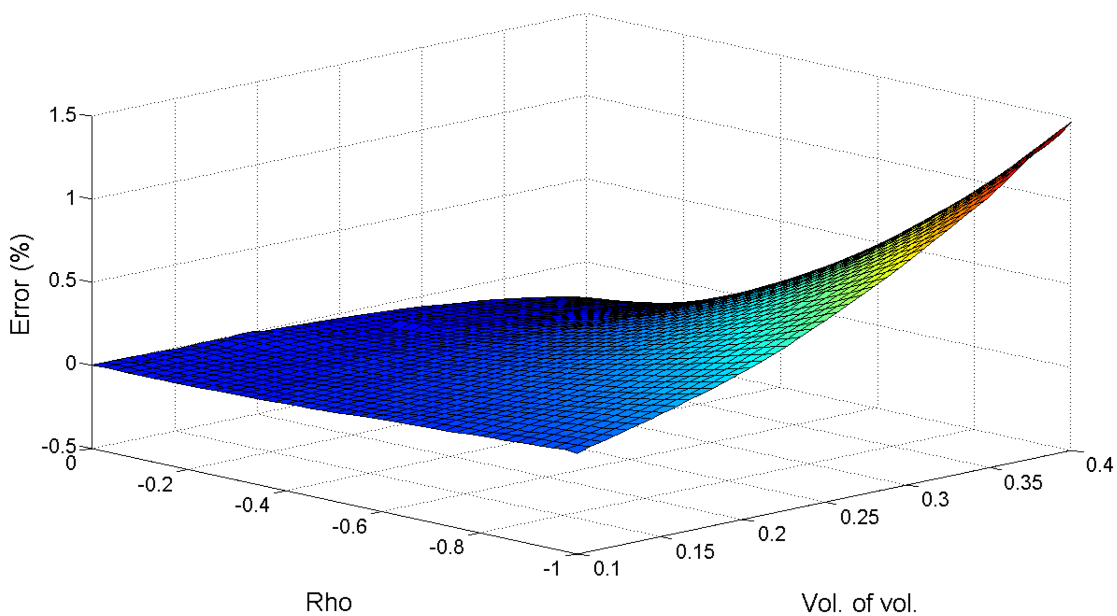

Figure 3.9: Error of approximation as a function of correlation ρ and volatility of volatility γ in the Heston model when $S_0 = 100$, $K = 97$, $r = 0.01$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $T = 0.5$.

3.5 Application in hedging barrier options

In addition to the dynamic hedging strategy that we derived on process D in the last chapter (2.7), which is constructed by a holding of bonds, stocks, realised as well as the instantaneous volatility process, we now propose an alternative hedging approach consisting of static and dynamic hedging portfolios, where the main part of risk is covered by the static position and the remaining risk replicated by dynamic trading on volatility derivatives.

3.5.1 Overview of the method

In the event $[\tau \leq T]$, $S_\tau = B$, whereas in the general self-duality theorem (2.23), D_τ is random which introduces a time-dependency in the claim (2.9). Therefore we propose a mixed strategy of static/dynamic hedging:

(i) We consider the same claim, but with S instead of D, so purchase a static position of

$$\left(G(S_T) + \frac{S_T}{B} G\left(\frac{B^2}{S_T}\right)\right) 1_{(\eta S_T \geq \eta B)}$$

as in (2.3).

(ii) In addition, we trade in a dynamic portfolio of

$$X_t = E_t^{\mathbb{Q}}\left[\frac{D_T}{D_t} G\left(\frac{B}{D_T/D_t}\right) 1_{(\eta \frac{D_T}{D_t} > \eta)}\right] - E_t^{\mathbb{P}}\left[e^{-r(T-t)} \frac{S_T}{S_t} G\left(\frac{B}{S_T/S_t}\right) 1_{(\eta \frac{S_T}{S_t} > \eta)}\right].$$

(iii) *If and when the barrier knocks in, exchange these claims for a $G(S_T)$ claim with zero cost.*

Here we claim that

$$E_{\tau\wedge T}^{\mathbb{P}}\left[\frac{S_T}{S_{\tau\wedge T}} G\left(\frac{B}{S_T/S_{\tau\wedge T}}\right) 1_{(\eta \frac{S_T}{S_{\tau\wedge T}} > \eta)}\right] = E_{\tau\wedge T}^{\mathbb{P}}\left[\frac{S_T}{B} G\left(\frac{B^2}{S_T}\right) 1_{(\eta S_T > \eta B)}\right].$$

Indeed, if the first hitting time is before maturity, i.e. $\tau \leq T$, we have $S_\tau = B$, yielding

$$E_{\tau}^{\mathbb{P}}\left[\frac{S_T}{S_\tau} G\left(\frac{B}{S_T/S_\tau}\right) 1_{(\eta \frac{S_T}{S_\tau} > \eta)}\right] = E_{\tau}^{\mathbb{P}}\left[\frac{S_T}{B} G\left(\frac{B^2}{S_T}\right) 1_{(\eta S_T > \eta B)}\right];$$

while if $\tau > T$,

$$\frac{S_T}{S_T} G\left(\frac{B}{S_T/S_T}\right) 1_{(\eta \frac{S_T}{S_T} > \eta)} = \frac{S_T}{B} G\left(\frac{B^2}{S_T}\right) 1_{(\eta S_T > \eta B)} = 0.$$

Hence the value at time $\tau \leq T$ of our static position plus dynamic portfolio matches with $\Gamma^{\mathbb{P}}(S_T) + \Gamma^{\mathbb{Q}}_{\tau}\left(\frac{D_T}{D_\tau}\right)$ in Theorem 2.21. Therefore, one could exchange it for a $G(S_T)$ claim with zero cost, and it is valueless at maturity as desired, hence it perfectly replicates the barrier option.

Let us consider the down-and-in barrier option as an example where

$$\frac{D_T}{D_t} G\left(\frac{B}{D_T/D_t}\right) 1_{(\eta \frac{D_T}{D_t} > \eta)} = K\left(\frac{B}{K} - \frac{D_T}{D_t}\right)^+$$

Remark 3.20 *Recall that the dynamic of modified price process D under measure $\mathbb{Q}$ is*

$$dD = D\left(-rdt - \rho\,\sigma(V)\,dW^{\mathbb{Q}} + \overline{\rho}\,\sigma(V)\,dW^{\perp}\right)$$

and price process S under $\mathbb{P}$ is

$$dS = S\left(\,rdt + \rho\,\sigma(V)\,dW + \overline{\rho}\,\sigma(V)\,dW^{\perp}\right)$$

where we notice that by changing the sign of the interest rate r, correlation ρ, and the corresponding strike, we arrive at the decomposition and approximation formula for the claim

$$E^{\mathbb{Q}}_t\left[K\left(\frac{B}{K} - \frac{D_T}{D_t}\right)^+\right]$$

as we showed in the last section for the claim

$$E^{\mathbb{P}}_t\left[(S_T - K)^+\right] \;.$$

3.5.2 Numerical simulation

We check the performance of the hedging portfolio by numerical simulation on a down-and-in barrier option in the Heston model and Stein & Stein model. Recall that when there is correlation between the price process and the instantaneous volatility process, PCS fails. Hence we calculate the price of the barrier option by the finite difference method and the value of hedging portfolio by the Monte-Carlo simulation.

First of all, let us state the approximation formula for the claim

$$E_t^{\mathbb{Q}}\left[K\left(\frac{B}{K}-\frac{D_T}{D_t}\right)^+\right]$$

by the following corollaries.

Corollary 3.21 *Assuming that the price process is given by the Heston model (3.9) with $2\kappa\theta > \gamma^2$ and $\tau < T$, the value under $\mathbb{Q}$ at time τ of a put option with payoff*

$$G(S_T) = \ K\left(\frac{B}{K}-\frac{D_T}{D_\tau}\right)^+$$

is approximately equal to

$$K \cdot P_{BS}(\tau, 0;\ \frac{B}{K}, T;\ v_\tau) + \rho\gamma \cdot \frac{K\exp\left(-\frac{d_+^2}{2}\right)\cdot d_-}{2\sqrt{2\pi}\kappa^{\mathbb{Q}}} \cdot \left[\frac{\left(V_\tau - 2\theta^{\mathbb{Q}}\right)\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)}{\kappa^{\mathbb{Q}}}\right.$$

$$\left.+(T-\tau)\left(\theta^{\mathbb{Q}}-(V_\tau-\theta^{\mathbb{Q}})e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)\right] + \gamma^2 \cdot \frac{K\exp\left(-\frac{d_+^2}{2}\right)}{8\sqrt{2\pi}(\kappa^{\mathbb{Q}})^2} \cdot \frac{d_+d_- - 1}{v_\tau\sqrt{T-\tau}} \cdot \left[(T-\tau)\right.$$

$$\cdot\left(\theta^{\mathbb{Q}} - 2(V_\tau-\theta^{\mathbb{Q}})e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right) + \frac{2(V_\tau - 2\theta^{\mathbb{Q}})}{\kappa^{\mathbb{Q}}}\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right) + \frac{(2V_\tau-\theta^{\mathbb{Q}})}{2\kappa^{\mathbb{Q}}}(1-$$

$$\left. e^{-2\kappa^{\mathbb{Q}}(T-\tau)}\ )\right] \tag{3.11}$$

where

$$v_\tau = \sqrt{\frac{1}{T-\tau}E^{\mathbb{Q}}\left(\int_\tau^T V_s ds\middle|\mathcal{F}_\tau\right)},$$

$$d_+ = \frac{\log\frac{K}{B} + (T-\tau)\left(\frac{1}{2}v_\tau^2 - r\right)}{v_\tau\sqrt{T-\tau}},\ d_- = d_+ - v_\tau\sqrt{T-\tau},$$

$$\kappa^{\mathbb{Q}} = \kappa - \gamma\rho,\ \theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa-\gamma\rho},$$

i.e., a function of the square root of the future cumulative variance process v_τ, the correlation ρ and the volatility of volatility γ.

Proof. Recall that the dynamics of modified price process D under measure $\mathbb{Q}$ are

$$dD = D\left(-r\,dt - \rho\sqrt{V}\,dW^{\mathbb{Q}} + \overline{\rho}\sqrt{V}\,dW^{\perp}\right)$$

and price process S under $\mathbb{P}$ is

$$dS = S\left(\,r\,dt + \rho\sqrt{V}\,dW + \overline{\rho}\sqrt{V}\,dW^{\perp}\right)\,.$$

Regarding the variance process V under measure $\mathbb{Q}$, we recall that

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = R' = \mathcal{E}\left(\rho\int\sqrt{V}\,dW\right).$$

By the Girsanov's theorem, we have

$$\begin{aligned} W^{\mathbb{Q}} &= W - \int\frac{1}{R'}\,d[R',W] \\ &= W - \rho\int\sqrt{V_u}\,du. \end{aligned}$$

Moreover, we have the volatility process

$$\begin{aligned} dV_t &= \kappa\left(\theta - V_t\right)\,dt + \gamma\sqrt{V_t}dW_t \\ &= \kappa\left(\theta - V_t\right)\,dt + \gamma\sqrt{V_t}dW^{\mathbb{Q}} + \gamma\rho V_t\,dt \\ &= \kappa^{\mathbb{Q}}\left(\theta^{\mathbb{Q}} - V_t\right)\,dt + \gamma\sqrt{V_t}dW^{\mathbb{Q}}, \qquad\qquad V_0 = v_0, \end{aligned}$$

where

$$\kappa^{\mathbb{Q}} = \kappa - \gamma\rho,\ \theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa - \gamma\rho}.$$

Therefore, the approximation formula for the claim

$$G(S_T) = \ K\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+$$

follows directly from Theorem 3.10 with changing in the strike, the parameters $\kappa^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$ and the sign of correlation ρ and interest rate r. ∎

Remark 3.22 *The value of the approximation formula 3.11 goes to zero as $(T-\tau)\downarrow 0$. See the Appendix A for the detailed proof.*

Corollary 3.23 *We assume the dynamics of the risky asset price is given by the Stein & Stein model (3.7), and $\tau \le T$, the value under $\mathbb{Q}$ at time τ of a put option with payoff*

$$G(S_T) = \; K\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+ \quad ,$$

approximately equals to

$$\begin{aligned}
&K \cdot P_{BS}(\tau, 0;\; \frac{B}{K}, T;\; v_\tau) + \rho\gamma \cdot \frac{K \cdot e^{-\frac{d_+^2}{2}} \cdot d_-}{\sqrt{2\pi} v_\tau^2\,(T-\tau)} \cdot \frac{1}{4(\kappa^{\mathbb{Q}})^2} \cdot \Bigg[\; \big[(\theta^{\mathbb{Q}})^2 (4\kappa^{\mathbb{Q}}(T-\tau) - 9) \\
&+V_\tau(V_\tau + 4\theta^{\mathbb{Q}}) + \frac{\gamma^2}{\kappa^{\mathbb{Q}}}(\kappa^{\mathbb{Q}}(T-\tau) - 1)\;\big] + [4 \cdot \theta^{\mathbb{Q}}\,\kappa^{\mathbb{Q}}(T-\tau)(\theta^{\mathbb{Q}} - V_\tau) + 4 \cdot \theta^{\mathbb{Q}} \\
&\cdot (3\theta^{\mathbb{Q}} - 2V_\tau)\,] \cdot e^{-\kappa^{\mathbb{Q}}(T-\tau)} + \;[\frac{\gamma^2}{\kappa^{\mathbb{Q}}}(\kappa^{\mathbb{Q}}(T-\tau) + 1) \; - (\theta^{\mathbb{Q}} - V_\tau)(3\theta^{\mathbb{Q}} - V_\tau) \\
&-2\kappa^{\mathbb{Q}}(T-\tau)(\theta^{\mathbb{Q}} - V_\tau)^2\big] \cdot e^{-2\kappa^{\mathbb{Q}}(T-\tau)} \Bigg] + \; \gamma^2 \cdot \frac{K \cdot e^{-\frac{d_+^2}{2}} \cdot (d_+ d_- - 1)}{\sqrt{2\pi}(v_\tau\sqrt{T-\tau})^{3/2}} \cdot \frac{1}{8(\kappa^{\mathbb{Q}})^3} \\
&\cdot \Bigg[\frac{1}{2} \cdot \; [-\frac{5\gamma^2}{4\kappa^{\mathbb{Q}}} + (V_\tau^2 + 6 \cdot \theta^{\mathbb{Q}} V_\tau - 19(\theta^{\mathbb{Q}})^2) + (T-\tau) \cdot (\, 8 \cdot (\theta^{\mathbb{Q}})^2 \cdot \kappa^{\mathbb{Q}} + \gamma^2)\;] \\
&+ \big[2\theta^{\mathbb{Q}} \cdot (7\theta^{\mathbb{Q}} - 3V_\tau) + 4\theta^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_\tau) \cdot \kappa^{\mathbb{Q}}(T-\tau)\,\big] \cdot e^{-\kappa^{\mathbb{Q}}(T-\tau)} + \; \big[2\theta^{\mathbb{Q}} \cdot (2V_\tau - 3\theta^{\mathbb{Q}}) \\
&+(T-\tau) \cdot (\,\gamma^2 - 2\kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}} - V_\tau)^2) \; + \frac{\gamma^2}{2\kappa^{\mathbb{Q}}} \;] \cdot e^{-2\kappa^{\mathbb{Q}}(T-\tau)} + [\,2 \cdot \theta^{\mathbb{Q}} \cdot (\theta^{\mathbb{Q}} - V_\tau)\,] \\
&\cdot e^{-3\kappa^{\mathbb{Q}}(T-\tau)} + \; \frac{1}{2} \cdot \left[\frac{\gamma^2}{4\kappa^{\mathbb{Q}}} - (\theta^{\mathbb{Q}} - V_\tau)^2\right] \; \cdot e^{-4\kappa^{\mathbb{Q}}(T-\tau)} \Bigg]
\end{aligned} \tag{3.12}$$

where

$$v_\tau = \sqrt{\frac{1}{T-\tau} E^{\mathbb{Q}}\left[\int_\tau^T V_t^2 dt \,\middle|\, \mathcal{F}_\tau\right]},$$

$$d_+ = \frac{\log\frac{K}{B} + (T-\tau)\left(\frac{1}{2}v_\tau^2 - r\right)}{v_\tau\sqrt{T-\tau}}, \;\; d_- = d_+ - v_\tau\sqrt{T-\tau},$$

$$\kappa^{\mathbb{Q}} = \kappa - \gamma\rho,\ \theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa - \gamma\rho}.$$

i.e., a function of square root of future cumulative variance process v_τ, correlation ρ and volatility of volatility γ.

Proof. Similar to the proof in Corollary 3.21, we have, by the Girsanov theorem,

$$\begin{aligned} W^{\mathbb{Q}} &= W - \int \frac{1}{R'}\, d[R', W] \\ &= W - \rho \int V_u\, du \end{aligned}$$

and

$$\begin{aligned} dV_t &= \kappa\left(\theta - V_t\right)\, dt + \gamma dW_t \\ &= \kappa\left(\theta - V_t\right)\, dt + \gamma dW^{\mathbb{Q}} + \gamma\rho V_t\, dt \\ &= \kappa^{\mathbb{Q}}\left(\theta^{\mathbb{Q}} - V_t\right)\, dt + \gamma dW^{\mathbb{Q}}, \qquad V_0 = v_0\ , \end{aligned}$$

where

$$\kappa^{\mathbb{Q}} = \kappa - \gamma\rho,\ \theta^{\mathbb{Q}} = \frac{\kappa\theta}{\kappa - \gamma\rho}.$$

Hence, the approximation formula for the claim

$$G(S_T) = K\left(\frac{B}{K} - \frac{D_T}{D_\tau}\right)^+$$

follows directly from Theorem 3.9 by changing in the strike, the parameters $\kappa^{\mathbb{Q}}$, $\theta^{\mathbb{Q}}$ and the sign of correlation ρ and interest rate r. ■

Example 3.24 *In fig. 3.10, we plot the performance of the replicating portfolios for a Heston model as a function of the correlation ρ and the volatility of volatility γ, where the surface in light color illustrates the hedging error if we hold only the static positions on puts as in the classic theorem and the surface in dark color represents the replicating error of the hedging portfolio we derived in* (3.11)*. It is easy to see that the relative error increases with increasing of $|\rho|$ and γ and the performance is improved by the mixed hedging strategy.*

Example 3.25 *In fig. 3.11, we plot the performance of the replicating portfolios for a Stein & Stein model as a function of the correlation* ρ *and the volatility of volatility* γ*, where the surface in light color illustrates the hedging error if we hold only the static positions on puts as in the classic theorem and the surface in dark color represents the replicating error of the hedging portfolio we derived in* (3.12)*. Similar to the case in Heston model, the relative error increases with increasing* $|\rho|$ *and* γ *and the performance is improved by the mixed hedging strategy.*

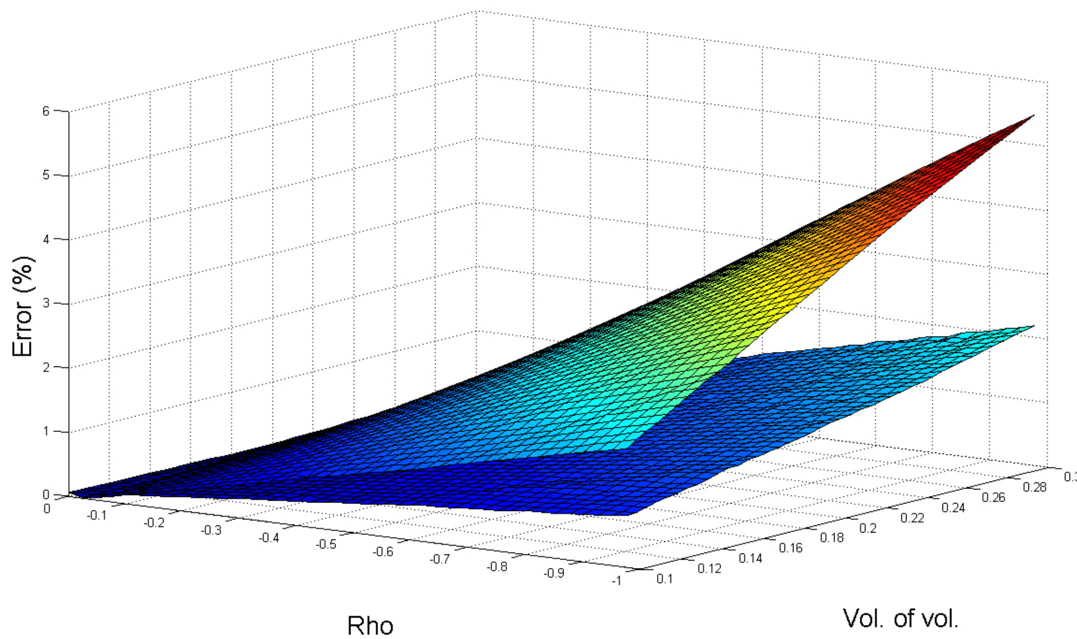

Figure 3.10: Error of hedging portfolios as a function of correlation ρ and volatility of volatility γ in the Heston model when $S_0 = 100$, $K = 97$, $B = 95$, $r = 0$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $T = 0.5$.

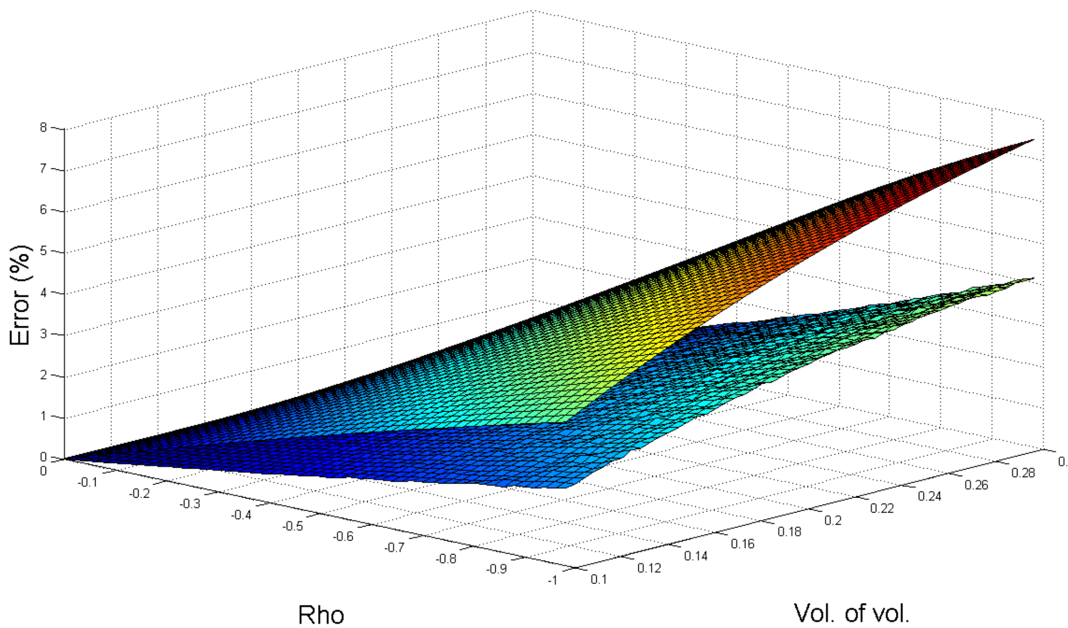

Figure 3.11: Error of hedging portfolios as a function of correlation ρ and volatility of volatility γ in the Stein & Stein model when $S_0 = 100$, $K = 97$, $B = 95$, $r = 0$, $\kappa = 4$, $\theta = 0.2$, $v_0 = 0.2$, $T = 0.5$.

Chapter 4

Reflection principle and application on pricing exotic options

4.1 Introduction

In this chapter, we adopt an alternative approach of pricing path-dependent options in stochastic volatility models. This work is inspired by Désiré André's reflection principle for Brownian motions [3] [22] and we apply it to the continuous stochastic volatility framework by means of changing of time technique and Ocone martingale argument.

Regarding to the work on pricing barrier options in stochastic volatility models, Lipton [26] derives a (semi-)analytical solutions for double barrier options in a reduced Heston framework (with zero correlation between underlying assets price and variance processes) via the bounded Green's function, while the price of a single barrier option would be implied by setting one of the two barriers to a extreme value. Nevertheless, Faulhaber [18] shows in his thesis that an extension of these techniques to the general Heston framework fails. Recently, Griebsch and Pilz [19] develop a (semi-) closed-form valuation formula for continuous barrier options in the reduced Heston framework and approximations for these types of options in the general Heston model. Chaumont and Vostrikova [12] work on the other direction of the problem. They characterize Ocone martingales by a sequence that satisfies the reflection principle.

We prove the reflection principle via a different approach involving the Ocone martingale argument. By conditioning on the filtration $\mathcal{F}^V$ generated by the entire information of the volatility process V, we show that the logarithm of the underlying price process is a Brownian motion with deterministic time-change and deterministic drift. Then we provide the joint density of the logarithm of the underlying process and its running maximum as well as the closed-form pricing formula for barrier and lookback options.

4.2 Time-changed Brownian motion and reflection principle

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space where the filtration satisfies the usual conditions with $\mathcal{F}_0$ being trivial up to $\mathbb{P}$-null sets, and fix a finite but arbitrary time horizon $T > 0$. All stochastic processes are RCLL and defined on $[0, T]$. We assume that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supports at least two independent Brownian motions W and $W^{\perp}$. Let $E_t^{\mathbb{P}}$ denote the $\mathcal{F}_t$-conditional $\mathbb{P}$-expectation. (In)equalities between stochastic processes are in the sense of indistinguishability, whereas between random variables they are to be understood in the a.s. sense (if the dependency on the measure can be dropped). A martingale measure for a process X is a probability measure $\mathbb{P}$ such that X is a local $\mathbb{P}$-martingale.

We consider the following stochastic volatility model on a time interval $[0, T]$ under a risk-neutral measure $\mathbb{P}$:

$$\begin{aligned} dS_t &= rS_t dt + \sigma(V_t)\, S_t\, dZ_t, \qquad S_0 = s_0 > 0, \\ dV_t &= \mu(V_t)\ dt + \gamma(V_t)\, dW_t, \qquad V_0 = v_0 > 0. \end{aligned} \tag{4.1}$$

Here $r \geq 0$ denotes the riskless interest rate, and Z, W are two Brownian motions with correlation $\rho \in [-1, 1]$. Let $Z = \rho W + \overline{\rho} W^{\perp}$, where $W^{\perp}$ and W are independent standard Brownian motions and $\overline{\rho} = \sqrt{1 - \rho^2}$. We assume that the functions σ, μ, γ are such that there exists a weak solution (S, V), and that $\sigma(V)$ is non-zero on $[0, T]$. The filtration is set to be $\mathbb{F} = \mathbb{F}^{S,V}$, the filtration generated by S and V. Moreover, we assume that our **standing assumption** (2.3.1) holds, i.e., σ is such that all stochastic exponentials of the form $\mathcal{E}\left(\lambda \int \sigma(V)\, d\omega\right)$, with $\lambda \in [-1, 1]$ and ω some Brownian motion adapted to $\mathbb{F}^{S,V}$, are true martingales.

Let us recall the definition of Ocone martingale.

Definition 4.1 *Let M be a continuous P-martingale vanishing at zero and such that $[M]_\infty = \infty$, and consider its Dambis-Dubins-Schwarz (DDS) representation $M = B_{[M]}$. The process M is called an* ***Ocone martingale*** *if B and $[M]$ are independent.*

Here we define

$$L_t := \int_0^t \sigma(V_s)\, dW_s^\perp, t \in [0,T]$$

and recall that $\mathcal{F}^V = \mathcal{F}^V_\infty$ where $\left(\mathcal{F}^V_t\right)_{t\geq 0}$ is the augmented filtration generated by V. Note that, given $\mathcal{F}^V$, $\left(\int_0^\cdot \sigma^2(V_s)\, ds\right)$ is deterministic. And by a changing of time technique, there exists a Brownian motion $\widehat{W^\perp}$ such that

$$L_t = \widehat{W^\perp}_{\int_0^t \sigma^2(V_s)\, ds},$$

i.e., a Brownian motion with deterministic time-change, hence L is a conditional Ocone martingale.

Lemma 4.2 *Given $\mathcal{F}^V$, the process L has the conditional strong Markov property.*

Proof. The proof follows the same as for Brownian motions, see [22], Theorem 6.15, Chapter 2. ∎

Definition 4.3 *An adapted process $(X_t)_{0\leq t\leq T}$ is process symmetric if $X \sim -X$ (the finite dimensional distributions of X and $-X$ are the same).*

Let us recall Lemma 2.8.

Lemma 4.4 *([32], Lemma 18) If L is a continuous Ocone martingale, then L is process symmetric.*

Theorem 4.5 *Given $\mathcal{F}^V$, let $(L_t)_{0\leq t\leq T}$ be defined as above and $Y_t := \sup_{s\leq t} L_s$, for $t \in [0,T]$. Then the reflection principle holds,*

$$\mathbb{P}(Y_t \geq y, L_t < x) = \mathbb{P}(L_t > 2y - x) \quad \text{for } t \in [0,T],\ y \geq x \vee 0.$$

Proof. Given $\mathcal{F}^V$, L is an Ocone martingale, therefore it's process symmetric, and L enjoys the conditional strong Markov property, then it follows the same as the proof of reflection principle of Désiré André for Brownian motions, e.g. see [22], section 2.6. ∎

4.3 Derivation of the joint density

To illustrate the application of the reflection principle, we firstly derive the conditional joint density of (Y, L) .

Proposition 4.6 *Given $\mathcal{F}^V$, for $t \in [0, T]$, the conditional joint p.d.f. of (Y_t, L_t) is given by*

$$f_{Y,L}(m,w) = \frac{2(2m-w)}{\sqrt{2\pi}\Sigma_t^{3/2}} \exp\left(-\frac{(2m-w)^2}{2\Sigma_t}\right) \qquad \text{for} \;\; w \le m, m > 0.$$

where we denote $\Sigma_\cdot = \int_0^\cdot \sigma^2(V_s)\, ds$.

Proof. Given $\mathcal{F}^V$, $\sigma(V_s)$ is deterministic, and we have $L_t = \int_0^t \sigma(V_s)\, dW_s^\perp \sim N(0, \Sigma_t)$, hence,

$$\mathbb{P}(L_t > 2m - w) = \frac{1}{\sqrt{2\pi}\Sigma_t^{1/2}} \int_{2m-w}^{\infty} \exp\left(-\frac{y^2}{2\Sigma_t}\right) dy$$

and by the reflection principle,

$$\mathbb{P}(L_t > 2m - w) = \mathbb{P}(Y_t \ge m, L_t < w) = \int_m^\infty \int_{-\infty}^{w} f_{Y,L}(y,x)\, dx\, dy \; .$$

Differentiation with respect to m and then w leads to the result. ∎

4.4 Closed-form valuation formula

4.4.1 Valuation formula for zero correlation and zero interest rate.

Barrier options

We are now ready to present the closed-form pricing formula for exotic options. Firstly, let's work on an example of a European up-and-in put option with zero correlation and zero interest rate, i.e., $\rho = 0$ and $r = 0$. Recall that Lipton (2001) [26] derives a (semi-)analytical solutions for double barrier options in Heston model via the bounded Green's function. And in Chapter 2, we show that when there is no correlation between the price process and the instantaneous volatility, the self-duality holds and one could semi-static hedge the barrier call option by holding a position of puts. Here we study an alternative pricing approach via the reflection principle theorem.

Theorem 4.7 *In the continuous stochastic volatility model* (4.1) *with* $\rho = 0$ *and* $r = 0$,*the value of a European up-and-in put option with maturity* T *and payoff*

$$(K - S_T)^+ 1_{\{\sup_{t \le T} S_t \ge B\}}$$

is given by

$$\int_0^\infty f^{\,\Sigma}(y) \int_{\ln \frac{B}{s_0}}^\infty \int_{-\infty}^{\ln \frac{K}{s_0}} (K - s_0 e^w) \cdot \frac{2(2m - w)}{\sqrt{2\pi} y^{3/2}} \cdot \exp\left(-\frac{(2m - w)^2}{2y} - \frac{w}{2} - \frac{y}{8} \right) dw\, dm\, dy \ ,$$

where we denote $f^{\,\Sigma}(y)$ *as the marginal density of* Σ_T *under measure* $\mathbb{P}$.

Proof. Recall that,

$$S_T = s_0 \exp\left(\int_0^T \sigma(V_s)\, dW_s^\perp - \frac{1}{2} \int_0^T \sigma^2(V_s)\, ds \right),$$

we define for $t \in [0, T]$,

$$\hat{L}_t := \int_0^t \sigma(V_s)\, dW_s^\perp - \frac{1}{2} \int_0^t \sigma^2(V_s)\, ds = \int_0^t \sigma(V_s)\, d\hat{W}_s,$$

where

$$d\hat{W}_s = dW_s^\perp - \frac{1}{2} \sigma(V_s)\, ds$$

and

$$\hat{Y}_t := \sup_{s \le T} \hat{L}_s \ .$$

We claim that $(\hat{L}, \hat{Y})$ enjoys the reflection principle theorem 4.5 as process $\hat{L}$ is an Ocone martingale given $\mathcal{F}^V$ under measure $\tilde{\mathbb{P}}$, where by Girsanov's theorem,

$$\begin{aligned}
\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} &= \mathcal{E}\left(\int \frac{1}{2}\sigma\left(V_s\right)\, dW_s^{\perp}\right)_T \\
&= \exp\left(\frac{1}{2}\int_0^T \sigma\left(V_s\right)\, dW_s^{\perp} - \frac{1}{8}\int_0^T \sigma^2\left(V_s\right)\, ds\right) \\
&= \exp\left(\frac{1}{2}\hat{L}_T + \frac{1}{8}\Sigma_T\right)
\end{aligned}$$

Then, by the tower property, we have

$$E^{\mathbb{P}}\left[(K - S_T)^{+} 1_{\left\{\sup_{t\leq T} S_t \geq B\right\}}\right] = E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[(K - S_T)^{+} 1_{\left\{\sup_{t\leq T} S_t \geq B\right\}}\middle| \mathcal{F}^V\right]\right]$$

and we firstly calculate the inner conditional expectation

$$\begin{aligned}
& E^{\mathbb{P}}\left[(K - S_T)^{+} 1_{\left\{\sup_{t\leq T} S_t \geq B\right\}}\middle| \mathcal{F}^V\right] \\
= \; & E^{\mathbb{P}}\left[(K - S_T)\cdot 1_{\left\{S_T \leq K,\ \sup_{t\leq T} S_t \geq B\right\}}\middle| \mathcal{F}^V\right] \\
= \; & E^{\mathbb{P}}\left[\left(K - s_0 e^{\hat{L}_T}\right)\cdot 1_{\left\{\hat{L}_T \leq \ln\frac{K}{s_0},\ \hat{Y}_T \geq \ln\frac{B}{s_0}\right\}}\middle| \mathcal{F}^V\right] \\
= \; & E^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\cdot\left(K - s_0 e^{\hat{L}_T}\right)\cdot 1_{\left\{\hat{L}_T \leq \ln\frac{K}{s_0},\ \hat{Y}_T \geq \ln\frac{B}{s_0}\right\}}\middle| \mathcal{F}^V\right] \\
= \; & \int_{\ln\frac{B}{s_0}}^{\infty}\int_{-\infty}^{\ln\frac{K}{s_0}} \exp\left(-\frac{w}{2} - \frac{\Sigma_T}{8}\right)\cdot\left(K - s_0 e^{w}\right)\cdot\frac{2(2m-w)}{\sqrt{2\pi}\Sigma_T^{3/2}} \\
& \cdot\exp\left(-\frac{(2m-w)^2}{2\Sigma_T}\right) dw\, dm
\end{aligned}$$

$\mathbb{P}-$ a.s. Then,

$$E^{\mathbb{P}}\left[(K - S_T)^{+} 1_{\left\{\sup_{t\leq T} S_t \geq B\right\}}\right]$$

$$
\begin{aligned}
&= E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[(K-S_T)^{+}1_{\left\{\sup_{t\le T} S_t \ge B\right\}}\middle|\,\mathcal{F}^V\right]\right] \\
&= \int_0^{\infty} f^{\,\Sigma}(y)\int_{\ln\frac{B}{s_0}}^{\infty}\int_{-\infty}^{\ln\frac{K}{s_0}} (K-s_0e^{w})\cdot\frac{2(2m-w)}{\sqrt{2\pi}y^{3/2}} \\
&\quad \cdot\exp\left(-\frac{(2m-w)^2}{2y}-\frac{w}{2}-\frac{y}{8}\right)dw\,dm\,dy\ ,
\end{aligned}
$$

where we denote $f^{\,\Sigma}(y)$ as the marginal density of Σ_T under $\mathbb{P}$. ■

Lookback options

Let us then work on the other typical application of reflection principle in lookback options.

Theorem 4.8 *In the continuous stochastic volatility model* (4.1) *with zero correlation and zero interest rate, the value of an European lookback call option with maturity T and payoff $(\sup_{t\le T} S_t - K)^{+}$ is given by*

$$
\begin{aligned}
&\int_0^{\infty} f^{\,\Sigma}(y)\int_{\ln\frac{K}{s_0}}^{\infty}\int_{-\infty}^{m} (s_0\,e^{m}-K)\cdot\frac{2(2m-w)}{\sqrt{2\pi}y_T^{3/2}} \\
&\cdot\exp\left(-\frac{(2m-w)^2}{2y}-\frac{w}{2}-\frac{y}{8}\right)dw\,dm\,dy\,.
\end{aligned}
$$

Proof. We have

$$S_T = s_0\exp\left(\hat{L}_T\right),$$

hence

$$
\begin{aligned}
& E^{\mathbb{P}}\left[\left(\sup_{t\le T} S_t - K\right)^{+}\middle|\,\mathcal{F}^V\right] \\
&= E^{\mathbb{P}}\left[\left(s_0e^{\hat{Y}_T}-K\right)\cdot 1_{\left\{\hat{Y}_T\ge\ln\frac{K}{s_0}\right\}}\middle|\,\mathcal{F}^V\right] \\
&= E^{\tilde{\mathbb{P}}}\left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}\cdot\left(s_0e^{\hat{Y}_T}-K\right)\cdot 1_{\left\{\hat{Y}_T\ge\ln\frac{K}{s_0}\right\}}\middle|\,\mathcal{F}^V\right]
\end{aligned}
$$

$$= \int_{\ln \frac{K}{s_0}}^{\infty} \int_{-\infty}^{m} (s_0 \, e^m - K) \cdot \frac{2(2m-w)}{\sqrt{2\pi}\Sigma_T^{3/2}} \; \exp\left(- \frac{(2m-w)^2}{2\Sigma_T} - \frac{w}{2} - \frac{\Sigma_T}{8} \right) dw \, dm.$$

$\mathbb{P}-$ a.s. Therefore,

$$\begin{aligned}
& E^{\mathbb{P}} \left[\left(\sup_{t \leq T} S_t - K \right)^+ \right] \\
= \; & E^{\mathbb{P}} \left[E^{\mathbb{P}} \left[\left(\sup_{t \leq T} S_t - K \right)^+ \middle| \mathcal{F}^V \right] \right] \\
= \; & \int_0^{\infty} f^{\,\Sigma}(y) \int_{\ln \frac{K}{s_0}}^{\infty} \int_{-\infty}^{m} (s_0 \, e^m - K) \cdot \frac{2(2m-w)}{\sqrt{2\pi}\Sigma_T^{3/2}} \\
& \cdot \exp\left(- \frac{(2m-w)^2}{2y} - \frac{w}{2} - \frac{y}{8} \right) dw \, dm \, dy \, .
\end{aligned}$$

■

Theorem 4.9 *In the continuous stochastic volatility model* (4.1) *with zero correlation and zero interest rate, the value of a European lookback put option with maturity T and payoff $\sup_{t \leq T} S_t - S_T$ is given by*

$$\begin{aligned}
& s_0 \int_0^{\infty} f^{\,\Sigma}(y) \int_0^{\infty} \int_{-\infty}^{m} (\, e^m - e^w) \cdot \frac{2(2m-w)}{\sqrt{2\pi} y^{3/2}} \\
& \cdot \exp\left(- \frac{(2m-w)^2}{2y} - \frac{w}{2} - \frac{y}{8} \right) dw \, dm \, dy.
\end{aligned}$$

Proof. We have

$$S_T = s_0 \exp\left(\hat{L}_T \right)$$

hence

$$\begin{aligned}
& E^{\mathbb{P}} \left[\sup_{t \leq T} S_t - S_T \middle| \mathcal{F}^V \right] \\
= \; & E^{\mathbb{P}} \left[s_0 \, e^{\hat{Y}_T} - s_0 \, e^{\hat{L}_T} \middle| \mathcal{F}^V \right]
\end{aligned}$$

$$= s_0\, E^{\tilde{\mathbb{P}}} \left[\frac{d\mathbb{P}}{d\tilde{\mathbb{P}}} \cdot \left(e^{\hat{Y}_T} - e^{\hat{L}_T} \right) \middle| \mathcal{F}^V \right]$$

$$= s_0 \int_0^{\infty} \int_{-\infty}^{m} (\, e^m - e^w) \cdot \frac{2(2m-w)}{\sqrt{2\pi}\Sigma_T^{3/2}} \exp\left(-\frac{(2m-w)^2}{2\Sigma_T} - \frac{w}{2} - \frac{\Sigma_T}{8} \right) dw\, dm,$$

$\mathbb{P}-$ a.s. Therefore,

$$\begin{aligned} & E^{\mathbb{P}}\left[\sup_{t\leq T} S_t - S_T \right] \\ = \;& E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[\sup_{t\leq T} S_t - S_T \middle| \mathcal{F}^V \right] \right] \\ = \;& s_0 \int_0^{\infty} f^{\,\Sigma}(y) \int_0^{\infty} \int_{-\infty}^{m} (\, e^m - e^w) \cdot \frac{2(2m-w)}{\sqrt{2\pi} y^{3/2}} \\ & \cdot \exp\left(-\frac{(2m-w)^2}{2y} - \frac{w}{2} - \frac{y}{8} \right) dw\, dm\, dy. \end{aligned}$$

■

4.4.2 Valuation formula in the general model

In this section, we work on more complicate cases of pricing exotic options in the continuous stochastic volatility model (4.1) with interest rate and non-zero correlation.

In the general model, we solve that

$$S_T = s_0\, \mathcal{E} \left(rT + \overline{\rho} \int \sigma(V)\, dW^{\perp} + \rho \int \sigma(V)\, dW \right)_T$$

and define

$$\mu_T := rT + \rho \int_0^T \sigma(V_s)\; dW_s - \frac{1}{2} \int_0^T \sigma^2(V_s)\, ds\ .$$

Hence we have

$$S_T = s_0 \exp(\overline{\rho} \int_0^T \sigma(V_s)\; dW_s^{\perp} + \mu_T)$$

with

$$\mu_T = rT + \rho\left(\int_0^T \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^T \frac{\sigma(V_s)\,\mu(V_s)}{\gamma(V_s)} ds\right) - \frac{1}{2}\int_0^T \sigma^2\left(V_s\right) ds.$$

We define for $t \in [0, T]$

$$\tilde{L}_t = \overline{\rho}\int_0^T \sigma\left(V_s\right)\, dW_s^{\perp} + \mu_T$$

and

$$\tilde{Y}_T := \sup_{s \leq T} \tilde{L}_s\ . \tag{4.2}$$

Note that the changing of measure technique from the last section does not apply since the Radon-Nikodym derivative involves terms such as: $\int_0^T \frac{r}{\sigma(V_s)} dW_s^{\perp}$ and $\int_0^T \frac{dV_s/ds}{\gamma(V_s)} dW_s^{\perp}$. Therefore we work directly on the joint density of $(\tilde{L}, \tilde{Y})$ via the reflection principle theorem and the Brownian motion with drift problem.

Firstly, let us recall the following well-known results on distribution of the maximum of Brownian motions.

Lemma 4.10 *Let B be the standard Brownian motion defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we have, for $m \geq 0$,*

$$\mathbb{P}\left(\sup_{s \leq T} B_s \geq m,\ B_T \in dw\right) = \begin{cases} \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(2m-w)^2}{2T}\right) dw, & w \leq m, \\ \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{w^2}{2T}\right) dw, & w > m. \end{cases}$$

Proof. The result follows directly from the reflection principle of Brownian motions. ■

Proposition 4.11 *Let B^{μ} denote a Brownian motion endowed with drift μ, we have, for $m \geq 0$,*

$$\begin{aligned} &\mathbb{P}\left(\sup_{s \leq T} B_s^{\mu} \geq m,\ B_T^{\mu} \in dw\right) \\ = &\begin{cases} \frac{1}{\sqrt{2\pi T}} \exp(2\mu\,(w-m)) \exp\left(-\frac{(2m-w-\mu T)^2}{2T}\right) dw, & w \leq m, \\ \frac{1}{\sqrt{2\pi T}} \exp\left(-\frac{(w-\mu T)^2}{2T}\right) dw, & w > m. \end{cases} \end{aligned} \tag{4.3}$$

Proof. The result can be easily obtained from Lemma 4.10 and by using the law of an absorbed Brownian motion with drift. ■

Now we are ready to present the conditional joint density of $(\tilde{L}, \tilde{Y})$.

Theorem 4.12 *Let* $(\tilde{L}, \tilde{Y})$ *be defined as in* (4.2)*, given* $\mathcal{F}^V$*, we have the conditional joint density*

$$\begin{aligned} & \mathbb{P}\left(\tilde{Y}_T \in dm,\ \tilde{L}_T \in dw\right) \\ = \; & \frac{2(2m-w)}{\sqrt{2\pi}((1-\rho^2)\,\Sigma_T)^{3/2}} \exp\left(\frac{\mu_T\, w}{(1-\rho^2)\,\Sigma_T} - \frac{\mu_T^2}{2\,(1-\rho^2)\,\Sigma_T}\right) \\ & \cdot \exp\left(-\frac{(2m-w)^2}{2\,(1-\rho^2)\,\Sigma_T}\right) \end{aligned}$$

for $w \leq m$, $m > 0$*, where*

$$\Sigma_T = \int_0^T \sigma^2\left(V_s\right) ds$$

and

$$\mu_T = rT + \rho\left(\int_0^T \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^T \frac{\sigma(V_s)\,\mu(V_s)}{\gamma(V_s)} ds\right) - \frac{1}{2}\int_0^T \sigma^2\left(V_s\right) ds.$$

Proof. Recall that

$$\tilde{L}_t = \overline{\rho} \int_0^T \sigma\left(V_s\right)\, dW_s^{\perp} + \mu_T$$

and by applying the Dambis-Dubins-Schwartz representation of $\overline{\rho}\,\int_0^T \sigma\left(V_s\right)\, dW_s^{\perp}$, we have

$$\begin{aligned} \tilde{L}_T &= \widehat{W^{\perp}}_{\overline{\rho}^2 \int_0^T \sigma^2(V_s)\, ds} + \mu_T \\ &= \widehat{W^{\perp}}_{(1-\rho^2)\,\Sigma_T} + \frac{\mu_T}{(1-\rho^2)\,\Sigma_T} \cdot \left(1-\rho^2\right)\Sigma_T \end{aligned}$$

Note that given $\mathcal{F}^V$, Σ_T and μ_T are deterministic. Therefore, it follows Proposition 4.11 with a deterministic change of time

$$T \to \left(1-\rho^2\right) \int_0^T \sigma^2(V_s)\, ds$$

and drift

$$\mu \to \frac{\mu_T}{(1-\rho^2)\ \Sigma_T}.$$

Hence we have for $w \leq m$, $m > 0$,

$$
\begin{aligned}
&\mathbb{P}\left(\tilde{Y}_T \geq m,\, \tilde{L}_T \in dw\right) \\
= \quad &\frac{1}{\sqrt{2\pi\left(1-\rho^2\right)\Sigma_T}} \exp\left(\frac{2\mu_T\,(w-m)}{\left(1-\rho^2\right)\,\Sigma_T}\right) \exp\left(-\frac{\left(2m\, -\, w\, -\, \mu_T\right)^2}{2\left(1-\rho^2\right)\,\Sigma_T}\right)\, dw.
\end{aligned}
$$

Differentiation with respect to m leads to the result. ■

Recall that the price of an option with payoff function $G(S_T)$ related to the maximum of the underlying price process, such as barrier options and lookback options, equals to (let $\mathbb{P}$ be the risk neutral pricing measure)

$$E^{\mathbb{P}}\left[G(S_T)\right] = E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[G(S_T)\middle|\, \mathcal{F}^V\right]\right]$$

Then we can firstly calculate the inner expectation $E^{\mathbb{P}}\left[G(S_T)\middle|\, \mathcal{F}^V\right]$ with the joint density derived in Theorem 4.12, while it generally requires the joint density of

$$\left(\int_0^T f(V_s)\, dV_s,\, \int_0^T g\left(V_s\right)\, ds\right)$$

so as to calculate the outer expectation, where f and g depend on the specific models.

Now let us present the closed-form pricing formula in the following classic stochastic volatility models:

Heston model

Assume that the price process is given by the Heston model:

$$dS_t = S_t\left(r\, dt + \sqrt{V_t}\, dZ_t\right),.$$

$$dV_t = \kappa(\theta - V_t)\, dt + \gamma\sqrt{V_t} dW_t, \tag{4.4}$$

for $t \in [0, T]$, $Z = \rho W + \overline{\rho} W^{\perp}$, $r \geq 0$, $\kappa \geq 0$, $\theta \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Note that, in the Heston model,

$$\Sigma_T = \int_0^T V_s ds,$$

$$\mu_T = rT + \frac{\rho}{\gamma}\left(V_T - V_0 - \kappa\theta T + \kappa\int_0^T V_s ds\right) - \frac{1}{2}\int_0^T V_s ds$$

and

$$\tilde{L}_T = \overline{\rho}\int_0^T \sqrt{V_s}\, dW_s^{\perp} + \mu_T.$$

Theorem 4.13 *In the Heston model* (4.4)*, the value of an European up-and-in put option with maturity T and payoff $(K - S_T)^+ 1_{\left\{\sup_{t\leq T} S_t \geq B\right\}}$, for $s_0 \leq K \leq B$, is given by*

$$e^{-rT}\int_0^\infty\int_0^\infty g^{V,\Sigma}(z,y)\int_{\ln\frac{B}{s_0}}^\infty\int_{-\infty}^{\ln\frac{K}{s_0}}(K - s_0 e^{\,w}\,)\cdot\frac{2(2m-w)}{\sqrt{2\pi}(\,(1-\rho^2)\;y\,)^{3/2}}$$
$$\cdot\exp\left(-\frac{(2m-w)^2 + 2\mu(z,y)\,w - \mu^2(z,y)}{2\,(1-\rho^2)\,y}\right)dw\,dm\,dy\,dz$$

where

$$\mu(z,y) = rT + \frac{\rho}{\gamma}\,(z - v_0 - \kappa\theta T + \kappa y) - \frac{1}{2}y\ ,$$

and the joint density of $\left(V_T, \int_0^T V_s ds\right)$ under $\mathbb{P}$ is denoted by $g^{V,\Sigma}(z,y)$.

Proof. Recall that

$$S_T = s_0\exp\left(\tilde{L}_T\right).$$

By the tower property, we have

$$E^{\mathbb{P}}\left[(K - S_T)^+ 1_{\left\{\sup_{t\leq T} S_t\geq B\right\}}\right] = E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[(K - S_T)^+ 1_{\left\{\sup_{t\leq T} S_t\geq B\right\}}\middle|\,\mathcal{F}^V\right]\right]$$

and the inner conditional expectation equals to

$$E^{\mathbb{P}}\left[(K - S_T)^+ 1_{\left\{\sup_{t\leq T} S_t\geq B\right\}}\middle|\,\mathcal{F}^V\right]$$

$$= \quad E^{\mathbb{P}}\left[(K - S_T)\cdot 1_{\left\{S_T\,\leq K,\ \sup_{t\,\leq\,T} S_t\,\geq\,B\right\}}\middle|\,\mathcal{F}^V\right]$$

$$= E^{\mathbb{P}}\left[\left(K - s_0 e^{\tilde{L}_T}\right)\cdot 1_{\left\{\tilde{L}_T \le \ln\frac{K}{s_0},\, \tilde{Y}_T \ge \frac{B}{s_0}\right\}}\middle|\, \mathcal{F}^V\right]$$

$$= \int_{\ln\frac{B}{s_0}}^{\infty}\int_{-\infty}^{\frac{K}{s_0}} (K - s_0 e^{w})\cdot \frac{2(2m-w)}{\sqrt{2\pi}((1-\rho^2)\,\Sigma_T)^{3/2}} \cdot \exp\left(-\frac{(2m-w)^2 + 2\mu_T\, w - \mu_T^2}{2\,(1-\rho^2)\,\Sigma_T}\right) dw\, dm$$

$\mathbb{P}-$ a.s.

Then, the up-and-in put price equals to

$$e^{-rT} E^{\mathbb{P}}\left[(K - S_T)^+ 1_{\{\sup_{t\le T} S_t \ge B\}}\right]$$

$$= e^{-rT} E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[(K - S_T)^+ 1_{\{\sup_{t\le T} S_t \ge B\}}\middle|\, \mathcal{F}^V\right]\right]$$

$$= e^{-rT}\int_0^{\infty}\int_0^{\infty} g^{V,\Sigma}(z,y)\int_{\ln\frac{B}{s_0}}^{\infty}\int_{-\infty}^{\ln\frac{K}{s_0}} (K - s_0 e^{w})\cdot \frac{2(2m-w)}{\sqrt{2\pi}(\,(1-\rho^2)\,\, y\,)^{3/2}} \cdot \exp\left(-\frac{(2m-w)^2 + 2\mu(z,y)\, w - \mu^2(z,y)}{2\,(1-\rho^2)\, y}\right) dw\, dm\, dy\, dz$$

where

$$\mu(z,y) = rT + \frac{\rho}{\gamma}(z - v_0 - \kappa\theta T + \kappa y) - \frac{1}{2}y$$

and we denote the joint density of $(V_T, \int_0^T V_s ds)$ under $\mathbb{P}$ by $g^{V,\Sigma}(z,y)$. ■

Now, let us work on the lookback options. Firstly, we derive the density of $\tilde{Y}$ in the following lemma.

Lemma 4.14 *In the Heston model (4.4), let $\tilde{Y}_T := \sup_{s\le T}\tilde{L}_s$ where $\tilde{L}_t = \overline{\rho}\int_0^t \sqrt{V_s}\, dW_s^{\perp} + \mu_t$, given $\mathcal{F}^V$, we have the density, for $m \ge 0$,*

$$\mathbb{P}\left(\tilde{Y}_T \in dm\right) = \frac{2}{\sqrt{2\pi\,(1-\rho^2)\,\Sigma_T}}\exp\left(\frac{-\left(m-\mu_T\right)^2}{2\,(1-\rho^2)\,\Sigma_T}\right) + \frac{2\mu_T}{(1-\rho^2)\,\Sigma_T}\exp\left(\frac{2\mu_T\, m}{(1-\rho^2)\,\Sigma_T}\right)\Phi\left(-\frac{m+\mu_T}{\sqrt{(1-\rho^2)\,\Sigma_T}}\right),$$

where Φ *denotes the standard normal CDF:*

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

$$\mu(z, y) = rT + \frac{\rho}{\gamma}(z - v_0 - \kappa\theta T + \kappa y) - \frac{1}{2} y \ .$$

Proof. By integrating (4.4) w.r.t. w, we have

$$\begin{aligned}\mathbb{P}\left(\tilde{Y}_T \geq m\right) &= \Phi\left(-\frac{m-\mu_T}{\sqrt{(1-\rho^2)\,\Sigma_T}}\right) \\ &\quad + \exp\left(\frac{2\mu_T\, m}{(1-\rho^2)\,\Sigma_T}\right)\Phi\left(-\frac{m+\mu_T}{\sqrt{(1-\rho^2)\,\Sigma_T}}\right)\end{aligned}$$

and then differentiating w.r.t. m leads to the result. ■

Theorem 4.15 *In the Heston model* (4.4)*, the value of an European lookback call option with maturity* T *and payoff* $\left(\sup_{t\leq T} S_t - K\right)^+$*, is given by*

$$e^{-rT}\int_0^\infty\int_0^\infty g^{V,\Sigma}(z,y)\int_{\ln\frac{K}{s_0}\vee 0}^{\infty}(s_0 e^m - K)\cdot\left(\frac{2}{\sqrt{2\pi\,(1-\rho^2)\,\Sigma_T}}\exp\left(\frac{-(m-\mu_T)^2}{2\,(1-\rho^2)\,\Sigma_T}\right)\right.$$

$$\left.+\frac{2\mu_T}{(1-\rho^2)\,\Sigma_T}\exp\left(\frac{2\mu_T\, m}{(1-\rho^2)\,\Sigma_T}\right)\Phi\left(-\frac{m+\mu_T}{\sqrt{(1-\rho^2)\,\Sigma_T}}\right)\right)dm\ dy\, dz$$

where

$$\mu(z, y) = rT + \frac{\rho}{\gamma}(z - v_0 - \kappa\theta T + \kappa y) - \frac{1}{2} y \ ,$$

and the joint density of $\left(V_T, \int_0^T V_s ds\right)$ *under* $\mathbb{P}$ *is denoted by* $g^{V,\Sigma}(z, y)$*.*

Proof.

$$S_T = s_0 \exp\left(\tilde{L}_T\right)$$

hence

$$E^{\mathbb{P}}\left[\left(\sup_{t\leq T} S_t - K\right)^+ \middle|\, \mathcal{F}^V\right]$$

$$= \quad E^{\mathbb{P}}\left[\left(s_0\, e^{\tilde{Y}_T} - K\right)\cdot 1_{\left\{\tilde{Y}_T \,\geq \ln\frac{K}{s_0}\right\}}\middle|\, \mathcal{F}^V\right]$$

$$= \quad \int_{\ln\frac{K}{s_0}\vee 0}^{\infty} \left(s_0 e^{\,m} - K\right)\cdot \left(\frac{2}{\sqrt{2\pi\left(1-\rho^2\right)\Sigma_T}}\exp\left(\frac{-\left(m-\mu_T\right)^2}{2\left(1-\rho^2\right)\Sigma_T}\right)\right.$$
$$\left. + \frac{2\mu_T}{\left(1-\rho^2\right)\Sigma_T}\exp\left(\frac{2\mu_T\, m}{\left(1-\rho^2\right)\Sigma_T}\right)\,\Phi\left(-\frac{m+\mu_T}{\sqrt{\left(1-\rho^2\right)\Sigma_T}}\right)\right) dm$$

$\mathbb{P}-$ a.s.

Then the price of the lookback option equals to

$$e^{-rT} E^{\mathbb{P}}\left[\left(\sup_{t\leq T} S_t - K\right)^+\right]$$
$$= \quad e^{-rT} E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[\left(\sup_{t\leq T} S_t - K\right)^+\middle|\, \mathcal{F}^V\right]\right]$$
$$= \quad e^{-rT}\int_0^\infty\int_0^\infty g^{V,\Sigma}(z,y)\int_{\ln\frac{K}{s_0}\vee 0}^{\infty}\left(s_0 e^{\,m} - K\right)\cdot\left(\frac{2}{\sqrt{2\pi\left(1-\rho^2\right)\Sigma_T}}\exp\left(\frac{-\left(m-\mu_T\right)^2}{2\left(1-\rho^2\right)\Sigma_T}\right)\right.$$
$$\left. + \frac{2\mu_T}{\left(1-\rho^2\right)\Sigma_T}\exp\left(\frac{2\mu_T\, m}{\left(1-\rho^2\right)\Sigma_T}\right)\,\Phi\left(-\frac{m+\mu_T}{\sqrt{\left(1-\rho^2\right)\Sigma_T}}\right)\right) dm\; dy\, dz$$

■

Theorem 4.16 *In the general Heston model* (4.4), *the value of an European lookback option with maturity T and payoff $\sup_{t\leq T} S_t - S_T$ is given by*

$$s_0 e^{-rT}\int_0^\infty\int_0^\infty g^{V,\Sigma}(z,y)\int_0^\infty\int_{-\infty}^{m}\left(e^{\,m} - e^{\,w}\right)\cdot\frac{2(2m-w)}{\sqrt{2\pi}\left(\,(1-\rho^2)\;y\,\right)^{3/2}}$$
$$\cdot\exp\left(-\frac{(2m-w)^2 + 2\mu(z,y)\,w - \mu^2(z,y)}{2\left(1-\rho^2\right)y}\right) dw\, dm\, dy\, dz\ .$$

Proof. We have

$$S_T = s_0\exp\left(\tilde{L}_T\right)$$

hence

$$E^{\mathbb{P}}\left[\sup_{t\le T} S_t - S_T \middle| \mathcal{F}^V\right] = E^{\mathbb{P}}\left[s_0\, e^{\tilde{Y}_T} - s_0\, e^{\tilde{L}_T} \middle| \mathcal{F}^V\right]$$

$$= s_0 \int_0^\infty \int_{-\infty}^m (e^{\,m} - e^{\,w}) \cdot \frac{2(2m-w)}{\sqrt{2\pi}((1-\rho^2)\,\Sigma_T)^{3/2}} \cdot \exp\left(-\frac{(2m-w)^2 + 2\mu_T\, w - \mu_T^2}{2\,(1-\rho^2)\,\Sigma_T}\right) dw\, dm$$

Therefore,

$$e^{-rT} E^{\mathbb{P}}\left[\sup_{t\le T} S_t - S_T\right] = e^{-rT} E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[\sup_{t\le T} S_t - S_T \middle| \mathcal{F}^V\right]\right]$$

$$= s_0 e^{-rT} \int_0^\infty \int_0^\infty g^{V,\Sigma}(z,y) \int_0^\infty \int_{-\infty}^m (e^{\,m} - e^{\,w}) \cdot \frac{2(2m-w)}{\sqrt{2\pi}(\,(1-\rho^2)\;y\,)^{3/2}} \cdot \exp\left(-\frac{(2m-w)^2 + 2\mu(z,y)\, w - \mu^2(z,y)}{2\,(1-\rho^2)\, y}\right) dw\, dm\, dy\, dz$$

■

4.4.3 The density for $\int_0^T V_s\, ds$ and V_T in the Heston model

We note that in the Heston model, the variance process V is constructed by a Cox-Ingersoll-Ross process with

$$dV_t = \kappa(\theta - V_t)\; dt + \gamma\sqrt{V_t}\, dW_t, \quad V_0 = v_0 > 0. \tag{4.5}$$

And the density of V_T, $\int_0^T V_s\, ds$, as well as $\left(V_T, \int_0^T V_s\, ds\right)$ is well-known, e.g., see [10], [13] and [14].

Laplace transform method

The Laplace transform of $\int_0^T V_s\, ds$ is (see [10]).

$$\mathcal{L}_f(p) = E\left[\exp\left(-\,p\int_0^T V_s ds\right)\right] = A\,(p,\, T)\exp\left(v_0\, B\,(p,\, T)\right)$$

for all $p \in \mathbb{C}$, where

$$\begin{aligned}
A\,(p,\,T) &= \frac{\exp\left(\frac{\kappa^2\theta T}{\gamma^2}\right)}{\left(\cosh\left(\frac{\xi T}{2}\right)+\frac{\kappa}{\xi}\sinh\left(\frac{\xi T}{2}\right)\right)^{2\kappa\theta/\gamma^2}} \\
B\,(p,\,T) &= \frac{-2\,p}{\kappa+\xi\coth\left(\frac{\xi T}{2}\right)} \\
\xi &= \sqrt{\kappa^2+2\,\gamma^2 p}
\end{aligned}$$

By applying the inverse Laplace transform, we have the marginal density of $\int_0^T V_s\,ds$,

$$f^{\,\Sigma}(y) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{yp}\mathcal{L}_f\,(p)\,dp$$

and the joint Laplace transform of $\left(V_T,\int_0^T V_s\,ds\right)$ is given in [23]:

$$\begin{aligned}
\mathcal{L}_g\,(u,\;p) &= E\left[\exp\left(-uV_T - \;p\int_0^T V_s\,ds\right)\right] \\
&= \exp\left(-\kappa\theta\,A\,(u,\,p,\,T)\right)\exp\left(-v_0\,B\,(u,\,p,\,T)\right)
\end{aligned}$$

for all $p \in \mathbb{C}$, where

$$\begin{aligned}
A\,(u,\,p,\,T) &= -\frac{2}{\gamma^2}\log\left(\frac{2\xi\exp\left(\frac{(\xi+\kappa)T}{2}\right)}{\gamma^2\,u\;\left(\exp\left(\xi T\right)-1\right)+\left(\xi-\kappa\right)+\left(\xi+\kappa\right)\exp\left(\xi T\right)}\right) \\
B\,(u,\,p,\,T) &= \frac{u\;\left(\xi+\kappa+\left(\xi-\kappa\right)\exp\left(\xi T\right)\right)+2p\;\left(\exp\left(\xi T\right)-1\right)}{\gamma^2\,u\;\left(\exp\left(\xi T\right)-1\right)+\left(\xi-\kappa\right)+\left(\xi+\kappa\right)\exp\left(\xi T\right)} \\
\xi &= \sqrt{\kappa^2+2\,\gamma^2 p}
\end{aligned}$$

By applying the inverse Laplace transform, we have the joint density of $\left(V_T, \int_0^T V_s ds\right)$,

$$g^{V,\Sigma}(z,y) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty}\int_{R-i\infty}^{R+i\infty} e^{y(u+p)}\mathcal{L}_g\left(u,\, p\right) du\, dp$$

Explicit form

Dassios and Nagaradjasarma derive the marginal density of $\int_0^T V_s\, ds$ and the joint density of $(\int_0^T V_s\, ds, V_T)$ in the Cox-Ingersoll-Ross framework in their paper ([13],[14]), which we apply to construct the closed-form solution of pricing the exotic options in the Heston model.

Theorem 4.17 *[14] In the Heston model, the marginal density of $\int_0^T V_s\, ds$ (denoted as Σ_T) is given by*

$$f^{\,\Sigma}(y) = \beta \exp\left(\frac{b(aT+v_0)}{\sigma^2} - b^2 y\beta\right)\sum_{k=0}^{\infty}\frac{f_k^{\,\Sigma}(y)}{2^k}$$

where

$$f_k^{\,\Sigma}(y) = \sum_{n=0}^{k}\sum_{m=n}^{k}\binom{k+\frac{2a}{\sigma^2}-1}{k-n}\binom{k-n}{m-n}\frac{(-2v_0)^n}{n!\sigma^{2n}}(-1)^m\, I_{k,k-n}(y,\varpi_m)$$

with

$$\varpi_m = \frac{aT+v_0}{\sigma^2} + mT \qquad \text{and} \qquad \beta = \frac{1}{2\sigma^2}$$

and a sequence $I_{p,q}(y,\varpi)$ in the following recursive way for positive integers p and q

For $q=0$

$$I_{p+1,0}(y,\varpi) = I_{p,0}(y,\varpi) - \sqrt{\frac{2}{\pi}}\frac{(b\varpi+p+1)He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+3}}}$$

For $q=1$

$$I_{p,1}(y,\varpi) = \sqrt{\frac{2}{\pi}}\frac{He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+1}}}$$

For $q=2$

$$I_{p+1,2}(y,\varpi) = \sqrt{\frac{2}{\pi}}\frac{He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+1}}} - bI_{p,2}(y,\varpi)$$

For $q = 3$

$$I_{p,3}(y,\varpi) = p1_{(p>0)}I_{p-1,2}(y,\varpi) - \varpi I_{p,2}(y,\varpi) + \sqrt{\frac{2}{\pi}}\frac{He_p\left(\frac{\varpi-2yb\beta}{\sqrt{2y\beta}}\right)e^{-\frac{\varpi^2}{4y\beta}}}{\sqrt{(2y\beta)^{p+1}}}$$

For $q > 3$

$$I_{p,q}(y,\varpi) = \frac{p1_{(p>0)}I_{p-1,q-1}(y,\varpi) + 2y\beta I_{p,q-2}(y,\varpi) - \varpi I_{p,q-1}(y,\varpi)}{q-2}$$

with the initial conditions

$$\begin{cases} I_{0,0}(y,\varpi) = \frac{\varpi}{2\sqrt{\pi(y\beta)^3}}e^{-\frac{\varpi^2}{4y\beta}} \\ I_{0,2}(y,\varpi) = \operatorname{erf}c\left(\frac{\varpi}{2\sqrt{y\beta}}\right) \end{cases}$$

and the Hermite polynomials

$$He_k(x) = \sum_{s=0}^{[k/2]}(-1)^s\frac{x^{k-2s}}{2^s}\frac{k!}{(k-2s)!s!}$$

Proof. See [14] Theorem 3.1. ■

Theorem 4.18 *[13] In the Heston model, the joint density of* V_T *and* $\int_0^T V_s ds$ *(denoted as* Σ_T*) is given by*

$$g^{V,\Sigma}(z,y) = \frac{(\frac{z}{\sqrt{2}})^{\frac{2\kappa}{\sigma^2}-1}}{2\sqrt{\pi}(\sqrt{y\alpha})^{\frac{2\kappa}{\sigma^2}+2}}e^{-\frac{\theta^2 y}{2\sigma^2}-\frac{\theta(z-v_0)}{\sigma^2}+\frac{\kappa\theta T}{\sigma^2}}\sum_{n=0}^{\infty}\frac{n!\alpha}{\Gamma(n+\frac{2\kappa}{\sigma^2})}N_n(y),$$

with the term $N_n(y)$ *defined as*

$$\begin{aligned} N_n(y) &= \sum_{p=0}^{n}\frac{\binom{n+\frac{2\kappa}{\sigma^2}-1}{n-p}}{p!}(\frac{-z}{\sqrt{2y\alpha}})^p \\ &\quad \sum_{q=0}^{n}\frac{\binom{n+\frac{2\kappa}{\sigma^2}-1}{n-q}}{p!}(\frac{-v_0}{\sqrt{2y\alpha}})^q D_\omega(\frac{\alpha_n}{\sqrt{2y\alpha}})e^{-\frac{\alpha_n^2}{8y\alpha}}, \\ \alpha &= \frac{\sigma^2}{8}, \\ \alpha_n &= \frac{z+v_0+(\kappa+n\sigma^2)T}{2}, \end{aligned}$$

$$\omega \;=\; p+q+\frac{2\kappa}{\sigma^2}+1.$$

and D_ω is the parabolic cylinder function of order ω.

Proof. See [13] Theorem 4.2. ■

Stein & Stein model

Assume that the price process is given by the Stein & Stein model:

$$dS_t = S_t\;(r\,dt + V_t\,dZ_t)\,,.$$

$$dV_t = \kappa(\theta - V_t)\;dt + \gamma\,dW_t,$$

for $t \in [0,T]$, $Z = \rho W + \overline{\rho} W^{\perp}$, $r \geq 0$, $\kappa \geq 0$, $\theta \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Note that, in the Stein & Stein model,

$$\Sigma_T = \int_0^T V_s^2 ds,$$

$$\begin{aligned}
\mu_T &= rT + \rho\left(\int_0^T \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^T \frac{\sigma(V_s)\,\mu(V_s)}{\gamma(V_s)} ds\right) - \frac{1}{2}\int_0^T \sigma^2(V_s)\,ds \\
&= rT + \frac{\rho}{\gamma}\left(\int_0^T V_s\,dV_s - \kappa\int_0^T (\theta - V_s)V_s ds\right) - \frac{1}{2}\int_0^T V_s^2 ds
\end{aligned}$$

and

$$\tilde{L}_T = \overline{\rho}\int_0^T V_s\,dW_s^{\perp} + \mu_T.$$

Then the pricing formula for barrier and lookback options remain the same form as in the Heston model except for different Σ_T, μ_T and one needs to derive the joint density of $\left(V_T^2,\; \int_0^T V_s\,ds,\; \int_0^T V_s^2 ds\right)$.

Remark 4.19 *When $\theta = 0$, the problem is easy to solve, that is, we only have to calculate joint density of $\left(V_T^2,\; \int_0^T V_s^2 ds\right)$ instead. And note that, in that case, by Itō's formula*

$$dV_s^2 \;=\; 2V_s\,dV_s + d[V]_s$$

$$
\begin{aligned}
&= -2\kappa V_s^2 ds + 2\gamma V_s dW_s + \gamma^2 ds \\
&= 2\kappa\left(\frac{\gamma^2}{2\kappa} - V_s^2\right) ds + 2\gamma\sqrt{V_s^2} dW_s \ .
\end{aligned}
$$

Therefore V^2 is a C.I.R. process and we are dealing the same problem as in the Heston model with new parameters $\kappa' = 2\kappa$, $\theta' = \frac{\gamma^2}{2\kappa}$ and $\gamma' = 2\gamma$.

Hull & White model

Assume that the price process is given by the Hull & White model:

$$
dS_t = S_t \left(r\, dt + \sqrt{V_t}\, dZ_t\right),.
$$

$$
dV_t = \mu V_t\, dt + \gamma V_t\, dW_t,
$$

for $t \in [0,T]$, $Z = \rho W + \overline{\rho} W^{\perp}$, $r \geq 0$, $\kappa \geq 0$, $\theta \geq 0$, $\gamma \geq 0$, $V_0 = v_0 > 0$, $S_0 = s_0 > 0$.

Note that, in the Hull & White model,

$$
\Sigma_T = \int_0^T V_s ds,
$$

$$
\begin{aligned}
\mu_T &= rT + \rho\left(\int_0^T \frac{\sigma(V_s)}{\gamma(V_s)} dV_s - \int_0^T \frac{\sigma(V_s)\,\mu(V_s)}{\gamma(V_s)} ds\right) - \frac{1}{2}\int_0^T \sigma^2(V_s)\, ds \\
&= rT + \frac{\rho}{\gamma}\left(\int_0^T V_s^{-\frac{1}{2}}\, dV_s - \mu\int_0^T \sqrt{V_s} ds\right) - \frac{1}{2}\int_0^T V_s ds
\end{aligned}
$$

and

$$
\tilde{L}_T = \overline{\rho}\int_0^T \sqrt{V_s}\, dW_s^{\perp} + \mu_T.
$$

By Itō's formula and integration by parts, we have

$$
dV_s^{-\frac{1}{2}} = -\frac{1}{2} V_s^{-\frac{3}{2}} dV_s + \frac{1}{2}\cdot\frac{3}{4} V_s^{-\frac{5}{2}} d\left[V\right]_s
$$

$$
\begin{aligned}
&= -\frac{1}{2}V_s^{-\frac{1}{2}}\left(\mu\, ds + \gamma\, dW_s\right) + \frac{3\gamma^2}{8}V_s^{-\frac{1}{2}}ds \\
&= V_s^{-\frac{1}{2}}\left(\left(\frac{3\gamma^2}{8} - \frac{\mu}{2}\right)ds - \frac{\gamma}{2}dW_s\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T V_s^{-\frac{1}{2}}\, dV_s &= \sqrt{V_T} - \sqrt{V_0} - \int_0^T V_s\, dV_s^{-\frac{1}{2}} - \int_0^T d\left[V^{-\frac{1}{2}}, V\right]_s \\
&= \sqrt{V_T} - \sqrt{v_0} - \int_0^T \sqrt{V_s}\left(\left(\frac{3\gamma^2}{8} - \frac{\mu}{2}\right)ds - \frac{\gamma}{2}dW_s\right) \\
&\quad + \frac{\gamma^2}{2}\int_0^T \sqrt{V_s}ds \\
&= \sqrt{V_T} - \sqrt{v_0} + \left(\frac{\gamma^2}{8} + \frac{\mu}{2}\right)\int_0^T \sqrt{V_s}ds + \frac{\gamma}{2}\int_0^T \sqrt{V_s}dW_s\ .
\end{aligned}
$$

Here we recall that

$$
\int_0^T V_s^{-\frac{1}{2}}\, dV_s - \mu\int_0^T \sqrt{V_s}ds = \gamma\int_0^T \sqrt{V_s}dW_s\ .
$$

Therefore, we solve that

$$
\gamma\int_0^T \sqrt{V_s}dW_s = \sqrt{V_T} - \sqrt{v_0} + \left(\frac{\gamma^2}{8} - \frac{\mu}{2}\right)\int_0^T \sqrt{V_s}ds + \frac{\gamma}{2}\int_0^T \sqrt{V_s}dW_s\ ,
$$

$$
\int_0^T \sqrt{V_s}dW_s = \frac{2}{\gamma}\left(\sqrt{V_T} - \sqrt{v_0} + \left(\frac{\gamma^2}{8} - \frac{\mu}{2}\right)\int_0^T \sqrt{V_s}ds\right)\ ,
$$

$$
\begin{aligned}
\mu_T &= rT + \rho\int_0^T \sqrt{V_s}dW_s - \frac{1}{2}\int_0^T V_s\, ds \\
&= rT + \frac{2\rho}{\gamma}\left(\sqrt{V_T} - \sqrt{v_0} + \left(\frac{\gamma^2}{8} - \frac{\mu}{2}\right)\int_0^T \sqrt{V_s}ds\right) - \frac{1}{2}\int_0^T V_s\, ds\ .
\end{aligned}
$$

Then the pricing formula for barrier and lookback options remain the same form as in the Heston model except for different Σ_T, μ_T and the joint density of $\left(\sqrt{V_T},\ \int_0^T \sqrt{V_s}ds,\ \int_0^T V_s ds\right)$, where V is a geometric Brownian motion.

4.5 Numerical Simulation

First, we study the performance of the pricing formula for up-and-in put options with payoff $(K-S_T)^+1_{\{\sup_{t\leq T} S_t \geq B\}}$ in the Heston model. Recall that when there is no correlation between the stock prices and the volatility, the value of the up-and-in put option equals to that of K/B units of vanilla calls with strike B^2/K by PCS and we calculate it via the closed-form formula derived in [24].

We show in table 4.1 and figure 4.1 the performance of our pricing formula with different strike prices. It's easy to see that the valuation formula works well as the relative error is very small and decreasing with the decrease of the difference between the barrier and strike price.

Table 4.1. Performance of the pricing formula as a function of K in a Heston model. We take $S_0 = 100$, $B = 110$, $r = 0$, $\kappa = 4$, $\theta = 0.04$, $\gamma = 0.2$, $v_0 = 0.04$, $\rho = 0$, $T = 0.5$.

K	**valuation formula**	**analytical value**	**error**	**error (in %)**
100.0	0.590890416	0.590653801	0.000236615	0.040
100.5	0.637396339	0.637161423	0.000234916	0.037
101.0	0.686801947	0.686544105	0.000257842	0.038
101.5	0.739114524	0.738917138	0.000197386	0.027
102.0	0.794581754	0.794396290	0.000185464	0.023
102.5	0.853252772	0.853097438	0.000155333	0.018
103.0	0.915284818	0.915136365	0.000148453	0.016
103.5	0.980763196	0.980628468	0.000134729	0.014
104.0	1.049816122	1.049688411	0.000127711	0.012
104.5	1.122535363	1.122429945	0.000105418	0.009
105.0	1.198927032	1.198965563	0.000038531	0.003
105.5	1.279376526	1.279406221	0.000029695	0.002
106.0	1.363784997	1.363859654	0.000074656	0.005
106.5	1.452355260	1.452437126	0.000081866	0.006
107.0	1.545152063	1.545239056	0.000086993	0.006
107.5	1.642264358	1.642368868	0.000104511	0.006
108.0	1.743824690	1.743925726	0.000101036	0.006
108.5	1.849906100	1.850005490	0.000099390	0.005
109.0	1.960556999	1.960700694	0.000143695	0.007
109.5	2.075956108	2.076100121	0.000144013	0.007
110.0	2.196148781	2.196288967	0.000140186	0.006

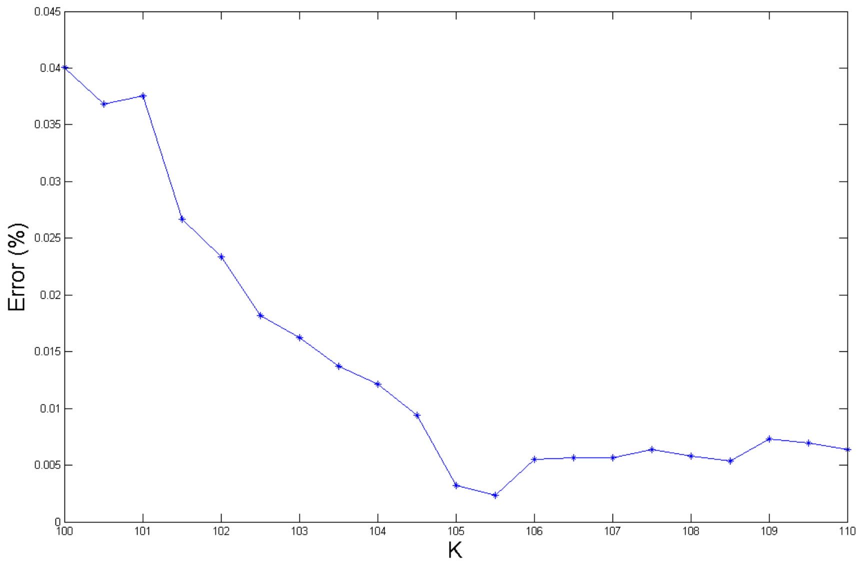

Figure 4.1: Error of pricing formula of a up-and-in put option as a function of strike price in the Heston model when $S_0 = 100$, $r = 0$, $= 110$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $\rho = 0$, $T = 0.5$.

Next, we plot the relative error of the pricing formula as a function of strike price K and barrier B in figure 4.2. Observe that the relative error is less than 0.05% and decreasing with the decrease of the difference between the barrier and strike price.

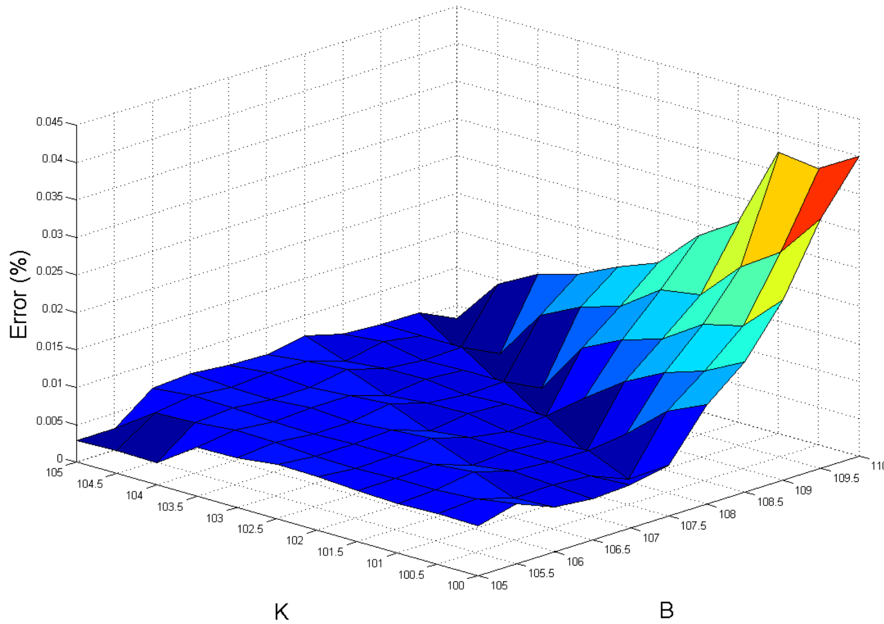

Figure 4.2: Error of pricing formula of a up-and-in put option as a function of strike price and barrier level in the Heston model when $S_0 = 100$, $r = 0$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $\rho = 0$, $T = 0.5$.

In order to check our valuation formula for the general model, we work on a Heston model with non-zero interest rate. We can see form figure 4.3 that our valuation formula performs well and the relative error is decreasing with the decrease of the difference between the barrier and strike price.

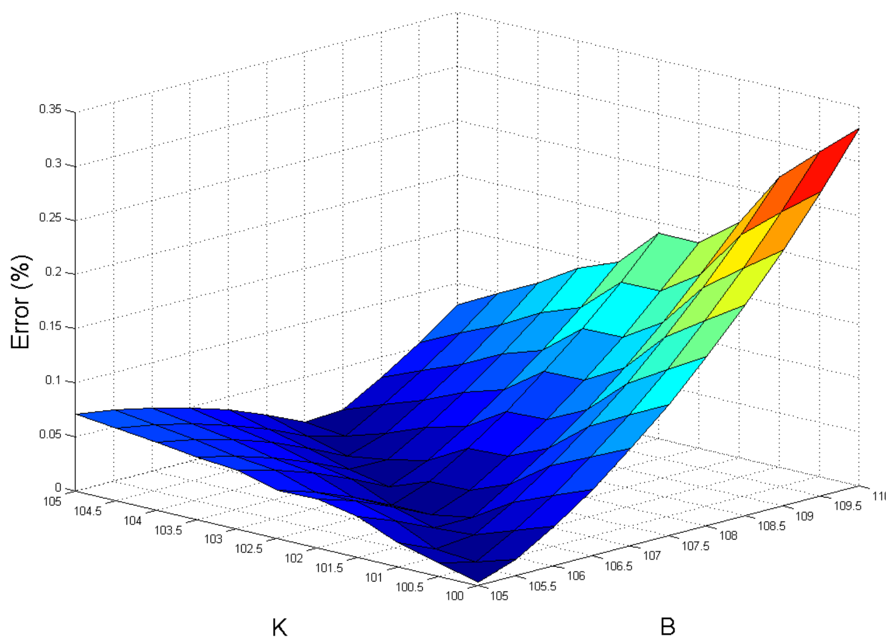

Figure 4.3: Error of pricing formula of a up-and-in put option as a function of strike price and barrier level in the Heston model when $S_0 = 100$, $r = 0.05$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $\rho = 0$, $T = 0.5$.

Finally, we study the performance of the pricing formula of a lookback options with payoff $\left(\sup_{t\leq T} S_t - K\right)^+$ in the Heston model, where we calculate the analytical value of the option by Monte Carlo simulation. Recall that the pricing formula is

$$e^{-rT}\int_0^\infty\int_0^\infty g^{V,\Sigma}(z,y)\int_{\ln\frac{K}{s_0}\vee 0}^\infty (s_0e^{\,m}-K)\cdot\left(\frac{2}{\sqrt{2\pi\left(1-\rho^2\right)\Sigma_T}}\exp\left(\frac{-\left(m-\mu_T\right)^2}{2\left(1-\rho^2\right)\Sigma_T}\right)\right.$$

$$\left.+\frac{2\mu_T}{\left(1-\rho^2\right)\Sigma_T}\exp\left(\frac{2\mu_T\,m}{\left(1-\rho^2\right)\Sigma_T}\right)\,\Phi\left(-\frac{m+\mu_T}{\sqrt{\left(1-\rho^2\right)\Sigma_T}}\right)\right)\,dm\;dy\,dz$$

where

$$\mu(z,y) = rT + \frac{\rho}{\gamma}\left(z - v_0 - \kappa\theta T + y\right) - \frac{1}{2}y\ .$$

We see in figure 4.4 that the option values are very close between our pricing formula (blue stars) and Monte-Carlo simulation (red line).

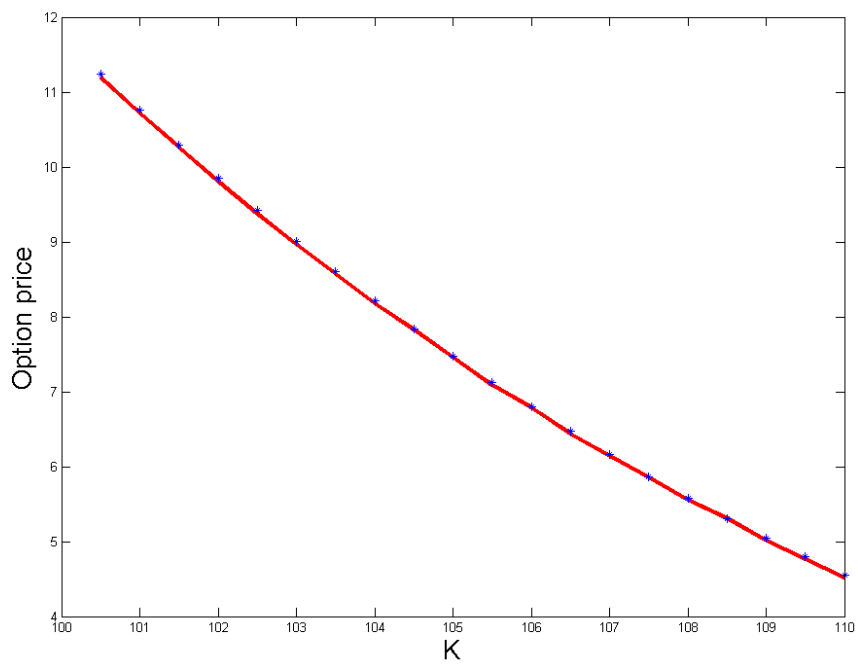

Figure 4.4: Price of the lookback options as a function of strike price in the Heston model when $S_0 = 100$, $r = 0$, $\kappa = 4$, $\theta = 0.04$, $v_0 = 0.04$, $\rho = 0$, $T = 0.5$.

Notations

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$	filtered probability space
$\mathbb{P}$	risk-neutral probability measure
$E_t^{\mathbb{P}}$	$\mathcal{F}_t$-conditional $\mathbb{P}$-expectation
S	underlying price process
M	self-dual part in the multiplicative decomposition of S
R	remaining part in the multiplicative decomposition of S
$\mathbb{Q}$	equivalent probability measure $\mathbb{Q}$
D	modified price process
V	volatility process
$\mathcal{F}^V$	filtration generated by the entire information of the volatility process V
r	risk-less interest rate
T	maturity date
ρ	correlation between the price and volatility process
W	standard Brownian motion under measure $\mathbb{P}$
$W^{\perp}$	orthogonal standard Brownian motion to W
G	option payoff function
τ	first passage time in the single barrier options
τ_U	first passage time to the upper bound in double barrier options
τ_L	first passage time to the lower bound in double barrier options
τ_{UL}	first passage time to the lower bound after firstly hitting the upper bound in sequential barrier options
P_{BS}	Black-Scholes put option price
v_t^2	squared time future average volatility

L	stochastic integral w.r.t. $W^{\perp}$
Y	maximum process of L
Σ	realized variance
f^{Σ}	marginal density of Σ_T under measure $\mathbb{P}$
$g^{V,\Sigma}$	joint density of $\left(V_T, \int_0^T V_s ds\right)$ under measure $\mathbb{P}$
Φ	standard normal C.D.F.

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Appendix A

Proof of Remark 3.22

Proof. We recall that

$$\begin{aligned} & E^{\mathbb{Q}}\left(\int_\tau^T V_s\,ds \middle| \mathcal{F}_\tau\right) \\ = \; & (T-\tau)\, v_\tau^2 \\ = \; & \frac{(V_\tau - \theta^{\mathbb{Q}})\left(1 - e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)}{\kappa^{\mathbb{Q}}} + \theta^{\mathbb{Q}}(T-\tau) \end{aligned}$$

therefore

$$(V_\tau - \theta^{\mathbb{Q}}) = \frac{\kappa^{\mathbb{Q}}(T-\tau)(v_\tau^2 - \theta^{\mathbb{Q}})}{1 - e^{-\kappa^{\mathbb{Q}}(T-\tau)}}.$$

Then we could rewrite the approximation formula as

$$\begin{aligned} & K \cdot P_{BS}(\tau, 0;\ \frac{K}{B}, T;\ v_\tau) + \rho\gamma \cdot \frac{K \exp\left(-\frac{d_+^2}{2}\right) \cdot d_-}{2\sqrt{2\pi}\kappa^{\mathbb{Q}}} \cdot \left[1 - \left(\frac{\kappa^{\mathbb{Q}}(v_\tau^2 - \theta^{\mathbb{Q}})(T-\tau)}{\left(1 - e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right) v_\tau^2}\right.\right. \\ & \left.\left. + \frac{\theta^{\mathbb{Q}}\left(1 - e^{-\kappa(T-\tau)}\right)}{\kappa^{\mathbb{Q}}(T-\tau)\, v_\tau^2}\right)\right] + \gamma^2 \cdot \frac{K \exp\left(-\frac{d_+^2}{2}\right)}{8\sqrt{2\pi}(\kappa^{\mathbb{Q}})^2} \cdot \frac{d_+ d_- - 1}{v_\tau\sqrt{T-\tau}} \cdot \left[1 - 2\left(\frac{\kappa^{\mathbb{Q}}(v_\tau^2 - \theta^{\mathbb{Q}})(T-\tau)}{\left(1 - e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right) v_\tau^2}\right.\right. \\ & \left.\left. + \frac{\theta^{\mathbb{Q}}\left(1 - e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)}{\kappa^{\mathbb{Q}}(T-\tau)\, v_\tau^2}\right) + \left(\frac{\theta^{\mathbb{Q}}\left(1 - e^{-2\kappa(T-\tau)}\right)}{2\kappa^{\mathbb{Q}}(T-\tau)\, v_\tau^2} + \frac{(v_\tau^2 - \theta^{\mathbb{Q}})\, e^{-\kappa^{\mathbb{Q}}(T-\tau)}}{v_\tau^2}\right)\right] \end{aligned}$$

By letting $(T-\tau) \downarrow 0$, we have

$$v_\tau^2 = \frac{1}{T-\tau} E^{\mathbb{Q}}\left(\int_\tau^T V_s ds \middle| \mathcal{F}_\tau\right) \to V_T$$

$$d_\pm = \frac{-\log\frac{K}{B} \pm \frac{1}{2}(T-\tau)v_\tau^2}{v_\tau\sqrt{T-\tau}} \to -\infty$$

hence,

$$P_{BS}(\tau, 0; \frac{K}{B}, T; v_\tau) = \Phi(-d_-) - \frac{K}{B}\Phi(-d_+) \to 0$$

$$\exp\left(-\frac{d_+^2}{2}\right) \cdot d_- = \frac{\exp\left(-\frac{d_+^2}{2}\right)}{1/d_-} \to 0$$

$$\exp\left(-\frac{d_+^2}{2}\right) \cdot \frac{d_+ d_- - 1}{v_\tau\sqrt{T-\tau}} \to 0$$

$$\frac{1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}}{\kappa^{\mathbb{Q}}(T-\tau)} \to 1,\ \frac{1-e^{-2\kappa^{\mathbb{Q}}(T-\tau)}}{2\kappa^{\mathbb{Q}}(T-\tau)} \to 1$$

yields

$$\begin{aligned} & 1 - \left(\frac{\kappa^{\mathbb{Q}}(v_\tau^2 - \theta^{\mathbb{Q}})(T-\tau)}{\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)v_\tau^2} + \frac{\theta^{\mathbb{Q}}\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)}{\kappa^{\mathbb{Q}}(T-\tau)\,v_\tau^2}\right) \\ \to \quad & 1 - \left(\frac{v_\tau^2 - \theta^{\mathbb{Q}}}{v_\tau^2} + \frac{\theta^{\mathbb{Q}}}{v_\tau^2}\right) = 0 \end{aligned}$$

and

$$\begin{aligned} & 1 - 2\left(\frac{\kappa^{\mathbb{Q}}(v_\tau^2 - \theta^{\mathbb{Q}})(T-\tau)}{\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)v_\tau^2} + \frac{\theta^{\mathbb{Q}}\left(1-e^{-\kappa^{\mathbb{Q}}(T-\tau)}\right)}{\kappa^{\mathbb{Q}}(T-\tau)\,v_\tau^2}\right) \\ & + \left(\frac{\theta^{\mathbb{Q}}\left(1-e^{-2\kappa^{\mathbb{Q}}(T-\tau)}\right)}{2\kappa^{\mathbb{Q}}(T-\tau)\,v_\tau^2} + \frac{(v_\tau^2-\theta^{\mathbb{Q}})\,e^{-\kappa^{\mathbb{Q}}(T-\tau)}}{v_\tau^2}\right) \\ \to \quad & 1 - 2\left(\frac{v_\tau^2 - \theta^{\mathbb{Q}}}{v_\tau^2} + \frac{\theta^{\mathbb{Q}}}{v_\tau^2}\right) + \left(\frac{\theta^{\mathbb{Q}}}{v_\tau^2} + \frac{v_\tau^2 - \theta^{\mathbb{Q}}}{v_\tau^2}\right) = 0 \end{aligned}$$

Hence, if the barrier is never touched, the value of the decomposition portfolio is zero as desired. ■

Appendix B

Matlab codes

B.1 Closed-form formula for vanilla call options in the Heston model

% Closed-form valuation formula for the vanilla call option in Heston model.
% Based on the paper by D.Lemmens , M. Wouters, J. Tempere and S. Foulon.

```
% S0      Currant stock price;
% v0      Instantaneous variance at time 0;
% r       Interest rate;
% kappa   Speed of mean reversion of the variance process;
% theta   Level of variance;
% sigma   Volatility of the variance process;
% rho     Correlation between the stock and variance process;
% T       Maturity date;
% K       Strike price;
```

```
% Integrand
function y=intfun(l,S0,r,K,T,v0,kappa,theta,rho,sigma)
omega = 0.5.*sigma.*sqrt((kappa./sigma+i.*l.*rho).^2+l.*(l-i));
N= (cosh(omega.*T)+0.5.*(kappa+i.*l.*rho.*sigma).*sinh(omega.*T)./omega).^(-1);
```

xe = log(K./S0);
a = v0 + kappa.*theta.*T;
vega = 0.5.*sigma.*sqrt((kappa./sigma+i.*l.*rho-rho).$^2 + l. * (l + i)$);
$M = (cosh(vega. * T) + 0.5. * (kappa + i. * l. * rho. * sigma - rho. * sigma). * sinh(vega. * T)./vega).\hat{(} - 1)$;
$Theta = 2. * vega. * v0. * (M - cosh(vega. * T))/(sigma.^2. * sinh(vega. * T)) + 2. * kappa. * theta. * log(M)./(sigma.^2)$;
$Gamma = 2. * omega. * v0. * (N - cosh(omega. * T))/(sigma.^2. * sinh(omega. * T)) + 2. * kappa. * theta. * log(N)./(sigma.^2)$;

y=(i./l).*(exp(i.*(rho.*a./sigma+xe-r.*T).*l+kappa.*a/(sigma.^{2})). $* (S0. * exp(Theta - rho. * a./sigma) - exp(-r. * T). * K. * exp(Gamma)) - S0 + exp(-r. * T). * K)./pi$;

% Closed-form formula.
function V = ClosedformHeston(S0,r,K,T,v0,kappa,theta,rho,sigma)
Vint=real(quad(@(l)intfun(l,S0,r,K,T,v0,kappa,theta,rho,sigma),0,999));
V=(S0-exp(-r*T)*K)/2+Vint;

B.2 Approximation formula for vanilla put options in the Heston model

% Calculating the put value via the approximation formula in the Heston model.
function Simu = HestonDe(S0,v0,r,kappa,theta,sigma,rho,T,K)
VSR = theta.*T+(v0-theta).*(1-exp(-kappa*T))/kappa;
% Expectation of the realised variance.
d1 = (log(S0/K)+r*T+0.5*VSR)./sqrt(VSR);
d2 = d1 - sqrt(VSR);
dNd1 = exp(-d1.*d1/2)./sqrt(2*pi);
dNd2 = exp(-d2.*d2/2)./sqrt(2*pi);

Simu = exp(-r*T)*K* normcdf(-d2)- S0*normcdf(-d1)-rho*sigma*S0*dNd1.*d2... .*((v0-2*theta).*(1-exp(-kappa*T))/kappa + T.*(theta-... (v0-theta).*exp(-kappa*T)))/kappa./(2*VSR)

$+ ...$ sigma$^2 * S0 * dNd1. * (d1. * d2 - 1)/8/kappa^2/VSR.^(3/2). * (theta * T + ...(v0 - theta). * (1 - exp(-kappa * T))/kappa - 2 * theta * (1 - exp(-kappa * T))/kappa... - 2 * T. * (v0 - theta). * exp(-kappa * T) + theta * (1 - exp(-2 * kappa * T))/kappa/2 + ...(v0 - theta). * (exp(-kappa * T) - exp(-2 * kappa * T))/kappa);$

B.3 Numerical inversion of Laplace transform of the realised variance in the Heston model

% Laplce transform of the realised variance in the Heston model.

function F = Hestonlaplacetransform (x)

kappa = 4; theta = 0.04; gamma = 0.2; v0 = 0.04; T = 0.5;

zeta = sqrt (kappa.$^2 + 2. * gamma.^2. * x);$

$B = -2. * x/(kappa + zeta. * coth(zeta. * T/2));$

$A = exp(kappa.^2. * theta. * T./gamma.^2)./(cosh(zeta. * T./2) + kappa./zeta. * sinh(zeta. * T./2)).\hat{(}2. * kappa. * theta./gamma.^2);$

$F = A * exp(v0 * B);$

% Fixed Talbot argorithm by P.Valco & J.Abate.

function G2 = HestonInverseLaplaceFT(x)

M=15; r=2*M/(5*x); Sum=0;

for j = 1:M-1

theta = j*pi/M;

S = r*theta*(1/tan(theta)+i);

sigma = theta+(theta/tan(theta)-1)/tan(theta);

Sum = Sum + real(exp(x*S)* (1+i*sigma) * Hestonlaplacetransform(S));

end

G2 = r/M* (0.5*exp(r*x)*Hestonlaplacetransform(r) + Sum);

B.4 Closed-form valuation for the up-and-in put options in the Heston model

% Value of the inner expectation.

function ans=ItgfuncfsrBintegral(x,K,r,B)

```
S0= 100;
ans = integral2(@(w,m)( K-S0.*exp(w) ).*exp(-0.5.*w-0.125.*x + 0.5.*r.*0.5 + r.*0.5.*w./x
-... 0.5.*r.^2.*0.25./x).*2.*(2*m−w)/sqrt(2.*pi)./(x.^(3/2)).*exp(−(2*m−w).^2./2./x), ...−
10, log(K/S0), log(B/S0), 10);
```

% Approximation of the valuation formula by the trapezoid rule.

function ans = HestonbarrierLaplaceFTrB (K,r,B)

```
Y=zeros(1,40);
for k=1:40
j = (k)/400;
Y(k) = HestonInverseLaplaceFT(j) .* ItgfuncfsrBintegral(j,K,r,B);
end

ans = 0.0025*trapz(Y)*exp(-r*0.5);
```