Essays in macroeconomic theory: informational frictions, market microstructure and fat-tailed shocks

by

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Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

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Statement of conjoint work

Chapter 1 of this thesis is based on research that I undertook while working as a Research Specialist at the Central Reserve Bank of Peru. This work was jointly co-authored with Dr. Carlos Montoro and I contributed a minimum of 50% of the work.

Chapter 2 of this thesis is based on research that I undertook while working as a Research Specialist at the Central Reserve Bank of Peru. This work was jointly co-authored with Dr. Carlos Montoro and I contributed a minimum of 50% of the work.

Abstract

This thesis is composed by five chapters. Chapter 1 presents a new Keynesian open economy model that includes risk-adverse foreign-exchange market dealers and foreign exchange intervention by the monetary authority. In this setup portfolio decisions made by dealers add an endogenous time variant risk-premium element to the traditional UIP that depends on FX intervention by the central bank and FX orders by foreign investors. We use the model to analyse the interactions between monetary policy and FX interventions.

Chapter 2 introduces information heterogeneity into the model presented in Chapter 1. As in Bacchetta and van Wincoop (2006), the "rational confusion" generated by the introduction of heterogeneous information magnifies the impact of the *unobservable* capital flows shocks on the exchange rate.

Chapter 3 introduces fat-tailed shocks in the model of Kato and Nishiyama (2005). This is a simple new Keynesian model where the central bank explicitly considers the zero lower-bound constraint on interest rates. We find that shocks with 'excess kurtosis' make monetary policy relatively more aggressive far away from the zero lower bound region though, this difference reverts when the economy is close to this constraint. Under our baseline calibration, the difference between optimal policies under Gaussian and fat-tailed shocks is not quantitatively significant.

Chapter 4 presents a model in which investors form their expectations in an adaptive way to price bonds, in the spirit of Adam, Marcet and Nicolini (2011). We follow different assumptions regarding the learning process followed by agents. In the case of finite maturity bonds, the knowledge of the pricing of the first maturity will act as an 'anchor, limiting the price volatility of bonds with short maturities. As the maturity increases,

the price volatility converges to the one of the consol bond.

Chapter 5 surveys the literature on imperfect information, learning and the yield curve.

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Contents

List of Tables

List of Figures

Chapter 1

Foreign exchange intervention and monetary policy design: a market microstructure analysis

1.1 Introduction

Interventions by central banks in foreign exchange (FX) markets have been common in many countries, and they have become even more frequent in the most recent past, in both emerging market economies and some advanced economies.[1] These interventions have been particularly large during periods of capital inflows, when central banks bought foreign currency to prevent an appreciation of the domestic currency. Also, they have been recurrent during periods of financial stress and capital outflows, when central banks used their reserves to prevent sharp depreciations of their currencies. For instance, in Figure 1 we can see that during 2009-12 the amount of FX interventions as a percentage of FX reserves minus gold was between 30% and 100% in some Latin American countries, and considerably more than 100% in Switzerland. Also, these FX interventions were sterilised in most cases, enabling central banks to keep short-term interest rates in line with policy rates.

[1] Mihaljek (2005) reports that the typical share of intervention in turnover in EMEs fell from 12% in 2002 to 8% in 2004 as a percentage of the average monthly holdings of FX reserves. Notwithstanding significant fluctuations over the years, these shares are significantly higher now than they were a decade ago. Filardo et al. (2011) document how the central banks of Chile and Poland, which were inactive in the FX market for years, decided to resume FX interventions during the 2010-2011 period.

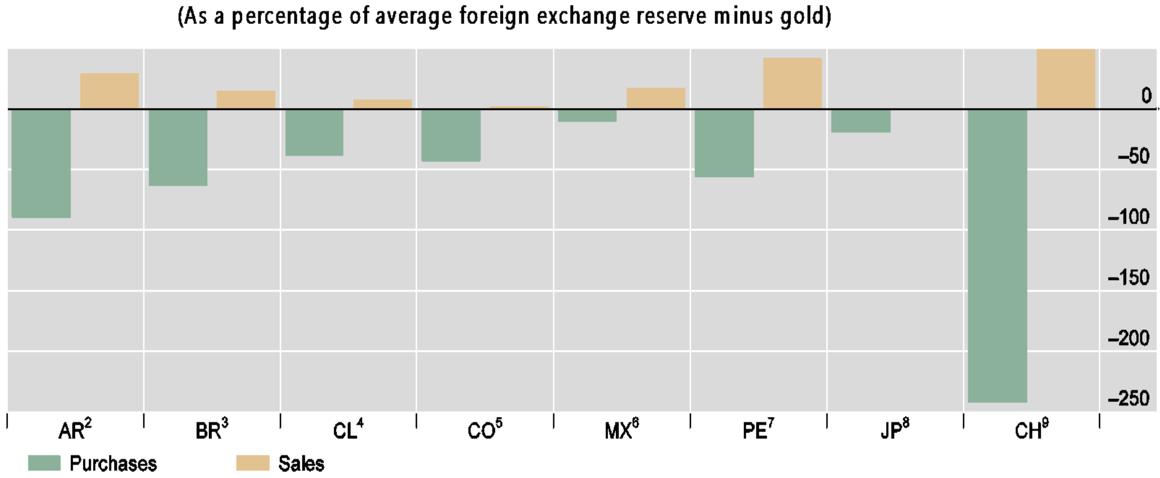
Given the scale of interventions in FX markets by some central banks, it should be important for them to include this factor in their policy analysis frameworks. A variety of questions need to be addressed, such as: How does sterilised intervention affect the transmission mechanism of monetary policy? Which channels are at work? Are there benefits to intervention rules? What should be the optimal monetary policy design in the context of FX intervention? To analyse these questions we need an adequate framework of exchange rate determination in macroeconomic models.

There is substantial empirical evidence that traditional approaches of exchange rate determination (e.g., asset markets) fail to explain exchange rate movements in the short-run, see Meese and Rogoff (1983) and Frankel and Rose (1995). This empirical evidence shows that most exchange fluctuations at short- to medium-term horizons are related to order flows - the flow of transactions between market participants - as in the microstructure approach presented by Lyons (2006), and not to macroeconomic variables. However, in most of the models used for monetary policy analysis, the exchange rate is closely linked to macroeconomic fundamentals, as in the uncovered interest rate parity (UIP) condition. Such inconsistency between the model and real exchange rate determination in practice could lead in some cases to incorrect policy prescriptions such as the overestimation of the impact of fundamentals and the corresponding underestimation of the impact of *liquidity* trading. The latter include, *inter alia*, current account transactions such as trade in good and services, transfers in capital income, remittances, and tourism related flows, which are not related to traditional macroeconomic fundamentals (i.a.: the interest rate differential).

Regarding the effectiveness of FX intervention, the empirical evidence remains inconclusive. Reviews by Menkhoff (2012) and Chamon et al. (2012) suggest that interventions in some cases have a systematic impact on the rate of change in exchange rates, while in other cases they have been able to reduce exchange rate volatility. Intervention appears to be more effective when it is consistent with monetary policy (Amato et al. (2005), Kamil (2008)). This evidence suggests that the impact of FX interventions depend on the specific episode and instrument used. Clearly, the effectiveness of central bank intervention also needs to be evaluated against its policy goal.

Benes et al. (2013) provide a framework for the joint analysis of hybrid inflation

Figure 1-1: Intervention in the foreign exchange market: 2009 - 2012[1]



AR = Argentina; BR = Brazil; CH = Switzerland; CL = Chile; CO = Colombia; JP = Japan; MX = Mexico; PE = Peru.

[1] The bars correspond to the sum of amounts allocated/transacted in the period. Positive (negative) values indicate a sale (purchase) of US dollars by the central bank, unless otherwise indicated. [2] Spot sales. [3] Swap market auctions; spot market auctions; repo market auctions; trade finance loans and forward market auctions. [4] Reserve accumulation programme and swap auctions. [5] Options for reserve accumulation; options for exchange rate volatility control and direct purchase auctions. [6] USD sales mechanism; auctions with minimum price; auctions without a minimum price; Fed swaps; direct sales and option auctions. [7] Auctions; CDR certificates and CDLD certificates. [8] Interventions by the Japanese Ministry of Finance. [9] Inferred by changes in the foreign currency investments of the Swiss National Bank by currency; adjusted for exchange rate changes.

Sources: National data; BIS calculations.

targeting (IT) regimes with FX interventions strategies (e.g., exchange rate corridors, pegged or crawling exchange rates, managed floats.), where the central bank can exercise control over the exchange rate as an instrument independent of monetary policy and the policy interest rate.[2] Their strategy consists of introducing imperfect substitutability between central bank securities - used for purposes of sterilization - and private sector bank loans in a model where banks hold local currency denominated assets and foreign currency liabilities. An increase in the supply of central bank securities pushes banks to increase their overall exposure to exchange rate risk. This has an effect on interest rates as banks charge a higher premium to compensate for the higher risk they bear. In a related work, which also assumes imperfect substitutability of assets, Vargas et al. (2013) find that sterilised FX interventions can have an effect on credit supply by changing the balance sheet composition of commercial banks.

Unlike previous research, we follow a market microstructure approach by intro-

[2] Chamon et al. (2012) discusses the use of hybrid IT schemes in emerging market economies (EME). Authors recommend the use of a two-instrument IT framework as a way to reinforce its commitment to a low inflation rate.

ducing risk-averse FX dealers and FX intervention by the monetary authority. These ingredients generate deviations from the uncovered interest parity (UIP) condition. More precisely, dealers' portfolio decisions endogenously add a time-variant exchange rate risk premium element to the traditional UIP that depends on FX intervention by the central bank and FX orders by foreign investors. Moreover, we explicitly account for the role that exchange rate volatility plays in the deviation from the UIP, and how FX intervention rules can impact the economy through their effect on this volatility. Our model shows how central bank FX intervention can affect exchange rate determination through two channels: the portfolio balance effect and a volatility effect. In the former, a sterilised intervention alters the value of the currency because it modifies the ratio between domestic and foreign assets held by the private sector; and according to the latter, central bank interventions have an impact on the volatility of exchange rates and consequently on the extent to which *liquidity* based trades affect the equilibrium exchange rate. Thus, in our model, the *trading mechanism* and the *players*, two of the three key elements in the microstructure approach according to Lyons (2006), affect the determination of the exchange rate.[3]

Our findings show that in general equilibrium, FX intervention can have important implications for central bank stabilization policies. In some cases, FX intervention can mute the monetary transmission mechanism through exchange rates, reducing the- impact on aggregate demand and prices, while in others it can amplify the impact. We also show that there are some trade-offs in the use of FX intervention, in line with the results in Benes et al. (2013). On the one hand, it can help isolate the economy from external financial shocks, but on the other it prevents some necessary adjustments of the exchange rate in response to nominal and real external shocks. Finally, regarding FX intervention policy design, we show that intervention rules can have stronger stabilisation power in response to shocks as they exploit the volatility channel.

In the next section we introduce the model, with a special focus on the FX market. In Section 1.3 we show results from the simulation of the model. In Section 1.4 we present some robustness exercises. The last section concludes.

[3]The third element mentioned by the author is *information.* We present a model where information across dealers is heterogeneous in Chapter 2.

1.2 The Model

The model describes a small open economy with nominal rigidities, in line with the contributions from Obstfeld and Rogoff (1995), Chari et al. (2002), Galí and Monacelli (2005), Christiano et al. (2005) and Devereux et al. (2006), among others. To maintain the concept of general equilibrium, we use a two-country framework taking the size of one of these economies close to zero, such that the small (domestic) economy does not affect the large (foreign) economy.[4]

In this setup, dealers in the small domestic economy operate the secondary bond market. They receive customer orders for the sale of domestic bonds from households and for the sale of foreign bonds from foreign investors and the central bank. Dealers invest each period in both domestic and foreign bonds, maximising their portfolio returns. This is a cashless economy. The monetary authority intervenes directly in the FX market selling or purchasing foreign bonds in exchange for domestic bonds. The central bank issues the domestic bonds and sets the nominal interest rates paid by these assets. The central bank can control the interest rate regardless of the FX intervention, that is we assume the central bank can always perform fully sterilised interventions.[5]

We assume the frequency of decisions is the same for dealers and other economic agents. Households consume final goods, supply labour to intermediate goods producers and save in domestic bonds. Firms produce intermediate and final goods. Additionally, we include monopolistic competition and nominal rigidities in the retail sector, price discrimination and pricing to market in the export sector, and incomplete pass-through from the exchange rate to imported good prices - characteristics that are important to analyse the transmission mechanism of monetary policy in a small open economy. We also consider as exogenous processes foreign variables such as output, inflation, the

[4] We acknowledge the general equilibrium perspective introduces a series of linear relationships among the foreign economy variables. The disadvantage of following this modelling strategy is that shocks to foreign variables will not be observed independently, as only combination of foreign variables will impact the domestic economy. This would not allow us to analyse the impact of shocks to foreign variables independently (and the impact would depend as well on the calibration of the foreign economy.) The literature favours the approach followed here. For examples see Adolfson et al. (2008).

[5] However, in practice sterilised interventions have limits. For example, the sale of foreign bonds by the central bank is limited by the level of foreign reserves. On the other hand, the sterilised purchase of foreign currency is limited by the availability of instruments to sterilise those purchases (e.g., given by the demand for central bank bonds or by the stock of treasury bills in the hands of the central bank). Also, limits to the financial losses generated by FX intervention can represent a constraint for intervention itself.

interest rate and non-fundamental capital flows.[6]

1.2.1 Dealers

In the domestic economy there is a continuum of dealers ι in the interval $[0,1]$. Each dealer ι receives ϖ_t^{ι} and $\varpi_t^{\iota,cb}$ in domestic bond sale and purchase orders from households and the central bank, and $\varpi_t^{\iota*}$ and $\varpi_t^{\iota*,cb}$ in foreign bond sale orders from foreign investors and the central bank, respectively. These orders are exchanged among dealers, that is $\varpi_t^{\iota} + \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right) = B_t^{\iota} + S_t B_t^{\iota*}$, where B_t^{ι} and $B_t^{\iota*}$ are the ex-post holdings of domestic and foreign bonds by dealer ι, respectively.[7] Each dealer receives the same amount of orders from households, foreign investors and the central bank. The exchange rate S_t is defined as the price of foreign currency in terms of domestic currency, such that a decrease (increase) of S_t corresponds to an appreciation (depreciation) of the domestic currency. At the end of the period, any profits -either positive or negative- are transferred to the households.[8]

Dealers are risk-averse and short-sighted. They select an optimal portfolio allocation in order to maximise the expected utility of their end-of-period returns, where their utility is given by a CARA utility function. The one-period dealer's horizon gives tractability and captures the feature that FX dealers tend to unwind their FX exposure at the end of any trading period, as explained by Vitale (2011). [9] The problem of dealer ι is:

$$\max_{B_t^{\iota*}} -E_t^{\iota} e^{-\gamma\Omega^{\iota}_{t+1}}$$

[6]There is an extensive empirical literature addressing the determinants of portfolio capital flows to emerging economies. Moreover, Arias et al. (mimeo) find that lagged FX interventions impact portfolio capital inflows, however this factor is significantly lower than 1, implying that FX interventions can still be an effective instrument to counter portfolio capital inflows.

[7]Recall these are one period bonds, hence the flows and stocks are equivalent. At the beginning of each period the stock of bonds in possession of dealers is zero.

[8]Under the present formulation FX transactions carried out for commercial purposes will only affect the exchange rate through their impact in the domestic interest rate though not through variations in the order flow faced by dealers.

[9]Notice that dealers are passive (market makers), as they are willing to accept any trade affecting their portfolio for the right compensation. They must absorb the aggregate change in their portfolio by the end of the period as they are not able to recompose their portfolio in the same period. This assumption can be motivated by the imperfect capital markets integration exhibited by some of the developing countries which intervene in FX markets.

subject to:

$$\varpi_t^{\iota} + \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right) = B_t^{\iota} + S_t B_t^{\iota*} \tag{1.1}$$

where E_t is the rational expectations operator, γ is the coefficient of absolute risk aversion and Ω_{t+1}^{ι} is the total investment after returns, given by:

$$\begin{aligned}\Omega_{t+1}^{\iota} &= (1+i_t)\, B_t^{\iota} + (1+i_t^*)\, S_{t+1} B_t^{\iota*} \\ &\approx (1+i_t)\left[\varpi_t^{\iota} + \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right)\right] + (i_t^* - i_t + s_{t+1} - s_t)\, B_t^{\iota*}\end{aligned}$$

where we have made use of the resource constraint of dealers, we have log-linearised the excess of return on investing in foreign bonds and $s_t = \ln S_t$. Since the only non-predetermined variable is s_{t+1}, assuming it is normal distributed with time-invariant variance, the first order condition for the dealers is:[10]

$$0 = -\gamma\,(i_t^* - i_t + E_t s_{t+1} - s_t) + \gamma^2 B_t^{\iota*} \sigma^2$$

where $\sigma^2 = var_t\,(\Delta s_{t+1})$ is the conditional variance of the depreciation rate. Then, the demand for foreign bonds by dealer ι is given by the following portfolio condition:

$$B_t^{\iota*} = \frac{i_t^* - i_t + E_t s_{t+1} - s_t}{\gamma \sigma^2} \tag{1.2}$$

According to this expression, the demand for foreign bonds will be larger the higher its return, the lower the risk aversion or the lower the volatility of the exchange rate.

FX market equilibrium

Foreign bonds equilibrium in the domestic market should sum FX market orders from foreign investors (capital inflows) and central bank FX intervention, that is:[11]

$$\int_0^1 B_t^{\iota*} d\iota = \int_0^1 \left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right) d\iota = \varpi_t^* + \varpi_t^{*,cb}.$$

[10] Conditions verified to be satisfied ex-post.

[11] Similar to other foreign variables in the model, holdings of foreign bonds in the domestic market are exogenous (i.e., it is not affected by domestic conditions). This is consistent with the small open economy assumption, meaning that domestic conditions do not affect foreign variables. The second part of this equation is an accounting relationship.

Dealers are passive and unable to rebalance their trading with foreigners. This assumption is in line with Lyons (2006), who explains how the risk that drives the portfolio balance effect is undiversifiable across dealers.[12] Replacing the FX market equilibrium condition in the aggregate demand for foreign bonds yields the following arbitrage condition:

$$E_t s_{t+1} - s_t = i_t - i_t^* + \gamma\sigma^2(\varpi_t^* + \varpi_t^{*,cb}) \tag{1.3}$$

Condition (1.3) determines the exchange rate, and differs from the traditional uncovered interest parity condition because of an endogenous risk premium component. According to it, an increase (decrease) in capital inflows or sales (purchases) of foreign bonds by the central bank appreciates (depreciates) the exchange rate s_t, *ceteris paribus*. This effect is larger, the more risk-averse dealers are (larger γ) or the more volatile the expected depreciation rate is (larger σ^2).[13]

Equation (1.3) is useful to understand both mechanisms through which FX intervention can affect the exchange rate. The last term on the right hand side captures the portfolio-balance channel. Given that dealers are risk-averse and hold domestic and foreign assets to diversify risk, FX intervention changes the composition of domestic and foreign asset held by the dealers. This will be possible only if there is a change in the expected relative rate of returns of these assets, which compensates for the change in the risk they bear. In other words, according to the portfolio-balance channel, a sale (purchase) of foreign bonds by the central bank augments (reduces) the ratio between foreign and domestic assets hold by dealers, inducing an appreciation (depreciation) of the domestic currency because dealers require a greater (smaller) risk premium to hold a larger (smaller) quantity of this currency.

The second mechanism at work is the volatility channel. When central banks intervene in the FX markets they can affect the conditional volatility of exchange rates, reducing the impact that shifts in portfolio have over the equilibrium exchange rate.

[12] These shocks imply that the market as a whole must hold a position that they would not otherwise hold, which entails an enduring risk premium. See Lyons (2006), Ch. 2.

[13] Sterilised intervention implies that a sale (purchase) of foreign bonds by the central bank is accompanied by purchases (sales) of domestic bonds by the monetary authority, such that the domestic interest rates are in line with the policy target rate. In our model, the central bank directly exchange domestic bonds in their balance for foreign ones. In this sense, interventions will have no impact on the interest rate as households' aggregate savings remain invariant.

Notice that the volatility effect, from (1.3), scales the portfolio channel as the variance of the changes in the exchange rate multiplies the aggregate order flow.

1.2.2 Monetary authority

The central bank in the domestic economy intervenes in the FX market by selling/buying foreign bonds to/from dealers in exchange for domestic bonds. Each period the central bank negotiates directly with dealers, such that every dealer receives the same amount of sales/purchases of foreign bonds from the central bank. Each period any dealer ι receives a market order $\varpi_t^{\iota*,cb}$ from the central bank, where $\varpi_t^{\iota*,cb} > 0 \left(\varpi_t^{\iota*,cb} < 0\right)$ when the central bank sells (purchases) foreign bonds in exchange of domestic bonds. The total customer flow of foreign bonds received by dealer ι equals $\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}$. We assume the central bank can always perform fully sterilised FX interventions, therefore it maintains control over the interest rate regardless of the intervention. Moreover, we further assume the central bank does not have to distribute profits/losses to the households. That is, the monetary authority is not constrained by its balance sheet to perform interventions in the FX market.[14][15]

FX intervention

We assume the central bank's purpose to intervene is to reduce the overall volatility caused by external shocks. As Mihaljek (2005) documents, central banks that intervene in foreign markets claim as one of the main reasons the need of stabilizing exchange rate markets, preventing exchange rate volatility to affect other sectors of the economy.[16]

[14]Sterilised intervention implies that a sale (purchase) of foreign currency by the central bank is accompanied by purchases (sales) of domestic bonds by the monetary authority such that the domestic interest rates are in line with the policy target rate. We implicitly assume an asymmetry between the FX market and the domestic currency bond markets. In the latter, non-fundamental sales (purchases) by the central bank have no impact on the price of the bond. In this way the bank intermediates between markets with a heterogeneous microstructure.

[15]The balance sheet of the central bank is the following: $S_t R_t^{cb} = B_t^{cb} + NW_t^{cb}$, where R_t^{cb}, B_t^{cb} and NW_t^{cb} are the central bank's reserves in foreign bonds, liabilities in domestic bonds and net worth, respectively. The first two components evolve according to: $R_t^{cb} = (1 + i_t^*) R_{t-1}^{cb} - \varpi_t^{*,cb}$ and $B_t^{cb} = (1 + i_t) B_{t-1}^{cb} - \varpi_t^{cb}$. Also, profits are given by: $P_t \Gamma_t^{cb} = \left[\frac{S_t\left(1+i_t^*\right)}{S_{t-1}} - 1\right] S_{t-1} R_{t-1}^{cb} - i_t B_{t-1}^{cb} - \left(S_t \omega_t^{*,cb} - \omega_t^{cb}\right)$

[16]Mihaljek (2005) presents a survey on 23 central banks from emerging markets. Out of the 18 banks in the sample which intervened during the 2002-2004 Q3 period, 16 claimed interventions were effective or sometimes effective calming disorderly exchange rate markets.

The central bank can have three different FX intervention strategies. First, it can perform pure discretional intervention:

$$\varpi_t^{*cb} = \varepsilon_t^{cb,0} \tag{1.4}$$

where the central bank intervenes via unanticipated or secret interventions. According to strategy (1.4), FX intervention by the central bank is not anticipated.[17]

As a second case, the central bank can perform rule based intervention taking into account the changes in the exchange rate. We call this strategy "the Δs rule".

$$\varpi_t^{*cb} = \phi_{\Delta s}\Delta s_t + \varepsilon_t^{cb,1} \tag{1.5}$$

According to this rule, when there are depreciation (appreciation) pressures on the domestic currency, the central bank sells (purchases) foreign bonds to prevent the exchange rate from fluctuating. $\phi_{\Delta s}$ captures the intensity of the response of the FX intervention to pressures in the FX market.

Finally, the monetary authority can take into account misalignments of the real exchange rate as a benchmark for FX intervention. We call this strategy "the RER rule".

$$\varpi_t^{*cb} = \phi_{rer} rer_t + \varepsilon_t^{cb,2} \tag{1.6}$$

where rer_t captures deviations of the real exchange rate with respect to its steady state. In the same vein as the previous case, under this rule the central bank sells (purchases) foreign bonds when the exchange rates depreciates (appreciates) in real terms from its long-run value. The Δs rule is expressed in nominal terms and takes into account only the change in the exchange rate, whilst the RER rule takes into account the deviations in the level of the exchange rate in real terms. The difference between both rules is similar to that between inflation targeting and price level targeting for the

[17] We contrast (comparable) discretional interventions with rule based interventions in order to gauge the impact of rules on expectations. The difference between discretional interventions and no intervention will be given by the effect of the variance of the discretional interventions shock on the overall exchange rate volatility.

case of shocks to the price level. Intuitively, under the Δs rule shocks to the exchange rate are accommodated, while under the RER rule, they are reversed.

We explicitly leave out a rule according to which intervention responds to *liquidity* trading, even though we acknowledge this type of rule will be the most effective against these shocks. The reason is twofold: (1) in practice it is difficult for central banks to determine which type of capital flows are affecting the exchange rate - fundamental or *liquidity* trading - and (2) the rules under study are in line with the goals some central banks claim to address through their FX intervention policies.[18]

Monetary policy

The central bank implements monetary policy by setting the nominal interest rate according to a Taylor-type feedback rule that depends on CPI inflation. The generic form of the interest rate rule that the central bank uses is given by:

$$\frac{(1+i_t)}{\left(1+\bar{i}\right)} = \left(\frac{\Pi_t}{\overline{\Pi}}\right)^{\varphi_\pi} \exp\left(\varepsilon_t^{MON}\right) \tag{1.7}$$

where $\varphi_\pi > 1$. $\overline{\Pi}$ and $\bar{i}$ are the levels in steady state of inflation and the nominal interest rate. The term ε_t^i is a random monetary policy shock distributed according to $N \sim \left(0, \sigma_i^2\right)$.

1.2.3 Households

Preferences

The world economy is populated by a continuum of households of mass 1, where a fraction n of them is allocated in the home economy, whereas the remaining $1-n$ is in the foreign economy. Each household j in the home economy enjoys utility from the consumption of a basket of final goods, C_t^j , and receives disutility from working, L_t^j. Households preferences are represented by the following utility function:

$$\mathcal{U}_t = E_t\left[\sum_{s=0}^{\infty} \beta^{t+s} U\left(C_{t+s}^j, L_{t+s}^j,\right)\right], \tag{1.8}$$

[18] We address this and other problems related to informational asymmetry in Chapter 2 of the present document.

where E_t is the conditional expectation on the information set at period t and β is the intertemporal discount factor, with $0 < \beta < 1$. In particular we assume the instantaneous utility is given by:

$$U\left(C_t, L_t\right) = \frac{C_t^{1-\gamma_c}}{1-\gamma_c} - \frac{L_t^{1+\chi}}{1+\chi}, \quad \text{if } \gamma_c \neq 1. \tag{1.9}$$

when $\gamma_c = 1$, this function becomes:

$$U\left(C_t, L_t\right) = \ln C_t - \frac{L_t^{1+\chi}}{1+\chi} \tag{1.10}$$

The consumption basket of final goods is a composite of domestic and foreign goods, aggregated using the following consumption index:

$$C_t \equiv \left[\left(\gamma^H\right)^{1/\varepsilon_H}\left(C_t^H\right)^{\frac{\varepsilon_H-1}{\varepsilon_H}} + \left(1-\gamma^H\right)^{1/\varepsilon_H}\left(C_t^M\right)^{\frac{\varepsilon_H-1}{\varepsilon_H}}\right]^{\frac{\varepsilon_H}{\varepsilon_H-1}}, \tag{1.11}$$

where ε_H is the elasticity of substitution between domestic (C_t^H) and foreign goods (C_t^M), and γ^H is the share of domestically produced goods in the consumption basket of the domestic economy. In turn, C_t^H and C_t^M are indices of consumption across the continuum of differentiated goods produced in the home country and those imported from abroad, respectively. These consumption indices are defined as follows:

$$C_t^H \equiv \left[\left(\frac{1}{n}\right)^{\frac{1}{\varepsilon}} \int_0^n C_t^H(z)^{\frac{\varepsilon-1}{\varepsilon}} dz\right]^{\frac{\varepsilon}{\varepsilon-1}}, C_t^M \equiv \left[\left(\frac{1}{1-n}\right)^{\frac{1}{\varepsilon}} \int_n^1 C_t^M(z)^{\frac{\varepsilon-1}{\varepsilon}} dz\right]^{\frac{\varepsilon}{\varepsilon-1}} \tag{1.12}$$

where $\varepsilon > 1$ is the elasticity of substitution across goods produced within the home economy, denoted by $C_t^H(z)$, and within the foreign economy, $C_t^M(z)$. Household's optimal demands for home and foreign consumption are given by:

$$C_t^H(z) = \frac{1}{n}\gamma^H \left(\frac{P_t^H(z)}{P_t^H}\right)^{-\varepsilon} \left(\frac{P_t^H}{P_t}\right)^{-\varepsilon_H} C_t, \tag{1.13}$$

$$C_t^M(z) = \frac{1}{1-n}\left(1-\gamma^H\right) \left(\frac{P_t^M(z)}{P_t^M}\right)^{-\varepsilon} \left(\frac{P_t^M}{P_t}\right)^{-\varepsilon_H} C_t \tag{1.14}$$

This set of demand functions is obtained by minimising the total expenditure on consumption $P_t C_t$, where P_t is the consumer price index. Notice that the consumption of each type of goods is increasing in the consumption level, and decreasing in their corresponding relative prices. Also, it is easy to show that the consumer price indices, under these preference assumptions, is determined by the following condition:

$$P_t \equiv \left[\gamma^H \left(P_t^H\right)^{1-\varepsilon_H} + (1-\gamma^H)\left(P_t^M\right)^{1-\varepsilon_H}\right]^{\frac{1}{1-\varepsilon_H}} \tag{1.15}$$

where P_t^H and P_t^M denote the price level of the home-produced and imported goods, respectively. Each of these price indexes is defined as follows:

$$P_t^H \equiv \left[\frac{1}{n}\int_0^n P_t^H(z)^{1-\varepsilon} dz\right]^{\frac{1}{1-\varepsilon}},\ P_t^M \equiv \left[\frac{1}{1-n}\int_n^1 P_t^M(z)^{1-\varepsilon} dz\right]^{\frac{1}{1-\varepsilon}} \tag{1.16}$$

where $P_t^H(z)$ and $P_t^M(z)$ represent the prices expressed in domestic currency of the variety z of home and imported goods, respectively.

Households' budget constraint

For simplicity, we assume domestic households save only in bonds.[19] The budget constraint of the domestic household (j) in units of home currency is given by:

$$\varpi_t^j = (1+i_{t-1})\,\varpi_{t-1}^j - \frac{\psi}{2}\left(\varpi_t^j - \overline{\varpi}\right)^2 + W_t L_t^j - P_t C_t^j + P_t \Gamma_t^j \tag{1.17}$$

where ϖ_t^j is wealth in domestic assets, W_t is the nominal wage, i_t is the domestic nominal interest rate, and Γ_t^j are nominal profits distributed from firms and dealers in the home economy to the household j. Each household owns the same share of firms and dealer agencies in the home economy. Households also face portfolio adjustment costs, for adjusting wealth from its long-run level.[20] Households maximise (1.8) subject to (1.17).

[19] This way the only portfolio decision is made by dealers, which simplifies the analysis.

[20] This assumption is necessary to provide stationarity in the asset position held by the households. See Schmitt-Grohe and Uribe (2003).

Consumption decisions and the supply of labour

The conditions characterising the optimal allocation of domestic consumption are given by the following equation:

$$U_{C,t} = \beta E_t \left\{ U_{C,t+1} \left[\frac{1+i_t}{1+\psi\left(\varpi_t^j - \overline{\varpi}\right)} \right] \frac{P_t}{P_{t+1}} \right\} \tag{1.18}$$

where we have eliminated the index j for the assumption of representative agent. $U_{C,t}$ denotes the marginal utility for consumption. Equation (1.18) corresponds to the Euler equation that determines the optimal path of consumption for households in the home economy, by equalising the marginal benefits of savings to its corresponding marginal costs. The first-order conditions that determine the supply of labour are characterised by the following equation:

$$-\frac{U_{L,t}}{U_{C,t}} = \frac{W_t}{P_t} \tag{1.19}$$

where $\frac{W_t}{P_t}$ denotes real wages. In a competitive labour market, the marginal rate of substitution equals the real wage, as in equation (1.19).

1.2.4 Foreign economy

The consumption basket of the foreign economy is similar to that of the domestic economy, and is given by:

$$C_t^* \equiv \left[\left(\gamma^F\right)^{1/\varepsilon_F} \left(C_t^X\right)^{\frac{\varepsilon_F - 1}{\varepsilon_F}} + \left(1-\gamma^F\right)^{1/\varepsilon_F} \left(C_t^F\right)^{\frac{\varepsilon_F - 1}{\varepsilon_F}} \right]^{\frac{\varepsilon_F}{\varepsilon_F - 1}} \tag{1.20}$$

where ε_F is the elasticity of substitution between domestic (C_t^X) and foreign goods (C_t^F), respectively, and γ^F is the share of domestically produced goods in the consumption basket of the foreign economy. Also, C_t^X and C_t^F are indices of consumption across the continuum of differentiated goods produced similar to C_t^H and C_t^M defined in equations (1.12). The demands for each type of good is given by:

$$C_t^X(z) = \frac{1}{n}\gamma^F \left(\frac{P_t^X(z)}{P_t^X}\right)^{-\varepsilon} \left(\frac{P_t^X}{P_t^*}\right)^{-\varepsilon_H} C_t^* \tag{1.21}$$

$$C_t^F(z) = \frac{1}{1-n}\left(1-\gamma^F\right) \left(\frac{P_t^F(z)}{P_t^F}\right)^{-\varepsilon} \left(\frac{P_t^F}{P_t^*}\right)^{-\varepsilon_H} C_t^* \tag{1.22}$$

where P_t^X and P_t^F correspond to the price indices of exports and the goods produced abroad, respectively. P_t^* is the consumer price index of the foreign economy:

$$P_t^* \equiv \left[\gamma^F \left(P_t^X\right)^{1-\varepsilon_F} + \left(1-\gamma^F\right)\left(P_t^F\right)^{1-\varepsilon_F}\right]^{\frac{1}{1-\varepsilon_F}} \tag{1.23}$$

The small open economy assumption

Following Sutherland (2005), we parameterise the participation of foreign goods in the consumption basket of home households, $\left(1-\gamma^H\right)$, as follows: $\left(1-\gamma^H\right) = (1-n)(1-\gamma)$, where n represents the size of the home economy and $(1-\gamma)$ the degree of openness. In the same way, we assume the participation of home goods in the consumption basket of foreign households, as a function of the relative size of the home economy and the degree of openness of the world economy, that is $\gamma^F = n\left(1-\gamma^*\right)$.

This particular parameterisation implies that as the economy becomes more open, the fraction of imported goods in the consumption basket of domestic households increases, whereas as the economy becomes larger, this fraction falls. This parameterisation allows us to obtain the small open economy as the limiting case of a two-country economy model when the size of the domestic economy approaches zero, that is $n \to 0$. In this case, we have that $\gamma^H \to \gamma$ and $\gamma^F \to 0$. Therefore, in the limiting case, the use in the foreign economy of any home-produced intermediate goods is negligible, and the demand condition for domestic, imported and exported goods can be re-written as follows:

$$Y_t^H = \gamma \left(\frac{P_t^H}{P_t}\right)^{-\varepsilon_H} C_t \tag{1.24}$$

$$M_t = (1-\gamma)\left(\frac{P_t^M}{P_t}\right)^{-\varepsilon_H} C_t \tag{1.25}$$

$$X_t = \left(1-\gamma^*\right)\left(\frac{P_t^X}{P_t^*}\right)^{-\varepsilon_F} C_t^* \tag{1.26}$$

Thus, given the small open economy assumption, the consumer price index for the home and foreign economy can be expressed in the following way:

$$P_t \equiv \left[\gamma \left(P_t^H\right)^{1-\varepsilon_H} + (1-\gamma)\left(P_t^M\right)^{1-\varepsilon_H}\right]^{\frac{1}{1-\varepsilon_H}} \tag{1.27}$$

$$P_t^* = P_t^F \tag{1.28}$$

Given the small open economy assumption, the foreign economy variables that affect the dynamics of the domestic economy are foreign output, Y_t^*, the foreign interest rate, i^*, the external inflation rate, Π^*, and capital inflows, ϖ_t^*. To simplify the analysis, we assume these four variables follow an autoregressive process in logs.

1.2.5 Firms

Intermediate goods producers

A continuum of z intermediate firms exists. These firms operate in a perfectly competitive market and use the following linear technology:

$$Y_t^{int}(z) = A_t L_t(z) \tag{1.29}$$

$L_t(z)$ is the amount of labour demand from households, A_t is the level of technology.

These firms take as given the real wage, W_t/P_t, paid to households and choose their labour demand by minimising costs given the technology. The corresponding first order condition of this problem is:

$$L_t(z) = \frac{MC_t(z)}{W_t/P_t} Y_t^{int}(z)$$

where $MC_t(z)$ represents the real marginal costs in terms of home prices. After replacing the labour demand in the production function, we can solve for the real marginal cost:

$$MC_t(z) = \frac{W_t/P_t}{A_t} \tag{1.30}$$

Given that all intermediate firms face the same constant returns to scale technology, the real marginal cost for each intermediate firm z is the same, that is $MC_t(z) = MC_t$.

Also, given these firms operate in perfect competition, the price of each intermediate good is equal to the marginal cost. Therefore, the relative price $P_t(z)/P_t$ is equal to the real marginal cost in terms of consumption unit (MC_t).

Final goods producers

Goods sold domestically Final goods producers purchase intermediate goods and transform them into differentiated final consumption goods. Therefore, the marginal costs of these firms equal the price of intermediate goods. These firms operate in a monopolistic competitive market, where each firm faces a downward-sloping demand function, given below. Furthermore, we assume that each period t final goods producers face an exogenous probability of changing prices given by $(1-\theta^H)$. Following Calvo (1983), we assume that this probability is independent of the last time the firm set prices and the previous price level. Thus, given a price fixed from period t, the present discounted value of the profits of firm z is given by:

$$E_t\left\{\sum_{k=0}^{\infty}\left(\theta^H\right)^k\Lambda_{t+k}\left[\frac{P_t^{H,o}(z)}{P_{t+k}^H}-MC_{t+k}^H\right]Y_{t,t+k}^H(z)\right\} \tag{1.31}$$

where $\Lambda_{t+k}=\beta^k\frac{U_{C,t+k}}{U_{C,t}}$ is the stochastic discount factor, $MC_{t+k}^H=MC_{t+k}\frac{P_{t+k}}{P_{t+k}^H}$ is the real marginal cost expressed in units of goods produced domestically, and $Y_{t,t+k}^H(z)$ is the demand for good z in $t+k$ conditioned to a fixed price from period t, given by

$$Y_{t,t+k}^H(z)=\left[\frac{P_t^{H,o}(z)}{P_{t+k}^H}\right]^{-\varepsilon}Y_{t+k}^H$$

Each firm z chooses $P_t^{H,o}(z)$ to maximise (1.31). The first order condition of this problem is:

$$E_t\left\{\sum_{k=0}^{\infty}\left(\theta^H\right)^k\Lambda_{t+k}\left[\frac{P_t^{H,o}(z)}{P_t^H}F_{t,t+k}^H-\mu MC_{t+k}^H\right]\left(F_{t,t+k}^H\right)^{-\varepsilon}Y_{t+k}^H\right\}=0$$

where $\mu\equiv\frac{\varepsilon}{\varepsilon-1}$ and $F_{t,t+k}^H\equiv\frac{P_t^H}{P_{t+k}^H}$.

Following Benigno and Woodford (2005), the previous first order condition can be written recursively using two auxiliary variables, V_t^D and V_t^N, defined as follows:

$$\frac{P_t^{H,o}(z)}{P_t^H}=\frac{V_t^N}{V_t^D}$$

where

$$V_t^N = \mu U_{C,t} Y_t^H MC_t^H + \theta^H \beta E_t \left[V_{t+1}^N \left(\Pi_{t+1}^H \right)^{\varepsilon} \right] \tag{1.32}$$

$$V_t^D = U_{C,t} Y_t^H + \theta^H \beta E_t \left[V_{t+1}^D \left(\Pi_{t+1}^H \right)^{\varepsilon - 1} \right] \tag{1.33}$$

Also, since in each period t only a fraction $\left(1 - \theta^H\right)$ of these firms change prices, the gross rate of domestic inflation is determined by the following condition:

$$\theta^H \left(\Pi_t^H \right)^{\varepsilon - 1} = 1 - \left(1 - \theta^H\right) \left(\frac{V_t^N}{V_t^D} \right)^{1-\varepsilon} \tag{1.34}$$

The equations (1.32), (1.33) and (1.34) determine the supply (Phillips) curve of domestic production.

Exported goods We assume that firms producing final goods can discriminate prices between domestic and external markets. Therefore, they can set the price of their exports in foreign currency. Also, when selling abroad they face an environment of monopolistic competition with nominal rigidities, with a probability $1 - \theta^X$ of changing prices.

The problem of retailers selling abroad is very similar to that of firms that sell in the domestic market, which is summarised in the following three equations that determine the supply curve of exporters in foreign currency prices:

$$V_t^{N,X} = \mu \left(Y_t^X U_{C,t} \right) MC_t^X + \theta^X \beta E_t \left[V_{t+1}^{N,X} \left(\Pi_{t+1}^X \right)^{\varepsilon} \right] \tag{1.35}$$

$$V_t^{D,X} = \left(Y_t^X U_{C,t} \right) + \theta^X \beta E_t \left[V_{t+1}^{D,X} \left(\Pi_{t+1}^X \right)^{\varepsilon - 1} \right] \tag{1.36}$$

$$\theta^X \left(\Pi_t^X \right)^{\varepsilon - 1} = 1 - \left(1 - \theta^X\right) \left(\frac{V_t^{N,X}}{V_t^{D,X}} \right)^{1-\varepsilon} \tag{1.37}$$

where the real marginal costs of the goods produced for export are given by:

$$\begin{aligned} MC_t^X &= \frac{P_t MC_t}{S_t P_t^X} \\ &= \frac{MC_t}{RER_t \left(\frac{P_t^X}{P_t^*} \right)} \end{aligned} \tag{1.38}$$

which depend inversely on the real exchange rate ($RER_t = \frac{S_t P_t^*}{P_t}$) and the relative price of exports to external prices $\left(\frac{P_t^X}{P_t^*}\right)$.

Retailers of imported goods

Those firms that sell imported goods buy a homogeneous good in the world market and differentiate it into a final imported good $Y_t^M(z)$. These firms also operate in an environment of monopolistic competition with nominal rigidities, with a probability $1 - \theta^M$ of changing prices.

The problem for retailers is very similar to that of producers of final goods. The Phillips curve for importers is given by:

$$V_t^{N,M} = \mu\left(Y_t^M U_{C,t}\right) MC_t^M + \theta^M \beta E_t\left[V_{t+1}^{N,M}\left(\Pi_{t+1}^M\right)^{\varepsilon}\right] \tag{1.39}$$

$$V_t^{D,M} = \left(Y_t^M U_{C,t}\right) + \theta^M \beta E_t\left[V_{t+1}^{D,M}\left(\Pi_{t+1}^M\right)^{\varepsilon-1}\right] \tag{1.40}$$

$$\theta^M\left(\Pi_t^M\right)^{\varepsilon-1} = 1 - \left(1-\theta^M\right)\left(\frac{V_t^{N,M}}{V_t^{D,M}}\right)^{1-\varepsilon} \tag{1.41}$$

where the real marginal cost for importers is given by the cost of purchasing the goods abroad ($S_t P_t^*$) to the price of imports (P_t^M):

$$MC_t^M = \frac{S_t P_t^*}{P_t^M} \tag{1.42}$$

where MC_t^M also measures the deviations from the law of one price.[21]

1.2.6 Market clearing

Total domestic production is given by:

$$P_t^{def} Y_t = P_t^H Y_t^H + S_t P_t^X Y_t^X \tag{1.43}$$

[21] See Galí and Monacelli (2005) for a similar formulation.

After using equations (1.24) and (1.25) and the definition of the consumer price index (1.27), equation (1.43) can be decomposed in:

$$P_t^{def} Y_t = P_t C_t + S_t P_t^X Y_t^X - P_t^M Y_t^M \tag{1.44}$$

To identify the gross domestic product (GDP) of this economy, Y_t, it is necessary to define the GDP deflator, P_t^{def}, which is the weighted sum of the consumer, export and import price indices:[22]

$$P_t^{def} = \phi_C P_t + \phi_X S_t P_t^X - \phi_M P_t^M \tag{1.45}$$

where ϕ_C, ϕ_X and ϕ_M are steady state values of the ratios of consumption, exports and imports to GDP, respectively. The demand for intermediate goods is obtained by aggregating the production for home consumption and exports:

$$\begin{aligned} Y_t^{int}(z) &= Y_t^H(z) + Y_t^X(z) \\ &= \left(\frac{P_t^H(z)}{P_t^H}\right)^{-\varepsilon} Y_t^H + \left(\frac{P_t^X(z)}{P_t^X}\right)^{-\varepsilon} Y_t^X \end{aligned} \tag{1.46}$$

Aggregating (1.46) with respect to z, we obtain:

$$Y_t^{int} = \frac{1}{n}\int_0^n Y_t^{int}(z)\,dz = \Delta_t^H Y_t^H + \Delta_t^X Y_t^X \tag{1.47}$$

where $\Delta_t^H = \frac{1}{n}\int_0^n \left(\frac{P_t^H(z)}{P_t^H}\right)^{-\varepsilon} dz$ and $\Delta_t^X = \frac{1}{n}\int_0^n \left(\frac{P_t^X(z)}{P_t^X}\right)^{-\varepsilon} dz$ are measures of relative price dispersion, which have a null impact on the dynamic in a first order approximation of the model. Similarly, the aggregate demand for labour is:

$$L_t = \frac{MC_t}{W_t/P_t}\left(\Delta_t^H Y_t^H + \Delta_t^X Y_t^X\right) \tag{1.48}$$

After aggregating household's budget constraints, firms' and dealers' profits, and including the equilibrium condition in the financial market that equates household wealth with the stock of domestic bonds, we obtain the aggregate resources constraint of the home

[22] Strictly this variable constitutes a first order approximation to the deflator, since weights change when the economy is outside of the steady-state.

economy:

$$\frac{B_t}{P_t} - \frac{B_{t-1}}{P_{t-1}} + \frac{\psi}{2}\left(\frac{B_t}{P_t} - \frac{\overline{B}}{P}\right)^2 = \frac{P_t^{def}}{P_t}Y_t - C_t + \left\{\frac{(1+i_{t-1})}{\Pi_t} - 1\right\}\frac{B_{t-1}}{P_{t-1}} + REST_t \tag{1.49}$$

Equation (1.49) corresponds to the current account of the home economy. The left-hand side is the change in the net asset position in terms of consumption units. The right-hand side is the trade balance, the difference between GDP and consumption which is equal to net exports, and the investment income. The last term, $REST_t \equiv \frac{P_t^M}{P_t} Y_t^M \left(1 - \Delta_t^M MC_t^M\right)$ is negligible and takes into account the monopolistic profits of retail firms.[23]

1.3 Results

1.3.1 Calibration

Instead of calibrating the parameters to a particular economy, we set the parameters to values that are standard in the new open economy literature, as shown in Table 1.1. The discount factor β is fixed at 0.9975, which implies a real interest rate of 1% in the steady state. The labour supply elasticity is set at 0.5 implying a relatively inelastic labour supply, though within the values found in empirical studies.[24] The parameter γ governing households' risk aversion is fixed at 1, which is the one corresponding to logarithmic utility. The value for the elasticity of substitution between home and foreign goods is a controversial parameter. We follow previous studies in the DSGE literature, which consider values between 0.75 and 1.5.[25] The share of domestic tradable goods in the CPI is set to 0.6, implying a participation of imported final and intermediate goods of 0.4 in the domestic CPI, in line with other studies for small open economies.[26] Regarding price stickiness, we set a higher value for domestic goods over imported and exported ones. For domestic goods, the assumed stickiness implies that firms keep their

[23] A complete set of the log-linearised equations of the model can be found in Appendix 1.B.

[24] See Chetty et al. (2011).

[25] See Rabanal and Tuesta (2006). Other authors in the trade literature find values for this elasticity around 5, see Lai and Trefler (2002).

[26] See Castillo et al. (2009).

prices fixed for 4 quarters on average.

Table 1.1: **Baseline Calibration**

$Parameter$	$Value$	$Description$
β	0.9975	Consumers time-preference parameter.
χ	0.5	Labour supply elasticity.
γ_c	1	Risk aversion parameter.
ε	0.75	Elasticity of substitution btw. home and foreign goods.
ε_F	0.75	Elasticity of substitution btw. exports and foreign goods.
ψ	0.6	Share of domestic tradables in domestic consumption.
θ_H	0.75	Domestic goods price rigidity.
θ_M	0.5	Imported goods price rigidity.
θ_X	0.5	Exported goods price rigidity.
ψ_b	0.01	Portfolio adjustment costs.
φ_π	1.5	Taylor rule reaction to inflation deviations.
γ	500	Absolute risk aversion parameter (dealers)
ϕ_ϖ	0.5	Net asset position over GDP ratio
ϕ_C	0.68	Consumption over GDP ratio
σ_x	0.01	S.D. of all shocks x
ρ_x	0.5	AR(1) coefficient for all exogenous processes

The parameter for portfolio adjustment costs is set a 0.01 to ensure that the cost of adjusting the size of the portfolio is small in the baseline calibration. For the central bank reaction function, we fixed a baseline reaction to inflation deviations of 1.5, which means that the central bank reacts more than one for one to inflation expectations, affecting the real interest rate. The coefficient of absolute risk aversion for dealers was set to 500 as in Bacchetta and Wincoop (2006).[27] Finally, The standard deviation of all exogenous processes was set to 0.01 and the autocorrelation coefficient to 0.5. In the benchmark case, we calibrate the FX intervention reaction to exchange rate changes and real exchange rate misalignments to 0.5 for the Δs rule and 0.3 for the RER rule, and analyse how results change with those parameters.

[27]Notice that this parameter must be corrected by the steady state consumption level to make it comparable with the CRRA case. Additionally, only the product of γ and the equilibrium value of the exchange rate volatility (of order $10^-3) matter for the model dynamics.$

1.3.2 Model dynamics

In this section we present our results. We first discuss briefly the existence of equilibrium.[28] Once we confirm the existence of an equilibrium, we study the effectiveness of different FX intervention strategies in reducing the macroeconomic volatility. We do this by contrasting the relative volatility of a sample of variables in the absence and under the presence of intervention. Next, we explore the reaction of the economy to external shocks under different intervention strategies through the calculation of impulse-response functions. We close this section studying how FX intervention affects the relative importance of shocks to fundamentals *vis-à-vis* liquidity based trading. We present robustness exercises to the parameters defining the pass-through of exchange rates to prices $(\varepsilon, \varepsilon_F)$ and domestic price rigidity (θ_H in Section 1.4).

Rational expectations (RE) equilibria

As shown in Section 1.2, the risk premium-adjusted uncovered interest parity condition (equation 1.3) depends, among other things, on the conditional variance of the change in the exchange rate. This, is an endogenous outcome of the RE equilibrium of the model. Solving for the RE equilibria entails solving for a fixed point problem in the conditional variance of the change in the exchange rate. In Figure 1-2, we plot the mappings of the conjectured and the implied conditional variance of the depreciation rate for different parameterisations of the FX intervention reaction function. Intersections with the 45-degree straight line correspond to fixed points for the conditional variance of the depreciation rate.

As shown in the left-hand panels, there are two RE equilibria in the case of no FX intervention, corresponding to a low-variance stable equilibrium and a high-variance unstable equilibrium.[29] This type of multiple equilibria is similar to the one found by Bacchetta and Wincoop (2006) in a model without FX intervention. However, as shown in the centre and right-hand panels, FX intervention helps to rule out the second unstable equilibrium. Under both rules of FX intervention there is only a unique and

[28] As in Vitale (2011), when solving for the equilibrium variance of the exchange rate, we are unable to rely on a theorem of existence, nor exclude the presence of multiple equilibria.

[29] A slope lower (higher) than one of the mapping of the conjectured and the implied conditional variance of the depreciation rate, evaluated at the intersection with the 45-degree straight line, indicates a stable (unstable) equilibrium.

stable equilibrium. Also, the intensity of FX intervention reduces the RE equilibrium variance of the exchange rate change.[30]

The RE equilibrium variance of the exchange rate change also affects the direct impact of FX intervention and capital flows on the exchange rate, as shown in equation (1.3). Therefore, a more intensive FX intervention strategy also reduces its effectiveness as the reduction in variance dampens the impact of interventions on the exchange rate.

Transmission of external shocks

In Table 2 we present unconditional relative variances of some main macroeconomic variables assuming only one source of volatility at the time for different FX intervention regimes.[31] For comparison, relative variances are normalised with respect to the no intervention case.

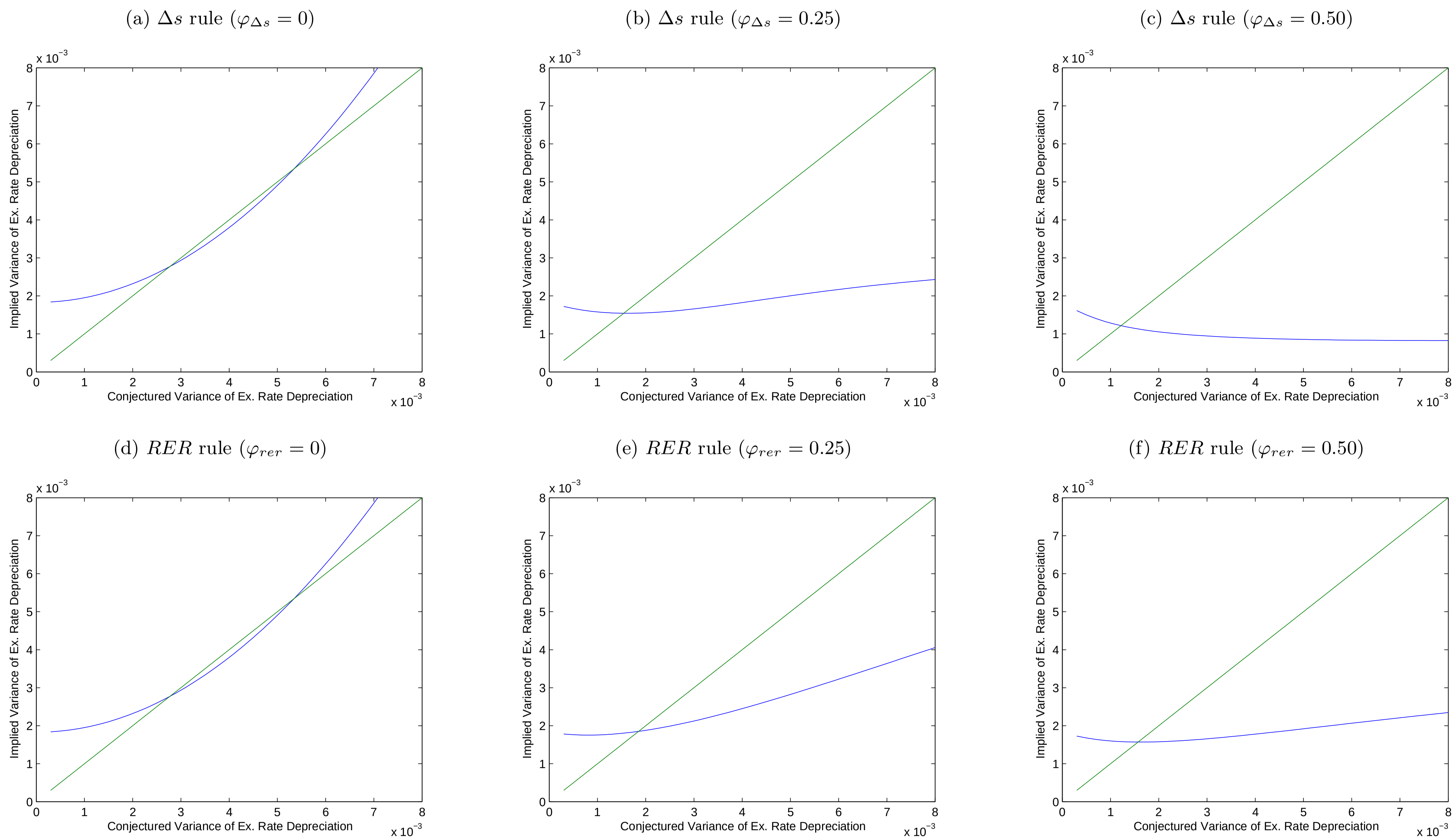
As shown, not surprisingly, FX intervention reduces the volatility of the change of the exchange rate in all cases. However, this exercise highlights some trade-offs in the use of FX intervention. In particular, the effects of FX intervention on the volatility of other macroeconomic variables will depend on the source of the shock. FX intervention helps to isolate domestic macroeconomic variables from financial external shocks, but amplifies fluctuations in some domestic variables from nominal and real external shocks.

For instance, the volatility of consumption, exports, output and inflation generated by foreign interest rate and capital flow shocks is reduced under both types of FX intervention regimes. However, the use of FX interventions to smooth the nominal exchange rate amplifies the volatility of inflation and output generated by foreign inflation shocks. Similarly, the use of a real exchange rate misalignment rule increases the volatility of consumption, exports, output and inflation generated by foreign output shocks. In this case, FX intervention prevents the adjustment of the real exchange rate as a macroeconomic stabiliser.

[30] This is a novel result, in stark contrast with the findings of Vitale (2011). We consider the author's setup different to ours as in his model, central bank FX interventions are always informative and can potentially increase information dispersion across agents.

[31] Exercises are simulated using the conditional variance of the depreciation rate in equilibrium in equation 1.3.

Figure 1-2: Existence of equilibria under FX intervention rules



(a) Δs rule ($\varphi_{\Delta s} = 0$)

(b) Δs rule ($\varphi_{\Delta s} = 0.25$)

(c) Δs rule ($\varphi_{\Delta s} = 0.50$)

(d) RER rule ($\varphi_{rer} = 0$)

(e) RER rule ($\varphi_{rer} = 0.25$)

(f) RER rule ($\varphi_{rer} = 0.50$)

Simulations involved 61 values for the conjectured variances of the change of the exchange rate. When the intervention parameter under both rules is zero, we replicate the values for the pure discretional intervention case.

Table 1.2: **Macroeconomic volatility under FX Intervention Rules (No intervention $\equiv 1$)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.31	0.69	0.48	0.92	0.44	0.12	0.41
$\varphi_{\Delta s} = 0.50$	0.10	0.56	0.26	0.89	0.21	0.03	0.17
$\varphi_{rer} = 0.25$	0.76	0.90	0.84	0.97	0.86	0.66	0.84
$\varphi_{rer} = 0.50$	0.64	0.85	0.75	0.96	0.78	0.49	0.74
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.25	0.23	0.31	0.28	0.34	0.32	0.31
$\varphi_{\Delta s} = 0.50$	0.15	0.13	0.21	0.17	0.24	0.21	0.21
$\varphi_{rer} = 0.25$	0.38	0.38	0.40	0.38	0.42	0.39	0.40
$\varphi_{rer} = 0.50$	0.24	0.23	0.25	0.24	0.27	0.25	0.26
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.90	0.78	0.99	0.92	1.08	1.02	1.06
$\varphi_{\Delta s} = 0.50$	0.84	0.67	0.96	0.85	1.13	1.01	1.08
$\varphi_{rer} = 0.25$	0.76	0.74	0.77	0.75	0.81	0.78	0.80
$\varphi_{rer} = 0.50$	0.63	0.62	0.66	0.63	0.71	0.66	0.69
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.28	0.93	0.89	0.99	0.72	0.92	0.63
$\varphi_{\Delta s} = 0.50$	1.42	0.85	0.85	0.98	0.59	0.89	0.46
$\varphi_{rer} = 0.25$	0.72	0.73	1.10	1.02	1.15	1.09	1.18
$\varphi_{rer} = 0.50$	0.59	0.62	1.15	1.02	1.24	1.14	1.30

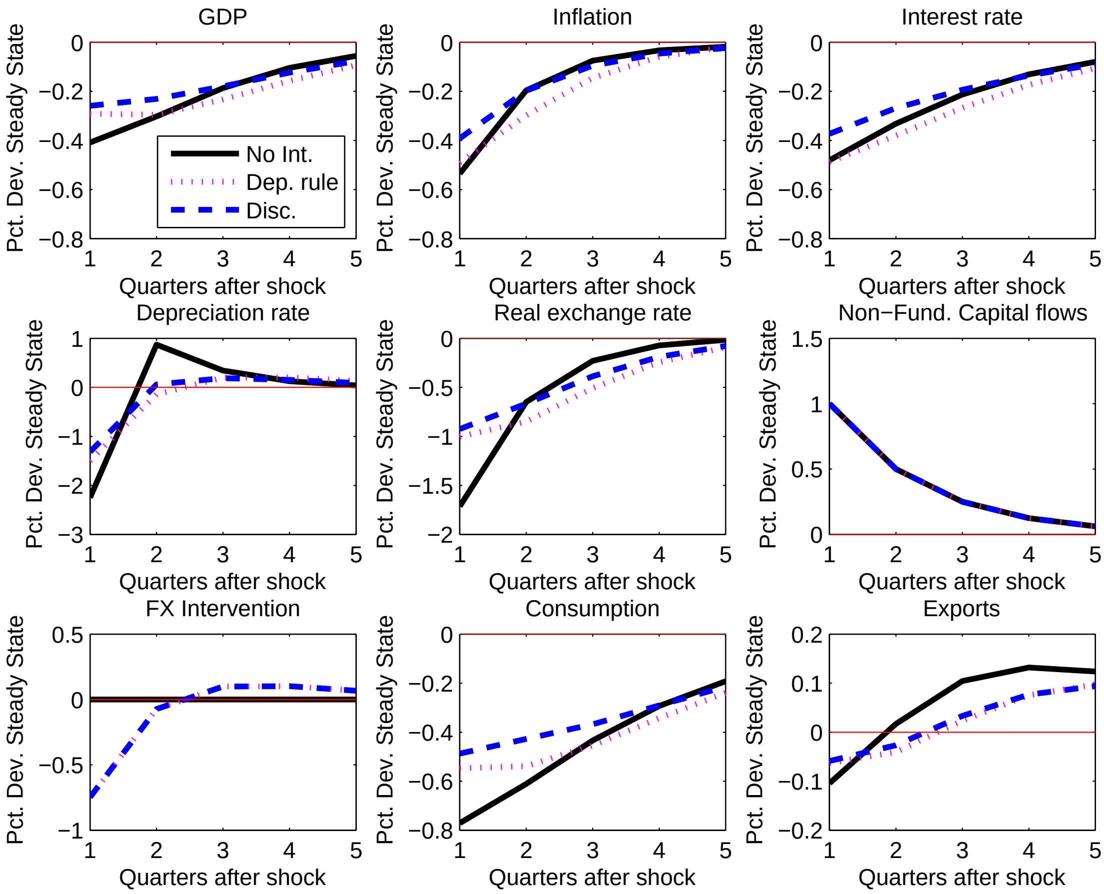
Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

In Figures 1-3, 1-4 and 1-5 we compare the dynamic effects of external shocks under discretion, the Δs_t rule and the case with no intervention.[32] Overall, the effectiveness of intervention rules is confirmed. In other words, given that it is known the central bank will enter the FX market to prevent large fluctuations in the exchange rate, the amount of intervention necessary to reduce fluctuations is smaller. This means that the FX sales and purchases by the central bank necessary to stabilise the exchange rate will be much higher under discretion because it does not influence expectations as in the case of an intervention rule.

In Figures 1-3 we show the reaction to a portfolio or non-fundamental capital flow shock. These inflows generate an appreciation of the exchange rate, that under no intervention affects the whole economy. In the case where the central bank intervenes through rules or discretion, the effects of these shocks are dampened, stabilising the economy. For the case of a foreign interest rate shock, in Figure 1-4 we show how interventions can ease the pressure of capital outflows on the exchange rate. It is interesting to see how interventions have similar effects when reacting to non-fundamental (order flow) and fundamental (interest rate) shocks. Finally, in Figure 1-5 we show the reaction to a foreign inflation shock. In this case, as in the previous ones, interventions provide a channel to counter the impact of external shocks on the economy. Foreign inflation will generate an exchange rate appreciation and a current account deficit. An active central bank is capable of reversing these effects through foreign exchange interventions, since the combination of a low nominal depreciation under the exchange rate smoothing rule with higher foreign inflation can generate a depreciation of the real exchange rate.

[32] The case of the RER rule is presented in figures 1-8, 1-9 and 1-10 in Appendix 1.A.

Figure 1-3: Reaction to a 1% portfolio shock - Δs_t rule.

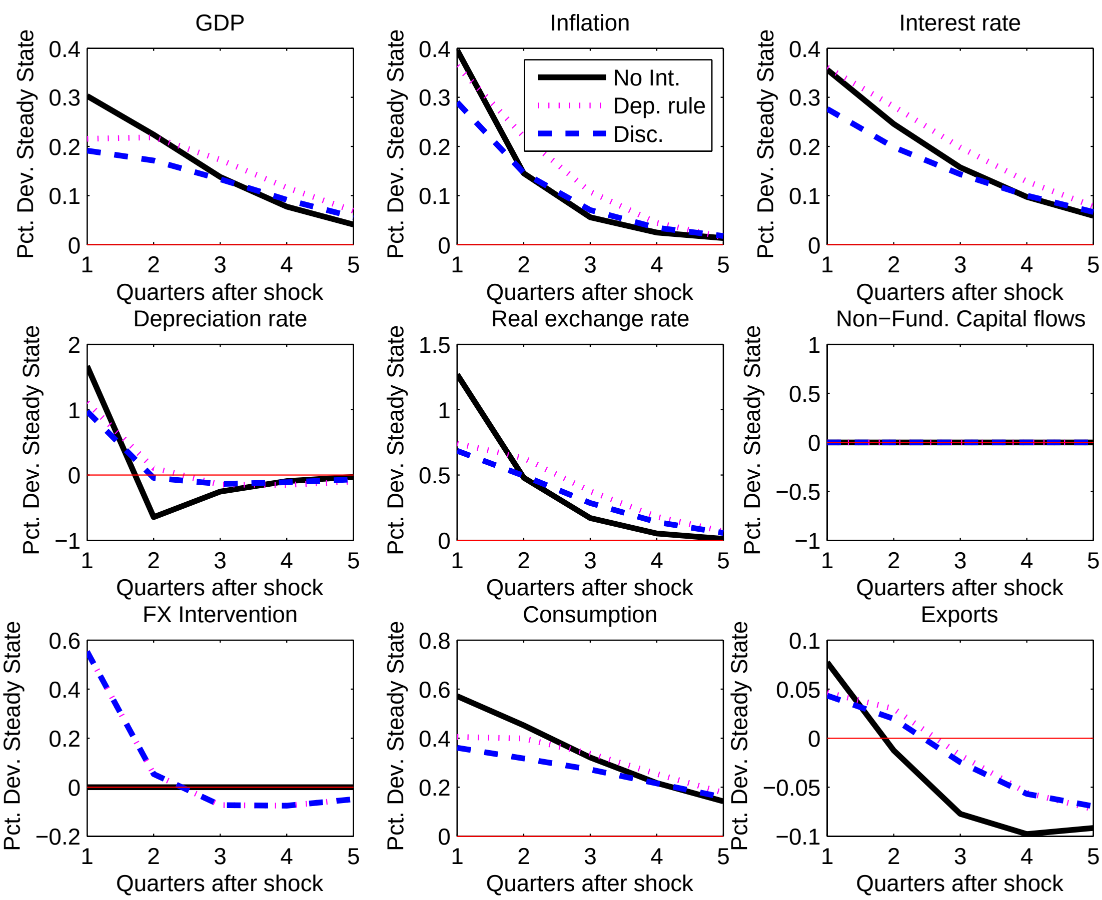


Note: Intervention under discretion normalised to the implied intervention path under rules.

Contribution of shocks and FX intervention

Up to now we have shown the effectiveness of FX interventions by the central bank as a mechanism to cope with the effects of external shocks. To show this we have kept the variance of the exchange rate constant across regimes, as a way to make results comparable. However, as shown by Figure 1-2, intervention rules reduce the equilibrium value of the exchange rate volatility. This is key to understanding an additional effect of interventions. The impact of portfolio shocks on the exchange rate value is a function of the risk dealers bear for holding more foreign currency in their portfolio. Hence, a lower volatility will reduce the risk and consequently the premia they charge for these holdings. This makes interventions less effective when dealing with most external shocks, as shown by Table 1.2, while improving the resilience of the economy to portfolio or non-fundamental capital flow shocks. Specifically, when we assume the only shocks in the

Figure 1-4: Reaction to a 1% foreign interest rate shock - Δs_t rule.



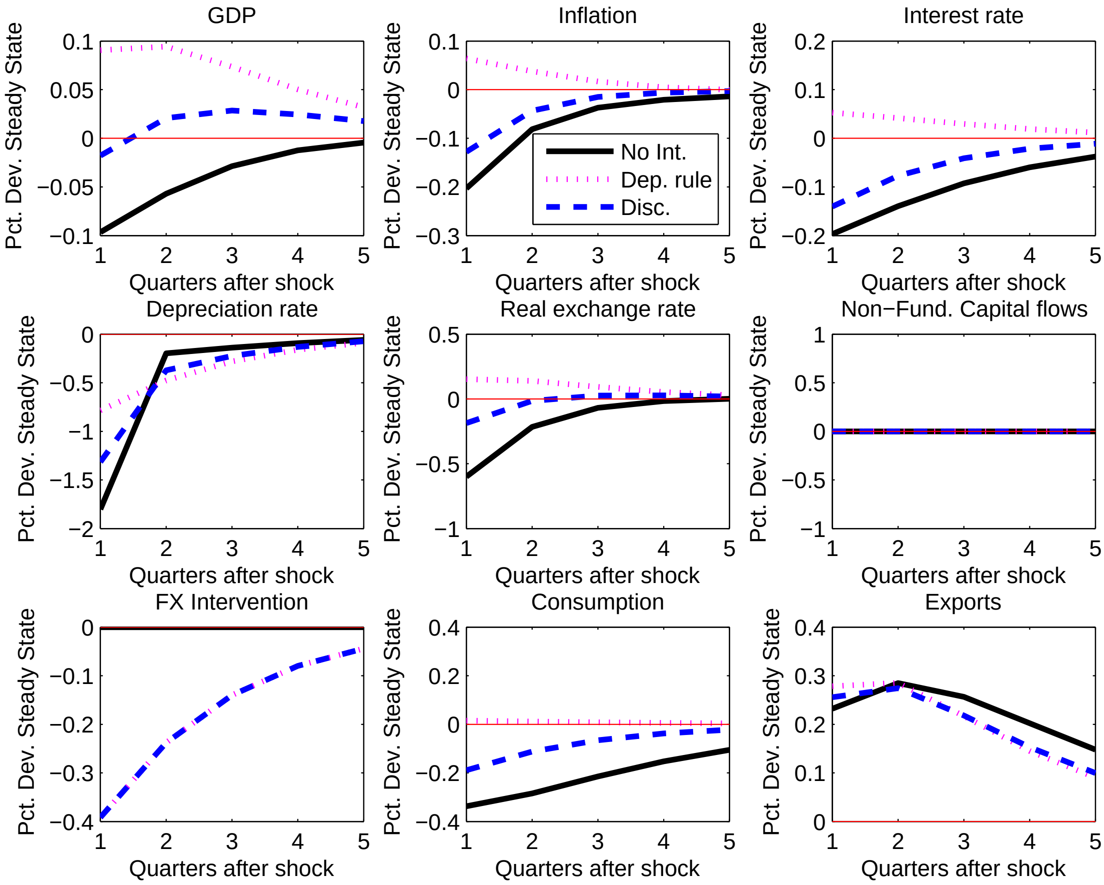
Note: Intervention under discretion normalised to the implied intervention path under rules.

economy are given by the portfolio capital flows shocks, the volatility of the real exchange rate and the change of the exchange rate fall up to 85 and 87 percent respectively, in comparison to the no intervention case. This implies that through FX interventions, it is possible to reduce significantly the response of the exchange rate to portfolio shocks.[33]

Thus, our simulations show that intervention rules that reduce the volatility of the exchange rate affect as well the relative importance that shocks have in explaining this variance. In Figure 1-6 we show the variance decomposition of the exchange rate variation under different shocks. Our result is robust to the intensity of intervention, when the central bank intervenes in the FX market through rules, capital flows shocks explain a smaller fraction of the fluctuations of the change of the exchange rate, while the effect of others, such as foreign interest rate shocks, become relatively more important.

[33]Since discretional interventions work in a similar way as these portfolio shocks, the ability of the central bank to affect the exchange rate through discretional sales or purchases, diminishes as well.

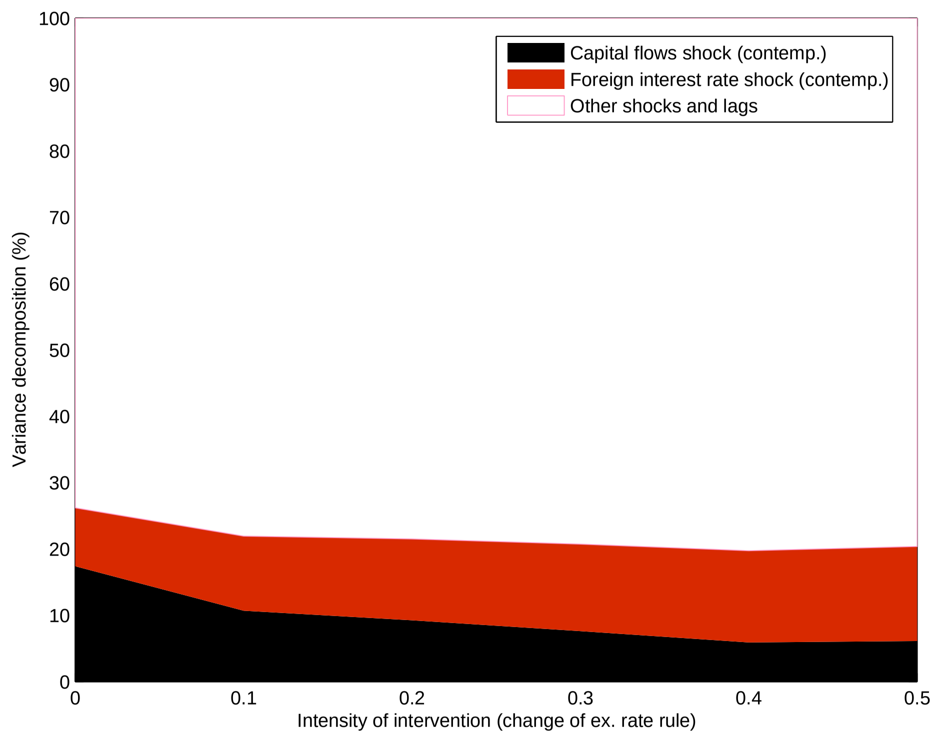
Figure 1-5: Reaction to a 1% foreign inflation rate shock - Δs_t rule.



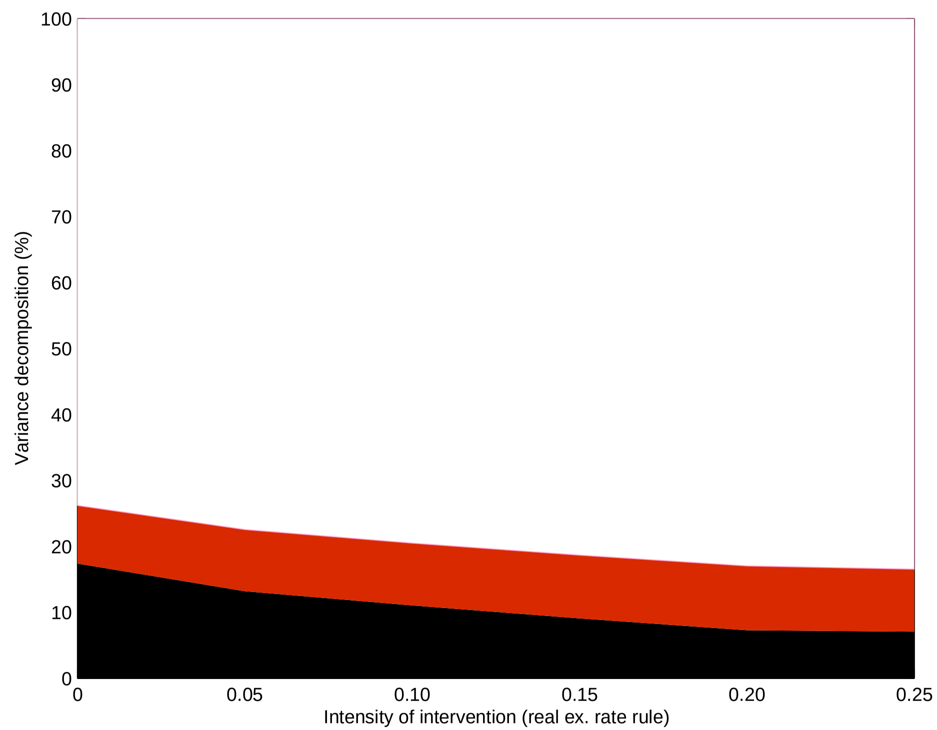
Note: Intervention under discretion normalised to the implied intervention path under rules.

Figure 1-6: Variance decomposition of the exchange rate changes (Δs_t)

(a) Δs rule



(b) RER rule



Graphs report the average R2 statistic of regressions of the change of the exchange rate over the specified series and a constant. The sample in each simulation is of 2500 observation (first 500 points dropped). Regressions are done on all 9 shocks of the model for their contemporaneous value and 5 lags. Sum of R2 statistics is normalized to 1.)

1.4 Robustness

We perform robustness exercises to several parameters related to the transmission mechanism of FX interventions into prices. Results are presented in Appendix 1.A. Results are robust to the assumed degree of elasticity between home and foreign goods (ε), as Tables 1.3 and 1.4 show. Tables 1.5 and 1.6 show results for changes in the elasticity of substitution between foreign and exports goods, (ε_F). This parameter has strong effects on the capacity of the central bank to reduce the relative volatility of consumption and production in the face of financial shocks. This result is not surprising since a lower elasticity of substitution means that shocks to the exchange rate will have a smaller impact on the quantities exported, but a higher impact on the country's income. As we observe, interventions are more effective reducing the volatility of consumption but less effective in the case of exports. The opposite occurs in the case of a high ε_F.

Tables 1.7 and 1.8 show results for the case of low and high domestic good price rigidity, respectively. We observe that price rigidity increases the effectiveness of FX intervention rules in isolating the economy from foreign interest rate shocks. Under low domestic good price rigidity, intervention rules imply a volatility of consumption between 43% and 96% of the no intervention benchmark. When price rigidity is high ($\theta_H = 0.95$), the relative volatility of consumption with intervention rules is between 12% and 64% of the no intervention benchmark. However, this result does not hold when the economy is hit by capital flows shocks. In this case, a central bank aiming to smooth the exchange rate can actually increase the volatility of variables such as consumption and production. The presence of high price stickiness, combined with a sluggish exchange rate - due to an active FX intervention policy - slows down corrections of the real exchange rate, increasing both consumption and GDP volatility.

1.5 Conclusions

In this chapter, we present a model to analyse the interaction between monetary policy and FX intervention by central banks, which also includes microstructure fundamentals in the determination of the exchange rate. We introduce a portfolio decision of risk-averse dealers, which adds an endogenous risk premium to the traditional uncovered interest rate condition. In this model, FX intervention affects the exchange rate through both a portfolio-balance and and a volatility channel.

Our results illustrate that FX intervention has strong interactions with monetary policy. Intervening to smooth real exchange rate misalignments can mute the monetary transmission mechanism through exchange rates, reducing the impact on aggregate demand and prices, while intervening to smooth nominal exchange rate fluctuations can amplify the impact. Also, FX intervention rules can be more powerful in stabilising the economy as they exploit the expectations channel. When we analyse the response to foreign shocks, we show that FX intervention rules have some advantages as a stabilisation tool, because they anchor expectations about future exchange rates. Therefore, the amount of FX intervention needed to stabilise the exchange rate under rules is much smaller than under discretion. We also show that there are some trade-offs in the use of FX intervention. On the one hand, it can help isolate the economy from external financial shocks, but on the other it prevents some necessary adjustments of the exchange rate in response to nominal and real external shocks.

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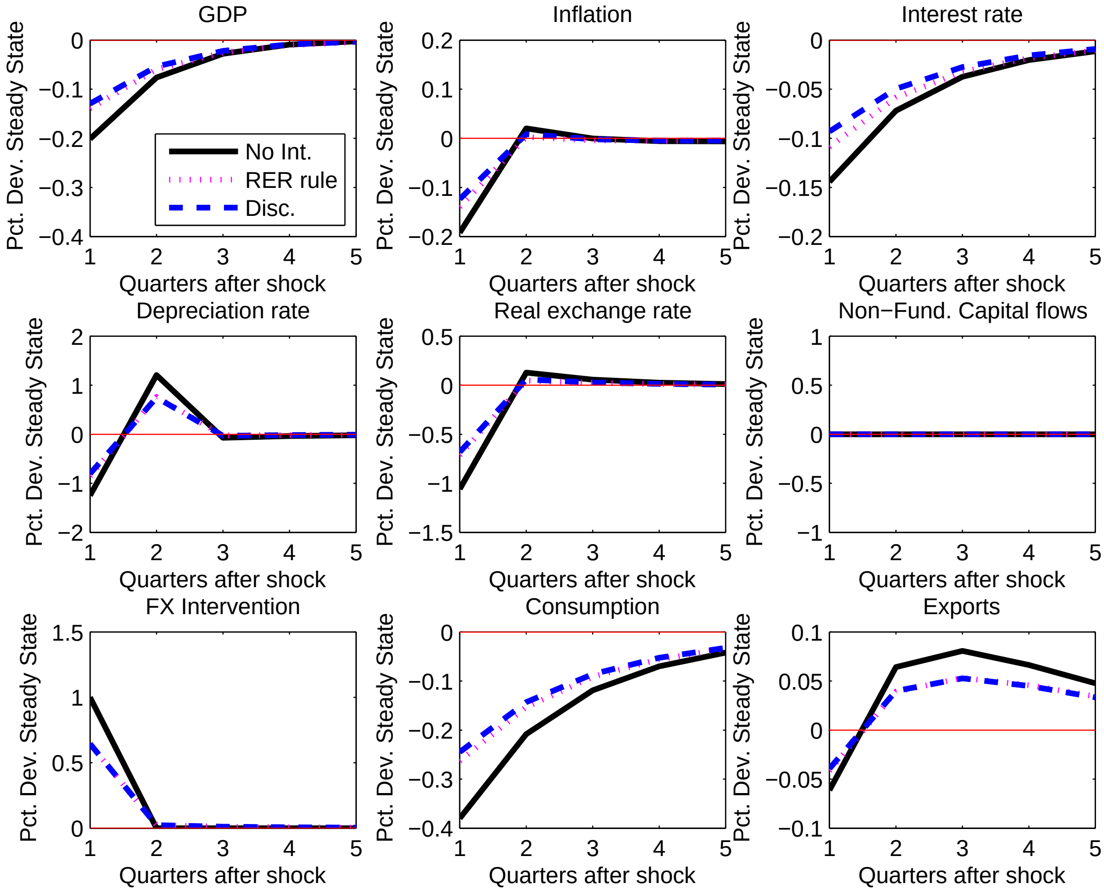
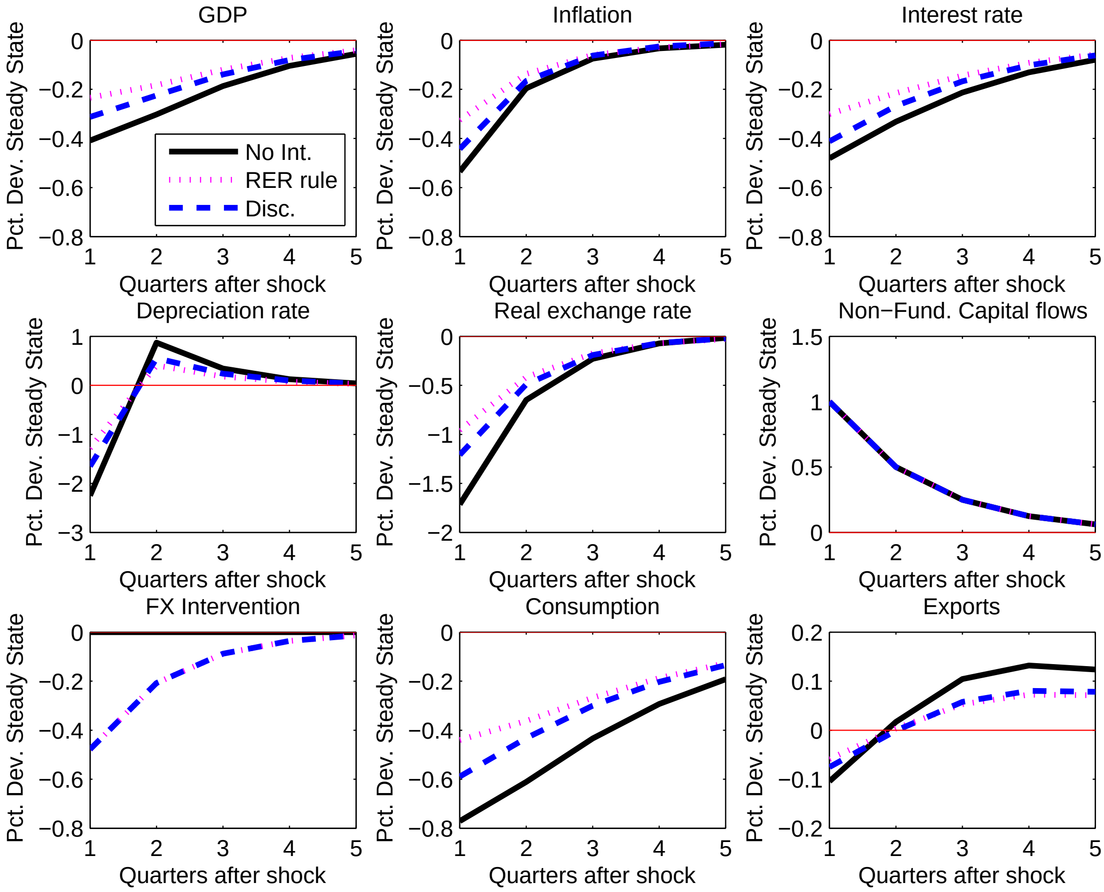
1.A Figures and tables

Figure 1-7: Reaction to a 1% FX intervention shock - *RER* rule.



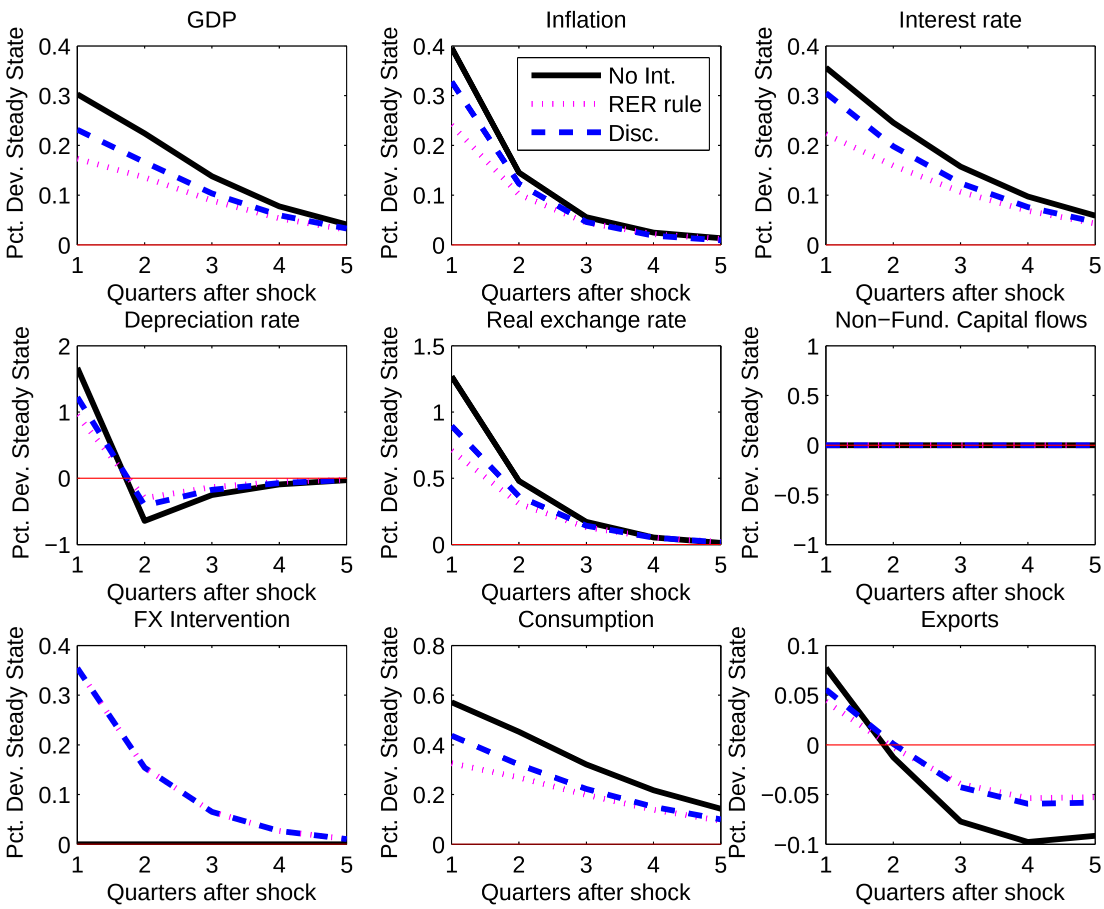
Note: Intervention under discretion normalised to the implied intervention path under rules.

Figure 1-8: Reaction to a 1% portfolio shock - *RER* rule



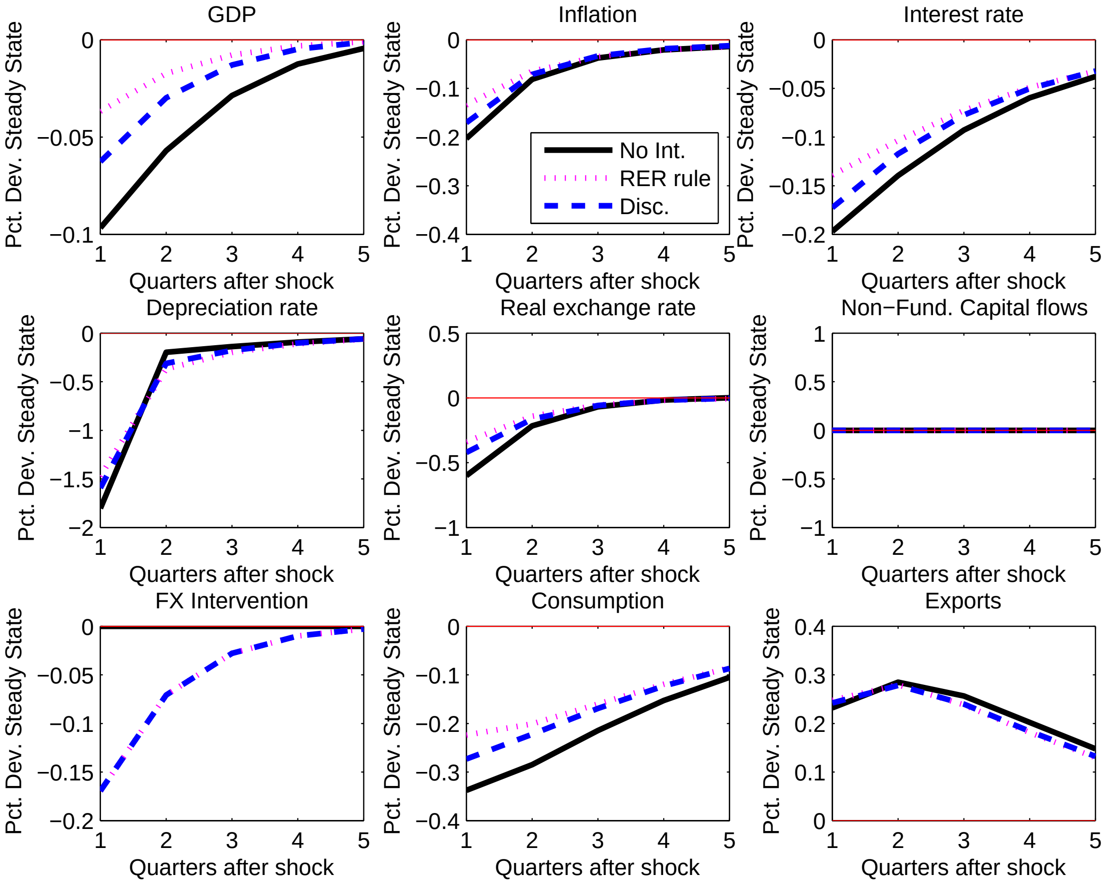
Note: Intervention under discretion normalised to the implied intervention path under rules.

Figure 1-9: Reaction to a 1% foreign interest rate shock - *RER* rule.



Note: Intervention under discretion normalised to the implied intervention path under rules.

Figure 1-10: Reaction to a 1% foreign inflation rate shock - RER rule.



Note: Intervention under discretion normalised to the implied intervention path under rules.

Table 1.3: **Macroeconomic volatility (No intervention $\equiv 1$), Low elasticity of subs. btw. home and foreign goods ($\varepsilon = 0.4$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.32	0.69	0.5	0.98	0.46	0.09	0.42
$\varphi_{\Delta s} = 0.50$	0.07	0.53	0.24	0.99	0.18	0.12	0.14
$\varphi_{rer} = 0.25$	0.66	0.86	0.77	0.98	0.8	0.49	0.76
$\varphi_{rer} = 0.50$	0.55	0.81	0.69	0.98	0.72	0.33	0.67
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.21	0.19	0.25	0.22	0.28	0.26	0.26
$\varphi_{\Delta s} = 0.50$	0.15	0.12	0.2	0.15	0.24	0.21	0.21
$\varphi_{rer} = 0.25$	0.23	0.22	0.24	0.22	0.26	0.24	0.24
$\varphi_{rer} = 0.50$	0.12	0.11	0.12	0.11	0.14	0.12	0.13
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.93	0.79	0.98	0.96	1.11	1.03	1.08
$\varphi_{\Delta s} = 0.50$	0.87	0.67	0.95	0.93	1.2	1.02	1.13
$\varphi_{rer} = 0.25$	0.66	0.64	0.67	0.64	0.71	0.67	0.7
$\varphi_{rer} = 0.50$	0.54	0.52	0.55	0.52	0.61	0.56	0.58
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.34	0.96	0.91	0.99	0.72	0.91	0.64
$\varphi_{\Delta s} = 0.50$	1.55	0.88	0.86	1	0.56	0.88	0.44
$\varphi_{rer} = 0.25$	0.6	0.63	1.13	1.03	1.22	1.13	1.28
$\varphi_{rer} = 0.50$	0.47	0.54	1.19	1.04	1.32	1.18	1.4

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

Table 1.4: **Macroeconomic Volatility (No intervention $\equiv 1$), High elasticity of subs. btw. home and foreign goods ($\varepsilon = 1.5$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.31	0.69	0.46	0.84	0.42	0.15	0.4
$\varphi_{\Delta s} = 0.50$	0.14	0.59	0.28	0.78	0.23	0.02	0.21
$\varphi_{rer} = 0.25$	0.68	0.87	0.78	0.94	0.8	0.59	0.78
$\varphi_{rer} = 0.50$	0.57	0.83	0.69	0.92	0.72	0.45	0.69
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.31	0.28	0.4	0.37	0.43	0.4	0.39
$\varphi_{\Delta s} = 0.50$	0.16	0.14	0.22	0.2	0.25	0.22	0.22
$\varphi_{rer} = 0.25$	0.35	0.34	0.37	0.37	0.4	0.37	0.38
$\varphi_{rer} = 0.50$	0.18	0.17	0.2	0.19	0.22	0.19	0.2
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.87	0.77	0.98	0.96	1.04	0.99	1.02
$\varphi_{\Delta s} = 0.50$	0.81	0.68	0.96	0.94	1.06	0.98	1.03
$\varphi_{rer} = 0.25$	0.68	0.66	0.71	0.72	0.76	0.71	0.74
$\varphi_{rer} = 0.50$	0.57	0.55	0.6	0.61	0.66	0.6	0.64
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.22	0.89	0.88	1	0.72	0.92	0.62
$\varphi_{\Delta s} = 0.50$	1.3	0.83	0.84	1	0.61	0.9	0.49
$\varphi_{rer} = 0.25$	0.65	0.67	1.14	1.01	1.2	1.12	1.25
$\varphi_{rer} = 0.50$	0.53	0.58	1.2	1.01	1.29	1.17	1.37

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

Table 1.5: **Macroeconomic volatility (No intervention $\equiv 1$), Low elasticity of subs. btw. exports and foreign goods ($\varepsilon_F = 0.4$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.43	0.7	0.3	0.95	0.34	0.7	0.42
$\varphi_{\Delta s} = 0.50$	0.17	0.54	0.04	0.94	0.05	0.26	0.13
$\varphi_{rer} = 0.25$	0.65	0.82	0.53	0.98	0.51	0.59	0.6
$\varphi_{rer} = 0.50$	0.53	0.76	0.38	0.97	0.36	0.75	0.46
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.21	0.19	0.26	0.21	0.31	0.26	0.26
$\varphi_{\Delta s} = 0.50$	0.14	0.12	0.21	0.14	0.28	0.2	0.22
$\varphi_{rer} = 0.25$	0.22	0.22	0.23	0.22	0.24	0.23	0.23
$\varphi_{rer} = 0.50$	0.11	0.11	0.12	0.11	0.13	0.12	0.12
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.9	0.78	1	0.88	1.15	1.02	1.09
$\varphi_{\Delta s} = 0.50$	0.82	0.65	0.97	0.77	1.25	1	1.14
$\varphi_{rer} = 0.25$	0.65	0.64	0.67	0.64	0.7	0.67	0.68
$\varphi_{rer} = 0.50$	0.53	0.52	0.55	0.52	0.59	0.55	0.57
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.32	0.96	0.91	0.99	0.79	0.94	0.69
$\varphi_{\Delta s} = 0.50$	1.51	0.88	0.86	0.99	0.65	0.91	0.49
$\varphi_{rer} = 0.25$	0.65	0.65	1.08	1.02	1.09	1.07	1.13
$\varphi_{rer} = 0.50$	0.53	0.55	1.11	1.03	1.14	1.1	1.2

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

Table 1.6: **Macroeconomic volatility (No intervention $\equiv 1$), High elasticity of subs. btw. exports and foreign goods ($\varepsilon_F = 1.5$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.32	0.68	0.6	0.9	0.46	0.28	0.39
$\varphi_{\Delta s} = 0.50$	0.3	0.53	0.39	0.84	0.21	0.04	0.15
$\varphi_{rer} = 0.25$	0.72	0.97	0.99	0.97	1.19	1	1.19
$\varphi_{rer} = 0.50$	0.61	0.96	0.98	0.96	1.26	0.98	1.26
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.32	0.29	0.39	0.36	0.37	0.4	0.37
$\varphi_{\Delta s} = 0.50$	0.22	0.18	0.31	0.27	0.3	0.32	0.3
$\varphi_{rer} = 0.25$	0.35	0.35	0.39	0.37	0.46	0.39	0.41
$\varphi_{rer} = 0.50$	0.18	0.18	0.21	0.19	0.26	0.21	0.23
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.91	0.79	0.98	0.94	0.99	1.02	1.01
$\varphi_{\Delta s} = 0.50$	0.84	0.67	0.96	0.88	1	1.02	1.02
$\varphi_{rer} = 0.25$	0.69	0.67	0.72	0.7	0.83	0.73	0.8
$\varphi_{rer} = 0.50$	0.57	0.55	0.62	0.59	0.75	0.62	0.71
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.2	0.87	0.89	0.99	0.63	0.91	0.57
$\varphi_{\Delta s} = 0.50$	1.29	0.76	0.85	0.98	0.46	0.88	0.38
$\varphi_{rer} = 0.25$	0.63	0.69	1.25	1.03	1.53	1.25	1.61
$\varphi_{rer} = 0.50$	0.51	0.62	1.35	1.04	1.79	1.36	1.93

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

Table 1.7: **Macroeconomic volatility (No intervention $\equiv 1$), Low domestic good price rigidity ($\theta_H = 0.25$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.35	0.76	0.62	0.95	0.39	1.1	0.37
$\varphi_{\Delta s} = 0.50$	0.14	0.64	0.43	0.92	0.16	1.17	0.15
$\varphi_{rer} = 0.25$	0.84	0.99	0.96	0.98	1.06	1	1.07
$\varphi_{rer} = 0.50$	0.71	0.98	0.93	0.96	1.12	1	1.13
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.48	0.46	0.57	0.56	0.54	0.54	0.53
$\varphi_{\Delta s} = 0.50$	0.31	0.28	0.4	0.39	0.37	0.36	0.36
$\varphi_{rer} = 0.25$	0.65	0.65	0.72	0.7	0.79	0.67	0.79
$\varphi_{rer} = 0.50$	0.55	0.55	0.66	0.63	0.79	0.57	0.79
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	0.8	0.76	0.88	0.87	0.82	0.88	0.81
$\varphi_{\Delta s} = 0.50$	0.69	0.65	0.81	0.8	0.73	0.81	0.72
$\varphi_{rer} = 0.25$	0.83	0.86	0.91	0.91	0.98	0.86	0.99
$\varphi_{rer} = 0.50$	0.7	0.75	0.83	0.82	0.96	0.74	0.96
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	1.03	0.83	0.98	0.98	0.69	1.02	0.59
$\varphi_{\Delta s} = 0.50$	1.04	0.73	0.99	0.97	0.57	1.03	0.43
$\varphi_{rer} = 0.25$	0.85	0.88	1.17	1.07	1.62	1.06	1.68
$\varphi_{rer} = 0.50$	0.74	0.86	1.34	1.13	2.35	1.11	2.55

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

Table 1.8: **Macroeconomic volatility (No intervention $\equiv 1$), High domestic good price rigidity ($\theta_H = 0.95$.)**

Endogenous variable	*RER*	Δ *Ex. Rate*	*Consumption*	*Exports*	*Int. Rate*	*Production*	*Inflation*
			Foreign interest rate shock (ε_{i^*})				
$\varphi_{\Delta s} = 0.25$	0.23	0.55	0.37	0.88	0.44	0.16	0.26
$\varphi_{\Delta s} = 0.50$	0.03	0.4	0.12	0.85	0.14	0	0.04
$\varphi_{rer} = 0.25$	0.55	0.76	0.64	0.95	0.66	0.48	0.56
$\varphi_{rer} = 0.50$	0.41	0.68	0.52	0.93	0.54	0.32	0.42
			Capital flows shock (ε^{ω^*})				
$\varphi_{\Delta s} = 0.25$	0.72	0.56	1.05	0.66	1.44	1.09	0.89
$\varphi_{\Delta s} = 0.50$	0.41	0.27	0.71	0.33	1.23	0.75	0.56
$\varphi_{rer} = 0.25$	0.35	0.34	0.36	0.34	0.38	0.36	0.35
$\varphi_{rer} = 0.50$	0.18	0.18	0.19	0.18	0.2	0.19	0.18
			Foreign inflation shock (ε^{π^*})				
$\varphi_{\Delta s} = 0.25$	1.27	0.89	1.48	0.92	2.03	1.58	1.45
$\varphi_{\Delta s} = 0.50$	1.5	0.85	1.84	0.84	3.17	2.04	1.78
$\varphi_{rer} = 0.25$	0.55	0.53	0.57	0.52	0.61	0.57	0.56
$\varphi_{rer} = 0.50$	0.41	0.39	0.43	0.39	0.47	0.43	0.42
			Foreign output shock (ε^{y^*})				
$\varphi_{\Delta s} = 0.25$	2.55	1.47	0.98	1	0.96	0.98	0.8
$\varphi_{\Delta s} = 0.50$	3.97	1.74	0.97	1	0.93	0.97	0.7
$\varphi_{rer} = 0.25$	0.5	0.52	1.01	1	1.01	1.01	1.06
$\varphi_{rer} = 0.50$	0.36	0.41	1.01	1	1.01	1.01	1.09

Note: The table shows normalised unconditional relative variances of the model assuming the only source of volatility is the shock in the table heading. We have considered changes in variance produced by intervention rules themselves, and how these affect the overall volatility of the economy.

1.B The log-linear version of the model

Aggregate demand

Aggregate demand (y_t)

$$y_t = \phi_C(c_t) + \phi_X(x_t) - \phi_M(m_t) + g_t \tag{1.50}$$

GDP deflator $\left(t_t^{def}\right)$

$$t_t^{def} = \phi_X(rer_t + t_t^X) - \phi_M t_t^M \tag{1.51}$$

Real exchange rate (rer_t)

$$rer_t = rer_{t-1} + \Delta s_t + \pi_t^* - \pi_t \tag{1.52}$$

Euler equation (λ_t)

$$\lambda_t = \hat{\imath}_t + E_t(\lambda_{t+1} - \pi_{t+1}) - \psi_b b_t \tag{1.53}$$

Marginal utility (λ_t)

$$\lambda_t = -\gamma_c c_t \tag{1.54}$$

Exports (x_t)

$$x_t = -\varepsilon_F(t_t^X) + y_t^*; \tag{1.55}$$

Relative price of exports $\left(t_t^X\right)$

$$t_t^X = t_{t-1}^X + \pi_t^X - \pi_t^*; \tag{1.56}$$

Imports (m_t)

$$m_t = -\varepsilon(t_t^M) + c_t; \tag{1.57}$$

Relative price of imports $\left(t_t^M\right)$

$$t_t^M = t_{t-1}^M + \pi_t^M - \pi_t; \tag{1.58}$$

Home produced goods demand $\left(y_t^H\right)$

$$y_t^H = -\varepsilon(t_t^H) + c_t; \tag{1.59}$$

Relative price of home produced goods $\left(t_t^H\right)$

$$t_t^H = -\left(\frac{1-\psi}{\psi}\right) t_t^M \tag{1.60}$$

Aggregate supply

Total CPI (π_t):

$$\pi_t = \psi \pi_t^H + (1-\psi)\, \pi_t^M + \mu_t \tag{1.61}$$

Phillips curve for home-produced goods (π_t^H):

$$\pi_t^H = \kappa_H \left(mc_t - t_t^H\right) + \beta E_t \pi_{t+1}^H \tag{1.62}$$

Real marginal costs (mc_t)

$$mc_t = wp_t - a_t; \tag{1.63}$$

Phillips curve for imported goods (π_t^M):

$$\pi_t^M = \kappa_M mc_t^M + \beta E_t \pi_{t+1}^M \tag{1.64}$$

Marginal costs for imports $\left(mc_t^M\right)$

$$mc_t^M = rer_t - t_t^M \tag{1.65}$$

Phillips curve for exports (π_t^X)

$$\pi_t^X = \kappa_X mc_t^X + \beta E_t \pi_{t+1}^X \tag{1.66}$$

Marginal costs for exports $\left(mc_t^X\right)$

$$mc_t^X = mc_t - rer_t - t_t^X \tag{1.67}$$

Labour market

Labour demand (l_t)

$$l_t = y_t - a_t; \tag{1.68}$$

Labour supply (wp_t)

$$wp_t = \gamma_c c_t + \chi l_t \tag{1.69}$$

FX markets and current account

Risk premium-adjusted UIP (Δs_t)

$$\overline{E}_t \Delta s_{t+1} = \hat{\imath}_t - \hat{\imath}_t^* + \gamma \sigma^2 \left(\varpi_t^* + \varpi_t^{*,cb} \right) \tag{1.70}$$

Current account (b_t)

$$\phi_{\varpi} \left(b_t - \beta^{-1} b_{t-1} \right) = t_t^{def} + y_t - \phi_C c_t + \frac{\phi_{\varpi}}{\beta} \left(i_{t-1} - \pi_t \right) \tag{1.71}$$

Monetary policy

Interest rate ($\hat{\imath}_t$)

$$\hat{\imath}_t = \varphi_\pi (\pi_t) + \varepsilon_t^{int} \tag{1.72}$$

FX intervention $\left(\varpi_t^{*,cb} \right)$

$$\varpi_t^{*,cb} = \varphi_{\Delta s} \Delta s_t + \varphi_{rer} rer_t + \varepsilon_t^{cb} \tag{1.73}$$

Foreign economy

Foreign output (y_t^*):

$$y_t^* = \rho_{y^*} y_{t-1}^* + \varepsilon_t^{y^*} \tag{1.74}$$

Foreign inflation (π_t^*):

$$\pi_t^* = \rho_{\pi^*} \pi_{t-1}^* + \varepsilon_t^{\pi^*} \tag{1.75}$$

Foreign interest rates (i_t^*):

$$i_t^* = \rho_{i^*} i_{t-1}^* + \varepsilon_t^{i^*} \tag{1.76}$$

Capital inflows-order flows (ϖ_t^*)

$$\varpi_t^* = \rho_{\varpi^*} \varpi_{t-1}^* + \varepsilon_t^{\varpi^*} \tag{1.77}$$

Domestic shocks

Productivity shocks (a_t):

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a \tag{1.78}$$

Demand shocks (g_t):

$$g_t = \rho_g g_{t-1} + \varepsilon_t^g \tag{1.79}$$

Mark-up shocks (μ_t):

$$\mu_t = \rho_\mu \mu_{t-1} + \varepsilon_t^\mu \tag{1.80}$$

Thus we have in total 31 equations, 24 from the original model and seven auxiliary equations. We have included two exogenous shock processes - demand (g_t) and mark-up/inflation (μ_t) shocks - to perform additional analysis. The variables in the model are: a_t, y_t, c_t, x_t, m_t, y_t^*, y_t^H, l_t, λ_t, ϖ_t^*, mc_t, mc_t^X, mc_t^M, t_t^{def}, t_t^X, t_t^M, t_t^H, π_t, π_t^H, π_t^X, π_t^M, π_t^*, rer_t, Δs_t, i_t, i_t^*, wp_t, b_t, $\varpi_t^{*,cb}$, g_t, μ_t. The minimum state variable (MSV) set is composed of 12 variables: a_t, y_t^*, b_t, ϖ_t^*, t_t^X, t_t^M, π_t^*, rer_t, i_t, i_t^*, g_t, μ_t.

The nine shocks comprise four foreign economy shocks $(\varepsilon_t^{y^*}, \varepsilon_t^{\pi^*}, \varepsilon_t^{i^*}, \varepsilon_t^{\varpi^*})$, three domestic economy shocks $(\varepsilon_t^a, \varepsilon_t^g, \varepsilon_t^\mu)$ and two policy shocks $\left(\varepsilon_t^{int}, \varepsilon_t^{cb}\right)$

Chapter 2

Information heterogeneity and the role of foreign exchange interventions

2.1 Introduction

The microstructure approach to exchange rates literature has given economists key insights to understand the behaviour of exchange rates. The limited explanatory power that observed macro fundamentals have on the exchange rate, coined 'the exchange rate determination puzzle', has found answers in the microstructure literature. As Evans and Lyons (2002) found, exchange rate movements can be explained largely by order flows. As we have seen in the previous chapter, orders received by dealers and the exchange rate have a direct connection through *portfolio balance* effects. Nonetheless, this channel is not the sole one through which order flows affect exchange rate prices. As Lyons (2006) explains, order flows convey private information. The author identifies at least three different channels at work. The first one, is related to information about transitory risk premia, as dealers possess better information about their own inventories and the inventories of other dealers. Using this information dealers possess an informative advantage over the general public. The second channel is related to the aggregate position of dealers, reflected in the portfolio balance, which in the eyes of the dealers, are changes in the aggregate position which are undiversifiable across themselves. As Vitale (2011) explains, heterogeneous information might emerge in relation to this channel as certain dealers can have superior information regarding the aggregate position of the market, such as the case when Central Banks intervene through a subset of dealers. The third one is related to asset payoffs. Dealers could have as well private information regarding future interest rate differentials or, perhaps closer to reality, dealers could either interpret this information in a different way or have access to different information regarding other dealers' expectations about the future differentials.

As Bacchetta and van Wincoop (2011) explain, typically in macroeconomic models foreign exchange (FX) market participants are assumed to: i) have identical information; ii)perfectly know the model; iii) use the available information at all times. Assumptions that are quite inconsistent with how FX markets operate in reality. The authors show that relaxing these assumptions allows explaining various exchange rate puzzles, such as the disconnection between exchange rates and fundamentals and the forward premium puzzle. In this line, we extend the model introduced in Chapter 1 by relaxing the first assumption, acknowledging that FX dealers can have access to different sources of information and can have different expectations about future macroeconomic variables. As shown by Bacchetta and Wincoop (2006) in a more tractable model, these characteristics magnify the response of the exchange rate to unobserved variables and generate a disconnection in the short run between the exchange rate and observed fundamentals. In a related work, Vitale (2011) extends Bacchetta and Wincoop (2006) model to analyse the impact of FX intervention on FX markets. The resulting model is useful to analyse how FX intervention influences exchange rates via both a portfolio-balance and an information related channel.

Thus, the goal of the chapter is twofold. The first one is to introduce information heterogeneity into a Neo-Keynesian general equilibrium model and verify the role it plays in the determination of exchange rates and the *disconnection puzzle*. The second objective is to understand the role of FX interventions in this setup. Different from previous research, we treat information heterogeneity in a model where the interest rate is endogenous and reacts to the exchange rate through the effects the latter has on inflation. In this way, there is an explicit channel through which the FX market microstructure, FX interventions and monetary policy interact; a channel we consider worthwhile studying in more detail.[1]

On the technical side, the presence of heterogeneous information poses a challenge in terms of the solution method. Now, the variance of exchange rate changes will not only affect the risk premium charged by dealers for holding foreign currency assets in their portfolio, but will be a key element in the information extraction exercise that dealers perform. For this reason we follow an approach in line with Townsend (1983) and Bacchetta and Wincoop (2006). We solve a signal extraction problem of the investors to calculate the average expected depreciation rate in the modified uncovered interest parity (UIP) condition with an endogenous risk premium, which feeds from the rational expectations solution of the model.

We are not the first ones to treat the *exchange rate disconnection puzzle* from a general equilibrium perspective. Wang (2007) studies the role that the home-bias effect in consumption has in the ratio of volatilities between the exchange rate and the macroeconomic variables, though the results are driven by ad-hoc UIP shocks. Evans and Lyons (2007) work a two-country general equilibrium model with initially not publicly observed information that is assimilated by the exchange rate at a slow pace. In this model dealers form heterogeneous expectations about central bank reactions to changes in the economy and revise their expectations using the information contained in the order flow.

The literature presents a few examples of DSGE models with heterogeneous information. We find our setup close to the one of Lorenzoni (2009), who presents a model with a symmetric information structure in which agents confuse noise shocks with fundamental ones, as the latter are not perfectly observable. Carboni and Ellison (2011) present a New Keynesian DSGE model with asymmetric and incomplete information where the central bank and private sector better information about a different sets of shocks. The authors show the importance of transparency as a stabilisation mechanism since not only the private sector benefits from better information, but in addition, the central bank is able to infer information from the private sector more easily. Finally, this

[1] In countries with a dollarized financial system and agents with dollarized liabilities an additional channel is present. In this case exchange rate fluctuations generate balance sheet effects on households and firms, with consequences for interest rates in the banking sector. For a discussion of this channel, see Céspedes et al. (2004).

chapter is connected to some other strands of the literature such as the models of noisy rational expectations (see Brunnermeier (2001) for a survey) and imperfect information (see Woodford (2003); Mankiw & Reis(2002), Sims (2003)).

In the next section, the model in Chapter 1 is extended to take into account information heterogeneity in the dealers' market. Section 2.3 discusses the solution method. In Section 2.4 we present the results of the model. The last section concludes.

2.2 The model

The model describes a small open economy with nominal rigidities, in line with the contributions from Obstfeld and Rogoff (1995), Chari et al. (2002), Gal and Monacelli (2005), Christiano et al. (2005) and Devereux et al. (2006), with the key difference that the exchange rate is determined in a market of risk adverse dealers.[2] Different from the model seen in the previous chapter, now dealers in the FX market will receive heterogeneous information, as in Bacchetta and Wincoop (2006), raising an information extraction problem that will affect exchange rate dynamics.

2.2.1 Dealers

As in the baseline model, there is a continuum of dealers in the interval $[0, 1]$ operating in the domestic economy. Each dealer ι receives ϖ_t^{ι} and $\varpi_t^{\iota,cb}$ in domestic bond sale and purchase orders from households and the central bank, and $\varpi_t^{\iota*}$ and $\varpi_t^{\iota*,cb}$ in foreign bond sale orders from foreign investors and the central bank, respectively. These orders are exchanged among dealers, that is $\varpi_t^{\iota} + \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right) = B_t^{\iota} + S_t B_t^{\iota*}$, where B_t^{ι} and $B_t^{\iota*}$ are the ex-post holdings of domestic and foreign bonds by dealer ι, respectively. The exchange rate S_t is defined as the price of foreign currency in terms of domestic currency, such that a decrease (increase) of S_t corresponds to an appreciation (depreciation) of the domestic currency. At the end of the period, any profits -either positive or negative- are transferred to the households.[3]

[2]We refer the reader to Chapter 1 for more details on the model.

[3]Under the present formulation FX transactions carried out for commercial purposes will only affect the exchange rate through their impact in the domestic interest rate though not through variations in the order flow faced by dealers.

Dealers are risk-averse and short-sighted. They select an optimal portfolio allocation in order to maximise the expected utility of their end-of-period returns, where their utility is given by a CARA utility function. The one-period dealer's horizon gives tractability and captures the feature that FX dealers tend to unwind their FX exposure at the end of any trading period, as explained by Vitale (2011). The problem of dealer ι is

$$\max_{B_t^{\iota*}} -E_t^\iota e^{-\gamma \Omega_{t+1}^\iota}$$

subject to:

$$\varpi_t^\iota - \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right) = B_t^\iota + S_t B_t^{\iota*} \tag{2.1}$$

where γ is the coefficient of absolute risk aversion and E_t^ι is the rational expectations operator conditional on the information that dealer ι possesses at time t, $\mathcal{I}_t^\iota$:

$$E_t^\iota[\cdot] \equiv E[\cdot \mid I_t^\iota]$$

Ω_{t+1}^ι represents total investment after returns, given by:

$$\begin{aligned}\Omega_{t+1}^\iota &= (1+i_t)\, B_t^\iota + (1+i_t^*)\, S_{t+1} B_t^{\iota*} \\ &\approx (1+i_t)\left[\varpi_t^\iota - \varpi_t^{\iota,cb} + S_t\left(\varpi_t^{\iota*} + \varpi_t^{\iota*,cb}\right)\right] + \left(i_t^* - i_t + s_{t+1} - s_t\right) B_t^{\iota*}\end{aligned}$$

where we have made use of the resource constraint of dealers. We have log-linearised the excess of return on investing in foreign bonds and $s_t = \ln S_t$. Since the only non-predetermined variable is s_{t+1}, assuming it is normal distributed with time-invariant variance, the first order condition for the dealers is:

$$0 = -\gamma\left(i_t^* - i_t + E_t^\iota s_{t+1} - s_t\right) + \gamma^2 B_t^{\iota*} \sigma^2$$

where $\sigma^2 = var_t\left(\Delta s_{t+1}\right)$ is the conditional variance of the depreciation rate. Then, the demand for foreign bonds by dealer ι is given by the following portfolio condition:

$$B_t^{\iota*} = \frac{i_t^* - i_t + E_t^\iota s_{t+1} - s_t}{\gamma \sigma^2} \tag{2.2}$$

2.2.2 FX market equilibrium

Foreign bonds equilibrium in the domestic market should sum FX market orders from foreign investors (capital inflows) and central bank FX intervention, that is:

$$\int_0^1 B_t^{\iota *} d\iota = \int_0^1 \left(\varpi_t^{\iota *} + \varpi_t^{\iota *, cb} \right) d\iota = \varpi_t^* + \varpi_t^{*,cb}.$$

Replacing the FX market equilibrium condition in the aggregate demand for foreign bonds yields the following arbitrage condition:

$$\bar{E}_t s_{t+1} - s_t = i_t - i_t^* + \gamma \sigma^2 (\varpi_t^* + \varpi_t^{*,cb}) \tag{2.3}$$

where $\bar{E}_t s_{t+1}$ is the average rational expectation of the next period exchange rate across all dealers. Given that dealers have access to different sets of information, expected exchange rate depreciation would differ among them as well. Condition (2.3) determines the exchange rate, and adds three new elements to the traditional uncovered interest rate parity condition.[4] On the right-hand side, we note the presence of central bank market orders, reflecting the *portfolio balance effect* of FX interventions. The second novel element, also studied in the previous chapter, is the presence of the exchange rate volatility, which scales the impact of interventions and portfolio capital flows shocks in the exchange rate. We call this the *volatility channel.*

Finally, on the left-hand side we find the average rational expectation of the next period exchange rate, reflecting the presence of heterogeneous information. In our model, dealers will form both conditional moments present in condition (2.3) through a signal extraction problem. As we discuss, the way in which the central bank intervenes could affect both in the manner in which information is processed and in the information available to agents.

[4] See Obstfeld and Rogoff (1995) for an example of the standard UIP condition.

2.2.3 Information structure

Two sources of information heterogeneity among dealers are considered: first, we assume dealers face idiosyncratic shocks in the amount of customer orders from foreign investors and, second, they will also receive noisy signals about some future shocks. The later assumption seems reasonable, since regularly dealers form their own forecasts from different models or own experiences, generating heterogeneity in spite of having access to the same data.

In particular, we assume the foreign investor exposure for each dealer is equal to the average plus an idiosyncratic term:

$$\varpi_t^{\iota *} = \varpi_t^* + \varepsilon_t^\iota \tag{2.4}$$

where ε_t^ι has an infinite support, so that knowing one's own foreign investor exposure provides no information about the average exposure as in Bacchetta and Wincoop (2006). ϖ_t^* is unobservable and follows an AR(1) process:

$$\varpi_t^* = \rho_{\varpi^*} \varpi_{t-1}^* + \varepsilon_t^{\varpi^*} \tag{2.5}$$

where $\varepsilon_t^{\varpi^*} \sim N\left(0, \sigma_{\varpi^*}^2\right)$. We consider the case in which this autoregressive process is known by all agents.

We assume that dealers observe past and current fundamental shocks, while they also receive private signals about some future shocks. More precisely, we assume dealers receive one signal each period about the foreign interest rate one period ahead.[5] That is, at time t dealer ι receives a signal

$$v_t^\iota = i_{t+1}^* + \varepsilon_t^{v\iota},\ \varepsilon_t^{v\iota} \sim N\left(0, \sigma_v^2\right) \tag{2.6}$$

where $\varepsilon_t^{v\iota}$ is independent from i_{t+1}^* and other agents' signals. This idiosyncratic signal can be reconciliated with the fact that dealers have different models to forecast future fundamentals, so each can imperfectly observe future variables with an idiosyncratic

[5] This assumption can be extended to the case where dealers receive each period a vector of signal of a set of fundamental variables.

noise. We also assume that the average signal received by investors is i^*_{t+1}, that is $\int_0^1 v^\iota_t d\iota = i^*_{t+1}$. The foreign interest rate follows an AR(1) process known by dealers:

$$i^*_t = \rho_{i^*} i^*_{t-1} + \varepsilon^{i^*}_t \tag{2.7}$$

where $\varepsilon^{i^*}_t \sim N\left(0, \sigma^2_{i^*}\right)$. Dealers solve a signal extraction problem for the unknown innovations $\left(\varepsilon^{\varpi^*}_t, \varepsilon^{i^*}_{t+1}\right)$, given the observed depreciation rate and signal $(\Delta s_t, v^\iota_t)$.[6]

As Bacchetta and Wincoop (2006), we consider a *common knowledge* (CK) benchmark. In this case, the signal about future interest rates becomes public, but remains noisy. Agents only extract information from this signal, since the equilibrium exchange rate stops being informative. For a detailed description see Section 2.B.3 in the appendix.

FX intervention

We describe two different FX intervention strategies for the central bank, aside of the no intervention scenario. First, the central bank can perform a rule based intervention taking into account the changes in the exchange rate. We call this strategy "the Δs rule".

$$\varpi^{*cb}_t = \phi_{\Delta s} \Delta s_t + \varepsilon^{cb,1}_t \tag{2.8}$$

According to this rule, when there are depreciation (appreciation) pressures on the domestic currency, the central bank sells (purchases) foreign bonds to prevent the exchange rate from fluctuating. $\phi_{\Delta s}$ captures the intensity of the response of the FX intervention to pressures in the FX market. Second, the monetary authority can take into account misalignments of the real exchange rate as a benchmark for FX intervention. We call this strategy "the RER rule".

$$\varpi^{*cb}_t = \phi_{rer} rer_t + \varepsilon^{cb,2}_t \tag{2.9}$$

The rest of the model describes the behaviour of households, firms, the external sector and a monetary policy authority, which participates actively in the FX market through

[6] As explained by Bacchetta and Wincoop (2006), if the $\mathcal{B}(L)$ polynomial in equation (2.15) is invertible, knowledge of the depreciation rate at times $t-1$ and earlier and of the interest rate shocks at time t and earlier, reveals the shocks ε^{ϖ^*} at times $t-1$ and earlier. That is, $\varepsilon^{\varpi^*}_{t-s}$ becomes observable at time t for $s \geq 1$.

discretionary or rule-based interventions. We refer the reader to Chapter 1, for a complete description of the model and a thorough explanation of the differences among these three FX intervention strategies.

2.3 Computational strategy

The computational strategy consists of dividing the system of log-linearized equations into two blocks. In the first block we take into account all the equations but the risk-premium adjusted UIP, which is included in the second block. Then, we solve for the rational expectations equilibrium of the first block taking the depreciation rate as an exogenous variable. This solution feeds into the second block to solve for the policy function of the depreciation rate. Note that with this computational strategy we are also eliminating any informational spillovers between dealers and other economic agents, such as households and firms. However, the segmented information in the FX market seems a reasonable assumption, since it takes into account that dealers have access to private information, which is not known by other economic agents.[7]

Accordingly, in the first block the depreciation rate only appears in the real exchange rate equation:

$$rer_t = rer_{t-1} + \Delta s_t + \pi_t^* - \pi_t \tag{2.10}$$

This system of equations can be written as:

$$A_0 \begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix} = A_1 \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + A_2 \Delta s_t + B_0 \epsilon_t \tag{2.11}$$

where $X_t = [rer_t, i_t, \pi_t^*, w_t^*, i_t^*, ...]'$ is a size n_1 vector of backward looking variables, $Y_t = [\pi_t, ...]'$ is a size n_2 vector of forward looking variables, such as $n_T = n_1 + n_2 + 1$ is the number of endogenous variables. ϵ_t is the vector of observable shocks in the model. $A_2 = [1, 0..0]'$ is a $(n_1 + n_2) \times 1$ matrix.[8]

[7] To further clarify this, agents in the economy are reacting as if the information heterogeneity does not affect the optimal response to shocks, taking, as a matter of fact, movements in the exchange rate as shocks. Relaxing this assumption will impose a recursive relationship between the first and second block of the solutions, complicating the numerical solution. We leave the implementation of an explicit recursivity between both blocks for future research.

[8] Since information heterogeneity only enters the model through the exchange rate, the unobservable shocks are excluded from the first step.

The second block corresponds to the risk-premium adjusted UIP condition:

$$\overline{E}_t \Delta s_{t+1} = i_t - i_t^* + \gamma\sigma^2 \left(\varpi_t^* + \varpi_t^{*,cb}\right) \tag{2.12}$$

In the first stage we find the rational expectations solution of the system in (2.11) using the perturbation method, taking as exogenous Δs_t.[9] That is, we find the policy functions:

$$Y_t = M_1 X_{t-1} \tag{2.13}$$

$$X_t = M_2 X_{t-1} + M_3 \Delta s_t + M_4 \epsilon_t \tag{2.14}$$

In the second stage we use the previous solution to find the policy function of Δs_t using Townsend (1983) method. More precisely, we conjecture a solution for Δs_t as a function of infinite lag polynomials of the shocks in the model.

$$\Delta s_t = \mathcal{A}(L)\varepsilon_{t+1}^{i^*} + \mathcal{B}(L)\varepsilon_t^{\varpi^*} + \mathcal{D}(L)\zeta_t \tag{2.15}$$

where $\varepsilon_{t+1}^{i^*}$ is an innovation to the future foreign interest rate (i_{t+1}^*), the fundamentals over which agents receive a signal, and $\varepsilon_t^{\varpi^*}$ is the shock to the unobservable capital flow (ϖ_t^*), which can be inferred with a lag. $\mathcal{A}(L)$ and $\mathcal{B}(L)$ are infinite lag polynomials, while $\mathcal{D}(L)$ is a vector of infinite lag polynomials operating ζ_t, the vector of remaining shocks. [10]

In the second stage we solve for the signal extraction problem of the dealers for the unobserved innovations $\left(\varepsilon_t^{\varpi^*}, \varepsilon_{t+1}^{i^*}\right)$, using both the depreciation rate and their private signal $\left(\Delta s_t, v_t^d\right)$, which serves to calculate the average expectation of the future depreciation rate and its conditional variance in equation (2.12) as functions of shocks.[11]

The next step involves relating the coefficients of (2.12) to those on the conjectured

[9] We use Dynare to solve for the rational expectations of the first block. More information see: Villemot (2011) and Adjemian et al. (2011).

[10] Notice that ϵ_t and ζ_t are not exactly the same, since the latter can also include FX intervention shocks.

[11] In turn, given the solution of the first block, we can express the endogenous variables in (2.12) as function of shocks as well.

solution (2.15). This yields a system of non-linear equations on the unknown coefficients of $\mathcal{A}(L)$, $\mathcal{B}(L)$ and $\mathcal{D}(L)$. Although this is an infinite-order set of equations, we can exploit the recursive pattern present among the coefficients. We are able to solve the system through a numerical approach that limits the number of lags affecting the solution, effectively imposing zeros after a certain lag. This lag is determined numerically, through an iterative process. See appendix B for details on the computational strategy.

2.4 Model Dynamics

Our interest lies in understanding first and foremost, how information heterogeneity affects the connection between the exchange rates and the "traditional" fundamentals.[12] These are the variables that affect the exchange rate determination in traditional monetary models (i.a., interest rate differentials). We follow Bacchetta and Wincoop (2006) by solving the model for different values for the parameters that govern the inference problem that dealers face.

2.4.1 Calibration

With respect to the baseline model studied in the previous chapter, this extension presents an additional parameter which affects the precision of the private signal (σ_ν). This value is set at 0.08 for the baseline calibration, the same standard deviation assumed by Bacchetta and Wincoop (2006). There are two key parameters for the signal extraction problem: the standard deviation of noise in the signal (σ_ν), the standard deviation of the capital flows shock (σ_{ω^*}). We study the properties of the simulated series under different values for these parameters. For a discussion on the calibrated values for the rest of parameters in the model see Section 1.3.1 in the previous chapter.

[12] Bacchetta and Wincoop (2006) treat portfolio flow shocks as "non-fundamental" variables. Vitale (2011) considers that, given the importance of order flows for the determination of exchange rates, these should be considered fundamentals too.

Table 2.1: **Baseline Calibration**

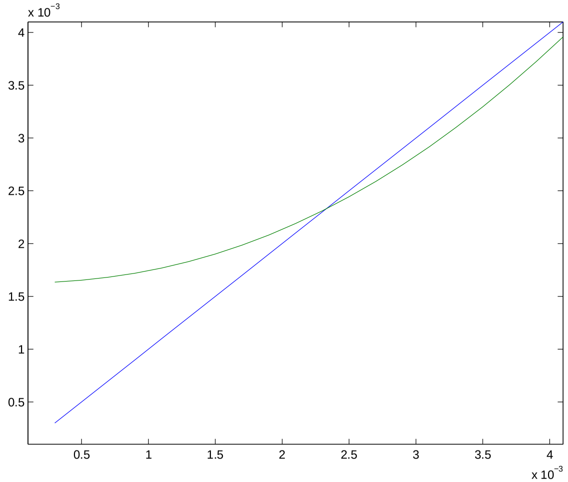
Parameter	*Value*	*Description*
β	0.9975	Consumers time-preference parameter.
χ	0.5	Labour supply elasticity.
γ_c	1	Risk aversion parameter.
ε	0.75	Elasticity of substitution btw. home and foreign goods.
ε_X	0.75	Elasticity of substitution btw. exports and foreign goods.
ψ	0.6	Share of domestic tradables in domestic consumption.
θ_H	0.75	Domestic goods price rigidity.
θ_M	0.5	Imported goods price rigidity.
θ_X	0.5	Exported goods price rigidity.
ψ_b	0.01	Portfolio adjustment costs.
φ_π	1.5	Taylor rule reaction to inflation deviations.
γ	500	Absolute risk aversion parameter (dealers)
ϕ_ϖ	0.5	Net asset position over GDP ratio
ϕ_C	0.68	Consumption over GDP ratio
σ_x	0.01	S.D. of all shocks x
ρ_x	0.5	AR(1) coefficient for all exogenous processes
σ_ν	0.08	S.D. of noise in signal.

2.4.2 Variance fixed-point problem

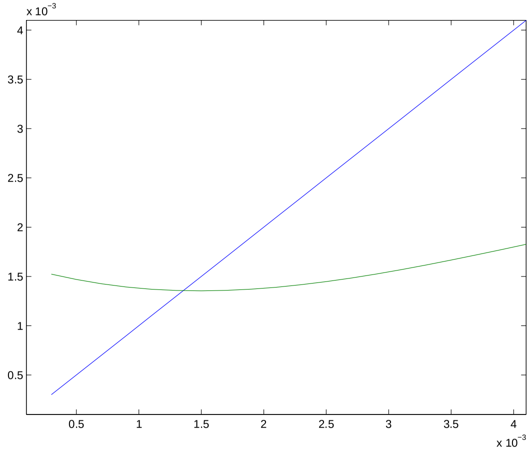
As in the previous chapter, the risk premium-adjusted uncovered interest parity condition (equation 2.3) is a function of the exchange rate variation. In turn, this variable depends on the RE equilibrium of the model. Different from the full information case studied in the previous chapter, the solution now involves a search for the undetermined coefficients in the lag polynomials $\mathcal{A}(L)$, $\mathcal{B}(L)$ and $\mathcal{D}(L)$, defined in (2.15). We conjecture a variance and solve for the unknown coefficients. In Figure 2-1 we plot the mappings of the conjectured and the implied conditional variance of the depreciation rate for different parameterisations of the FX-intervention reaction function. Intersections with the 45-degree straight line correspond to fixed points for the conditional variance of the depreciation rate. The results found under full information carry over to the heterogeneous information case, as under both rules of FX intervention there is only a unique and stable equilibrium. Also, the intensity of FX intervention reduces the RE equilibrium variance of the exchange rate change. The value of the variances is similar as well, although this model presents an extra shock given by the noise in the public signal.

Figure 2-1: Existence of equilibria under FX intervention rules (HI)

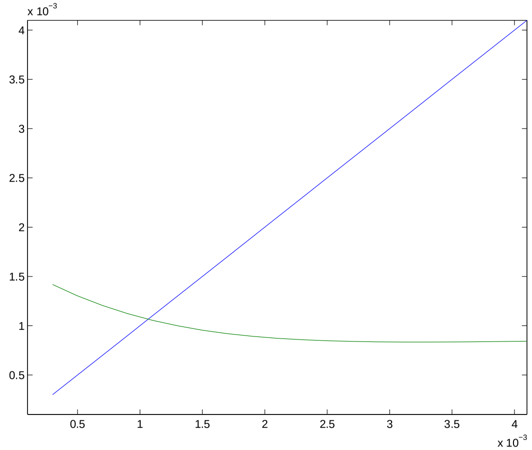
(a) Δs rule ($\varphi_{\Delta s} = 0$)



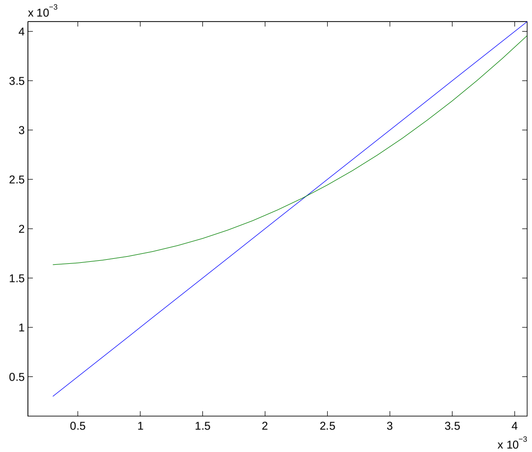
(b) Δs rule ($\varphi_{\Delta s} = 0.25$)



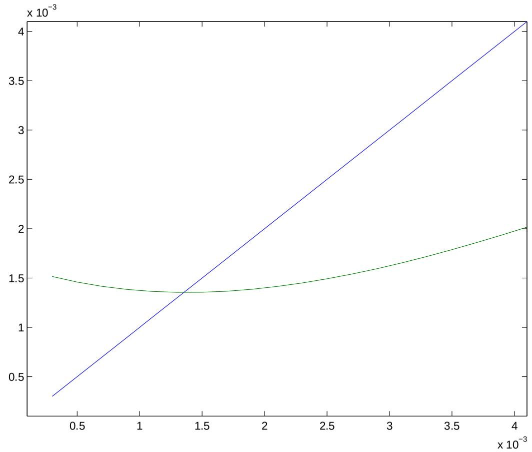
(c) Δs rule ($\varphi_{\Delta s} = 0.50$)



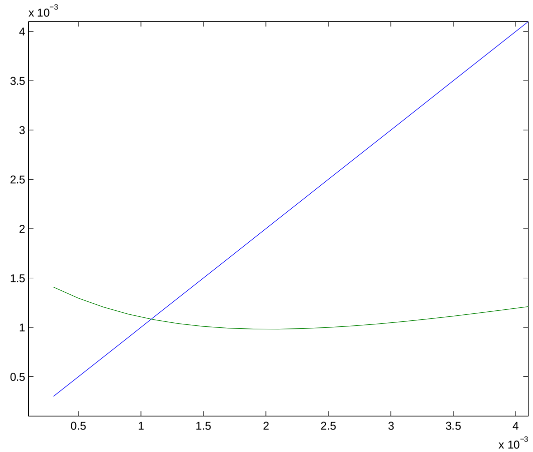
(d) RER rule ($\varphi_{rer} = 0$)



(e) RER rule ($\varphi_{rer} = 0.15$)



(f) RER rule ($\varphi_{rer} = 0.30$)



Note: Simulations involved 61 values for the conjectured variances of the change of the exchange rate. When the intervention parameter under both rules is zero, we replicate the values for the pure discretional intervention case.

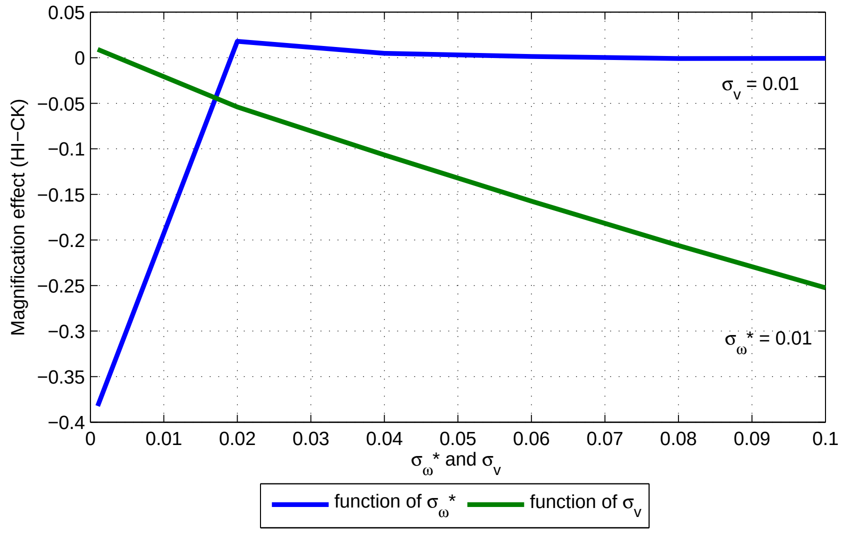
2.4.3 The effects of heterogeneous information

Bacchetta and Wincoop (2006) proved that by adding heterogeneous information in an exchange rate determination model it is possible to account for the short-run disconnection between the exchange rate and observed fundamentals. Instead, the exchange rate becomes closely associated to order flow, which the author associates to the private information component of total market orders. The mechanism at work is a magnification effect of unobserved fundamentals, such as portfolio capital flows in our model, on the exchange rate. Under heterogeneous information, there is rational confusion since when the exchange rate changes dealers do not know whether this is driven by unobserved fundamentals (e.g.: portfolio capital flows) or by information about future macroeconomic fundamentals held by other dealers (e.g.: foreign interest rates).

The rational confusion magnifies the impact of the unobserved capital flows on the exchange rate, an effect Bacchetta and Wincoop (2006) called *the magnification effect.* As we have explained, agents will have now two different signals. The first is the private information about the fundamental. The second is the equilibrium exchange rate - more precisely the unknown component of this rate. As unobservable fundamental capital flows impact the exchange rate, agents will confuse them with changes in observable fundamentals and will react to them, amplifying the effect of capital flows. This magnification effect depends on the precision of the public signal (the exchange rate) relative to the precision of the private signal (v_t^d). Figure 2-2 shows the difference in the contemporaneous response to capital flow shocks between the HI and CK cases. The magnification effect increases with σ_v and decreases with σ_{ω^*}. This is in line with our previous observations. As the private signal becomes noisier, dealers will rely more on the equilibrium exchange rate as a source of information. Thus, liquidity based capital inflows and outflows effects in the exchange rate will be amplified. By contrast, an increase in σ_{ω^*} will reduce the magnification effect. In this case, the exchange rate loses power as a signal, since its dynamics will be more affected by capital flows instead of traditional macro fundamentals.

To show the role of heterogeneous information and the magnification effect in explaining the disconnection effect in our setup we perform simulations of the model and the calculate the R2 of the regression between the exchange rate and the observable vari-

Figure 2-2: Magnification effect for different values of σ_{ω^*} and σ_ν



Note: Figure shows the difference in the contemporaneous response of the variation in the exchange rate s to a one standard deviation capital flows shock ($\varepsilon_t^{\omega^*}$) under heterogeneous information (HI) and common knowledge (CK), for different values of σ_{ω^*} and σ_ν.

ables in the model and contrast it with the imperfect common knowledge case. Under imperfect common knowledge agents will fully observe the aggregate capital flows and follow the same signal, however, this signal will be a noisy one. Hence, their forecast error will have an effect on the equilibrium exchange rate. [13]

2.4.4 FX intervention under heterogeneous information

FX intervention can affect the magnification effect and the connection of the exchange rate with observed fundamentals. We show in figure 2-3 the response on impact of the exchange rate change to future foreign interest rate shocks (i^*_{t+1}) and unobserved capital flows shocks (ϖ^*_t) , that is coefficients a_1 and b_1 respectively. We show in the first column the responses in a model with common knowledge, defined as one in which all dealers have access to the same information, and in the second column the responses in a model with heterogeneous information.[14] In the last column we present the differences between the responses in heterogeneous information and the common knowledge models. These responses are plotted for different values of the standard deviation of unobserved

[13]For a detailed explanation of how the model works under imperfect common knowledge see section 2.B.3 in the appendix.

[14]Therefore, in a common knowledge model capital flows become an observable variable and all dealers observe signal shock (ε_t^υ)

capital flow shocks $\left(\sigma_{\varpi_t^*}\right)$, for three degrees of FX intervention intensity under the Δs rule (no intervention, $\phi_{\Delta s} = 0.25$, and $\phi_{\Delta s} = 0.25$).

The following things are important to notice: i) In both the imperfect common knowledge and heterogeneous information cases, FX intervention dampens the impact of both unobserved capital flow shocks and future foreign interest rate shocks. ii) the standard deviation of unobserved capital flow shocks $\left(\sigma_{\varpi_t^*}\right)$ affects the responses under heterogeneous information, but not under common knowledge. This is because the response of the exchange rate depends on the precision of the signals only in the former model. iii) There is evidence of a magnification effect. That is, the response to unobservable capital flow shocks is much stronger in the heterogeneous information than in common knowledge model. The opposite is true for the response to future foreign interest rate shocks. iv) The magnification effect is larger when the intensity of FX intervention is stronger. The main mechanism for this result is that, when FX intervention reduces the exchange rate volatility it also increases the precision of the public signal, which amplify the magnification effect[15] [16]

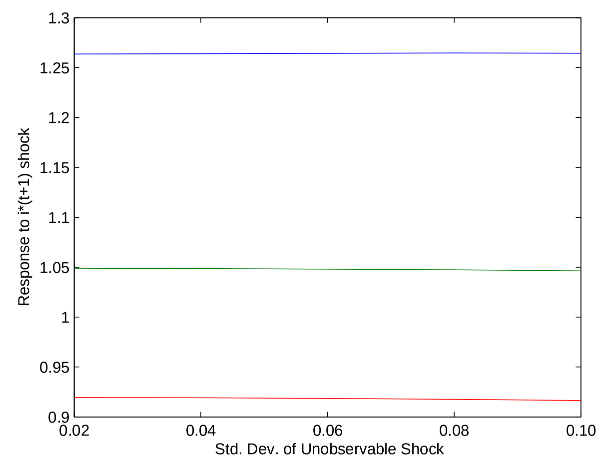
These results shed light of an additional effect that intervention can have in the FX market, that is the magnified response of the exchange rate to unobservable shocks, such as capital flows. However, the magnification effect is not strong enough to increase the disconnection between the exchange rate and observed fundamentals. Figure 2-4 reports the R^2 of regressions of Δs_t on unobserved capital flows shocks (ϖ_t^*) and future interest rates (i_{t+1}^*). As shown, FX intervention reduces the contribution of unobserved capital flow shocks to exchange rate changes, and as a counterpart increases the connection between observed fundamentals and the exchange rate.

[15] On the other hand, as shown in figure 2-2f, the magnification effect is larger when the unobservable capital flows are more volatile, because that increases the exchange rate volatility.

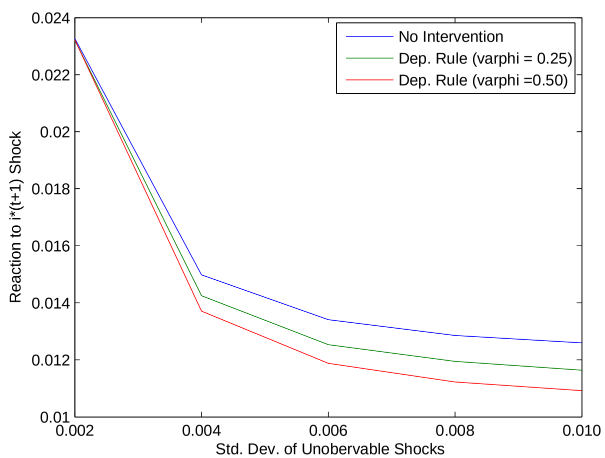
[16] However, this result could change if intervention can bring additional information about future fundamentals to the FX market, as analysed by Vitale (2011).

Figure 2-3: Reaction to unobservable and fundamental shocks under Heterogeneous Information and Common Knowledge

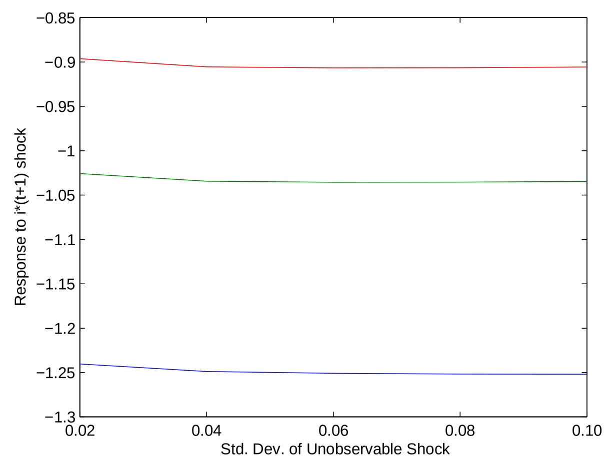
(a) Reaction to a i^*_{t+1} - CK



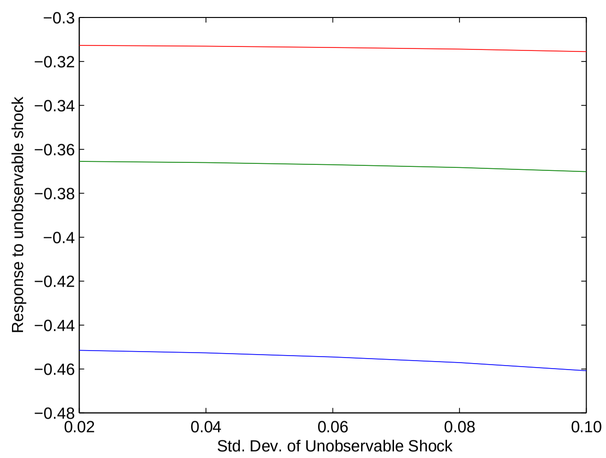
(b) Reaction to a i^*_{t+1} - HI



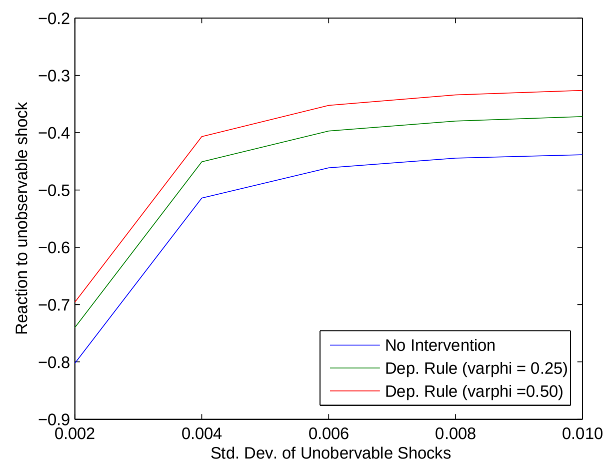
(c) Difference (HI - CK)



(d) Reaction to a ω^*_t - CK



(e) Reaction to a ω^*_t - HI



(f) Magnification Effect (HI - CK)

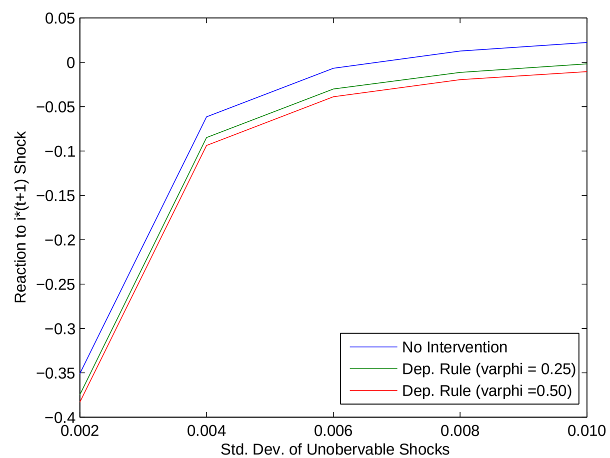
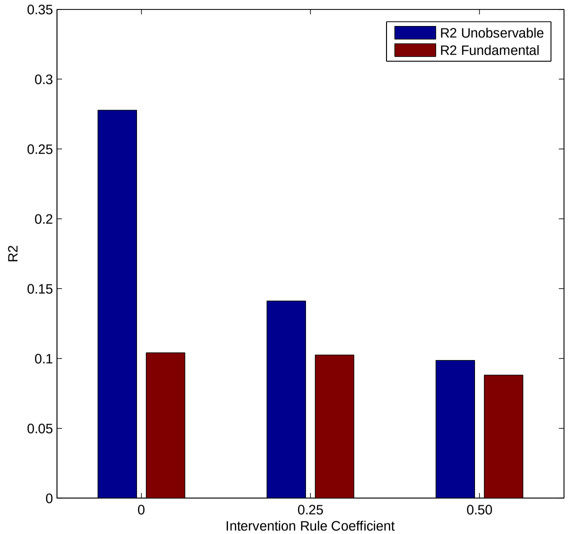
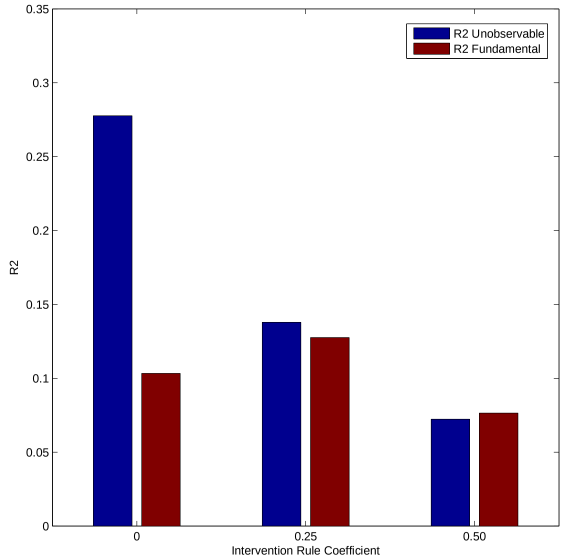


Figure 2-4: Regression of Δs on unobservable and fundamental shocks - Δs rule

(a) Reaction to a i^*_{t+1} - CK



(b) Reaction to a i^*_{t+1} - HI



Note: Figure shows the R2 statistic of the regressions of Δs_t on $\varepsilon^{\omega^*}_t$ and on $\varepsilon^{i^*}_{t+1}$. Average of 20 simulations reported. Each simulation involve a sample of 1000 observations for the model. Regression includes an intercept.

2.5 Conclusions

In this chapter we extend the model of Chapter 1 by introducing heterogeneous information across dealers in the FX markets in line with Bacchetta and Wincoop (2006). We confirm that the *magnification effect*, which amplifies the contemporaneous impact of capital flow shocks, is still present in our framework. This effect is generated by the rational confusion emerging as dealers are unable to identify the source of shocks. The presence of the endogenous response of interest rates to changes in the exchange rate generates a channel between monetary policy and the information extraction problem of agents. Moreover, this framework allows us to study the interaction between exchange rate interventions by the central bank and the magnification effect observed under heterogeneous information. We find that FX interventions can reduce the contribution of unobserved capital flows shocks to the exchange rate, also increasing its connection with observed fundamentals. Despite these findings, the relationship between the degree of FX intervention and the connection to fundamentals is not monotonic. Finally, on the technical side, we propose an extension of Townsend (1983) that can be useful to solve DSGE models with heterogeneous information.

Further research should introduce richer dynamics in the information setup, such as central banks operating in a hidden way as in Vitale (2011), increasing the information dispersion through FX interventions, or central banks that reveal public signals through interventions. We consider that the setup presented here is capable of handling these problems. We leave these extensions for future research.

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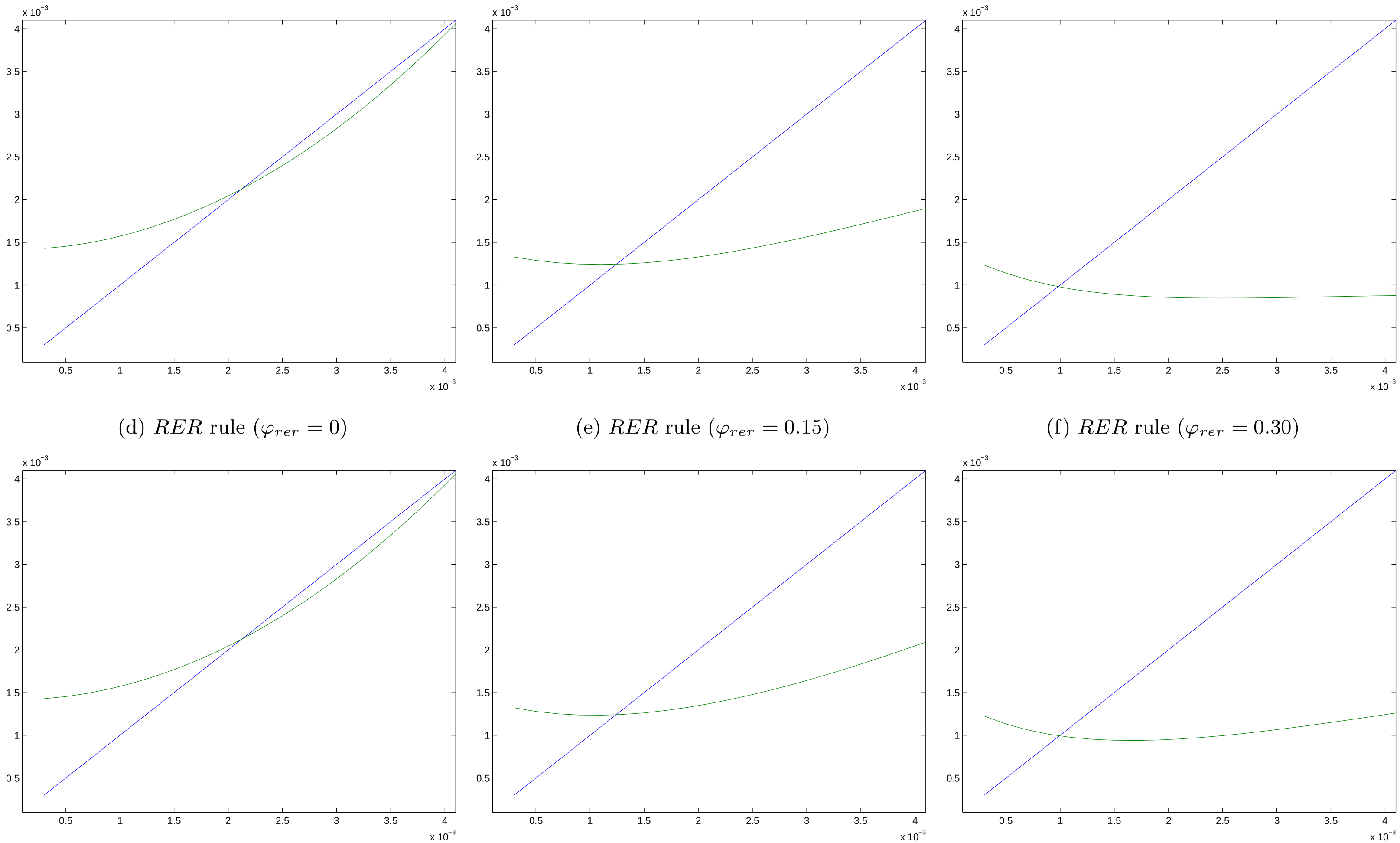
2.A Figures

Figure 2-5: Existence of equilibria under FX intervention rules (CK)



(a) Δs rule ($\varphi_{\Delta s} = 0$)

(b) Δs rule ($\varphi_{\Delta s} = 0.25$)

(c) Δs rule ($\varphi_{\Delta s} = 0.50$)

(d) RER rule ($\varphi_{rer} = 0$)

(e) RER rule ($\varphi_{rer} = 0.15$)

(f) RER rule ($\varphi_{rer} = 0.30$)

Note: Simulations involved 61 values for the conjectured variances of the change of the exchange rate. When the intervention parameter under both rules is zero, we replicate the values for the pure discretional intervention case.

2.B Details of the computational strategy

The log-linearised system of equations of the model can be written as:

$$A_0 \begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix} = A_1 \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + A_2 \Delta s_t + B_0 \epsilon_t \tag{2.16}$$

and

$$\overline{E}_t \Delta s_{t+1} = i_t - i_t^* + \gamma\sigma^2 \left(\varpi_t^* + \varpi_t^{*,cb} \right) \tag{2.17}$$

where $A_2 = [1, 0..0]'$ is a $(n_1 + n_2) \times 1$ matrix and the definitions of the other matrices and vectors are in Section 2.3. This is the state space form of the model.

2.B.1 Solving the first block

As an illustration, we will solve the system in (2.16) under some simplifying assumptions. For a more general solution, see Villemot (2011). The system in (2.16) can be written as:[17]

$$\begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix} = A_0^{-1} A_1 \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + A_0^{-1} A_2 \Delta s_t + A_0^{-1} B_0 \epsilon_t$$

or

$$\begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix} = A \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + \begin{bmatrix} a_{11} \Delta s_t \\ 0_{(n_1+n_2-1)\times 1} \end{bmatrix} + B\epsilon_t$$

after making $A = A_0^{-1} A_1$, $B = A_0^{-1} B_0$ and a_{11} the $(1,1)$ element of A_0^{-1}. Using the Jordan decomposition of $A = P\Lambda P^{-1}$, it becomes:

$$P^{-1} \begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix} = \Lambda P^{-1} A \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + \begin{bmatrix} p_{11} a_{11} \Delta s_t \\ 0_{(n_1+n_2-1)\times 1} \end{bmatrix} P^{-1} B \Delta s_t + P^{-1} C \epsilon_t$$

Making $R = P^{-1}B, \Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, $R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}$ and p_{11} the $(1,1)$ element of P^{-1}. Λ_1 (Λ_2) is the diagonal matrix of stable (unstable) eigenvalues

[17] Assuming A_0 is invertible, otherwise we can generalise this for the case of non-invertible matrix.

of size n_1 (n_2). The system of equations become:

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} X_t \\ E_t Y_{t+1} \end{bmatrix}$$
$$= \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} X_{t-1} \\ Y_t \end{bmatrix} + \begin{bmatrix} p_{11} a_{11} \Delta s_t \\ 0_{(n_1+n_2-1)\times 1} \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \epsilon_t.$$

Making $\widetilde{X}_{t-1} = P_{11}X_{t-1} + P_{12}Y_t$, $\widetilde{Y}_t = P_{21}X_{t-1} + P_{22}Y_t$, the system becomes:

$$\begin{bmatrix} \widetilde{X}_t \\ E_t\widetilde{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \begin{bmatrix} \widetilde{X}_{t-1} \\ \widetilde{Y}_t \end{bmatrix} + \begin{bmatrix} p_{11} a_{11} \Delta s_t \\ 0_{(n_1+n_2-1)\times 1} \end{bmatrix} + \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \epsilon_t$$

According to Blanchard & Kahn, given that Λ_2 is the diagonal of unstable eigenvalues, the only stable solution is given by: $\widetilde{Y}_t = 0 = P_{21}X_{t-1} + P_{22}Y_t$.

Then, the solution for the forward looking variables is:

$$Y_t = (P_{22})^{-1} P_{21} X_{t-1}. \tag{2.18}$$

The solution for the system of stable (backward looking) equations is:

$$\widetilde{X}_t = \Lambda_1 \widetilde{X}_{t-1} + \begin{bmatrix} p_{11} a_{11} \Delta s_t \\ 0_{(n_1-1)\times 1} \end{bmatrix} + R_1 \epsilon_t \tag{2.19}$$

2.B.2 Solving the second block

The $MA(\infty)$ representation of the first block

Now we change the classification of endogenous variables in the block 1 to focus in the ones which are part of the minimum state variables (MSV) set. We call these variables Z_t, while the rest of endogenous variables is referred as Z_t^-. In our case the Z_t is formed by *12* variables as defined in Appendix 1.B, in the previous chapter.

The transition and policy functions can be written as:

$$\begin{bmatrix} Z_t \\ Z_t^- \end{bmatrix} = \begin{bmatrix} W \\ W^- \end{bmatrix} Z_{t-1} + \begin{bmatrix} V \\ V^- \end{bmatrix} \epsilon_t^* \tag{2.20}$$

where $\epsilon_t^{*\prime} = [\epsilon_t^\prime, \Delta s_t]$ appends the depreciation rate in the vector of shocks. Evaluating the transition function in $t-1$ and replacing it in (2.20), we have:

$$\begin{bmatrix} Z_t \\ Z_t^- \end{bmatrix} = \begin{bmatrix} W \\ W^- \end{bmatrix} \left(W Z_{t-2} + V \epsilon_{t-1}^*\right) + \begin{bmatrix} V \\ V^- \end{bmatrix} \epsilon_t^*$$

Repeating this process many times, we get:

$$\begin{bmatrix} Z_t \\ Z_t^- \end{bmatrix} = \begin{bmatrix} W \\ W^- \end{bmatrix} \left[(W)^n Z_{t-n-1} + (W)^{n-1} V \epsilon_{t-n}^* + ... + W V \epsilon_{t-2}^* + V \epsilon_{t-1}^*\right] + \begin{bmatrix} V \\ V^- \end{bmatrix} \epsilon_t^*$$

Which allows us to write the solution as a $MA(\infty)$:

$$\begin{bmatrix} Z_t \\ Z_t^- \end{bmatrix} = \begin{bmatrix} W \\ W^- \end{bmatrix} \sum_{i=1}^{\infty} (W)^{i-1} V \epsilon_{t-i}^* + \begin{bmatrix} V \\ V^- \end{bmatrix} \epsilon_t^* \tag{2.21}$$

Given the form of matrix W, the impact of shocks diminish over time, allowing us to approximate the solution using a fixed number of lags. We focus in the solution for i_t in this step and replace it back into (2.17). In our setup i_t^* follows an exogenous process which is easy to express as a function of shocks. Finally, the last term, $\gamma\sigma^2\left(\varpi_t^* + \varpi_t^{*,cb}\right)$ is a combination of the conditional volatility term σ^2, the first order autoregressive process of ϖ_t^* and other endogenous variables in the policy rule for $\varpi_t^{*,cb}$, that can also be expressed as function of shocks.

Conditional moments and solution method

In order to calculate the the conditional volatility of the depreciation rate, we need to make use of the strategy proposed by Bacchetta and van Wincoop (2006), based on

Townsend (1983).

First we conjecture a solution for the depreciation of exchange rate of the form:

$$\Delta s_t = \mathcal{A}(L)\varepsilon_{t+1}^{i^*} + \mathcal{B}(L)\varepsilon_t^{\varpi^*} + \mathcal{D}(L)'\zeta_t \tag{2.22}$$

where $\mathcal{A}(L)$ and $\mathcal{B}(L)$ are infinite order lag polynomials, while $\mathcal{D}(L)$ is an infinite order lag polynomials vector operating ζ_t, the vector all other shocks in the model. Writing $\mathcal{A}(L) = a_1 + a_2 L + a_3 L^2 + ...$ (and a similar definition for $\mathcal{B}(L)$ and $\mathcal{D}(L)$), we evaluate forward the conjecture (2.22) to obtain the value in $t+1$.

$$\Delta s_{t+1} = a_1\varepsilon_{t+2}^{i^*} + b_1\varepsilon_{t+1}^{\varpi^*} + d_1'\zeta_{t+1} + \vartheta'\xi_t + \mathcal{A}^*(L)\varepsilon_t^{i^*} + \mathcal{B}^*(L)\varepsilon_{t-1}^{\varpi^*} + \mathcal{D}^*(L)'\zeta_t \tag{2.23}$$

where $\xi_t = \left(\varepsilon_{t+1}^{i^*}, \varepsilon_t^{\varpi^*}\right)'$ contains the unobservable innovations, $\vartheta' = (a_2, b_2)$ stands for the parameters associated to these shocks, $\mathcal{A}^*(L) = a_3 + a_4 L + ...$ (similar definition for $\mathcal{B}^*(L)$) and $\mathcal{D}^*(L) = d_2 + d_3 L + ...$ The last three terms in 2.23, $\mathcal{A}^*(L)\varepsilon_t^{i^*} + B^*(L)\varepsilon_{t-1}^{\omega^*} + \mathcal{D}^*(L)'\zeta_t$ represent all observable and past known shocks. Taking expectations for dealer ι over the previous equation yields:

$$E_t^\iota(\Delta s_{t+1}) = \vartheta' E_t^\iota(\xi_t) + \mathcal{A}^*(L)\varepsilon_t^{i^*} + \mathcal{B}^*(L)\varepsilon_{t-1}^{\varpi^*} + \mathcal{D}^*(L)\zeta_t \tag{2.24}$$

while the conditional variance as a function of unobservable innovation is given by:

$$var_t(\Delta s_{t+1}) = a_1^2 var_t(\varepsilon_{t+2}^{i^*}) + b_1^2 var_t(\varepsilon_{t+1}^{\varpi^*}) + (d_1)' var_t(\zeta_{t+1}) d_1 + \vartheta' var_t(\xi_t)\vartheta. \tag{2.25}$$

Here $\sigma^2 \equiv var_t(\Delta s_{t+1})$ is constant given that $var_t(\xi_t)$ is also constant. In order to obtain the conditional moments we need to obtain the conditional expectation and variance of the unobservable component ξ_t. The computation of the conditional moments is then obtained following Townsend (1983) and Bacchetta and Wincoop (2006).

FX traders extract information from the observed variation of the exchange rate Δs_t and the signal v_t^ι. To focus on the informational content of observable variables, we subtract the known components from these observables and define these new variables as Δs_t^* and $v_t^{\iota *}$. We follow the notation in Bacchetta and Wincoop (2006). The measurement equation on this part of the problem is given by:

$$Y_t^{\iota} = H'\xi_t + w_t^{\iota} \tag{2.26}$$

where $Y_t^{\iota} = (\Delta s_t^*, v_t^{\iota *})'$, $w_t^{\iota} = (0,\, \varepsilon_t^{v\iota})'$, and

$$H' = \begin{bmatrix} a_1 & b_1 \\ 1 & 0 \end{bmatrix}$$

The unconditional means of ξ_t and w_t^{ι} are zero, while we define their unconditional variances as $\widetilde{P}$ and R respectively. Following Townsend (1983), we can write:

$$E_t^{\iota}(\xi_t) = MY_t^{\iota} \tag{2.27}$$

where:

$$M = \widetilde{P}H\left(H'\widetilde{P}H + R\right)^{-1}.$$

For the conditional variance of the unobservable component we have, $P \equiv var_t(\xi_t)$, where

$$P = \widetilde{P} - MH'\widetilde{P}. \tag{2.28}$$

Substituting (2.26) and (2.27) in (2.24) and averaging over dealers gives the average conditional expectation of the variation of the exchange rate in terms of the shocks:

$$\overline{E}_t \Delta s_{t+1} = \vartheta' MH'\xi_t + \mathcal{A}^*(L)\varepsilon_t^{i^*} + \mathcal{B}^*(L)\varepsilon_{t-1}^{\varpi^*} + \mathcal{D}^*(L)\zeta_t. \tag{2.29}$$

Replacing the FX intervention policy strategy in equation (2.17), the $MA\,(\infty)$ representation of the endogenous variables (2.21), and the definition of σ^2 from (2.25), we obtain:

$$\overline{E}_t \Delta s_{t+1} = \hat{\imath}_t - \hat{\imath}_t^* + \gamma\sigma^2\left(\varpi_t^* + \varphi_{\Delta s}\Delta s_t + \varphi_{rer} rer_t + \varepsilon_t^{cb}\right) \tag{2.30}$$

$$\begin{aligned} \overline{E}_t \Delta s_{t+1} = {} & \mathcal{F}_i(L)\epsilon_t^* - \mathcal{G}(L)\varepsilon_t^{i^*} + ... \\ & + \gamma\left[(a_1^2 var_t(\varepsilon_t^{i^*}) + b_1^2 var_t(\varepsilon_t^{\varpi^*}) + (d_1)' var_t(\zeta_t) d_1 + \vartheta' P \vartheta\right] \\ & \times \left[\mathcal{J}(L)\varepsilon_t^{\omega^*} + \varphi_{\Delta s}\Delta s_t + \varphi_{rer}\mathcal{F}_{rer}(L)\epsilon_t^* + \varepsilon_t^{cb}\right] \end{aligned} \tag{2.31}$$

where $\mathcal{F}_z(L)\epsilon_t^*$ stands for $z_t = \{i_t, rer_t\}$, $\mathcal{G}(L)\varepsilon_t^{i^*}$ for i_t^*, and $\mathcal{J}(L)\varepsilon_t^{\omega^*}$ for ω_t^*. This is the "fundamental equation" $MA(\infty)$ representation.

To solve for the parameters of $\mathcal{A}(L)$, $\mathcal{B}(L)$ and $\mathcal{D}(L)$ we need to match the coefficients from equations (2.29) and (2.31).

Solution of parameters

Now we go through the algebra. Define $z_j^{y,x} \equiv \frac{dy_t}{dx_{t-j+1}}$ as the linear impulse response in the first step of the endogenous variable y_t with respect to the exogenous variable x_{t-j+1}. With this auxiliary variable we identify the parameters multiplying each shock. For this, we use the method of undetermined coefficients comparing equations (2.29) and (2.31).

Solution without rule-based FX intervention For simplicity, we solve first for the parameters assuming first there is no rule-based FX intervention, that is: $\varphi_{\Delta s} = \varphi_{rer} = 0$.

We start taking derivatives to the right hand side of equations (2.29) and (2.31) with respect to $\varepsilon_t^{i^*}, \varepsilon_{t-1}^{i^*}, ..., \varepsilon_{t-s+3}^{i^*}$, respectively**:**

$$
\begin{aligned}
a_3 &= \frac{di_t}{d\varepsilon_t^{i^*}} + \left(\frac{di_t}{d\Delta s_t}\frac{d\Delta s_t}{d\varepsilon_t^{i^*}} + \frac{di_t}{d\Delta s_{t-1}}\frac{d\Delta s_{t-1}}{d\varepsilon_t^{i^*}}\right) - \frac{di_t^*}{d\varepsilon_t^{i^*}}\\
a_4 &= \frac{di_t}{d\varepsilon_{t-1}^{i^*}} + \left(\frac{di_t}{d\Delta s_t}\frac{d\Delta s_t}{d\varepsilon_{t-1}^{i^*}} + \frac{di_t}{d\Delta s_{t-1}}\frac{d\Delta s_{t-1}}{d\varepsilon_{t-1}^{i^*}} + \frac{di_t}{d\Delta s_{t-2}}\frac{d\Delta s_{t-2}}{d\varepsilon_{t-1}^{i^*}}\right) - \frac{di_t^*}{d\varepsilon_{t-1}^{i^*}}\\
&\vdots\\
a_s &= \frac{di_t}{d\varepsilon_{t-s+3}^{i^*}} + \sum_{j=1}^{s-1}\left(\frac{di_t}{d\Delta s_{t+1-j}}\frac{d\Delta s_{t+1-j}}{d\varepsilon_{t-s+3}^{i^*}}\right) - \frac{di_t}{d\varepsilon_{t-s}^{i^*}}
\end{aligned}
$$

In this case the direct effect is zero, because i_t^* only appears in the risk-premium adjusted UIP condition, that is $\frac{di_t}{d\varepsilon_{t-s+3}^{i^*}} = 0$. Then the solution for a_3, a_4, ...is given by:

$$
a_s = \sum_{j=1}^{s-1} z_{s-j}^{i,\Delta s} a_j - \rho_{i^*}^{s-3} \qquad \text{for } s \geqslant 3 \tag{2.32}
$$

Similarly, taking derivatives with respect to $\varepsilon_{t-1}^{\varpi^*}, \varepsilon_{t-2}^{\varpi^*}, ..., \varepsilon_{t-s+2}^{\varpi^*}$, yields:

$$b_3 = \frac{di_t}{d\varepsilon^{\varpi^*}_{t-1}} + \left(\frac{di_t}{d\Delta s_t}\frac{d\Delta s_t}{d\varepsilon^{\varpi^*}_{t-1}} + \frac{di_t}{d\Delta s_{t-1}}\frac{d\Delta s_{t-1}}{d\varepsilon^{\varpi^*}_{t-1}}\right) + \gamma\sigma^2 \frac{d\varpi^*_t}{d\varepsilon^{\varpi^*}_{t-1}}$$

$$\vdots$$

$$b_s = \frac{di_t}{d\varepsilon^{\omega^*}_{t-s+2}} + \sum_{j=1}^{s-1}\left(\frac{di_t}{d\Delta s_{t+1-j}}\frac{d\Delta s_{t+1-j}}{d\varepsilon^{i^*}_{t-s+2}}\right) + \gamma\sigma^2 \frac{d\varpi^*_t}{d\varepsilon^{i^*}_{t-s+2}}$$

Similarly to the previous case, the direct effect is zero here, that is $\frac{di_t}{d\varepsilon^{\varpi^*}_{t-s+2}} = 0$. Then the solution for b_3, $b_4, ...$ is given by:

$$b_s = \sum_{j=1}^{s-1} z^{i,\Delta s}_{s-j} b_j + \gamma\sigma^2 \rho^{s-2}_{\omega^*} \qquad \text{for } s \geqslant 3 \tag{2.33}$$

Using the same approach, we take derivatives with respect to $\varepsilon_t, \varepsilon_{t-1}, ..., \varepsilon_{t-s}$ for $\varepsilon \in \zeta$:

$$d^{\varepsilon}_2 = \frac{di_t}{d\varepsilon_t} + \left(\frac{di_t}{d\Delta s_t}\frac{d\Delta s_t}{d\varepsilon_t}\right) + \gamma\sigma^2\left(\mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}}\right)$$
$$d^{\varepsilon}_3 = \frac{di_t}{d\varepsilon_{t-1}} + \left(\frac{di_t}{d\Delta s_t}\frac{d\Delta s_t}{d\varepsilon_{t-1}} + \frac{di_t}{d\Delta s_{t-1}}\frac{d\Delta s_{t-1}}{d\varepsilon_{t-1}}\right) + \gamma\sigma^2\left(\rho_\varepsilon \mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}}\right)$$
$$\vdots$$
$$d^{\varepsilon}_s = \frac{di_t}{d\varepsilon_{t-s+2}} + \sum_{j=1}^{s-1}\left(\frac{di_t}{d\Delta s_{t+1-j}}\frac{d\Delta s_{t+1-j}}{d\varepsilon_{t-s+2}}\right) + \gamma\sigma^2\left(\rho^{s-2}_{\varepsilon} \mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}}\right)$$

where $\mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}}$ is an indicator value of 1 when the shock ε equals $\varepsilon^{\omega^*,cb}$. This system is summarised by:

$$d^{\varepsilon}_s = z^{i,\varepsilon}_{s-1} + \sum_{j=1}^{s-1} z^{i,\Delta s}_{s-j} d_j + \gamma\sigma^2\left(\rho^{s-2}_{\varepsilon} \mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}}\right) \tag{2.34}$$

which is valid for $s \geqslant 2$. Note also that $\frac{di_t}{d\varepsilon_{t-s+2}} = 0$ when $\varepsilon = \varepsilon^{\omega^*,cb}$.

This set of equations (2.32), (2.33) and (2.34) allows us to express the whole system as a function of parameters a_1, a_2, b_1, b_2 and the vector of parameters d_1.

Taking derivatives with respect to the two unobservable shocks $\left\{\varepsilon_{t+1}^{i^*}, \varepsilon_t^{\varpi^*}\right\}$ we get:

$$(\vartheta' M H')_1 = z_1^{i,\Delta s} a_1, \tag{2.35}$$

$$(\vartheta' M H')_2 = z_1^{i,\Delta s} b_1 + \gamma\sigma^2. \tag{2.36}$$

By substituting back the values for the matrices, we obtain a non-linear system of equations on the unknowns:

$$[a_2\ b_2]\, M \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = z_1^{i,\Delta s} a_1 \tag{2.37}$$

$$[a_2\ b_2]\, M \begin{bmatrix} b_1 \\ 0 \end{bmatrix} = z_1^{i,\Delta s} b_1 + \gamma\sigma^2 \tag{2.38}$$

Note that considering (2.37) and (2.38) we have two equations and four unknowns, which impedes us to solve for the system. Bacchetta and Wincoop (2006) overcome this problem by proving that the coefficients in the lag polynomials follow a recursive pattern. Assuming non-explosive coefficients, they are able to obtain additional restrictions on the values of the coefficient in the lag-polynomial. In our case, the interest rate is endogenous, meaning a feedback is present from the effect of unobservable shocks into the exchange rate and from there into the interest rate. This feedback effect makes the relationship across the coefficients in the lag polynomials a function of the solution in the first block and of the assumed FX intervention strategy. For this reason we follow instead a numerical approach that limits the number of lags affecting the solution. We set up the non-linear system of equations on the first elements of both infinite lag polynomials and search for a numerical solution using the trust-region-dogleg method implemented by MATLAB. The extra restrictions in our case are given by selecting a limit to the lags and setting the parameters associated with this lag at zero.[18] Since these are functions of the first parameters (the unknowns), we can solve the system and obtain the solution. We change sequentially this limit and derive new solutions in each step. The algorithm stops when a fixed point is achieved, revealing that the inclusion of additional lags has

[18]Note that Bacchetta and Wincoop (2006) guess a solution for the *level* of the exchange rate, while we solve for its *first difference*. Our method implicitly assumes the first difference of the exchange rate is stationary. We consider that in our setup our assumption is less restrictive.

a negligible effect on the result.[19]

The system of equations: We can represent the system of equations using some auxiliary matrices.

The A system The set of equations in (2.32) can be written as:

$$\begin{bmatrix} a_3 \\ a_4 \\ \vdots \\ a_{n+1} \\ a_{n+2} \end{bmatrix} = \begin{bmatrix} z_1^{i,\Delta s} & 0 & \dots & 0 & 0 \\ z_2^{i,\Delta s} & z_1^{i,\Delta s} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_{n-1}^{i,\Delta s} & z_{n-2}^{i,\Delta s} & \dots & z_1^{i,\Delta s} & 0 \\ z_n^{i,\Delta s} & z_{n-1}^{i,\Delta s} & \dots & z_2^{i,\Delta s} & z_1^{i,\Delta s} \end{bmatrix} \begin{bmatrix} a_2 \\ a_3 \\ \vdots \\ a_n \\ a_{n+1} \end{bmatrix} - \begin{bmatrix} 1 \\ \rho_{i^*} \\ \vdots \\ (\rho_{i^*})^n \\ (\rho_{i^*})^{n-1} \end{bmatrix} + \\ + a_1 \begin{bmatrix} z_2^{i,\Delta s} \\ z_3^{i,\Delta s} \\ \vdots \\ z_n^{i,\Delta s} \\ z_{n+1}^{i,\Delta s} \end{bmatrix} \tag{2.39}$$

These equations can be written in the matrix form, after assuming that the value of $a_{n+2} \longrightarrow 0$:

$$Z_1 A = Z_2^i A - X_{i^*} + a_1 Z_3^{i,\Delta s} \tag{2.40}$$

where $Z_1 = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0 & 0_{1\times(n-1)} \end{bmatrix}$, $A = [a_2, ..., a_{n+1}]'$ is a $n \times 1$ vector, Z_2^i is the lower triangular matrix that pre-multiplies A, $X_{i^*} = \left[1, \rho_{i^*}, ..., (\rho_{i^*})^{n-1}\right]'$, and $Z_3^{i,\Delta s} = \left[z_2^{i,\Delta s}, z_3^{i,\Delta s}, ..., z_{n+1}^{i,\Delta s}\right]'$

The B system:

Similarly, equations (2.33) can be written as:

$$Z_1 B = Z_2^i B + \gamma\sigma^2 \rho_{\varpi^*} X_{\varpi^*} + b_1 Z_3^{i,\Delta s} \tag{2.41}$$

[19] We set the fixed-point algorithm convergence criterion over the maximum difference in the values of the coefficients associated with the unobservable shocks.

where $B = [b_2, b_3, .., b_{n+1}]'$ and $X_{\varpi^*} = \left[1, \rho_{\varpi^*}, ..., (\rho_{\varpi^*})^{n-1}\right]'$.

The D system

In the same vein, the system for $D^{\varepsilon} = [d_1^{\varepsilon}, d_2^{\varepsilon}, .., d_n^{\varepsilon}]'$ is the following

$$\begin{aligned} Z_1 D^{\varepsilon} &= Z_2^i D^{\varepsilon} + Z_3^{i,\varepsilon} \qquad \text{when } \varepsilon \neq \varepsilon^{\varpi^*,cb} \\ Z_1 D^{\varpi^*,cb} &= Z_2^i D^{\varpi^*,cb} + \gamma\sigma^2 X_{\varpi^*,cb} \;\; \text{otherwise} \end{aligned}$$

where $Z_3^{i,\varepsilon} = \left[z_1^{i,\varepsilon}, z_2^{i,\varepsilon}, ..., z_n^{i,\varepsilon}\right]'$ and $X_{\varpi^*,cb} = \left[1, \rho_{\varpi^*,cb}, ..., (\rho_{\varpi^*,cb})^{n-1}\right]'$.

The complete system of equations.

Then, after making use of $Z = Z_1 - Z_2^i$, the total system of non-linear equations becomes:

$$\begin{aligned} [a_2 \; b_2]\, M \begin{bmatrix} a_1 \\ 1 \end{bmatrix} &= z_1^{i,\Delta s} a_1 \\ [a_2 \; b_2]\, M \begin{bmatrix} b_1 \\ 0 \end{bmatrix} &= z_1^{i,\Delta s} b_1 + \gamma\sigma^2 \\ A &= -Z^{-1}\left(X_{i^*} - a_1 Z_3^{i,\Delta s}\right) \\ B &= Z^{-1}\left(\gamma\sigma^2 \rho_{\varpi^*} X_{\varpi^*} + b_1 Z_3^{i,\Delta s}\right) \\ D^{\varepsilon} &= Z^{-1} Z_3^{i,\varepsilon} \\ D^{\varpi^*,cb} &= \left(\gamma\sigma^2\right) Z^{-1} X_{\varpi^*,cb} \\ \sigma^2 &= a_1^2 var_t(\varepsilon_{t+2}^{i^*}) + b_1^2 var_t(\varepsilon_{t+1}^{\varpi^*}) + (d_1)' var_t(\zeta_{t+1}) d_1 + \vartheta' var_t(\xi_t)\vartheta \end{aligned} \tag{2.42}$$

Note the system has $n \times \#$ of shocks $+3$ equations and unknowns, which only $n \times 2 + 3$ are non-linear equations (those corresponding to the B and $D^{\varpi^*,cb}$ system and the equations for a_1, b_1 and σ^2).

Solution with FX intervention rules

When we allow for FX intervention, the equations (2.32), (2.33), (2.34), (2.37) and (2.38) are replaced by:

$$a_s = \sum_{j=1}^{s-1} z_{s-j}^{i,\Delta s} a_j - \rho_{i^*}^{s-3} + \gamma\sigma^2 \left[\varphi_{\Delta s} a_{s-1} + \varphi_{rer} \sum_{j=1}^{s-1} z_{s-j}^{rer,\Delta s} a_j \right] \tag{2.43a}$$

$$b_s = \sum_{j=1}^{s-1} z_{s-j}^{i,\Delta s} b_j + \gamma\sigma^2 \left[\rho_{\omega^*}^{s-2} + \varphi_{\Delta s} b_{s-1} + \varphi_{rer} \sum_{j=1}^{s-1} z_{s-j}^{rer,\Delta s} b_j \right] \tag{2.43b}$$

$$d_s^{\varepsilon} = z_{s-1}^{i,\varepsilon} + \sum_{j=1}^{s-1} z_{s-j}^{i,\Delta s} d_j + \gamma\sigma^2 \left[\begin{array}{c} \rho_{\varpi^*,cb}^{s-2} \mathcal{I}_{\varepsilon=\varepsilon^{\varpi^*,cb}} + \\ \varphi_{\Delta s} d_{s-1} + \varphi_{rer} \sum_{j=1}^{s-1} z_{s-j}^{rer,\Delta s} d_j^{\varepsilon} \end{array} \right] \tag{2.43c}$$

$$[a_2 \ b_2]\, M \begin{bmatrix} a_1 \\ 1 \end{bmatrix} = z_1^{i,\Delta s} a_1 + \gamma\sigma^2 \left(\varphi_{\Delta s} a_1 + \varphi_{rer} z_1^{rer,\Delta s} a_1 \right) \tag{2.43d}$$

$$[a_2 \ b_2]\, M \begin{bmatrix} b_1 \\ 0 \end{bmatrix} = z_1^{i,\Delta s} b_1 + \gamma\sigma^2 \left(1 + \varphi_{\Delta s} b_1 + \varphi_{rer} z_1^{rer,\Delta s} b_1 \right) \tag{2.43e}$$

We can also express this with linear algebra. For example, the A system can be written as:

$$Z_1 A = Z_2^i A + \gamma\sigma^2 \left(\varphi_{\Delta s} I_n + \varphi_{rer} Z_2^{rer} \right) A - X_{i^*} + a_1 \left(Z_3^{i,\Delta s} + \gamma\sigma^2 \varphi_{rer} Z_3^{rer,\Delta s} \right)$$

Then, after making use of $Z^{FX} = Z_1 - Z_2^i - \gamma\sigma^2 \left(\varphi_{\Delta s} I_n + \varphi_{rer} Z_2^{rer} \right)$, where Z_2^{rer} is a lower triangular matrix, analogous to Z_2^i, with $z_1^{rer,\Delta s}$ as elements of its main diagonal, the total system of non-linear equations becomes:

$$
\begin{aligned}
[a_2\ b_2]\, M \begin{bmatrix} a_1 \\ 1 \end{bmatrix} &= z_1^{i,\Delta s} a_1 + \gamma\sigma^2 \left(\varphi_{\Delta s} + \varphi_{rer} z_1^{rer,\Delta s} \right) a_1 \\
[a_2\ b_2]\, M \begin{bmatrix} b_1 \\ 0 \end{bmatrix} &= z_1^{i,\Delta s} b_1 + \gamma\sigma^2 \left(1 + \varphi_{\Delta s} b_1 + \varphi_{rer} z_1^{rer,\Delta s} b_1 \right) \\
A &= -\left(Z^{FX}\right)^{-1} \left[X_{i^*} - a_1 \left(Z_3^{i,\Delta s} + \gamma\sigma^2 \varphi_{rer} Z_3^{rer,\Delta s} \right) \right] \\
B &= \left(Z^{FX}\right)^{-1} \left(\gamma\sigma^2 \rho_{\varpi^*} X_{\varpi^*} + b_1 \left(Z_3^{i,\Delta s} + \gamma\sigma^2 \varphi_{rer} Z_3^{rer,\Delta s} \right) \right) \\
D^{\varepsilon} &= \left(Z^{FX}\right)^{-1} Z_3^{i,\varepsilon} \\
D^{\varpi^*,cb} &= \left(\gamma\sigma^2\right) \left(Z^{FX}\right)^{-1} X_{\varpi^*,cb} \\
\sigma^2 &= a_1^2 var_t(\varepsilon_{t+2}^{i^*}) + b_1^2 var_t(\varepsilon_{t+1}^{\varpi^*}) + (d_1)' var_t(\zeta_{t+1}) d_1 + \vartheta' var_t(\xi_t) \vartheta
\end{aligned}
\tag{2.44}
$$

where $M = \frac{1}{(a_1)^2 \sigma_{i^*}^2 \sigma_v^2 + (b_1)^2 \sigma_{\omega^*}^2 \left(\sigma_{i^*}^2 + \sigma_v^2\right)} \begin{bmatrix} a_1 \sigma_{i^*}^2 \sigma_v^2 & (b_1)^2 \sigma_{\omega^*}^2 \sigma_{i^*}^2 \\ b_1 \sigma_{\omega^*}^2 \left(\sigma_{i^*}^2 + \sigma_v^2\right) & -a_1 b_1 \sigma_{i^*}^2 \sigma_{\omega^*}^2 \end{bmatrix}$

and $P = var_t(\xi_t) = \frac{\sigma_{i^*}^2 \sigma_v^2 \sigma_{\omega^*}^2}{(a_1)^2 \sigma_{i^*}^2 \sigma_v^2 + (b_1)^2 \sigma_{\omega^*}^2 \left(\sigma_{i^*}^2 + \sigma_v^2\right)} \begin{bmatrix} (b_1)^2 & -a_1 b_1 \\ -a_1 b_1 & (a_1)^2 \end{bmatrix}$

2.B.3 The problem with common knowledge (CK)

In the common knowledge benchmark, investors share the same signal about the future fundamental values. Information is common but incomplete. All investors receive:

$$
v_t = i_{t+1}^* + \varepsilon_t^v, \qquad \varepsilon_t^v \sim N(0, \sigma_v^2)
$$

Under common knowledge ϖ_t^* becomes observable, because we get rid of the idiosyncratic shocks. Thus, capital flows shocks will only affect the economy through the portfolio balance channel, as in Chapter 1. In the signal extraction problem dealers have to infer information only for $\xi_t^{CK} = \varepsilon_{t+1}^{i^*}$. We must assume now that the equilibrium exchange rate depends directly on the ε_t^v shock, the noise of the signal common to all agents. We guess a solution of the type:

$$\Delta s_t = \mathcal{A}(L)\varepsilon_{t+1}^{i^*} + \mathcal{B}(L)\varepsilon_t^{\omega^*} + \mathcal{D}(L)\zeta_t + \Psi(L)\varepsilon_t^{\upsilon} \tag{2.45}$$

Notice that now the $\varepsilon_t^{\omega^*}$ shock is observable and we have a new term in the solution for the 'now relevant' signal noise.

The only relevant signal for the problem under common knowledge is given by υ_t.[20] Following Townsend (1983), we obtain:

$$E_t(\varepsilon_{t+1}^{i^*}) = \hat{M}\upsilon_t^*,$$

where

$$\hat{M} = \frac{\sigma_{i^*}^2}{\sigma_{i^*}^2 + \sigma_\upsilon^2}$$

and

$$\upsilon_t^* = \varepsilon_{t+1}^{i^*} + \varepsilon_t^{\upsilon} \tag{2.46}$$

is the unknown component of the signal υ_t at time t. We first obtain an expression for s_{t+1}, using (2.45):

$$\begin{aligned}\Delta s_{t+1} = a_1\varepsilon_{t+2}^{i^*} + b_1\varepsilon_{t+1}^{\omega^*} + \psi_1\varepsilon_{t+1}^{\upsilon} + d_1\zeta_{t+1}' + \vartheta^{CK}\xi_t^{CK} + ...\\ ... + \mathcal{A}^*(L)\varepsilon_t^{i^*} + \mathcal{B}^*(L)\varepsilon_t^{\omega^*} + \mathcal{D}^*(L)\zeta_t + \Psi^*(L)\varepsilon_{t-1}^{\upsilon}\end{aligned} \tag{2.47}$$

where $\vartheta^{CK} = [a_2]$ and we have grouped the shocks known at t in the lag polynomials denoted with (*). Now, taking expectations over (2.47):

$$E(\Delta s_{t+1}) = a_2 E(\varepsilon_{t+1}^{i^*}) + A^*(L)\varepsilon_t^{i^*} + B^*(L)\varepsilon_t^{\omega^*} + D^*(L)\zeta_t + \Psi^*(L)\varepsilon_{t-1}^{\upsilon} \tag{2.48}$$

where we have used the fact that $E_t(\varepsilon_t^{\upsilon}) = 0$. Now we take the second moment:

$$var_t(\Delta s_{t+1}) = a_1^2\sigma_{i^*}^2 + b_1^2\sigma_{\omega^*}^2 + \psi_1^2\sigma_\upsilon^2 + (d_1)'var_t(\zeta_{t+1})d_1 + \vartheta_{CK}'P^{CK}\vartheta_{CK}$$

[20] It is straightforward to verify this. The unknown part of the equilibrium variation of the exchange rate is given by $\Delta s_t^* = a_1\varepsilon_{t+1}^{i^*} + \psi_1\varepsilon_t^{\upsilon}$. Since a_1 would be equal to ψ_1, it is clear the equilibrium exchange rate brings no additional information.

Note that:

$$
\begin{aligned}
E(\varepsilon^{i^*}_{t+1}) = E_t(\xi^{CK}_t) &= \hat{M} Y_t \\
&= \hat{M}\left(\varepsilon^{i^*}_{t+1} + \varepsilon^{\upsilon}_t\right)
\end{aligned}
$$

Then, the equation (2.48) becomes:

$$
E(\Delta s_{t+1}) = a_2 \frac{\sigma^2_{i^*}}{\sigma^2_{i^*} + \sigma^2_{\upsilon}} \left(\varepsilon^{i^*}_{t+1} + \varepsilon^{\upsilon}_t\right) + A^*(L)\varepsilon^{i^*}_t + B^*(L)\varepsilon^{\omega^*}_t + D^*(L)\zeta_t + \Psi^*(L)\varepsilon^{\upsilon}_{t-1}
$$

this equation is equivalent to (2.29) in the heterogeneous information case. We compare the coefficients with respect to (2.31).

Equations (2.43d) and (2.43e) now become:

$$
a^{CK}_2 \frac{\sigma^2_{i^*}}{\sigma^2_{i^*} + \sigma^2_{\upsilon}} = z^{i,\Delta s}_1 a^{CK}_1 + \gamma\sigma^2_{CK}\left(\varphi_{\Delta s} a^{CK}_1 + \varphi_{rer} z^{i,\Delta s}_1 a^{CK}_1\right) \tag{2.49}
$$

$$
a^{CK}_2 \frac{\sigma^2_{i^*}}{\sigma^2_{i^*} + \sigma^2_{\upsilon}} = z^{i,\Delta s}_1 \psi^{CK}_1 + \gamma\sigma^2_{CK}\left(\varphi_{\Delta s} \psi^{CK}_1 + \varphi_{rer} z^{i,\Delta s}_1 \psi^{CK}_1\right) \tag{2.50}
$$

from this equations we obtain that $a^{CK}_1 = \psi^{CK}_1$. Since agents only observe the sum of both the fundamental an noise shock, it stands to reason that the contemporaneous reaction to both shocks must be the same.

Additionally, we have a set of equations for $\Psi^*(L)$:

$$
\psi_s = \sum_{j=1}^{s-1} z^{i,\Delta s}_{s-j}\psi_j + \gamma\sigma^2\left[\varphi_{\Delta s}\psi_{s-1} + \varphi_{rer}\sum_{j=1}^{s-1} z^{rer,\Delta s}_{s-j}\psi_j\right] \tag{2.51}
$$

The system of equations then becomes:

$$
\begin{aligned}
A_{CK} &= -\left(Z^{FX}\right)^{-1}\left[X_{i^*} - a_1^{CK}\left(Z_3^{i,\Delta s} + \gamma\sigma^2\varphi_{rer}Z_3^{rer,\Delta s}\right)\right] \\
B_{CK} &= \left(Z^{FX}\right)^{-1}\left[\gamma\sigma^2\rho_{\varpi^*}X_{\varpi^*} + b_1^{CK}\left(Z_3^{i,\Delta s} + \gamma\sigma^2\varphi_{rer}Z_3^{rer,\Delta s}\right)\right] \\
\Psi_{CK} &= \left(Z^{FX}\right)^{-1}\left[\psi_1^{CK}\left(Z_3^{i,\Delta s} + \gamma\sigma^2\varphi_{rer}Z_3^{rer,\Delta s}\right)\right] \\
D_{CK}^{\varepsilon} &= \left(Z^{FX}\right)^{-1} Z_3^{i,\varepsilon} \\
D_{CK}^{\varpi^*,cb} &= \left(\gamma\sigma^2\right)\left(Z^{FX}\right)^{-1} X_{\varpi^*,cb} \\
a_2^{CK}\frac{\sigma_{i^*}^2}{\sigma_{i^*}^2+\sigma_v^2} &= z_1^{i,\Delta s}a_1^{CK} + \gamma\sigma_{CK}^2\left(\varphi_{\Delta s}a_1^{CK} + \varphi_{rer}z_1^{i,\Delta s}a_1^{CK}\right) \\
\psi_1^{CK} &= a_1^{CK} \\
b_2^{CK} &= z_1^{i,\Delta s}b_1^{CK} + \gamma\sigma^2\left(1 + \varphi_{\Delta s}b_1^{CK} + \varphi_{rer}Z_1^{rer}b_1^{CK}\right) \\
var_t(\Delta s_{t+1}) &= a_1^2\sigma_{i^*}^2 + b_1^2\sigma_{\omega^*}^2 + \psi_1^2\sigma_v^2 + (d_1)'var_t(\zeta_{t+1})d_1 + \vartheta_{CK}'P^{CK}\vartheta_{CK}
\end{aligned}
$$

Once again, the remaining restrictions come from imposing zeros at a given lag for the whole model, since the rest of the elements in the lag polynomials can be expressed as a function of ones associated with the unknowns.

Chapter 3

Fat-tailed shocks and the central bank reaction

"[...] we would expect policy-makers to take action when the mean and variance of forecast distributions are likely to stay the same, while the probability of some extreme bad event increases. [...] even if the variance is unchanged, an increase in the possibility of a severe economic downturn is likely to prompt action."
Cecchetti (2000).

3.1 Introduction

According to Mishkin (2011), one of the main lessons from the financial crisis is that key elements in the "science of monetary policy" need to be revisited. In particular those related to the non-linearities emerging in presence of the zero lower bound (ZLB), tail risk, and non-standard utility functions - such as agents' aversion to very negative outcomes. As the author points out, previous to the 2008 financial crisis, economists were aware of the presence of potential negative shocks with 'excess kurtosis' hitting the economy with a higher tail risk probability than the one implied by a Gaussian distribution.[1] In spite of acknowledging the presence of these shocks, little was done to

[1] A fact reflecting this concern was the emergence of Financial Stability Reports as a regular publication by Central Banks where the risks that the financial system put into the economy were discussed, see Mishkin (2011).

study the importance of excess kurtosis in monetary policy design.

The presence of non-linearities is obvious when monetary policy is affected by the non-negativity constraint on nominal interest rates. If the policy rate falls below zero, agents will prefer to keep their resources in cash, which pays a zero interest rate. For this reason, the space for the policy interest rate is bounded from below, with consequences for the policy decisions. Moreover, Kato and Nishiyama (2005) and Adam and Billi (2007) show how the presence of the zero lower bound makes the (discretionary) optimal monetary policy reaction to be non-linear *outside of the constrained region* as well. In particular, central banks should become more expansionary and more aggressive as they approach the ZLB, compared to what a linear Taylor rule type of policy predicts. This result is in line with the suggestions in Blinder (2000):

"... make the response function non-linear. In particular, the coefficient a [the coefficient in the Taylor rule that controls the response of the policy rate to inflation] *- and perhaps b* [the coefficient in the Taylor rule that controls the response of the policy rate to the output gap] *as well - could be higher when inflation is low. (...) such a modification would make monetary policy looser whenever inflation was very low, thus buying more insurance against getting stuck in the liquidity trap at i = 0."*

Central to the non-linearities generated outside of the ZLB region is the hazard of falling in it. For this reason, when the economy faces shocks from a fat-tailed distribution or increased kurtosis, the reaction should be more aggressive. However, it is not clear how this excess kurtosis impacts optimal policy rules. The present document tackles this question by introducing fat-tailed shocks in the model of Kato and Nishiyama (2005). This is a simple Neo-Keynesian model where the central bank reacts in a "pre-emptive" manner as the probability of falling into the ZLB increases, generating non-linear responses *outside* of the zero lower bound region.[2]

We perform this exercise to gauge the extent to which excess kurtosis affects the optimal behaviour of central banks outside the zero lower bound and, if this effect is significant, analyse to what extent excess kurtosis may be behind the reported change in

[2]In this model the rest of the economy is characterized by a set of linear equations. Introducing agents that change their behaviour in the presence of the zero lower bound increases greatly the computational costs. See Fernández-Villaverde et al. (2012).

the behaviour of central banks before and during the crisis.[3]

We focus exclusively in the role of excess kurtosis, assuming a time-invariant distribution of shocks. Our main findings are as follows: (1) under fat-tailed shocks, monetary policy becomes more aggressive further away from the zero lower bound region, compared to the model under Gaussian shocks. (2) As the economy approaches the ZLB, this pattern reverts and monetary policy is relatively *less aggressive* under shocks with excess kurtosis. (3) *Quantitatively*, these differences are not very significant as the largest differential between the optimal rates, under our baseline calibration, is lower than 10 basis points.

There is a small but growing literature related to the presence of fat-tailed shocks in macroeconomics. Fagiolo et al. (2008) pursue the hypothesis of non-normal shocks and fit via maximum likelihood the growth rate distributions for a series of OECD countries to the exponential-power (EP) family of densities, rejecting the hypothesis of normality in these series. In related work, Ascari et al. (2012) show that non-normality and fat tails characterize not only the time-series properties for GDP in the U.S, but also those for consumption, investment, employment, inflation and real wage.

By contrast, the literature on liquidity traps and the optimal policy at ZLB is extensive. The theoretical question regarding the effectiveness of monetary policy at low rates can be found in Keynes (1936). More recently, the subject received a lot of attention from policy-makers and academics as the lower inflation experienced during the early 1990s in advanced economies brought with it episodes of near-zero interest rates.[4] In October 1995, the Bank of Japan (BOJ) set its policy interest rates at 50 basis points in the midst of a deflationary crisis. A few years later, the federal funds rate in the US experienced a sharp fall, going from 6.50 percent in November 2000 to only one percent on July 2003. To date, both the Federal Reserve and the Bank of Japan maintain their policy interest rates effectively at zero.

Fuhrer and Madigan (1997) constitute one of the first efforts to analyse the dynamics

[3]Authors such as Taylor (2007) and Calani et al. (2010) estimate Taylor rules type of policies for the pre-crisis period and simulate the paths of interests rates provided those rules would have continued during the years of turmoil, finding very large differences between the actual path of interest rates and the projected paths. They conclude that the cuts in rates represent "deviations" from the pre-crises Taylor rules.

[4]In the advanced economies, the median inflation rate fell from 7% in the 1980s to 2% in the 1990s. See Kroszner (2007).

of the economy in a model with forward looking agents and an explicit ZLB constraint. The authors find that after a negative shock to the economy, the recovery of the inflation rate and output takes longer when monetary policy becomes ineffective due to the ZLB. Regarding the optimal policy under the ZLB, Reifschneider and Williams (1999) find that the standard Taylor rule is suboptimal in this scenario. Orphanides and Wieland (2000) add to this result by showing that the optimal policy under the ZLB constraint will become a non-linear function of the inflation rate. The literature considers as well the idea of monetary policy being affected by the ZLB before the constraint becomes binding (Hunt and Laxton (2004), Goodfriend (2001)). More recently, Kato and Nishiyama (2005) studied the importance of this pre-emptive motive, showing how optimal monetary policy should become more aggressive and expansionary as the economy approaches the ZLB. Nakov (2006) relaxes the assumption of perfect foresight in the Kato and Nishiyama (2005) and studies an optimal "censored" Taylor rule, which is the best lineal response conditional on the presence of the ZLB. Eggertsson and Woodford (2003) study the implications of the ZLB for monetary policy in a model that assumes a 2-state Markov chain for an exogenous disturbance. They find support for a price-level targeting type of policy, though lose the pre-emptive motive that emerges under a more general distribution for the exogenous disturbance. Finally, Fernández-Villaverde et al. (2012) adopt a fully non-linear approach in a New Keynesian model with an explicit ZLB and explore the role of fiscal policy when the economy hits the constraint. Authors relax the assumption of a time-invariant distribution and study the role of skewness and time-varying volatility for endogenous variables when the economy hits the ZLB.

The present document is structured as follows: Section 3.2 reviews the model of Kato and Nishiyama (2005) and explains the mechanism behind the results. In the next section, we discuss the computational strategy. Section 3.4 discusses the calibration of parameters and presents the results. Section 3.5 concludes.

3.2 The Model

In the current section we review the model of Kato and Nishiyama (2005). We assume the Central Bank minimizes a loss function in the spirit of Svensson (1997), Svensson (2002) and Ball (1999), namely:

$$L_t = \frac{1}{2}\{y_t^2 + \lambda(\pi_t - \pi_t^*)^2\}; \tag{3.1}$$

here π stands for inflation; y for the output gap and π^* the inflation target, which we assume constant. The parameter λ controls the relative importance that the central bank puts on the inflation rate deviations from the target, relative to the output gap. Following Woodford (2003), the economy is described by the following IS-AS framework:

$$y_{t+1} = \rho y_t - \delta(i_t - E_t\pi_{t+1}) + \nu_{t+1} \tag{3.2}$$

$$\pi_{t+1} = \pi_t + \alpha y_t + \varepsilon_{t+1} \tag{3.3}$$

where ν and ε are random disturbances. ρ stands for the degree of inertia over the business cycle. δ is a parameter reflecting the impact of real interest rates on the next period output - thus monetary policy affects the economy with a lag. Finally, α represents the impact of the output gap on future inflation.

Equations (3.2) and (3.3) represent the investment-savings (IS) and aggregate supply (AS) equations respectively. We substitute the expectation of inflation by a combination of the current inflation rate and the output gap, namely:

$$E_t\pi_{t+1} = \pi_t + \alpha y_t. \tag{3.4}$$

The inter-temporal problem of the monetary authority will be given by:

$$\min_{\{i_{t+j}\}_{j=0}^{\infty}} E_t \sum_{j=0}^{\infty} \beta^j L_{t+j}, \tag{3.5}$$

subject to the laws of motion for inflation and output gap given by equations (3.3) and (3.2), and an explicit zero lower constraint on the interest rate introduced through the Karush-Kuhn-Tucker approach. β reflects the time-preference of the central banker, or equivalently, the importance they assign to future losses relative to losses in the current period. This framework allows us to set up a Bellman equation with three Lagrange multipliers:

$$
\begin{aligned}
V(y_t, \pi_t) = \min_{i_t}[\frac{1}{2}\{y_t^2 + \lambda(\pi_t - \pi^*)^2\} - E_t\phi_{t+1}\{(\rho + \alpha\delta)y_t - \delta i_t + \delta\pi_t - y_{t+1}\} \\
- E_t\mu_{t+1}(\pi_t + \alpha y_t - \pi_{t+1}) - \psi_t i_t + \beta E_t V(y_{t+1}, \pi_{t+1})]. \qquad (3.6)
\end{aligned}
$$

where ψ_t is the Lagrange multiplier in the non-negativity constraint for the policy interest rate. The first order condition with respect to the interest rate yields:

$$E_t\phi_{t+1}\delta = \psi_t; \qquad (3.7)$$

which measures the "shadow cost" produced by monetary policy ineffectiveness at the zero lower bound. The first order conditions with respect to inflation and the output gap are given by the following two equations:

$$E_t\mu_{t+1} = -\beta\left[\lambda E_t(\pi_{t+1} - \pi^*) - \delta E_t\phi_{t+2} - E_t\mu_{t+2}\right] \qquad (3.8)$$

$$E_t\phi_{t+1} = -\beta\left[E_t y_{t+1} - (\rho + \alpha\delta)E_t\phi_{t+2} - \alpha E_t\mu_{t+2}\right], \qquad (3.9)$$

By combining equations (3.7), (3.8), and (3.9). It is possible to get some intuition about the restrictions for monetary policy that the ZLB imposes. In the case the ZLB is not binding we know that $\psi_t = 0$. This means, , from Equation (3.7), that ϕ_{t+1}, the Lagrange multiplier associated with Equation (3.2) - the IS equation - is zero as well. Thus, the only restriction that matters for the central bank will be the one associated with Equation (3.3), which represents the trade-off between stabilizing the inflation rate deviations and the output gap. In other words, the bank can fully neutralize the shocks coming from the IS equation. However, when the ZLB is binding, then $E_t\phi_{t+1} > 0$, meaning that the central bank can no longer offset the shocks coming from the IS equation. In this scenario, the central bank needs to balance the need of offsetting both the AS and IS shocks.

Kato and Nishiyama (2005) obtain an analytical derivation of the optimal interest rate.[5]

$$i^*(\pi_t, y_t) = \pi_t + \left(\alpha + \frac{\rho\theta_1 + \theta_1 - 1}{\delta\theta_1}\right) y_t + \left(\frac{\theta_1 - 1}{\alpha\delta\theta_1}\right)(\pi_t - \pi^*) + \left(\frac{1}{\delta\theta_1}\right)\sum_{i=0}^{\infty}\theta_2^i E_t\psi_{t+i} \tag{3.10}$$

Equation (3.10) represents the optimal reaction function outside of the zero lower bound region. The values of θ_1 and θ_2 are combinations of the "deep parameters" α, β, and λ. The first three terms of this expression are linear in the output gap and the inflation rate. The last term is the one generating the non-linearities, which stem from the shadow cost represented by the sequence of Lagrange multipliers associated with the non-negativity restriction $\{E_t\psi_{t+i}\}_{i=0}^{\infty}$. As we already mentioned, Equation (3.7) tells us that when the value of this multiplier is different from zero, the central bank is unable to offset the shocks coming from the IS equation.[6] In other words, the non-linearities are associated with the probability that the ZLB restriction becomes binding in the future. Given the difficulty of obtaining a closed-form solution for the optimal policy as a function of the inflation rate deviations and the output gap, the solution is obtained through a numerical procedure.

3.3 Computational strategy

The numerical strategy follows Kato and Nishiyama (2005). It is based on collocation methods.[7] The Bellman equation in (3.6) imposes a series of restrictions that must hold in every point of the state-space. This defines an infinite-dimensional fixed-point problem that can be discretized by approximating the value function as the sum of a finite set of basis functions. Since it is important to capture the non-linear behaviour of optimal rates, the value function is approximated through cubic splines. Obtaining the value function involves the calculation of expectations, for which we use numerical integration techniques. In particular, a Gaussian quadrature technique is used to approximate the integrals.[8] For the case of Gaussian shocks, we use the Gaussian-Hermite

[5] We refer the reader to the paper for the derivations.

[6] It is important to mention that it is possible to express $\sum_{i=0}^{\infty}\theta_2^i E_t\psi_{t+i}$ as a function of the states (π_t, y_t). This means that we can still characterize the optimal response as a (potentially non-linear) function of these two variables.

[7] See Judd (1998).

[8] See Press et al. (1992) for a detailed description of this procedure.

quadrature method, for which tables with the values of nodes and abscissas are easily found.

For the case of fat-tailed shocks we need the use of a distribution exhibiting "excess kurtosis".[9] Additionally, this distribution must exhibit finite moments (at least up to the 4th order) that are stable functions of the distribution parameters, such that we are able to control the lower moments. For this purpose, we use the Exponential Power family of distributions, attributed to Subbotin (1923). The functional form of this distribution reads:[10]

$$f(x;b,a,m) = \frac{1}{2ab^{\frac{1}{b}}\Gamma\left(1+\frac{1}{b}\right)} e^{-\frac{1}{b}\left|\frac{x-m}{a}\right|^{b}} \tag{3.11}$$

where the kurtosis depends on a shape parameter (b). An interesting feature of this family of distributions is that it encompasses both the Gaussian distribution ($b = 2$) and the Laplace distribution ($b = 1$). Whenever $b < 2$ the distribution will exhibit tails fatter than the Gaussian ones (or "super-Normal" tails).[11] Due to the numerical solution followed in the present chapter, the use of Gaussian quadrature for approximating the distribution of shocks would require the calculation of quadrature weights and abscissas for each value of the shape parameter. In our case, we decided to focus on Laplace shocks, which exhibit an excess kurtosis of 3, for the following two reasons. First, Fagiolo et al. (2008) find strong support for this distribution when analysing the distribution of a set of macroeconomic series in OECD economies. Second, quadrature rules can be calculated for Laplace distribution weights through a modification of the Laguerre-Quadrature rules.

3.4 Calibration and Results

3.4.1 Calibration

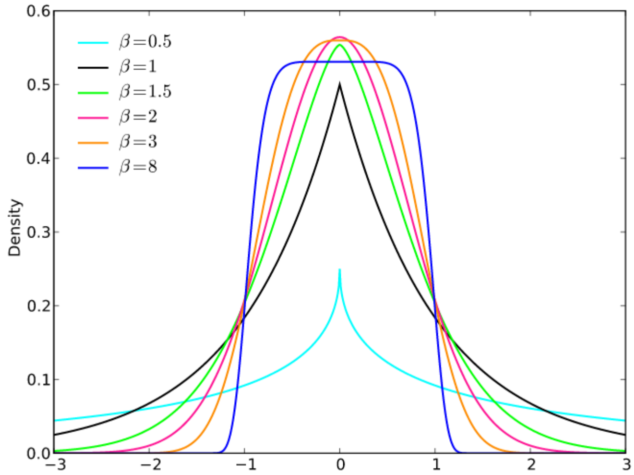
Before moving forward with the numerical exercises, we need to set values for the model parameters. Table 3.1 shows the baseline calibration, based in Woodford (2003).

[9]The "excess kurtosis" refers to the case when a distribution exhibits a kurtosis higher than 3, which is the kurtosis of the normal distribution.

[10]For instance, a problem we would faced using a t-student distribution is that the one period forward variables with t-distributed shocks will not follow a t-distribution, due to the non-zero mean. In addition, low degrees of freedom generate unbounded moments.

[11]For a detailed discussion of the properties of this family of distributions, see Fagiolo et al. (2008).

Figure 3-1: Exponential Power family of distributions



Note: Figure shows plots of the exponential power family of distributions for different values of the shape parameter (β). It includes the Gaussian ($\beta = 2$) and Laplace distributions ($\beta = 1$). Source: `http://en.wikipedia.org/wiki/File:Generalized_normal_densities.svg`.

From there we take values for ρ, δ and α. The parameters for the standard deviations are taken from Adam and Billi (2007), who estimate these parameters following the approach of Rotemberg and Woodford (1998). We keep the value of the time-preference parameter relatively low, at 0.6, for the baseline calibration. This value comes from Kato and Nishiyama (2005), who find that a lower value of β is needed in order to guarantee the existence of a stationary optimal policy reaction function. We set the inflation rate target at 0%. The value of λ is set at 20 which is taken from Rotemberg and Woodford (1998). We perform robustness exercises on this value since it has been documented that monetary policy becomes more *dovish* during periods of low inflation, which is the region of the state-space associated with the ZLB.[12] We perform robustness exercises for the slope of the Phillips curve (α), the real rate elasticity of output (δ), the central banker's time-preference parameter (β), and the standard deviations of the AS and IS shocks (σ_ν and σ_ε). The results are reported in Section 3.4.4.

[12] Orphanides and Wilcox (2002) calls this behaviour *the opportunistic approach to monetary policy.* Martin and Milas (2007) provide empirical evidence supporting this hypothesis for the case of monetary policy in the U.S. during the 1983 to 2004 period.

Table 3.1: **Baseline Calibration**

Parameter	*Value*	*Description*
λ	20	Relative weight on inflation-deviations variability.
β	0.6	Central banker's time-preference parameter.
ρ	1	Persistence of output dynamics.
δ	0.5	Real rate elasticity of output.
α	-0.02	Slope of the Phillips curve (negative).
σ_ν	1.5	S.D of AS shock.
σ_ε	0.15	S.D of IS shock.
π^*	0	Target inflation rate.

3.4.2 Results under Gaussian shocks

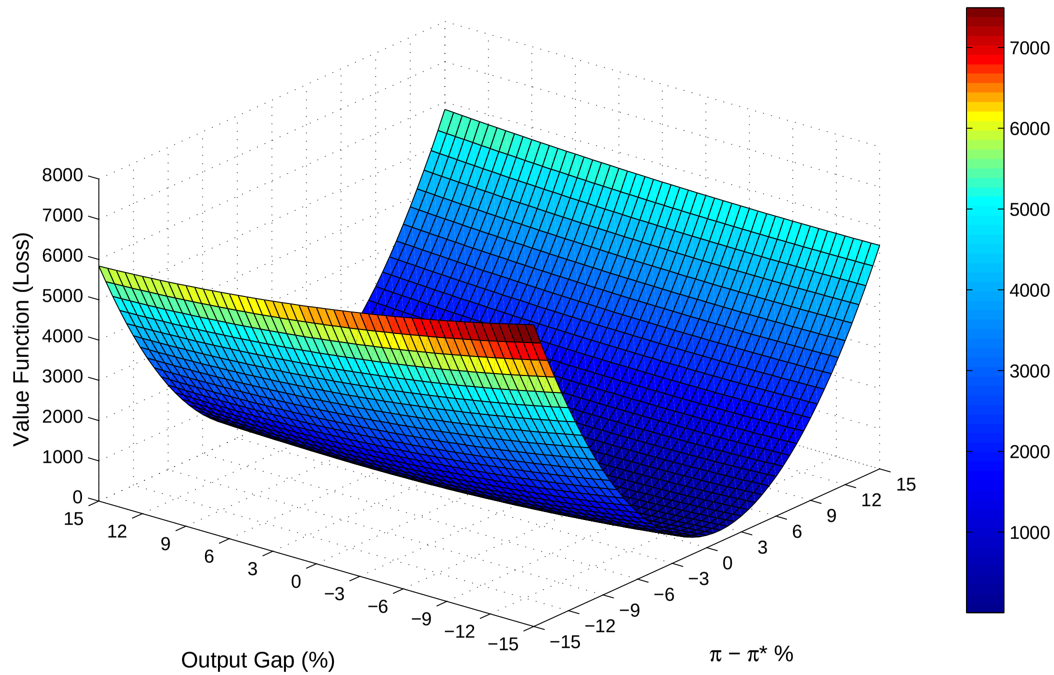
We explore first the results under Gaussian shocks. As we can observe from Fig. 3-2, the value function under the presence of the ZLB will not be quadratic. It can be noted that when the inflation rate and output gap are negative, the loss for the central bank increases. In other words, the cost of stabilization that the central bank faces increases sharply when the economy is in this state, which the literature associates with a *deflationary spiral*. The reason can be traced back to Equation (3.6). When the ZLB constraint is binding, the hazard of remaining in the same region is high. Therefore, the slackness condition over the non-negativity of interest rates restriction calls for an expected positive value for ψ, the Lagrange multiplier associated with this constraint. Thus, inside the ZLB, the central bank would not be able to offset the shocks coming from the IS equation. Its ineffectiveness to stabilize the economy will be reflected in a higher variability of aggregate output and inflation rate variations, and consequently, a higher welfare loss.[13]

The optimal reaction function ceases to be linear. As we can observe from Fig. 3-3, the optimal reaction exhibits the pre-emptive motive. Now the interest rate outside of the ZLB region is non-linear. In Figure 3-4 we can compare the reaction to inflation rate deviations from the target and the output gap under a standard Taylor rule and when the ZLB restriction is taken into consideration. Panels 3-4a to 3-4c show how

[13] Woodford (2003) Ch. 6, explores the problem of monetary policy under the ZLB in a model where the nonnegativity constraint is replaced by a constraint in the interest rate variability. The author results follow the same intuition. The constraint (or the additional objective) makes the stabilization of the inflation rate and the output gap harder to achieve, increasing their variability and, consequently, the welfare losses.

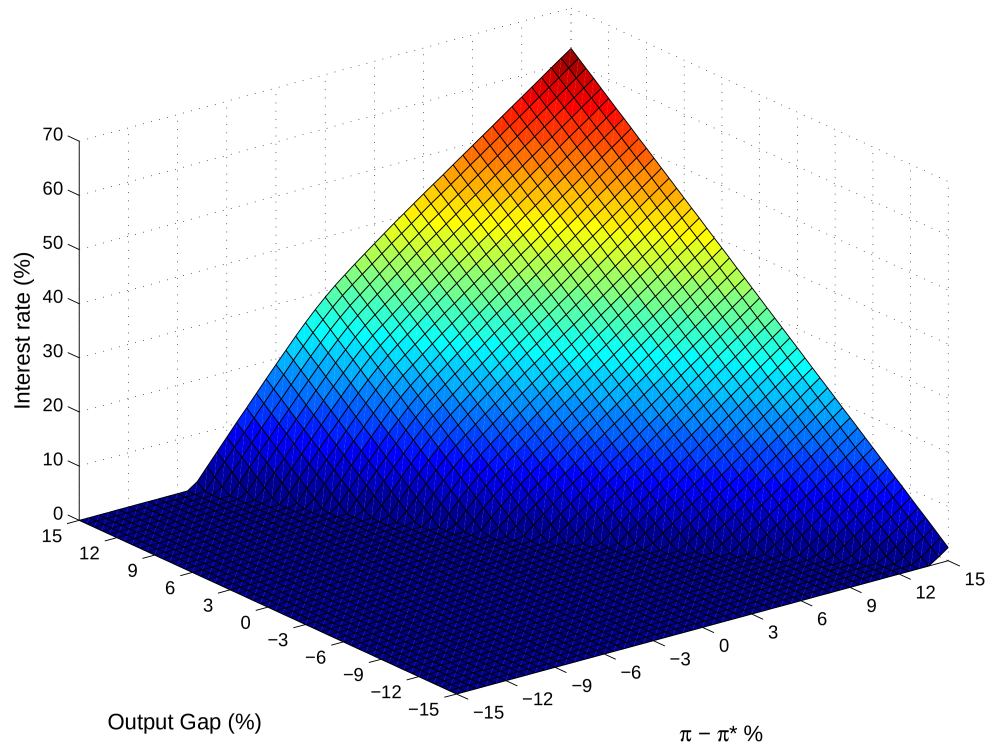
the optimal policy deviates from a linear policy rule as the economy approaches the ZLB region, becoming concave. As the probability of being restricted by the ZLB in future periods increases, the central bank becomes more aggressive in its response to inflation deviations. Panels 3-4d to 3-4f show how monetary policy becomes also more expansionary. As Kato and Nishiyama (2005) explain, this effect is related to the threat of a *deflationary spiral.* Under this threat, it is in the interest of the central bank to be more expansionary in comparison to the standard Taylor rule.

Figure 3-2: Value Function with Zero Lower Bound (baseline calibration)



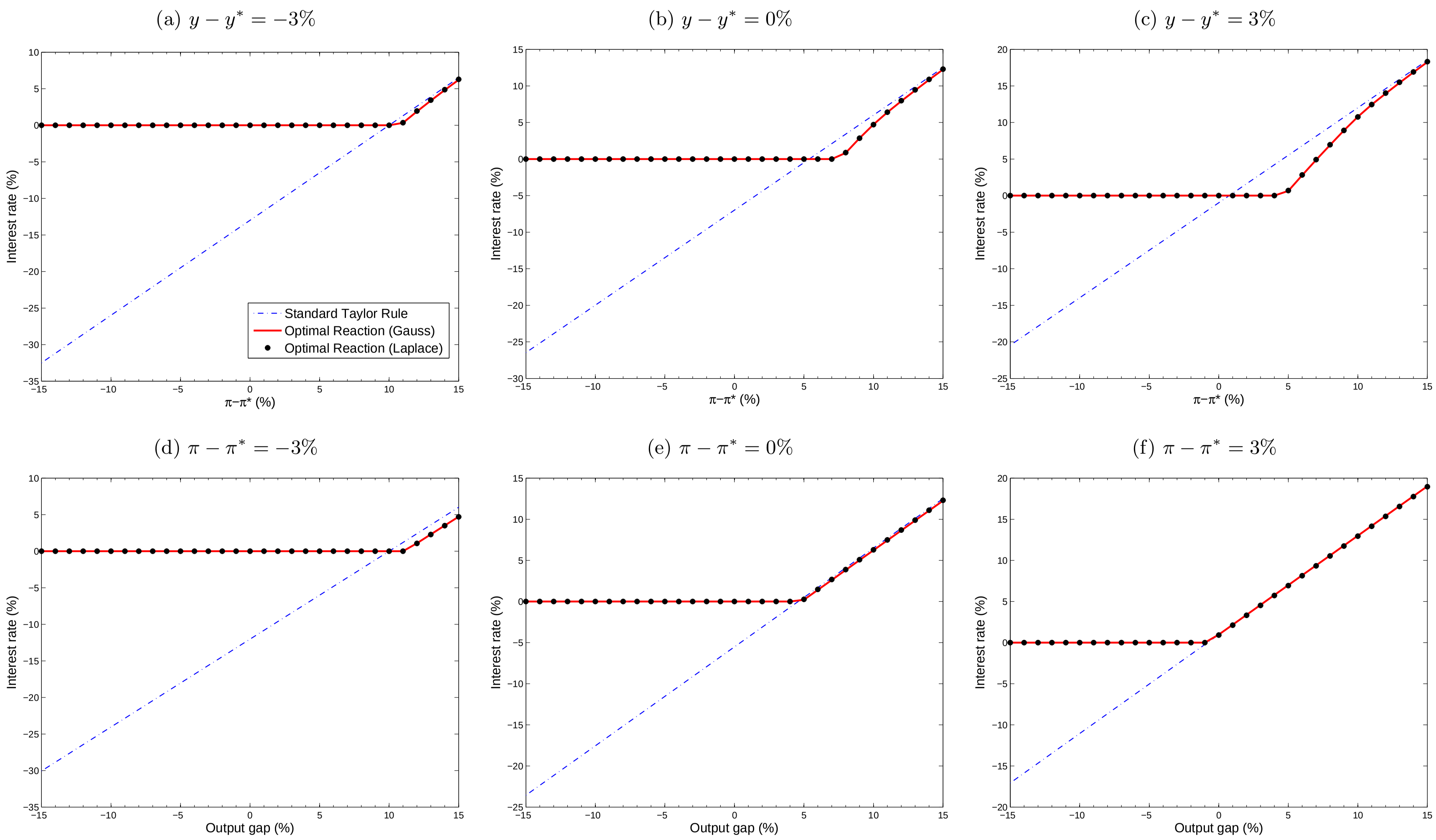
Note: Value function for central bank under Gaussian shocks and baseline calibration. Approximation performed over 51 points for output gap and 51 points for the inflation rate deviations. Calibration follows values in Table 3.1.

Figure 3-3: Optimal Policy Reaction Function (baseline calibration)



Note: Optimal reaction for central bank under Gaussian shocks and baseline calibration. Approximation performed over 51 points for output gap and 51 points for the inflation rate deviations. Calibration follows values in Table 3.1.

Figure 3-4: Optimal reaction and Taylor rule under Gaussian and Laplace Shocks

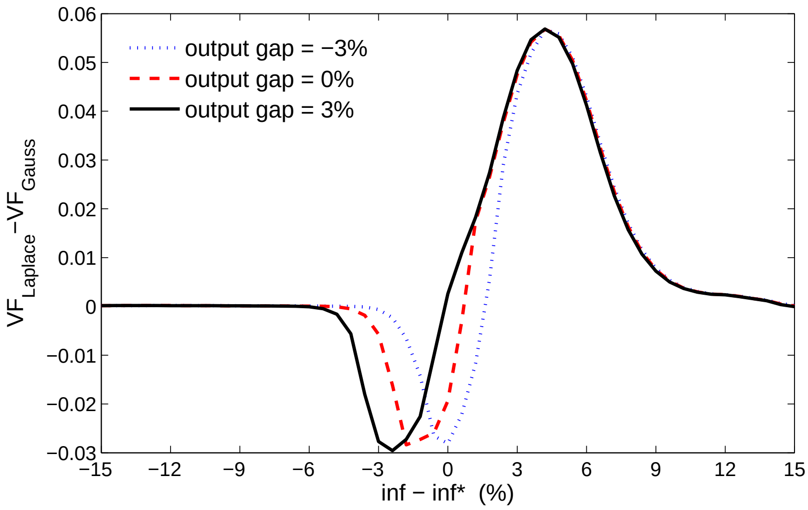


Note: Upper row shows the interest rate for different the inflation rate, leaving the output gap constant. Lower row shows the interest rate for different values of the output gap, leaving the inflation rate constant.

3.4.3 The role of fat-tailed shocks

Now we study how the optimal monetary policy predicted by the model changes under the presence of fat-tailed shocks. As previously discussed, we would like to assess how *excess kurtosis*, which modifies the probabilities of falling into the ZLB region, affects the optimal behaviour of central banks.

Figure 3-5: Central Bank's loss function, (Laplace - Gaussian)

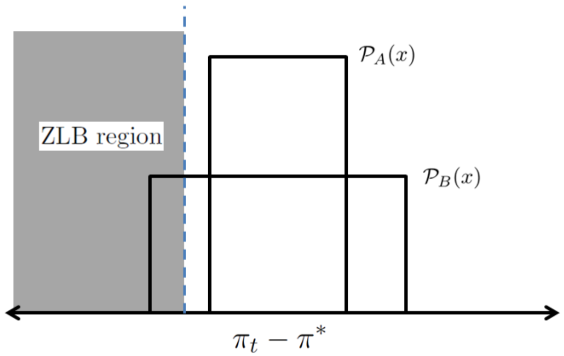


Note: Figure shows the difference between the central bank's loss function in Eq. 3.5 under Laplacian and Gaussian-distributed shocks for different values of inflation deviations, keeping the output gap constant.

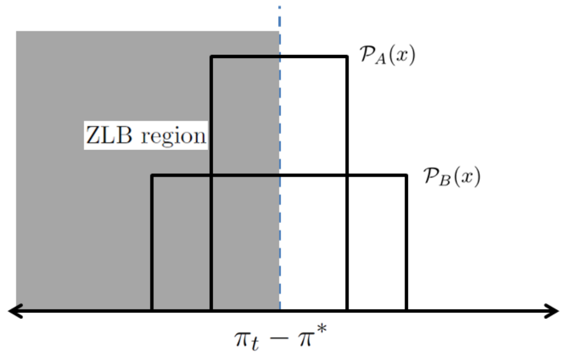
Since we use a global solution method we can obtain the solution to the problem for the central banker at different points of the state-space. In Figure 3-5 we show the differences between the loss of the central bank under both assumed distributions for different values for the inflation deviations, keeping the output gap constant. We find that, away from the zero lower bound region, the loss under fat-tailed shocks is higher. As the economy approaches the ZLB, this pattern first increases and then reverts. Inside the constrained region the difference turns negative, which means that the central bank is worse off under Gaussian shocks. Notice that for lower values of the output gap, the difference between value functions reverts faster. In order to explain this result we make use of Figure 3-6, which presents a simple case of how fat tails interact with the hazard of falling or staying in the ZLB in the following period.

Figure 3-6: Distribution of shocks and the ZLB

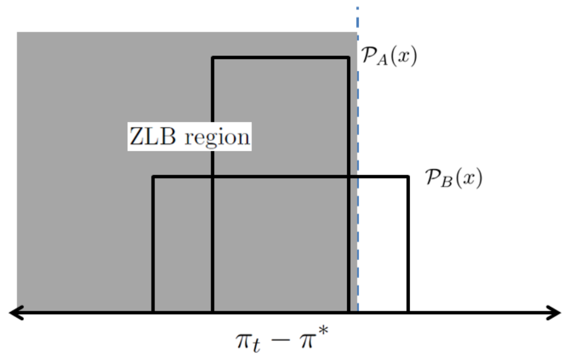
(a) *Case A*



(b) *Case B*



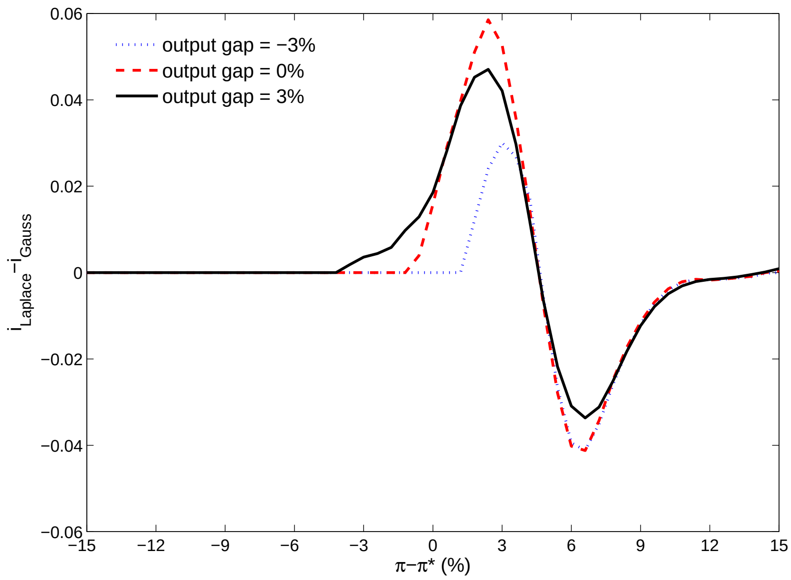
(c) *Case C*



Note: Diagram shows how fat-tails affect the hazard of being in the ZLB under shocks following two different distributions. As the economy gets closer to the ZLB from high values of the inflation rate (Case A), the hazard of falling into the ZLB is higher under the relative heavy-tailed distribution ($\mathcal{P}_B$). In Case B, the probability of being in the ZLB in the next period is the same under both distributions. Finally, Case C shows that when the economy gets inside the ZLB, fat-tailed distributions might imply a higher probability of leaving the constrained region.

From the viewpoint of the central bank there is a reason to become more aggressive under Laplace shocks far away from the ZLB. As the economy approaches this region, this result reverts, as the central bank anticipates that getting near to the ZLB will be more costly under Gaussian shocks. We observe this pattern holds for the optimal interest rates, presented in Figure 3-7.

Figure 3-7: Difference in optimal monetary policy, (Laplace - Gaussian)



Note: Figure shows the difference between the optimal interest rates of the problem in Eq. 3.5 under Laplacian and Gaussian-distributed shocks for different values of inflation deviations, keeping the output gap constant.

The introduction of super-normal tails generates an interesting result as monetary policy will become relatively *less aggressive* under fat-tailed shocks near the ZLB. From a *quantitative* point of view, the difference between both cases is not significant. Figure 3-4, suggests that the optimal central bank's reaction is almost unaffected by the change in the assumed distribution of the shocks. Figure 3-7, shows that the difference between interest rates, for the cases considered, ranges between 0 and 6 basis points, far from the 25 basis point step central banks use when monetary policy changes are announced. Clearly we would require higher excess kurtosis in order to generate effects of a significant magnitude.[14]

[14]Due to the complexity in the construction of quadrature rules for distributions with higher excess kurtosis we leave these exercises for future research.

3.4.4 Robustness

Alternative parameterizations are considered. Table 3.2 reports the maximum differences found between the optimal (discretionary) monetary policy under Gaussian and Laplace-distributed shocks. Results are not particularly sensitive to changes in most parameters. For the case of δ, which is associated with the impact monetary policy has on aggregate demand, we find a maximum difference between optimal polices of 34 basis points. When δ is low, it is harder for monetary policy to steer the economy away from the constrained region. For this reason the level of pre-emptive behaviour will be stronger and the interest rate will be more sensitive to the distribution of shocks. We confirm that loss functions follow the same pattern observed in Figure 3-5. Similarly, the results found in Figure 3-4 hold under the parameter values considered in the robustness exercises, this is, the optimal reaction is barely affected by the change in the assumed distribution of shocks.

3.5 Conclusions

We introduce shocks with 'super-normal tails' into the simple NK model with a monetary authority that explicitly considers the ZLB in their optimal policy design, as in Kato and Nishiyama (2005). When the central bank considers this restriction explicitly, the optimal policy ceases to be linear outside of the ZLB. These non-linearities represent a pre-emptive motive, as the central bank becomes more aggressive, in an attempt to avoid falling into a region in which monetary policy becomes ineffective. Central to this decision is the hazard of falling into the ZLB region, which is affected by the distribution of the shocks hitting the economy.

Under shocks with higher kurtosis, non-linearities in the reaction function will emerge further away from the zero interest rate region, relative to the Gaussian shocks case. However, as the economy approaches the ZLB region, this pattern reverts, as the central bank anticipates that under Gaussian shocks, it will be harder to leave the ZLB region, once the economy is inside it. This means monetary policy would actually be relatively *less aggressive* near the ZLB under fat-tailed shocks. Nonetheless, the effects of excess kurtosis are *quantitatively* very limited as the largest difference in optimal interest

Table 3.2: **Robustness to alternative parameterisations**

$Parameter$	$Value$	$Max \mid i^*_{Laplace} - i^*_{Gauss} \mid$
	0.01	6.13
α	0.02 (*baseline*)	7.27
	0.03	10.54
	0.10	34.44
δ	0.25	13.08
	0.5 (*baseline*)	7.27
	5	6.29
λ	10	6.78
	20 (*baseline*)	7.27
	0.5	6.37
β	0.55	6.05
	0.6 (*baseline*)	7.27
	0.5	1.60
σ_ν	1	4.17
	1.5 (*baseline*)	7.27
	0.1	7.47
σ_ε	0.15 (*baseline*)	7.27
	0.5	6.02

Note: Table shows the maximum distance between discretionary optimal policies under Gaussian and Laplace-distributed shocks. Values are reported in basis points. In each exercise the indicated parameter value is changed, keeping the rest at the baseline calibration values in Table 3.1. Optimal interest rates are calculated for values of inflation and output gap in the range $[-15, 15]$ for both variables. Approximation is performed for 31 points for the output gap and 31 points for the inflation rate. α stands for slope of the Phillips curve (negative). δ is the real rate elasticity of output. λ represents the relative weight on inflation-deviations variability. β is the central banker's time-preference parameter. Finally, σ_ν and σ_ε are the standard deviations of the AS and IS shocks, respectively.

rates found is of 34 basis points. Changes in the baseline calibration confirm results are robust to variations in parameter values.

Our findings suggest that, in the current setup, the presence of fat-tailed shocks does not produce significant effects on the optimal monetary policy design.

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3.A. Numerical Algorithm

For the numerical solution we followed Kato and Nishiyama (2005), using a collocation method for solving the Bellman equation problem. The Bellman equation, given by Equation (3.6) follows:

$$V(\pi, y) = \min_{i}\{f(\pi, y) + \beta EV(g(\pi, y, i, \nu, \varepsilon))\} \tag{3.12}$$

where $f(\pi, y)$ represents the instantaneous loss of the Central Bank. The function $g(\pi, y, x, \nu, \varepsilon)$ represent the laws of motion for the state variables $\{\pi, y\}$, which are given by equations (3.2) and (3.3).

$$g(\pi, y, i, \nu, \varepsilon) = \begin{bmatrix} \rho + \alpha\delta & \delta \\ \alpha & 1 \end{bmatrix} \begin{bmatrix} y \\ \pi \end{bmatrix} + \begin{bmatrix} \delta \\ 0 \end{bmatrix} i + \begin{bmatrix} \nu \\ \varepsilon \end{bmatrix} \tag{3.13}$$

After setting the Bellman equation we proceed with the discretization of the state space. In this case we focus on the interval $[-15, 15]$ for both state variables and set a number interpolation nodes, which we choose to be equally distributed. We need to find approximate the form of the value function on both sides, hence we will ask the algorithm to hold the equality in equation (3.12) at every point of the grid. The LHS will be given by:

$$LHS_{n_y n_\pi}(c) = \sum_{i=1}^{N_\pi} \sum_{j=1}^{N_y} c_{ij} \gamma_i^\pi(\pi_{n_\pi}) \gamma_j^y(y_{n_y}) \text{ for each } (\pi_{n_\pi}, y_{n_y}) \in Node. \tag{3.14}$$

Here, the functions $\gamma_i^\pi(\pi_{n_\pi})$ and $\gamma_j^y(y_{n_y})$ form the basis for the splines. Hence we can form a continuous function that is a piecewise polynomial, though, smooth over the connecting points.[15]

Now, the RHS of the equation has a similar structure, however, the result is affected by the shocks ν and ε, for which we assume a known distribution. As described above, we follow two cases, in the first one we assume a Normal distribution for shocks, while in the second, we follow a Laplace or double-exponential distribution. We follow Gaussian Quadrature for the treatment of both shocks. In the first case, we use a Gaussian-Hermite

[15] See Judd (1998), Ch 6 for a thorough description of the use of splines.

quadrature, which is associated with weights that are normally distributed. In the second, we modify the Gaussian-Laguerre quadrature, used for exponential distributions. By re-weighting the quadrature weights we can approximate an exponential distribution, for an even number of abscissa. Hence, the RHS of equation (3.12), is given by:

$$RHS_{n_y n_\pi}(c) = \min_{i \geq 0} \left[f(\pi_{n_\pi}, y_{n_y}) + \beta \sum_{h_\nu=1}^{M_\nu} \sum_{h_\varepsilon=1}^{M_\varepsilon} \sum_{i=1}^{N_\pi} \sum_{j=1}^{N_y} w_{h_\nu h_\varepsilon} c_{ij} \gamma_{ij}(g(\pi, y, i, \nu, \varepsilon)) \right] \tag{3.15}$$

where the value function represented by b-splines is the same as in Equation (3.14), for consistency. Now however, we evaluate its value at the abscissa and nodes generated by the Gaussian quadrature. We perform a value function iteration looking for a fixed point. Convergence is attained when:

$$\max \mid V_k(\pi_{n_\pi}, y_{n_y}) - V_{k+1}(\pi_{n_\pi}, y_{n_y}) \mid < \tau, \tag{3.16}$$

where τ is the tolerance parameter, set at $1e-4$ in our exercise. With the values of i that minimize the solution we construct a cubic spline approximation for the mapping from the states to the control. This will yield the optimal policy function.

Chapter 4

Learning Through the Yield Curve

4.1 Introduction

The adaptive learning approach proposes the modelling of agents as econometricians, who use all the available data to construct forecasting functions, revised over time as new data becomes available. Adam et al. (2011) use this approach to successfully explain a series of equity pricing puzzles. Modelling agents who learn about the behaviour of equity prices is key to their results. This chapter explores whether this particular mechanism is useful for the study of bond yield dynamics.

Carceles-Poveda and Giannitsarou (2008) provide a thorough analysis of the introduction of *self-referential* learning general equilibrium framework. Self-referential learning means that agents' beliefs affect the behaviour of economic variables, which in turn affect agents' beliefs and so on. In the models presented by Carceles-Poveda and Giannitsarou (2008), agents are assumed to know the functional form of the law of motion relating the variable of interest to the state variables, but they do not know the value of the coefficients in this law of motion. The authors conclude that the effects of learning in their setup are modest, providing very little improvements over the rational expectations (RE) case. Timmermann (1996) analyzes self-referential learning when agents also learn about the exogenous dividend process. This mechanism is capable of increasing the volatility of simulated stock prices but, as in the case of Carceles-Poveda and Giannitsarou (2008), its impact is modest. The key difference between these results and the ones obtained by Adam et al. (2011) is that in the latter, agents are not endowed with knowledge of the mapping between dividends and prices. One of the critical implications of this assumption is that agents are unable to express the value of the asset as a discounted sum of payoffs. Moreover, in a related paper, Adam and Marcet (2011) show that endowing agents with this knowledge involves a large set of strong assumptions about agents' beliefs. This means that agents have to form beliefs about the law of motion of stock prices directly. These beliefs are used to forecast next period's stock price. Since this period's stock price is affected by expectations of next period's stock price, there is a very direct link between beliefs about stock prices and the actual behaviour of stock prices.

In this chapter, we follow Adam et al. (2011) by presenting a model in which agents learn about bond prices. Applying the idea of Adam et al. (2011) to bond prices means that learning is strictly speaking no longer self-referential, unless a consol bond is considered. That is beliefs about the price of an n-th period bond do not affect the behaviour of the price of this n-th period bond, but they do affect the behaviour of the prices of bonds with a higher maturity. We obtain three key results: First, learning about price dynamics affects the relative volatility of bonds in a heterogeneous way, as the prices of higher maturity assets become more volatile relative to the rational expectations (RE) case. Second, as maturity increases, the price volatility converges to the one implied by the consol bond case, which we treat as a benchmark since, similar to equity, this instrument has no redemption date.[1] Finally, we perform numerical exercises that suggest this mechanism can be useful for explaining the pattern of volatilities observed in the term structure.

The present chapter is organized in the following way. In Section 4.2 we present the baseline model and the proposed cases for study. In Section 4.3 we discuss the learning rule and contrast analytically the case of consol bonds, in which learning is *self-referential* and the case of finite-maturity bonds. In Section 4.4, we present numerical simulations and report findings regarding the implied behaviour of price and yield volatility in both

[1] In Chapter 5 we study, through numerical simulations, how the introduction of learning can help reproducing some of the stylized facts exhibited by consol bonds.

cases. Section 4.5 presents our conclusions.

4.2 Setup

4.2.1 The Economy

The economy is composed by infinite-lived consumers-investors who face the following problem:

$$\max_{c_t} E_t \sum_{t=0}^{\infty} \delta^t \ln c_t, \tag{4.1}$$

Agents can save their resources in nominal assets. In particular, we assume they have access to a series of $\mathcal{T}$ risk-free bonds with a payoff at maturity and a series of coupons in each period, denoted by ϕ_t. $Q_t^{(\tau)}$ stands for the nominal price of the bond with τ periods to maturity in period t, while $B_{t+1}^{(\tau)}$ is the quantity of bonds. We assume $\mathcal{T}$ is an arbitrary high number. Additionally, agents have access to a consol type bond, which price we denote by Q_t^{∞}, and pays the same coupon as the finite maturity bonds.

The representative investor budget constraint will be given by:

$$P_t c_t + \sum_{\tau=1}^{\mathcal{T}} Q_t^{(\tau)} B_{t+1}^{(\tau)} + Q_t^{\infty} B_{t+1}^{\infty} \leq P_t y_t + \sum_{\tau=1}^{\mathcal{T}} \left(Q_t^{(\tau)} + \phi_t \right) B_t^{(\tau)} + (Q_t^{\infty} + \phi_t) B_t^{\infty}, \quad \forall_t.$$

The first order conditions over the bonds yield the following set of Euler equations:

$$Q_t^{(i)} = \delta E_t \left[\left(\frac{P_t}{P_{t+1}} \frac{c_t}{c_{t+1}} \right) \left(Q_{t+1}^{(i-1)} + \phi_{t+1} \right) \right], \ \forall_{i \in [1, \mathcal{T}]} \tag{4.2}$$

$$Q_t^{(\infty)} = \delta E_t \left[\left(\frac{P_t}{P_{t+1}} \frac{c_t}{c_{t+1}} \right) \left(Q_{t+1}^{(\infty)} + \phi_{t+1} \right) \right], \tag{4.3}$$

where Q_t^0 is the principal paid by the bond at maturity, in period t. The principal is given by $Q_t^{0,I} = \phi_t \bar{Q}^{0,I}$. Thus, coupons and principal will share the same growth rate.

4.2.2 Cases

The cases we present in this chapter will follow the pricing equations given by 4.2 and 4.3, however they will differ in the assumptions we make regarding the process for inflation and the coupon payments, as well as in the way agents form their beliefs, defined by the *perceived law of motion* (PLM).

Case I: Stochastic coupon growth and learning about nominal bond price growth rates.

We begin as close as possible to Adam et al. (2011) to provide a benchmark. This exercise will show us how the introduction of a finite maturity alters the results obtained by the aforementioned authors for the case of equity. We make the pay-off structure stochastic by defining the following process for the coupon and principal:

$$\phi_{t+1} = \gamma_\phi \phi_t \varepsilon_t, \;\; \varepsilon_t \sim N(1, \sigma_\phi^2), \;\; \gamma_\phi > 0. \tag{4.4}$$

Under stochastic coupons, a consol-type bond will have the same pay-off structure as equity. Since here we focus on the coupon as the main driver of these assets dynamics, consumption will be parameterized as random walk process, following Cochrane and Campbell (1999):

$$\frac{c_t}{c_{t+1}} = \mu_c + \varepsilon_t^c \tag{4.5}$$

where $\varepsilon_t^c \sim N(0, \sigma_c^2)$.[2] We assume as well the consumption growth covariance with the rest of the processes in the model is zero.[3] Prices are assumed constant. We follow Adam et al. (2011) by specifying agents with the following beliefs:

$$Q_{t+1}^{(i,I)} = \beta_t^i Q_t^{(i,I)} \epsilon_t^{(i)}, \; \forall_{i>1} \tag{4.6}$$

where $Q_{t+1}^{(i,I)}$ stands for the price of the bond with i periods to maturity, in the Case I setup, and $\epsilon_t \sim i.i.d \, N(1, \sigma_\epsilon^2)$. Note that agents will hold an individual PLM for each maturity.[4]

[2] The literature has not reached a definitive consensus on the process consumption growth follows. While some authors find that the implications for consumption growth in the model of Hall (1978) - consumption follows a random walk - cannot be completely rejected by the data, others claim that the series fits a process with high serial autocorrelation. For a discussion see Carroll et al. (2011).

[3] Introducing a covariance would complicate the analysis as the actual growth rate of prices would stop being constant. Throughout this Chapter we will assume agents only care about first moments.

[4] We simply follow the proposed learning rule in Adam et al. (2011). Alternatively, we could have set a model where agents iterate 4.2 forward and obtain a yield curve where all maturities depend on the same factors. This would be equivalent to the learning mechanism proposed by Carceles-Poveda

Case II: Stochastic inflation and learning about bond price real growth rates The case of stochastic coupon payments provides a good approximation to the effects of different maturities for the learning mechanism in Adam et al. (2011). However, bonds exhibit a deterministic path for coupons (usually flat). For this reason we change this assumption and introduce now a deterministic growth rate for coupons.

$$\phi_{t+1} = \gamma_\phi \phi_t; \tag{4.7}$$

which considers the case of flat coupon-payment schedule when $\gamma_\phi = 1$. The processes for consumption remains the same as in the previous case. However, inflation will now be stochastic:

$$\frac{P_t}{P_{t+1}} = \gamma_\pi \varepsilon_t^\pi, \;\; \varepsilon_t^\pi \sim N(1, \sigma_\pi^2), \;\; \gamma_\pi \geq 0. \tag{4.8}$$

We assume inflation is uncorrelated with the rest of the processes in the economy.[5] Agents will focus on learning the real growth rate of bonds. For this reason, the PLM will be given by:

$$\frac{Q_{t+1}^{(i,II)}}{P_{t+1}} = \beta_t^i \frac{Q_t^{(i,II)}}{P_t} \epsilon_t^{(i)}, \; \forall_{i>1} \tag{4.9}$$

In this case, we have chosen to use inflation as the process behind bond price dynamics. Assuming similar dynamics for the consumption growth rate will generate equivalent results.[6]

and Giannitsarou (2008), which yields modest results. Adam and Marcet (2011) discuss this element of arbitrariness, which is often present in learning models, and propose a microfounded framework where the *Law of iterated expectations* ceases to hold and learning about price dynamics arises as an optimal behaviour. This type of learning can be rationalized in several ways: (1) as a model where agents possess short-term buy and sell strategies; (2) as a model where agents do not know they are the marginal pricer; and (3) as a model of vanishing heterogeneity across agents. For a discussion see Adam and Marcet (2011).

[5] As we show in Section 4.A in the appendix, under rational expectations, the inflation rate and bond prices will be uncorrelated. However the covariance will not be zero *along the learning path*. For this reason we need agents who hold *linear* beliefs as a condition for convergence to the rational expectations equilibrium.

[6] In the case consumption growth is chosen as the source of dynamics agents will learn about the *risk-adjusted* price growth rate. For an example see Adam et al. (2011).

4.3 Learning

4.3.1 Analytic results

In this section we study analytically how the introduction of the learning mechanisms proposed in Section 4.2 affects the dynamics of asset prices. This is important since in order to stay as close to Adam et al. (2011), we restrict our attention to learning mechanisms that comply with two desirable properties: (1) the laws of motion under a least-squares learning rule converge to those corresponding to the rational expectations equilibrium, and (2) learning should be *reasonable*. To check whether the first condition is satisfied, we analyse the ordinary differential equations (ODE) associated with the stochastic recursive algorithm (SRA) that describes the dynamics of the processes under learning. For this purpose we construct T-maps and check if the rational expectations equilibrium constitutes an equilibrium of the system of ODE. For the second condition, we verify that the PLM presented in Section 4.2 are not *misspecified*, in the sense that PLMs used cannot possibly converge to a REE.[7]

We start by defining a general learning function.[8] For the case of nominal price growth learning, beliefs will be updated following:

$$\hat{\beta}_t^{(i)} = \hat{\beta}_{t-1}^{(i)} + g_t \left(\frac{Q_{t-1}^{(i)}}{Q_{t-2}^{(i)}} - \hat{\beta}_{t-1}^{(i)} \right), \; \forall_{i \in [1,\mathcal{T}], i=\infty} \tag{4.10}$$

where $g_t(0) = 0$ and $g'(\cdot) > 0$. These conditions define a learning process that adjusts beliefs in the same direction as the prediction error.[9]

Since both cases share the same properties, we focus on Case I for our analytical results.[10] First we focus on the consol bond. Substituting assumptions 4.4, 4.5, together

[7]It is important differentiate this from econometric misspecification. Most learning models are misspecified from an econometric viewpoint, since agents fail to recognize the *self-referential* nature of the process they estimate. See Evans and Honkapohja (2001), Ch.13.

[8]We acknowledge it would be possible to use $\frac{Q_{t-1}^{i}}{Q_{t-2}^{i+1}}$ as an argument for updating. In this case, agents would learn about the growth rate of an specific asset over time. We leave this case for future research.

[9]The assumption that agents use lagged information to update their beliefs is standard in the learning literature as simultaneous updating gives rise to a series of difficulties. For a discussion see Evans and Honkapohja (2001).

[10]We refer the reader to Section 4.B in the appendix for derivations for Case II.

with the PLM for this case into Eq. 4.3, we obtain:

$$Q_t^{(\infty,I)} = \delta\mu_c\gamma_\phi\phi_t + \delta\mu_c\hat{\beta}_t^\infty Q_t^{(\infty,I)},$$

where $\hat{\beta}_t^\infty$ is the belief agents hold about β^∞. Solving for Q_t^∞ yields:

$$Q_t^{(\infty,I)} = \frac{\delta\mu_c\gamma_\phi}{1-\delta\mu_c\hat{\beta}_t^\infty}\phi_t \tag{4.11}$$

From this result we can derive the actual behaviour of the consol bond growth rate, which is given by:

$$\frac{Q_t^{(\infty,I)}}{Q_{t-1}^{(\infty,I)}} = \frac{1-\delta\mu_c\hat{\beta}_{t-1}^\infty}{1-\delta\mu_c\hat{\beta}_t^\infty}\frac{\phi_t}{\phi_{t-1}} \tag{4.12}$$

Using 4.4 and collecting terms we obtain:

$$\frac{Q_t^{(\infty,I)}}{Q_{t-1}^{(\infty,I)}} = \left(\gamma_\phi + \frac{\gamma_\phi\delta\mu_c\Delta\hat{\beta}_t^\infty}{1-\delta\mu_c\hat{\beta}_t^\infty}\right)\varepsilon_t = T\left(\hat{\beta}_t^\infty, \Delta\hat{\beta}_t^\infty\right)\varepsilon_t \tag{4.13}$$

where:

$$T\left(\beta^\infty, \Delta\hat{\beta}^\infty\right) \equiv \gamma_\phi + \frac{\gamma_\phi\delta\mu_c\Delta\hat{\beta}_\infty}{1-\delta\mu_c\hat{\beta}^\infty} \tag{4.14}$$

is the T-mapping, which summarizes the actual behaviour of the infinite-maturity asset growth rate for given values of β and $\Delta\beta$. Therefore, the dynamics of consol bond prices are not only defined by the beliefs agents hold on the growth rate, $\hat{\beta}$, but also by the change in these beliefs. This generates *momentum* in the consol price dynamics, which is the key element explaining the low frequency "ups and downs" observed in the data.[11] This result is obtained because learning is *self-referential*. When a positive shock to beliefs occur, it increases the observed future growth rates, which in turn, pushes up future beliefs. Notice as well that the specified PLM allows for *reasonable* learning, since provided beliefs converge, consol bond prices will follow the behaviour implied by rational expectations.[12]

[11]See Adam et al. (2011). The limiting behaviour is associated with ordinary differential equation $\dot{\beta} = g_t\left(T(\beta)\varepsilon - \beta_t\right)$. Authors show the conditions for the support of ε needed to guarantee E-stability.

[12]See Appendix 4.A for derivations.

For the case of finite-maturity assets, we obtain from Eq. 4.2:

$$Q_t^{(i,I)} = \delta\mu_c\gamma_\phi\phi_t + \delta\mu_c\hat{\beta}_t^i Q_t^{(i-1,I)}, \ \forall_{i\in[2,\mathcal{T}]} \tag{4.15}$$

while the one period to maturity asset price will be given by:

$$Q_t^{1,I} = \delta\mu_c\gamma_\phi\left(\phi_t + \phi_t\bar{Q}^{0,I}\right) \tag{4.16}$$

Substituting the actual growth rates for each maturity, it is possible to express the actual growth rate as:[13]

$$\frac{Q_t^{(i,I)}}{Q_{t-1}^{(i,I)}} = \left(\gamma_\phi + \frac{\gamma_\phi\Delta\hat{\beta}_t^{(i-1)} + \gamma_\phi\hat{\beta}_t^{(i-1)}\Omega_t^{(i-1)}}{\gamma_\phi\frac{\phi_{t-1}}{Q_{t-1}^{(i-1,I)}} + \hat{\beta}_t^{(i-1)}}\right)\varepsilon_t, \ \forall_{i\in[2,\mathcal{T}]}, \tag{4.17}$$

where:

$$\Omega_t^{(i)} = \frac{\Delta\hat{\beta}_t^{(i-1)} + \hat{\beta}_t^{(i-1)}\Omega_t^{(i-1)}}{\gamma_\phi\frac{\phi_{t-1}}{Q_{t-1}^{(i-1,I)}} + \hat{\beta}_{t-1}^{(i-1)}}, \ \forall_{i\in[2,\mathcal{T}]}. \tag{4.18}$$

and:

$$\Omega_t^{(1)} = 0, \tag{4.19}$$

Equation 4.17 represents the actual dynamics followed by the price growth rate of each maturity. We notice several interesting features. First, learning stops being *self-referential*. In this case the shocks affecting the beliefs do not have a feedback effect. An increase in the estimated growth rate for the price of maturity i, (i.e.: $\Delta\hat{\beta}_t^{(i)} > 0$), will affect prices for maturities higher than i but, given the family of learning rules defined by 4.10, not maturity i. When agents believe the price of the i periods to maturity asset will increase over the next period, the price of the subsequent maturity $(i+1)$ would increase today. This affects the perceived dynamics of this maturity and consequently the price of the $i+2$ periods to maturity asset. In this sense, agents learn *through* the yield curve.

Second, we verify that these *perceived laws of motion* are well specified, as the

[13]See the appendix, section 4.B for derivations.

change in the bond price would converge under least squares learning to the predicted growth rate under rational expectations for all maturities.[14] [15] If $\Delta\hat{\beta}^{(i)} = 0, \ \forall_i$, then:

$$\hat{\beta}_t^i = \gamma_\phi, \ \forall_i \tag{4.20}$$

The third result is related in the way in which pricing errors in previous maturities affect new ones. As in the case of rational expectations, a shock in the coupon growth rate will increase the price of all maturities. However, due to the presence of price learning, an additional effect emerges as higher maturities carry changes in expectations from lower maturities. This effect will not necessarily push the price of bonds in the same direction as the shock, since the updating of beliefs occurs using lagged information. Given the non-linearities and interaction among different maturities, we analyse these dynamics in Section 4.4 through numerical simulations.

Finally, it is important to stress that when agents learn the growth rate for each maturity, the growth rates they observe will actually differ across maturities, even if their priors are the correct ones.[16]

4.3.2 Learning

In order to complete the characterization of bond price dynamics it is necessary to define how agents update their beliefs. The literature presents several alternatives. The most popular ones are: (a) *least-squares learning* (LSL) and (b) *constant-gain learning* (CGL). The former depicts agents who put the same weight on each observation. Therefore, it is a *decreasing-gain* learning mechanism. By contrast, under CGL, agents always put a higher weight on new observations relative to previous ones. As Sargent (1993) points out, the use of this type of learning may reflect agents' concerns with regime changes or a preference for adaptability.[17] In this section we consider the CGL case.

[14]Strictly, we need to define additional conditions for convergence and stability. In specific, we would have to make use of the *Projection Facility*, which imposes bounds on the values of the $\beta' s$. Some additional assumptions must be made on the support of ε. For a discussion see Adam et al. (2011).

[15]See Appendix 4.A. for derivations of the rational expectations equilibrium growth rates.

[16]This result is related to the *self-fulfilled* dynamics observed in other applications of learning models. For an example see Branch and Evans (2013).

[17]On the other side, the presence of constant-gain learning can give rise to unexpected dynamics. For discussion see Williams (2001). See Evans and Honkapohja (2001), Ch. 14 for a general discussion of the properties of models of constant-gain learning.

Branch and Evans (2006) find ample empirical evidence supporting the use of this type of learning when modelling agents expectations.[18] Now we define the beliefs updating equations. For the case of simple growth rates, as in Case I:

$$\hat{\beta}_t^{(i)} = \hat{\beta}_{t-1}^{(i)} + \alpha \left[\frac{Q_{t-1}^{(i)}}{Q_{t-2}^{(i)}} - \hat{\beta}_{t-1}^{(i)} \right], \ \forall_{i \in [1,\mathcal{T}], i=\infty} \tag{4.21}$$

where α is a constant and positive gain parameter.[19] When agents learn about the price real growth rate, as in Case II, beliefs follow:

$$\hat{\beta}_t^{(i)} = \hat{\beta}_{t-1}^{(i)} + \alpha \left[\frac{P_{t-2}}{P_{t-1}} \frac{Q_{t-1}^{(i)}}{Q_{t-2}^{(i)}} - \hat{\beta}_{t-1}^{(i)} \right], \ \forall_{i \in [1,\mathcal{T}], i=\infty} \tag{4.22}$$

4.4 Numerical Exercises

4.4.1 Baseline Calibration

Each model involves five parameters, reported in tables 4.1 and 4.2 for cases I and II. We set a value of 0.994 for the time-preference parameter, in order to match the United States ex-post 3-month treasury bill average real return rate for the 1969-2013 period. The growth rates of consumption and inflation are taken from the *Bureau of Economic Analysis* NIPA tables, for the years 1969 to 2013. In the stochastic coupon case we set the growth rate and standard deviation of coupons to match the behaviour of consumption. Finally, the gain parameter is set at 0.001 for the case of stochastic coupons and to 0.01 for the case of learning over the real growth rate. We set this parameter to avoid falling into the projection facility upper and lower bounds, for the simulated maturities. This means that all dynamics are generated by the learning process and not by additional constraints. We remind the reader that our emphasis is not into matching any empirical moments, but simply to analyse how maturity plays a role under the proposed learning mechanism.

[18] Even if we assume CGL, we consider PLMs that allow for convergence to the rational expectations' beliefs when LSL is followed.

[19] We make use of the standard timing assumption in order to avoid the joint determination of beliefs and observed prices. For a discussion see Evans and Honkapohja (2001), Ch. 3.

Table 4.1: **Baseline Calibration: Case I**

Parameter	*Value*	*Description*
δ	0.994	consumers time-preference parameter.
μ_c	0.993	inverse of consumption growth factor.
γ_ϕ	1/0.993	growth rate of coupons and principal.
σ_ϕ	0.005	standard deviation of coupon growth rate.
α	0.001	gain parameter (fixed).

Table 4.2: **Baseline Calibration: Case II**

Parameter	*Value*	*Description*
δ	0.994	consumers time-preference parameter.
μ_c	0.993	inverse of consumption growth factor.
γ_π	0.989	inverse of inflation factor.
σ_π	0.007	standard deviation of inflation factor.
α	0.01	gain parameter (fixed).

4.4.2 Results

We perform numerical simulations for both cases. First, we confirm our analytical results. As the maturity increases, the volatility of the price/coupon ratio under learning rises. Figure 4-1, shows the results for Case I, the model with stochastic coupon growth. We report the sample standard deviations for the learning case for both the finite maturity and consol bonds. We observe a pattern that increases with maturity, approximating the consol bond. Notice that under rational expectations the price/coupon ratio is constant.

The numerical results confirm the insights obtained from our analytical derivations. Learning affects volatilities in a heterogeneous way across maturities. There are two reasons for this: First, higher maturities carry on pricing errors from previous ones. Due to the fact that the one period to maturity bond is not subject to pricing errors, the number of terms to maturity limits the pricing error of a given asset. Second, as maturity increases, the importance of capital gains, relative to coupon income, rises. The fact that agents know perfectly how to price next period coupons, makes learning about prices more important for higher maturities. As maturity increases, this effect vanishes and the volatility of the asset starts converging to the once implied by the consol bond case.

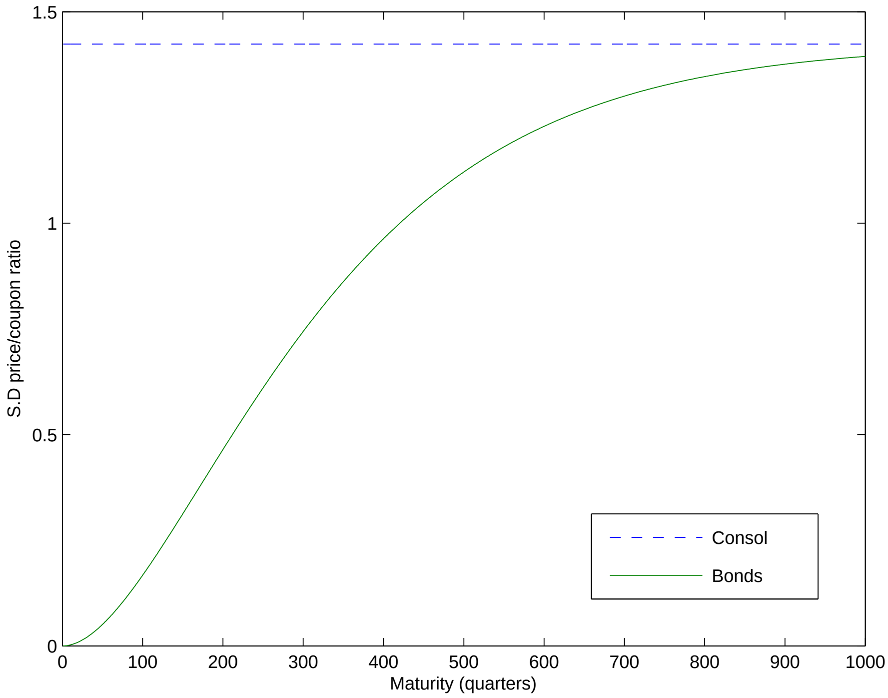
Figure 4-2 shows the evolution of beliefs for three particular maturities. As we can observe, the behaviour of beliefs for the highest maturity approximates the one of the infinite-lived asset.

For Case II we observe similar dynamics. This is not surprising since, from our analytical derivations, we know that the way in which changes in beliefs affect higher maturities is analogous to the case with an stochastic coupon growth rate. Because under rational expectations prices would be fixed, we report the price standard deviation for each maturity. Figure 4-3 shows the results. As we observe, simulated standard deviations of prices approximate the one of the consol bond as maturity increases. Beliefs show a similar pattern as the dynamics of the beliefs for the highest maturity approximate the ones of the consol bond in each simulation. Note that only at very high maturities is the volatility of bond prices of finite-maturity bonds similar to that of the consol bond. Learning the price of a one-period bond is simple since it is only involves learning about an exogenous variable. Learning about a two-period bond involves learning about an exogenous variable and the law of motion of a one-period bond. As the maturity increases the learning exercise involves more endogenous variables, but this only gradually leads to higher volatility.

Finally, we use these prices to calculate the yield to maturity. We report the simulated statistics for a selection of maturities. The pattern generated is similar to the one found in the data: yields decrease slowly through the term structure.[20] The volatility for the first maturity is zero since it is priced assuming rational expectations. Although the purpose of the exercises in this chapter is not to match the empirical data, the results suggests this learning mechanism can help the model capture the observed empirical dynamics.

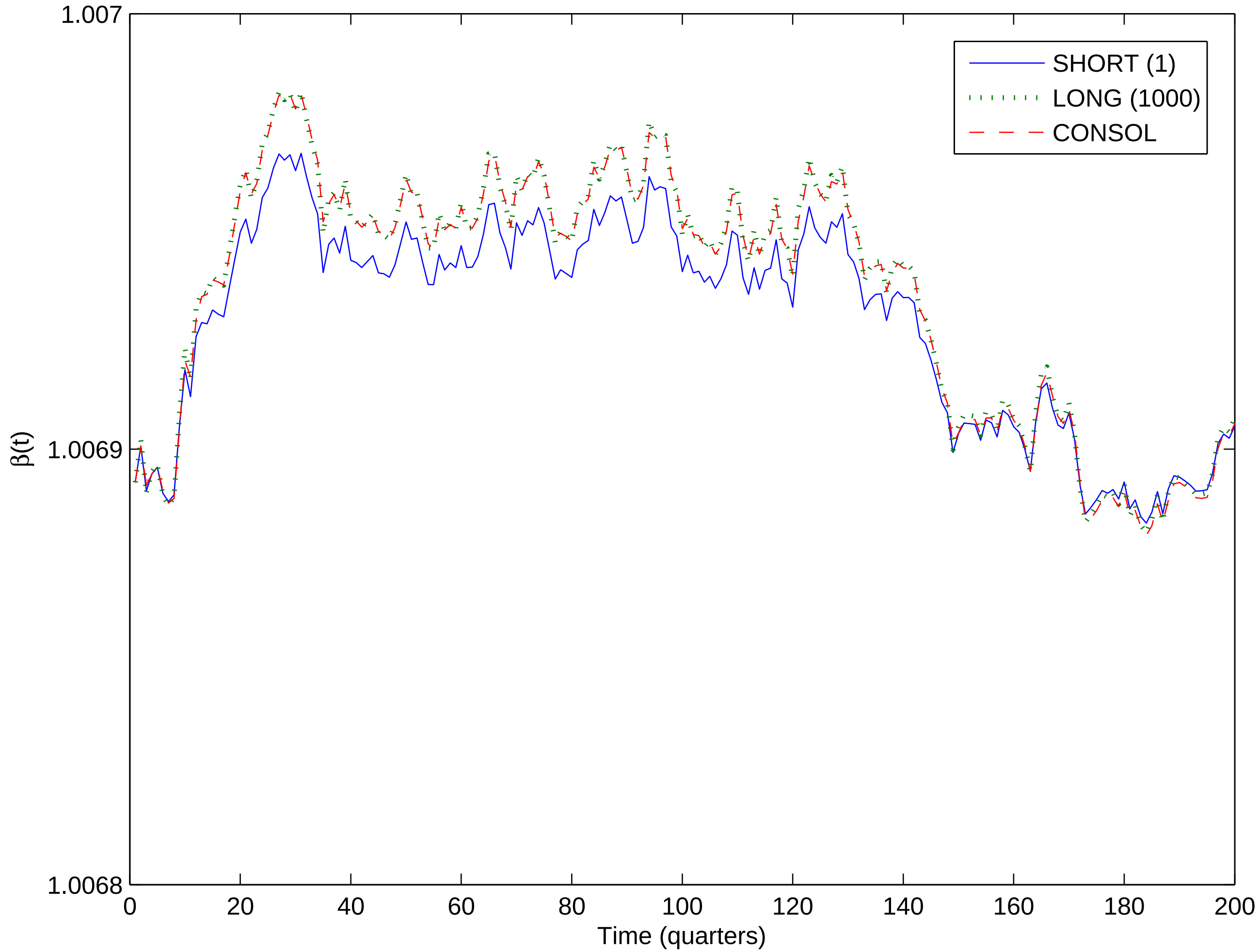
[20]For a discussion of this stylized fact observed in the behaviour of the yield curve, see Chapter 5, Section 5.3. It must be noticed that here yields correspond to the ones of non-zero coupon bonds. Nonetheless, volatility arises from the behaviour of expected capital gains, while coupons are fixed.

Figure 4-1: Case I: Std. Dev. of price/coupon ratio



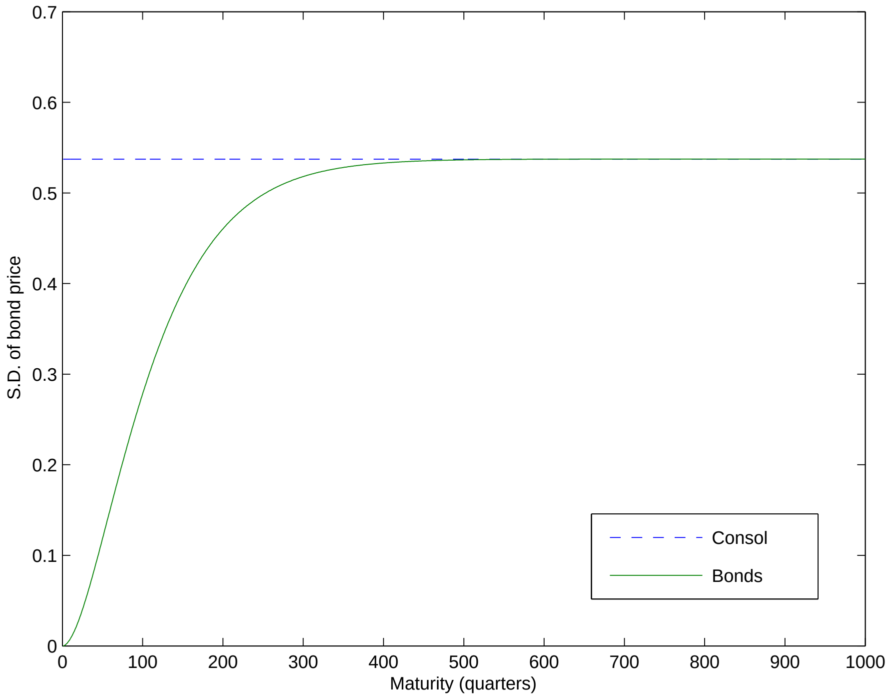
Note: Figure reports the standard deviation of the price/coupon ratio of the simulated prices under learning. Simulations follow the calibration in Table 4.1. Sample size in each draw is 1000. We report averages of 20 draws. Simulated values never hit the *projection facility*.

Figure 4-2: Case I: Evolution of beliefs



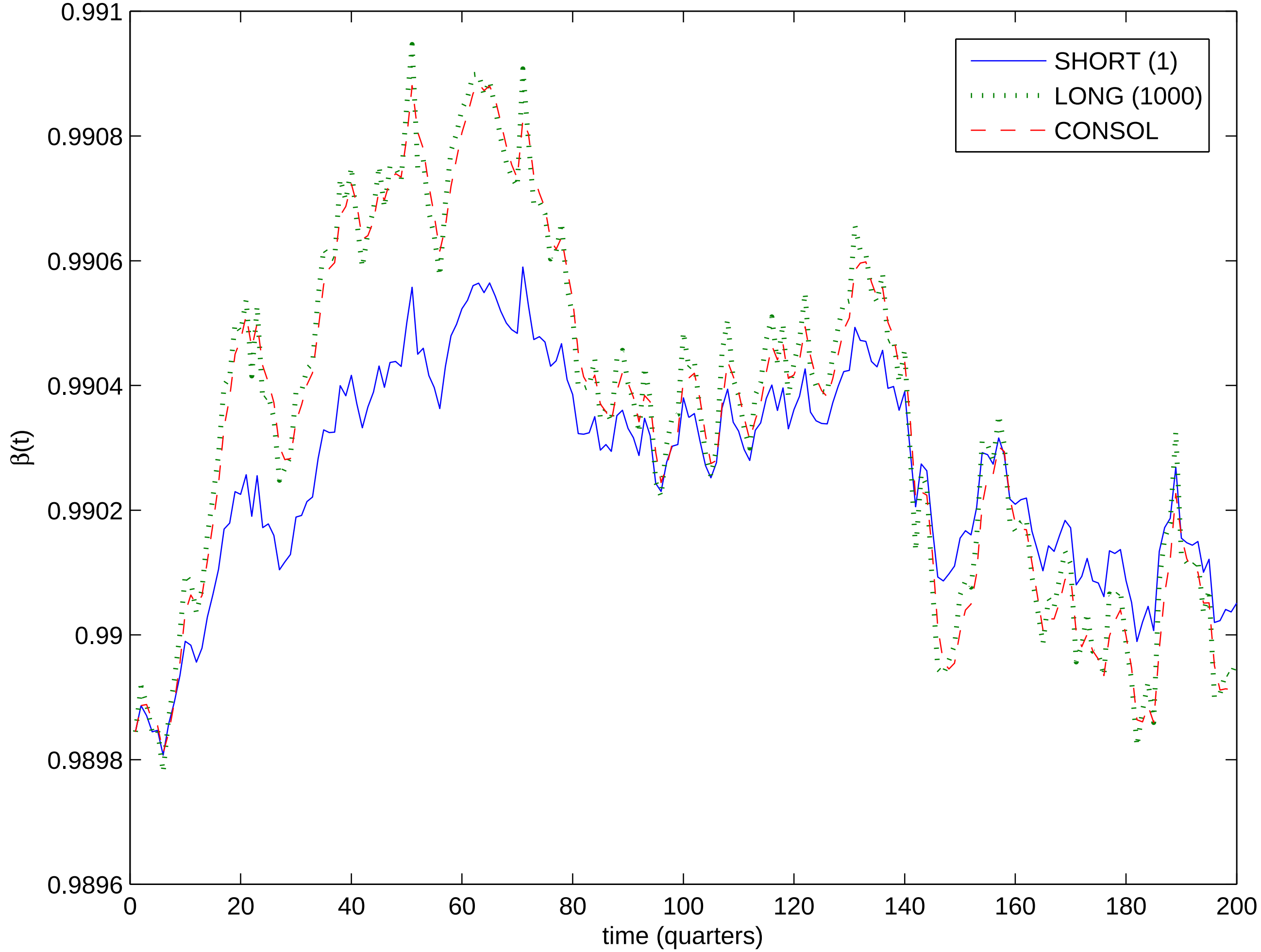
Note: Figure reports the beliefs of the price growth for each of the reported maturities under learning. Simulations follow the calibration in Table 4.1. Sample size of the draw is 1000. We report first 200 observations

Figure 4-3: Case II: Standard deviation of bond prices under learning



Note: Figure reports the standard deviation of the simulated prices under learning and rational expectations. Simulations follow the calibration in Table 4.1. Sample size in each draw is 1000. We report first 200 observations.

Figure 4-4: Case II: Evolution of beliefs



Note: Figure reports the beliefs of the price growth for each of the reported maturities under learning. Simulations follow the calibration in Table 4.1. Sample size of the draw is 1000. We report first 200 observations.

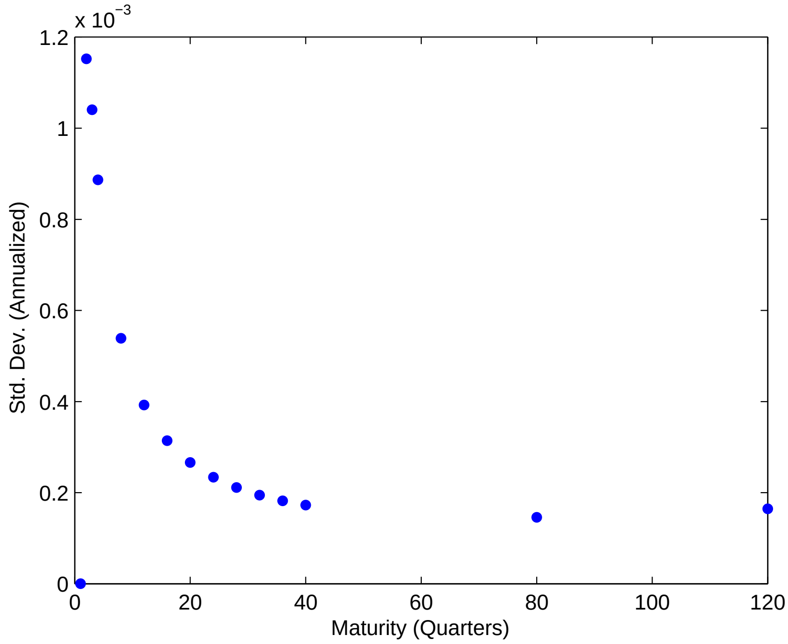
4.5 Conclusions

The present paper presents a model in which agents learn about the growth rate of bonds adaptively, following Adam et al. (2011). The proposed learning mechanism, when applied to finite-maturity assets, ceases to be *self-referential*, although, it generates interesting dynamics. First, changes in beliefs are carried over to higher maturities. This generates an amplification of shocks *through* the yield curve. Second, the impact of learning *through* the yield curve affects maturities in an heterogeneous way. Higher maturities exhibit a larger increase in their volatilities relative to the rational expectations results. Finally, the numerical results suggest that as maturity increases, the volatility of the asset converges to the one of the consol bond, which is subject to a *self-referential* learning mechanism as in Adam et al. (2011).

In addition, we present a model in which coupons are fixed. In this case the volatility will come from agents updating their beliefs about the real growth rate of bond prices. Even if under rational expectations the nominal price of these bonds would be fixed, the introduction of learning is capable of generating a slow decaying volatility pattern across maturities, similar to the one observed in the data. We confirm that the learning mechanism presented by Adam et al. (2011) for the case of equity, can help in the understanding of the behaviour of finite-maturity assets.

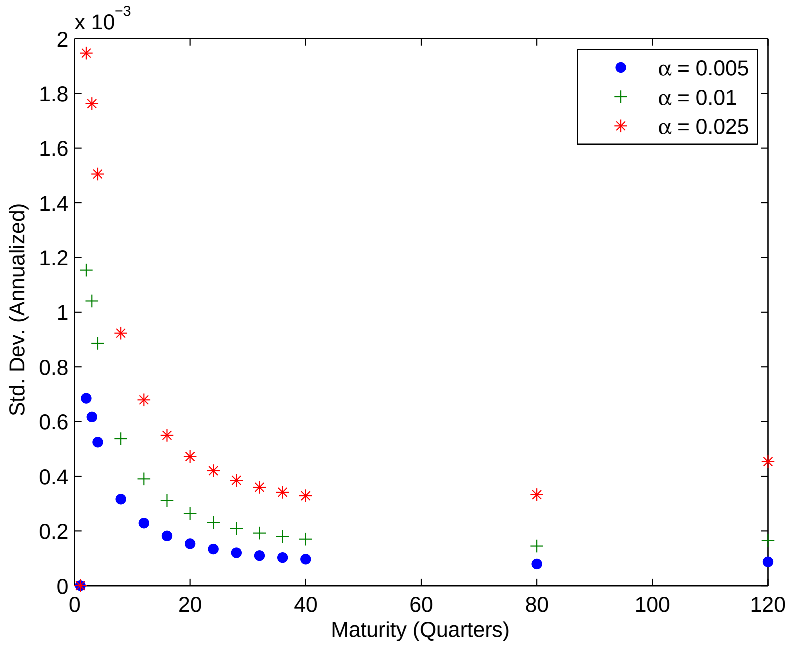
Even though these results are promising, we consider that the adaptive learning literature still has some open issues regarding its microfoundations. Although Adam and Marcet (2011) address this subject by proposing a model in which the PLM arises from a well-specified agent-based problem, their formulation still has problems in terms of determining the existence and uniqueness of equilibrium prices. This is one of the key topics that must be tackled by future research.

Figure 4-5: Case II: Yield volatility under learning



Note: Figure reports the average standard deviation of calculated yield to maturity for simulated coupon-paying bonds prices under learning. Simulations follow the calibration in Table 4.1. Sample size in each draw is 1000. We report averages of 20 draws.

Figure 4-6: Case II: Sensitivity to gain parameter



Note: Figure reports the average standard deviation of calculated yield to maturity for simulated coupon-paying bonds prices under learning. Simulations follow the calibration in Table 4.1 except for the value of α, which stands for the constant-gain parameter. Sample size in each draw is 1000. We report averages of 20 draws.

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4.A Rational expectations

Here we derive the rational expectations behaviour of assets for each case. For this we use equations 4.2 and 4.3 together with the assumptions for each case.

Case I: Stochastic coupon growth and learning about nominal bond price growth rates.

In this case, the euler equation for consol bonds (Eq. 4.3) becomes:

$$Q_t^{\{(\infty,I),RE\}} = \delta\mu_c E_t\left[\gamma_\phi\phi_t + Q_{t+1}^{\{(\infty,I),RE\}}\right] \tag{4.23}$$

solving forward, we obtain:[21]

$$Q_t^{\{(\infty,I),RE\}} = \frac{\delta\mu_c\gamma_\phi}{1-\delta\mu_c\gamma_\phi}\phi_t \tag{4.24}$$

The rational expectations growth rate is be given by:

$$\frac{Q_t^{\{(\infty,I),RE\}}}{Q_{t-1}^{\{(\infty,I),RE\}}} = \frac{\phi_t}{\phi_{t-1}} = \gamma_\phi\varepsilon_t \tag{4.25}$$

For finite-maturity assets:

$$Q_t^{\{(i,I),RE\}} = \left(\sum_{j=1}^{i}(\delta\mu_c\gamma_\phi)^j + (\delta\mu_c\gamma_\phi)^i\bar{Q}_0\right)\phi_t \tag{4.26}$$

The rational expectations growth rate is given by:

$$\frac{Q_t^{\{(i,I),RE\}}}{Q_{t-1}^{\{(i,I),RE\}}} = \frac{\phi_t}{\phi_{t-1}} = \gamma_\phi\varepsilon_t,\ \forall_{i\in[1,\mathcal{T}]} \tag{4.27}$$

Notice that under rational expectations the price of nominal bonds remains constant, thus, the covariance between bond nominal prices and the actual inflation rate will be zero, as the expected inflation is the same in every period.

[21] We assume parameter values guarantee that the price of the consol bond remains positive and finite.

Case II: Stochastic inflation and learning about bond price real growth rates

The euler equation for consol bonds (Eq. 4.3) becomes:

$$Q_t^{\{(\infty,II),RE\}} = \delta\mu_c\gamma_\pi\gamma_\phi + E_t\left[\bar{\phi} + Q_{t+1}^{\{(\infty,II),RE\}}\right] \tag{4.28}$$

solving forward we obtain:[22]

$$Q_t^{\{(\infty,II),RE\}} = \frac{\delta\mu_c\gamma_\pi\gamma_\phi}{1-\delta\mu_c\gamma_\pi\gamma_\phi}\bar{\phi} \tag{4.29}$$

The rational expectations real growth rate of bonds will be given by:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{\{(\infty,II),RE\}}}{Q_{t-1}^{\{(\infty,II),RE\}}} = \frac{P_{t-1}}{P_t} = \gamma_\pi\varepsilon_t^\pi \tag{4.30}$$

For finite-maturity assets:

$$Q_t^{\{(i,II),RE\}} = \left(\sum_{j=1}^{i}(\delta\mu_c\gamma_\pi\gamma_\phi)^j + (\delta\mu_c\gamma_\pi\gamma_\phi)^i\bar{Q}_0\right)\bar{\phi} \tag{4.31}$$

The rational expectations real growth rate is given by:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{\{(i,II),RE\}}}{Q_{t-1}^{\{(i,II),RE\}}} = \frac{P_{t-1}}{P_t} = \gamma_\pi\varepsilon_t^\pi,\ \forall_{i\in[1,\mathcal{T}]} \tag{4.32}$$

4.B Properties of proposed learning mechanisms

Here we study the properties of the proposed learning mechanisms.

Case I: Stochastic coupon growth and learning about nominal bond price growth rates.

We derive the observed behaviour of finite-maturity bonds. Starting from the one-period to maturity asset in Eq. 4.16:

$$\frac{Q_t^{1,I}}{Q_{t-1}^{1,I}} = \gamma_\phi\varepsilon_t \tag{4.33}$$

[22] As in the previous case, we assume parameter values are such that the price of the consol bond remains positive and finite.

now, for the two-period to maturity we have from Eq. 4.4 and our result for the one-period maturity in Eq. 4.15:

$$\frac{Q_t^{2,I}}{Q_{t-1}^{2,I}} = \frac{\gamma_\phi \phi_t + \hat{\beta}_t^1 Q_t^{1,I}}{\gamma_\phi \phi_{t-1} + \hat{\beta}_{t-1}^1 Q_{t-1}^{1,I}}$$

Substituting the process for the coupon in Eq. 4.4 and our result for the one-period to maturity asset in Eq. 4.33:

$$\begin{aligned}
\frac{Q_t^{2,I}}{Q_{t-1}^{2,I}} &= \frac{\gamma_\phi(\gamma_\phi \phi_{t-1}\varepsilon_t) + \hat{\beta}_t^1(\gamma_\phi Q_{t-1}^{1,I}\varepsilon_t)}{\gamma_\phi \phi_{t-1} + \hat{\beta}_{t-1}^1 Q_{t-1}^{1,I}} \\
&= \left(\gamma_\phi + \frac{\gamma_\phi \Delta\hat{\beta}_t^1}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{1,I}} + \hat{\beta}_{t-1}^1}\right)\varepsilon_t
\end{aligned} \tag{4.34}$$

which yields the actual law of motion for the growth rate of two-period to maturity asset. Now for the next maturity we have from Eq. 4.15:

$$\frac{Q_t^{3,I}}{Q_{t-1}^{3,I}} = \frac{\gamma_\phi \phi_t + \hat{\beta}_t^2 Q_t^{2,I}}{\gamma_\phi \phi_{t-1} + \hat{\beta}_{t-1}^2 Q_{t-1}^{2,I}}$$

Once again, from the assumed process for the coupon in Eq. 4.4 and our result for the two-period to maturity asset in Eq. 4.34:

$$\begin{aligned}
\frac{Q_t^{3,I}}{Q_{t-1}^{3,I}} &= \frac{\gamma_\phi(\gamma_\phi \phi_{t-1}\varepsilon_t) + \hat{\beta}_t^2 \left[\left(\gamma_\phi + \frac{\gamma_\phi \Delta\hat{\beta}_t^1}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{1,I}} + \hat{\beta}_{t-1}^1}\right)\right] Q_{t-1}^{2,I}\varepsilon_t}{\gamma_\phi \phi_{t-1} + \hat{\beta}_{t-1}^2 Q_{t-1}^{2,I}} \\
&= \left(\gamma_\phi + \frac{\gamma_\phi \Delta\hat{\beta}_t^2 + \gamma_\phi \hat{\beta}_t^2 \left[\left(\frac{\Delta\hat{\beta}_t^1}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{1,I}} + \hat{\beta}_{t-1}^1}\right)\right]}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{2,I}} + \hat{\beta}_{t-1}^2}\right)\varepsilon_t
\end{aligned} \tag{4.35}$$

which is the actual law of motion for the growth rate of the three-period to maturity asset. Following this process we arrive to:

$$\frac{Q_t^{(i,I)}}{Q_{t-1}^{(i,I)}} = \left(\gamma_\phi + \frac{\gamma_\phi \Delta\hat{\beta}_t^{(i-1)} + \gamma_\phi \hat{\beta}_t^{(i-1)} \Omega_t^{(i-1)}}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{(i-1,I)}} + \hat{\beta}_{t-1}^{(i-1)}}\right)\varepsilon_t, \quad \forall_{i \in [2,\mathcal{T}]}, \tag{4.36}$$

where:

$$\Omega_t^{(i)} = \frac{\Delta\hat{\beta}_t^{(i-1)} + \hat{\beta}_t^{(i-1)}\Omega_t^{(i-1)}}{\gamma_\phi \frac{\phi_{t-1}}{Q_{t-1}^{(i-1,I)}} + \hat{\beta}_{t-1}^{(i-1)}}, \quad \forall_{i\in[2,\mathcal{T}]}. \tag{4.37}$$

which are the equations 4.17 and 4.18 in the main text. Additionally:

$$\Omega_t^{(1)} = 0, \tag{4.38}$$

As in the case of Adam et al. (2011) an equilibrium of the model corresponds to the case of rational expectations.

Case II: Stochastic inflation and learning about bond price real growth rates

We start describing the consol bond equations. Substituting assumptions in 4.7, 4.8 and 4.9 into Eq. 4.3 we obtain:

$$Q_t^{(\infty,II)} = \delta\gamma_\pi\mu_c\gamma_\phi\bar{\phi} + \delta\mu_c\hat{\beta}_t Q_t^{(\infty,II)} \tag{4.39}$$

without loss of generality, we assume a flat structure for coupons and normalize their value to 1. Thus, substituting $\gamma_\phi = 1$ and $\bar{\phi} = 1$ into Eq. 4.39 yields:

$$Q_t^{(\infty,II)} = \delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t Q_t^{(\infty,II)} \tag{4.40}$$

solving for $Q_t^{(\infty,II)}$:

$$Q_t^{(\infty,II)} = \frac{\delta\gamma_\pi\mu_c}{1-\delta\mu_c\hat{\beta}_t} \tag{4.41}$$

thus the which is the *actual law of motion* for the consol bond price. We calculate the observed real growth rate:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{(\infty,II)}}{Q_{t-1}^{(\infty,II)}} = \left(1 + \frac{\delta\mu_c\Delta\beta_t}{1-\delta\mu_c\hat{\beta}_t}\right)\gamma_\pi\varepsilon_t^\pi = T(\beta,\Delta\beta)\varepsilon_t^\pi \tag{4.42}$$

where we have used the process for prices given by Eq. 4.8. The T-map shows properties similar to the ones in the previous case. A change in agents' beliefs about the real growth rate of prices will generate self-exciting dynamics.

Now for the case of finite maturity bonds, we use 4.2.

$$Q_t^{(i,II)} = \delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t^{(i-1)}Q_t^{(i-1,II)},\ \forall_{i\in[2,\mathcal{T}]} \tag{4.43}$$

which is the *actual law of motion* for the i-period to maturity bond. The behaviour of the actual real price growth is given by:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{(i,II)}}{Q_{t-1}^{(i,II)}} = \frac{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t^{(i-1)}Q_t^{(i-1,II)}}{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_{t-1}^{(i-1)}Q_{t-1}^{(i-1,II)}}\gamma_\pi\varepsilon_t^\pi \tag{4.44}$$

Now we start from the first maturity:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{(1,II)}}{Q_{t-1}^{(1,II)}} = \frac{\delta\gamma_\pi\mu_c + \delta\mu_c\bar{Q}^{(0,II)}}{\delta\gamma_\pi\mu_c + \delta\mu_c\bar{Q}^{(0,II)}}\gamma_\pi\varepsilon_t^\pi = \gamma_\pi\varepsilon_t^\pi \tag{4.45}$$

The growth rate for the price of the second maturity will be given by:

$$\frac{P_{t-1}}{P_t}\frac{Q_t^{(2,II)}}{Q_{t-1}^{(2,II)}} = \frac{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t^{(1)}Q_t^{(1,II)}}{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_{t-1}^{(1)}Q_{t-1}^{(1,II)}}\gamma_\pi\varepsilon_t^\pi \tag{4.46}$$

Replacing the result in 4.45:

$$\begin{aligned}
\frac{P_{t-1}}{P_t}\frac{Q_t^{(2,II)}}{Q_{t-1}^{(2,II)}} &= \frac{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t^{(1)}\left(\frac{P_t}{P_{t-1}}Q_{t-1}^{(1,II)}\gamma_\pi\varepsilon_t^\pi\right)}{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_{t-1}^{(1)}Q_{t-1}^{(1,II)}}\gamma_\pi\varepsilon_t^\pi & (4.47)\\
&= \frac{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_t^{(1)}Q_{t-1}^{(1,II)}}{\delta\gamma_\pi\mu_c + \delta\mu_c\hat{\beta}_{t-1}^{(1)}Q_{t-1}^{(1,II)}}\gamma_\pi\varepsilon_t^\pi & (4.48)\\
&= \left(1 + \frac{\Delta\hat{\beta}_t^{(1)}}{\frac{\gamma_\pi}{Q_{t-1}^{(1,II)}} + \hat{\beta}_{t-1}^{(1)}}\right)\gamma_\pi\varepsilon_t^\pi & (4.49)
\end{aligned}$$

which is similar to the one found for the previous case. Further substitutions leads to:

$$\frac{P_t}{P_{t-1}}\frac{Q_t^{(i,II)}}{Q_{t-1}^{(i,II)}} = \left(1 + \frac{\Delta\hat{\beta}_t^{(i-1)} + \hat{\beta}_t^{(i-1)}\Omega_t^{(i-1,II)}}{\frac{\gamma_\pi}{Q_{t-1}^{(i-1,II)}} + \hat{\beta}_{t-1}^{(i-1)}}\right)\gamma_\pi\varepsilon_t^\pi,\ \ \forall_{i\in[2,\mathcal{T}]}, \tag{4.50}$$

where:

$$\Omega_t^{(i,II)} = \frac{\Delta\hat{\beta}_t^{(i-1)} + \hat{\beta}_t^{(i-1)}\Omega_t^{(i-1,II)}}{\frac{\gamma_\pi}{Q_{t-1}^{(i-1,II)}} + \hat{\beta}_{t-1}^{(i-1)}}, \quad \forall_{i\in[2,\mathcal{T}]}. \tag{4.51}$$

Agents will carry over forecast errors from shorter maturities.

Chapter 5

Imperfect information, learning and the yield curve: a survey

5.1 Introduction

Many of the empirical facts related to the behaviour of the yield curve and long interest rates in the data are easier to understand once we take into account the processes through which agents acquire information and form their beliefs. Standard macroeconomic models face difficulties explaining stylized facts such as the volatility of long-term rates, the size and volatility of the risk premia, the sensitivity of the yield curve, and the return predictability puzzles.[1] As these models demand agents to have a deep knowledge about the economy, relaxing the information endowment of agents by limiting the information set available to them or picturing them as econometricians who learn the model and parameter values over time, constitute natural alternatives to rational expectations models with perfect information.

The use of learning mechanisms in macroeconomics and asset pricing has grown intensively over the last decades.[2] Sargent (1993) identifies five areas in which the application of bounded rationality can help economists improve their models and understanding of reality. These are: (1) equilibrium selection; (2) new sources of dynamics; (3) analyses of regime changes; (4) re-evaluation of rational expectations models; and (5)

[1]See Section 5.3 for a detailed discussion of these puzzles.

[2]See Evans and Honkapohja (2008) for a survey on the use of learning in macroeconomics.

new optimization and estimation methods. Most of the papers analysed in this survey are related to the second line of research, namely, analysing new dynamics. Regularly, these dynamics will occur out of the equilibrium as convergence to the rational expectations (RE) equilibrium is regarded as a positive result. In other models, the use of constant-gain learning, precludes the convergence to the RE equilibrium, though it can improve the model-predicted dynamics.

Asset pricing models with imperfect information also provide interesting implications for yield curve dynamics. For instance, under imperfect information, agents may confuse transitory shocks with persistent ones. This will increase the reaction of long interest rates to temporary shocks as agents update their long-term inflation beliefs. Imperfect common knowledge is also instrumental in the study of the connection between forecast errors and the volatility of bond returns.

The models reviewed in the present survey attempt to explain - often simultaneously - stylized facts observed in the yield curve that appear puzzling from the (full information) rational expectations perspective. The literature presents a handful of survey papers about learning and financial markets, however, none focus explicitly on the yield curve.[3] Our objective is to lay bare important theoretical mechanisms and highlight the connections and differences across these approaches.

The present chapter is organized as follows. Section 5.2 examines the mechanisms. In section 5.3 we review a series of stylized facts about the yield curve that represent a challenge for standard macroeconomic models. Section 5.4 presents models where imperfect information about a state variable drives the yield curve dynamics. Section 5.5 reviews models where parameter uncertainty and *self-referential* learning are able to generate new dynamics outside of the equilibrium. We conclude by presenting the main avenues in which the literature keeps developing and the challenges that lie ahead.

5.2 A review of the schemes

The imperfect information and learning literature comprises a vast array of schemes which differ, *inter alia*, in the information available to agents, the way it is updated

[3] For example, Pastor and Veronesi (2009) present a thorough survey of Bayesian learning models to explain a variety of facts found in finance.

and the "degree of rationality" of agents. Although limited information and learning are related concepts, the former refers to the case in which agents fully know the structure of the economy, but are unable to observe a subset of state variables directly. In the learning literature these assumptions are flipped, allowing agents to observe all realizations, but with incomplete knowledge about the structure of the economy.[4]

5.2.1 Learning

In learning models, agents are not endowed with knowledge about certain aspects of the economy but they are given the tools to learn about it. While agents learn, the perceived mapping from states to equilibrium values could change over time. In this literature, the concept of *self-referential* learning is a key one. It refers to the feedback that learning introduces into future realizations of endogenous variables. Alternatively, there are cases in which learning does not present a feedback effect. Here agents will learn the *data-generating process* (DGP) of the exogenous process. When agents can learn the DGP, learning adds little to the dynamics, as the true law of motion of the exogenous process and prices converge quickly to those of the rational expectations equilibrium.[5] The literature presents several alternatives to this setup that preclude agents to learn the true law of motion. Among them we find cases where agents use a misspecified PLM (e.g.: Evans (2010)) or do not use information efficiently (e.g.: Piazzesi and Schneider (2006)). In each case, learning is used with a specific purpose, and its effects vary according to the way the learning mechanism is introduced.

It is also important to distinguish between fully and boundedly rational learning.[6] Under rational learning, agents regard unknown parameters as random variables.

[4]Nimark (2010) uses the following example. Suppose we study a model that can represented by the following set of equations:

$$X_t = AX_{t-1} + Cu_t$$
$$Z_t = DX_t + v_t.$$

Models of imperfect information deal with agents that know the structure of the economy - i.e. the values for A, C, D and Σ_{vv}, the matrix of variances and covariance of the vector of shocks v_t. Nonetheless, agents are unable to directly observe the state variable X_t. In the learning literature agents can observe X_t, but do not completely know the structure of the economy - i.e. the do not know the matrices A, C, D and Σ_{vv}. Moreover, the learning literature consider models in which this system of equations represents the beliefs agents hold about how the variables in the model are related. For this reason, when agents hold a misspecified model, they might never learn the true form (or structure) of the economy.

[5]See Evans and Honkapohja (2001) for a discussion.

[6]We refer the reader to Cogley and Sargent (2008) for an extensive discussion of the different types of learning.

Namely, agents consider how their beliefs interact with the rest of the model and influence the generation of new observations. Agents are aware that the unknown parameters should be treated as (hidden) state variables, meaning their maximization problem is under a recursive learning path.[7] In words of Blume and Easley (1993), *"fully rational learning would require each trader to take into account the effect of his learning (...) on equilibrium prices."*

Boundedly rational learning deals with agents who learn about laws of motion as if they are econometricians who do not consider themselves to be part of the model economy. They estimate the parameters of the model using new data every period. As agents of the model, they treat these estimated beliefs as non-random when forming expectations and taking decisions. Consequently, these beliefs are used to form intertemporal optimal rules without considering that in the very next period these beliefs will vary once again. As Cogley and Sargent (2008) put it, *"their decisions reflect a pretence that this is the last time they will update their beliefs, a pretence that is falsified at the beginning of every subsequent period."*[8]

The cobweb model of Bray (1982) is a useful way to understand this difference. This is a simple model of self-referential learning where production decisions are made one period in advance. The market-clearing condition is given by:

$$p_t = a + bE_t p_{t+1} + u_t \tag{5.1}$$

where $E_t p_{t+1}$ is the price that market participants expect for period $t+1$, while u_t is distributed i.i.d with mean zero and variance σ_u^2. Bray (1982) assumes agents determine next period's price using the simple average of past prices. Thus, agents will set: $E(p_{t+1}) = \beta_t$, where:

$$\beta_t = \beta_{t-1} + \frac{1}{t}(p_{t-1} - \beta_{t-1}),$$

[7] Rational learning is closely linked to the concept of Bayesian learning, though, as Guidolin and Timmermann (2007), rationality imposes some restrictions over the prior beliefs on the probability distribution of the unknown parameter, which may not be present under Bayesian learning.

[8] This behaviour is related to the concept of 'anticipated utility' in Kreps (1998).

and the *actual law of motion* for prices will be given by:

$$p_t = a + b\beta_t + u_t, \tag{5.2}$$

The key difference between this specification, where agents learn in an adaptive manner, and one with fully rational agents is that here agents incorrectly believe that prices follow (5.2) for all subsequent time periods, when actually they should treat β_t as a hidden state variable, which dynamics affect the behaviour of observed prices. As Sargent (1993) indicates, agents should consider instead a dynamic system like:

$$\begin{aligned} p_t &= a + \frac{b}{t}p_{t-1} + \frac{b(t-1)}{t}\beta_{t-1} + u_t \\ \beta_t &= \frac{1}{t}p_{t-1} + \left(\frac{t-1}{t}\right)\beta_{t-1} \end{aligned}$$

when agents use this system for forecasting the value of p_{t+1}, they actually obtain the rational expectations forecast, which generates a lower mean-square forecasting error than the one obtained in the adaptive learning case. By solving forward (5.1), we obtain a set of expected inflation rates at different horizons. Combined with the assumption of a constant real interest rate, we would get a model of the yield curve, that is:

$$i_t^\tau = r + \frac{1}{\tau}(p^e_{t+\tau}/p_t)$$

Note that without a learning mechanism the expected price will be given by $\frac{a}{1-b}$ for all horizons, yielding a zero expected inflation rate and no volatility in the nominal yield curve. Bray (1982) shows that when parameter $b < 1$, the price will converge to the rational expectations value with probability one, making the rational expectations equilibrium a *learnable equilibrium*.[9]

5.2.2 Imperfect information

Imperfect information models present agents who know the exact form of the economy but are unable to directly observe a subset of state variables. Instead, agents receive noisy signals of these states and optimize extracting information through a filtering ex-

[9] The *learnability* of an equilibrium is used as a selection criterion in models that exhibit multiple equilibria. See McCallum (2009) for an application of learning in equilibrium selection.

ercise.

We present an example, based on Muth (1959), on how we can apply imperfect common knowledge to study the interest rates and the yield curve. We start from a simple Fisher equation:

$$i_t = \pi_t + \bar{r} \tag{5.3}$$

where i_t is the one-period nominal return on a bond, π_t stands for the one period inflation rate and $\bar{r}$ is the real return rate. We can regard the inflation and the real return rate as the fundamentals over which agents form beliefs. For the sake of simplicity, the real return rate is assumed fixed.[10] We characterize the dynamics of inflation as:

$$\pi_t = \pi_t^* + u_t \tag{5.4}$$

where u_t is distributed *i.i.d.* with mean zero and variance σ_u^2, and π_t^* is the long run level of inflation, which can be interpreted as the central bank's inflation target. Eq. (5.4) is the observation equation. We assume the following process for π_t^*:

$$\pi_t^* = \pi_{t-1}^* + \varepsilon_t \tag{5.5}$$

where ε_t is distributed *i.i.d.* with mean zero and variance σ_ε^2. Eq. (5.5) is the state equation. Thus, the inflation target is characterized by a random walk process with innovation variance σ_u^2. This assumption merits further discussion, as the presence of a non-stationary process would be sufficient to explain several volatility puzzles for the yield curve.[11] Most of the imperfect information models reviewed in the present survey introduce non-stationary processes in order to replicate the volatility observed by long term interest rates.[12] Despite this fact, imperfect common knowledge proves to be instrumental along other dimensions. As an example, here we show how imperfect common

[10]We will relax this assumption when we take the model to the data.

[11]As Flavin (1983) and Marsh and Merton (1986) state, variance bounds tests in LeRoy and Porter (1981), Shiller (1979) and Singleton (1980) should be reformulated when short rates are non-stationary.

[12]To be precise, the imperfect information models based in a Cox et al. (1985) production economy, such as the one proposed in Feldman (1989), assume a Ornstein-Uhlenbeck-process for production, which is mean-reverting and stationary. Here the assumption of a process with time-varying volatility helps the model replicate the observed pattern followed by interest rates. As we will see, imperfect information can improve the fit of the model in other dimensions.

knowledge can reconcile the high sensitivity of long interest rates to transitory shocks with the expectations hypothesis of the yield curve.[13]

In order to construct a yield curve, we first derive the expectations for inflation and interest rates for different horizons. Under full information we obtain:

$$\pi_{t+j/t}^{FI} = \pi_{t+j/t}^{*,FI} = \pi_t^*, \;\; \forall j \geq 1 \tag{5.6}$$

where $\pi_{t+j/t}^{FI}$ denotes the expectation of the inflation in period $t+j$ conditional on information in period t, under full information. Under the EH of the yield curve, we obtain the following expression for the conditional expectations on interest rates:

$$i_{t+j/t}^{FI} = \pi_t^* + \bar{r}, \;\; \forall j \geq 1. \tag{5.7}$$

Under limited information:

$$\begin{aligned} \pi_{t+j/t}^{LI} &= \pi_{t+j/t}^{*,LI} = \pi_{t/t}^{*,LI}, \;\; \forall j \geq 1 \\ i_{t+j/t}^{LI} &= \pi_{t/t}^{*,LI} + \bar{r}, \;\; \forall j \geq 1. \end{aligned} \tag{5.8}$$

In this case, agents face an inference problem. The application of the optimal Kalman filter algorithm yields the following updating process for the unobservable inflation target:

$$\pi_{t/t}^{*,LI} = \pi_{t-1/t-1}^{*,LI} + \kappa\left(\pi_t - \pi_{t/t-1}^{*,LI}\right) \tag{5.9}$$

where $\kappa = 1 - \phi$ and:

$$\phi = \phi\left(\frac{\sigma_\varepsilon^2}{\sigma_u^2}\right)$$

Eq. (5.9) states that agents will update their beliefs by using the difference between their prediction for period $t-1$ and the actual realization. The parameter κ is the steady state Kalman gain, which determines the degree of 'rational confusion'.[14] This is the extent to which changes in the inflation forecast are attributed to changes in the inflation target. As innovations to the underlying process become noisier relative to

[13]For a discussion of this stylized fact see Section 5.3.

[14]See Hamilton (1994), Ch. 13 for a detailed explanation of the Kalman filter.

Table 5.1: **Fisher equation model - maximum likelihood estimation**

	μ_y	κ_y	σ_y^2
r	1.3716	0.4366	1.5828
	-	(0.0648)	(0.0738)
π	0.8279	0.5627	1.4012
	-	(0.0680)	(0.0654)

Note: Yield data is taken from the CRSP Fama-Bliss discount bond files on forward rates for U.S. Treasury maturities. Inflation data is obtained from the *Bureau of Economic Analysis* NIPA tables. Quarterly data sample includes the period 1952:2 to 2010:4. Table contains estimates for the system in (5.10). Numbers in brackets indicate the maximum-likelihood asymptotic standard errors obtained from the Hessian matrix. Variances correspond to those of the forecast errors.

shocks to the observables, agents will attribute more of the variations in the observable variable to changes in the persistent component. As agents are unable to discern between permanent and transitory shocks, the latter will affect the prediction of the inflation target, generating an impact in the long term interest rates.

Now we relax the assumption of a fixed real interest rate. Let y_t be the 2-dimensional vector of the observed variables (real rate of return and inflation). They will be a function of the unobservable variables $\xi_t = [r_t^*, \pi_t^*]$, which form the state vector. The state space representation of the system defining y_t is given by:

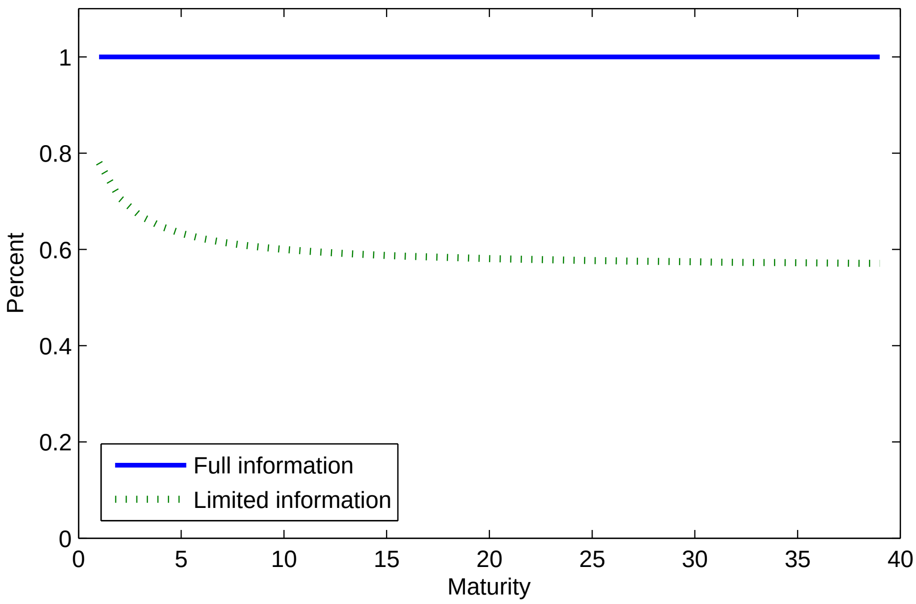
$$\begin{aligned} \xi_t &= \xi_{t-1} + \mathrm{e}_t \\ y_t &= \mu_y + \xi_t + \upsilon_t \end{aligned} \tag{5.10}$$

Where υ_t is a 2 by 1 vector of $i.i.d$ shocks with mean zero and variance-covariance matrix Q, and e_t is a 2 by 1 vector of $i.i.d$ shocks with mean zero and variance-covariance matrix R. We also assume that the disturbances are uncorrelated at all lags. We estimate the model via maximum likelihood. For simplicity, we use for the ex-post real interest rate the one-quarter yield minus inflation. For prices we take the index for personal consumption expenditures reported by the the Bureau of Economic Analysis' National Income and Product Account (NIPA) tables. Yield data is taken from the CRSP Fama-Bliss discount bond files, which provides data on forward rates for U.S. Treasury maturities up to 5 years. Our sample goes from 1952:2 to 2010:4. We use the sample mean and variance as initial values for the system.

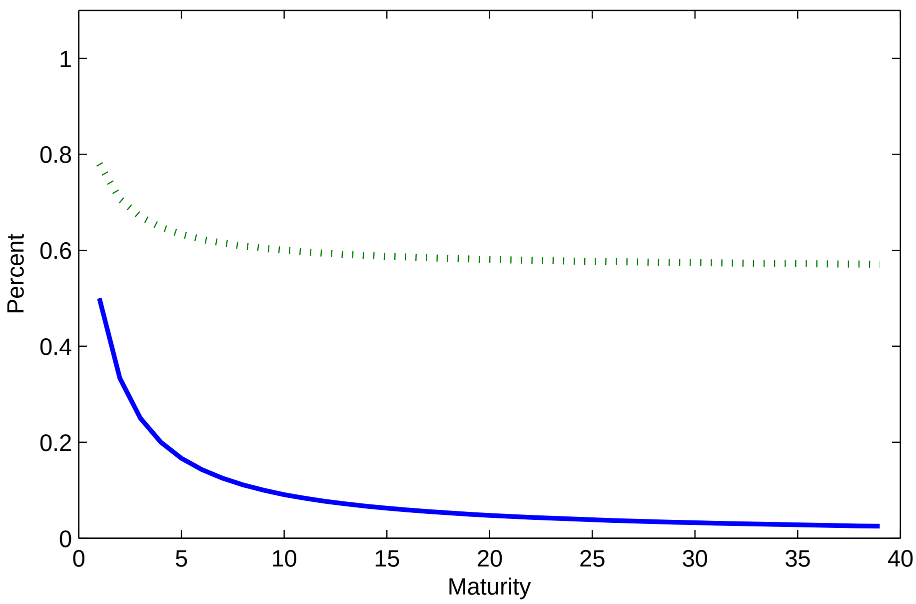
Table 5.1 reports the estimation results for both the Kalman gain parameters and

Figure 5-1: Yields reaction to transitory and permanent shocks to inflation

(a) Reaction to a permanent shock



(b) Reaction to a transitory shock



Note: Simulations correspond to a contemporaneous reaction of yields from two periods to maturity onwards to a 1 percent transitory ($\upsilon_{2,t}$) and 1 percent permanent ($e_{2,t}$) shock under full information and limited information. For the limited information case a value of 0.5627 for the gain parameter is used, following the results of the estimation reported in Table 5.1.

the variance of the forecast errors. For the case of inflation, we obtain a gain-parameter of 0.5627. In simple terms, when agents observe a change in the inflation rate, they attribute 56% of this variation to a shift in the inflation target. Figure 5-1 shows the impulse response functions of forward rates to a transitory ($\upsilon_{2,t}$) and a permanent shock ($\mathrm{e}_{2,t}$) to inflation in the full information and the imperfect common knowledge cases. Under full information, agents can observe changes to the inflation target, which have a one for one impact on forward rates, causing a parallel shift in the yield curve. Under imperfect common knowledge, agents are unable to discern between a transitory or a permanent shock. In this case they will only observe a change in the inflation rate and attribute a fraction of this variation to a permanent shock. The remainder will be attributed to a transitory shock. The yield curve will be affected in two ways. First, all future rates will shift as the inflation target belief is updated. Second, the change in the inflation rate attributed to a transitory shock will have a decreasing effect as the maturity increases, since this shock has no effect on future rates.

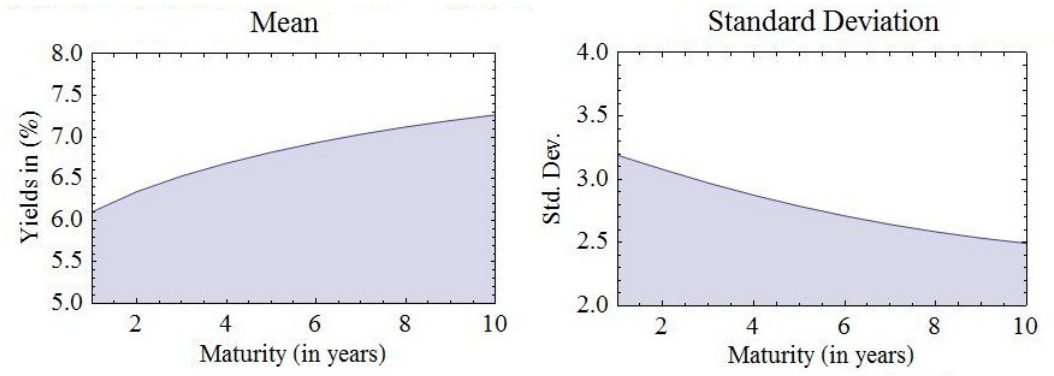
This simple example highlights how the introduction of imperfect common knowledge helps models of the yield curve to predict a high sensitivity of long rates to transitory shocks. As we will review, imperfect common knowledge generates as well other effects. For instance, under imperfect common knowledge, permanent shocks will be slowly reflected into agents expectations and, consequently, into the yield curve. Additionally, it has effects in the volatility of forecast errors. We will review these effects in more detail in section 5.4.

5.3 Stylized facts

This section presents some of the stylized facts of long-term US treasury bond rates and why these are hard to explain with a standard consumption-based asset-price model. We describe four empirical facts found in the data:

1. On average, the yield curve is upward sloping. Thus, yields increase with the maturity of the bond.
2. The standard deviation of yields decreases slightly with the maturity of the bond.
3. Future expected long yields fall when the yield spread between longer and shorter

Figure 5-2: U.S. nominal yield curve



Note: Data corresponds to the average nominal rates (monthly basis), obtained from the calculations for the US Yield curve in Gurkaynak et al. (2007).

maturities rises.

4. The long yields react significantly and frequently to current or "temporary" events.

These four facts are difficult to explain with standard consumption-based asset pricing models. The first two facts are studied by Backus et al. (1989). Den Haan (1995) extends the analysis to stochastic general equilibrium models. The third fact, closely related to the *predictability of returns*, is analysed by Campbell and Shiller (1991). Finally, the sensitivity to "news" is documented by Kuttner (2001) and Gurkaynak et al. (2003), among others.

5.3.1 Fact 1: upward-sloping yield curve

The term structure is upward-sloping on average. Figure 5-2 (left), shows the mean yield curve for the period 1972-2010 using monthly data. Even this widely known empirical fact poses a challenge for representative consumer asset pricing models. Den Haan (1995) studies the implications of these models for the yield curve. Here we present some of his derivations for exposition. Assume an endowment economy where agents maximize the utility extracted from consumption, given by:

$$\max_{\{c_t\}} E_t \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\phi}}{1-\phi}$$

subject to:

$$c_t + \sum_{t=1}^{\infty} q_t^k B_{t+1}^k = y_t + \sum_{t=1}^{\infty} q_t^k B_t^k, \forall_t$$

q_t^k stands for the price of a bond that pays one unit of consumption in k periods. B_t represents bond quantities. The endowment y_t follows an autoregressive process of the form:

$$\ln(y_{t+1}/y_t) = \ln(\bar{y}) + A(L)\varepsilon_t,$$

where ε_t follows a normal distribution with mean zero and variance $sigma_\varepsilon^2$. As Backus et al. (1989) show, the price of bonds will be a function of the endowment growth rate:

$$q_t^k = \beta^t E_t(y_{t+k}/y_t)^{-\phi}.$$

Following Den Haan (1995), we use the continuous time formula for the yield to maturity:

$$R_t^k = -\ln\beta - \frac{-\phi E_t \ln(y_{t+k}/y_t) + \phi^2 VAR(\ln(y_{t+k}/y_t))^{ue}/2}{k} \tag{5.11}$$

where the term $\ln(y_{t+k}/y_t)^{ue}$ stands for the prediction error conditional on period t information. Assuming homoskedasticity, this variance will not depend on t but can depend on the bond maturity k.

For the average term structure, we need to calculate the unconditional expectation of the interest rates in Eq. (5.11):

$$E(R_t^k) = -\ln\beta + -\phi\ln\bar{y} - \frac{\phi^2 VAR(\ln(y_{t+1}/y_t))^{ue}/2}{k} \tag{5.12}$$

an increase in k will cause both the numerator and denominator in Eq. (5.12) to increase. Thus, the slope of the yield curve will depend on which of the two terms increases faster. The result hinges on the process followed by the endowment. When consumption growth is positively autocorrelated, the persistence of shocks generates the variance of the prediction error to increase faster, yielding a negative slope. By contrast, a negatively autocorrelated process for the endowment generates a positive slope in the yield curve.

The intuition is related to the insurance effect that a long-term bond provides against shocks to the endowment. When the consumption is positively autocorrelated, a negative consumption shock depresses interest rates and increases the price of long-term bonds. In this sense, long-term bonds are desirable as a hedge against endowment shocks and command a negative term premium.

Moving from an endowment economy to a production one does not help resolving this issue. As Den Haan (1995) demonstrates, in the case of a production economy with *i.i.d.* shocks to productivity, the persistence of shocks increases through capital accumulation. This leads to a lower rate of decay of the volatility of interest rates against maturity, which helps explaining fact 2. However, the introduction of capital generates a downward-sloping yield curve, which stresses the difficulties of obtaining an upward-sloping yield curve in equilibrium.

5.3.2 Fact 2: 'excess' volatility

In standard consumption-based asset pricing models, long-term interest rates are calculated as averages of expected short-term rates. As a result, long-term rates are in fact conditional averages that should only change dramatically when important new information arrives. Shiller (1979) studied the relationship implied by the Expectations Hypothesis (EH) between the volatility of one period holding returns (derived from the long run rates) and the volatility of the short rates in a general class of expectations models. We follow the author's derivations, for the consol bond case. Let's define:

$$R_t = E_t R_t^* + \Phi, \tag{5.13}$$

where R_t is the return rate of a consol type bond, Φ is a constant liquidity premium and R_t^* represents the 'ex-post rational rate', defined as:

$$R_t^* = (1-\gamma)\sum_{i=0}^{\infty} \gamma^i r_{t+i}, \tag{5.14}$$

where r_t is the one period interest rate, and $\gamma = 1/(1+\bar{R})$. In this model, R_t^* is in fact a weighted moving average of the short-term interest rates, with geometrically declining weights.[15] Provided expectations are rational, we can obtain the forecast error from (5.13):

$$R_t^* - E_t(R_t^*) = R_t^* - R_t + \Phi_t \tag{5.15}$$

This term, under the expectations hypothesis, must be uncorrelated with the information known at time t. This is:

$$\begin{aligned} &E[(R_t^* + \Phi - R_t) \cdot R_{t-\tau}] = 0 \\ &E[(R_t^* + \Phi - R_t) \cdot r_{t-\tau}] = 0; \ \forall \, \tau \geq 1. \end{aligned} \tag{5.16}$$

As Shiller (1979) states, it is not possible to test for this relationship by directly regressing $R_t^* - R_t$ onto a constant and the lagged rates. As we can observe in (5.14), the residual errors of such regressions are serially correlated. For this reason, the author proposes using the one-period holding return $H_t^{(n)}$, given by:

$$H_t^{(n)} = \frac{P_{t+1}^{(n-1)} - P_t^{(n)} + C}{P_t^{(n)}} \tag{5.17}$$

where $P_t^{(n)}$ stands for the price of $n-$period bond at time t and C, the coupon payment ant the end of the period. The price is given by the present value of coupons, discounted by $R_t^{(n)}$, the yield to maturity of an n-period bond:

$$P_t^{(n)} = \frac{C}{R_t^{(n)}} + \frac{R_t^{(n)} - C}{R_t^{(n)} \left[1 + R_t^{(n)}\right]^n} \tag{5.18}$$

Replacing (5.17) into (5.18), it is possible to write the an expression for the one-period holding return in terms of the yield to maturity and the coupon:

$$H_t^{(n)} = \frac{C + \frac{C}{R_{t+1}^{(n-1)}} + \frac{R_{t+1}^{(n-1)} - C}{R_{t+1}^{(n-1)}\left[1 + R_{t+1}^{(n-1)}\right]^{n-1}}}{\frac{C}{R_t^{(n)}} + \frac{R_t^{(n)} - C}{R_t^{(n)}\left[1 + R_t^{(n)}\right]^n}} - 1. \tag{5.19}$$

[15]This is due the fact that the bonds studied by Shiller (1979) have non-zero coupons.

Finally we can relate the expected one-period holding returns to the short rate:

$$E_t[H_t^{(n)}] = r_t + \phi^{(n)}, \tag{5.20}$$

where $\phi^{(n)}$ is a constant. Equation (5.20) says that the expected one-period holding returns should be equal to the one period interest rate plus a constant term, reflecting a risk premium. Substituting directly (5.20) into (5.19) results in a non-linear expression relating $R^{(n)}$ and r. Given the author focuses on the linear relationships, a linear approximation of $H_t^{(n)}$ is obtained through a Taylor expansion, which yields:

$$\tilde{H}_t^{(n)} = \frac{R_t^{(n)} - \gamma_n R_{t+1}^{(n-1)}}{1 - \gamma_n} \tag{5.21}$$

where $\gamma_n = (\gamma(1 - \gamma^{(n-1)})/(1 - \gamma^n))$. The author shows that from (5.16) and the assumption that the processes for interest rates are stationary:

$$cov(\tilde{H}_t - r_t, R_t) = 0$$

Combining this result with (5.21), gives the following expression:

$$cov(R_{t+1}, R_t) = \frac{1}{\gamma} var(R_t) - \frac{1-\gamma}{\gamma} \rho_{rR} \sqrt{var(R_t)} \sqrt{var(r_t)},$$

where ρ_{rR} is the correlation coefficient between r_t and R_t. The next step is to replace the value of $var(\tilde{H}_t)$ into the last equation, then take the derivative of this expression with respect to $var(R_t)$ to maximize its value and set the left hand side expression equal to zero. The solution imposes a maximum bound on the variance of the excess period holding returns as a scalar multiple of the variance of the short-term rate. This maximum bound is given by:

$$V_{\tilde{H}} = \frac{var(r)\rho_{r,R}^2}{(1 - \gamma^2)}, \tag{5.22}$$

Shiller (1979) tests this upper bound against sample variances of the holding period returns on a set of corporate bonds. In most data sets the condition in (5.22) is violated, presenting evidence against models based on the EH. In a related approach, Singleton (1980) calculates the upper and lower bounds for the variance of long-term rates implied by a linear rational expectations model. The author defines:

$$y_t = \sum_{t=0}^{\infty} \beta_s' ({}_t x_{t+s}^e)$$

where ${}_t x_{t+s}^e = E_t(x_{t+s} \mid \Phi_t)$, $\beta_s' = (\beta_1^s, ..., \beta_K^s)$. When $K = 1$, the system reduces to (5.14), where $y_t = R_t^n$ and $x_t = r_t$. Now let's define y_t^* to be:

$$y_t^* = \sum_{t=0}^{\infty} \beta_s' x_{t+s},$$

and $\hat{y}_t$ as:

$$\hat{y}_t = \sum_{t=0}^{\infty} \beta_s' ({}_t \hat{x}_{t+s}),$$

where ${}_t\hat{x}_{t+s} = E_t(x_{t+s} \mid x_t, x_{t-1}, ...)$. LeRoy and Porter (1981) show the following result when $K = 1$:

$$var(y_t^*) > var(y_t) \geq var(\hat{y}_t). \tag{5.23}$$

The first part of the of the inequality in (5.23) is explained as follows. Let $\delta_t = \sum_{t=0}^{\infty} \beta_s'(x_{t+s} -_t x_{t+s}^e)$. Then, by construction,

$$y_t^* = y_t + \delta_t.$$

Since δ_t represents innovations happening 'after' period-t, both y_t and δ_t are contemporaneously uncorrelated which means:

$$var(y_t^*) = var(y_t) + var(\delta_t).$$

The second inequality in (5.23) is derived from the fact that the covariances of $\hat{y}_t$ and the forecast error $(x_{t+s} -_t \hat{x}_{t+s})$ are zero, so:

$$var(y_t^*) = var(\hat{y}_t) + var(\xi_t);$$

where $\xi_t = \sum_{t=0}^{\infty} \beta'_s (x_{t+s} -_t \hat{x}_{t+s})$, and $var(\xi_t) \geq var(\delta_t)$. Singleton (1980) estimates the variance of these three indicators. The results show that the first inequality is consistently violated. The author also confirms the results of Shiller (1979) regarding the excessive volatility of the expected holding period returns.

It is important to mention that the tests presented were criticized for the implied assumption of stationarity of financial data, as we will discuss later, results for tests of stationarity in interest rates and inflation rates remain inconclusive.[16]

5.3.3 Fact 3: predictability of returns

The Expectations Hypothesis of the term structure implies that the spread between long and short rates forecasts: (1) the change in yield of the long-term bond over the life of the shorter term bond and (2) a weighted average of the changes in shorter terms over the life of the longer term bond. Campbell and Shiller (1991) found that the first relationship is not present in the data. Focusing on the case of pure discount bonds, we have:

$$R_t^{(n)} = (1/k) \sum_{i=0}^{k-1} E_t(R_{t+mi}^{(m)}) + c$$

where $k = n/m$. This simply states that the long rate is a constant plus the weighted sum of expected future shorter term interest rates. This equation holds true when we assume that expected continuously compounded yields to maturity on all discount bonds are equal, up to a constant - the approach taken by Fama (1984) - or if we take it as a linear approximation to a nonlinear expectations theory of the term structure.

Let's now define the spread between the n and m-period rate as:

$$S_t^{(n,m)} = R_t^{(n)} - R_t^{(m)}.$$

[16] Results vary depending on the country, period of analysis and methodology. Cox et al. (1985) concluded that short term interest rates follow a stationary process, while Campbell and Shiller (1987) use a non-stationary process to characterize interest rates. Newbold et al. (2001) find evidence for non-stationary nominal interest rates using US and UK data. Cerrato et al. (2013) find support for the hypothesis of (break) stationary nominal interest rates for the US and Canada. King et al. (1991) fail to reject the null hypothesis of a unit root in the nominal interest rate. Regarding real interest rates, Rose (1988) finds evidence of the presence of a unit root in short-term nominal interest rates using date for 18 countries in the Organisation for Economic Co-operation and Development (OECD). Others, such as Rapach and Wohar (2005) and Bai and Perron (2003), contest this result finding support for a regime-wise stationary process. For a detailed discussion, see Neely and Rapach (2008).

The spread is a measure of the slope of the yield curve. The EH implies that the spread is a forecast of changes in future interest rates (plus a constant risk premium.) Therefore, the exercise consists of evaluating whether the spread resembles an optimal forecast of the changes in interest rates. According to the EH:

$$\frac{m}{n-m} S_t^{(n,m)} = E_t R_{t+m}^{(n-m)} - R_t^{(n)} \tag{5.24}$$

Eq. (5.24) simply states that the spread between two rates at different maturities (with m being the shorter maturity and n the longer one) should be equivalent to the difference between the expected rate of the period comprised from m to n and the long rate. In such a manner both alternatives become equivalent for the investor.

Campbell and Shiller (1991) test this hypothesis by regressing $R_{t+m}^{n-m} - R_t^n$ onto a constant and its predicted value $s_t^{n,m}$. Theoretically, the slope coefficient should be one, representing a 'rational' prediction error. Their results show that, for the sample considered, the term structure between almost any two maturities (m and n, where $n > m$) gives the wrong direction of the forecast. The authors consider this finding a *resounding failure of the expectations hypothesis of the term structure.*

5.3.4 Fact 4: 'excess' sensitivity

"While I was at the Fed, I asked the staff to use daily data to compute the correlation between changes in the current one-year interest rate and changes in the implied one-year forward rate 29 years in the future. Using 1994 as an example, the answer was 0.54! (...) you have to be a pretty devout believer in efficient markets to claim that the daily flow of news really has that much durable significance."
Blinder (1999)

The phenomenon dubbed 'excess sensitivity' of long-term bonds refers to the fact that, on average, long-term bond rates react positively and significantly to current events. In reference to monetary policy innovations, Cook and Hahn (1989) built a database of monetary policy announcements by the FED during the period of 1974 to 1979. They found a positive and significant response at all maturities, but smaller at the long end of the yield curve. Later on, Roley and Sellon (1995) followed the same approach, but found a statistical insignificant response of bond rates to changes in the target policy rate.

Kuttner (2001) claimed that a possible explanation for the negative results was an improved capacity of agents to predict monetary policy actions, making these changes an imperfect proxy for the 'surprise' component of monetary policy. The author tests this hypothesis by first deriving a 'surprise' type component of monetary policy using information from the futures markets, and finds a significant response of yields to this component (from 50.4 at the 3 month maturity to 19.4 at the 30 year one). These results show that the response of the yield curve over the month prior to the FED's decision is significant and larger for both the anticipated and unanticipated component. These responses are larger than those predicted by the EH, since reactions to surprises vanish 'slowly' through the yield curve. Here the hypothesis, also followed by Rudebusch (1998), is that the high persistence of FED's reaction generates movements in the short future rates - because they encompass changes in short-term interest rates in the near future. However not much is said about the long end of the curve, where unexpected shocks have effects of 51.9 basis points to 1 percent changes in the short rate.

Gurkaynak et al. (2003) present evidence on the same type of highly sensitive reaction of long run rates to news about inflation and output. In models where it is assumed that the long-run characteristics of the economy, such as the level of inflation and real interest rates, are constant over time and perfectly known by all economic agents, expectations of far enough rates in the future should remain relatively fixed. The authors present linearized versions of two Neo-Keynesian models. Both can be represented by:

$$\pi_t = \mu E_t \pi_{t+1} + (1-\mu) A_\pi(L)\pi_t + \gamma y_t + \varepsilon_t^\pi$$
$$y_t = \mu E_t y_{t+1} + (1-\mu) A_y(L) y_t - \beta(i_t - E_t \pi_{t+1}) + \varepsilon_t^y$$

where π denotes the inflation rate, y the output gap, i the short-term nominal interest rate, and ε^π and ε^y are both *i.i.d.* shocks. When $\mu = 1$ we have the pure New Keynesian model of Clarida et al. (2000). However, a value of $\mu = 0.3$ has been advocated by Fuhrer (1997), Rudebusch (2001) and Estrella and Fuhrer (2002), to match the degree of persistence in inflation present in U.S. data. They use parameter values of Rudebusch

(2001) for this second model. Both models are closed by a Taylor-type interest rate rule:

$$i_t = (1-c)\left[\bar{\pi}_t + a(\bar{\pi}_t - \pi_t^*) + by_t\right] + ci_{t-1} + \varepsilon_t^i$$

where $\bar{\pi}$ represents the trailing four-quarter moving average of inflation, π^* stands for the constant inflation target and ε^i is an i.i.d. shock. The authors use this setup to simulate impulse response functions of the interest rate (short term) to 1 percent inflation, output and interest rate shocks. In the pure forward-looking case of Clarida et al. (2000), shocks disappear after four quarters. However, when the persistence parameter is calibrated following Rudebusch (2001), these shocks persist for up to ten years in the simulations. These results suggest that in order to obtain a persistent reaction to innovations, agents must update their beliefs in an adaptive way.

The authors regress the daily changes in forward rates on the surprise component of macroeconomic data releases and monetary policy announcements - using a technique similar to the one in Kuttner (2001). A proxy for expectations is constructed by taking the median market forecast obtained from *Money Market Services*. They find significance at the 5-percent level for the impact of the 'surprise' components of 8 indicators (of a total of 13 studied) on short-term rates up to 10 years after the shock occurs. The surprises were also consistent with the Taylor rule - in other words, when procyclical variables had a positive surprise, short interest rates increased and the opposite in the case of countercyclical variables. Moreover, consistent with Campbell and Shiller (1991) results, short end rates reacted positively to monetary policy surprises - in line with a perceived persistent federal funds rate - while long run rates reacted in the opposite direction.

5.4 Imperfect information and the yield curve

This section discusses models in which agents know the true form of the economy, but are unable to observe the behaviour of a subset of the model state variables. A key difference with the models in the learning literature is that even if these variables remain unobservable, agents are endowed with full knowledge of the economy, this is, they are aware of the law of motion followed by the unobservable variables.

The review in Section 5.3, shows how standard macroeconomic models face a series of problems trying to replicate the observed yield curve dynamics. One of these key stylized facts relates to the volatility of long term interest rate. One of the approaches to reconcile this particular feature of the yield curve with the EH is to directly introduce non-stationary processes into the model. Fuhrer (1997) explains how the introduction of shifts across different monetary policy regimes can reconcile the EH with the pattern observed by long-term interest rates. Kozicki and Tinsley (2001) follow this idea by proposing a model in which agents do not observe the central bank's inflation target, which is subject to a series of regime shifts. Agents ignore both the timing and size of these shifts, which leads to an inference problem. It is important to notice that in the case of an underlying process subject to structural breaks, imperfect information plays a secondary role explaining the dynamics of the yield curve, since as we mentioned, it is sufficient to introduce a non-stationary process for one of the determinants of long-interest rates to obtain the desired volatility levels. However, as the authors state, the data of long term inflation rate expectations obtained from the Survey of Professional Forecasters (SPF) is at odds with the assumption of discrete and perfectly observed regime shifts. In the case that the policy reaction function or the inflation target of the FED were subject to perfectly observable discrete changes, these would be immediately reflected into agents' expectations, and consequently, into the yield curve. As Kozicki and Tinsley (2001) state: *"First, long-horizon expectations are not subject to the dramatic fluctuations in monthly inflation rates. Second, the downward path of inflation expectations in the 1980s appears to lag considerably the tend movements in historical inflation."* For these reasons, introducing imperfect common knowledge helps this model as it allows for a smoother behaviour of inflation expectations.

The absence of full rationality in the model with structural breaks of Kozicki and Tinsley (2001) is made evident by the model of Timmermann (2001).[17] As Sargent (1993) points out, models that calculate rational expectations equilibria under different regimes and then analyse the changes in the behaviour of agents between one regime and the next fall into 'a bit of a contradiction' since agents who are unable to factor in the probability

[17]Timmermann (2001) follows a non-recurrent regime-switching approach to explain the volatility of equity prices. A particular feature of this model is the presence of a 'meta-distribution' for the mean growth rate of the dividend process. When a new regime arrives the mean growth rate of the dividend process is drawn from this meta-distribution. Although agents infer its value from observed realizations of dividend prices, the probability of abandoning the regime is factored in when pricing assets.

of a regime change are in essence exhibiting boundedly rational behaviour. Kozicki and Tinsley (2001) presented an alternative model, in which the underlying process is not subject to low-frequency structural breaks. Instead, they assume a unit root process for the unobservable inflation target. Once again, the presence of imperfect information will not be key to explain the volatility of long-term interest rates, but will play the role explaining the pattern followed by long-run inflation expectations. Dewachter and Lyrio (2006) follow this idea by introducing imperfect information into an affine-factors term structure macro-finance model linking macroeconomic dynamics to the yield curve.[18] Similar to Kozicki and Tinsley (2001), agents hold subjective expectations for the time-varying endpoints of exogenous variables. As these are non-stationary processes, the system is described in terms of a Vector of Error Corrections Model (VECM). Dewachter et al. (2011) extend this model by relaxing the absence of arbitrage opportunities and the use of endogenous constant prices of risk. Additionally, they allow for differences between the beliefs of private agents and those of the central bank. As the use of survey data is common in this literature, modelling agents who update their long-run expectations on key macroeconomic variables gradually improves the fit of these models to the data.

Imperfect common knowledge does play a key role helping models explaining the high-sensitivity of long term rates to transitory shocks. As discussed in the previous section, Gurkaynak et al. (2003) present evidence for a significant and persistent reaction of long term rates to macroeconomic news. The authors elaborate a potential explanation for this feature, based in imperfect common knowledge. In their setup, agents are unable to tell when changes to the policy interest rate rule are a product of transitory or permanent shocks - i.e., changes in the inflation target. When the central bank changes its policy rate, agents update their estimate of the (unobservable) inflation target by means of the Kalman filter. Thus, transitory monetary policy shocks affect agents expectations of the central bank's inflation target. Although this mechanism generates a shift in the long term rates, as we saw in section 5.2, it is not enough to obtain enough persistence. For this reason the authors introduce a feedback effect from recent values of inflation into the inflation target. In this manner, transitory shocks to the economy are capable of generating persistent shifts in long term rates.[19]

[18]See Bekaert et al. (2010), Cogley (2005) and Dewachter and Lyrio (2006) for other examples of macro-finance models of the term structure with unobservable variables and time-varying beliefs.

[19]Gurkaynak et al. (2003) motivate this assumption by depicting a central bank which finds it easier

Beechey (2004) develops a similar model, but provides analytical proofs regarding how the introduction of imperfect common knowledge affects the dynamics of the yield curve. As the author shows, besides affecting the response of long term rates to transitory shocks, imperfect information affects long term rates along other dimensions. First, the variance of forecast errors increases. Second, measures for bond return volatility are heightened. Lastly, the conditional covariance between short and long rates increases.

In order to explore these findings, we reproduce some of the author's results with the simple model presented in Section 5.2. Under full information, we can obtain the inflation forecast error by combining (5.4), (5.5) and (5.6):

$$\pi_{t+1} - \pi^{FI}_{t+1/t} = \pi^*_{t+1} + u_t - \pi^*_t = \varepsilon_{t+1} + u_{t+1}. \tag{5.25}$$

This means, from (5.3):

$$i_{t+1} - i^{FI}_{t+1/t} = \varepsilon_{t+1} + u_{t+1}. \tag{5.26}$$

The one-period-ahead interest rate forecast variance is given by:

$$var(i_{t+1} - i^{FI}_{t+1/t}) = \sigma_\varepsilon + \sigma_u. \tag{5.27}$$

Now, let's consider the case under limited information. Here the inflation forecast error is given by:

$$\pi_{t+1} - \pi^{*,LI}_{t+1/t} = \varepsilon_{t+1} + (\pi^*_t - \pi^{*,LI}_{t/t}) + u_{t+1}, \tag{5.28}$$

and:

$$i_{t+1} - i^{LI}_{t+1/t} = \varepsilon_{t+1} + (\pi^*_t - \pi^{*,LI}_{t/t}) + u_{t+1}. \tag{5.29}$$

as we can see, now the inflation target forecast error term $(\pi^*_t - \pi^{LI,*}_{t/t})$ enters the forecast error of interest rates adding noise. We can obtain the value of this expression by

to change the inflation target after the actual inflation rate has deviated from it over the alternative of pushing the inflation rate back to the target.

recursive substitution:

$$i_{t+1} - i^{LI}_{t+1/t} = \varepsilon_{t+1} - u_t + \phi k_t, \tag{5.30}$$

where $\phi k_t = \pi_t - \pi^*_{t/t}$, $k_t - \phi k_{t-1} = \varepsilon + u_t - u_{t-1}$ and:

$$\phi = \sqrt{\frac{1}{1 + \frac{\sigma^2_\varepsilon}{\sigma^2_u}}} \tag{5.31}$$

Notice that $1 - \phi$ is the steady state Kalman gain, which has a value between 0 and 1. Substituting recursively, we obtain:

$$i_{t+1} - i^{LI}_{t+1/t} = \sum_{\tau=0}^{\infty} \phi_\tau \varepsilon_{t+1-\tau} - (1 - \phi) \sum_{\tau=0}^{\infty} \phi^\tau u_{t-\tau} \tag{5.32}$$

From which the variance is calculated:

$$var(i_{t+1} - i^{LI}_{t+1/t}) = \sigma^2_\varepsilon \left(\frac{1}{1 - \phi^2} \right) + \sigma^2_u \left(\frac{(1 - \phi)^2}{1 - \phi^2} \right). \tag{5.33}$$

For $\frac{1}{1-\phi^2} > 1$ and $\frac{(1-\phi)^2}{1-\phi^2} > 0$, we obtain a higher volatility for the prediction error. It is also possible to confirm the results regarding the sensitivity of interest rates explored in Section 5.2. A yield curve obtained following the EH:

$$i^\tau_t = \frac{1}{\tau} \sum_{j=0}^{\tau-1} i_{t+j/t}. \tag{5.34}$$

where $i_{t+j/t}$ is the expected one period interest rate j periods ahead with the information available at period t. In the case of full information, from (5.7):

$$i^{\tau,FI}_t = \bar{r} + \pi^*_t + \frac{1}{\tau} u_t. \tag{5.35}$$

where $i^{\tau,FI}_t$ is the nominal yield of the τ periods to maturity bond, under full information. As we see, given the process we have assumed for the inflation target, the variance of interest rate is unbounded as t goes to infinity. Thus, the introduction of imperfect information is not central for explaining the puzzles related to the volatility of long-term rates in these models.[20] We can calculate the variation of yields over time, by differencing

[20] Although this is true, Beechey (2004) shows how the introduction of imperfect information increases

the expression in (5.34):

$$i_t^{\tau,FI} - i_{t-1}^{\tau,FI} = \varepsilon_t + \frac{1}{\tau}(u_t - u_{t-1}) \tag{5.36}$$

Taking derivatives with respect to both the permanent and transitory shock we obtain:

$$\frac{\partial(i_t^{\tau,FI} - i_{t-1}^{\tau,FI})}{\partial\varepsilon_t} = 1$$
$$\frac{\partial(i_t^{\tau,FI} - i_{t-1}^{\tau,FI})}{\partial u_t} = \frac{1}{\tau}$$

This result is the one presented in Fig. 5-1. Under full information, a shock to the inflation target will generate a parallel shift in the yield curve, as yields of all maturities react one for one to this shock. In turn, the reaction to transitory shocks diminishes as maturity (τ) increases.

In the limited information case, we obtain the τ-period to maturity yield by replacing (5.4) and (5.8) into (5.34):

$$i_t^{\tau,LI} = \bar{r} + \frac{1}{\tau}\left(\pi_t^* + u_t + (\tau-1)\pi_{t/t}^{*,LI}\right) \tag{5.37}$$

Now, the nominal component will be a combination of the actual inflation target π_t^*, affecting the one period interest rate, and its inferred value $\pi_{t/t}^{*,LI}$. The variation of the τ-periods to maturity yield is given by:

$$i_t^{\tau,LI} - i_{t-1}^{\tau,LI} = \frac{1}{\tau}\left(\varepsilon_t + (\tau-1)(\pi_{t/t}^{*,LI} - \pi_{t-1/t-1}^{*,LI}) + u_t - u_{t-1}\right) \tag{5.38}$$

Using the Kalman equation in (5.9), we can express the previous equation as:

$$i_t^{\tau,LI} - i_{t-1}^{\tau,LI} = \frac{1}{\tau}\left(\varepsilon_t + (\tau-1)(1-\phi)(\pi_{t-1}^* - \pi_{t-1/t-1}^{*,LI} + \varepsilon_t + u_t) + u_t - u_{t-1}\right) \tag{5.39}$$

where we have substituted the value of the actual inflation using (5.3) and (5.4). The

the period-to-period volatility of returns, measured as the variance of $i_t^\tau - i_{t-1}^\tau$.

reaction to each type of shock is obtained by taking derivatives to the previous expression:

$$\frac{\partial(i_t^{\tau,LI} - i_{t-1}^{\tau,LI})}{\partial \varepsilon_t} = 1 - \frac{\tau - 1}{\tau}\phi$$
$$\frac{\partial(i_t^{\tau,LI} - i_{t-1}^{\tau,LI})}{\partial u_t} = 1 - \frac{\tau - 1}{\tau}\phi$$

Rates will react equally to both transitory and permanent shocks, as agents are unable to distinguish among the two. Relative to the full information case, the term structure under-reacts to permanent changes in the target and over-reacts to transitory shocks. As τ increases, the reaction of yields converges to $1 - \phi$, the steady state Kalman gain parameter. Intuitively, agents assign a fraction of the total variation in the inflation rate to a transitory shock, which impact declines as maturity increases. Thus, long term interest rates will be mainly affected by the inferred shifts in the inflation target. When a transitory shock hits the economy, agents will confuse it with a permanent one, generating the effect observed in long interest rates.

In the case in which transitory and persistent shocks are uncorrelated at all leads, it is possible to show that the value of the conditional covariance between the one period interest rate and the rest of yields falls as τ increases under limited information.[21]

Regarding the slope of the yield curve, the imperfect information literature presents a handful of studies with varying degrees of success replicating this pattern. As we reviewed in the previous section, the process followed by consumption, either in an endowment or production economy, is one of the key determinants of the yield curve slope. Dothan and Feldman (1986), Detemple (1986), Feldman (1989) and Detemple (1991) characterize the term structure of interest rates under imperfect information in a Cox et al. (1985) framework, augmented with an stochastic 'productivity' factor affecting the growth rate of the economy.[22] When this factor is unobservable, new information changes the conditional distribution used by agents to price assets, which impacts the behaviour of the yield curve.[23] Although this is an important result, it hinges in the

[21] Intuitively, short and long interest rates will be affected by a forecast error component, which increases the comovement across rates. For a proof, see Beechey (2004).

[22] Detemple and Murthy (1994) present an extension of this model with heterogeneous beliefs.

[23] This is one of the ways in which a non-constant dispersion in the filter density is obtained. A time-varying dispersion on the posterior density, suggests periods of greater or lower confidence about the value of the state. Detemple (1991) obtains this result by assuming beliefs agents hold are not Gaussian, even if the state follows a Gaussian diffusion process. This results in interest rates exhibiting a stochastic variance. David (1997), also in a Cox-Ingersoll-Ross production economy, shows that it is possible to

assumption of an stochastic growth rate which, in this case, is prevalent even under full information. This is shown by Riedel (2000), who presents a model where the growth rate is constant but unknown. In this case, the yield curve becomes downward-sloping.

We conclude that the introduction of imperfect information in the form of imperfect common knowledge can help models of the yield curve improve in several dimensions. Nonetheless, in the end, strong assumptions regarding the process followed by the exogenous (unobservable) states are needed in order to explain the yield curve related puzzles. Moreover, the fact many of these models rely in a non-stationary process in order to improve the fit to the data could be regarded as an undesirable feature, since the debate on the stationarity of nominal interest rates and inflation rate remains open.[24] [25]

5.5 Self-referential learning and the yield curve

In this section we present models of *self-referential* learning. In these models agents follow a recursive learning algorithm to updated their beliefs. Asset prices dynamics hinge on the *self-referential* property of learning. Self-referential learning introduces new dynamics because expectations will now depend of current realizations of the variables, which are themselves a product of expectations held on previous periods. By learning from endogenously-generated observations, the introduction of learning can provide new dynamics without the need for non-stationary driving processes or structural breaks.

Now, we present an extension of the model of Adam et al. (2011) to the case of bonds. Different from the exercises in Chapter 4, we focus our attention to consol bonds and the empirical predictions of the model. We present an economy where infinite-lived investors with the following problem:

$$\max_{C_t} E_t \Sigma_{s=0}^{\infty} \delta^{t+s} U(C_{t+s})$$

obtain the same result by assuming a non-Gaussian process for the unobservable state variable. Veronesi (1999) applies this idea to study equity prices under incomplete information. See Bidarkota et al. (2005) for a discussion.

[24]Naturally, results of tests for stationarity of the inflation rate vary from country to country and depend on the particular statistical methods adopted. Ng and Perron (2001), apply a wide variety of unit root tests to quarterly inflation data of the G7 countries and are unable to reach a definitive conclusion on the stationarity of inflation rates.

[25]Furthermore, assuming it is possible to show that the inflation target or other parameters in the monetary policy rule shift over time at high frequencies, a natural question arises regarding the reason for which central banks would behave in such a manner.

subject to:

$$P_tC_t + Q_tB_t + Q_t^{\infty}B_t^{\infty} = P_tY_t + (Q_t^{\infty} + 1)B_{t-1}^{\infty} + B_{t-1}$$

where C_t is consumption at time t, δ is the discount factor and $U(.)$ is strictly increasing and concave. Agents receive an endowment Y_t, and can save their resources in two different assets, a one-period risk free bond, which end-of-period holdings are denoted by B_t and a infinite-lived consol-type bond, denoted by B_t^{∞}. The prices of these assets are denoted by Q_t and Q_t^{∞} respectively. We have standardized the coupons paid by the consol bond to 1. Under rational expectations (RE), the equilibrium prices for both assets are given by:

$$\begin{aligned} Q_t^{RE} &= \delta E_t\left[\frac{U'(C_{t+1})}{U'(C_t)}\frac{P_t}{P_{t+1}}\right] \\ Q_t^{\infty,RE} &= \delta E_t\left[\frac{U'(C_{t+1})}{U'(C_t)}\frac{P_t}{P_{t+1}}(Q_{t+1}^{\infty,RE} + 1)\right] \end{aligned} \tag{5.40}$$

which defines the mapping asset prices to the consumption and inflation processes. Following the discussion in Chapter 4, we assume that price of consol bonds admits the following representation.

$$E_t\left[\frac{U'(C_{t+1})P_t}{U'(C_t)P_{t+1}}Q_{t+1}^{\infty,RE}\right] = \beta_t^{RE}Q_t^{\infty,RE} \tag{5.41}$$

where β_t^{RE} represents the risk-adjusted real growth rate of the consol price under rational expectations. Substituting (5.41) into (5.40), we get:

$$Q_t^{\infty,RE} = \frac{\delta E_t\left[x_{t+1}\right]}{1 - \delta\beta_t^{RE}}$$

where:

$$\beta_t^{RE} = E_t\left(x_{t+1}\frac{Q_{t+1}^{\infty,RE}}{Q_t^{\infty,RE}}\right) \tag{5.42}$$

and $x_t = \frac{U'(C_t)}{U'(C_{t-1})}\frac{P_{t-1}}{P_t}$. Assuming x_t is distributed i.i.d with mean μ_x and variance σ_x^2, we obtain:

$$Q_t^{\infty,RE} = \frac{\delta\mu_x}{1 - \delta\beta_t^{RE}} \tag{5.43}$$

Provided the price of the consol bond remains bounded, the rational expectation solution yields:

$$\beta_t^{RE} = \mu_x \ \ \forall t.$$

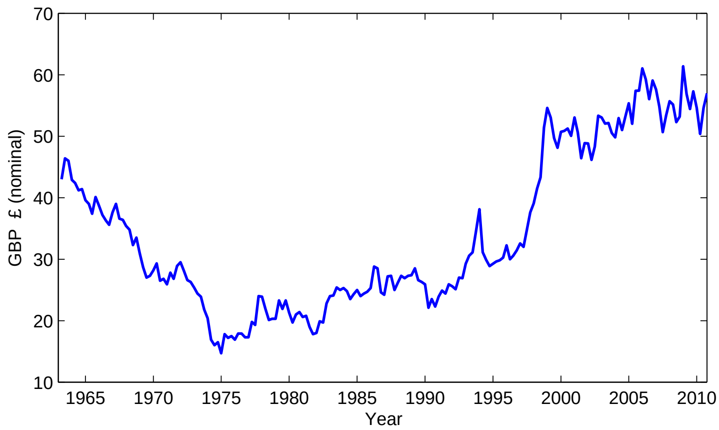
Replacing this result in (5.43), we obtain the value for the consol bond price under rational expectations:

$$Q_t^{\infty,RE} = Q^{\infty,RE} = \frac{\delta\mu_x}{1-\delta\mu_x}, \quad \forall t.$$

which is time-invariant.[26]

Figure 5-3 presents the real price of the United Kingdom (UK) 2.5 percent consol bond for the 1963 to 2010 period. These bonds, popular during the 18th century, represent only a small fraction of the current total government outstanding debt in the UK, however their yields are highly correlated with those of long term bonds.[27]

Figure 5-3: UK 2.5% consol bond price (nominal, expressed in GBP £)



Note: United Kingdom 2.5% consol bond clean nominal price for the 1963:Q1 to 2010:Q3 period. Data obtained from *Global Financial Data* databases.

As we can appreciate, the observed price of the consol bond exhibits large fluctuations around its sample mean. Now we show how self-referential learning can help improving the model-predicted dynamics. Under learning, agents will form a linear pre-

[26]To see this, notice that replacing (5.43) into (5.42), yields the following expression for β_t^{RE}

$$\beta_t^{RE} = E_t\left[x_{t+1}\left(1+\frac{\delta\Delta\beta_t^{RE}}{1-\delta\beta_{t+1}^{RE}}\right)\right].$$

where $\Delta\beta_t^{RE} \equiv \beta_{t+1}^{RE} - \beta_t^{RE}$. Assuming $\Delta\beta_t = 0$, we obtain the result.

[27]By 2012, a total of eight perpetual bonds were being traded, with a total worth combined of £2.7 bn., representing less than half percent of the outstanding government debt.

dictor for the real stochastic discount factor-adjusted consol price growth rate. Following Cogley and Sargent (2008), we relax the assumption that investors know the true value of β^{RE} in (5.43). Instead, agents will learn it from the data. The dynamics of prices under learning are given by:[28]

$$Q_t^{\infty,LE} = \frac{\delta \mu_x}{1 - \delta \hat{\beta}_t} \tag{5.44}$$

where $\hat{\beta}_t$ represents the belief agents hold about parameter β at time t. Agents will learn the value of β from the behaviour of consol bond prices. We assume they follow a statistical approach to learning, based on an econometric model specification. In this case, agents will use *Recursive Least Squares* (RLS) to forecast the future expectations of the (SDF-adjusted) consol price growth rate.

$$\hat{\beta}_t = \hat{\beta}_{t-1} + \frac{1}{\alpha_t}\left[x_{t-1}\frac{Q_{t-1}^{\infty,LE}}{Q_{t-2}^{\infty,LE}} - \hat{\beta}_{t-1}\right] \tag{5.45}$$

We assume agents initiate the learning process with the rational expectations prior ($\beta_0 = \mu_x$). The gain parameter, α_t, stands for the confidence that agents have on their prior. Under RLS:

$$\alpha_t = \alpha_{t-1} + 1 \tag{5.46}$$

Following Adam et al. (2011) and Evans and Honkapohja (2001), the assumptions about the learning process guarantee the convergence the rational expectations equilibrium.

Our parameter vector is given by $\theta \equiv (\delta, \gamma, 1/\alpha_1, \mu_x, \sigma_x)$, where δ is the discount factor, γ the relative risk aversion coefficient, $1/\alpha_1$ the initial confidence that agents hold in their priors, and μ_x and σ_x are the two parameters defining the distribution of the real consumption growth rate. We use the two last parameters to match the mean and variance of consumption and inflation processes. The risk aversion parameter is calibrated using $\gamma = 3$ as an upper bound, in order to keep low levels of risk aversion. We calibrate the remaining two parameters $(\delta, 1/\alpha_1)$ to match six statistics related to the consol bond price. These statistics are: (1) the mean consol bond yield rate, (2)

[28] For simplicity, we maintain the assumption that agents price correctly the value of the coupons. The introduction of learning about the stochastic discount factor (SDF) adjusted growth rate value of the coupon will not change the model predicted dynamics significantly, neither, it will preclude agents for learning the growth rate of the price as knowledge of the mapping of coupons and prices has not been endowed to agents. For a discussion see Adam and Marcet (2011).

Table 5.2: **Learning models - results of simulations**

height*Statistics*	*UK Data*	*std*	*RLS learning*	$t-ratio$	*Constant gain*	$t-ratio$
$E(r^q)$	8.52	5.13	8.54	0.00	7.43	0.05
σ_{r^q}	3.00	2.94	3.00	0.00	2.07	0.05
$E(hpr)$	9.87	26.13	9.23	0.01	8.32	0.03
σ_{hpr}	26.48	24.67	25.85	0.01	29.5	-0.06
$\rho_{r^q,-1}$	0.98	0.79	0.98	0.00	0.97	0.01
$E(r^s)$	7.74	5.53	6.58	0.05	6.58	0.05

Parameters	*RLS Learning*	*Constant gain*
δ	0.991	0.991
γ	2	2
$1/\alpha$	1/25	1/65
$\mathcal{P}(pf)$	0%	0%

Note: Table show results for simulations of learning models. Data for United Kingdom 2.5% consol bond clean nominal price for the 1963:Q1 to 2010:Q3 period. Data of yields obtained from *Global Financial Data* databases. Data on inflation and consumption obtained from the *Office for National Statistics*. Moments reported - (1) $E(r^q)$: mean consol bond yield rate, (2) σ_{r^q}: standard deviation of the consol yield rate, (3) $E(hpr)$: mean holding period return rate, (4) σ_{hpr}: standard deviation of holding period returns, (5) $\rho_{r^q,-1}$: persistence of the consol bond yield, and (6) $E(r^s)$: mean short rate. Column 3 reports the standard deviation for each moment estimated from data, building a sample statistic function as in Adam et al. (2011). Least-squares learning model selection follows a grid-search process over the values of α_1 and δ for a simulation of 190 observations to minimize the distance between simulated and observed statistics. We report the average moments of 5000 draws for the x_t process. Learning model is formed by equations 5.44-5.46. δ stands for the stochastic discount factor, α stands for the initial confidence in the *RLS* case and for the constant gain parameter in the *CG* case. γ is the risk-aversion parameter in assumed CRRA preferences, and $\mathcal{P}(pf)$ is the percentage of periods in which simulated prices are at the projection facility bounds.

the volatility of the consol yield rate, (3) the mean holding period return rate, (4) the volatility of the holding period returns, (5) the persistence of the consol bond yield and (6) the mean short rate. We use a distance minimization criterion based on the sample standard deviation of a set of six statistics, including the standard deviation of the return rate.[29] Here we present the results of the simulations.[30]

Results are reported in Table 5.2. The model with learning fares considerable well replicating several of the features found in the data. This is a strong result given the relatively low degrees of freedom the model has to replicate these moments. In the RLS

[29]We follow the procedure in Adam et al. (2011) for the construction of a sample statistic function.

[30]As it is common in the literature, we restrict expectations to the ones that keep the price of the consol bonds bounded. For this we make use of the *Projection Facility*, as in Timmermann (1996), Marcet and Sargent (1989), Adam et al. (2011) and Cogley and Sargent (2008). For this, we define bounds for β such that the expected return to the consol bond and the price remain positive. For the upper bound we use the *Smoothed Projection Facility* (SPF), proposed by Adam and Marcet (2011). The values we use for the SPF are 85 and 90.The consol bond prices in the sample are between 14.7 and 61.4, with a mean of 34.1 and a s.d of 12.9. Hence we are placing the SPF at 4 standard deviations away from the sample mean. The SPF assures us continuity of the consol price along the possible values of the parameter set, which is desirable given the calibration procedure we follow.

learning case, an initial confidence level of 96 percent is able to generate a significant improvement in the model predicted volatility. In this model, beliefs about future growth of bond prices will influence the current price, reinforcing the initial beliefs and generating a low frequency boom and bust type of behaviour in prices. For this reason, the main gains from the use of this learning mechanism come in the form of increased model predicted volatilities for both the consol yield and the holding period returns.

The *self-referential* learning literature includes as well models that characterize the yield curve through the Expectation Hypothesis, similar to the ones reviewed in the previous section. Sinha (2009) presents a NK model where the presence of learning helps explaining the predictability puzzle. Under time-varying beliefs, the regression proposed by Campbell and Shiller (1991), reviewed in Section 5.3, is biased downwards, affecting the estimated slope coefficient. Although the model is able to capture this feature and predict an increase in long rates volatility with respect to the full information rational expectations case, it fails in generating a positive slope in the yield curve.

5.6 Concluding comments

The present survey reviews the role of imperfect information and learning in models of the yield curve. Models of imperfect information in which agents infer the value of non-stationary unobservable variables have been proved useful to match the dynamics of observed asset prices. In these models, volatility is generated by introducing a non-stationary process in the model (i.e.: unit root or regime shifting process). Imperfect common knowledge improves these models along other dimensions, such as the sensitivity of long interest rates to transitory shocks. Despite their apparent empirical success, they leave unexplained the reason why regimes keep shifting or why agents believe this is the case. Models of *self-referential* learning address this problem. In this case, the observed dynamics are generated in the convergence path towards the equilibrium. This learning path generates self-reinforcing dynamics that help models improve their empirical predictions. Nonetheless, these models still need to introduce an arbitrary initial point or justify why agents follow a particular PLM or use a constant gain learning mechanism.

Recent work in learning presents other interesting approaches that could be applied to model yield curve dynamics. Adam and Marcet (2011) develop a framework where the idea of agents taking asset prices as exogenous is formalized. Cogley et al. (2012) study the evolution of the market price of risk in models where heterogeneous beliefs interact with the market structure. In a related paper, Branch and Evans (2010) present a framework where agents use misspecified models to update their beliefs, which yields endogenous regime-switching returns and volatility on equity markets. Berardi and Duffy (2010) present an application of adaptive learning using the parameterized expectations approach, that could be applied to non-linear learning models of the yield curve. Future work in the study of the yield curve dynamics can benefit from these advances.

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