The London School of Economics and Political Science

Inequity-Averse Decisions in Operational Research

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Declaration

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I confirm that Chapters 2 and 5 were jointly co-authored with Professor Morton and I contributed 90% of this work. Chapter 4 was jointly co-authored with Professor Morton and Dr Argyris and I contributed 90% of this work.

Abstract

This thesis is on inequity-averse decisions in operational research, and draws on concepts from economics and operational research such as multi-criteria decision making (MCDM) and mathematical modelling. The main focus of the study is developing systematic methods and modelling to help decision makers (DMs) in situations where equity concerns are important. We draw on insights from the economics literature and base our methods on some of the widely accepted principles in this area.

We discuss two equity related concerns, namely equitability and balance, which are distinguished based on whether anonymity holds or not. We review applications involving these concerns and discuss alternative ways to incorporate such concerns into operational research (OR) models. We point out some future research directions especially in using MCDM concepts in this context. Specifically, we observe that research is needed to design interactive decision support systems.

Motivated by this observation, we study an MCDM approach to equitability. Our interactive approach uses holistic judgements of the DM to refine the ranking of an explicitly given (discrete) set of alternatives. The DM is assumed to have a rational preference relation with two additional equity-related axioms, namely anonymity and the Pigou-Dalton principle of transfers. We provide theoretical results that help us handle the computational difficulties due to the anonymity property. We illustrate our approach by designing an interactive ranking algorithm and provide computational results to show computational feasibility.

We then consider balance concerns in resource allocation settings. Balance concerns arise when the DM wants to ensure justice over entities, the identities of which might affect the decision. We propose a bi-criteria modelling approach that has efficiency (quantified by the total output) and balance (quantified by the imbalance indicators) related criteria. We solve the models using optimization and heuristic algorithms. Our extensive computational experiments show the satisfactory behaviour of our algorithms.

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Chapter 1

Introduction

Many real life settings involve equity concerns. It is of great importance to address these concerns because a solution which fails to handle equity concerns is often abandoned by the stakeholders on the grounds of unfairness.

Various real-life applications exist where equity concerns play a significant role. The problems considered so far in the literature include but are not limited to allocation, assignment, location, vehicle routing, scheduling and transportation network design settings.

Addressing equity concerns in operational research (OR) is interesting and challenging because:

- Equity is an ethical concept, on which many discussions exist in various areas such as philosophy, sociology and economics. Engineering applications tend to "quantify" equity using some inequality measures but this approach requires sufficient knowledge on the underlying theory of different measures.

- The underlying motivation for equity and the specifics of the problem may result in different equity-related concerns, such as equitability and balance. These two concerns are substantially different from each other and hence may require different methods. Moreover, what is considered as fair depends on the context and the decision maker (DM)'s understanding of a fair allocation. Different settings may require different methods to handle equity concerns.

- Equity rarely appears as the sole concern in applications. Real life applications often involve considering tradeoffs between equity and other relevant criteria such as efficiency. Therefore, many mathematical modelling approaches to equity use multi-criteria decision making problems, which are considerably more challenging than their single objective counterparts in terms of both computational and cognitive effort needed.

The challenge to address equity concerns in mathematical modelling approaches raises many stimulating questions some of which are addressed in this work. We mainly aim to develop good decision support tools for decision makers in such settings by finding solutions that would be acceptable to many inequity-averse decision makers.

Research Questions:

- What are the mostly used equity-related concepts in OR? How can we incorporate equity concerns in decision making models? What are the approaches that have been used so far in the OR literature? What are the advantages and limitations of these approaches?

We attempt to address this question in the paper called "Inequity-averse optimisation in operational research", which is under review at the European Journal of Operational Research an invited review. As we will discuss later in detail, there is a broad range of applications where equity concerns are discussed and incorporated into the mathematical models in an explicit way. The equity discussions are mostly on two equity-related concepts: equitability and balance. The key difference between these two concepts is the anonymity assumption: equitability concern arises in settings where the recipients are indistinguishable (hence anonymity holds) whereas balance concern occurs in settings where the recipients are distinguishable based on for example claims, needs or preferences (hence anonymity may not hold).

From a modelling point of view, three main approaches are used to handle equitability concerns:

- - The first is a Rawlsian approach, where the focus is only on the worst-off entity.
  - The second is based on using inequality indices in mathematical models.
  - The third is a holistic (multi-criteria decision making) approach where equity is addressed by imposing person anonymity and Pigou-Dalton principle of transfers (PD) conditions on the model of the DM's preferences. We call the corresponding utility functions representing such preference models equitable aggregations.

Balance concerns are handled in similar ways, either by using an imbalance indicator in the model or by converting the problem to an equitability problem by scaling the alternatives in an appropriate way.

The simplest and crudest of the equitability-handling approaches is the Rawlsian approach, which assesses equity by the amount allocated to the worst-off entity in a distribution. In another approach to equitability, one uses well defined inequality indices (based on relatively restrictive assumptions), which are able to provide a DM with a single alternative that is the "best" in terms of equity. When equity is the only concern, such approaches present the DM with a single solution. However, using such inequality indices in multicriteria decision making environments would involve tradeoffs between multiple criteria and selecting the "best" solution would require using appropriate decision support. The third approach is a more general approach to equitability, based on commonly accepted convexity and symmetry axioms (equitable efficiency) but is a more complicated method and may remain inconclusive since the resulting models always involve multiple criteria. In such cases one would call for decision support to find the "best" solution unless there is a single alternative that is better than others in all criteria.

The problem of obtaining many inconclusive comparisons in multicriteria decision making settings can be mitigated by taking into account value judgements which the DM has provided. That is, we can incorporate information which the DM gives us (for example that she prefers one distribution to another) into the preference model. Our attempt to operationalise this idea of incorporating DM's preference information for the third approach to equitability, where we assume anonymity and PD axioms on the preference model of the DM, led to the following question:

- How can one assist a decision maker who is able to provide limited preference information in terms of holistic judgements when there are equitability concerns and hence anonymity property holds in the preference model?

The second part of the thesis focuses on incorporating preference information from the DM for the methods where equity is addressed using anonymity and quasi-concavity (convexity) conditions on the model of the DM's preferences. This addresses the problem of obtaining many inconclusive comparisons and can be used to find the best alternative, or a set of "good" alternatives, to sort the alternatives into some predefined groups, or to rank the alternatives.

Our attempt to answer this research question led to the following two papers:

- - "Using Holistic Multicriteria Assessments: The Convex Cones Approach", which has been published in Wiley Encyclopedia of Operations Research and Management Science. In this article, we discuss the convex cones approach, a well-known approach in the multi-criteria decision making literature that uses holistic preference information in the decision support process. We discuss this approach within the classical setting, i.e. for problems where there is no equity concern.
  - "Incorporating Preference Information in Multi-criteria Problems with Equitability Concerns", where we suggest an approach for problems with equitability concerns that takes into account value judgements which the DM has provided. We extend the theory of convex cones discussed in the previous paper so that it is applicable to problems where there is anonymity. We also provide a ranking algorithm to illustrate the computational significance of these ideas in a practical setting.

Taking our motivation from resource allocation settings, where the DM has equity (mostly balance) and efficiency concerns we try to address the following question in the third paper:

- How can we design models for resource allocation problems where balance is a concern alongside efficiency and address the tradeoff between these two concerns?

One possible answer to this question is provided in the paper "Incorporating balance concerns in resource allocation decisions: A bi-criteria modelling approach". In this paper we propose a means to handle balance concerns alongside efficiency concerns in allocation problems and hence provide a bi-criteria framework to think about trading balance off against efficiency. This article is published in Omega.

Our contributions are the following:

- Providing a comprehensive review of the recent OR literature along with a thorough discussion of equity-related concepts considered in this area
- Providing a classification of the approaches that are used to address equity concerns in optimisation settings and discussing advantages and limitations of these different approaches
- Proposing a novel interactive approach, which involves substantial theoretical results that can be used to find the best alternative, rank or sort the alternatives in settings where equitability concerns hold
- Designing an interactive ranking algorithm for the settings where the DM has equitability concerns and providing computational results. To the best of our knowledge, this is the first multi-criteria ranking algorithm designed in the literature for the problem settings we consider.
- Defining and classifying balance line-based imbalance indicators
- Proposing a bicriteria modelling approach for handling efficiency and balance concerns in various resource allocation settings
- Performing computational experiments on the epsilon constraint approach and a TS algorithm, which are suggested to solve the resulting bicriteria and multicriteria models

To sum up, the main body of the thesis is composed of the following chapters, which are based on the papers discussed above:

Chapter 2: Inequity-averse decisions in operational research

Chapter 3: Using holistic multicriteria assessments: the convex cones approach

Chapter 4: Incorporating preference information in multi-criteria problems with equitability concerns

Chapter 5: Incorporating balance concerns in resource allocation decisions: A bi-criteria modelling approach

We conclude the thesis in Chapter 6.

Individual chapters are intended to be read as self-contained papers: as such, decisions about the optimal notation to use have been made individually within chapters. Notation is reintroduced in each individual chapter and notations may subtly differ between different chapters. The definition of commonly used terms will be repeated.

Chapter 2

Inequity Averse Optimisation in Operational Research

2.1 Introduction

There are various real life applications where equity concerns naturally arise and it is important to address these concerns for the proposed solutions to be applicable and acceptable. As a result, there exist many articles cited in the operational research (OR) literature that consider classical problems, such as location, scheduling or knapsack problems, and extend available models so as to accommodate equity concerns. These models are used across a broad range of applications including but not limited to airflow traffic management, resource allocation, workload allocation, disaster relief, emergency service facility location and public service provision. This broad range of applications indicates that considering these classical models with an emphasis on equity is practically relevant in addition to being technically interesting.

In this paper we present a literature review on inequity aversion in operational research and a classification of the modelling approaches used to incorporate concerns about equity in optimisation problems. The equity concept is often studied in an allocation setting, where a resource or good is allocated to a set of entities. The concern for equity involves treating a set of entities in a "fair" manner in the allocation. The allocated resource or outcome can be a certain good, a bad or be a chance of a good or bad. The entities can be for example organizations, persons or groups of individuals which are at different locations or are members of different social classes.

At this point it may be helpful to look at three small examples. Let us start with a simple example in which we have two people who are allocated some money. Consider the following two allocations to these people, who are no different in terms of claim: (100,50) and (80,70). Common sense suggests that the second allocation is more equitable than the first one. The *Pigou-Dalton principle of transfers (PD)* formalizes this intuition. The PD states that any transfer from a richer person to a poorer person, other things remaining the same, should always lead to a more equitable allocation.

PD allows us to compare allocations that have the same aggregate amount as is the case in our simple example. However, things get more complicated when we have allocations that differ in terms of the aggregate amount. In many situations an increase in equity results in a decrease in efficiency, which is usually measured by the total amount of the good (bad) that is allocated. As an example, consider a case where an emergency service facility is going be located. Suppose that a number of potential sites for the facility is already determined and the problem is to choose one of them. The facility will be serving different customers and it is important for the decision maker (DM) to ensure an equitable service to them. The DM evaluates how good a service is by the distance the customers have to travel to reach the facility: the shorter the distance between a customer and the facility, the better it is. One can consider choosing an alternative that minimizes the total distance that all the customers travel to the facility to evaluate how good each potential site is. However, in such a solution some of the customers may be significantly under-served. Figure 2.1 shows a small example with 3 customers located at the nodes of a network (C_1, C_2 and C_3). Suppose that there are two alternative locations for the emergency service facility (P_1 and P_2, respectively). We will represent the two alternative locations using distance distributions that show the distance that each customer has to travel. The first location (P_1) results in distance distribution (3,4,4) and the second one (P_2) results in distribution (0,5,5). We see that the second alternative is more efficient in

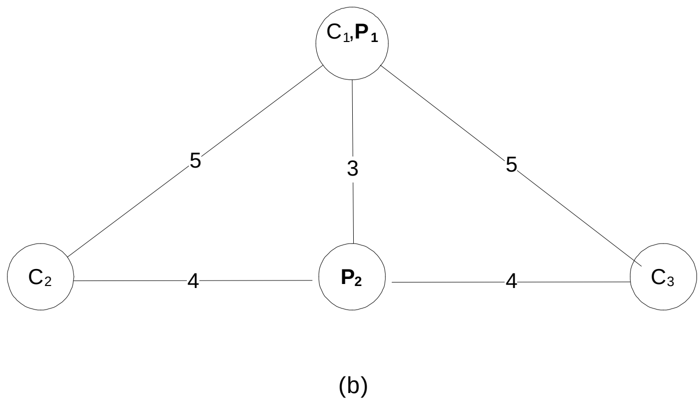

Figure 2.1: Two alternative locations for an emergency service facility

the sense that the total distance travelled is less. However, this efficiency is obtained at the expense of customers C_2 and C_3 who have to travel 5 units of distance. In the first alternative, the total distance travelled is larger but the distance travelled by the customers C_2 and C_3 is reduced. This is a typical example of the trade-off between efficiency and equity, which occurs in many real life situations. The DM's preferences would determine the better alternative in such cases: there is no "objective" way to determine which distribution is better, and reasonable people may take different views. For example the DM may argue that the first alternative is better claiming that the maximum distance travelled is smaller, or s/he may argue that the second alternative is better as it saves on total distance travelled.

The above examples show cases where anonymity holds; that is, the identities of the entities are not important. However, as we will see in the next example, there may be situations where the entities have different characteristics and hence anonymity may not make sense. Suppose that you are the head of an academic department and you have to decide on the allocation of the next year's studentship budget to the PhD students. Which of the following rules would you use as a base for your decisions?

-Allocate every student the same amount regardless of any other factor

-Allocate the budget proportional to the students' declared needs, which are measured as the shortfall from target income (students that need more get more)

Different people would give different answers to this question. The first rule respects person anonymity and hence is equitable. However, there are other sensible arguments that would favor other rules, as anonymity may be inappropriate when we have entities with different characteristics, such as different needs. These two rules involve two different dimensions of equity, "horizontal" and "vertical" equity. Horizontal equity is concerned with the extent to which entities within a class are treated similarly ([1]); hence *giving equal amounts to the students with the same need* would satisfy concerns on horizontal equity. Vertical equity is concerned with the extent to which members of different classes are treated differently. *Giving different amounts to students with different needs* is a decision regarding vertical equity.

As seen in this example, a reasonable equity concept might involve "unlike treatment of unlikes", such as giving different amounts to students with different needs. We call this equity concept that involves entities which are distinguished by an attribute such as need, claim or preferences *balance*.

2.1.1 Review Methodology

The search methodology we use for this review is as follows: We used the "Web of Science" database for our search and used the keyword "equit*" to be able to include the words such as "equity", "equality", and "equitable". We narrowed down the search by area (Operational Research/Management science) and we limited the search to "Journal Articles". We note that the results of the search highly depend on which journals are classified as Operational Research or Management Science journals. For example many articles in the telecommunications area dealing with equity are not included since those appear in journals not classified as OR/MS. As our focus is on current practice we surveyed the 10 years from 2003 to the time of analysis and mid way through 2013. We have identified 392 articles. Screening by title, we eliminated the irrelevant ones, most of which use "equity" as a financial term, and obtained 181 articles. We further screened them by abstract. We focused on the studies that either report a modelling approach that incorporates equity concerns or discuss equity measures that have been used in the OR literature. We obtained 89 articles this way. Scanning the references of these articles we added 27 articles to our review list. In Table 2.1 we report the the journals that contribute to the literature with 2 or more publications. Around 16% of the articles were published in European Journal of Operational Research, followed by 9% and 8% in Computers and Operations Research and Transportation Science, respectively. In total there were 40 journals, which shows that equity considerations arise in various settings and are discussed in publications in a variety of journals with different audiences and scopes.

Table 2.1: Number of articles by journal

Journal	**Frequency**
European Journal of Operational Research	17
Computers and Operations Research	10
Transportation Science	9
Annals of Operations Research	7
Interfaces	6
Journal of the Operational Research Society	6
Operations Research	6
Transportation Research Part B	5
Networks	4
Omega	4
Naval Research Logistics	3
Transportation Research Part E	3
Discrete Applied Mathematics	2
IIE Transactions	2
INFORMS Journal on Computing	2
International Journal of Production Research	2
International Transactions in Operational Research	2
Management Science	2
Mathematical Methods of Operations Research	2
Networks and Spatial Economics	2

The rest of the paper is as follows: Section 2.2 discusses the two main equity related terms, which are equitability and balance. We mention some of the applications involving equity concerns cited in the OR literature. For such problems, we summarize the motivation for equity, the outcome distribution used in assessing equity and the entities for which equity is sought. In this section we do not attempt to give technical details on how the equity concerns are incorporated into mathematical models; we rather want to show that there is a wide range of applications and that equity is regarded as an important concern in the modelling process. Section 2.3 includes a more detailed discussion of different approaches taken in the literature to incorporate equitability and balance concerns in mathematical models. We conclude the discussion in section 2.4, where we point out future research directions that would be interesting to explore.

2.2 Equitability and Balance

In this section we discuss two equity related concepts, namely *equitability* and *balance*. Equitability is used for comparing allocations across a set of indistinguishable entities. Balance concerns occur when we allocate goods over entities with different needs, claims or preferences. In such situations, ensuring justice might require treating different entities differently. We discuss these concepts in an order based on the frequency of appearance in our review.

2.2.1 Equitability Concerns

Around two thirds of the articles in this review deal with equitability concerns. Equitability concerns occur when the set of entities are indistinguishable and hence anonymity holds. The first two examples used in the introduction (Section 2.1) show two important settings in which equitability can be a concern. The first setting is where a fixed amount of resource is being allocated and distributions can be quasi-ordered using PD. The second setting is where we have allocations with different total amounts which are not comparable using PD. This second setting makes things more interesting and complicated as there is often a tradeoff between efficiency and equitability.

We also gave an example regarding horizontal and vertical equity, which we relate to equitability and balance concepts, respectively. Alongside horizontal and vertical equity, equity can be quantified in other dimensions such as spatial equity and temporal equity ([1]). Spatial equity is concerned with the extent to which the good is distributed equally over space, i.e. over the entities at different locations. Temporal equity, which is also referred to as *longitudinal* or *generational* equity, is the extent to which the good is distributed to the present or future recipients, i.e. to entities are distinguished by temporal aspects such as different generations who are the beneficiaries of a road investment or entities that use an emergency service system at different times.

Let us introduce some notation that will be used throughout the paper. Suppose that we have an outcome distribution (allocation) $y = (y_1, y_2, ..., y_m)$ where y_i is the outcome level of entity $i \in I$, I being the entity set. Without loss of generality, we assume that the more the outcome level, the better, i.e. the problem is a maximization problem. Note that it is possible to define the outcome distribution in multiple ways using different scales. For example, in a resource allocation problem two possible outcome definitions are the following: one can define the outcome distribution in terms of the absolute resource amounts allocated to different entities (y_i) or as the shares of the total resource allocated to different entities ($y_i / \sum_{i \in I} y_i$). An inequality index can be defined for either of the two distributions. The difference stems from the outcome definition rather than the index itself as seen in the example. In this work we do not distinguish the inequality indices based on how the distributions are scaled (see [2] for detailed information and a categorization of the inequality indices used in location theory).

We now provide a list of some of the many applications cited in the literature along with a discussion of the motivation for equity in such cases. We classify the applications based on the underlying technical problem. This classification is summarized in Table 2.2.

Allocation Problems: An equitable allocation of the good or resource over multiple entities is sought in such problems ([3]). Applications include bandwidth allocation ([4], [5], [6], [7], [8], [9], [10], [11], [12]), water rights allocation ([13]), health care planning ([14], [15], [16]), WIP (Kanban) allocation ([17]) in production systems, fixed cost allocation ([18], [19], [20]), and public resource allocation such as allocating voting machines to election precincts ([21]). There are also studies that consider general resource allocation settings such as [22], [23], [24] and [25].

One classical problem in this group is the discrete knapsack problem. The discrete knapsack problem selects a set of items such that the total value of the set is maximized subject to capacity constraints. In some applications equity is a concern as well as efficiency (total output maximization). A linear knapsack problem with profit and equity objectives is considered in [26]. [25] introduce the lexicographically minimum and maximum load linear programming problems in order to achieve equitable resource allocations.

In resource allocation problems equity may be defined as spatial equity but other definitions are also possible such as space-time equity across members of the public in terms of the allocated amount. In water distribution problems, spatial and temporal equity across demand points is considered. One example of temporal equity concerns is averting high variation in water deficits in a region over multiple periods to avoid extreme deficits [13].

[22] discuss different fairness concepts that are used to ensure fair allocation of resources in an abstract environment. The authors derive bounds for the price of fairness, which is the loss in efficiency when a "fair" resource allocation is pursued. [23] also focus on balancing efficiency and equity in resource allocation settings.

Another classical OR problem is the assignment problem which involves allocation of workload over agents. These problems may involve concerns on fairness among agents. Equity can be sought in terms of the assigned workload as in [27]. In air traffic management, when a foreseen reduction in a destination airport's landing capacity is anticipated, ground delay programs (GDP) are used as the primary tool for traffic flow management. In a GDP, the departure times of the affected flights are coordinated and hence the aircraft is delayed on ground. [28] and [29] model the GDP as an assignment problem and incorporate equity concerns.

Location Problems: One of the main concerns in facility location models is ensuring an equitable service to the population. Especially in essential public service facility location models, geographic equity of access to the service facilities is considered as one of the main requirements for an applicable solution. The access level can be measured in different terms such as the distance between demand points (customers) and the facilities (as in [30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45]) or the time required to access the facility from the demand points as in [46], [47], and [48]. [49] considers the generic location problem from a multicriteria perspective and formulates a model where each individual access level is minimized.

[32] define equity over demand points in different groups, which are not aggregated based on geographic location and which may be scattered in the whole area. The demand points are categorized based on a common attribute value and equity across these categories is sought. An example would be ensuring equitable service to different wealth classes.

If the facilities are not essential service facilities, which can serve customers within a limited distance, the amount of population covered at each facility can be used as an indicator for which an equitable distribution is sought [33]. A related problem is the equitable load problem, where ensuring an equitable service load distribution over the service facilities is of concern ([50], [51], [52], [53], [54]).

Other problems include location-price setting problems, where equitable profit sharing between competing firms is addressed [55]. [56] consider the problem of locating warehouses and try to ensure equity in holding inventory among all supply chain members, because equity in inventory is argued to have a great impact on the future throughput of the company through competitiveness issues. Realizing that the solution which minimizes the total inventory often treats some retailers in an inequitable way, the authors seek equity across retailers in terms of the amount of inventory.

Vehicle Routing Problems: Vehicle routing problems are used in many applications such as pick-up and delivery service, disaster relief, hazardous material shipment and reverse logistics (e.g. waste collection).

One of the outcomes over which equity is sought in vehicle routing problems is vehicle workload ([57]). In an effort to ensure an equitable workload distribution among vehicles in a multi vehicle pick-up and delivery problem, the expected length of the longest route is minimized in [58]. Similarly, [59] consider a routing problem, and propose a model that guarantees that lottery sales representatives travel roads of similar length on different days. This ensures an equitable distribution of workload over a time period. Workload balance is also considered in [60] in a periodic vehicle routing model used to optimize periodic maintenance operations. [61] deals with an equitable partitioning problem that ensures a balanced workload distribution to vehicles. [62] consider a reverse logistics network problem in which the service areas for multiple depots are defined. Equitable workload distribution to depots is considered in one of the objectives of their model. The workload of a depot is measured in terms of the hours needed to serve the service area it is assigned to.

Equity concerns naturally arise in vehicle routing problems arising in disaster relief contexts ([63]). In such problems, one of the concerns of the decision makers is ensuring equitable service distribution to different affected areas (nodes). Equity of service to demand nodes is defined in various ways. For example, if all the demand is satisfied when a node is visited then the arrival time is used to measure service ([64]).

[65] develop a multiobjective location-routing model, to model a home-to-work bus service, and try to achieve an equitable extra time distribution across customers. Extra time is defined as the difference between the bus transport time and the time of a direct trip from home to work.

An interesting variation on this sort of problem is provided by [66], who introduce the so called *balanced path problem* in acyclic networks, which finds paths from an origin to a destination in an acyclic network such that the cost difference between the longest and shortest paths is minimized. Similarly, [67] consider the balanced travelling salesman problem, which finds a Hamiltonian cycle (a spanning cycle in a graph, i.e. a cycle through every vertex [68]) with the minimum cost range over its edges.

Scheduling: In personnel scheduling, equitable systems attempt to distribute the workload fairly and evenly among employees [69]. As an example of this, [70] formulate a bidline scheduling problem as a set partitioning problem and consider workload equity over pilots in their model.

Fairness across patients is one of the factors considered while designing appointment systems ([71]). For appointment scheduling for clinical services [72] introduce a model which includes equity related constraints in order to find uniform schedules for the patients assigned to different slots. The proposed unfairness measures are based on the expected waiting times at each slot and the number of patients in the system at the beginning of each slot.

[73] propose bicriteria models to schedule ambulance crews, the two criteria being the aggregate expected coverage and the minimum expected coverage over every hour. The second criterion is included to incorporate temporal equity concerns into the model.

Other examples include [74] who consider reentrant hybrid flow shop scheduling problem, which allows the products to visit certain machines more than once. In this paper, the equity concept is used with a different underlying motive. The authors propose a bi-criteria model and use equity in order to generate solutions which are good enough in both criteria. That is, solutions that perform very well in one criterion while performing very badly in the other are avoided. This idea is explained in Section 2.3.

Transportation Network and Supply Chain Design Problems: In transportation network design, equity over network users is considered (as in [75], [76], and [77]).

Equity over users is considered while designing access control policies, in which meters are installed at on-ramps to control entry traffic flow rates. Different equity concepts are reported such as temporal equity and spatial equity: "The temporal equity measures the difference of travel time, delay and speed among users who travel on the same route but arrive at the ramp at different times while the spatial equity concerns the difference among users arriving at difference ramps at the same time" [78].

Equitable approaches are also used in congestion pricing schemes to ensure "fair" treatment of the travelers that are categorized for example by income or geographic locations ([79], [1]). [79] consider a pricing scheme more equitable if it leads to a more uniform distribution of wealth across different groups of population.

Equitable capacity utilizations among the participating warehouses and manufacturers is considered in collaborative supply chain design ([80]).

Other Integer/Linear Programming Problems, Combinatorial Optimization Problems and Stochastic Models: In an effort to form equitable student case study teams, [81] proposes an integer programming model which ensures that each team has similar a priori academic performance. By creating teams that are as equally capable as possible (i.e. by equitably distributing capability over teams) the author seeks to ensure that final grades of the students are not influenced by the assignment system.

Punnen and Aneja [82] introduce the lexicographic balanced optimization problem. Given a finite set E of elements $e : e \in E$ each with a certain cost value c_e, it is assumed that a a family of subsets of E are defined as feasible solutions and denoted as F (F is assumed to have a compact representation and the members of F are not listed explicitly). The problem finds a feasible subset $S \in F$ that lexicographically minimizes the absolute difference between $max_{e\in S}\ c_e$ and the kth minimum of $\{c_e : e \in S\}$. This problem is a lexicographic extension of the balanced optimization problem that minimizes the range $(max_{e\in S}c_e - min_{e\in S}c_e)$. Solution algorithms and generalizations for the problem are discussed. Turner et al. [83] consider the generalized balanced optimization problem, which involves finding the feasible subset S that minimizes $max_{e\in S}\ |c_e - k\max(S)|$, where $k\max(S)$ is the kth largest cost coefficient in S. This problem reduces to the balanced optimization problem, which minimizes the range, for $k = 1$ and $k = |S|$.

[84] study the dispersion problem with equity based objectives, i.e. the equitable dispersion problem. The dispersion problem selects a subset (of a certain cardinality m) of items from a given set such that a function of the interelement distances (d_{ij} for any two elements i and j) of the selected elements is maximized. The equitable dispersion problem focuses on equity-based objectives in this context, which are argued to achieve an equitable dispersion among the elements in the selected subset. This approach is an alternative to the studies that consider efficiency-based objectives such as maximizing the total dispersion.

[85], [86], [87], [88] approach equity from a multicriteria perspective and hence formulate multicriteria decision making models.

Markov decision process (MDP) models can also be considered with additional equity concerns. [89] develop an LP model with side constraints on equity to model the dispatch of emergency medical servers to patients in an MDP framework. Different equity constraints are used to ensure both service and resource allocation equity over patients and workload and job satisfaction equity over servers.

2.2.2 Balance Concerns

About one third of the articles in our review deal with balance concerns. Balance is a special type of equity concern in which the entities are not necessarily treated anonymously since they differ in some equity-relevant characteristics such as needs, claims or preferences. Such problems do not have anonymity and an ideal solution may not give each entity the same proportion of the total allocation.

Heterogeneity of Needs (or size)

The social equity concept quantifies equity based on the extent to which any good received is proportional to the need ([1]).

As an example, [90] considers equity related concerns in a public policy problem faced by a municipality which has to select a portfolio of foreclosed homes to purchase to stabilize vulnerable neighborhoods. A spatial equity based objective is incorporated into the corresponding knapsack model, which minimizes the maximum disparity between the fraction of all purchased homes in a neighborhood and the number of available foreclosed houses in that neighborhood across all neighborhoods. In this example, the need of a neighborhood is quantified by the number of available foreclosed houses in that neighborhood.

In disaster relief settings the demand points have different needs. If partial satisfaction of demand is possible, the proportion of demand satisfied is used as a measure of service. Such measures are used by [91] in an inventory management model and by [92] and [93] in multi-objective transportation/distribution models. [91] propose a stochastic programming model for placing commodities and distributing supplies in a humanitarian logistics network. There are studies that use more complicated service functions combining timing and proportion of demand satisfied (see e.g. [94], which consider vehicle routing and supply allocation decisions in disaster relief). Similarly [95] and [96] consider a drug allocation setting and provide each clinic with a fraction of drug supply which is proportional to their demand. [97] propose an integer programming model to optimize siding rosters and ensure that growers with different amounts of cane maintain approximately the same percentage of cane harvested throughout the harvest season.

In locating undesirable facilities such as waste disposal facilities, geographic equity in the distribution of nuisance effects or social rejection is one of the concerns that is incorporated into the models [98], [99]. In such problems the towns have different nuisance parameters since they have different sizes. A tenant-based subsidized housing problem is considered in [100], where subsidy recipients are allocated to regions and equity across the potential host communities, which differ in size, has to be considered.

Heterogeneity of Claims

In some settings the entities are distinguishable based on their claims for a resource. The claims may be as a result of a previous legal agreement or on agreed upon rules. For example, in GDPs spreading delay or delay-related costs equitably among multiple airlines (flights or flight types) is one of the main concerns while assigning landing slots to airlines. In such settings the schedule which is generated before the disruptions is taken as a reference solution and hence may provide airlines with a basis to construct claims regarding the new schedule. For example a flight which was supposed to land first in the previous schedule would find it unfair if assigned as the last one in the new schedule.

[101], [102] develop an airspace planning and collaborative decision making model, which is a mixed integer programming model. The model is developed for a set of flights and selects a flight plan for each flight from a set of proposed plans. Each alternative plan consists of departure and arrival times, altitudes and trajectories for the flight. The suggested model addresses the equity issues among airline carriers in absorbing the costs due to rerouting, delays, and cancellations. [103] extends this model by integrating slot exchange mechanisms that allow airlines to exchange the assigned slots under a GDP. [104] propose an air traffic flow management model that assigns ground and air-borne delays to flights subject to both en route sector and airport constraints. The model is described as a *macroscopic* version of a previous model by [105], with a different objective function, which is argued to "spread" the delay in an equitable way across affected flights. Similarly, [106] propose integer programming models that are based on the models discussed in [105] and [107]. The models assign ground holding delays to flights in a multiresource traffic flow environment that also take equity in delay distribution into account. By considering

the en route sector capacity constraints, these models differ from the GDP models that only consider arrival airport capacity. [108] consider the runway scheduling problem in airport transportation, which finds a schedule and corresponding arrival and departure times for aircraft. Equity among aircraft is ensured by the constraint position shifting approach. This approach requires that there is no significant deviation between positions of the aircraft in the optimized sequence and the first-come-first-served sequence. A similar approach is used in [109]. [110] use a stochastic programming model that assigns ground delays to flights under uncertainty. The model minimizes expected delay and incorporates balance concerns among flights using a balance-related constraint.

Another application is scheduling commercials in broadcast television, in which balance concerns over clients are incorporated into a mathematical model in [111]. Similarly, [112] propose a bicriteria modelling framework that considers both efficiency and balance concerns in resource allocation problems.

Heterogeneity of Preferences

In some problems entities have different preferences which make them distinguishable from each other. For example, [113] considers (as they call it) the minimum-envy location problem, where the customers have ordinal preference orderings for the candidate sites. The problem is opening a certain number of facilities to which the customers will be assigned. Each customer is assigned to his most preferred facility among those which are open and the envy between a pair of customers is measured as the difference between the ranks of the facilities.

Diversity Concerns

Another concept which is related to equity but in an indirect or orthogonal way is diversity. Around 4% of the reviewed papers use the diversity concept. To see the motivation for this concept, suppose that you are going to select a set of candidates for a degree programme. You have concerns on *diversity* in the sense that you want certain population groups to have a certain degree of representation in the selected set. These groups may, for example, consist of people with an inferior socioeconomic background. A common way of achieving this is to use quotas or proportion targets, i.e. ensuring that a certain proportion of the selected people will be from the specific group of concern. This approach involves treating people with different characteristics differently such that the selected team is diversified enough. For example, in an applicant selection model [114] ensure diversity in the selected team in order to represent certain population groups.

[115] consider the problem of forming teams of service personnel with different skills. To treat customers served by different teams equitably, the author introduces a diversity measure and ensures that the diversity is above a threshold for all the teams. To take another example of diversity, in hazardous material shipment, spreading risk over population groups in an equitable way is one of the main concerns [116], [117], [118]. In some studies the concept of equity of risk is handled by determining spatially dissimilar paths. These studies incorporate equity concerns by selecting a set of paths to carry the hazardous material, which are as dissimilar as possible. Two examples are due to [116] and [118], who consider the problem of selecting of k routes in multiobjective hazardous material route planning. They use a measure of spatial dissimilarity and obtain an equitable distribution of risk over the related region by choosing spatially dissimilar paths to ship the hazardous material.

We do not devote a separate section to diversity and discuss it in this section under balance concerns. That is because although these studies address equity in a relatively indirect way which is based on creating diversity, it is possible to conceptualize diversity as a balance concern in such settings. For example when selecting candidates for a degree program, the underlying problem can be considered as allocating admission to the degree program to population subgroups. Although there is no way in which degree admission can be allocated equally across people - out of N people, only n can be accepted onto the programme, and the remaining $N - n$ will have to be rejected- admission can be allocated in a balanced way across the population subgroups by ensuring that the set of admitted candidates is diverse. Similarly, when selecting routes in hazardous material shipment settings, the membership of the selected route(s), i.e. being a node on the route, is allocated to different population centres. Diversity ensures an equitable allocation of membership over different nodes avoiding inequitable solutions such as a solution in which

Table 2.2: Classical problems in OR re-considered with equity concerns

Problem	**Examples**
Allocation	[4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [112], [90], [26], [27], [28], [29], [114], [119], [115], [120], [121], [95], [96], [94],
Location	[30], [31], [32], [33], [34], [35], [36], [37], [38], [39], [40], [41], [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [113], [98], [99], [55], [100], [56], [122]
Vehicle Routing	[58], [59], [60], [62], [61], [64], [92], [94], [65], [66], [67], [116], [117], [118]
Scheduling	[70], [72], [111], [97], [108], [73], [74], [109]
Transportation Network Design	[75], [76], [77], [78], [79]
Other	[81], [80], [101], [102], [103], [104], [106], [82], [83], [84], [85], [86], [87], [88], [93], [123], [124], [110], [125], [91], [89], [126]

most of the routes pass through the same set of nodes exposing these nodes to much higher risk than the rest.

2.3 Different Approaches to Handle Equity Concerns

2.3.1 Different Approaches to Incorporate Equitability

Equity has been widely discussed in the economics literature where it is generally accepted that there is no *one-size-fits-all* solution and that special methods are required to handle equity concerns in particular cases (see e.g. Sen [127], and Young [128], who discusses different concepts of equity). Nevertheless, using transparent and explicit rules that determine what is equitable and what is not or how equitable a given distribution is on a cardinal or sometimes ordinal scale can be useful in ensuring that the decisions are applicable and acceptable.

Similarly, in operational research there are many different ways of incorporating equitability in the decision process since its precise interpretation depends on both the structure of the problem at hand and the decision maker's understanding of a "fair" distribution. In this section, we discuss the operational research approaches that incorporate equitability concerns in mathematical models.

One of the most common and simplest ways to incorporate equitability concerns is focusing on the min (max) level of outcomes across persons. This approach is called the Rawlsian principle ([129]). The Rawlsian principle is justified using a *veil of ignorance* concept, which assumes that the entities do not know what their positions (the worst-off, the second worst-off etc.) will be in the distribution. To illustrate, suppose that you are given two distributions over two people generically named A and B, such as (5,50) and (30,25). You have to choose one of the allocations and then will learn whether you are A or B. You would seriously consider choosing (30,25) as you might be the worse-off person in a distribution and would get only 5 units if you choose (5,50). This ignorance is a reason to consider the worst-off entities in the distribution as any entity should find the distribution acceptable after learning its position. This approach, however, fails to capture the difference between distributions that give the same amount to the worst-off entity: two distributions such as (1,1,9) and (1,5,5) are indistinguishable in terms of inequity from a Rawlsian point of view although the latter is significantly more equitable from a common sense point of view. This drawback can be avoided by using a lexicographic extension, which will be discussed later in detail.

A more sophisticated approach to incorporate equitability concerns would be using summary inequality measures in the model. We call such approaches *inequality index based approaches*. These approaches can be further categorized based on whether the index is employed in a constraint while defining the feasible region or is used as one of the criteria in the objective function.

A more general, and hence more complicated approach would be to use a (inequity-averse) aggregation function and to maximize it. We refer to such approaches as *aggregation function based approaches*. Some studies optimize a specific function of the distribution and obtain a single equitable solution while others use a multi-criteria approach and obtain a set of equitable solutions.

The above classification is summarized in Table 2.3. We will discuss these approaches further in the following sections.

Table 2.3: Solution approach framework

Main Approaches				
Rawlsian	**Inequality index based**		**Aggregation function based**	
[44], [43], [45], [51],	Range	[98], [53], [19], [67],	Social welfare functions	[122], [29], [16], [23]
[91], [64], [31], [55],		[26], [72], [84], [89],	Equitable Efficiency	[85], [87], [9], [10],
[98], [36], [50], [47],		[66], [34], [46], [62],		[74], [86], [88]
[100] ,[99], [51], [41],		[82], [83]		
[93], [65], [77], [17],	Mean Deviation	[46], [49], [18], [27],		
[15], [13], [90], [56],		[59], [54]		
[58], [84], [14], [73],				
[22], [39], [21], [89],	Variance	[46], [40], [34], [70],		
[20], [30], [92]		[38], [72]		
Lexicographic Extension	Standard Deviation	[80], [60]		
[28], [11], [7], [25],	Gini Coefficient	[35], [79]		
[12], [8], [120], [121],	Sum of pairwise deviations	[46], [42], [37], [113]		
[6], [5], [4], [24]	Sum of square deviations	[95], [96], [76], [32]		
	Deviation from a reference	[81], [61],[97]		

The Rawlsian Approach ($\min_i\ y_i$)

These methods represent equity preference by focusing on the worst-off entity, hence the minimum outcome level in a distribution ([129]). Some studies try to maximize the minimum outcome while others restrict it in a constraint that makes sure that it is above a predefined value. The studies encountered that use a Rawlsian approach to equitability are [44], [43], [45], [51], [91], [64], [31], [55], [98], [36], [50], [47], [100], [99], [51], [41], [93], [65], [77], [17], [15], [13], [90], [56], [58], [84], [14], [73], [39], [22], [21], [89], [20] and [30]. Clearly, this is an easy to implement and popular approach.

The Rawlsian approach is the one of the oldest approaches in OR used to incorporate a fairness concept into the models. Many classical OR problems such as assignment, scheduling and location have also been studied with "bottleneck" objectives. For example, the facility location problems that locate p facilities such that the maximum distance between any demand point and its nearest facility is minimized are known as *p-center* problems. These models assign each demand point to its nearest facility, hence full coverage of customers is always ensured. *p-center* location problems are widely considered in location theory, especially in public sector applications ([130]).

The Rawlsian approach can be extended to a lexicographic approach, which in addition to the worst outcome maximizes the second worst (provided that the worst outcome is as large as possible), third worst (provided that the first and second worst outcomes are as large as possible) and so on [87]. Lexicographic maximin approach is a regularization of the Rawlsian maximin approach such that it satisfies strict monotonicity and PD principle of transfers. Lexicographic approaches are used in [28], [11], [7], [25], [12], [8], [120], [121], [6], [5], [4] and [24]. Lexicographic approaches are very inequality averse and considered by some studies as the "most equitable" solution.

Inequality Index Based Approaches

In many studies equitability concerns are incorporated into the model through the use of inequality indices $I(y) : \mathbb{R}^m \to \mathbb{R}$, which assign a scalar value to any given distribution showing the degree of inequality. Many inequality measures are studied in the economics literature (see Sen [127]). Some of them are also used in the operational research literature when dealing with problems that involve equity concerns alongside efficiency concerns. As inequality indices are used to assess the disparity in a distribution, they are related to several mathematical concepts of dispersion and variance. They respect the anonymity property ([131]) and have a value of 0 when perfect equity occurs. They assign a scalar value to the distribution ([131]) and are "complete" in the sense that every pair of distributions can be compared under these measures ([127]).

The indices are used to address equitability concerns and do not incorporate any concerns on efficiency. When efficiency concerns are also relevant, the corresponding models that use an inequality index to handle equity concerns are either designed as multicriteria models (two of the criteria usually being efficiency and equity related, respectively) or as single objective models that maximize an efficiency metric and use the index in a constraint. For example, Ogryczak [49] works on location problems and develops bicriteria mean/equity models as simplified approaches. These models deal with the equity concern by adapting the inequality measures to the location framework and trying to minimize them. He discusses different ways to find efficient solutions to these bicriteria models. Other bi(multi)-criteria examples include [98], [26], [72], [62], [59], [54], [80], [60], [79] and [37]. There are also single objective models where equity is handled via constraints which set minimum levels of allocation for each entity and an efficiency metric is maximized ([89]).

The examples that use an inequality index as the only criterion are [53], [19], [67], [84], [34], [46], [82], [83], [18], [40], [38], [32], [35], [42], [113] and [81]. [70] minimize a weighted sum of two inequality measures.

Using an explicit inequality measure has some advantages such as bringing transparency to the decision making process, making the equitability concept computationally tractable, and hence making it possible to optimize the system with respect to these equality measures once a suitable measure is agreed upon or to tradeoff equity and efficiency. On the other hand, using an inequality index to incorporate equitability concerns implies a certain approach to fairness dictated by the axioms underlying the selected index and sometimes may result in oversimplification of the discussion on equity. Moreover, different indices are based on different concepts of equity, hence may provide different rankings for the same set of alternatives. Selecting an index in line with the DM's understanding of fairness requires some extra knowledge on the underlying theoretical properties of different indices.

Recall that the widely-accepted Pigou-Dalton principle of transfers (PD) states that any transfer from a poorer person to a richer person, other things remaining the same, should always increase the inequality index value. That is, for any inequality index $I(y)$: $\mathbb{R}^m \rightarrow \mathbb{R}$ satisfying PD the following holds: $y_j > y_i \Rightarrow I(y) < I(y + \varepsilon \mathbf{e}_j - \varepsilon \mathbf{e}_i)$, for all $y \in \mathbb{R}^m$,where $\varepsilon > 0$, where $\mathbf{e}_i$, $\mathbf{e}_j$ are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$. A weak version of this principle requires such a transfer not to decrease the inequality index value. This weak version can be considered as the minimal property to be expected from an inequality index. All the indices discussed below satisfy the weak PD. We will indicate the indices that additionally satisfy (the strong version of) PD.

We now discuss the most commonly used inequality indices. All the indices except the last one are familiar from the economics literature.

1) The range between the minimum and maximum levels of outcomes ($\max_i\, y_i - min_i\, y_i$*):* This is the difference between the maximum and minimum outcomes in a distribution. This index is used in [98], [53], [19], [67], [26], [72], [84], [89], [66], [34], and [46]. [62] minimize the function $(\frac{\max_i y_i - min_i y_i}{min_i y_i}) * 100$, hence use a range function normalized by the minimum outcome.

In this method the equity level of an allocation is assessed by considering the two extremes; hence this index fails to distinguish allocations that have same level of extremes but different levels of the other values. In that sense, this index is rather crude but is used in many applications owing to its being simple and easy to understand.

[82] consider the lexicographic extension of the range measure, hence in addition to the difference between the maximum and the minimum outcome (range), they minimize the difference between the maximum and the second minimum outcome (given that the range

is as small as possible) and so on. [83] consider a generalization of the range measure in the sense that $|\max_i y_i - k^{th}\min_i y_i|$ is minimized, where $k^{th}\min_i y_i$ is the k^{th} minimum outcome value.

2) *(Relative) Mean Deviation:* This is the deviation from the mean. Note that in many cases the mean of the distribution is not known beforehand and is derived endogenously in the model. It is possible to use the total absolute deviations from the mean ($\sum_{i\in I}|y_i - \overline{y}|$, where $\overline{y} = \frac{\sum_{i\in I} y_i}{m}|$) ([18], [46], [49], [27]) or to use the positive or negative deviations only, as in [49]. The mean deviation does not satisfy strong PD because it is not affected by transfers between two entities which are both above the mean or both below it.

[59] use the mean square deviation ($\sum_{i\in I}(y_i - \overline{y})^2$). [46] and [54] use the maximum componentwise deviation from average as a measure of inequity ($Max_{i\in I}\ |y_i - \overline{y}|$).

3) *Variance* ($\sum_{i\in I}(y_i - \overline{y})^2/m$): [46], [40], [34], [70], [38], and [72] use variance as a measure of fairness in their models. Variance satisfies PD. Equivalently, the standard deviation is also used in some studies ([80], [60]).

4) *Gini Coefficient*: One of the widely used income inequality measure used by the economists is the Gini coefficient owing to its respecting the PD ([132]). The Gini coefficient has the following formula: $\frac{\sum_{i\in I}\sum_{j\in J}|y_i - y_j|}{2m\sum_{i\in I} y_i}$. Two examples are [35] and [79], who use the Gini coefficient in location of service facilities and in design of more equitable congestion pricing schemes, respectively.

5) *Sum of pairwise (absolute) differences* ($\sum_{i\in I}\sum_{j\in J}|y_i - y_j|$): Sum of absolute differences between all pairs is considered in [46], [42], [37] and [113]. Like the Gini coefficient and variance, this measure satisfies the PD. A closely related measure is the sum of square deviations between all pairs which is used in [76]. [32] consider facility location problems where they classify demand points into different demand groups over which equity is sought. They use the (weighted) sum of squares of differences between service distances to different demand groups to measure inequality. This formulation allows demand groups to include multiple demand points and service distances between any two demand point that belong to the same group are not considered in the measure. They show that minimizing this measure is equivalent to minimizing variance when each demand point is a group by

itself.

The measures discussed so far are also discussed in the economics literature especially for assessing income inequality. The first two measures (range and relative mean deviation) are relatively crude measures and hence not as popular as the others for assessing income inequality. However, they are used in OR models arguably because these indices have simpler formulations than the others and so lead to more tractable optimisation problems.

Some other examples that minimize *deviation from a point of perfect equality* are due to [81] and [61]. [81] propose an integer programming model to form student teams to undertake a case study. For equity purposes the teams are required to have similar a priori academic performance, hence the maximum deviation of a team's academic performance from the class average is minimized in the proposed model. The squared deviation functions are also used ([52]). [61] partitions a region into subregions such that a subregion is served by a vehicle. He ensures that the workload distribution to the vehicles is equitable, i.e. all vehicles have (asymptotically) equal load.

As discussed above, there are many different inequality indices and selecting one implies certain assumptions on the decision maker's attitude to equity. For example, in a resource allocation environment, if the range is used then the focus is on the most and least deprived parties.

Inequity-averse Aggregation Function Based Approaches

One natural way to achieve an equity-efficiency trade-off without specifying an inequity index is to use an aggregation function of the distribution vector in the model that would encourage equitable distributions. An example would be a symmetric function under which a convex combination of two distributions which have the same functional value would achieve a higher value than these distributions (e.g. if the function is symmetric (40,50) has a higher value than (30,60) or (60,30)). Such a function is inequity averse in the sense that the averaging operation improves the distribution. By maximizing such aggregation (value) functions, we can avoid distributions that give some entities too much while depriving some others.

We call these approaches *aggregation function based approaches.* Unlike an inequality index which only focuses on the inequity in a distribution, an inequity-averse aggregation function reflects concerns for both equity and efficiency. There are several approaches to how the equity should be captured. There are studies that use value functions which are Schur-concave, (symmetric) quasi-concave or concave with the aim of obtaining equitable solutions. Note that when allocating a bad, a Schur-convex, quasi-convex or convex aggregation (cost) function is minimized.

In these approaches, one uses an aggregation function $U : \mathbb{R}^m \to \mathbb{R}$, and modifies the original problem as follows: $\max\{U(y) : y \in Y\}$ where $Y \in \mathbb{R}^m$ is the feasible outcome space. For a specified function form to be inequity-averse, it has to satisfy some properties. First of all, such a function should be symmetric to respect anonymity and should reflect concerns in terms of inequity-aversion and the equity-efficiency trade-off. We call the set of symmetric functions that satisfy the strict Pigou-Dalton principle of transfers and strict monotonicity *equitable aggregation functions.*

Definition 1 *An equitable aggregation function is a function $U : \mathbb{R}^m \to \mathbb{R}$ for which the following hold:*

$y^1 < y^2$ *then* $U(y^1) < U(y^2)$, *for all* $y^1, y^2 \in Y$, *i.e. U is strictly increasing with respect to every coordinate.*

$U(y) = U(\Pi^l(y))$, *where $\Pi^l(y)$ is an arbitrary permutation of the y vector, i.e. U is symmetric.*

$y_j > y_i \Rightarrow U(y) < U(y - \varepsilon \mathbf{e}_j + \varepsilon \mathbf{e}_i)$, *for all $y \in \mathbb{R}^m$, where $0 < \varepsilon < y_j - y_i$, where $\mathbf{e}_i$, $\mathbf{e}_j$ are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$, i.e. U satisfies PD.*

All equitable aggregation functions are strictly Schur-concave [87]. Similarly, in a minimization environment, for example in cost distribution, equitable aggregations are Schur-convex functions. We now give the definition of Schur-concave functions. Let us first give the definition of a *bistochastic matrix.*

Definition 2 *A bistochastic (doubly stochastic) matrix (Q) is a square matrix which has all nonnegative entries and each row and column of the matrix adds up to 1.*

Permutation matrices, which reorder the elements of a vector, are special cases of bistochastic matrices.

The well-known Birkhoff–von Neumann theorem [133] states that the set of doubly stochastic matrices of order m is the convex hull of the set of permutation matrices of the same order. Moreover, the vertices of this polytope are the permutation matrices. That is, a bistochastic matrix of order m is a convex combination of the set of permutation matrices of the same order.

Definition 3 *A function f is strictly Schur-concave (Schur-convex) if and only if for all bistochastic matrices Q that are not permutation matrices, $f(Qx) > f(x)$ $(f(Qx) < f(x))$.*

Schur-concave functions are symmetric by definition. Schur-concavity relates to more familiar concavity concepts in the following way: All symmetric (strictly) quasi-concave and symmetric (strictly) concave functions are (strictly) Schur-concave.

Maximizing (minimizing) a specific (strictly) Schur-concave (convex) function that aggregates the outcomes is discussed in a number of papers in the literature. Ball et al. [29] investigate a class of models for assigning flights to slots in ground delay problems and discuss the use of Schur-convex aggregation functions as a way of obtaining equitable solutions within this setting.

Marin et al. [122] use “ordered median functions” as objective functions of discrete location problems. Ordered median functions are weighted total cost functions, in which the weights are rank-dependent. As the weights are rank dependent, these functions are symmetric and if the weights are chosen appropriately, ordered median functions can be inequity-averse in the sense that they are strictly concave. They show that both the *range* and *sum of pairwise differences* functions can be modeled using this approach, hence are particular cases of their model.

In communication engineering, one of the commonly used fairness concepts is *proportional fairness*, which can be obtained by maximizing $\sum_{i \in I} \log(y_i)$. An allocation y is proportionally fair if for any other feasible allocation y' the total proportional change $(\sum_{i \in I}(y'_i - y_i)/y_i)$ is zero or negative when all outcomes are nonnegative. The proportional fairness concept can be advocated from a game theoretic point of view as a proportionally fair allocation is also the Nash bargaining solution, satisfying certain axioms of fairness ([22], [134], [135]). Proportional fairness is a specific case of a more general fairness scheme called $\alpha - fairness$, which maximize the following parametric class of utility functions for $\alpha \geq 0$ ([23]):

$$U_\alpha(y) = \begin{cases} \sum_{i=1}^{m} \frac{y_i^{1-\alpha}}{1-\alpha} \text{ for } \alpha \geq 0, \alpha \neq 1 \\ \sum_{i=1}^{m} \log(y_i) \text{ for } \alpha = 1 \end{cases} .$$

Lexicographic maximin approach, which is a regularization of the Rawlsian maximin approach such that it satisfies strict monotonicity and PD, is another example.

There are also approaches that use a Schur-concave function and hence respect the weak version of the PD only while failing to satisfy the strong version. For example, Hooker and Williams [16] consider allocation of utilities to individuals (or classes of individuals) and propose a weakly Schur-concave aggregation function to be maximized. The function is based on the idea of combining objectives of equity -they use a Rawlsian approach- and efficiency. The authors provide a mixed integer linear programming formulation of the allocation problem and apply the formulation to a healthcare planning example.

A difficulty with equitable aggregation functions is that the decision maker or modeller has to select a specific aggregation function. In most settings there may not be a natural choice of equitable aggregation. A set of approaches based on the concept of a unanimity order have been developed to address this issue. Given a set F of functions $f \in F$, the unanimity order with respect to F is the binary relation $<^*$over outcome vectors and defined as follows: for any two allocation vectors y^1 and $y^2 \in Y$, $y^1 <^* y^2 \iff f(x) < f(y)$ for all $f \in F$.

Note that unanimity order is a quasiorder. The approaches discussed so far in this section maximize a particular concave, quasi-concave and Schur-concave function in their models. We note that rather than using specific functions, if we consider the unanimity order for the set of all concave, quasi-concave or Schur-concave functions, there is no difference between the resulting order. This important result is summarized in the following theorem.

Theorem 4 *For two allocation vectors y^1 and y^2, the following cases are equivalent:*

1. $U(y^1) \leq U(y^2)$ for all $U : U$ is increasing and Schur-concave [136]. (Note that [136] uses a strict version of the PD; hence strictly Schur-concave functions)

2. $U(y^1) \leq U(y^2)$ for all $U : U$ is symmetric, increasing and quasi-concave ([137])

3. $U(y^1) \leq U(y^2)$ for all $U : U$ is symmetric, increasing and concave ([137])

4. $U(y^1) \leq U(y^2)$ for all $U : U$ is additive. That is, $U(g) = \sum_{i \in I} u(y_i)$ where u is increasing and concave ([136], [137])

Parts of Theorem 4 for the special case where $\sum_{i \in I} y_i^1 = \sum_{i \in I} y_i^2$ are proven by Atkinson [138] and Dasgupta, Sen and Starrett [139] based on the results by Hardy, Littlewood, and Polya [140] on majorization (see also [141]). The results for the more general case ($\sum_{i \in I} y_i^1 \neq \sum_{i \in I} y_i^2$) can be found in Rothschild and Stiglitz [137] and Shorrocks [136]. This theorem states that the unanimity ordering of a given set of alternatives under the set of all Schur-concave functions is equivalent to the unanimity ordering under the set of all quasiconcave, concave functions or additive functions of concave functions.

A multicriteria perspective: Equitable Efficiency and Schur-concavity

The above approaches use particular functions in order to capture equity concerns. The specific functional forms used are context dependent and different forms are adopted in different studies. Two common properties of these functions are that they are increasing or nondecreasing (in a maximization problem) and inequity-averse in the sense that they satisfy PD, though sometimes in a weak way as in [16]. Considering the aggregation function approach from a multicriteria perspective, one can relate such functions to the DM's preferences and specify a set of properties that an equity-averse DM's preference model should satisfy. [142] and [87] take this point of view and introduce the concept of *equitable efficiency*. Given two distributions, the "more equitable" one is distinguished based on a set of axioms defined on the DM's preference model. They call a social welfare function which is in line with this specific set of axioms an *equitable aggregation function* and a solution which maximizes an equitable aggregation function *equitably efficient*. This multicriteria decision making perspective is based on defining each element of the outcome vector as a separate criterion to be maximized as explained below. This discussion is based on the theory introduced in [142].

Consider the following problem: $\max\{f(x) : x \in Q\}$ where $X \in \mathbb{R}^n$ is the decision space, $Y \in \mathbb{R}^m$ is the outcome space and $f(x)$ is a vector function that maps X to Y and Q is the feasible set. A typical outcome vector is $y^k = (y_1^k, y_2^k, ..., y_m^k)$, where y_i^k is the outcome value corresponding to entity $i \in I$ $(i = 1, 2, ..., m)$ and k is the index of the alternative.

We denote the weak preference relation of the DM as $\preceq$ (the corresponding strict and indifference relations are denoted by $\prec$ and $\sim$, respectively). Assume that the DM has a preference model in which the preference relation satisfies the following axioms [87]:

1.Reflexivity (R): $y \preceq y$ for all $y \in Y$.

2.Transitivity (T): $(y^1 \preceq y^2$ and $y^2 \preceq y^3) \Rightarrow y^1 \preceq y^3$, for all $y^1, y^2, y^3 \in Y$.

3.Strict monotonicity (SM): $y^1 < y^2$ then $y^1 \prec y^2$, for all $y^1, y^2 \in Y$.

4.Anonymity (A): $(y) \sim \Pi^l(y)$ for all $l = 1, ..., m!$, for all $y \in \mathbb{R}^m$, where $\Pi^l(y)$ stands for an arbitrary permutation of the y vector.

5.Pigou-Dalton principle of transfers (PD): $y_j > y_i \Rightarrow y \prec y - \varepsilon \mathbf{e}_j + \varepsilon \mathbf{e}_i$, for all $y \in \mathbb{R}^m$,where $0 < \varepsilon < y_j - y_i$, where $\mathbf{e}_i$, $\mathbf{e}_j$ are the i^{th} and j^{th} unit vectors in $\mathbb{R}^m$.

The anonymity axiom states that the corresponding preference relation should treat all the permutations of a vector as indifferent. That is, the identities of the entities are irrelevant. This is in contrast to what we have called balance problems. The preference for equity is stated by the PD axiom. The preference relations that satisfy R, T, SM, A and PD are called *equitable rational preference relations.* Using equitable rational preference relations, the relations of *equitable dominance*, *equitable indifference* and *equitable weak dominance* can be defined as follows:

Definition 5 *For any two outcome vectors* y^1 *and* y^2,

$y^1 \prec_e$ $(/ \preceq_e / \sim_e)$ y^2 *(*y^2 *equitably dominates/ equitably weakly dominates/is equitably indifferent to* y^1*) iff* $y^1 \prec$ $(/ \preceq / \sim)$ y^2 *for all equitable rational preference relations* $\preceq$.

Note that "rational dominance", i.e. the normal dominance concept, which is the intersection relation of all preference relations satisfying R, T and SM, implies equitable dominance but not vice versa. This is clear from the fact that the set of axioms used to define rational dominance, which is the standard multicriteria dominance concept, is a subset of the axioms used to define equitable dominance.

Equitable dominance is also called generalized Lorenz dominance (see [136]). Generalized Lorenz dominance is an extension of the well-known Lorenz dominance concept used in the economics literature to the cases where the means of the distributions are not necessarily equal. An alternative is *equitably efficient* if there is no alternative that equitably dominates it. Note that the set of equitably efficient solutions is a subset of the Pareto efficient set.

We have already defined (see definition 1) *equitable aggregation functions.* It so happens that the equitable aggregations, i.e. Schur-concave functions are the functions that respect axioms 1-5. That is, if an equitable rational preference relation is representable by a utility function, the function has to be increasing strictly Schur-concave in a maximization problem [87]. The equitably efficient set is the set of alternatives each of which maximizes at least one increasing strictly Schur-concave function.

There are two possible equity modelling approaches using such aggregations: The first approach is choosing a suitable equitable aggregation function (Schur-concave function) and optimizing it in the model. Optimizing a predefined aggregation function will return one of the (possibly many) equitably efficient solutions. The aggregation function based approaches discussed previously, which optimize a strictly Schur-concave (Schur-convex) function are in this category.

The second approach is finding the set of equitably efficient solutions without specifying the aggregation function further. This way one would obtain a set of alternatives that is guaranteed to include the DM's most preferred alternative as long as her utility function is (strictly) Schur-concave. This approach is discussed in ([142], [85]) and [87] for multiple criteria linear problems and nonlinear problems, respectively. Baatar and Wiecek [88] define the equitable preference structure using a cone-based approach and propose a two step method including two single objective nonlinear programs in order to find equitably efficient solutions.

As an application example, Ogryczak et al. [10] consider equitable optimization in bandwidth allocation. For practical purposes, they consider a restricted set of criteria and find equitable solutions for the restricted model using the reference point approach. A similar approach is taken in [9].

Mut and Wiecek [86] generalize the concept of equitability. They define two different relations which are more general than $\preceq_e$ and investigate the axioms that these new relations satisfy. They derive the conditions under which the new preferences satisfy the original and modified axioms of equitable preference.

In most of the above approaches the whole set of nondominated points or a subset of it is found; hence the algorithms return multiple alternatives without using an interactive setting. The studies we encountered that consider interactive approaches are [87], [9] and [10], which use a reference point approach and Karsu et al. [143], which use the convex cones approach to incorporate DM's preference information to guide the selection or ranking process.

The classical multicriteria decision making problem settings include criteria that do not have the same range, hence it is not appropriate to use equitable aggregation over the original criteria values. However, in the reference point method, the outcome vectors are converted to achievement vectors using *scalarizing functions*. The scalarizing function transforms the outcomes into a uniform scale, which makes it possible to apply an equitable aggregation on the transformed achievement scores. Kostreva et al. [87] make this observation and discuss the use of equitable aggregations for the reference point method. Using the same idea, Dugardin et al. [74] use the equitable dominance concept in a well-known multi-criteria solution approach (Non-Dominated Sorting Genetic Algorithm 2) to discard the alternatives which are competitive in only one criterion. The authors introduce a function which scales different components of the objective vector. This is an application where the equity concept is used in order to choose "good" alternatives in a multi-criteria problem that does not have the impartiality property. These applications show the two way link between the Pareto efficiency and the equitable efficiency concept. In addition to generating equitably efficient solutions using the classical MCDM solution methods designed to generate Pareto efficient solutions, one can also use the equitable efficiency concept to come up with Pareto efficient solutions once the outcome vectors are modified using appropriate scalarizations.

2.3.2 Handling Balance

Most of the approaches handle balance concerns by using an imbalance indicator, which measures *deviation from a predefined level,* which is chosen e.g. based on claims, needs or preferences. This approach is similar to an inequality index based approach to equitability, however an imbalance indicator does not necessarily achieve its minimum at a distribution where each entity receives the same amount.

Examples of applications handling the balance concept using this approach are as follows. In a heterogeneous server system model, [126] consider equity over servers with different service rates. They formulate the problem as a Markov decision process and solve a related LP model, in which the customer waiting time is minimized along with a fairness constraint on the workload division over servers with different skill levels. Specifically, they use a constraint set that controls the fraction of the idle time that the server groups with different paces have. These fractions are ensured to have pre-determined values, which are set by the decision maker. Cook and Zhu [119] allocate a fixed cost among the existing Decision Making Units (DMU). In order to treat the DMUs in an equitable way, the authors ensure that the efficiencies of the DMUs remain unchanged after the allocation. [48] and [33] incorporate balance concerns over users of a public service provision system by minimizing weighted negative and positive deviations from a standard service level specified by the DM.

In ground delay programs, the ration-by-schedule (RBS) rule is used as a reference. This rule assigns the landing slots to unassigned flights on a First Scheduled First Served (FSFS) basis based on the arrival times submitted at the beginning of the daily operations. The studies that use the deviation from the FSFS solution as a measure of inequity (imbalance) in arrival slot allocations are [110], [108], [106], and [125]. Karsu and Morton [112] propose a two dimensional framework to trade balance off against efficiency in resource allocation problems motivated by problems in R&D project selection. They use imbalance indices which measure the deviation of an allocation from an ideally balanced allocation the DM provides.

The deviation (cost) function, i.e. the imbalance indicator, can be the total absolute deviation ([125]) or the sum of negative or positive deviations. There are also studies that minimize the maximum componentwise deviation ([110], [92]) or use a constraint which ensures that maximum componentwise deviation is below a pre-defined level ([109], [108]). In some models designed to improve an existing system (e.g. the current transportation network) any negative deviation from the status quo is forbidden by constraints as in [75]. They propose a transportation network improvement model, which ensures that no origin-destination pair gets worse than the status quo in terms of consumer surplus, i.e. the difference between what travelers would be willing to pay for travel and what they actually pay. There are also studies that use a weighted total deviation from the weighted mean such as [101], [102], [103] ($\sum_{i\in I} w_i|y_i - \sum_{i\in I} w_i y_i|$, where $\sum_{i\in I} w_i = 1$).

The above studies focus on keeping the total deviation from a predefined level at minimum, which may result in some componentwise deviations to be significantly larger than others. Similar to equitable aggregation functions, convex functions are optimized in some models to handle balance concerns. Such convex functions encourage fairness in the distribution of deviation (cost) and hence avoid some entities deviate significantly for the sake of minimizing total deviation. In that sense, convex functions can be considered as special types of imbalance indicators, which measure deviation using a convex function. Exponential (cost) functions and squared deviation functions are examples of such convex functions ([124], [52]). Mukherjee and Hansen [124] propose a dynamic stochastic integer programming model for the GDP that allows one to revisit the assignment in case of a change in airport operating conditions. They use a convex ground delay cost function in their objective in order to ensure a uniform spread of ground delay across different flight categories. Kotnyek and Richetta [123] consider the stochastic GDP and ensure that the

FSFS holds by using convex ground-hold cost functions. [104] use the same idea in an Air Traffic Flow Management model, where an equitable distribution of delay is achieved by using objective function cost coefficients that are a convex function of the tardiness of a flight. Similarly, Barnhart et al. [106] use an exponential delay penalty function. For each flight a worst-case FSFS delay is calculated and each interval delay beyond this worst-case FSFS delay is penalized by an exponentially increasing amount. Similarly, [111] minimizes a piecewise linear penalty function of deviations from goals. In an access control policy design problem, Zhang and Shen [78] incorporate spatial equity into the model by using the weighted square sum of the average delay over different entry points. [94] use convex disutility functions of unsatisfied demand percentages of each node in a relief routing model. Hence, the whole demand of each node is not necessarily supplied so as to save supply for other nodes.

It is also possible to use a *scaling approach* and define the outcome distribution as the per capita allocation, i.e. y_i/n_i where n_i is an attribute value, such as a measure showing the size or need of an entity. For example in disaster relief models, the proportion of demand satisfied in different demand nodes is used as a measure of service. This scaling approach allows one to assume anonymity over the scalarized outcome distribution and hence handle the balance concerns in an equitability environment. Examples that use this scaling approach are used in different settings including public policy ([90], [100]), disaster relief ([91], [92], [93]), drug allocation to clinics ([95], [96]), water resources allocation ([120], [121]), transportation network design ([76]) and scheduling ([97]).

2.4 Conclusion

Although most (of the early) attempts in operational research focused on efficiency concerns, there is a vast amount of applications where equity is an additional, sometimes the sole, concern. The need for equity is appreciated by the OR practitioners and academicians as can be observed by the recent increase in the number of OR papers, which re-consider some of the well-known problems such as knapsack, assignment and location problems with an effort to incorporate equity concerns. The applications that require explicit consideration of equity appear in a broad range of situations both in the public and private sector.

In this paper, we provide a review of the approaches that are used to handle equity concerns by optimizing mathematical models. We first discuss two equity related concepts: equitability, and balance. We discuss the differences between these two concepts along with their applications. Most of the approaches in our review can be classified as either being concerned with equitability, i.e. assuming anonymity or with balance, i.e. distinguishing entities with respect to an attribute indicating for example need, claim or preference. Handling equity by promoting diversity is an indirect approach which is discussed only in a few papers and it is possible to define such diversity concerns as a special case of balance concerns.

We provide a detailed discussion of the solution approaches designed to incorporate equitability and balance concerns. We categorise the solution approaches to problems involving equitability concerns into three categories. The first and the crudest approach is the Rawlsian (maxmin) approach, which compares alternative distributions based on the amount the worst-off entity receives. In the second approach, an inequality measure is used either in a constraint or as a criterion so as to quantify equity. When the inequality index is used in a constraint in the model, inequity is kept below a certain threshold by this constraint. The inequality measure can also be defined as a separate criterion alongside other efficiency related criteria, resulting in a multi-criteria model. Bicriteria equity/efficiency models defined this way are easy to solve. Inequality indices are useful as summary measures but should be used with caution as they may lead to oversimplification of the equity concept. Understanding the strengths and weaknesses of each index and choosing the most appropriate one requires some knowledge of the underlying theory of inequality measurement.

The last approach to equitability is based on using inequity-averse aggregation functions of the outcome distributions. Some studies using this approach maximize specific inequity-averse functions in their models. Multicriteria decision making concepts provide us with a means to relate a set of inequity-averse functions with a set of axioms on the underlying preference relation of a DM. Two equity-related axioms are additionally assumed for a rational DM's preference relation: anonymity and the Pigou-Dalton principle of transfers. The set of functions that represent such preference relations are called equitable aggregations and all such functions are Schur-concave in a maximization problem. These aggregations can be used as scalarizing functions to obtain the set of nondominated (equitably efficient) solutions or as the single objective function to be optimized to obtain a specific nondominated solution.

Balance concerns are handled in two main ways. The first one is based on using imbalance indicators, which measure the deviation from a reference solution which is considered as balanced. These indicators can be functions of various forms including convex deviation functions, which distribute deviation in an equitable way across the entities. The second way to handle balance concerns is to convert balance problems into equitability problems by normalising allocations, hence making it possible to use any of the equitability-handling approaches.

Among the approaches used to handle equitability concerns, finding the set of equitably efficient solutions can be used as a "gold standard" for other approaches owing to its reasonably weak assumptions on the underlying preference relation (the DM can have any type of Schur-concave function). This multicriteria approach is more attractive than an inequality index based approach as specifying an inequality index may be difficult for the DM. On the other hand, the approaches that find the set of equitably efficient solutions are computationally complex. One way to choose from these two extremes would be relying on the equitable aggregation concept when the underlying optimization problem at hand is relatively simple and easy to solve; and using an inequality index when the problem is less tractable.

We see great potential for further research in improving the decision support process in multicriteria problems where equity is a concern. Further research on guiding the DM through the set of candidate alternatives (e.g. the nondominated alternatives) could be usefully performed. This applies to multicriteria models in both inequality (or imbalance) index based and aggregation function based approaches. Selecting the "best" alternative requires information on the tradeoff between the criteria unless there is a single alternative which is better than the others in terms of all criteria, which is unlikely. Hence, in most of the multicriteria mathematical modelling approaches which address equity concerns using inequality or imbalance indices, a single alternative is obtained by maximizing a weighted sum of the criteria with predetermined weights. A more robust approach would be presenting the DM with a subset of solutions or using an interactive procedure rather than predefined weights. Which approach is more appropriate depends on the problem context. In some cases, presenting the DM a subset of "good" solutions for further evaluations may be required from the analyst whereas in some others decision support may be required until the decision maker makes the final selection. Similarly, in equitable aggregation based multicriteria models, even if some or all the equitably efficient solutions are found and presented to the DM, it may be difficult for him to choose from this set. Appropriate decision support would be required if the decision maker wants to obtain a single solution. This renders interactive approaches relevant and necessary in such settings.

Most of the problems in OR can be categorized into one of three classes based on what is required from the decision support. These are finding the best solution (or a subset of good solutions), ranking and sorting ([144], [145]). All the papers in our review of the operational research literature consider the first type although there may be ranking or sorting problems in which equity should be considered. An example of a ranking problem involving equity concerns arises naturally in intercountry comparisons based on income inequality and social welfare. This is one of the classical topics in the theory of equity as it has been discussed in economics. MCDM optimisation tools can be relatively easily adapted for ranking and sorting problems that involve equity concerns: See Sen [127] for a discussion and Karsu et al. [143] for an interactive ranking algorithm that is based on the equitable efficiency concept. An interesting application would be finding ways to sort different countries in terms of social welfare, or to sort different policy decisions in terms of the resulting social welfare.

In many cases addressing fairness concerns results in a decrease in efficiency. A relevant question is how much one sacrifices from efficiency when a “fair” solution is adopted. Observing the tradeoff between efficiency and equity would make the DMs more comfortable when making decisions and communicating the decisions to the stakeholders. For example, if the efficiency loss is negligible, the DM would find it easier to support a solution that ensures fairness. On the other hand if the efficiency loss is significant, a compromise solution can be selected. There are studies in the literature that analyze the price of fairness, i.e. the efficiency difference between the following two cases: selecting a very aggressive inequality averse approach and not using an inequality averse approach [22], [23]. This concept can be generalized to see the extend to which selecting the “wrong” inequality approach affects the solutions. Analyzing robustness of solutions with respect to different inequity-averse approaches awaits further attention. There are some initial attempts to explore the similarities of different inequality measures used in the location context (see e.g. [146], [42] and other references therein) but there is still more research to be done. As pointed out in [42] an axiomatic introduction of the equality (and imbalance) measures could throw some light on the question of how similar different measures are. Even when an inequality or imbalance index is chosen and used in a constraint, which controls its value by a threshold, sensitivity analysis can be performed to see the effect of the threshold value on the optimal solution. Such an analysis would help us to suggest more robust solutions but was not discussed in most of the studies (see Batta et al. [30] for an analysis in the context of a *p-median* problem on a network, where the authors try to find how bad a locational choice can be provided that the decision makes use dispersion, population and equity criteria).

To sum up, we believe that being a practically relevant and theoretically challenging concept, equity can stimulate a number of research questions for operational researchers especially in the areas of decision support, different problem types, and robustness.

Chapter 3

Using Holistic Multi-criteria Assessments: the Convex Cones Approach

3.1 Introduction

Consider a general multi-criteria decision making (MCDM) problem which can be formulated as follows:

$$\begin{aligned}&\text{“}Max\text{”}\ z = f(x) = (f_1(x), ..., f_p(x)) \\ &s.t.\ x \in X\end{aligned} \tag{3.1}$$

where x is the *decision vector*, $X \subseteq \mathbb{R}^n$ is the *feasible decision space, $f_j(.)$* is the *jth criterion (objective) function* and z is the *criterion vector.* The above formulation uses the decision space representation of an MCDM problem. One can also formulate MCDM problems in the criterion space as follows:

$$\text{"}Max\text{"}\ \{z_1, ..., z_p\} \qquad (3.2)$$
$$s.t.\ z \in Z$$

where $Z = \{z \in \mathbb{R}^p : z = f(x) : x \in X\}$ is called the *(feasible) criterion space*. That is, Z is the image of the feasible decision set (X) in the criterion space. Throughout the text we refer to both x and z as the solutions (alternatives) of the MCDM problem.

A classification of the MCDM problems can be made based on whether the solutions (alternatives) are explicitly or implicitly defined. Problems where a finite set of alternatives is explicitly given are called *multiple-criteria evaluation problems* (or *multi-criteria evaluation problems*) and problems where the set of alternatives is implicitly defined by constraints are called *multiple-criteria design problems* or *multiple objective (mathematical programming) problems (MOPs)* [147]. When the problem considered is an MOP, $X \in \mathbb{R}^n$ can be discrete or continuous.

Note that the maximization of a vector in models 3.1 and 3.2 is not a well defined operator. Therefore, solving an MCDM model may refer to different things depending on the context. Most MCDM approaches try to identify the best alternative, i.e. to find the alternative that is most preferred by the decision maker (DM). Some other cases are also possible. For example, three kinds of *problematiques* are reported to be generally used in practice in order to support decision makers in multi-criteria evaluation problems ([144], [145]). These are as follows:

1) Identify the best alternative or a small subset of good alternatives
2) Rank the alternatives from the best to the worst
3) Classify / sort the alternatives into predefined homogeneous groups

If the MCDM problem is an MOP, i.e. the set of solutions is implicitly defined by constraints, the number of solutions can be infinite (in the continuous case) or prohibitively large (in the discrete case) hence the ranking and sorting problematiques are not typically considered. In such cases one may want to identify the best alternative or a small subset of good alternatives.

Unless there is a single alternative which is better than the others in terms of all the criteria, which typically is not the case when there are conflicting objectives, the need to distinguish different solutions from each other makes some information on the DM's preferences necessary. This makes MCDM theory closely connected to the theory on preference relations and utility. Basic notation and definitions used in the theory of preference relations are provided in the next section.

In this chapter we provide a review of the literature on interactive MCDM approaches which use convex cones as a means of representing the DM's preference structure. In the next section we discuss the relation between preference relations and value (utility) functions. Specifically, we provide the assumptions made on the DM's underlying preference relation by the MCDM solution methods using the convex cones approach. In Section 3.3 we cover the basic theory on the convex cones approach and show how the cones can be used to obtain the best solution in a given set. We also discuss a related concept: the use of polyhedra whose vertices are the cone generators in MCDM ranking or sorting problems. We then provide a review of the MCDM approaches that use the convex cones and we conclude the discussion in Section 3.4.

3.2 Preference Relations and Value Functions

We first define and discuss the properties of a weak preference relation, denoted by $\preceq$, which completely characterizes the preference model in the criterion space. Next, we introduce the term *rational preference* along with its underlying properties and discuss the relation between preference relations and value functions. We then provide the assumptions on the DM's preference relation that allow us to use convex cones in MCDM approaches.

For two alternatives z^1 and z^2 the statement $z^1 \preceq z^2$ is used to symbolize that z^2 *is weakly preferred to* z^1 .

Definition 6 *Given a relation of weak preference, the corresponding relations of strict* *preference and indifference are defined as follows:*

$z^1 \prec z^2$ (z^2 *is strictly preferred to* z^1) $\Leftrightarrow$ ($z^1 \preceq z^2$ *and not* $z^2 \preceq z^1$)

$z^1 \sim z^2$ (z^2 *is indifferent to* z^1) $\Leftrightarrow$ ($z^1 \preceq z^2$ *and* $z^2 \preceq z^1$).

We now define some properties for a preference relation.

Definition 7 *A preference relation* $\preceq$ *is complete if either* $z^1 \preceq z^2$ *or* $z^2 \preceq z^1$*, for all* $z^1, z^2 \in Z$.

A preference relation $\preceq$ *is transitive if* ($z^1 \preceq z^2$ *and* $z^2 \preceq z^3$) $\Rightarrow z^1 \preceq z^3$*, for all* $z^1, z^2, z^3 \in Z$.

Preference relation $\prec$ *satisfies strict monotonicity if* $z^1 < z^2$ *then* $z^1 \prec z^2$*, for all* $z^1, z^2 \in Z$.

The relation $\preceq$ is called a *rational preference relation* if it is complete, transitive and its strict part ($\prec$) is strictly monotonic. Based on rational preferences we can define (rational) dominance, which is the intersection relation of all rational preference relations [142].

Definition 8 *For any two alternatives* z^1 *and* z^2,

$z^1 \prec_r z^2$ (z^2 *(rationally) dominates* z^1) *if* $z^1_i \leq z^2_i$ *for all* $i \in \{1, 2, ..., p\}$ *where at least one strict inequality holds.*

$z^1 \preceq_r z^2$ (z^2 *weakly dominates* z^1) *if* $z^1_i \leq z^2_i$ *for all* $i \in \{1, 2, ..., p\}$.

Having defined these preference relations we can now discuss their relation with value functions.

Definition 9 *The preference relation* $\preceq$ *is said to be represented by a value function* $v(.)$ *if* $\forall z^1, z^2$, $v(z^1) \leq v(z^2)$ *if and only if* $z^1 \preceq z^2$.

A preference relation is not necessarily representable by a value function. However, one can derive the conditions under which a preference relation is representable by a specified form of value function. We provide below one of the well-known results on representability by Debreu [148], [149].

Definition 10 *A set A is closed if and only if it contains all of its limit points. A preference relation is continuous if for all $z \in Z$, $\{y \in Z \mid y \preceq z\}$ and $\{y \in Z \mid z \preceq y\}$ are closed.*

The well-known representation theorem by Debreu [148], [149] is as follows.

Theorem 11 *If a preference relation on a set $Z \subseteq \mathbb{R}^p$ is complete, transitive and continuous, then it is representable by a continuous utility (value) function.*

In an MCDM problem, if the DM has a rational preference relation representable by a value function, then this function is strictly increasing. Further structural assumptions on the value function imply further assumptions on the underlying preference relation of the DM.

MCDM solution strategies under the *value maximization approach* assume that the DM's preferences are representable by an underlying value function ($v : Z \to \mathbb{R}$) and involve maximizing this value function. Different forms of value functions have been studied such as linear ([150], [151]), quasiconcave [152], and monotonicly increasing [153]. There are also algorithms for partially ranking [154] or sorting [155] alternatives based on an implicit quasiconcave value function assumption.

The MCDM approaches can also be categorized based on when the information is taken from the DM as follows (see [145], Chapter 16):

- *Methods based on the prior articulation of preferences:* In such methods, the preference information from the DM is taken at the beginning.

- *Methods based on the progressive articulation of preferences:* These approaches are called *interactive approaches*. In such methods we iteratively reduce the solution space and approach the best solution (or a subset of good solutions), eliciting preference information from the DM at each iteration. Interactive value maximization strategies assume an implicit value function and employ an interactive search process that makes use of the structural assumptions on the value function (see [156] and [157] for two reviews).

- *Methods based on the posterior articulation of preferences:* These methods try to find a good approximation of the nondominated frontier and present it to the DM.

Preference information can be gathered in different forms. The DM can be asked to provide pairwise comparisons of alternatives, provide reference points [158], reference directions, trade-offs, information on strength of preferences [159] etc.

A review of the literature on interactive MCDM approaches using convex cones as a means of representing the DM's preference structure is provided in this study. A convex cone is a convex polyhedral set in the objective function (criterion) space that contains solutions that are less preferred by the DM than a given set of solutions [152].

In most of the approaches using convex cones, the DM's preference model is assumed to be representable by a value function which is (strictly) quasiconcave and increasing. A function $g(.)$ is strictly quasiconcave if for all $z^1, z^2 : z^1 \neq z^2$ and $\alpha \in (0, 1)$ we have $g(\alpha z^1 + (1-\alpha)z^2) > \min\{g(z^1), g(z^2)\}$. Similarly g is quasiconcave if $g(\alpha z^1 + (1-\alpha)z^2) \geq \min\{g(z^1), g(z^2)\}$.

The quasiconcavity assumption of the DM's value function implies that the DM's rational (i.e. complete, transitive and strictly monotonic) preference relation also satisfies an additional convexity assumption.

Definition 12 *A preference relation $\preceq$ satisfies (weak) convexity if for all $z^1, z^2, z^3 \in Z$ such that $z^1 \preceq z^2$ and $z^3 \in (z^1, z^2]$, we have $z^1 \preceq z^3$. ($z^3 \in (z^1, z^2]$ means that there exists a real α, $0 < \alpha \leq 1$ such that $z^3 = \alpha z^1 + (1-\alpha)z^2$. That is, z^3 is a convex combination of z^1 and z^2).*

The quasiconcavity assumption for the value function corresponds to requiring the indifference curves (contours) to be convex to the origin. This assumption is quite reasonable since it corresponds to decreasing marginal rate of substitution, which is a commonly accepted property underlying consumer preferences in economics literature [160]. As the name implies, *marginal rate of substitution*, is the maximum amount of one good a consumer would be willing to give up in order to obtain an additional unit of another. The marginal rate of substitution between two goods usually depends on the amount of goods the consumer currently has. For example, consider a case where the consumer has two types of goods: Good 1 and Good 2. When the consumer has a large amount of Good 1 and a very small amount of Good 2, she would probably be willing to give up quite a large amount of Good 1 (as she already has plenty) to obtain one more unit of Good 2 (as Good 2 is scarce). In the opposite case where the consumer has a small amount of Good 1 and plenty of Good 2, s/he would probably be willing to give up only a very small amount of Good 1 to obtain more of Good 2. This is called the *decreasing marginal rate of substitution (DMRS).* As seen in the example, DMRS implies that, as the consumption of one good increases and the other decreases, the consumer would be willing to give up smaller quantities of the latter in exchange of a further unit of the former [160].

3.3 Convex Cones and Polyhedra

We now provide a review of relevant results and studies from the literature on the use of convex cones in solving MCDM problems.

3.3.1 Basic Theory

We start with a discussion of two studies by Korhonen et al. [152] and Hazen [161], who introduce the basic theoretical results underlying the convex cones approach in independent works. The two results slightly differ in the underlying assumptions; hence, we will consider them separately. The main difference is that, the first assumes a value function exists whereas the second one relaxes this assumption. We start with the first results by [152] and then provide the more general case which is given by [161].

Assuming an implicit value function

Given a set of k vectors such that $z^1, ..., z^k \in \mathbb{R}^p$ and an increasing quasiconcave function $f(.)$ defined on $\mathbb{R}^p$ such that $f(z^k) < f(z^i)$ for all $i \neq k$ the following definitions are made:

Definition 13 *We define the cone $C(z^1, ..., z^{k-1}; z^k)$ where $z^i : i \neq k$ are the upper generators and z^k is the lower generator as follows:*

$C(z^1, ..., z^{k-1}; z^k) = \{z \mid z = z^k + \sum_{i \neq k} \mu_i (z^k - z^i),\ \mu_i \geq 0\}$.

The cone dominated region of $C(z^1, ...z^{k-1}; z^k)$ is denoted by $CD(z^1, ..., z^{k-1}; z^k)$ and defined as follows:

$CD(z^1, ..., z^{k-1}; z^k) = \{z' \mid z' \leq z \text{ where } z \in C(z^1, ..., z^{k-1}; z^k)\}$.

Lemma 14 *For any $z \in C(z^1, ..., z^{k-1}; z^k)$, $f(z) \leq f(z^k)$. Moreover, for any $z' \in CD(z^1, ..., z^{k-1}; z^k)$, $f(z') \leq f(z^k)$.*

Definition 15 *We define the polyhedron spanned by the vectors $z^1, ..., z^k$ as follows:*

$P(z^1, ..., z^k) = \{z \mid z = \sum \mu_i z^i,\ \sum \mu_i = 1,\ \mu_i \geq 0 \text{ for all } i\}$.

The upper side of $P(z^1, ..., z^k)$ is denoted by $UP(z^1, ..., z^k)$ and defined as follows:

$UP(z^1, ..., z^k) = \{z' \mid z \leq z' \text{ where } z \in P(z^1, ..., z^k)\}$.

Lemma 16 *For any $z \in P(z^1, ..., z^k)$, $f(z^k) \leq f(z)$. Moreover, for any $z' \in UP(z^1, ..., z^k)$, $f(z^k) \leq f(z')$.*

Assume that we have k points such that $z^1, ..., z^k \in \mathbb{R}^p$ and $z^k \prec z^i$ for all $i \neq k$, where $\prec$ is the DM's preference relation. We assume that $\prec$ is a rational preference relation. If the DM has a quasiconcave value function $f(.)$, by Lemma 14, for any $z \in C(z^1, ..., z^{k-1}; z^k)$ we have $f(z) \leq f(z^k)$, hence $z \preceq z^k$. Moreover, for each point $z' : z' \leq z$ where $z \in C(z^1, ..., z^{k-1}; z^k)$, $z' \preceq z^k$, i.e., the points in the cone dominated region will be at most as preferred as z^k. We will call each such point $z' \in CD(z^1, ..., z^{k-1}; z^k)$ *cone dominated*. Note that if we assume strict quasiconcavity we have $z \prec z^k$ for $z \neq z^k$ and $z' \prec z^k$ for $z' \neq z^k$.

Similarly, by Lemma 16, for any $z \in P(z^1, ..., z^{k-1}, z^k)$, $f(z^k) \leq f(z)$, hence $z^k \preceq z$. Moreover, for each point $z' : z \leq z'$ where $z \in P(z^1, ..., z^k)$, $z^k \preceq z'$, i.e. the points lying on the upper side of the polyhedron spanned by the k points will be at least as preferred as the lower generator, z^k. If we assume strict quasiconcavity we have $z^k \prec z$ and $z^k \prec z'$.

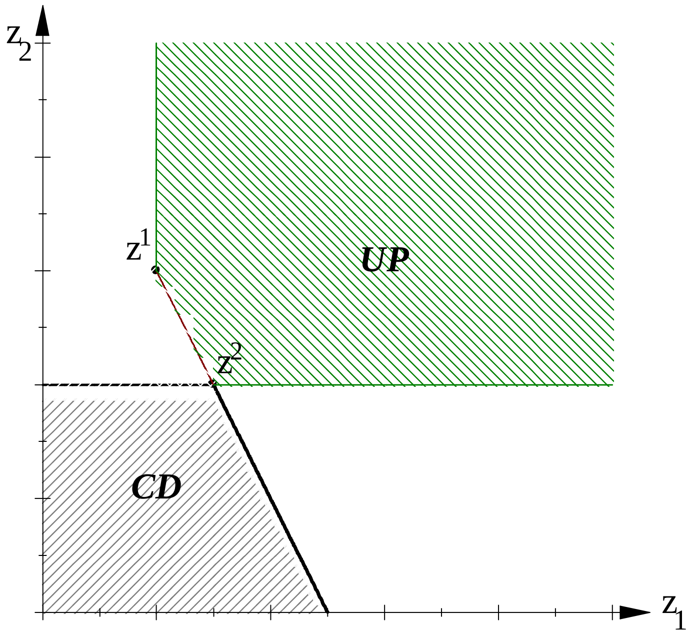

Figure 3.1: Illustration of a 2-point cone in $\mathbb{R}^2_+$

Figure 3.1 shows an example of a 2-point cone in $\mathbb{R}^2_+$. The thick solid line is the cone $C(z^1; z^2)$ and the area with diagonal grey lines is the cone dominated region, $CD(z^1; z^2)$. The line segment between z^1 and z^2 is the polyhedron $P(z^1, z^2)$ and the region above this line segment is the upper side of the polyhedron, $UP(z^1, z^2)$.

Figure 3.2 shows an example of a 3-point cone in $\mathbb{R}^2_+$. The region filled with the diagonal lines is $C(z^2, z^3; z^1)$ and the grey region including the dark grey part and the diagonal lines is the cone dominated region. We refer the interested reader to Figure 1.b in [162] for an illustration of a 3-point cone in $\mathbb{R}^3$.

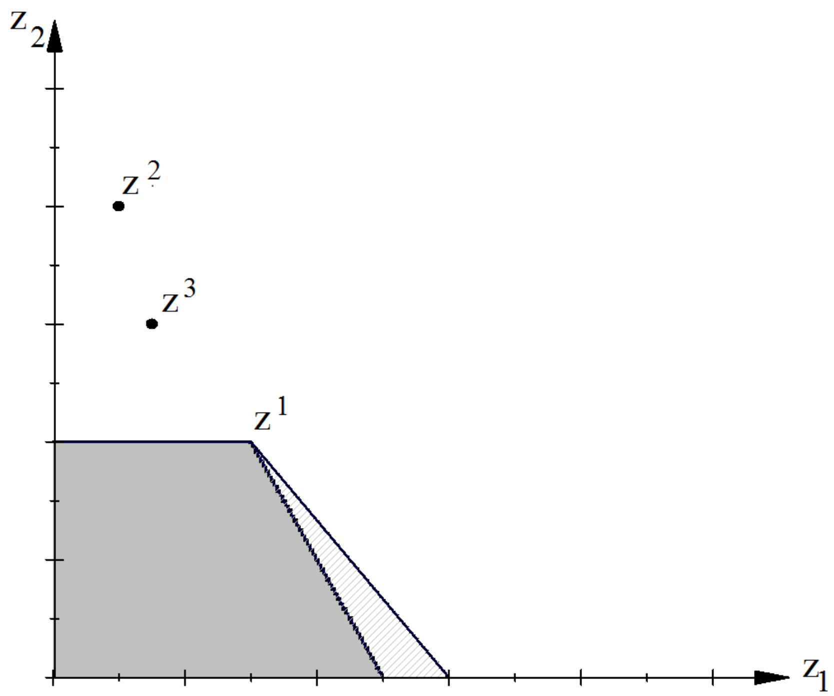

Figure 3.2: Illustration of a 3-point cone in $\mathbb{R}^2_+$

We will visualize the additional information implied by the quasiconcavity using an example. Recall that a quasiconcave value function implies the convexity assumption on the DM's preference relation. See Figure 3.3 for the criteria space of a bicriteria example where two of the alternatives (z^1 and z^2) are seen. Suppose that the DM has a rational preference relation which can be represented by a value function and prefers z^2 to z^1 ($z^1 \prec z^2$). By strict monotonicity one can say that the points in region A are less preferred to z^1. By convexity (quasiconcavity), we gain information about region B, which is cone dominated; hence the points in that region are less preferred to z^1.

Similarly, using monotonicity, we can conclude that any alternative in regions C and D are preferred to z^1. Since $z^1 \prec z^2$, transitivity ensures that the points in region E, which are vector dominating z^2, are also preferred to z^1. Using convexity (quasiconcavity), we are able to say that the points in region F are preferred to z^1. Therefore, we gain information about regions F and B by assuming a quasiconcave value function. Observe that the amount of the additional information gained depends on the two alternatives selected. One line of research in convex cones theory in MCDM focuses on finding smart

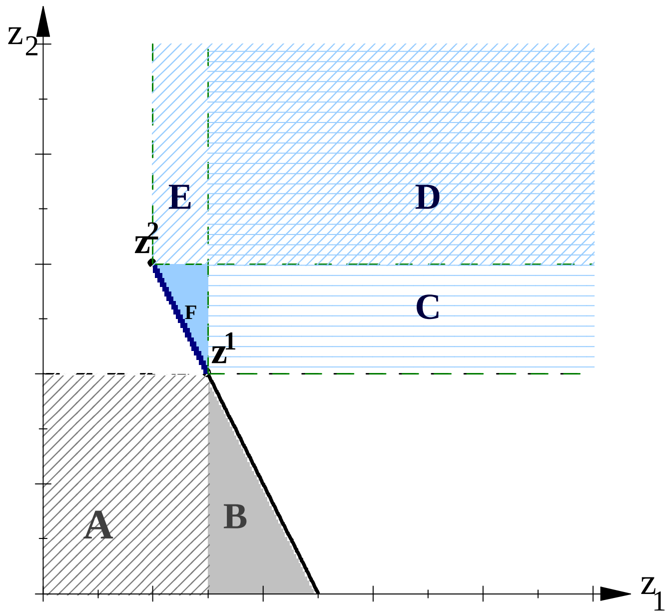

Figure 3.3: Using quasiconcavity

ways of selecting the sets of alternatives to ask the DM.

A more general result

Hazen [161] discusses the same results by showing that consideration of explicit responses (preference information) in the presence of increasing quasiconcave utility yields a stronger order than the strict componentwise vector order ($<$). He directly works on the DM's preference relation defined over the set of alternatives rather than assuming that it is represented by a value function.

Hazen assumes the following axioms for the DM's preference relation.

For a relation $\prec$:

1. Irreflexivity (I): not $z \prec z$, for all $z \in \mathbb{R}^p$.
2. Transitivity (T): $(z^1 \prec z^2$ and $z^2 \prec z^3) \Rightarrow z^1 \prec z^3$, for all $z^1, z^2, z^3 \in \mathbb{R}^p$.
3. Strict Monotonicity (SM): $z^1 < z^2 \Rightarrow z^1 \prec z^2$, for all $z^1, z^2 \in \mathbb{R}^p$.
4. Weak convexity (WC): $z^1 \preceq z^2$ and $z^3 \in (z^1, z^2] \implies z^1 \preceq z^3$, for all $z^1, z^2 \in \mathbb{R}^p$.
5. Convexity (C) : $z^1 \prec z^2$ and $z^3 \in (z^1, z^2] \implies z^1 \prec z^3$, for all $z^1, z^2 \in \mathbb{R}^p$.
6. Preference Data (P): $z^1 \prec z^2 \prec ... \prec z^m$ is provided by the DM.

Note that if the binary relation $\prec$ is representable by a value function, weak convexity and convexity axioms correspond to the quasiconcavity and strict quasiconcavity of the function, respectively.

The preference data axiom stands for the preference information provided by the DM, which is in the form of ranking m alternatives.

Hazen uses the concept of *unanimity order* which is analogous to the dominance concept in MCDM and defines it as follows.

Definition 17 *Assume that the decision maker's preference relation satisfies I, T, SM, WC, C and P; and denote the set of binary relations satisfying these properties with B. The unanimity order with respect to B is the binary relation $<^*$ over outcome vectors x and y and defined as follows: $x <^* y \iff x \prec y$ for all $\prec \in B$.*

Note that rational dominance is another unanimity order where B is the set of binary relations satisfying axioms reflexivity, transitivity and strict monotonicity.

The concept of unanimity order is important in MCDM as it allows us to infer results without knowing the exact preference relation of the DM over the whole decision space. If we know that the preference relation of the DM satisfies a set of properties, any result that holds for the corresponding unanimity order will hold for the DM's preference relation.

Hazen's main result is the following theorem. Note that we changed the terminology to ensure consistency in presentation. (See the original paper for the original notation and the corresponding proof).

Theorem 18 *Suppose $z^1, z^2, ..., z^m : z^1 \prec z^2 \prec ... \prec z^m$ are distinct elements of $\mathbb{R}^p$. Define the binary relation $\prec_m$ on $\mathbb{R}^p$ as follows:*

$x \prec_m y \iff x < y$ or $\exists j < m$ such that $x \in CD(z^{j+1}, z^{j+2}, ..., z^m;\ z^j)$ and $y \in UP(z^{j+1}, z^{j+2}, ..., z^m;\ x)$ and $y \neq x$.

Then $<^* = \prec_m$ unless $z^j \in P(z^{j+1}, z^{j+2}, ..., z^m)$ for some j, in which case, $<^* = \mathbb{R}^p \times \mathbb{R}^p$. (In which case, the DM is inconsistent).

This theorem is valid for any case where we replace the componentwise strict vector partial order $<$ by any irreflexive conical order (see the original paper [161]). For an arbitrary vector space A, $\prec$ is a conical order if it is a binary relation on A represented by a convex cone K. That is, for all $x, y \in A$; $x \prec y \iff y - x \in K$.

Given preference data $z^1 \prec z^2 \prec ... \prec z^m$ and the assumptions on the DM's preference relation, Theorem 18 provides a necessary and sufficient condition to conclude that a solution y is strictly preferred to solution x by the DM (without any further information): Either $x < y$ or there is an alternative z^j in the given preference set, for which x lies in $CD(z^{j+1}, z^{j+2}, ..., z^m;\ z^j)$ and y lies $UP(z^{j+1}, z^{j+2}, ..., z^m;\ x)$.

When the DM is consistent we have $x <^* y \iff x < y$ or $\exists j < m$ such that $x \in CD(z^{j+1}, z^{j+2}, ..., z^m;\ z^j)$ and $y \in UP(z^{j+1}, z^{j+2}, ..., z^m;\ x)$ and $y \neq x$.

It is easy to see that the sufficiency part of this proof holds as follows:

- If $x < y$ then $x <^* y$ (hence $x \prec y$). This is trivial from strict monotonicity.
- If $z \in CD(z^{j+1}, z^{j+2}, ..., z^m;\ z^j)$ for some z^j then $x \prec z^j$ by convexity, hence $x \prec z^{j+k}$ for $k = 1, ..., m-j$ by transitivity. Therefore any point $y : y \in UP(z^{j+1}, z^{j+2}, ..., z^m;\ x)$ and $y \neq x$ will be strictly preferred to x due to convexity.

The necessary part of the statement is not obvious. By proving the necessary condition, Hazen shows that under given assumptions on the DM's preference relation, for any two solutions x and y we cannot state that $x \prec y$; unless at least one of the two conditions is satisfied ($x < y$ or $\exists j < m$ such that $x \in CD(z^{j+1}, z^{j+2}, ..., z^m;\ z^j)$ and $y \in UP(z^{j+1}, z^{j+2}, ..., z^m;\ x)$ and $y \neq x$). This implies that under stated assumptions, by checking whether the conditions hold for two alternatives, a decision analyst will be making maximum use of the preference information available and if the conditions do not hold, s/he can be sure that it is not possible to conclude that one is preferred to the other without additional information.

This result is more general than the results provided in Lemmas 14 and 16 since the transitivity axiom allows indifference to be nontransitive, i.e. it includes some cases where the DM's preference relation is not representable by a value function. Moreover, Theorem

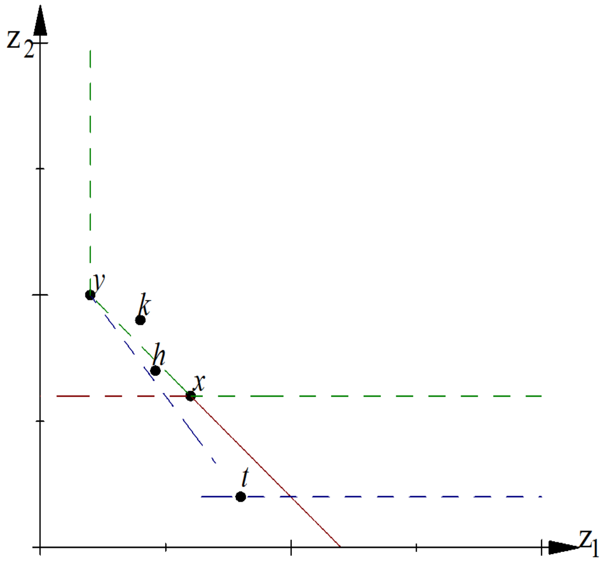

Figure 3.4: Example case

18 applies to any irreflexive conical order. Therefore, Lemmas 14 and 16 are special cases of Theorem 18.

An example

We will illustrate the results discussed so far using an example. We will show that under the same set of assumptions, the results provided in Lemmas 14, 16 and Theorem 18 provide us the same information. In the example, we assume that the DM's preference structure is representable by a strictly quasiconcave value function and use the componentwise strict vector partial order $<$ for Theorem 18.

Figure 3.4 shows the criteria space of a bicriteria problem where the DM strictly prefers y to x, i.e., $x \prec y$.

By Lemmas 14 and 16 we have the following:

1. $t \in CD(y; x)$ so $t \prec x$.

2. $k \in UP(y, x)$ hence $x \prec k$. We have $t \prec x \prec y$ and $t \prec x \prec k$. Based on the first ranking one can generate $P(y, x, t)$.

3. Observing that $h \in P(y, x, t)$ we say that h is preferred to t. So we have $t \prec x \prec y$, $t \prec x \prec k$ and $t \prec h$.

According to Theorem 18, given $x \prec y$;

1. $t \in CD(y;x)$ and $x \in UP(y;\, t)$ hence $t \prec x$. Now we have $t \prec x \prec y$.

2. $x \in CD(y;x)$ and $k \in UP(y;\, x)$ hence $x \prec k$. We have $t \prec x \prec y$ and $t \prec x \prec k$.

3. Finally, $t \in CD(y;\, x)$ and $h \in UP(y;\, t)$ hence $t \prec h$. We obtain the same results: $t \prec x \prec y$, $t \prec x \prec k$ and $t \prec h$.

3.3.2 Operationalizing the Theory

In this section we discuss how to operationalize the theory of convex cones. The research seeking to operationalize the convex preference cone ideas originated from the team Korhonen, Wallenius and Zionts, and their students and co-workers.

Checking cone dominance

The MCDM methods using convex cones gather preference information from the DM, usually in terms of pairwise comparisons. Using this information, more information is extracted about the other feasible alternatives which are not in the preference subset the DM provided. Given preference information, one can generate cones and check for each candidate point whether it is in a cone dominated region. If the objective is finding the best alternative out of a set, the cone dominated alternatives can be eliminated from further consideration. If we have a ranking or sorting problem, the information that an alternative is inside or in the upper side of the corresponding polyhedron is also useful. Therefore, one can also check the status of an alternative with respect to the generated polyhedra based on the preference information.

We can perform these checks using Linear Programming (LP) problems as follows.

Suppose that we want to check whether alternative z is dominated by $C(z^1, ..., z^{k-1}; z^k)$. Then we solve the following LP:

$$
\begin{aligned}
& Max\ \epsilon && (LP_1)\\
& s.t.\ \sum_{i=1}^{k-1} \mu_i(z^k - z^i) - \epsilon \geq z - z^k \\
& \mu_i \geq 0 \ \text{ for } i = 1, ..., k-1.
\end{aligned}
$$

Let us rewrite the first constraint set as follows: $z^k + \sum_{i=1}^{k-1} \mu_i(z^k - z^i) \geq z + \epsilon$. For each feasible μ_i value, the left hand side of the constraint corresponds to a point in the cone. If z is not dominated by any of the points in the cone, the maximum value that ϵ can take should be negative. Otherwise, if $\epsilon^* \geq 0$, the first constraint set implies $z^k + \sum_{i=1}^{k-1} \mu_i(z^k - z^i) \geq z$. Then for any nondecreasing quasiconcave function $f(.)$, we have $f(z^k + \sum_{i=1}^{k-1} \mu_i(z^k - z^i)) \geq f(z)$. That is, z is at most as preferred as a point in the cone, i.e., $z \in CD(z^1, ..., z^{k-1}; z^k)$ and $f(z^k) \geq f(z)$ by Lemma 14.

To check whether $x \in UP(z^1, ..., z^k)$ we solve the following LP:

$$
\begin{aligned}
& Max\ \epsilon && (LP_2)\\
& s.t.\ \sum_{i=1}^{k} \mu_i z^i + \epsilon \leq x \\
& \sum_{i=1}^{k} \mu_i = 1 \\
& \mu_i \geq 0 \text{ for } i = 1, ..., k.
\end{aligned}
$$

One can see that if $\epsilon^* \geq 0$ then $x \in UP(z^1, ..., z^k)$.

In each iteration we gain new information leading to new cones and polyhedra.

Tree representation of preferences and size of cones

Interactive approaches are based on gathering preference information from the DM throughout the process. Hence, such algorithms must keep track of the preference information gathered so far and generate the corresponding cones. A tree representation for preferences

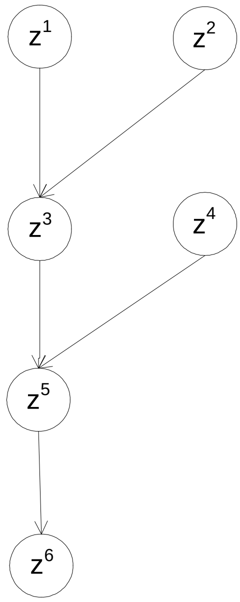

Figure 3.5: Tree representation

is suggested for this purpose in [163]. An example tree is given in Figure 3.5 where each node represents an alternative and each arc represents the preference relation between these alternatives. Using the tree representation it is possible to identify all distinct cones and polyhedra that can be generated given preference information.

Note that, the size of each cone as well as the number of total cones generated will affect the number of alternatives eliminated by the cones and hence the amount of information required from the decision maker. The computational time will also be affected. Different strategies can be used as follows:

- One might choose to generate all the possible cones with maximum number of generators. That is, for each alternative which is a candidate to be a lower generator (less preferred to at least one solution in the tree) one can generate a cone using all the alternatives preferred to that alternative as the upper generators. By doing so, one will be making maximum use of the information provided by the DM and hence less information is required from the DM ([152], [164]). In the example given in Figure 3.5 this corresponds to generating $C(z^1, z^2, z^3, z^4, z^5; z^6)$, $C(z^1, z^2, z^3, z^4; z^5)$, and $C(z^1, z^2; z^3)$. This is called *the minimal set of cones.*

- As the above method may be cumbersome and may result in high computational effort, one might choose to generate smaller cones ([152],[163]). In the literature in most cases only 2-point cones are used. In the example case in Figure 3.5 the distinct 2-point cones are $C(z^1; z^3)$, $C(z^2; z^3)$, $C(z^3; z^5)$, $C(z^4; z^5)$, and $C(z^5; z^6)$. Note that it is also possible to generate more 2-point cones using transitivity: e.g. $C(z^1; z^5), C(z^1; z^6), C(z^2; z^5), C(z^2; z^6)$. [152] report that in their experiments the computation time saved in using 2-point cones instead of the minimal set of cones was minimal and suggest using the the minimal set of cones. The use of cones with the most number of generators possible is suggested also in [162] owing to the decrease in the preference information required from the DM in that case.

Selecting the candidate solutions

Another issue to be considered in any interactive algorithm using convex cones is selecting the candidate solutions to be asked to the DM. The method used to select the alternatives depends on the characteristics of the problem. There are studies that use an estimated value function in an effort to select good candidates. At each iteration the parameters of the value function are updated and a solution that maximizes this value function (see e.g. [152]) is found and the DM is asked to compare this solution with its adjacent efficient alternatives (for an alternative z in the criterion space, adjacent efficient alternatives are the ones whose convex combinations with z are not dominated by any other convex combination generated by the rest of the alternatives). There are also studies selecting the candidates based on their distance to an ideal point (see e.g. [165]).

Interaction type

The interaction type refers to the type of the questions asked to the DM.

In the solution frameworks designed to select the best alternative, the DM is usually asked for pairwise comparisons of alternatives. Other strategies are also possible like providing the DM with m alternatives and ask him to select the best/ worst alternative in the set or rank these alternatives.

In the solution frameworks designed for sorting alternatives, the DM can also be asked to assign some alternatives to the predetermined groups. These alternatives are then used to generate cones and polyhedra.

3.3.3 Related Works in Literature

Many interactive MCDM methods using convex cones are proposed in the literature. Although not as common as convex cones, polyhedra are also used in sorting and partial ranking. The studies mainly differ in terms of the nature of the problem and the nature of the solution procedure, i.e, based on the way to elicit the preference information, the way to select the alternatives that will be presented to the DM for information gathering and the size of the generated cones.

The first problems considered are mostly multi-criteria evaluation problems where there is a finite set of alternatives. A series of papers that extend the application of convex cones to other problem types e.g. Multi-criteria Linear Problems (MCLP), Multi-criteria Integer Problems (MCIP) follow. There are also studies on increasing the efficiency of the solution procedures by finding ways to increase the region eliminated by cones and determining the number of the cone generators to be used. Recently, there has been an interest in using convex cones and polyhedra to sort and partially rank a finite set of alternatives.

We now review these applications of convex cones and polyhedra reported so far for MCDM problems.

Korhonen et al. [152] design an interactive algorithm for finding the best alternative in multi-criteria evaluation problems. They use convex cones to represent the preference structure. They generate cones and eliminate the alternatives which are inferior to these cones. By doing so, they successively restrict the solution space and try to obtain the best alternative after a number of iterations. In order to determine the alternatives to be asked to the decision maker, they use a composite linear value function of the form $\sum \delta_j x_{ij}$ where $\delta_j > 0$ are multipliers and x_{ij} is the performance score of alternative i in criterion j. They find the alternative that maximizes this function and ask the DM to compare it with other selected alternatives (the adjacent efficient alternatives). They start with an arbitrary set of multipliers and use the DM's responses to update the (feasible) set of multipliers. Note that this method approximates a nonlinear value function by a composite linear value function, hence it is possible that there are no set of multipliers consistent with the DM's preferences. This is handled by deleting the restriction on multipliers that correspond to the oldest information from the DM. In the proposed algorithm the information from the DM is gathered in terms of pairwise comparisons. The authors conduct experiments and report statistics on the number of pairwise judgements the DM has to make for various problem sizes. They first try generating all the cones and then repeat the experiments using only 2-point cones. Based on the results they conclude that the savings in the computational time when one is using only 2-point cones does not justify the increase the judgements DM has to make and hence suggest generating the minimal set of cones.

A few variations of the above algorithm is proposed by Köksalan et al. [166], Köksalan and Taner [167] and Köksalan and Sagala [168]. These studies address the same problem as in [152] and propose improvements in the solution algorithm. They suggest using dummy alternatives as one of the cone generators in order to increase the region eliminated by the convex cones. They discuss different ways to generate and select appropriate dummy alternatives. The authors report improvements in results compared to the algorithm used in [152] in terms of the total number of pairwise comparisons required to find the best alternative. In [168], which generalizes the results of the previous two papers ([166], [167]), two dummy alternatives are used simultaneously, one of which is (hopefully) less preferred to a real alternative so that it can be used as an upper generator of the cone. Another dummy alternative is generated as a potential lower generator. The dummy points are selected with the help of an estimated value function which is updated at each iteration.

Malakooti [169] also discusses different ways to select the cone generators as in [166], including the idea of using dummy alternatives to increase the area that is eliminated by the cone. He also discusses the use of local gradients, i.e., tradeoff information, at a point to eliminate worse alternatives and find better ones.

Ramesh et al. [163] study the underlying theory of the convex cones and the representation structure for the DM's preferences based on convex cones. They define rules to detect redundant cones, the cones which are already implied by other existing cones, in order to avoid unnecessary computations. They discuss two ways to represent the DM's preferences. The first one is an explicit representation, which is used in methods that use a single composite function of the multiple objectives. In these methods the preferences are represented using linear inequalities on the weights of this composite function thus the set of feasible weights is successively reduced. The second representation scheme is based on the convex cones. They conclude that representation via cones is more accurate than representation via linear constraints on weights of the objectives. This is due to the fact that by using a composite objective function one fails to accurately represent nonlinear value functions. The authors conduct experiments on the use of convex cones in solving both multi-criteria design (MCLP, MCIP (in the experiments bicriteria problems are solved)) and multi-criteria evaluation problems and report the percentage of questions saved using convex cones.

Ramesh et al. [164] incorporate the convex cones representation method into the algorithm of Zionts and Wallenius [170] for multicriteria linear programming problems. The authors assume that the DM has a pseudoconcave value function. Convex cones are used to obtain an accurate and robust representation of the DM's preferences. The DM is asked for pairwise comparisons of alternatives, one of them is an alternative that maximizes a composite linear value function and the other is an adjacent efficient alternative. Only 2-point cones are generated.

Ramesh et al. [171] develop an algorithm for the multi-criteria integer problems where the underlying value function of the DM is assumed to be pseudoconcave. They use the method previously proposed by Zionts and Wallenius [170] for MCLP in a branch and bound framework and use convex cones for the preference structure representation. The cones are used for fathoming candidate nodes in the branch and bound tree. They only generate 2-point cones and report computational results for bicriteria problems with up to 80 variables and 40 constraints.

Köksalan [165] proposes an interactive method using convex cones to identify and rank a most preferred subset of alternatives in multi-criteria evaluation problems. The

DM is asked for pairwise comparisons. Initially, candidate solutions are selected based on their distance to an ideal point and throughout the algorithm new candidate solutions are generated using a weighted quadratic value function as in [166].

Taner and Köksalan [172] conduct an experimental study on how to determine the number of cone generators, to select the cone generators and determine the order of pairwise comparisons to ask the DM. They also propose an algorithm based on the results of the experiments. In the proposed algorithms, $m(m = 1, 2..., 7)$ alternatives are selected by three methods, which use equal weighted linear, estimated linear and quadratic value functions, respectively. Then the least/most preferred alternative in the selected set is found by asking the DM a number of pairwise comparisons. Based on the preference information elicited from the DM, all the possible cones are generated. They conclude that the version where $m = 3$ and the least preferred alternative is found provides the best results in terms of the average number of comparisons the DM has to make. They point out that the results of the experiments are not very conclusive hence a more elaborate and detailed study awaits further attention.

Prasad et al. [162] observe that in the computations, although most of the solutions are not cone dominated, quite a few of them are nearly cone dominated. Motivated by this observation, they introduce the concept of "near cone dominance" or "p cone efficiency". A p value is used to show how close an alternative is to being dominated by the generated cone. They suggest using this measure to choose the challengers of the incumbent that will be presented to the DM for pairwise comparison. That is, the alternatives are presented to the DM for comparison in an order based on how close they are to be inferior. This heuristic extension is suggested to accelerate the search by reducing preference information requirement. Another use of this idea is for early termination. That is, for an incumbent solution, the p cone efficiencies of its adjacent solutions are calculated and the algorithm is terminated if maximum of these p cone efficiencies is below a given threshold. They illustrate the idea by incorporating it within a solution framework for solving MOLP problems.

Ulu and Köksalan [155] propose interactive approaches to partition a set of discrete alternatives into acceptable and unacceptable sets. Assuming the DM has a quasiconcave value function they use the convex cones and polyhedra in a sorting algorithm. Note that the proposed sorting algorithm is for the special case where the number of classes is two, rather than for the general case with more than two classes.

Another study that includes a sorting approach based on convex cones is Fowler et al. [173]. They propose an evolutionary algorithm, a genetic algorithm, for MCDM problems assuming that the DM has a quasiconcave preference function. Genetic algorithm is a widely studied method for approximately solving NP-hard problems. In a genetic algorithm the output is obtained by evolving an initial population of solutions through multiple generations by breeding and mutation. Hence, how the parents are selected becomes a critical issue to ensure offspring with good quality. The authors suggest partially sorting the population with the help of convex cones and polyhedra. The information taken from the DM is in terms of finding the best and worst in a given set of six solutions. Using this information they generate four 2-point cones (consisting of the best as the upper generator and each of the other points except the worst) and one 6-point cone (having the worst as the lower generator and the others as upper generators). They report that using convex cones in sorting is effective and improves the output solution.

Recently, Dehnokhalaji et al. [154], propose an approach which uses convex cones and polyhedra to partially order a finite set of alternatives. Similar to the p cone efficiency concept discussed in [162], an alternative is classified as *surely better than*, *surely worse than*, or *possibly better/possibly worse than* the lower generator of the cone. This makes the suggested approach flexible in the sense that it can be used as an exact or approximate approach by adjusting some parameters. They generalize the idea used in [162] and employ it to obtain a strict partial order. This approach is suggested to be used to partially rank alternatives or as a supplementary method in other solution approaches such as Evolutionary Multi-Objective Optimization as discussed in [173]. Developing an interactive solution algorithm based on the approach is pointed out as a subject of future research.

3.4 Conclusion

In this paper we consider the use of convex cones in interactive MCDM approaches, which is based on the DM's holistic assessments. We describe the assumptions on the DM's preference relation and the value function that allow us to implement the convex cones approach in solution algorithms for MCDM problems. We summarize the basic theoretical results from the literature, which show that convex cones can be used to iteratively approach the most preferred solution(s) by eliminating the ones that are less preferred. The results also show that, in multi-criteria evaluation problems, polyhedra can be used alongside the convex cones for sorting or obtaining a quasiordering of the alternatives.

We provide a review of the studies that implement the convex cones approach over the last three decades. A large body of literature exists that use the convex cones approach in various algorithms designed to solve both multi-criteria evaluation and multi-criteria design problems. There are also studies on how to select good candidate solutions (cone generators) to be asked to the DM. We mention below a few challenges regarding convex cones approach awaiting further research.

- Further research can be performed on using convex cones to solve more difficult problems, like Multi-Objective Combinatorial Optimization (MOCO) problems.

- Asking too many or relatively difficult questions increase the cognitive burden on the DM and may make the decision support system less attractive. Therefore, one needs to choose the type of interaction with the DM carefully. Further research can be conducted on finding the best way to minimize the cognitive burden while ensuring an effective decision support. Different types of questions can be asked to the DM. These include making pairwise comparisons, determining the best (worst) alternative in a given subset, or ranking a small subset of alternatives. Experimental studies that compare these different interaction modes and discuss their advantages and disadvantages in different problem settings can be performed.

- As the size and number of cones used in an algorithm increase, more alternatives are eliminated but the computational time may increase as well. By generating merely two-point cones one may not be utilizing the available information fully. To the best of our knowledge, it is not studied in detail how well this approach works compared to generating the *minimal set of cones* in terms of the information required from the DM throughout the procedure. In other words, it is still an open question how to determine the cone sizes that best balance the computational time with the amount of information required from the DM for different problem settings.

- Using the information provided by the polyhedra in ranking and sorting environments is also a promising area. More research on the use of polyhedra as well as convex cones for sorting and ranking purposes awaits further attention.

Chapter 4

Incorporating Preference Information in Multicriteria Problems with Equitability Concerns

4.1 Introduction

Multicriteria Decision Making (MCDM) problems deal with evaluation of alternatives based on a number of criteria. In most of the MCDM problems that have been studied so far, we do not measure the criteria using the same type of measure. We will call such criteria *unlike* and the MCDM problems with heterogenous criteria *classical MCDM* problems.

In this study we discuss a special type of multicriteria decision making problem where we have equitability concerns. These problems are motivated by the cases where we are concerned with finding an equitable allocation of a good or bad among multiple parties who are anonymous. That is, the identities of the parties are irrelevant and do not affect the decisions. In these problems, we have a single type of good (or bad), and the multiple parties receive an amount of this good (bad). For example, we may define a location problem where the parties are location points and the good is distance or time. Similarly, we may have a problem where the parties are individuals or groups of individuals and the allocated good is income, service etc. One can define such problems as MCDM problems. In an alternative, each criterion value corresponds to the amount of good allocated to a party. Hence, unlike a classical MCDM problem, the criteria in these problems are *like*, i.e., each criterion is measured in the same scale with the same measure.

Problems with equity concerns are encountered especially in the public sector. The problems include location problems where we try to find the best location of public service facilities such as hospitals or fire stations. In these location problems, the good is usually taken as the distance of the service facility to the demand points and we try to reach a feasible solution in which the demand points are treated as equitably as possible [174]. Income distribution problems are another example. In such income distribution problems the decision maker (DM) is faced with different policies that will result in different income distributions among the population. These income distribution profiles are analogous to the allocation profiles used in the health economics literature, where the outcome is measured in terms of the Quality Adjusted Life Years (QALY) that an individual is expected to live as a result of a health policy [175]. Other application areas include distribution of funds [176] and financial portfolio optimization [177], where the rate of return for each security considered is given by a finite discrete distribution and the investor has a risk-averse attitude.

Since those problems have different properties than a classical MCDM problem, they require different approaches than the classical methods. In this study, we discuss an approach that is appropriate for such problems.

The rest of the paper is organized as follows. In Section 4.2 we introduce the basic concepts and discuss the existing theory. We especially discuss the case where equity concerns are incorporated in the preference relation. Next, we present our problem formulation and discuss relevant results and methods from the literature. In Section 4.3, we discuss our contributions and present our work on how to extend the convex cones theory for problems that have the anonymity property. The results we provide in this section are general in the sense that they can be used in different settings where the objective is finding the best out of a set, ranking or sorting. We present an interactive ranking algorithm in Section 4.4 and report the results of our computational experiments in Section 4.5. Section 4.6 concludes the study with our main findings and future research plans.

4.2 Problem Definition

In a typical multicriteria evaluation problem we have a (finite) set of alternatives (defined in the criteria region) denoted by Z with a typical member as $z^i = (z^i_1, ..., z^i_p)$, where z^i_j is the performance score of solution z^i on criterion j and p is the number of criteria. We evaluate the alternatives based on these criteria. Different problematiques can be defined depending on the desired outcome: finding the best alternative or a limited set of the best alternatives, ranking the alternatives or sorting them into predefined groups [145].

In this study we focus on multicriteria problems with like criteria. In our setting, each alternative $z^i \in Z$ shows a distribution of a good among p parties and hence z^i_j is the amount of good that party j gets in alternative i. Although our theoretical results on convex cones are applicable to any type of problematique (selecting the best, ranking or sorting), we will focus on the ranking problematique. This is because ranking can be considered as a generalization of finding the best: once you rank the alternatives, you obtain the best alternative in the set. Moreover, if we have a ranking of the alternatives, it may be easier to sort them into groups.

4.2.1 Equity and Efficiency

We consider the problem settings where we try to rank the alternatives considering both efficiency and equity (fairness). For a problem where we have a desirable good, equity and efficiency preferences are our preferences for having a more equitable allocation and a higher mean, *ceteris paribus.* We will try to explain the efficiency and equity concepts and the tradeoff between them by using a small example. Note that, although we have a general problem formulation which is applicable to different settings with different types of goods and parties, from now on, we will be giving examples of income distribution problems to explain our ideas.

Example 19 *Suppose that we have a 2-person population and we have a set of alternatives, each corresponding to an income distribution between these two people. Assume that equity is taken into account by obtaining an index by using inequality measure.*

If we increase efficiency while keeping the inequality level the same, we obtain a better distribution. For example if we have $(3,3)$ *and* $(4,4)$ *as two alternatives, we can directly say that* $(4,4)$ *is preferred to* $(3,3)$*, since both alternatives have complete equality while in* $(4,4)$ *the total amount of income is higher. This corresponds to our efficiency preference.*

If we have a more equitable alternative while keeping the efficiency level constant, we obtain a better distribution. For two distributions that have the same total income, the efficiency levels are the same. Hence, we will base our decisions on the inequality levels. For example alternative $(3,5)$ *is preferred to* $(2,6)$ *since it is more equitable. This corresponds to our equity preference.*

One cannot make such judgements when the distributions concerned do not have the same level of (in)equality or efficiency. Moreover, even the measurement of inequality is a problem in itself; so when we use inequality measures to incorporate equity preferences, we will get different results for different types of measures. For instance, we cannot say anything about how the preference should be between $(4,4)$ *and* $(3,6)$*. In this example the alternative with the higher total income,* $(3,6)$*, is also more unequal. We can observe the tradeoff between equity and efficiency here.*

There are different ways to incorporate both of these preferences into a decision model. Throughout this work we assume that an increase in any individual's income is desirable as long as none of the other individuals is worse off. This assumption will allow us to draw conclusions about the alternatives that are vector dominating each other. For example, when faced with alternatives $(4,4)$ and $(4,5)$; $(4,5)$ should be preferred to $(4,4)$. While this may seem as a simplifying assumption ignoring or undermining the effect of the increase in the envy of the people whose incomes did not increase on the choice made, this is a standard assumption used in many solution approaches. This assumption helps us to compare some alternatives where the inequality level is not the same. By increasing the number of comparisons that we can make, this assumption will help us to come up with a practical solution procedure as discussed in Shorrocks [136].

The equitability concern is usually considered using the well-known Pigou-Dalton (PD) principle in inequality theory, which ensures that transferring a small amount of money from a person to a relatively worse off one without changing their relative positions to each other, results in a more preferred alternative. To illustrate, if we have $z = (3, 5)$ transferring 1 unit of income from the second person to the first one, we obtain $(4, 4)$, which is equally efficient but more equitable, thus is considered as a better alternative. Those ideas will give shape to the axioms of the preference model that we will assume.

4.2.2 Equity and Impartiality in Preference Relations

In the problems that we study, dealing with uniform criteria brings the property of anonymity to the preference model, i.e. the corresponding preference relation should treat all the permutations of a vector as indifferent. The rational preference relations satisfying anonymity (impartiality) are called *impartial rational preference relations* by [142].

The equity preference can be taken into account by an axiom based on the Pigou-Dalton principle of transfers.

We assume that the DM has a preference model in which the preference relation satisfies the following axioms [87]: Reflexivity (R), Transitivity (T), Strict Monotonicity (SM), Anonymity (Impartiality) (A) and Pigou-Dalton principle of transfers (PD).

1.Reflexivity (R): $z \preceq z$ for all $z \in Z$.

2.Transitivity (T): $(z^1 \preceq z^2$ and $z^2 \preceq z^3) \Rightarrow z^1 \preceq z^3$, for all $z^1, z^2, z^3 \in Z$.

3.Strict monotonicity (SM): $z^1 < z^2$ then $z^1 \prec z^2$, for all $z^1, z^2 \in Z$.

4.Anonymity (Impartiality, Symmetry) (A): $(z) \sim \Pi^l(z)$ for all $l = 1, ..., p!$, for all $z \in \mathbb{R}^p$, where $\Pi^l(z)$ stands for an arbitrary permutation of the z vector.

5.Pigou-Dalton principle of transfers (PD): $z_j > z_i \Rightarrow z \prec z - \varepsilon \mathbf{e}_j + \varepsilon \mathbf{e}_i$, for all z

$\in \mathbb{R}^p$,where $0 < \varepsilon < z_j - z_i$, where $\mathbf{e}_i$, $\mathbf{e}_j$ are the i^{th} and j^{th} unit vectors in $\mathbb{R}^p$.

The preference relations that satisfy R, T, SM, A and PD are called *equitable rational preference relations* and the relation of *equitable dominance* ($\preceq_e$) is the intersection relation of all equitable rational preference relations. Equitable dominance implies rational dominance, which is the intersection relation of all rational preference relations (satisfying R, T and SM), but not vice versa. This is clear from the fact that the axioms used to define rational dominance is a subset of the axioms used to define equitable dominance.

Following Kostreva and Ogryczak [87], we can introduce the ordered vector and cumulative ordered vector for an alternative z as follows:

Definition 20 *Given $z \in \mathbb{R}^p$, let $\overrightarrow{z}$ denote the permutation of z such that $\overrightarrow{z} : \overrightarrow{z}_1 \leq \overrightarrow{z}_2 \leq ... \leq \overrightarrow{z}_p$. $\overrightarrow{z}$ is called the ordered vector of z and $\overrightarrow{\mathbb{R}}^p = \{\overrightarrow{z} : z \in \mathbb{R}^p\}$ is called the ordered space.*

Definition 21 *Given $z \in \mathbb{R}^p$, let $\overline{\Theta} : \mathbb{R}^p \to \mathbb{R}^p$ be the cumulative ordering map defined as follows:*

$\overline{\Theta}(z) = (\overline{\theta}_1(z), \overline{\theta}_2(z), ..., \overline{\theta}_p(z))$ where $\overline{\theta}_i(z) = \sum_{j=1}^{i} \overrightarrow{z}_j$ for $i = 1, 2, ..., p$. $\overline{\Theta}(z)$ is called the cumulative ordered vector of z.

The following result is proved by [142]:

Theorem 22 *For any two alternatives z^1 and z^2,*

$z^1 \prec_e z^2 \Longleftrightarrow \overline{\theta}_i(z^1) \leq \overline{\theta}_i(z^2)$ for all $i \in \{1, 2, ..., p\}$ where at least one strict inequality holds.

$z^1 \preceq_e z^2 \Longleftrightarrow \overline{\theta}_i(z^1) \leq \overline{\theta}_i(z^2)$ for all $i \in \{1, 2, ..., p\}$, that is $\overline{\Theta}(z^1) \leq \overline{\Theta}(z^2)$.

This relation will allow us to use rational dominance concept, hence vector inequality, on the cumulative ordered vectors to check equitable dominance relation for two alternatives.

Note that the PD axiom is an axiom of convexity, but defined only for the alternatives that have the same total amount. One can define a convex preference by replacing the

PD axiom with the convexity axiom that involves alternatives with different sums. We use the following convexity axiom:

6.Convexity (C): $z^1 \preceq z^2$ and $z^3 \in (z^1, z^2) \implies z^1 \prec z^3$. Here $z^3 \in (z^1, z^2)$ means that there exists a real α, $0 < \alpha < 1$ such that $z^3 = \alpha z^1 + (1-\alpha)z^2$.

We will call relations satisfying R, T, SM, A and C *impartial convex rational preference relations.* The corresponding dominance relation, *impartial convex dominance*, will be implied by rational dominance and equitable dominance. That is because the axioms used for rational dominance is a subset of the axioms used for impartial convex dominance and C is a more general (restrictive) condition than PD. In fact, given impartiality, C reduces to PD for the alternatives with the same total.

If an equitable rational preference relation is representable by a utility function, the function has to be increasing strictly Schur-concave. If we assume that the DM has an impartial convex rational preference relation which is representable by a utility function, then the function has to be symmetric increasing strictly quasiconcave.

Definition 23 *A function f is quasiconcave if for all $z^1, z^2 : z^1 \neq z^2$ and $\alpha \in [0,1]$ we have $f(\alpha z^1 + (1-\alpha)z^2) \geq \min\{f(z^1), f(z^2)\}$. Similarly f is strictly quasiconcave if $f(\alpha z^1 + (1-\alpha)z^2) > \min\{f(z^1), f(z^2)\}$ for all $f(z^1) \neq f(z^2)$ and $\alpha \in (0,1)$.*

Each symmetric quasiconcave function is a Schur-concave function. On the other hand a Schur-concave function is not necessarily a symmetric quasiconcave function. For example, the following function:

$f(z_1, z_2) = z_1 z_2$ is Schur-concave although it is not quasiconcave in $\mathbb{R}^2$ [141].

Note that, if we assume monotonicity rather than strict monotonicity then the corresponding utility function is nondecreasing rather than increasing. Moreover, if we relax the convexity condition, PD or C, and replace the term "strictly better than" with "at least as good as", the corresponding function does not have to be strictly Schur-concave (strictly quasiconcave) but Schur-concave (quasiconcave).

4.2.3 Problem Definition

We consider the following problem:

Given $Z \subseteq R^p$, provide a ranking for all $z \in Z$ based on $\preceq$, where $\preceq$ is an impartial convex rational preference relation representing DM's preferences. Recall that equitable preference implies impartial convex preference.

$\preceq$ is a relation of weak preference, satisfying A and C axioms and we will denote the corresponding strict preference and indifference relations as $\prec$ and $\sim$, respectively. In this study, we do not require that $\preceq$ is represented by a utility function but if it is so, the function is symmetric increasing strictly quasiconcave.

4.3 Convex Cones in MCDM Problems with Equitability Concerns

A large body of literature dealing with classical multi-criteria problems is based on the use of convex cones to incorporate preference information in the model. However, all the problems discussed are classical MCDM problems, hence do not consider equity issues. To the best of our knowledge, the concept of equitability in multi-criteria problems is relatively new. Kostreva and Ogryczak [142] are the first ones who introduce the equitability concept in the MCDM environment. Other studies are due to Ogryczak [174], Kostreva et al. [87], Baatar and Wiecek [88], and Mut and Wiecek [86]. These studies do not provide us with a direct way to incorporate DM's preference information in the model. To the best of our knowledge, there are not many approaches that incorporate DM's preference information in the model for an MCDM problem with equitability concerns. There are a few studies that mention the possibility of using the reference point method (e.g. [10]) with very limited computational experiments and discussion. Moreover, to the best of our knowledge there are no studies that apply the convex cones approach in a symmetric environment. We try to fill this gap by analyzing the use of convex cones in that context.

In this section we discuss the use of convex cones in multicriteria problems where we have equitability concerns, hence anonymity and convexity properties. We extend the current theoretical framework for convex cones by introducing the anonymity property.

4.3.1 Definitions and Notations

Definition 24 *Given* $z \in Z$ *define* $\Pi^s(.)$ *as the permutation function* : $\Pi^s(z) = I_s z$, $s = 1, ..., p!$ *where* I_s *is a matrix rearranging elements of a vector.*

Note that each vector of size p will have $p!$ permutations and the DM is indifferent to all these permutations. If a utility function exists, each will have the same utility value.

Definition 25 *We will define the lower section of* z *as follows:* $L(z) = \{y \mid y \prec_e z\}$ *and the upper section of* z *as* $U(z) = \{y \mid z \prec_e y\}$.

We have $U(\Pi^s(z)) = U(z)$ and $L(\Pi^s(z)) = L(z)$, $\forall s$.

See Figure 4.1 for a two dimensional (2D) example. The green and blue regions include the alternatives that equitably dominate $(4, 3)/(3, 4)$ and are dominated by $(4, 3)/(3, 4)$; respectively.

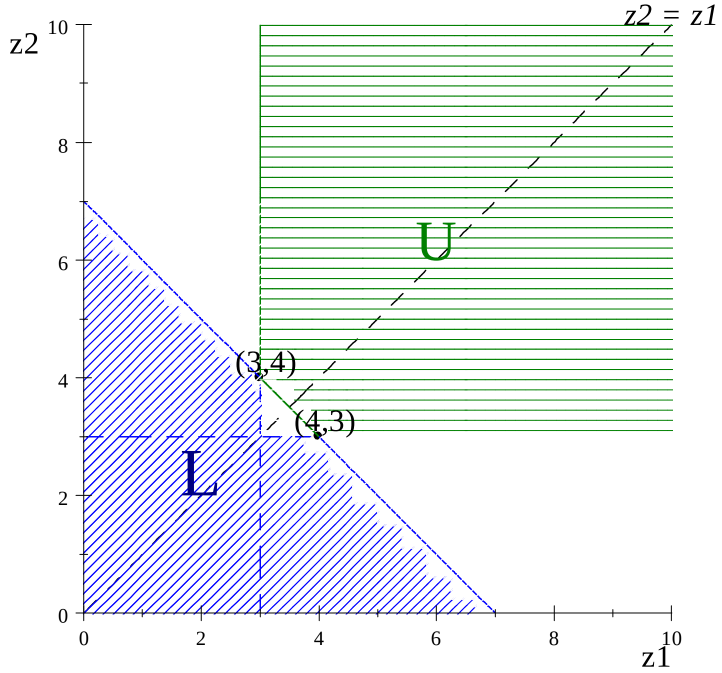

Figure 4.1: Upper and lower sections of $(3, 4)/(4, 3)$

Given a set of k vectors $z^1, ..., z^k \in \mathbb{R}^p$ such that $z^i \succ z^k$ for all $i \neq k$, where $\succ$ denotes an impartial convex rational preference relation, we will define the following:

We define the cone $C(z^1, ...; z^k)$ where $z^i : i \neq k$ are the *upper generators* and z^k is the *lower generator* as follows:

$$C(z^1, ...; z^k) = \{z \mid z = z^k + \textstyle\sum_{i \neq k} \mu_i (z^k - z^i),\ \mu_i \geq 0\ \forall i \neq k\}.$$

We define the corresponding cone dominated region $CD(z^1, ...; z^k)$ as follows:

$$CD(z^1, ...; z^k) = \{z' \mid z' \preceq_e z \text{ where } z \in C(z^1, ...; z^k)\}.$$

Note that $CD(z^1, ...; z^k)$ includes $C(z^1, ...; z^k)$. Let us denote the set of all impartial convex rational preference relations consistent with the given preference information as *IC (**I**mpartial **C**onvex).* For any $z \in CD(z^1, ...; z^k)$, $z \preceq z^k$ for all preference relations in set *IC*. Each such point is *equitably dominated by* $C(z^1, ...; z^k)$ and will be called as *equitably cone dominated.* Note that we use weak dominance here, i.e. when we say an alternative is equitably cone dominated, that means it is equitably weakly dominated.

Let $I = \{1, 2, ..., k\}$.The polyhedron spanned by the vectors $z^1, ..., z^k$ is defined by the following expression:

$$P(z^1, ..., z^k) = \{z \mid z = \textstyle\sum_{i \in I} \mu_i z^i,\ \sum \mu_i = 1,\ \mu_i \geq 0 \text{ for all } i \in I\}.$$

We define its upper side as follows:

$$UP(z^1, ..., z^k) = \{z' \mid z \preceq_e z' \text{ where } z \in P(z^1, ..., z^k)\}.$$

We again note that $UP(z^1, ...; z^k)$ includes $P(z^1, ...; z^k)$. For any $z \in UP(z^1, ..., z^k)$, $z^k \preceq z$ for all weak preference relations in set *IC*. This is a direct result of the convexity axiom.

Figures 4.2 and 4.3 below show a 2-point cone, $C((2, 6); (3, 4))$ with its equitably dominated region and $P((2, 6); (3, 4))$ with its upper side, respectively. In Figure 4.2 the line between points A and B is $C((2, 6); (3, 4))$ and the blue region with diagonal lines is the equitably cone dominated area. In Figure 4.3 the black line is $P((2, 6); (3, 4))$ and the upper side is the green region with diagonal lines.

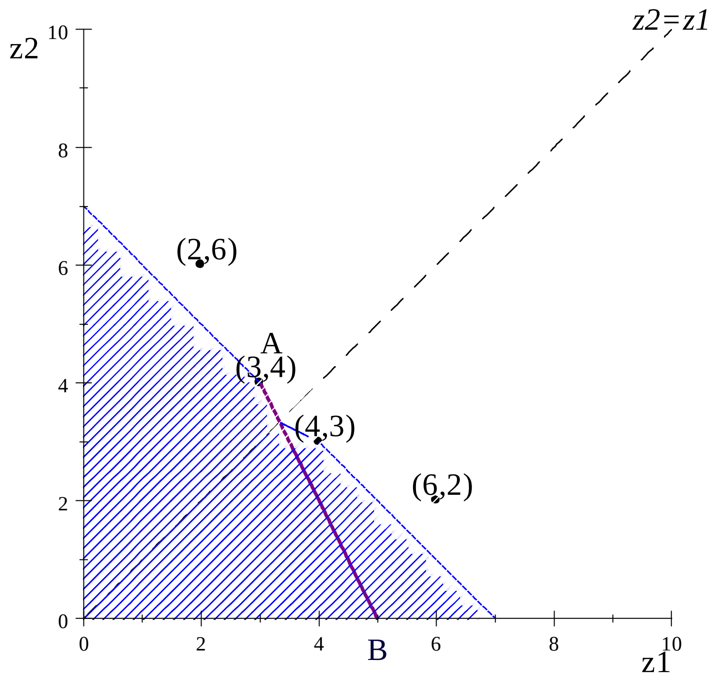

Figure 4.2: $C((2,6);(3,4))$ and its equitably dominated region

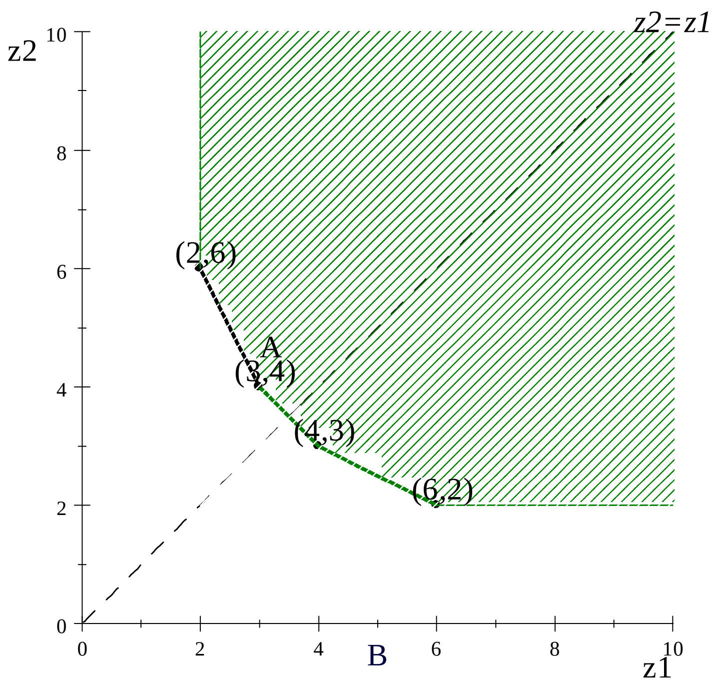

Figure 4.3: $P((2,6);(3,4))$ and its upper side

Based on the reported satisfactory performance of the convex cones approach in non-symmetrical MCDM settings, we propose using convex cones and polyhedrons in solving MCDM problems with equitability concerns, i.e. in a symmetrical setting. The main idea is the same: we obtain some preference information from the DM, generate the corresponding cones and use them to determine the alternatives that are equitably dominated by the cone, hence less preferred than the cone generators. Similarly, by using the polyhedron, we can determine the alternatives that are preferred to the lower generator of the polyhedron. However, the impartiality property brings computational difficulties. This is because once we get preference information about a set of alternatives, we have information on the relation of all permutations of these alternatives, each leading to a different cone. To illustrate, given $z^1, z^2 \in \mathbb{R}^p$ such that $z^1 \succ z^2$, we can generate $p! * p!$ 2-point cones such that $C(\Pi^r(z^1); \Pi^s(z^2))$ $\forall r = 1, ..., p!$ and $\forall s = 1, ..., p!$. We call each $C(\Pi^r(z^1); \Pi^s(z^2))$ a *permutation cone.*

Table 4.1 below illustrates the complexity that anonymity brings.

Table 4.1: Computational complexity due to impartiality

pD 2-point cones		**pD k-point cones**	
# of binary comparisons	# of cones	# of k-ary comparisons	# of cones
1	$p!p!$	1	$(p!)^k$
n	$n * p!p!$	n	$n * (p!)^k$

Example: Suppose that we have only two people in our population (that is, $p = 2$) and suppose that the DM has an impartial convex rational preference relation.

We have a number of possible income distributions for these two people, two of which are $(6, 2)$ and $(3, 4)$. Note here that $(6, 2)$ is more unequal than $(3, 4)$ in the sense that the gap is bigger between the two income levels, but the total income in this distribution is larger than the latter one. Hence, these two distributions are not equitably dominating one another. We need some extra information about the DM's preferences and we will gather it by asking him/her to choose one of the alternatives. Based on the preference information that the DM provides, we can generate the corresponding cones.

Suppose that the DM prefers $(6, 2)$ to $(3, 4)$. Thanks to our anonymity (symmetry, impartiality) axiom, this preference is valid for all the permutations of our alternatives. That is we have, $(6, 2) \succ (3, 4)$; $(6, 2) \succ (4, 3)$; $(2, 6) \succ (3, 4)$ and $(2, 6) \succ (4, 3)$. Figure 4.4 below shows the four 2-point cones that are generated based on this information.

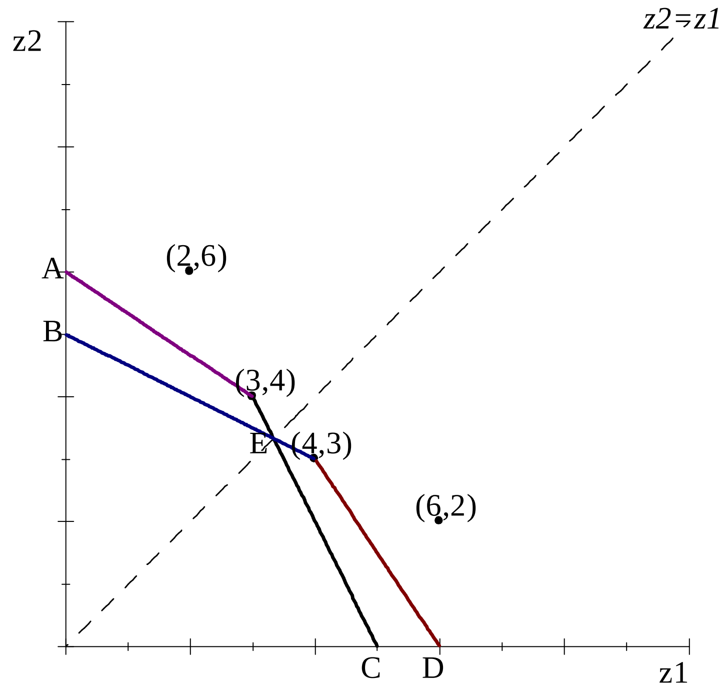

Figure 4.4: Cones generated based on $(6, 2) \succ (3, 4)$

$C((6, 2); (3, 4))$ is the purple line connecting $(3, 4)$ and point A. $C((6, 2); (4, 3))$ is the yellow line connecting $(4, 3)$ and point B etc.

In this 2D example, one can easily see that $C((2, 6); (3, 4))$ and $C((6, 2); (4, 3))$ are symmetric with respect to the equality line ($z_2 = z_1$). The same holds for $C((2, 6); (4, 3))$ and $C((6, 2); (3, 4))$.

In the next part we provide some theoretical results that will help us to deal with the computational complexity due to symmetry. We start by analyzing the case where we only generate 2-point cones. Next, we provide some results for the case where we have k-point cones where $k > 2$.

4.3.2 Results for the 2-point Cones Case

Recall our PD axiom which is also called Pigou-Dalton principle of transfers. In an alternative, we call a transfer that takes an amount of good from a party and gives it to a poorer party without changing their relative positions to each other a *Pigou-Dalton (P-D) transfer.*

Lemma 26 *If $z \preceq_e z'$ then there is a $z'' \in \mathbb{R}^p$ such that $z'' \leq z'$, and z'' is obtainable from z by a finite number of P-D transfers.*

Proof. See Ok [178], Lemma 1 for the proof. ■

Lemma 27 *Let z and $z' \in \mathbb{R}^p$ such that $z_i = z'_i \ \forall i \neq h, h+1$. Then $z \preceq_e z'$ if and only if $Min\{z_h, z_{h+1}\} \leq Min\{z'_h, z'_{h+1}\}$ and $z_h + z_{h+1} \leq z'_h + z'_{h+1}$.*

Proof. In this proof, we use the basic axioms that define our preference model. Recall that we assume A, SM and C, which implies PD.

Necessity:

This proof comes from the definition of equitable dominance. $Min\{z_h, z_{h+1}\} \leq Min\{z'_h, z'_{h+1}\}$ and $z_h + z_{h+1} \leq z'_h + z'_{h+1}$ imply $\overline{\Theta}(z) \leq \overline{\Theta}(z')$, hence $z \preceq_e z'$.

Sufficiency:

From Lemma 26, if $z \preceq_e z'$ then there is a $z'' \in \mathbb{R}^p$ such that $z'' \leq z'$, and z'' obtainable from z by a finite number of P-D transfers. Suppose that we have obtained a z'' such that $z'' \leq z'$ holds. Without loss of generality suppose that $Min\{z_h, z_{h+1}\} = z_h$ and $Min\{z'_h, z'_{h+1}\} = z'_h$. If this is not the case, we can arrange them accordingly since we have anonymity.

Suppose that at least one of the following holds:

$$z_h > z'_h \text{ or } z_h + z_{h+1} > z'_h + z'_{h+1} \tag{A1}$$

$z'_i = z_i \ \forall i \neq h, h+1,$ so for $z'' \leq z'$ to hold, the P-D type transfer in distribution z to obtain z'' should be from z_{h+1} to z_h.

That is, $z''_h = z_h + \epsilon$, $z''_{h+1} = z_{h+1} - \epsilon$, where $0 \leq \epsilon \leq z_{h+1} - z_h$.

$z'_i = z_i = z''_i \; \forall i \neq h, h+1$ and $z'' \leq z' \Rightarrow z'_h \geq z''_h = z_h + \epsilon$ and $z'_{h+1} \geq z''_{h+1} = z_{h+1} - \epsilon$.

That is, $z'_h \geq z_h$ and $z'_h + z'_{h+1} \geq z_h + z_{h+1}$, which is a contradiction to our initial assumption A1. ■

Lemma 28 *Given z^1, $z^2 \in \mathbb{R}^p_+$, if $z \in C(z^1; \overrightarrow{z^2})$ then $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$ (i.e. z is equitably dominated by $C(\overrightarrow{z^1}; \overrightarrow{z^2})$).*

Proof. Let $z^1 \neq \overrightarrow{z^1}$ (Otherwise, the result is immediate). Let h be the minimum value for which $z^1_h > z^1_{h+1}$ holds. Define $z^{1\prime}$ as the permutation obtained from z^1 by swapping z^1_h and z^1_{h+1}. That is, $z^1 = (z^1_1, z^1_2, ..., z^1_h, z^1_{h+1}, ..., z^1_p)$ and $z^{1\prime} = (z^1_1, z^1_2, ..., z^1_{h+1}, z^1_h, ..., z^1_p)$ where $z^1_h > z^1_{h+1}$. We will show the following holds:

If $z \in C(z^1; \overrightarrow{z^2})$ then $z \in CD(z^{1\prime}; \overrightarrow{z^2})$.

Suppose for an arbitrary $\mu \geq 0$ we have a point $z : z = \overrightarrow{z^2} + \mu(\overrightarrow{z^2} - z^1)$, that is $z \in C(z^1; \overrightarrow{z^2})$. Define $z' \in C(z^{1\prime}; \overrightarrow{z^2}) : z' = \overrightarrow{z^2} + \mu(\overrightarrow{z^2} - z^{1\prime})$.

One can easily show that z and z' have the same elements except the h^{th} and $h+1^{th}$ elements, which are as follows:

$z_h = \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h)$;
$z_{h+1} = \overrightarrow{z^2}_{h+1} + \mu(\overrightarrow{z^2}_{h+1} - z^1_{h+1})$;
$z'_h = \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_{h+1})$;
$z'_{h+1} = \overrightarrow{z^2}_{h+1} + \mu(\overrightarrow{z^2}_{h+1} - z^1_h)$

From Lemma 27 we know that $z \preceq_e z'$ if $Min\{z_h, z_{h+1}\} \leq Min\{z'_h, z'_{h+1}\}$ and $z_h + z_{h+1} \leq z'_h + z'_{h+1}$. Let us check (Recall that $z^1_h > z^1_{h+1}$):

$$\begin{aligned} Min\{z_h, z_{h+1}\} &= Min\{[\overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h)], [\overrightarrow{z^2}_{h+1} + \mu(\overrightarrow{z^2}_{h+1} - z^1_{h+1})]\} \\ &= \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h) = z_h. \end{aligned}$$

We do not know what $Min\{z'_h, z'_{h+1}\}$ is, hence we will compare z_h with both z'_h and z'_{h+1}.

$z_h = \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h) \leq \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_{h+1}) = z'_h$

$z_h = \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h) \leq \overrightarrow{z^2}_{h+1} + \mu(\overrightarrow{z^2}_{h+1} - z^1_h) = z'_{h+1}$. Hence,

$$Min\{z_h, z_{h+1}\} \leq Min\{z'_h, z'_{h+1}\} \tag{4.1}$$

$$\begin{aligned} z_h + z_{h+1} &= \overrightarrow{z^2}_h + \mu(\overrightarrow{z^2}_h - z^1_h) + \overrightarrow{z^2}_{h+1} + \mu(\overrightarrow{z^2}_{h+1} - z^1_{h+1}) \\ &= z'_{h+1} + z'_h. \text{ That is,} \end{aligned}$$

$$z_h + z_{h+1} \leq z'_h + z'_{h+1} \tag{4.2}$$

From 4.1 and 4.2 the conditions of Lemma 27 is satisfied so $z \preceq_e z'$. Since μ is arbitrary, this result is valid for every $z \in C(z^1; \overrightarrow{z^2})$.

We showed that if $z \in C(z^1; \overrightarrow{z^2})$, then $z \in CD(z^{1\prime}; \overrightarrow{z^2})$, where $z^{1\prime}$ is the permutation obtained by a single swap of two consecutive elements of z^1 as defined above. Note that any permutation of vector z^1 will result in $\overrightarrow{z^1}$ if we apply a finite number of such binary contiguous swaps. Starting from the first element which is higher than its consecutive element, these type of swaps will eventually result in $\overrightarrow{z^1}$. Hence, we have the following result:

For any z^1, $z^2 \in \mathbb{R}^p_+$, if $z \in C(z^1; \overrightarrow{z^2})$ then z is *equitably dominated by* $C(\overrightarrow{z^1}; \overrightarrow{z^2})$. $\blacksquare$

Let us show this result on our simple example. We claim that if $z \in C((6,2);(3,4))$ then z is equitably dominated by $C((2,6);(3,4))$. In this example this can be verified by simple observation since we can see from Figure 4.5 below that $C((6,2);(3,4) \in L((3,4))$ so any $z \in C((6,2);(3,4))$ is equitably dominated by $(3,4)$ itself.

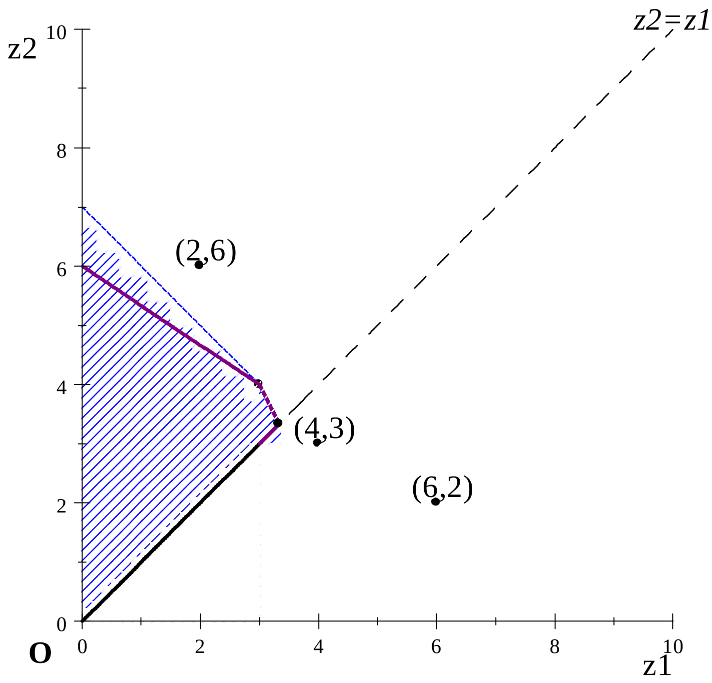

Figure 4.5: Relation between permutation cones

Theorem 29 *For any $z^1, z^2, z \in \mathbb{R}^p_+$, if $z \in CD(z^1; z^2)$ then $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$.*

Proof. If $z \in CD(z^1; z^2)$ there exists a $z' \in C(z^1; z^2) : z \preceq_e z'$.

Let $z^1 = \Pi^s(\overrightarrow{z^1})$ and $z^2 = \Pi^q(\overrightarrow{z^2})$. Then $z' = \Pi^q(\overrightarrow{z^2}) + \mu(\Pi^q(\overrightarrow{z^2}) - \Pi^s(\overrightarrow{z^1}))$. Let the inverse permutation of Π^q be Π^r and let $\Pi^r(\Pi^s) = \Pi^t$. Then $\Pi^r(z') \in C(\Pi^t(\overrightarrow{z^1}); \overrightarrow{z^2})$.

If $\Pi^r(z') \in C(\Pi^t(\overrightarrow{z^1}); \overrightarrow{z^2})$ then $\Pi^r(z')$ is equitably dominated by $C(\overrightarrow{z^1}; \overrightarrow{z^2})$ $\forall t$, implied by Lemma 28 proved above. Then from transitivity, $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$. Recall that equitable dominance is the intersection relation of all equitable rational preference relations hence it satisfies the transitivity axiom. ■

For our example, this means that if an alternative is equitably dominated by any of the other permutation cones it will also be equitably dominated by $C((2,6);(3,4))$. Note here that the reverse condition does not hold. That is, not all the points equitably dominated by $C((2,6);(3,4))$ are also equitably dominated by $C((6,2);(3,4))$. See Figure 4.6, the region ADFE which is in $CD((2,6);(3,4))$ is not in $CD((6,2);(3,4))$.

Theorem 29 shows that we can check the status of any point z with respect to any $C(z^1; z^2)$ by checking the status of it with respect to $C(\overrightarrow{z^1}; \overrightarrow{z^2})$. So instead of generating

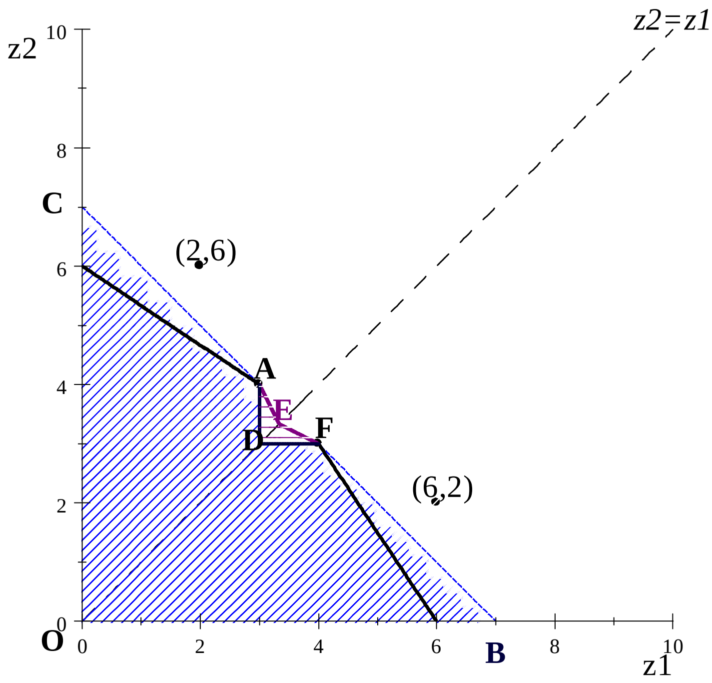

Figure 4.6: Inferior regions of permutation cones

all the permutation cones, it is sufficient to generate this single cone.

It is possible to provide similar results for the polyhedrons. See Appendix A Section 4.A.

4.3.3 Checking Equitable Dominance in 2-point Cones Case

In MCDM problems without impartiality property the common practise is to solve Linear Programming (LP) models to check the status of an alternative with respect to a cone or a polyhedron. These LPs are for the rational dominance check, i.e. checking whether an alternative is rationally dominated by a cone, hence we should make some modifications in these LP formulations to check equitable dominance. We now discuss a mathematical model that can be used to check equitable dominance by a cone.

Remark 30 *Equitable dominance is symmetric, hence if* $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$ *then* $\overrightarrow{z} \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$.

Define $P = \{1, 2, ..., p!\}$. From Theorem 29 we know that if $z \in \bigcup_{\substack{i \in P \\ j \in P}} CD(\Pi^i(\overrightarrow{z^1}), \Pi^j(\overrightarrow{z^2}))$ then $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$. And by Remark 30 $z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2}) \Longrightarrow \overrightarrow{z} \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$.

Based on the above results we suggest working on $\overrightarrow{\mathbb{R}}^p$, i.e. the space where all the alternatives are ordered from minimum to maximum. Recall that we refer to this space as the *ordered space.*

Working on $\overrightarrow{\mathbb{R}}^p$ involves mapping all the alternatives in $\mathbb{R}^p$ to $\overrightarrow{\mathbb{R}}^p$. That is, the cone (polyhedron) generators and the points that we check with respect to the cones (polyhedrons) will be the ordered vectors. For each alternative z we check whether $\overrightarrow{z} \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$, i.e. there exists $z' \in C(\overrightarrow{z^1}; \overrightarrow{z^2}) : \overrightarrow{z} \preceq_e z'$.

Before proposing our method we introduce a few models discussed in [85]:

Proposition 31 *For any $z \in \mathbb{R}^p$, $\overrightarrow{z}_n$ (n^{th} minimum) is the optimal value of the following LP problem:*

*MOD-N-MIN (**Mod**el **n**th **Min**imum)*

$$\overrightarrow{z}_n = Max\ r_n$$

subject to

$$r_n - z_i \le Mt_{ni} \text{ for } i = 1, ..., p$$
$$\sum_{i=1}^{p} t_{ni} \le n - 1$$
$$t_{ni} \in \{0, 1\} \text{ for } i = 1, ..., p$$

The above model can be extended to find the cumulative sum of the first n terms of $\overrightarrow{z}$ as follows.

Proposition 32 *[85]For any $z \in \mathbb{R}^p, \overline{\Theta}_n(z)$ ($= \sum_{i=1}^{n} \overrightarrow{z}_i$) is the optimal value of the* *following LP problem:*

*MODCUM-1(**Mod**el **Cum**ulative-1)*

$$\overline{\Theta}_n(z) = \max nr_n - \sum_{i=1}^{p} d_{ni} \tag{4.3}$$

subject to

$$r_n - d_{ni} - z_i \leq 0 \quad for\ i = 1, ..., p \tag{4.4}$$

$$d_{ni} \leq Mt_{ni}\ for\ i = 1, ..., p \tag{4.5}$$

$$\sum_{i=1}^{p} t_{ni} \leq n - 1 \tag{4.6}$$

$$d_{ni} \geq 0,\ t_{ni} \in \{0, 1\}\ for\ i = 1, ..., p \tag{4.7}$$

Proof. Denote the optimal r_n value as r_n^*. From the model *MOD-N-MIN* we know that $r_n^* = \overrightarrow{z}_n$. Note that at most $n-1$ of the t_{ni} variables can be 1 in a feasible solution. Minimizing $\sum_{i=1}^{p} d_{ni}$ with the constraint sets 4.5 and 4.6 ensure that at optimality the following hold:

$d_{ni}^* = 0$ for all $i : z_i \geq \overrightarrow{z}_n$

$d_{ni}^* > 0$ and $d_{ni}^* = \overrightarrow{z}_n - z_i$ for all $i : z_i < \overrightarrow{z}_n$.

Hence at optimality $\overline{\Theta}_n(z) = \sum_{i=1}^{n} \overrightarrow{z}_i = n\overrightarrow{z}_n - \sum_{i=1}^{n-1}(\overrightarrow{z}_n - \overrightarrow{z}_i) = nr_n^* - \sum_{i=1}^{p} d_{ni}^*$.

■

Theorem 33 *[85] For any $z \in \mathbb{R}^p, \overline{\Theta}_n(z)$ is the optimal value of the following LP problem:*

MODCUM-2

$$\overline{\Theta}_n(z) = \max nr_n - \sum_{i=1}^{p} d_{ni}$$

subject to

$$r_n - d_{ni} - z_i \leq 0 \quad for\ i = 1, ..., p$$

$$d_{ni} \geq 0 \quad for\ i = 1, ..., p$$

Proof. The proof is based on showing that the model *MODCUM-2* has the same optimal value with the model *MODCUM-1*. First of all, ignoring the t_{ni} variables, it is clear that each feasible solution to *MODCUM-1* is also a feasible solution to *MODCUM-2*. The feasible solutions to *MODCUM-2* which have less than n positive d_{ni} variables are feasible for *MODCUM-1* as well. In *MODCUM-2* we do not restrict the number of positive d_{ni} variables to $n-1$. However, one can show that for any feasible solution $(r_n,\mathbf{d})$ to this problem with $s \geq n$ positive d_{ni} variables, another feasible solution $(r'_n,\mathbf{d}')$ can be found with $s-1$ positive d_{ni} variables as follows:

Set $\alpha = Min\{d_{ni} : d_{ni} > 0\}$. $d'_{ni} = d_{ni} - \alpha$ for $d_{ni} > 0$.

Set $d'_{ni} = d_{ni}$ for $d_{ni} = 0$.

Set $r'_n = r_n - \alpha$.

Then $nr'_n - \sum_{i=1}^{p} d'_{ni} = n(r_n - \alpha) - (\sum_{i=1}^{p} d_{ni} - s\alpha) = nr_n - \sum_{i=1}^{p} d_{ni} + (s-n)\alpha \geq nr_n - \sum_{i=1}^{p} d_{ni}$. ■

Based on this result we propose the following model to check whether $z \in CD(\overrightarrow{z^1};\overrightarrow{z^2})$:

(LP_3)

$Max\ 0$

subject to

$$z'_i - \mu(\overrightarrow{z_i^2} - \overrightarrow{z_i^1}) = \overrightarrow{z_i^2} \quad for\ i = 1, ..., p \tag{4.8}$$

$$nr_n - \sum_{i=1}^{p} d_{ni} \geq \sum_{j=1}^{n} \overrightarrow{z}_j \quad for\ n = 1, ..., p \tag{4.9}$$

$$r_n - d_{ni} - z'_i \leq 0 \quad for\ i, n = 1, ..., p \tag{4.10}$$

$$\mu \geq 0 \tag{4.11}$$

$$d_{ni} \geq 0\ for\ i, n = 1, ..., p \tag{4.12}$$

This model checks whether there exists $z' \in C(\overrightarrow{z^1};\overrightarrow{z^2})$ such that $\overline{\Theta}(z) \leq \overline{\Theta}(z')$. Constraint sets 4.8 and 4.9 ensure that $z' \in C(\overrightarrow{z^1};\overrightarrow{z^2})$ and $\overline{\Theta}(z) \leq \overline{\Theta}(z')$, respectively. Constraint set 4.10 is used to ensure that $\overline{\Theta}_n(z') = \sum_{i=1}^{n} \overrightarrow{z'}_i = nr_n^* - \sum_{i=1}^{p} d_{ni}^*$, where r_n^* and

d^*_{ni} are the optimal values of these decision variables based on Theorem 33.

This is an LP problem with p^2+2p+1 variables and p^2+2p constraints excluding the set constraints.

4.3.4 Results for the k-point Cones Case

In this section we provide some results for k-point cones case, where $k > 2$. We first show that the previous result given in Theorem 29 is not generalizable to cases where $k > 2$ by providing a counter example. That is, for any k vectors $z^1, ..., z^k \in \mathbb{R}^p_+$ such that $z^i \succ z^k$ for all $i \neq k$ and $z \in \mathbb{R}^p_+$ we *cannot* claim that if $z \in CD(z^1, z^2, ...; z^k)$ then $z \in CD(\overrightarrow{z^1}, \overrightarrow{z^2}, ...; \overrightarrow{z^k})$.

Example 34 *Suppose that we have a case where $k = 3$ and $p = 3$, that is we have 3-point cones and we work in R^3_+. Suppose that the DM has the following utility function:*

$f(x) = x_1x_2x_3$.

Suppose that we present the following alternatives to the DM for him to compare:

$z^1 = (25, 4, 15)$

$z^2 = (7, 11, 27)$

$z^3 = (6, 7, 33)$

The corresponding utility values are $f(z^1) = 1500$, $f(z^2) = 2079$, $f(z^3) = 1386$. Hence the DM will provide us with the information that $z^2 \succ z^3$ and $z^1 \succ z^3$. Based on this we can generate the corresponding 3-point cones.

We will show that there exists a point $z : z \in C(z^1, z^2; z^3)$ and $z \notin CD(\overrightarrow{z^1}, \overrightarrow{z^2}; \overrightarrow{z^3})$. $z = (4.82, 4.65, 37.2)$ is such an example.

$z \in C(z^1, z^2; z^3)$ *since* $z = z^3 + \sum_{i=1}^{2} \mu_i(z^3 - z^i)$ *where* $\mu_1 = 0.03$ *and* $\mu_2 = 0.62$.

Let us check whether $z \in CD(\overrightarrow{z^1}, \overrightarrow{z^2}; \overrightarrow{z^3})$. We solve the following LP which is the LP_3 discussed in the previous section.

$Max\ 0$

subject to

$z'_1 - 2\mu'_1 + 1\mu'_2 = 6$

$z'_2 + 8\mu'_1 + 4\mu'_2 = 7$

$z'_3 - 8\mu'_1 - 6\mu'_2 = 33$

$r_1 - d_{11} - d_{12} - d_{13} \geq 4.65$

$2r_2 - d_{21} - d_{22} - d_{23} \geq 9.47$

$3r_3 - d_{31} - d_{32} - d_{33} \geq 46.67$

$r_n - z'_i - d_{ni} \leq 0 \; i, n = 1, 2, 3$

$\mu'_1, \mu'_2 \geq 0$

$d_{ni} \geq 0 \; i, n = 1, 2, 3$

The above problem is infeasible, which shows that there is no $z' \in C(\overrightarrow{z^1}, \overrightarrow{z^2}; \overrightarrow{z^3}) : z \preceq_e z'$ *(i.e.,* $\overline{\Theta}(z) \leq \overline{\Theta}(z')$*). Hence* $z \notin CD(\overrightarrow{z^1}, \overrightarrow{z^2}; \overrightarrow{z^3})$.

Having shown this counterexample, it is clear now that we have to find another way to deal with symmetry in k-point cones case. We try to do it by defining a region which encompasses all the information provided by the equitably dominated region of all the permutation cones. First, let us discuss some observations.

Given the information $z^i \succ z^k$ for $i \in I \setminus \{k\}$, for each permutation of $\overrightarrow{z^k}$, say $\Pi^s(\overrightarrow{z^k})$, we can generate a permutation cone of the form

$$C(\Pi^1(\overrightarrow{z^1}), ..., \Pi^{p!}(\overrightarrow{z^1}), ..., \Pi^1(\overrightarrow{z^{k-1}}), ..., \Pi^{p!}(\overrightarrow{z^{k-1}}), \Pi^1(\overrightarrow{z^k}), ..., \Pi^{p!}(\overrightarrow{z^k}); \Pi^s(\overrightarrow{z^k}))$$

Note that this cone contains all permutations of the lower generator as upper generators as we have $\Pi^s(\overrightarrow{z^k}) \preceq \Pi^l(\overrightarrow{z^k}) \; \forall s, k \in P = \{1, 2, ..., p!\}$.

For notational simplicity from now on we denote the above cone as follows

$$C(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \Pi^s(\overrightarrow{z^k})) \; \forall l \in P \text{ and } i \in I \setminus \{k\}$$

This is the largest cone that we can generate for $\Pi^s(\overrightarrow{z^k})$ as the lower generator given this preference information. We have $p!$ such cones each having a different permutation of $\overrightarrow{z^k}$ as the lower generator. The following remark shows that all these cones have the same equitably dominated region since they are reflections of each other.

Remark 35 *$CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \overrightarrow{z^k}) = CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \Pi^s(\overrightarrow{z^k}))$ for any $s \in P$, $\forall l \in P$ and $i \in I \setminus \{k\}$.*

Proof. $CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \overrightarrow{z^k}) = \{z : z \preceq_e z' \text{ and } z' \in C(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \overrightarrow{z^k})\}$.

$z' = \overrightarrow{z^k} + \sum_{j=1}^{p!} \sum_{i=1}^{k-1} \mu_{ji}(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^i})) + \sum_{j=1}^{p!} \beta_j(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k}))$ for $\mu_{ji} > 0$ and $\beta_j > 0$ by definition.

Apply Π^s so that

$\Pi^s(z') = \Pi^s(\overrightarrow{z^k}) + \sum_{j=1}^{p!} \sum_{i=1}^{k-1} \mu_{ji}(\Pi^s(\overrightarrow{z^k}) - \Pi^j(\overrightarrow{z^i})) + \sum_{j=1}^{p!} \beta_j(\Pi^s(\overrightarrow{z^k}) - \Pi^j(\overrightarrow{z^k}))$.

That is, $\Pi^s(z') \in C(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \Pi^s(\overrightarrow{z^k}))$ by definition.

$z \preceq_e z' \Longrightarrow z \preceq_e \Pi^s(z')$ hence $z \in CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \Pi^s(\overrightarrow{z^k}))$.

Hence all the points that are equitably dominated by any of the permutation cones lie in $CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \overrightarrow{z^k})$. ■

From now on we denote $CD(\Pi^l(\overrightarrow{z^i}), \Pi^l(\overrightarrow{z^k}); \overrightarrow{z^k})$ as U_1.

Definition 36 *For $z^i \in \mathbb{R}^p : z^i \succ z^k$ for all $i \in I \setminus \{k\}$, $U_1 = \overrightarrow{z^k} + \sum \lambda_t r_t$ such that $\lambda_t \geq 0$ and r_t are the rays in R, where R is the set of all rays $\overrightarrow{z^k} - \Pi^l(\overrightarrow{z^k})$ $\forall l : \Pi^l(\overrightarrow{z^k}) \neq \overrightarrow{z^k}$[1] and $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^i})$ for all $j, l \in P$ and for all $i \in I \setminus \{k\}$.*

U_1 is a convex set, defined by an extreme point $(\overrightarrow{z^k})$ and the extreme rays in set R. In our 2D example U_1 corresponds to the region ABOC seen in Figure 4.7.

[1] A special case occurs when $\Pi^l(\overrightarrow{z^k}) = \overrightarrow{z^k}$ and $\Pi^l(.)$ is not the permutation provided by the identity matrix. That is $\overrightarrow{z^k}$ has at least two elements that have the same value.

In such cases we can not talk about ray $\overrightarrow{z^k} - \Pi^l(\overrightarrow{z^k})$. Instead, we use points from the close neighborhood of $\overrightarrow{z^k}$ in order to represent the ray corresponding to the PD axiom. We define the following:

For each i and j such that $i \neq j$ and $\overrightarrow{z^k}_i = \overrightarrow{z^k}_j$ define

$z^{k\prime} : z^{k\prime}_h = \overrightarrow{z^k}_h$ for all $h \neq i, j$ and $z^{k\prime}_i = \overrightarrow{z^k}_i - \epsilon$ and $z^{k\prime}_j = \overrightarrow{z^k}_j + \epsilon$

$z^{k\prime\prime} : z^{k\prime\prime}_h = \overrightarrow{z^k}_h$ for all $h \neq i, j$ and $z^{k\prime\prime}_i = \overrightarrow{z^k}_i + \epsilon$ and $z^{k\prime\prime}_j = \overrightarrow{z^k}_j - \epsilon$

where ϵ is a small positive value and use the rays $\overrightarrow{z^k} - z^{k\prime}$ and $\overrightarrow{z^k} - z^{k\prime\prime}$ instead.

See Figure 4.8 for a 2D example.

From now on, we assume that this special case is considered when we talk about $\overrightarrow{z^k} - \Pi^l(\overrightarrow{z^k})$.

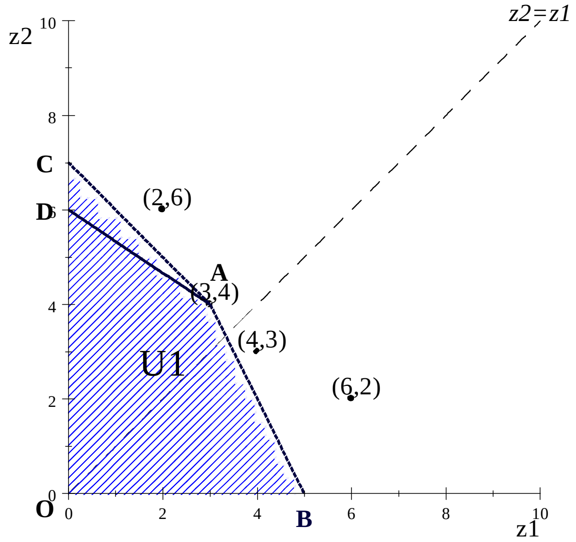

Figure 4.7: Region U_1

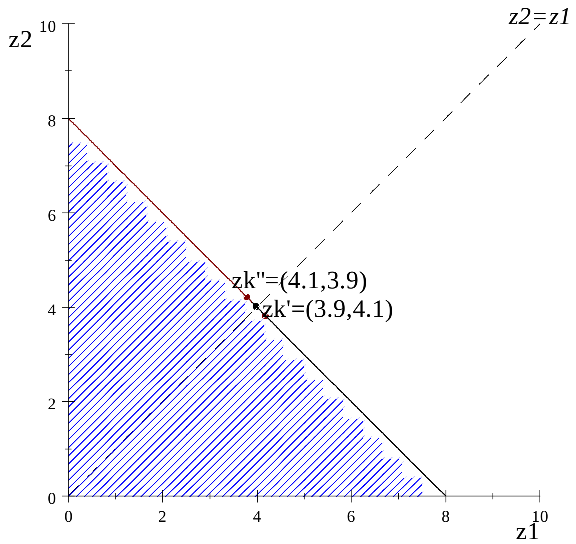

Figure 4.8: Special case in 2D ($\overrightarrow{z^k}_1 = \overrightarrow{z^k}_2$) Rays $\overrightarrow{z^k} - z^{k\prime}$ and $\overrightarrow{z^k} - z^{k''}$ where $\epsilon = 0.1$

Definition 37 *For a set A, an alternative z is equitably dominated by A if there exists an alternative $z' \in A : z \preceq_e z'$.*

Given $z^i \succ z^k$ for $i \in I \setminus \{k\}$ Remark 35 shows that every point that is equitably dominated by any of the permutation cones will also be equitably dominated by U_1. On the other hand, since U_1 is a convex cone having $\overrightarrow{z^k}$ as the lower generator it consists of points which are at most as preferred as $\overrightarrow{z^k}$.

To sum up, U_1 is a region that encompasses all the information provided by all the permutation cones generated based on $z^i \succ z^k$, $i \in I \setminus \{k\}$. Based on these results we propose to use U_1 and for each alternative z we check whether z is equitably dominated by U_1.

We now analyze U_1 in more detail. We claim that in region U_1, the rays given by $z^k - \Pi^j(\overrightarrow{z^i})$ for $i \in I \setminus \{k\}$, where $\Pi^j(\overrightarrow{z^i}) \neq \overrightarrow{z^i}$ are not extreme, hence can be written as a nonnegative combination of the other rays in R. In other words the cones $C(\Pi^j(\overrightarrow{z^i}), \overrightarrow{z^k})$: $\Pi^j(\overrightarrow{z^i}) \neq \overrightarrow{z^i}$ do not lie on the boundary of the region U_1. In our 2D example tis corresponds to claiming that the ray $(3,4) - (6,2)$, i.e. $(-3,2)$, is not an extreme ray for U_1 and this is clearly seen in Figure 4.7.

Lemma 38 *In set R, the rays given by $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^i})$, where $\Pi^j(\overrightarrow{z^i}) \neq \overrightarrow{z^i}$ can be written as a nonnegative combination of the rays $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})$ $\forall j : \overrightarrow{z^k} \neq \Pi^j(\overrightarrow{z^k})$ and $\overrightarrow{z^k} - \overrightarrow{z^i}$.*

Proof. We will prove this for an arbitrary element $i \in I \setminus \{k\}$.

Let $z^i \neq \overrightarrow{z^i}$ as assumed. Let h be the minimum value for which $z^i_h > z^i_{h+1}$ holds. Define $z^{i\prime}$ as the permutation obtained from z^i by swapping z^i_h and z^i_{h+1}. That is, $z^i = (z^i_1, z^i_2, ..., z^i_h, z^i_{h+1}, ..., z^i_p)$ and $z^{i\prime} = (z^i_1, z^i_2, ..., z^i_{h+1}, z^i_h, ..., z^i_p)$ where $z^i_h > z^i_{h+1}$. We first show the following holds:

$\overrightarrow{z^k} - z^i = \sum \lambda_t r_t$ where r_t are in the set $\{\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})$ $\forall j : \overrightarrow{z^k} \neq \Pi^j(\overrightarrow{z^k})$ and $\overrightarrow{z^k} - z^{i\prime}\}$. That is, $\overrightarrow{z^k} - z^i$ can be written as a nonnegative combination of $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})$ for all j and $\overrightarrow{z^k} - z^{i\prime}$.

For $z^i, z^{i\prime}$ as defined above the following holds:

$$\overrightarrow{z^k} - z^i = \overrightarrow{z^k} - z^{i\prime} + \left(\frac{z^i_h - z^i_{h+1}}{\overrightarrow{z^k}_{h+1} - \overrightarrow{z^k}_h}\right)(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k}))$$

where $\Pi^j(\overrightarrow{z^k})_i = \overrightarrow{z^k}_i \ \forall i \neq h, h+1$ and $\Pi^j(\overrightarrow{z^k})_h = \overrightarrow{z^k}_{h+1}$; $\Pi^j(\overrightarrow{z^k})_{h+1} = \overrightarrow{z^k}_h$ (all the elements of $\overrightarrow{z^k}$ are the same in $\Pi^j(\overrightarrow{z^k})$ except for h^{th} and $h+1^{th}$ being swapped).

It is clearly seen when we analyze the vectors in detail as below:

$$\begin{pmatrix} \overrightarrow{z^k}_1 - z^i_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - z^i_h \\ \overrightarrow{z^k}_{h+1} - z^i_{h+1} \\ \\ ... \\ \\ \overrightarrow{z^k}_p - z^i_p \end{pmatrix} = \begin{pmatrix} \overrightarrow{z^k}_1 - z^i_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - z^i_{h+1} \\ \overrightarrow{z^k}_{h+1} - z^i_h \\ \\ ... \\ \\ \overrightarrow{z^k}_p - z^i_p \end{pmatrix} + \left(\frac{z^i_h - z^i_{h+1}}{\overrightarrow{z^k}_{h+1} - \overrightarrow{z^k}_h}\right) \begin{pmatrix} \overrightarrow{z^k}_1 - \overrightarrow{z^k}_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - \overrightarrow{z^k}_{h+1} \\ \overrightarrow{z^k}_{h+1} - \overrightarrow{z^k}_h \\ \\ ... \\ \\ \overrightarrow{z^k}_p - \overrightarrow{z^k}_p \end{pmatrix}$$

In the above equation if $\overrightarrow{z^k}_{h+1} > \overrightarrow{z^k}_h$ then $\left(\frac{z^i_h - z^i_{h+1}}{\overrightarrow{z^k}_{h+1} - \overrightarrow{z^k}_h}\right) \geq 0$, that is we are able to write the ray that corresponds to $\overrightarrow{z^k} - z^i$ as a nonnegative combination of the rays $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})$ $\forall j : \overrightarrow{z^k} \neq \Pi^j(\overrightarrow{z^k})$; $\overrightarrow{z^k} - z^{i\prime}$.

Note that a special case occurs when $\overrightarrow{z^k}_{h+1} = \overrightarrow{z^k}_h$, hence $\overrightarrow{z^k} = \Pi^j(\overrightarrow{z^k})$. As discussed before we use $\overrightarrow{z^k} - z^{k\prime}$ and $\overrightarrow{z^k} - z^{k\prime\prime}$ instead of $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})$ where $z^{k\prime}$ (and $z^{k\prime\prime}$) are the vectors obtained from $\overrightarrow{z^k}$ by subtracting (adding) ϵ from (to) h^{th} element and adding (subtracting) ϵ to (from) the $h+1^{th}$ element.

$$\overrightarrow{z^k} - z^i = \overrightarrow{z^k} - z^{i\prime} + 2\left(\frac{z^i_h - z^i_{h+1}}{z^{k\prime\prime}_h - z^{k\prime\prime}_{h+1}}\right)(\overrightarrow{z^k} - z^{k\prime\prime}).$$ Note that $z^{k\prime\prime}_h - z^{k\prime\prime}_{h+1} = 2\epsilon$.

$$\begin{pmatrix} \overrightarrow{z^k}_1 - z^i_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - z^i_h \\ \overrightarrow{z^k}_h - z^i_{h+1} \\ \\ ... \\ \\ \overrightarrow{z^k}_p - z^i_p \end{pmatrix} = \begin{pmatrix} \overrightarrow{z^k}_1 - z^i_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - z^i_{h+1} \\ \overrightarrow{z^k}_h - z^i_h \\ \\ ... \\ \\ \overrightarrow{z^k}_p - z^i_p \end{pmatrix} + 2\left(\frac{z^i_h - z^i_{h+1}}{2\epsilon}\right) \begin{pmatrix} \overrightarrow{z^k}_1 - \overrightarrow{z^k}_1 \\ \\ ... \\ \\ \overrightarrow{z^k}_h - (\overrightarrow{z^k}_h + \epsilon) \\ \overrightarrow{z^k}_{h+1} - (\overrightarrow{z^k}_{h+1} - \epsilon) \\ \\ ... \\ \\ \overrightarrow{z^k}_p - \overrightarrow{z^k}_p \end{pmatrix}$$

Note that any permutation of vector z^i will result in $\overrightarrow{z^i}$ if we apply a finite number of such binary contiguous swaps. At each such step we will be able to write the first ray

$(\overrightarrow{z^k} - z^i)$ as a nonnegative combination of the rays $(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})\ \forall j : \overrightarrow{z^k} \neq \Pi^j(\overrightarrow{z^k});\ \overrightarrow{z^k} - z^{i\prime})$. Starting from the first element which is higher than its consecutive element, these type of swaps will eventually result in $\overrightarrow{z^i}$.

Hence, we have the following result:

In set R the rays given by $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^i})$, where $\Pi^j(\overrightarrow{z^i}) \neq \overrightarrow{z^i}$ can be written as a nonnegative combination of the rays $\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})\ \forall j : \overrightarrow{z^k} \neq \Pi^j(\overrightarrow{z^k})$ and $\overrightarrow{z^k} - \overrightarrow{z^i}$. ■

Corollary 39 *In U_1 the rays ($\overrightarrow{z^k} - z^i$): $z^i \neq \overrightarrow{z^i}$ where $i \in I \setminus \{k\}$ are not extreme rays.*

Proof. By Lemma 38, $\overrightarrow{z^k} - z^i$ where $z^i \neq \overrightarrow{z^i}$ can be written in terms of the other rays in R. Hence such $(\overrightarrow{z^k} - z^i)$s are not extreme rays of U_1. ■

Definition 40 *We change the definition of U_1 as follows:*

$U_1 = \{z : z = \overrightarrow{z^k} + \sum_{i=1}^{k-1} \mu_i(\overrightarrow{z^k} - \overrightarrow{z^i}) + \sum_{j=1}^{p!} \beta_j(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k})), where\ \mu_i \geq 0, \beta_j \geq 0\ \forall i \in I,\ j \in P\}$

The following theorem is our main result which is based on the results that we have provided so far. For notational simplicity we will define a region ED (*Equitably Dominated*) which is the inferior region that we can obtain through the convex cones approach given $z^i \succ z^k$ where $i \in I \setminus \{k\}$ in $\mathbb{R}^p_+$.

Definition 41 $ED = \{z : z \in \bigcup_{s \in P} CD(\Pi^l(\overrightarrow{z^i}); \Pi^s(\overrightarrow{z^k}))\}$.

Theorem 42 *If $z \in ED$ then z is equitably dominated by U_1.*

Proof. $z \in ED \implies z \in CD(\Pi^l(\overrightarrow{z^i}); \overrightarrow{z^k})$ due to Remark 35. Hence z is equitably dominated by U_1. ■

Theorem 42 states that all the points that are equitably inferior to the permutation cones will be equitably dominated by region U_1. That is, all the information that we can infer using cones is provided by equitably dominated region of U_1. Moreover, definition of U_1 (see Definition 40) provides us a way to partially handle the permutational complexities since it does not include all the permutations of the upper generators (z^is); we just use the corresponding ordered vectors ($\overrightarrow{z^i}$s). Note that we still use all the permutations of the lower generator ($\overrightarrow{z^k}$) to define U_1. This issue will be discussed in detail in the next section.

For similar results that apply to the corresponding polyhedrons see Appendix Section 4.B.

4.3.5 Checking Equitable Dominance by a Point in U_1

Recall that by Theorem 42 if $z \in ED$ then z is equitably dominated by U_1. Hence, for each alternative z we have to check whether there exists $z' \in U_1 : z \preceq_e z'$. Using Corollary 39 we can define each point $z' \in U_1$ using the equation $z' = \overrightarrow{z^k} + \sum_{i=1}^{k-1} \mu_i(\overrightarrow{z^k} - \overrightarrow{z^i}) + \sum_{j=1}^{p!} \beta_j(\overrightarrow{z^k} - \Pi^j(\overrightarrow{z^k}))$.

Suppose that we work on $\mathbb{R}^p_+$. The following model will be used for checking equitable dominance of an alternative z by U_1 :

$$(LP_4)$$

$$\max 0$$

subject to

$$z'_h - \sum_{i=1}^{k-1} \mu_i(\overrightarrow{z^k_h} - \overrightarrow{z^i_h}) - \sum_{j=1}^{p!} \beta_j(\overrightarrow{z^k_h} - \Pi^j(\overrightarrow{z^k})_h) = \overrightarrow{z^k_h} \quad for\ h = 1, ..., p \tag{4.13}$$

$$nr_n - \sum_{h=1}^{p} d_{nh} \geq \sum_{h=1}^{n} \overrightarrow{z}_h \quad for\ n = 1, ..., p \tag{4.14}$$

$$r_n - d_{nh} - z'_h \leq 0 \quad for\ h, n = 1, ..., p \tag{4.15}$$

$$\mu_i \geq 0\ for\ i = 1, ..., k-1 \tag{4.16}$$

$$\beta_j \geq 0\ for\ j = 1, ..., p! \tag{4.17}$$

$$d_{nh} \geq 0 \quad for\ h, n = 1, ..., p \tag{4.18}$$

This model checks whether there exists $z' \in U_1$ such that $\overline{\Theta}(z) \leq \overline{\Theta}(z')$. Constraint sets 4.13 and 4.14 ensure that $z' \in U_1$ and $\overline{\Theta}(z) \leq \overline{\Theta}(z')$, respectively. The objective function and constraint set 4.15 are used to ensure that $\overline{\Theta}_n(z') = \sum_{i=1}^{n} \overrightarrow{z}'_i = nr^*_n - \sum_{h=1}^{p} d^*_{nh}$, where r^*_n and d^*_{nh} are the optimal values of these decision variables based on Theorem 33.

It is an LP problem with $p!+p^2+2p+k-1$ variables and p^2+2p constraints excluding the set constraints. Hence it is practical to solve this LP for cases where p is small (≤ 5). Recall that the permutational term ($p!$) is due to the necessity of using all permutations of the lower generator. This seems to be restricting the size of problems that can be solved by an algorithm that uses this result. However, note that, we work on problems where we ask the DM to compare alternatives, i.e. vectors. Requesting the DM to compare vectors is only meaningful when the size of the vectors (p) is reasonable. Hence we already have a natural limit on p due to the cognitive limitations of the DM.

One can show similar results for the polyhedrons. See Appendix Section 4.C.

Table 4.2 below summarizes our main theoretical results (See Appendix for LP_5).

Table 4.2: Summary of the main results

2-point cones case	**k-point cones case**
$z \in CD(z^1; z^2) \implies z \in CD(\overrightarrow{z^1}; \overrightarrow{z^2})$	$z \in CD(z^1, ..., z^{k-1}; z^k) \implies \exists\, z' \in U_1 : z \preceq_e z'$
$z \in UP(z^1; z^k) \implies z \in UP(\overrightarrow{z^1}; \overrightarrow{z^k})$	$z \in UP(z^1, ..., z^{k-1}; z^k) \implies \overrightarrow{z} \in UP(\overrightarrow{z^1}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$
Work on $\overrightarrow{\mathbb{R}}^p$. Use LP_3 and LP_5.	Use LP_4 and LP_5.

In the next section we propose an interactive ranking algorithm which is based on our theoretical results.

4.4 An Interactive Ranking Algorithm

In this section we introduce a solution algorithm that uses the idea of convex cones for the ranking problematique. In this algorithm we test our method to deal with symmetry. We first provide a general description of the approach, followed by a simple numerical example. Next, we explain the algorithm in detail.

4.4.1 General Overview of the Algorithm

We propose an algorithm that can be used to obtain a ranking of a discrete set of alternatives given. In the algorithm, we gather preference information from the DM by presenting him/her some pairs. Using the information he/she provides, it is possible to generate cones and polyhedrons as discussed before. For each alternative in the set we check whether it is cone dominated or lies in the upper side of the polyhedron generated. We perform these checks by using models LP_4 and LP_5 discussed in the previous section. Note that, at the beginning of the algorithm, before asking questions to the DM, we will perform an initial check to see whether there are alternatives equitably dominating each other. Theorem 22 in Section 4.2.2 provides us the link between vector dominance and equitable dominance. Based on this theorem, in order to check equitable dominance, we will check rational dominance for the cumulative ordered vectors of the alternatives ($\overline{\Theta}(.)$). Hence, for practical reasons, we will find $\overline{\Theta}(.)$ vector for each alternative in our set at the beginning of the algorithm. This information will also be used in the LPs to check the status of an alternative z, since these LPs use $\sum_{h=1}^{i} \overrightarrow{z}_h$, which is $\overline{\Theta}_i(z)$, as a parameter.

Suppose that we are given a finite number of alternatives each showing a distribution profile for p parties. We can summarize our algorithm with the following steps:

S.1. Map the alternatives to $\overline{\Theta}(\mathbb{R}^p)$ and check whether any alternative is equitably dominated by the other for each pair of alternatives.

S.2. Select k alternatives ($k \geq 2$) based on a predetermined rule. Get the preference information from the DM by asking him to compare these alternatives. Denote the least preferred alternative as z^k and the rest as z^i for $i = 1, 2, .., k-1$.

S.3. Based on the preference information obtained, check for each alternative z whether $z \preceq z^k$ by solving LP_4. If not, then check whether $z^k \preceq z$ by solving LP_5.

S.4. Update the result accordingly. If the result is not satisfactory, continue with Step 2.

4.4.2 Numerical Example

Let us show the general idea using a 2D example. Note that our results are valid for any p-dimensional case, we are providing a 2D example for simplicity.

Example 43 *Suppose that a DM is trying to reach a partial ordering for the following*

Table 4.3: Example problem step 1. mapping the alternatives to the cumulative ordered space

Alternative (z)	Ordered Vector ($\overrightarrow{z}$)	Cumulative Ordered Vector ($\overline{\Theta}(z)$)
z^1	(1,2)	(1,3)
z^2	(2,3)	(2,5)
z^3	(2,2)	(2,4)
z^4	(3,4)	(3,7)
z^5	(2,6)	(2,8)
z^6	(0.5,8)	(0.5,8.5)
z^7	(0,10)	(0,10)
z^8	(3.5,3.5)	(3.5,7)
z^9	(2.5,5)	(2.5,7.5)
z^{10}	(4,6)	(4,10)

income distributions in a 2-person population:

$z^1 = (1,2)$, $z^2 = (3,2)$, $z^3 = (2,2)$, $z^4 = (3,4)$, $z^5 = (6,2)$, $z^6 = (0.5,9)$, $z^7 = (10,0)$, $z^8 = (3.5,3.5)$, $z^9 = (5,2.5)$, $z^{10} = (6,4)$. *Let us apply the algorithm.*

S.1. Map the alternatives to $\overline{\Theta}(\mathbb{R}^2)$ *and check whether each alternative is equitably dominated by the other for each pair of alternatives.*

Based on Theorem 22 we check equitable dominance by checking rational dominance of the corresponding cumulative ordered vectors. Through inspection one can see that the following holds:

$z^1 \prec z^3 \prec z^2 \prec z^5 \prec z^{10}$

$z^1 \prec z^3 \prec z^2 \prec z^4 \prec z^8 \prec z^{10}$

$z^1 \prec z^3 \prec z^2 \prec z^9 \prec z^{10}$

$z^6 \prec z^{10}$

$z^7 \prec z^{10}$

Figure 4.9 shows this information in a tree like form. $\preceq$ $(\prec)$ *is represented by an arrow from the preferred alternative to the less preferred one.*

Example 44 *S.2. Now suppose that we have preference information from the DM that* $z^4 \succ z^5$*.Then, we know that* $z^1 \prec z^3 \prec z^2 \prec z^5 \prec z^4 \prec z^8 \prec z^{10}$.

S.3. Based on the preference information obtained, check for each alternative z whether $z \preceq z^5$ *by solving* LP_4*. If not, then check whether* $z^5 \preceq z$ *by solving* LP_5.

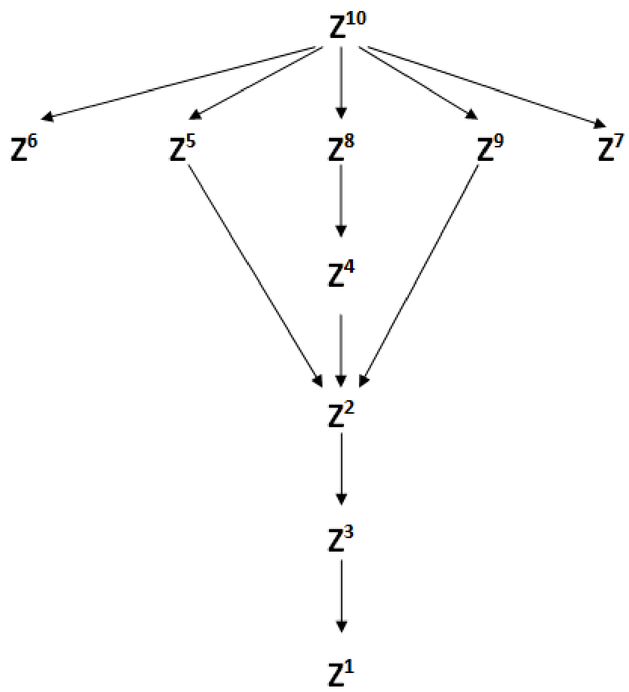

Figure 4.9: Preference tree after S1

Since we already know the statuses of z^1, z^2, z^3, z^4, z^8 *and* z^{10} *with respect to* z^5; *we perform the checks for* z^6, z^7 *and* z^9.

S.3.1 Check whether $z^6 \preceq z^5$*:*

To do this we will solve the following LP:

$$Max\ 0$$

$$z'_1 - \mu(\overrightarrow{z}^5_1 - \overrightarrow{z}^4_1) - \beta(\overrightarrow{z}^5_1 - \overleftarrow{z}^5_1) = \overrightarrow{z}^5_1$$

$$z'_2 - \mu(\overrightarrow{z}^5_2 - \overrightarrow{z}^4_2) - \beta(\overrightarrow{z}^5_2 - \overleftarrow{z}^5_2) = \overrightarrow{z}^5_2$$

$$r_1 - d_{11} - d_{12} \geq \overline{\Theta}(z^6)_1$$

$$2r_2 - d_{21} - d_{22} \geq \overline{\Theta}(z^6)_2$$

$$r_1 - d_{11} - z'_1 \leq 0$$

$$r_1 - d_{12} - z'_2 \leq 0$$

$$r_2 - d_{21} - z'_1 \leq 0$$

$$r_2 - d_{22} - z'_2 \leq 0$$

$$\mu, \beta \geq 0$$

$$d_{ni} \geq 0\ \forall i, n$$

That is, we solve the following LP:

$$Max\ 0$$

$$z'_1 + \mu + 4\beta = 2$$

$$z'_2 - 2\mu - 4\beta = 6$$

$$r_1 - d_{11} - d_{12} \geq 0.5$$

$$2r_2 - d_{21} - d_{22} \geq 8.5$$

$$r_1 - d_{11} - z'_1 \leq 0$$

$$r_1 - d_{12} - z'_2 \leq 0$$

$$r_2 - d_{21} - z'_1 \leq 0$$

$$r_2 - d_{22} - z'_2 \leq 0$$

$$\mu, \beta \geq 0$$

$$d_{ni} \geq 0\ \forall i, n$$

The above LP is feasible. Hence $z^6 \preceq z^5$.

S.3.2 Check whether $z^7 \preceq z^5$.

When the above LP is solved for z^7*, a feasible solution can be found. Hence* $z^7 \preceq z^5$.

S.3.3 Check whether $z^9 \preceq z^5$.

The corresponding LP model is not feasible. So we can not conclude that $z^9 \preceq z^5$*. Now we will check whether* $z^9 \in UP(\overrightarrow{z}^4, \overrightarrow{z}^5)$ *by solving the following LP:*

$$Max\ \epsilon$$

$$z'_1 - \mu(\overrightarrow{z}^5_1) - \mu_1(\overrightarrow{z}^4_1) = 0$$

$$z'_2 - \mu(\overrightarrow{z}^5_1) - \mu_1(\overrightarrow{z}^4_1) = 0$$

$$z'_1 + \epsilon \leq \overline{\Theta}(z^9)_1$$

$$z'_1 + z'_2 + \epsilon \leq \overline{\Theta}(z^9)_2$$

$$\mu + \mu_1 = 1$$

$$\mu, \mu_1 \geq 0$$

Writing down the parameters explicitly, we have the following LP:

$$Max\ \epsilon$$

$$z'_1 - 2\mu - 3\mu_1 = 0$$

$$z'_2 - 6\mu - 4\mu_1 = 0$$

$$z'_1 + \epsilon \leq 2.5$$

$$z'_1 + z'_2 + \epsilon \leq 7.5$$

$$\mu + \mu_1 = 1$$

$$\mu, \mu_1 \geq 0$$

Since $\epsilon^* = 0 \geq 0$, $z^9 \in UP(\overrightarrow{z}^4, \overrightarrow{z}^5)$*, hence* $z^5 \preceq z^9$.

S.4. Using the preference information $(z^4 \succ z^5)$ *that the DM provided and using the convex cones, we obtained the information that* $z^6 \preceq z^5; z^7 \preceq z^5$ *and* $z^5 \preceq z^9$*. Now we* *know the following:*

$z^1 \prec z^3 \prec z^2 \prec z^5 \preceq z^9 \prec z^{10}$

$z^1 \prec z^3 \prec z^2 \prec z^5 \prec z^4 \prec z^8 \prec z^{10}$

$z^6 \preceq z^5 \preceq z^9 \prec z^{10}$

$z^7 \prec z^5 \preceq z^9 \prec z^{10}$

See Figure 4.10 for a tree representation of these preferences.

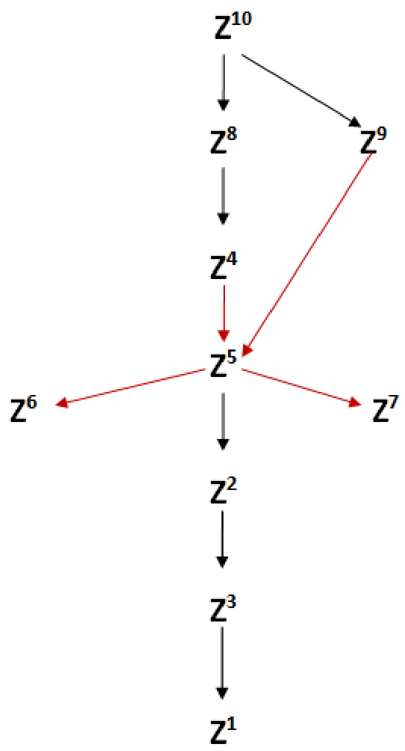

Figure 4.10: Preference tree after S4

Example 45 *If this quasi-ordering is not satisfactory, one can go back to Step 2, choose another pair and ask the DM for new information. For example we can ask her to compare* z^8 *and* z^9*.*

4.4.3 Detailed Description of the Algorithm

Example 43 shows the idea that our ranking algorithm is based on. The algorithm that we propose differs from the above sketch in a number of aspects. Unlike the example case, as long as new information is available, which allows us to generate new cones, we repeat Step 3. That is, we try to use the available information as much as possible before presenting the DM a new sample. After all the checks are performed, we continue with Step 4. We now provide a detailed description of the algorithm using subroutines.

For a problem including n alternatives showing allocation to p parties, that is with p criteria, the pseudocode of our algorithm is as follows:

Algorithm 46 *Initialize the parameters and generate data using **Initialization** subroutine*

*Check equitable dominance relation between each pair of alternatives using **Dominancecheck** subroutine*

Repeat

*Get preference information from the DM using **Getinfo** subroutine*

newinfo=1 //This parameter is used to check whether any new information is obtained that can allow us to generate new cones

Repeat

*Perform the checks related to the cones and polyhedrons using **Conegeneration** subroutine*

***Until** newinfo=0*

*Count the number of alternatives whose ranks are known using **Countassigned** subroutine*

***Until** n-unassigned<n or CPUtime>1800*

Display results and performance measure values

Let us now explain each subroutine in more detail.

Initialization

This subroutine is used to initialize the parameters and generate the set of alternatives. At the end of this subroutine, the cumulative ordered vectors for all the alternatives are also found and kept in memory.

Dominancecheck

As discussed before (see Theorem 22) equitable dominance is checked by checking rational dominance of the corresponding cumulative ordered vectors. We store the information on dominance relations in an $n * n$ matrix called *Dominancemat*. *Dominancemat(i,j) =1* if alternative i equitably dominates alternative j; 0 otherwise.

Getinfo

This subroutine gathers information from the DM by providing him with a set of alternatives. The size of this set is controlled by a parameter called *samplesize*. The alternatives are selected according to a predetermined rule. In the first iteration we rank the alternatives according to their Euclidean distances to an ideal point (IP) whose coordinates are defined as follows:

$IP_i = \underset{\overline{\Theta}(z)}{Max}\ \overline{\Theta}(z)_i\ \forall i = 1, ..., p.$

We select the ones having the least distances to the IP.

In the following iterations, we select the alternatives on whose ranks we have the least information. We keep track of the information on an alternative's possible ranks using an $n * 2$ matrix called *boundmatrix*. In this matrix each row is dedicated to an alternative and the two values in each row show the minimum and maximum possible ranks of the alternative, respectively. At the beginning these values are set to 1 and n for all the alternatives. Whenever new information is available, this matrix is updated accordingly. While asking the DM for preference information, we choose the alternatives for which the difference between the maximum and minimum rank is larger.

The preference information gathered is in form of ranking of the alternatives in the sample. If *samplesize* is two, this corresponds to a pairwise comparison. For samples with more than two alternatives, the DM ranks these alternatives from the best to the worst. When preference information is obtained, it is stored in an $n * n$ matrix called *Userpreference*. This matrix keeps the information for cone and polyhedron generation. When new information is gathered, the *Userpreference* matrix is updated based on transitivity. For example, if from previous iterations we know that alternative i is preferred to alternative j and in the current iteration we are given j is preferred to alternative k, then we update the matrix setting *Userpreference(i,k)=1*. This allows us to generate the largest cone for a given lower generator.

Conegeneration

Given available information on the preferences, in this subroutine we perform the corresponding checks related to the cones and polyhedrons. The two LPs, LP_4 and LP_5, are generated and solved in this subroutine. Whenever possible, redundant cones/polyhedrons and checks are avoided. We do not generate a cone/polyhedron that we generated before. Moreover, we do not solve these LPs for an alternative if we already know that it is equitably dominated by/equitably dominates or less preferred/more preferred to the cone's lower generator. Since some of the new information obtained through these checks leads us to new cones and polyhedrons, we repeat this subroutine until there is no useful new information. We check this condition by using a binary variable called *newinfo*. A flowchart of this subroutine is provided below in Figure 4.11:

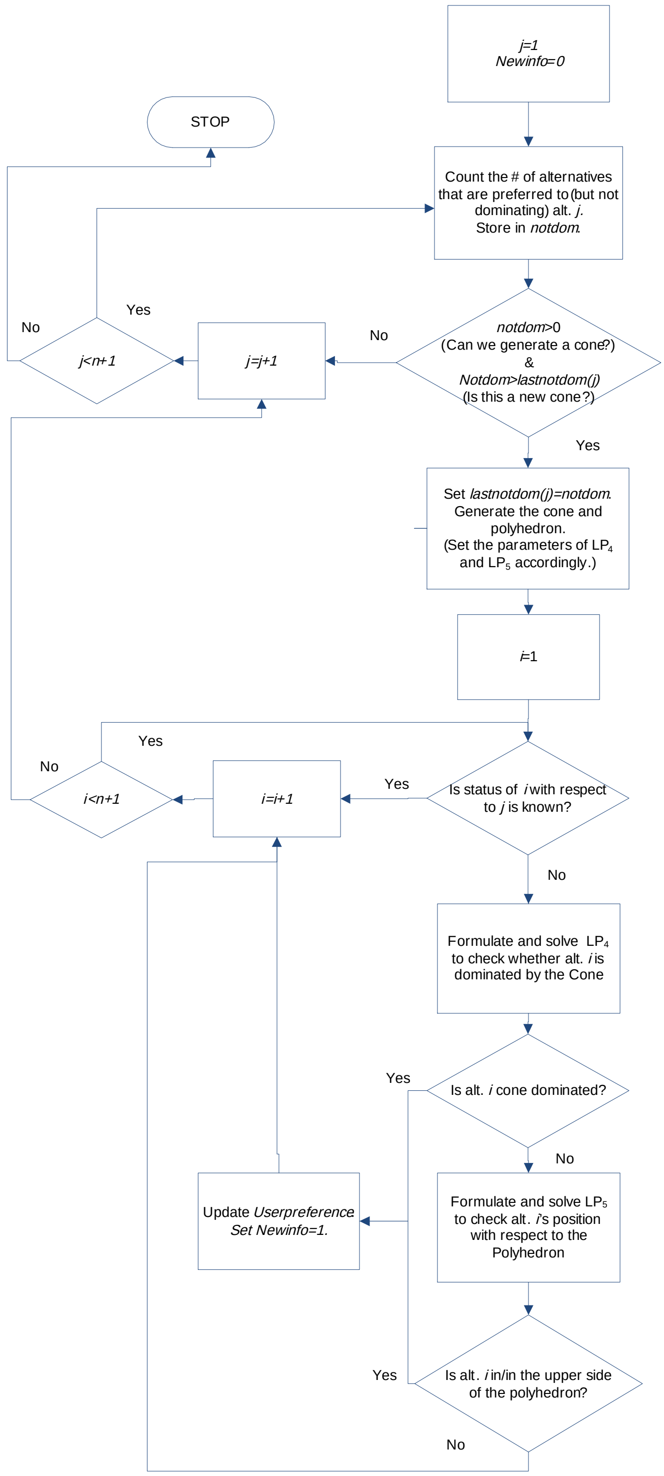

Figure 4.11: Flowchart of *Conegeneration* subroutine

Countassigned

Recall that we keep the lowest and highest possible ranks for an alternative in *boundmatrix*. At the end of each iteration, for each alternative, we count the alternatives that it dominates/is preferred to and the ones that it is dominated by/less preferred than. We update the information on *boundmatrix* accordingly. We then count the number of alternatives whose rank we know, i.e, whose maximum and minimum possible ranks are equal. This information is then used to decide whether to terminate the algorithm.

4.5 Computational Experiments

In this section we provide the results of our computational experiments on the performance of the algorithm. We present our experimental setting, state our performance measures and discuss the results of the experiments.

4.5.1 Experimental Setting

Two different data settings are used in the experiments. The first setting is based on real life data on income distributions of different countries. In the second setting the alternatives are generated using the random number generator of MATLAB.

In the first data set we use income distribution information of different countries from the World Bank [179] and UNU-WIDER (United Nations University- World Institute for Development Economics Research) [180] databases. We use the quintile values to represent a country's income distribution. For each country we take the percentage share of income that accrues to subgroups of population indicated by quintiles. Let us denote these percentage shares as S_i $i = 1, ..., 5$, where $S_i\%$ is the income share held by the i^{th} 20% of the population. Given these percentage shares, for each country, we can find mean income levels for each quintile, $\mu_i : i = 1, ..., 5$ as follows:

$$\mu_i = \frac{TI * S_i}{TP * 20} \quad i = 1, ..., 5$$

where TI and TP are the total income and population of the corresponding country.

Note that TI/TP is the mean income level for that country. We use *GNI (Gross National Income)* [181] values to estimate TI/TP, hence use the following equation:

$$\mu_i = GNI * \frac{S_i}{20} \quad i = 1, ..., 5$$

Hence for each country we use a distribution vector of size 5 consisting of the mean income levels of each quintile. One can think of these μ_i values as the income levels of 5 representative people in the population. In this setting we could be able to obtain samples of size (n) 14, 15, 26, 39, 54 and 66. These samples differ from each other in terms of income sharing unit and unit of analysis reported in the database. Note that since we work on quintiles $p = 5$.

In the second setting the alternatives are generated randomly from a uniform distribution using MATLAB's random number generator. In this set only equitably efficient alternatives are generated. This is ensured by generating the cumulative ordered vectors rather than the alternatives themselves. Note that for a vector to be a cumulative ordered vector, the difference between its consecutive elements should be increasing. We generate cumulative ordered vectors which are Pareto efficient so that the alternatives will be equitably efficient. Pareto efficiency is ensured by generating the vectors on a quarter circle (or on the boundary of a sphere) in the nonnegative orthant. If the generated vector does not correspond to an ordered vector, we repair it. We then derive the original set of alternatives by applying an inverse cumulative function to the cumulative order.

We refer to these two data sets as *Income Distribution (ID) and Equitably efficient (EE),* respectively. One can see that the difficulty of the problem is expected to increase in the latter one.

The DM's responses are simulated using an underlying value function. Three types of underlying value functions are used in the experiments:

1. Linear value function

$$\max_i \textstyle\sum_{j=1}^{p} w_j \overrightarrow{z}_j^i$$

where w_j, $j = 1, ..., p$ are generated from a uniform distribution between 0 and 1.

2. Product function

$$\underset{i}{max} \prod_{j=1}^{p} z_j^i$$

3. Tchebycheff value function

$$\underset{i}{max}\{\min_j \overrightarrow{z}_j^i\} = \underset{i}{max}\{\overrightarrow{z}_1^i\}$$

Another parameter used in the algorithm is the sample size, denoted by s. We use an s value of 2 in the experiments, that is we only ask for binary comparisons.

In the EE data set, for each combination of the settings discussed above, we generate instances starting with $n = 10$ and $p = 2$, increasing them in increments of 10 and 1, respectively. For each such combination we generate 10 problem instances.

4.5.2 Performance Measures

We now discuss the performance measures used to evaluate the algorithm and the performance of convex cones approach. We use the following measures:

1. CPU time in seconds
2. Number of LP_4 problems solved
3. Number of LP_5 problems solved
4. Number of binary comparisons gathered from the DM
5. Ratio of the binary comparisons gathered from the DM
6. Ratio of the binary comparisons gained through convex cones

The definitions of measures 1,2 and 3 are clear. Let us explain measures 4, 5 and 6.

Number of binary comparisons gathered from the DM: The calculation of this measure is obvious when we only gather pairwise comparisons from the DM. In the cases where we present more than two alternatives, we report the information that we gain in terms of the underlying pairwise comparisons made by the DM. For example, if we present the DM three alternatives for him to rank, we say that he provides $\binom{3}{2} = 6$ pairwise comparisons. Hence for a sample size of s, at each iteration, the number of binary comparisons gathered is $\binom{s}{2}$. In t iterations, we have $t * \binom{s}{2}$.

Ratio of the binary comparisons gathered from the DM *(qratio)* : This is another way to measure the amount of information taken from the DM. In order to achieve a complete ranking of n alternatives, one has to know the relation between each pair of alternatives in this set. Hence, we should know the result of $\binom{n}{2}$ binary comparisons. The ratio is then calculated as follows:

$$qratio = t * \binom{s}{2} / \binom{n}{2}$$

Recall that in *Dominancecheck* subroutine we find the number of pairwise equitable dominance relations. Let us denote it by d. In the output we also report the ratio of the equitable dominance relations which is a property of the problem set rather than a performance measure. We call it *dratio* and calculate as follows:

$$dratio = d / \binom{n}{2}.$$

Ratio of the binary comparisons gained through convex cones *(gainratio)*: Similar to *qratio*, this measure is used to see the amount of information that we gain by using convex cones. It is calculated as follows:

$$gainratio=1\text{-}qratio\text{-}dratio$$

The optimal solutions of the LPs are found by using CPLEX 12.2. We set a termination limit of 30 minutes to the algorithm. All experimentations are done in Intel Core i5 2.27 GHz, 4 GB RAM. The algorithm is coded with MATLAB.

4.5.3 Experiments

In this part we present the results of our experiments.

Table 4.4 shows our results for the first data set where we use income distribution information. For the linear utility function case for each n value we generate 10 problem instances, each with randomly generated objective function weights. We then report the average values over all these 10 instances. In the other cases, we solve the algorithm once.

Table 4.4: Results for WorldBank and UNU-WIDER Data

n	Utility Function	dratio	LP4	LP5	Solution Time (CPU seconds)	gainratio	qratio	Number of comparisons gained	Number of questions asked
14	1	0.901	3	3	0.443	0.022	0.077	2.00	7.01
	2	0.901	3	3	0.490	0.022	0.077	2.00	7.01
	3	0.901	6	6	0.890	0.044	0.055	4.00	5.01
15	1	0.876	10.6	9.9	1.423	0.054	0.070	5.69	7.33
	2	0.876	7	7	1.020	0.057	0.067	5.99	7.04
	3	0.876	12	12	1.740	0.019	0.105	2.00	11.03
26	1	0.895	43.2	43.1	5.480	0.032	0.073	10.37	23.76
	2	0.895	43	43	6.530	0.028	0.077	9.10	25.03
	3	0.895	45	44	6.280	0.040	0.065	13.00	21.13
39	1	0.916	73.3	72.4	9.721	0.030	0.054	21.86	40.09
	2	0.916	73	72	10.580	0.028	0.055	20.75	40.76
	3	0.916	58	54	8.300	0.050	0.034	37.05	25.19
54	1	0.932	125	123.2	16.786	0.028	0.040	40.07	57.24
	2	0.932	156	156	23.160	0.022	0.045	31.48	64.40
	3	0.932	115	113	16.800	0.038	0.030	54.38	42.93
66	1	0.898	594.1	590.6	80.617	0.056	0.046	119.05	99.31
	2	0.898	608	605	85.530	0.054	0.048	115.83	102.96
	3	0.898	480	470	66.270	0.068	0.034	145.86	72.93

It is observed that in this data set, *dratio* is very high, with a minimum value of 0.895. That is in at least 89.5% of all the pairs in the set, we can observe equitable dominance before asking to the DM for preference information. Hence, there is not much to ask the DM. This fact is observed in the average number of questions asked, in the *qratio* and the average number of the two LPs solved. Consequently, *gainratio* is quite low, the maximum is observed as 0.068 in the case where $n = 66$ and the DM has a Tchebycheff type of utility function.

One interesting observation is made regarding the ratio of gainings through cones to the questions asked (gainratio/qratio). This gives us how many extra binary comparisons we can infer, given a binary comparison from the DM. This ratio increases as the problem size, n, increases. In the set with 14 alternatives and the DM has a linear utility function this ratio is 0.29 while it increases to 2 in the set with 66 alternatives and the DM has a Tchebycheff type of utility function. This indicates that the gain that we obtain through cones tends to increase as the problem gets larger.

Note here that, these results are preliminary results for real life income distribution data. Since we are representing the income distributions with a vector of size 5, it becomes harder to capture the underlying distribution and the results are sensitive to changes in the data quality. A number of assumptions are made regarding the income distributions of different countries while collecting the data which may affect the results significantly (see [179] and [180] for more information). More detailed experiments in this area await further attention.

We now discuss the performance of our algorithm for the EE setting. For each combination of utility function, p and n values, 10 problem instances are generated. The average performance measure values over the 10 instances are shown in Table 4.5. Recall that in each problem instance none of the alternatives are equitably dominating each other, i.e, $dratio$=0. Hence this set consists of more difficult problems in that sense.

Table 4.5: Results for the EE set

Utility Function	**p**	**n**	**Number of LP4**	**Number of LP5**	**Solution Time (CPU seconds)**	**gainratio**	**qratio**	**Number of questions asked**
1	2	10	32.60	18.80	3.32	0.95	0.05	2.40
		20	192.40	149.30	21.66	0.96	0.04	7.90
		30	589.70	506.20	85.97	0.97	0.03	11.40
		40	944.20	813.00	121.55	0.98	0.02	13.70
		50	1714.20	1543.40	248.86	0.98	0.02	19.80
		60	2258.50	2005.80	341.17	0.99	0.01	17.80
		70	2159.40	1847.70	313.64	1.00	0.00	10.60
	3	10	100.90	94.10	12.27	0.68	0.32	14.50
		20	707.80	694.40	113.45	0.77	0.23	44.60
		30	2061.00	2011.50	320.35	0.84	0.16	70.60
		40	3446.10	3351.40	612.62	0.89	0.11	88.60
		50	6397.10	6308.70	1438.71	0.91	0.09	111.90
	4	10	129.50	128.70	16.43	0.54	0.46	20.90
		20	884.80	874.10	124.20	0.73	0.27	52.00
		30	2778.30	2757.50	472.12	0.78	0.22	95.80
		40	5737.00	5706.60	1278.61	0.82	0.18	140.90
	5	10	133.40	132.70	16.95	0.49	0.51	23.00
		20	1070.80	1065.30	149.77	0.64	0.36	67.70
		30	3100.40	3089.80	533.22	0.75	0.25	109.20
		40	6895.75	6881.38	1483.24	0.81	0.19	145.88
2	2	10	46.70	33.40	5.07	0.91	0.09	4.10
		20	260.80	220.70	31.30	0.95	0.05	9.90
		30	739.70	669.60	96.78	0.96	0.04	17.50
		40	1399.70	1296.20	197.30	0.97	0.03	22.00
		50	2617.40	2466.60	418.67	0.98	0.02	28.10
		60	4026.70	3826.20	733.58	0.98	0.02	34.40
		70	5528.70	5267.70	1110.43	0.98	0.02	38.50
	3	10	98.50	92.90	12.11	0.64	0.36	16.20
		20	752.30	729.20	101.44	0.77	0.23	43.00
		30	2075.00	2035.50	325.55	0.81	0.19	82.20
		40	4332.30	4272.80	843.19	0.85	0.15	119.60
	4	10	123.50	122.40	15.80	0.52	0.48	21.70
		20	971.80	967.10	137.55	0.68	0.32	60.40
		30	2946.70	2934.10	508.64	0.76	0.24	104.70
		40	6290.67	6265.67	1213.64	0.80	0.20	152.67
	5	10	127.20	126.40	16.37	0.50	0.50	22.50
		20	1092.20	1088.70	153.25	0.66	0.34	64.70
		30	3380.00	3375.30	611.02	0.73	0.27	119.20
3	2	10	27.20	11.80	2.51	0.96	0.04	1.80
		20	106.30	58.70	10.48	0.98	0.02	4.50
		30	263.00	157.40	26.79	0.99	0.01	3.20
		40	467.30	320.60	51.32	1.00	0.00	3.50
		50	573.20	375.60	73.15	1.00	0.00	3.40
		60	943.60	692.20	112.40	1.00	0.00	3.10
		70	1286.00	966.10	159.81	1.00	0.00	3.10
	3	10	96.00	90.30	11.77	0.67	0.33	14.90
		20	717.80	696.90	95.15	0.77	0.23	43.60
		30	2063.30	2019.90	323.04	0.84	0.16	71.70
		40	4189.10	4118.70	795.93	0.87	0.13	103.30
	4	10	128.20	127.00	16.56	0.49	0.51	23.10
		20	985.20	978.90	142.83	0.70	0.30	56.30
		30	2758.30	2740.40	562.76	0.78	0.22	95.10
		40	6040.00	6009.90	1324.71	0.81	0.19	147.10
	5	10	128.30	127.60	62.92	0.50	0.50	22.70
		20	1055.70	1052.00	531.56	0.66	0.34	64.20
		30	3275.90	3269.90	586.79	0.74	0.26	113.30
		40	4606.70	4594.50	1832.07	0.94	0.06	47.70

We can find complete rankings for problems with up to 70 alternatives when the number of parties, p, is two, and up to 40 alternatives when p is three, four and five in our time limit of 30 minutes.

These results reveal the contribution of convex cones. The minimum average *gainratio* value is 0.49, that is, at least about 50% of the binary comparisons are provided by the convex cones. This indicates a satisfactory performance for the convex cones approach. Note here that in some instances *gainratio* is seen as 0, which is due to rounding.

We can see the effect of problem size on the performance of the algorithm and on the amount of information gained from cones.

As can be observed from the table when the number of alternatives, n, increases *gainratio* increases. Hence for constant p, the contribution of convex cones approach to the solution increases as n increases. Note that although *qratio* decreases with increasing n, the actual number of questions increases, resulting in an increase in the number of cones/polyhedrons generated. Moreover as n increases so does the the number of LPs solved per cone/polyhedron. As a result, we observe an increase in the number of LPs solved and the solution time.

The effect of number of parties, p, is also notable in the performance of the convex cones. As p increases the number and ratio of the comparisons required from the DM increase. As a result of the increase in the ratio of the comparisons required, the ratio of information gained through cones decreases. Moreover the increase in the number of comparisons provided by the DM leads to an increase in the number of LP models solved and in turn an increase in solution time.

It is also observed that the effects of p and n are consistent over the three types of utility functions used.

4.6 Conclusion and Further Research

In this study we consider a method to incorporate preference information for multicriteria problems with equity concerns. We are motivated by the fact that problems involving equity concerns are widely encountered in real life, especially in public sector. Such problems include, facility location, income distribution and resource/service allocation problems.

We analyze a method based on convex cones that is frequently used to represent DM's preferences in multi criteria decision making environment. Based on the reported satisfactory performance of convex cones in reducing the amount of information required from the DM to solve different MCDM problems, we consider extending their use for problems with equity concerns. We provide theoretical results and discuss a way to partially handle the computational complexities due to symmetry. This allows us to use convex cones approach in problems with impartiality without significantly increasing the computational efforts. Related to the cones, we also discuss the use of polyhedrons in a ranking problematique.

We check the performance of the suggested approach by using it in a ranking algorithm. In the most difficult setting, where all the alternatives in the set are equitably efficient, our algorithm provides complete rankings for problems with up to 70 alternatives when the number of parties is two, and up to 40 alternatives when the number of parties is three, four and five in less than 30 minutes requiring a reasonable number of comparisons from the DM. It is observed that this satisfactory performance of the algorithm is mostly due to the high ratio of information gained by convex cones and polyhedrons. At least 50% of the information is obtained through cones and polyhedrons in this setting. We observe that the number of parties and the number of alternatives affect the problem complexity and the percentage of the information gained through cones.

To the best of our knowledge, this study is the first extensive study that attempts to incorporate DM's preference information in MCDM problems with equity concerns where the utility function is not assumed to be linear. This is also the first discussion on the convex cones approach in a symmetric environment and the first study that reports results for a ranking algorithm that uses the information from cones and polyhedrons.

In the near future, this study can be extended by working more on the MCDM models and generalizing the use of convex cones in this context. This includes four main areas: Searching for alternative ways to handle computational complexity due to symmetry, using convex cones in selecting the best and sorting problematiques, using convex cones in different feasible sets and performing an experimental study to use convex cones approach as efficiently as possible. These potential research topics are discussed below.

1. More on Handling the Computational Complexity: Recall that the method we propose to handle computational complexities due to symmetry reduces many of the permutational computations. However, in LP_4 we still have to find all the permutations of the lower generator of each cone. In the near future, more theoretical studies may be performed to see whether there exists a way to obtain the information without any permutational calculations.

2. Interactive Algorithms for Selecting the Best and Sorting Problematiques: Selecting the best alternative in an MCDM setting involving equity concerns is a problematique that is encountered in the public sector as discussed before. It has applications in location and public service/resource allocation decisions. We use different ways to gather and use DM's preference information in different problematiques. For example, in a sorting environment, instead of taking pairwise comparisons or rankings, we may request him/her to assign the alternatives into the classes.

3. Use of Convex Cones for Problems with Different Feasible Sets One can study different problem environments where the feasible region is defined by constraints. More research can be done to generalize the use of convex cones in such environments.

4. Experimental Study on Convex Cones: While designing an algorithm the analyst makes various decisions regarding convex cones that may affect the performance of the proposed method. These decisions are mostly related to the ways to collect preference information from the DM. The performance of the algorithm may vary based on the size of the sample used for gathering preference information, the selection rule applied to select the alternatives in the sample and the form of the information the DM provides. For example, given a set of k alternatives, we may require the DM to rank them or select the best/worst alternatives in the sample. We can perform an experimental study to see the effects of such decisions that are made in the

approaches using convex cones theory.

4.A Results for the 2-point Polyhedrons Case

Lemma 47 *For z^1, $z^2 \in \mathbb{R}^p_+$, if $z \in P(z^1; \overrightarrow{z^2})$ then $z \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$.*

Proof. Let $z^1 \neq \overrightarrow{z^1}$ (Otherwise, the result is immediate). Let h be the minimum value for which $z^1_h > z^1_{h+1}$ holds. Define $z^{1\prime}$ as the permutation obtained from z^1 by swapping z^1_h and z^1_{h+1}. That is, $z^1 = (z^1_1, z^1_2, ..., z^1_h, z^1_{h+1}, ..., z^1_p)$ and $z^{1\prime} = (z^1_1, z^1_2, ..., z^1_{h+1}, z^1_h, ..., z^1_p)$ where $z^1_h > z^1_{h+1}$.

Suppose for an arbitrary $\mu \geq 0$ we have a point $z \in P(z^1; \overrightarrow{z^2}) : z = \mu\overrightarrow{z^2} + (1-\mu)z^1$. Define $z' \in P(z^{1\prime}; \overrightarrow{z^2}) : z' = \mu\overrightarrow{z^2} + (1-\mu)z^{1\prime}$.

z and z' have the same elements except the h^{th} and $h+1^{th}$ elements, which are as follows:

$z_h = \mu\overrightarrow{z^2}_h + (1-\mu)z^1_h;$

$z_{h+1} = \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_{h+1};$

$z'_h = \mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1};$

$z'_{h+1} = \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_h.$

From Lemma 27 we know that $z' \preceq_e z$ if $Min\{z'_h, z'_{h+1}\} \leq Min\{z_h, z_{h+1}\}$ and $z'_h + z'_{h+1} \leq z_h + z_{h+1}$. Let us check (Recall that $z^1_h > z^1_{h+1}$):

$$\begin{aligned} Min\{z'_h, z'_{h+1}\} &= Min\{\mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1}, \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_h\} \\ &= \mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1} = z'_h. \end{aligned}$$

We do not know what $Min\{z_h, z_{h+1}\}$ is, hence we will compare z'_h with both z_h and z_{h+1}.

$z'_h = \mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1} \leq \mu\overrightarrow{z^2}_h + (1-\mu)z^1_h = z_h$

$z'_h = \mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1} \leq \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_{h+1} = z_{h+1}$. Hence,

$$Min\{z'_h, z'_{h+1}\} \leq Min\{z_h, z_{h+1}\} \tag{4.19}$$

$z_h + z_{h+1} = \mu\overrightarrow{z^2}_h + (1-\mu)z^1_h + \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_{h+1}$

$$= \mu\overrightarrow{z^2}_h + (1-\mu)z^1_{h+1} + \mu\overrightarrow{z^2}_{h+1} + (1-\mu)z^1_h$$
$$= z'_h + z'_{h+1}.$$ That is,

$$z'_h + z'_{h+1} \ \leq z_h + z_{h+1} \tag{4.20}$$

From 4.19 and 4.20 the conditions of Lemma 27 is satisfied so $z' \preceq_e z$. Since μ is arbitrary, this result is valid for every $z \in P(z^1; \overrightarrow{z^2})$.

We showed that if $z \in P(z^1; \overrightarrow{z^2})$, then $\exists z' \in P(z^{1\prime}; \overrightarrow{z^2})$ such that $z' \preceq_e z$, where $z^{1\prime}$ is the permutation obtained by a single swap of two consecutive elements of z^1 as defined above.

Any permutation of vector z^1 will result in $\overrightarrow{z^1}$ if we apply a finite number of such binary contiguous swaps. Starting from the first element which is higher than its consecutive element, these type of swaps will eventually result in $\overrightarrow{z^1}$. Hence, we have the following result:

For any z^2, $z^1 \in \mathbb{R}^p_+$, if $z \in P(z^1; \overrightarrow{z^2})$ then $\exists z' \in P(\overrightarrow{z^1}; \overrightarrow{z^2})$ such that $z' \preceq_e z$. That is, $z \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$. ■

Proposition 48 *For any z^2, $z^1 \in \mathbb{R}^p_+$, if $z \in P(z^1; z^2)$ then $z \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$.*

Proof. Let $z^1 = \Pi^s(\overrightarrow{z^1})$ and $z^2 = \Pi^q(\overrightarrow{z^2})$. Then $z = \mu(\Pi^q(\overrightarrow{z^2})) + (1-\mu)(\Pi^s(\overrightarrow{z^1}))$. Let the inverse permutation of Π^q be Π^r and let $\Pi^r(\Pi^s) = \Pi^t$.

We can rewrite the condition as follows: *If* $\Pi^r(z) \in P(\Pi^t(\overrightarrow{z^1}); \overrightarrow{z^2})$ *then* $\Pi^r(z) \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$ $\forall t$, implied by Lemma 47 proved above. ■

Theorem 49 *For any $z^2, z^1, z \in \mathbb{R}^p_+$, if $z \in UP(z^1; z^2)$ then $z \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$.*

Proof. If $z \in UP(z^1; z^2)$ there exists a $z' \in P(z^1; z^2)$: $z' \preceq_e z$. From Proposition 48 z' $\in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$. Then from transitivity of $\preceq_e$, $z \in UP(\overrightarrow{z^1}; \overrightarrow{z^2})$. ■

4.B Generalization for the k-point Polyhedrons Case

Lemma 50 *Every point in a k-point polyhedron is a convex combination of $k-1$ points which are in the $k-1$ distinct 2-point polyhedrons generated by one of the upper generators and the lower generator. That is, given z^i such that $z^i \succ z^k, \forall i \in I \setminus \{k\}$ we have the following:*

If $z \in P(z^1, z^2, ..., z^{k-1}; z^k)$ then there exists λ_i and $y^i \in P(z^i; z^k)$ for $i = 1, ..., k-1$ such that $z = \sum_{i=1}^{k-1} \lambda_i y^i$, $\sum_{i=1}^{k-1} \lambda_i = 1$.

Proof. *$z = \mu z^k + \sum_{i=1}^{k-1} \mu_i z^i$ such that $\mu + \sum_{i=1}^{k-1} \mu_i = 1$.*

Let $y^i = (1 - \mu_i')z^k + \mu_i' z^i$ $\forall i$.

Now we will show that there exist λ_i $i = 1, ..., k-1$ such that $z = \sum_{i=1}^{k-1} \lambda_i y^i$. Given μ_i corresponding to vector z, we will show that $\lambda_i s$ and $\mu_i' s$ exist as defined so that we can write z as a convex combination of $y^i s$. Suppose that we have λ_i values for $i = 1, .., k-2$ such that $\lambda_i > 0$ and $\sum_{i=1}^{k-2} \lambda_i < 1$ and we set $\lambda_{k-1} = 1 - \sum_{i=1}^{k-2} \lambda_i$. Given these λ_i and μ_i, we can set μ_i' values as follows:

$\mu_i = \lambda_i \mu_i' \quad for\ i = 1, .., k-1$

$\mu_i' = \mu_i / \lambda_i$. Since $\lambda_i > 0$ and $\mu_i \geq 0$, we have $\mu_i' \geq 0$.

$$
\begin{aligned}
z &= \mu z^k + \sum_{i=1}^{k-1} \mu_i z^i \\
&= (1 - \sum_{i=1}^{k-1} \lambda_i \mu_i') z^k + \sum_{i=1}^{k-1} \lambda_i \mu_i' z^i \ (Since\ \mu = 1 - \sum_{i=1}^{k-1} \mu_i = 1 - \sum_{i=1}^{k-1} \lambda_i \mu_i') \\
&= \sum_{i=1}^{k-1} \lambda_i (1 - \mu_i') z^k + \sum_{i=1}^{k-1} \lambda_i \mu_i' z^i. \\
&= \sum_{i=1}^{k-1} \lambda_i [(1 - \mu_i') z^k + \mu_i' z^i] \\
&= \sum_{i=1}^{k-1} \lambda_i y^i
\end{aligned}
$$

■

Remark 51 *If an alternative $z \in UP(z^i, z^k)$ then it is in the upper side of any k-point* *polyhedron having z^i and z^k as two of the generators (z^k being the lower generator).*

Theorem 52 *If $z \in P(z^1, z^2, ..., z^{k-1}; z^k)$ then $z \in UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$.*

Proof. From Lemma 50 if $z \in P(z^1, z^2, ..., z^{k-1}; z^k) \Longrightarrow$we can find $y^i \in P(z^i; z^k)$: $z = \sum_{i=1}^{k-1} \lambda_i y^i$ for $i = 1, ..., k-1$where $\sum_{i=1}^{k-1} \lambda_i = 1$.

By Theorem 49, $y^i \in P(z^i; z^k) \Rightarrow y^i \in UP(\overrightarrow{z^i}; \overrightarrow{z^k})$;

By Remark 51 $y^i \in UP(\overrightarrow{z^i}; \overrightarrow{z^k}) \Longrightarrow y^i \in UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$ for all $i = 1, ..., k-1$.

Since y^is, as defined above, are all in $UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$ and $UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$ is convex; any convex combination of them will also be in $UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$. Hence, $z \in UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$. ■

4.C Checking for Polyhedrons

If $z \in P(z^1, z^2, ..., z^{k-1}; z^k)$ then $z \in UP(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$. Hence, for each alternative z we have to check whether there exists $z' \in P(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k}) : z' \preceq_e z$.

The following model will be used for that purpose:

(LP_5)

$Max\ \epsilon$

subject to

$$z'_h - \mu \overrightarrow{z^k}_h - \sum_{i=1}^{k-1} \mu_i (\overrightarrow{z^i}_h) = 0 \quad for\ h = 1, ..., p \tag{4.21}$$

$$\sum_{h=1}^{n} z'_h + \epsilon \leq \sum_{h=1}^{n} \overrightarrow{z}_h \quad for\ n = 1, ..., p \tag{4.22}$$

$$\sum_{i=1}^{k-1} \mu_i + \mu = 1 \tag{4.23}$$

$\mu_i \geq 0\ \forall i$

$\mu \geq 0$

Constraint sets 4.21 and 4.23 ensure that $z' \in P(\overrightarrow{z^1}, ..., \overrightarrow{z^i}, ..., \overrightarrow{z^{k-1}}; \overrightarrow{z^k})$ and constraint set 4.22 ensures that $z' \preceq_e z$ by ensuring $\overline{\Theta}(z') \leq \overline{\Theta}(z)$.

It is an LP problem with $p + k + 1$ variables and $2p + 1$ constraints excluding the set constraints.

Chapter 5

Incorporating Balance Concerns in Resource Allocation Decisions: A Bi-criteria Modelling Approach

5.1 Introduction

Resource allocation (distribution) is a process by which resources (inputs) are allocated to different entities such as activities, projects or departments [182]. The inputs are usually allocated in a way that maximizes some output value.

A common goal in resource allocation in organizations alongside maximization of output (efficiency) is "balance" [182]. Balance can be sought in terms of various attributes such as risk (high risk vs. sure bets), internal vs. outsourced work, distribution of resources across industries, various markets the business is in, different project types etc. [183]. Failure to achieve a balanced portfolio is often revealed by a decision maker (DM) who claims that there is "too much" or "too little" resource going to activities of a particular type.

A related concept considered in many allocation decisions is equity (fairness). However, as we use the term, a "perfectly balanced distribution" is not necessarily a distribution where each category receives the same amount. We define balance as a more general concept, of which equity might be considered as a special case. We assume that the DM has a balance distribution based on which she evaluates the balance in a given distribution. We refer to a distribution that has the "desired proportions" shown by the balance distribution as a "perfectly balanced distribution" even if this distribution gives some categories more than the others. Equity concerns can be represented as the special case where the DM's balance distribution gives each category an equal amount.

The contributions of the current study are as follows:

- We propose a means to handle balance concerns alongside efficiency concerns in allocation problems and hence provide a bi-criteria framework to think about trading balance off against efficiency.
- We discuss ways to measure the deviation from a distribution which the DM considers as balanced and hence define and classify *imbalance indicators*.
- We propose formulations and algorithms which provide insight to the decision makers in general resource allocation settings.

Section 5.2 discusses an example allocation setting. Section 5.3 discusses alternative ways in which balance concerns have been handled in mathematical programming models and provides a brief review of related works from the literature. Section 5.4 introduces some imbalance indicators, which can be used to assess the degree of balance in a distribution. We introduce these indicators as another criterion to be optimized in the classical *maximize output* setting and provide bi-criteria models in section 5.5. In section 5.6 we discuss a way to solve the bi-criteria models and obtain nondominated solutions. We provide the results of our computational experiments on the performance of the suggested approach in section 5.7. We also provide results for 3-criteria extensions of the approach as well as a tabu search algorithm that can be used to solve large-sized problems. We conclude our discussion in Section 5.8.

5.2 The Balance Concept

Consider a setting where a DM is faced with m R&D projects and s/he will decide which ones to initiate given an available budget, B, which typically is not sufficient to initiate all projects. Each project i incurs a cost (input) c_i and returns an output value b_i. Suppose that it is possible to categorize the projects into n categories (e.g. based on the technological area or based on the department they are proposed by) and each project belongs to one (and only one) of these categories.

Each feasible portfolio corresponds to two portfolio-related distributions: a distribution of the budget B to different project categories and a distribution that shows the contribution of each category in terms of the output. Suppose that the DM wants to ensure balance in one or both of these distributions as well as having a high total output from the selected portfolio.

This is an example of an allocation problem in which the DM has concerns about ensuring balance. In this study we provide a general framework that can be used for many allocation problems. To have a structured discussion, we will illustrate the general idea using this R&D project selection example and discuss possible generalizations in the conclusion.

We distinguish the cases based on the space balance is sought, i.e. based on whether balance is sought in the input distribution or the output distribution. Which balance concept is more appropriate depends on the nature of the problem. For example in healthcare, the policy maker may want a balanced input allocation on the grounds that people should be responsible for their own health and the policy makers can only be responsible for providing them with a balanced allocation of inputs. On the other hand, the policy maker may prefer a balanced distribution of health (the output) on the grounds that health policy should aim at equal health for all.

5.3 Literature Review on Incorporating Balance into Models

In this section we mention some noteworthy studies that consider the balance concept in portfolio selection and allocation decisions in an explicit way. We refer the interested reader to [184] for a more detailed discussion on balance in project portfolio selection problems. There is also a wide range of applications in which equity concerns are incorporated into mathematical models, including but not limited to drug allocation [95], HIV prevention funds allocation [14], water allocation [120], bandwidth allocation [10], workload allocation [27] problems, and location-allocation problems in homeland defense [36].

Efficiency concerns are reflected to the model by maximizing the total output. From a modelling point of view, balance concerns may be handled in two ways:

- Modifying the feasible region by introducing constraints: In this approach the analyst changes the feasible region of the problem so that the feasible allocations will ensure a certain degree of balance.

[185] considers selecting a portfolio of solar energy projects using multiattribute preference theory. As a way of ensuring a balanced portfolio, they use lower bounds on the number (or monetary value) of the projects of a certain type that are included in a portfolio. Similarly, [186] uses linear programming to maximize the total technical score of funded projects on a smoking intervention study. Balance related constraints are used to ensure geographic equity in project proposal fundings and to ensure "a spread" of changes across different quartiles of the population with respect to smoking preference and decline in smoking rate. [187] proposes an integer programming model for selecting and scheduling an optimal project portfolio. The balance related constraints enforce an upper limit on the percentage of total investment made on different project categories, such as high risk and long term projects. The authors illustrate their approach by solving a small-size problem with 12 projects. [188] considers a multi-dimensional integer knapsack problem and introduces constraints to incorporate balance concerns into the model.

The constraints are used to apply upper and lower bounds on the fraction of the resources allocated to different project categories. The authors, however, mostly focus on the linear programming relaxation of the integer programming formulation and hence assume that partial resource allocation to projects is possible. [189] develops a nonlinear integer programming model to optimize a portfolio of (possibly interdependent) product development improvement projects over multiple periods. The projects are categorized based on the strategic objectives that they support and balance over different objectives is ensured by incorporating a constraint that shows the minimum number of projects from each category. [190] discusses a multi-criteria decision analysis (MCDA) framework to allocate fishing rights to candidates in South Africa. As part of their decision support system they provide an integer formulation for the candidate selection problem in which balance concerns are reflected using constraints. These constraints ensure that the proportion of the number of candidates selected from a designated group exceeds a minimum desired level for this group. [183] propose a DEA (Data Envelopment Analysis) based methodology to construct and evaluate balanced portfolios of R&D projects with binary interactions. As part of their proposed methodology, they compute indices of risk, efficiency and balance for each project. They use a maximum threshold for risk index and minimal thresholds for efficiency and balance indices and screen the initial list of candidate projects. Only the ones that satisfy the requirements set by the indices are considered further. A similar approach is used in [191] to evaluate R&D projects in different stages of their life cycle.

[192] develop a fuzzy R&D project selection model in which balance in spending between different strategic goals is enforced in constraints. These constraints specify upper and lower bounds on the spending for each strategic goal (see also [193]).

- Modifying the objective: In this approach the analyst increases the number of criteria of the corresponding model; turning it into a bi/multi criteria problem. The approach we take falls into this category. We use this approach as it is possible to observe the trade-off between different criteria by finding different solutions to the problem.

Modifying the objective typically relies on the use of a balance indicator, $z_I(x)$, which assigns a value that shows the level of balance in a distribution x. Using the indicator one can define a balance criterion along with the efficiency criterion ($z_T(x)$).

Note that if balance is considered over different aspects such as technology areas, markets etc., it is possible to use a balance indicator for each aspect and hence generate a multi criteria model. [194], [195], [196] and [197] use multiattribute models which tackle balance concerns over multiple attributes and then use Multiattribute Value/Multiattribute Utility models to aggregate the set of attributes into a single index. One of the restrictions on the generality of the proposed models is the assumption that the number of items in the subset is constant. Moreover, an additive value function may not always be appropriate, and even when it is appropriate, determining weights may not be easy.

[198] models the concern for balance as a separate set of criteria which minimize the deviation from the ideal allocation of manpower to different project categories and also to different client categories. With additional criteria which are not balance related, he formulates a multi-criteria decision making (MCDM) model for the project portfolio selection problem. The reference point approach (see [158]) is used, which involves solving nonlinear integer programming problems subject to linear resource constraints. The approach is used in an interactive setting. For various reasons including the technical difficulty due to nonlinearity, a heuristic method is used to solve the resulting optimization problems. The same approach is also used in [199].

As it expresses balance criteria as measures of deviation from a desired allocation, this approach is similar to the approach used to incorporate balance in this paper. We, however, mostly focus on a bi-criteria setting and use linear (integer) models whenever possible. The underlying reason for this choice is the ease of presentation. The ideas proposed here are easily generalizable to multidimensional settings where balance is desired over multiple attributes. Balance concern for each such attribute can be reflected as a criterion to the model and appropriate multicriteria optimization or heuristic methods can be implemented to obtain solutions. The emphasis of this paper is to introduce the idea of balance distribution based balance indicator as a way of handling balance concerns in an explicit and tractable way.

Unlike [198], we do not assume an interactive setting; we rather present the DM with a dispersed subset of nondominated portfolios. This is an alternative approach to the one used in [198]; empirical research should be performed to see which one is more appropriate in different problem environments. We provide graphical displays of the set of nondominated solutions which visualize the tradeoff between efficiency and balance. These graphs can be used as a starting point for further discussion with decision makers.

We also provide an explicit link between inequality measurement literature by making an analogy between the perfect balance line and the perfect equality line. This will be explained in detail in the next section. We discuss different indicators that can be used to assess imbalance. Our solution approach allows one to incorporate different imbalance indicators into the same model and hence observe the tradeoff between them.

5.4 Imbalance Indicators

In this section we propose imbalance indicators that measure how different a distribution is from an ideally balanced distribution. The indicators rely on a *balance distribution* which is provided by the DM. This balance distribution shows how the DM would allocate a certain amount of the input/output across the categories involved. This might be for example, the *status quo* or previous year's allocation.

We will use the following terminology and notation to frame our discussion:

We refer to the entities over which the balance is sought as categories. $J = \{1, 2, ..., n\}$ is the set of the categories. The vector $x \in \mathbb{R}^m$ is used to show the decision vector related to input allocation. Note that m is not necessarily equal to n unless we make explicit allocation decisions to categories themselves. For example, in the project selection problem m is the number of projects and x is the corresponding binary decision vector and we expect $n < m$ unless each project is considered as a different category.

x can be continuous or discrete and includes the decision variables which dictate the input allocation to categories (this dictation can be indirect as in the project selection problem: in that case, x shows the portfolio of projects, from which we can infer the allocation to categories). Any function defined over the input allocation is a function of the decision vector x. Let $a(x) \in \mathbb{R}^n$ be the distribution over which the balance is sought, hence it can either show the input or the output distribution to categories.

We denote the balance distribution of either input or output by $r \in \mathbb{R}^n$ where r_j is the amount allocated to category j in the balance distribution. Which of these (input/output) is intended will be clear from context. For notational simplicity we will normalize the balance distribution so as to obtain *balance shares (proportions)* for each category. Let us denote the balance share of category j as α_j. By definition $\alpha_j = r_j / \sum_{j \in J} r_j$. Hence, $\alpha \in \mathbb{R}^n$ is the balance distribution in terms of shares.

Suppose that given α, we want to assess how balanced a distribution $a(x)$ is. Using α, we can obtain a target point $\overline{r}(x)$ as follows: $\overline{r}(x)_j = \alpha_j * \sum_{j \in J} a(x)_j$.One can think of the elements of $\overline{r}(x) \in \mathbb{R}^n$ as target (desired) amounts for the different categories involved.

We denote the componentwise deviations of the distribution $a(x)$ from the corresponding $\overline{r}(x)$ as $d(x)_j$ $\forall j \in J$. That is, $d(x)_j = |a(x)_j - \overline{r}(x)_j| = \left|a(x)_j - \alpha_j * \sum_{j \in J} a(x)_j\right|$ $\forall j \in J$.

Figure 5.1 visualizes componentwise deviations in a 2 dimensional environment. Note that except r, all the terms are functions of the input allocation x. d_1 and d_2 are the componentwise distances of point a to the inflated balance point.

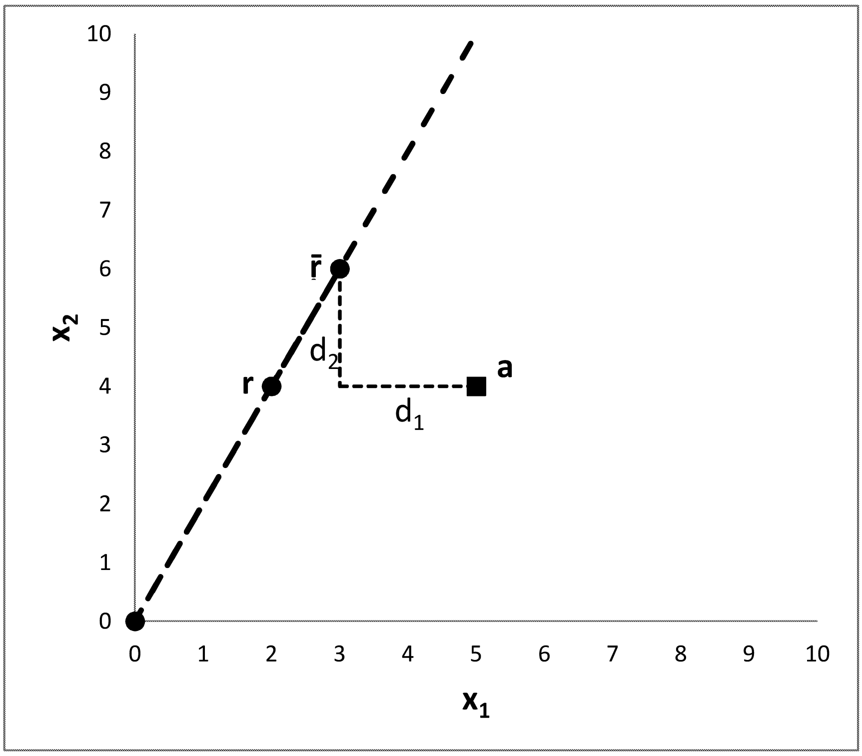

Figure 5.1: Distances from the balance point in 2D

It does not seem appropriate to capture balance using a distance measure from the balance distribution itself (r). The rescaling of r, i.e. generating $\overline{r}(x)$, is necessary to obtain an appropriate evaluation. Consider, for example, the case where $r = (1, 2)$, that is the DM considers $(1, 2)$ as a balanced distribution, and we want to assess how balanced distribution $a(x) = (2, 4)$ is. Using just a distance measure would mislead us by concluding that $(2, 4)$ is not balanced (as the componentwise absolute deviations between $(1, 2)$ and $(2, 4)$ are not equal to zero). However if $(1, 2)$ is balanced, it would seem natural to suppose that $(2, 4)$ is also balanced. This is clearly seen when we rescale $r = (1, 2)$ with respect to $(2, 4)$. We find $\overline{r}(x) = (1/3, 2/3)a(x) = (2, 4)$. Since $a(x) = \overline{r}(x)$, the componentwise deviations are all zero hence we capture that $a(x)$ has perfect balance. That is why, we avoid using just a distance measure from the given balance distribution to account for balance and instead generate a *balance line* based on the balance distribution.

The intuition behind our approach generalizes the *perfect equality line* concept used in inequality measurement theory ([127]). The *perfect equality line* consists of points whose components are equal in all dimensions, i.e, it consists of the distributions where everyone gets the same income. Despite being different in the total income, all points on this line are considered to have perfect equality, i.e. zero inequality. Similarly, we derive a line of *perfect balance* passing through the origin and the balance point (see Figure 5.1) and derive our balance indicators accordingly.

We now define the four imbalance indicators as follows.

Indicator 1: The total proportional deviation from the target.

$$I_1(x) = \frac{\sum_{j\in J} d(x)_j}{\sum_{j\in J} a(x)_j} = \frac{\sum_{j\in J} \left|a(x)_j - \alpha_j * \sum_{j\in J} a(x)_j\right|}{\sum_{j\in J} a(x)_j} = \sum_{j\in J} \left|\frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j\right|$$

This is the sum of the absolute differences between the actual share and the desired share for each category. Taken in its input oriented sense, this indicator is the fraction of input which is misallocated. Taking the proportional deviation also implies the following: of two alternative distributions with the same total absolute distance from the balance line, the one that has a larger sum will have a smaller imbalance value, hence will be favoured. Special cases of $I_1(x)$ where the balance distribution is the one with perfect equality, i.e., each category receives an equal share, have been used in the literature (e.g., in [111]).

Indicator 2: The maximum proportional deviation from the target. Unlike $I_1(x)$ this indicator focuses only on the worst-off deviation.

$$I_2(x) = \frac{Max_{j\in J}\{d(x)_j\}}{\sum_{j\in J} a(x)_j} = Max_{j\in J}\left|\frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j\right|$$

Indicator 3: The total componentwise proportional deviation. Compared to the first two indicators this is a more individual oriented measure as it is the sum of fractional misallocations to each party.

$$I_3(x) = \sum_{j\in J} \frac{d(x)_j}{\overline{r}(x)_j} = \sum_{j\in J} \frac{\left|a(x)_j - \alpha_j * \sum_{j\in J} a(x)_j\right|}{\alpha_j * \sum_{j\in J} a(x)_j} = \sum_{j\in J} \frac{1}{\alpha_j} \left|\frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j\right|$$

This measure is a weighted sum of the absolute differences between the actual share and the desired share for each category where weight for category j is $\frac{1}{\alpha_j}$. This allows one to penalize the deviations from the categories that are already assigned a low target share value. We note that for this measure to be meaningful, one should have $\alpha_j > 0$ for all j.

Indicator 4: The maximum proportional deviation from the corresponding target value over all elements of the distribution. Unlike $I_3(x)$ this indicator focuses only on the worst-off deviation.

$$I_4(x) = \underset{j\in J}{Max}\{\frac{d(x)_j}{\overline{r}(x)_j}\} = \underset{j\in J}{Max}\{\frac{1}{\alpha_j}\left|\frac{a(x)_j}{\sum_{j\in J} a(x)_j} - \alpha_j\right|\}$$

Which indicator one chooses to use, might have material significance for the solutions which are bi-criteria efficient in the biobjective formulations. However, when n is low as in Proposition 53, whichever indicator one chooses, one will get the same ordering and thus the same efficient frontier. Hence in this case, which indicator one chooses does not matter: one can choose any indicator and be confident of getting the same result.

Proposition 53 *For $n \leq 3$ we have $I_1(x) = 2 * I_2(x)$. Moreover when $n = 2$ the four indices provide us with the same order. That is, for any two distributions x^1 and x^2 where $n = 2$ (that is $a(x^1), a(x^2) \in \mathbb{R}^2$), the following holds: $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2) \iff I_3(x^1) \geq I_3(x^2) \iff I_4(x^1) \geq I_4(x^2)$.*

Proof is provided in Appendix A.

Remark 54 *In general, Proposition 53 no longer holds for $I_1(x)$; $I_3(x)$; $I_4(x)$ and $I_2(x)$; $I_3(x)$; $I_4(x)$ in problems where $n > 2$. As for $I_1(x)$ and $I_2(x)$ it no longer holds when $n > 3$.*

Proof is provided in Appendix A.

Table 5.1 summarizes the classification of the imbalance indicators.

Table 5.1: Imbalance indicators

Imbalance Indicators		
Objective\Focus	Collective	Individual Oriented
Sum	$I_1(x) = \frac{\sum_{j \in J} d(x)_j}{\sum_{j \in J} a(x)_j}$	$I_3(x) = \sum_{j \in J} \frac{d(x)_j}{\overline{r}(x)_j}$
Bottleneck	$I_2(x) = \frac{Max_{j \in J}\{d(x)_j\}}{\sum_{j \in J} a(x)_j}$	$I_4(x) = \underset{j \in J}{Max}\{\frac{d(x)_j}{\overline{r}(x)_j}\}$

5.5 Bi-criteria Models

In this section we develop bi-criteria models for allocation problems with objectives of maximizing total output and minimizing an imbalance indicator.

Although all the models we discuss are based on the same general idea, they differ in technical aspects depending on the problem type, i.e., based on whether the allocation is discrete or continuous and whether the balance is sought in the output or input distribution.

For our project selection problem we provide mixed integer formulations for the bi-criteria models, which exploit the fact that the decision variables are 0-1 variables to tackle nonlinearity due to the imbalance indicators.

We first provide a complete analysis for the case the DM desires a balanced input distribution as this problem naturally arises in many situations. It is straightforward to develop models for the case where a balanced output distribution is desired when we have a discrete setting.

5.5.1 Discrete Allocation

The general method proposed in this paper is applicable to different combinatorial problems that can be formulated as a binary integer problem (BIP), like some location problems. We use project selection problems as an example.

Consider the project selection problem discussed in Section 2. Suppose that the DM wants to have a portfolio where input is allocated to different project types in a balanced way and gives an example input allocation $r \in \mathbb{R}^n$, which he considers balanced. The corresponding proportional allocation is denoted as $\alpha \in \mathbb{R}^n$ as before.

We use an $m \times n$ incidence matrix G with elements g_{ij} for $i \in I$ and $j \in J$ as follows:

$$g_{ij} = \begin{cases} 1 \text{ if project } i \text{ belongs to category } j \\ 0 \text{ otherwise} \end{cases}$$

The binary variable associated with each project is as follows:

$$x_i = \begin{cases} 1 \text{ if project } i \text{ is initiated} \\ 0 \text{ otherwise} \end{cases} \quad \text{for } i \in I$$

Note that we seek balance in the input space and $a(x)_j = \sum\limits_{i \in I} c_i g_{ij} x_i$ for all $j \in J$, that is, the input allocated to a certain category is the sum of the costs of the initiated projects in that category. In what follows, we assume that at least one of the projects will be initiated in a feasible solution. We also have $\sum_{j \in J} a(x)_j = \sum_{i \in I} c_i x_i$.

We now provide an example model that uses the indicator $I_1(x)$ as the second objective. For the project selection problem $I_1(x)$ is as follows:

$$I_1(x) = \frac{\sum_{j \in J} \left| \alpha_j * \sum_{j \in J} a(x)_j - a(x)_j \right|}{\sum_{j \in J} a(x)_j} = \frac{\sum_{j \in J} \left| \sum_{i \in I} \alpha_j c_i x_i - \sum_{i \in I} c_i g_{ij} x_i \right|}{\sum_{i \in I} c_i x_i}.$$

We have the following model where we use variables Z_T and Z_I to denote $z_T(x)$ and $z_I(x)$, respectively.

$$Max \ \{Z_T, -Z_I\} \tag{5.1a}$$

$$s.t. \ \sum_{i \in I} c_i x_i \leq B \tag{5.1b}$$

$$Z_T = \sum_{i \in I} b_i x_i \tag{5.1c}$$

$$Z_I = \frac{\sum_{j \in J} \left| \sum_{i \in I} \alpha_j c_i x_i - \sum_{i \in I} c_i g_{ij} x_i \right|}{\sum_{i \in I} c_i x_i} \tag{5.1d}$$

$$x_i \in \{0, 1\} \ \forall i \in I \tag{5.1e}$$

The above model is nonlinear due to constraint set 5.1d. We linearize it by introducing auxiliary variables d_j, y_j and t_j and obtain the following MIP model (See [200] and [201] for more information on such linearizations):

Model 1

$$Max\ \{Z_T, -Z_I\} \tag{5.2a}$$

$$s.t.\ \sum_{i \in I} c_i x_i \leq B \tag{5.2b}$$

$$Z_T = \sum_{i \in I} b_i x_i \tag{5.2c}$$

$$\sum_{i \in I} c_i(\alpha_j - g_{ij}) x_i \leq d_j\ \forall j \in J \tag{5.2d}$$

$$\sum_{i \in I} c_i(g_{ij} - \alpha_j) x_i \leq d_j\ \forall j \in J \tag{5.2e}$$

$$d_j - \sum_{i \in I} c_i(\alpha_j - g_{ij}) x_i \leq 2 * d^{UB} * y_j\ \forall j \in J \tag{5.2f}$$

$$d_j - \sum_{i \in I} c_i(g_{ij} - \alpha_j) x_i \leq 2 * d^{UB} * (1 - y_j)\ \forall j \in J \tag{5.2g}$$

$$Z_I^{LB} x_i \leq t_i \leq Z_I^{UB} x_i \quad \forall i \in I \tag{5.2h}$$

$$Z_I^{LB}(1 - x_i) \leq Z_I - t_i \leq Z_I^{UB}(1 - x_i) \quad \forall i \in I \tag{5.2i}$$

$$\sum_{j \in J} d_j = \sum_{i \in I} c_i t_i \tag{5.2j}$$

$$x_i \in \{0, 1\}\ \forall i \in I \tag{5.2k}$$

$$y_j \in \{0, 1\}\ \forall j \in J \tag{5.2l}$$

$$t_i \geq 0\ \forall i \in I \tag{5.2m}$$

Constraint set 5.2b ensures that the total budget is not exceeded and the constraint set 5.2c defines Z_T, total output of the portfolio. We define new variables d_js that show the absolute distances, i.e. $d_j = \left|\sum_{i \in I} c_i(\alpha_j - g_{ij}) x_i\right|\ \forall i$. Constraint sets 5.2d, 5.2e, 5.2f, 5.2g and auxiliary binary variables (y_js) are used to define the absolute distances (d_js) and tackle the nonlinearity due to the absolute function. d^{UB} is an upper bound for the continuous d_j variables. We use the same upper bound for all the d_j variables and calculate the bound as follows: $d^{UB} = \sum_{i \in I} c_i$. Constraint sets 5.2h, 5.2i and 5.2j are used to tackle the nonlinearity due to the ratio terms in the definition of Z_I (see constraint set 5.1d) as follows: In constraint set 5.1d we have $\sum_{j \in J} d_j = Z_I * \sum_{i \in I} c_i x_i = \sum_{i \in I} c_i Z_I x_i$. We define auxiliary continuous variables t_i such that $t_i = Z_I * x_i \; \forall i \in I$ hence obtain constraint set 5.2j. Constraint sets 5.2h and 5.2i ensure that $t_i = Z_I * x_i \; \forall i$ hold. Z_I^{UB} and Z_I^{LB} are upper and lower bound parameters for Z_I, respectively. From the definition of Z_I, $Z_I^{LB} = 0$. We define Z_I^{UB} as follows: $Z_I^{UB} = n * d^{UB} / \underset{i}{Min}\{c_i\}$.

Model 1 has $2m+2n+2$ variables and $2m+4n+3$ constraints excluding set constraints.

Remark 55 *d^{UB} is an upper bound for all d_j.*

Proof. $d_j = \left|\sum_{i \in I} c_i(\alpha_j - g_{ij}) x_i\right| = \sum_{i \in I} c_i \left|\alpha_j - g_{ij}\right| x_i$. Since both $0 \leq \alpha_j \leq 1 \; \forall j$ and $0 \leq g_{ij} \leq 1 \; \forall i, j$ we have $|\alpha_j - g_{ij}| \leq 1$. Hence $\left|\sum_{i \in I} c_i(\alpha_j - g_{ij}) x_i\right| \leq \sum_{i \in I} c_i x_i \leq \sum_{i \in I} c_i$. ■

It is possible to include additional constraints in cases where certain projects are mutually exclusive for some underlying technical reasons. We note that these are easily handled computationally, hence for ease of presentation we do not include such constraints into the formulation explicitly. The models involving $I_2(x)$, $I_3(x)$ and $I_4(x)$ are very similar hence are provided in Appendix B.

It is straightforward to develop models for the case where a balanced output distribution is desired when we have a discrete setting. The model will be the same except the following: We use b_i instead of c_i in constraint sets 5.2d, 5.2e, 5.2f, 5.2g and 5.2j and change d^{UB} and Z_I^{UB} accordingly.

5.5.2 Continuous Allocation

Suppose that a DM should decide how to allocate a given input B among m projects but this time the allocation can be performed in a continuous manner. We use the same notation as in the discrete case with a difference in the decision variable and output definition.

Let x_i be the allocated input to project i and let $f_i(x_i)$ be the resulting output. The input allocated to category j denoted as $a(x)_j$ is a linear function of x such that $a(x)_j = \sum_{i \in I} g_{ij} x_i$ for all $j \in J$. In such cases the total input allocation is always B, i.e., $\sum_{j \in J} a(x)_j = B$. Note that the properties of the production functions $f_i(x_i)$ will affect the complexity of the problem and the resulting models may be difficult to solve when e.g. these functions are nonlinear. However, if production functions are concave it is possible to use piecewise linearization and obtain a linear problem as we show in the example in the next section.

Recall that the indicators in the discrete setting have decision variables in the denominator and hence require linearization. As Remark **??** shows the balance criterion no longer requires such linearization in the input oriented continuous setting.

Remark 56 *For the continuous allocation the indicators ($I_1(x)$,$I_2(x)$,$I_3(x)$ and $I_4(x)$) in the input oriented setting reduce to linear functions of deviations.*

Proof. Given a balance resource distribution α, $I_1(x)$ is as follows: $I_1(x) = \frac{\sum_{j \in J} |\alpha_j B - a(x)_j|}{B} = \frac{\sum_{j \in J} d(x)_j}{B}$. Hence minimizing $I_1(x)$ is equivalent to minimizing $\sum_{j \in J} d(x)_j$. Similarly, it is possible to show that minimizing $I_2(x)$, $I_3(x)$ and $I_4(x)$ are equivalent to minimizing $\underset{j \in J}{Max}\{d(x)_j\}$, $\sum_{j \in J} \prod_{i \neq j} \alpha_i d(x)_j$, and $\underset{j}{Max}\{\prod_{i \neq j} \alpha_i d(x)_j\}$, respectively. Also note that one does not need the auxiliary binary variables (e.g. $y_j s$ in model 1) to linearize the nonlinearity due to the absolute function as we directly minimize linear functions of the absolute distances. ■

5.6 Solution Approach

Our models are bi-criteria versions of the knapsack problem. In the discrete case knapsack problem is considered to be a nondeterministic polynomial-time hard (NP-hard) problem ([202]).

We define set Z as follows: $Z = \{(Z_T, Z_I) : Z_T = z_T(x) \text{ and } Z_I = z_I(x), x \in X\}$.

Definition 57 *For two points (Z_T, Z_I) and (Z'_T, Z'_I), (Z_T, Z_I) dominates (Z'_T, Z'_I) if*

$Z_T \geq Z'_T$ and $Z_I \leq Z'_I$ with strict inequality holding at least once.

Definition 58 *A point (Z_T, Z_I) is nondominated and the corresponding solution (x) is efficient if there is no other point in Z that dominates it.*

We call all the nondominated solutions for a problem the *nondominated set.*

We use the epsilon constraint method to obtain nondominated (/efficient) solutions for the bi-criteria problems considered here. This method is based on sequentially solving single objective problems in which the value of the second objective is controlled using a constraint (see [203] and [204] for a discussion of the epsilon constraint method).

The general algorithm is as follows (note that lex max refers to lexicographic maximization).

Step 0. Solve lex max $(z_T(x), -z_I(x))$

s.t. $x \in X$

Let the optimal value for $z_I(x)$ be Z_I^*

Step 1. If $Z_I^* \leq Z_I^{LB}$ Stop.

Otherwise, set $k = Z_I^* - Stepsize$.

Step 2. Solve lex max $(z_T(x), -z_I(x))$

s.t. $x \in X$

$z_I(x) \leq k$

Let the optimal value for $z_I(x)$ be Z_I^*

Go to Step 1.

When the objective function values are integer, it is possible to generate all nondominated points with this method. In this paper we use the method to generate a subset of the nondominated set as our objective function values are not necessarily integer. We first generate the solution that has the maximum output (Z_T) value and obtain a nondominated solution at each iteration until we generate the one that has the minimum imbalance (Z_I) value. We use a parameter *Stepsize* to control the maximum difference between two consecutively generated nondominated points in terms of their imbalance values. The smaller the *Stepsize*, the higher the number of nondominated solutions found. On the other hand, the higher the computational time is. Note that it is also possible to modify the algorithm such that it starts with the solution that has the minimum imbalance and moves toward the ones with higher total output values by controlling $z_T(x)$ by a constraint.

5.6.1 An Example Problem

We now provide a real life example for the input oriented discrete case based on data given to us by a public sector agency whose R&D portfolio selection problem provided the immediate motivation for the current work. The problem is a project selection problem subject to the available budget. The cost and value figures for each project are tabulated below (see Table 5.2). The values are a weighted average of performances of each project over multiple criteria. Note that the projects are of three types and the cost values are normalized to protect confidentiality. The budget and value correspond to input and output, respectively.

Table 5.2: Data for the example problem

Project Index	**Project Type**	**Cost**	**Overall Value**	**Project Index**	**Project Type**	**Cost**	**Overall Value**
1	Type 1	0.19	1.39	21	Type 2	0.88	1.71
2	Type 1	0.16	1.13	22	Type 2	0.86	1.34
3	Type 1	0.30	1.67	23	Type 3	0.05	2.15
4	Type 1	0.29	1.48	24	Type 3	0.18	2.47
5	Type 1	0.55	2.13	25	Type 3	0.16	1.96
6	Type 1	0.57	1.43	26	Type 3	0.31	3.42
7	Type 1	0.96	1.50	27	Type 3	0.43	3.92
8	Type 1	0.99	1.44	28	Type 3	0.42	3.42
9	Type 1	0.74	0.99	29	Type 3	0.42	2.97
10	Type 1	0.67	0.85	30	Type 3	0.33	2.29
11	Type 2	0.21	3.13	31	Type 3	0.37	1.67
12	Type 2	0.28	2.52	32	Type 3	0.59	2.60
13	Type 2	0.28	2.11	33	Type 3	0.42	1.79
14	Type 2	0.40	2.43	34	Type 3	0.96	4.08
15	Type 2	0.24	1.49	35	Type 3	0.54	2.11
16	Type 2	0.58	2.91	36	Type 3	0.54	2.08
17	Type 2	0.95	3.15	37	Type 3	0.90	3.25
18	Type 2	0.89	2.82	38	Type 3	0.75	2.20
19	Type 2	0.91	2.47	39	Type 3	0.81	2.06
20	Type 2	0.61	1.57				

Suppose that the agency has a budget (total input) of 9.31 units, which is about 45% of the total cost of all the projects available. Given this budget, the portfolio that maximizes the total output has a total output of 59.32 and requires an input of 9.2 units. The allocation of this total input to the three type of projects are 1.49 units, 1.99 units, 5.72 units, for types 1, 2 and 3 respectively.

Suppose that the DM considers an input allocation that has equal percentages as balanced. That is, in a perfectly balanced portfolio the total amount allocated to each project type should be 33% of the overall input.

For explanatory purposes we will use one of the indicators, $I_3(x)$, and show the portfolios obtained by solving the corresponding bi-criteria problems. A subset of the efficient portfolios obtained using $I_3(x)$ using *Stepsize* of 0.05 are visualized in Figure 5.2. The figure shows 13 portfolios each of which is obtained through one iteration of the algorithm.

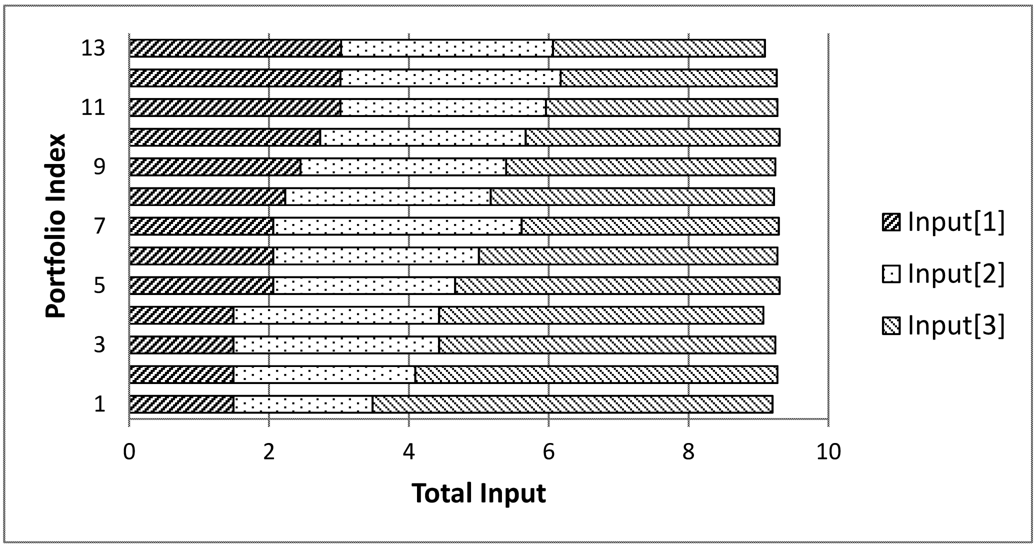

Figure 5.2: Solutions obtained using $I_3(x)$

The first portfolio is the one that gives the maximum total output and type 3 projects are allocated more input than the other two types in this portfolio. It is seen that in each new solution the algorithm returns a portfolio where the three types are closer in input usage. In the first iterations, balance is increased via increasing the input allocation to type 2 projects. As we restrict the solution to become more and more balanced, the allocation to type 1 projects increases. One can also see the amount of sacrifice from efficiency (total output) by moving towards more balanced portfolios in Figure 5.3, which is the total output vs. imbalance graph.

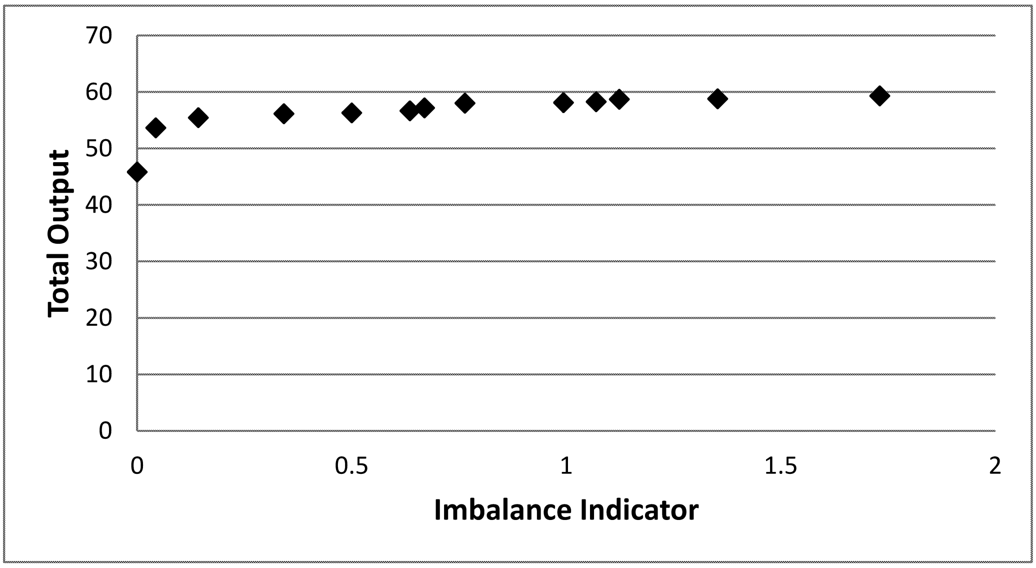

Figure 5.3: Efficiency vs. balance

The epsilon constraint approach allows us to visit the whole nondominated set in a uniform way, i.e., we provide representative portfolios for different parts of the whole nondominated set. Seeing such a uniform subset of the nondominated set has advantages in terms of clarity and transparency. The results show the tradeoff between the efficiency and balance criteria. For example, moving from the first solution to the second one sacrifices from efficiency around 0.8% and this increases balance around 20%. On the other hand, it is seen that as we restrict the solution to become more and more balanced, the efficiency sacrifice that we have to make may increase significantly. In addition to seeing the tradeoff between the two criteria, one can have more information about the solution structure. A quick review would give us an idea about the more "powerful" projects, the ones that occur in most of the nondominated solutions. In the above example we observed that the projects 1,3,4,5; 11,16; and 23, 24, 25, 26, 28, 29 are included in most of the portfolios. As a way of simplifying the decision making process, these projects can be fixed and the others might be analyzed in more detail.

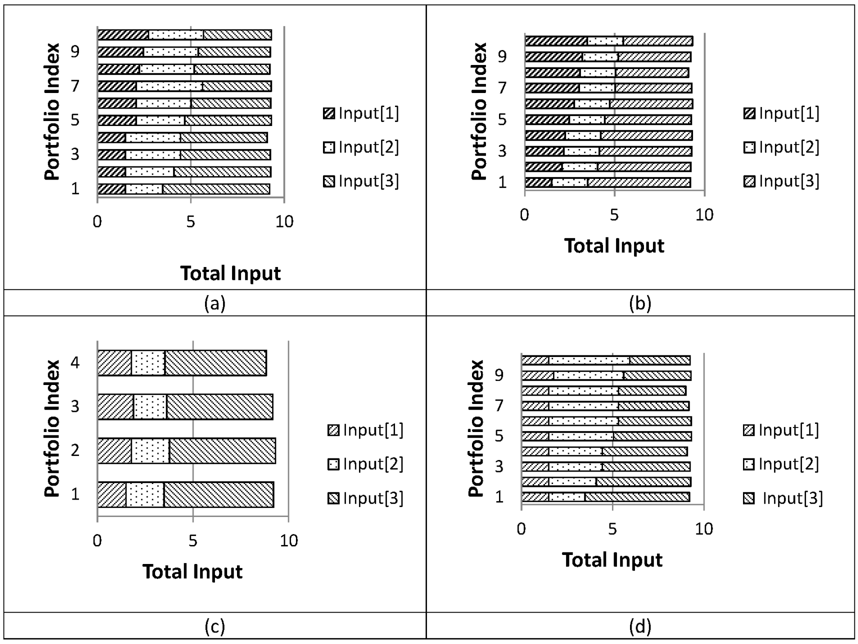

Figure 5.4: Different balance points

We have also generated solutions by using $I_1(x)$, $I_2(x)$ and $I_4(x)$ using a *Stepsize* of 0.05. We observed a similar trend as in Figure 5.2.

It is also possible to use other balance distributions and see the resulting solutions. Figure 5.4 (a)-(d) show the first 10 solutions obtained using the balance distributions (33, 33, 33); (60, 20, 20); (20, 20, 60); (20, 60, 20) respectively with $I_3(x)$. Note that in the third case we report only 4 solutions as these were the only solutions obtained using a *Stepsize* of 0.05 in the algorithm. This is because the output maximizing solution already has low imbalance with respect to the given balance distribution. No significant adjustment was necessary in this case.

As another example, we consider a linearized version of the above problem to analyze the case where we have continuous allocations. We assume that the production functions $f_i(x_i)$ are concave of the form $f_i(x_i) = s_i * x_i^{\theta_i}$ for all i. We generate s_i and θ_i values from uniform distributions $U(0, 5)$ and $U(0, 0.5)$, respectively. We solve the problem using piecewise linear approximation for the concave production functions. Figure 5.5 shows a

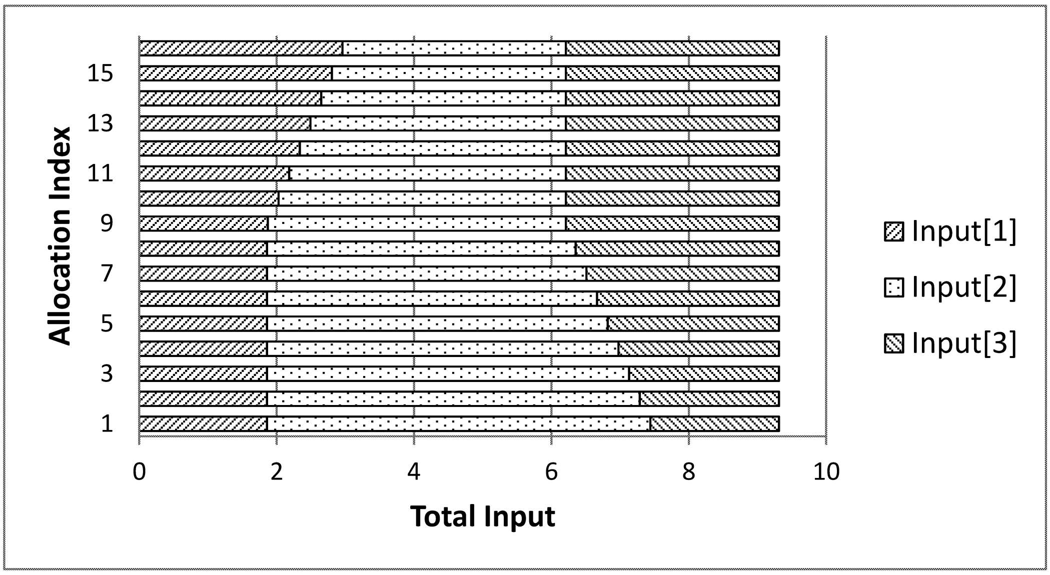

Figure 5.5: Continuous case using $I_3(x)$ with balance distribution (0.33,0.33,0.33)

set of efficient allocations which are obtained setting *Stepsize* to 0.1. One can clearly see that we move to more balanced allocations towards the end of the spectrum. Also note that unlike the discrete case, all of the allocations have the same total input value. The tradeoff between the two criteria is clearly seen in Figure 5.6, which shows the total output value vs. imbalance value for this continuous case.

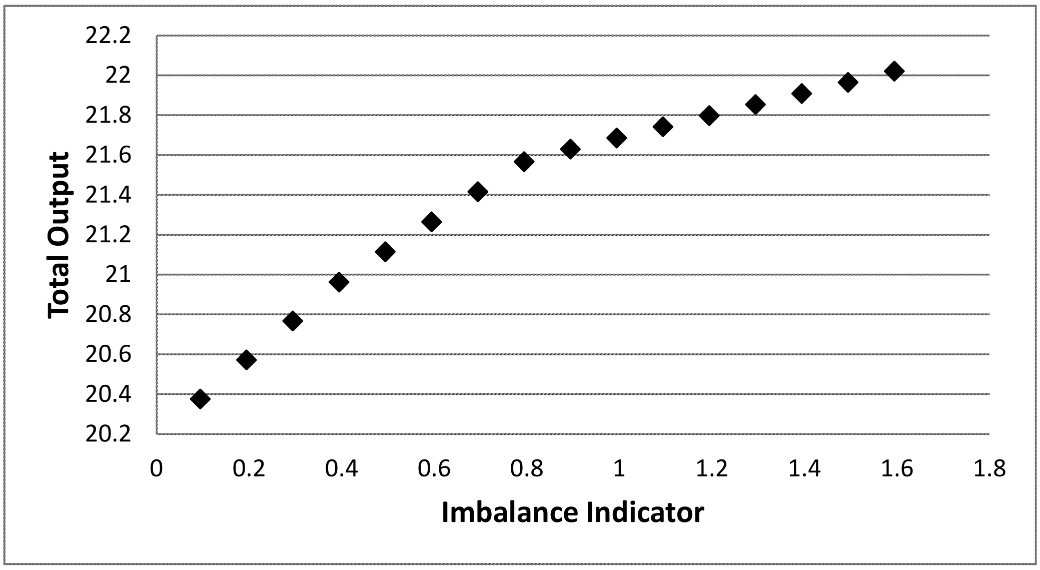

Figure 5.6: Efficiency vs. balance in the continuous case

5.7 Computational Study

In this part we discuss the computational aspects of the recommended epsilon constraint approach by providing the results of an experimental study. For the experimental study we again use the project selection problem. The aim of this section is to see the size of problems for which we can obtain a subset of the nondominated solutions to present the DM in reasonable time, using the formulations developed in the previous sections of this paper.

We consider the (discrete) project portfolio selection problem where m and n denote the number of projects and the number of project categories, respectively. As in [204] the output (b_i) and the input (c_i) values are randomly generated integers between 10 and 100. We set $B = 0.5 \sum c_i$. We start with m =50 increasing in increments of 50. As for n, we use 3 and 5. For each m and n combination, we generate 10 problem instances.

We use the (adaptive) epsilon constraint approach discussed in [204], which is a generalization of the scheme we discussed in section 6 for arbitrary numbers of objectives. The algorithms are coded in Visual C++ and solved by a dual core (Intel Core i5 2.27 GHz) computer with 4 GB RAM. The optimal solutions are found by CPLEX 12.2. The solution times are expressed in central processing unit (CPU) seconds. We set a time limit of 1 hour for the execution of the epsilon constraint approach.

We first discuss the results for problems where we seek balance in the input space as in our experience most applications involve concerns about ensuring balance in the input distribution to categories. In most cases we report the results for the models using I_3 ($I_3(x)$) as the imbalance indicator. That is because I_3 is likely to be computationally more complex than the others. We also report results for the cases where we introduce 2 imbalance indicators, in which case we use I_3 and I_2 ($I_2(x)$) as the two indicators. The balance distributions (r) are taken as $(50, 30, 20)$ and $(50, 30, 20, 10, 80)$ for the $n = 3$ and $n = 5$ settings, respectively. Hence α is $(0.5, 0.3, 0.2)$ and $(0.26, 0.16, 0.11, 0.05, 0.42)$ for these two settings.

We ran extensive experiments and we show a sample of the more interesting results in Table 5.3. In this table we report the average and maximum values for solution times and number of calls to CPLEX. We also report the average and minimum number of non-dominated solutions (|ND|) returned by the algorithm. Note that the number of instances for which the algorithm could not terminate in 1 hour are indicated in parenthesis for the settings where the maximum solution time is 3600 seconds and these settings are reported in italics. The table also reports the value of the parameter *Stepsize*, which is used to adjust the right hand side of the constraint restricting the criterion value that is treated in the constraint for the bi-criteria problems. We report results for problems with a single type of input (indicated as Inp=1) and with two types of inputs (indicated as Inp=2).

For the instances where we obtain the nondominated set we set the optimality gap (θ) to 0.01% for CPLEX. We optimize the model that minimizes the imbalance while restricting the total output value with a constraint. As we assume integer values for the total output, setting *Stepsize* to 1 ensures that none of the nondominated points is missed.

Table 5.3: Performance results for the epsilon constraint approach

Inp=1

				CPU Time		|ND|		*Stepsize*		Calls to CPLEX	
Criteria	Gap	m	n	Avg	Max	Avg	Min	Avg	Max	Avg	Max
2	0	50	3	363.92	930.81	64	31	1.0	1.0	128	219
			5	*2024.80*	*3600 (1)*	*103*	*70*	*1.0*	*1.0*	*207*	*283*
	0.01	50	3	4.31	13.61	13	5	9.9	15.9	27	37
			5	39.65	249.96	10	5	50.1	56.5	21	45
		100	3	6.46	10.70	15	8	13.1	29.3	30	39
			5	*831.51*	*3600 (2)*	*16*	*2*	*53.1*	*95.0*	*40*	*51*
		150	3	8.04	11.78	16	13	16.3	24.8	33	43
	0.05	50	3	0.69	0.98	7	4	9.9	15.9	14	19
			5	3.98	16.07	6	3	50.1	56.5	14	21
		100	3	1.94	3.77	10	4	13.1	29.3	21	41
			5	10.01	24.21	12	5	44.4	57.1	26	39
		150	3	3.88	5.48	12	6	16.3	24.8	25	39
			5	25.15	74.90	15	9	55.4	81.6	31	43
		50	3	1.24	1.86	11	8	1.0	1.0	24	37
			5	20.68	46.29	35	22	1.0	1.0	71	103
		100	3	4.31	6.53	20	5	1.0	1.0	40	69
			5	94.50	271.34	70	26	1.0	1.0	142	257
		150	3	8.82	14.97	26	15	1.0	1.0	53	95
			5	181.16	664.59	77	23	1.0	1.0	154	333
3	0.01	50	3	10.11	30.51	10	2	-	-	62	136
			5	332.66	1060.22	40	8	-	-	261	779
		100	3	49.27	372.90	11	2	-	-	56	84
			5	*2507.664*	*3600 (4)*	*65*	*9*	-	-	*426*	*785*
	0.05	50	3	6.21	38.75	8	1	-	-	47	83
			5	612.30	3311.74	31	7	-	-	190	444
		100	3	7.93	11.36	9	3	-	-	60	83
			5	*461.85*	*3600 (1)*	*29*	*9*	-	-	*163*	*286*
		150	3	12.09	23.48	7	2	-	-	55	100
			5	395.25	1363.90	40	9	-	-	224	585
Inp=2											
3	0.05	50	3	138.80	1230.46	21	3	-	-	97	263
			5	*2885.29*	*3600 (7)*	*84*	*11*	-	-	*992*	*2095*
		100	3	23.44	143.64	15	4	-	-	87	342
			5	*3253.64*	*3600 (9)*	*61*	*4*	-	-	*328*	*328*
		150	3	15.02	31.25	9	5	-	-	51	84

As seen from Table 5.3 the cardinality of the nondominated set and the solution time required to solve the single-objective subproblems increase as the number of categories increases. These results also indicate that for larger problems, the size of the nondominated set or the solution time of the single-objective subproblems may be prohibitively large to allow us to obtain the whole nondominated frontier in reasonable time. However, in many cases presenting a large number of solutions to the decision maker may be neither necessary nor desirable. We rather suggest obtaining a moderate number of solutions which approximate the nondominated solutions and spread over the nondominated region in a uniform way.

To determine an appropriate $Stepsize$ value we find the two nondominated solutions at the two ends of the nondominated frontier: The solution that has the largest Z_T value and the solution that has the smallest Z_I value. These solutions provide us the maximum and minimum total output values in the nondominated set: $Z_{TMax} = Max\{Z_T : (Z_T, Z_I) \in Z\}$ and $Z_{TMin} = Min\{Z_T : (Z_T, Z_I) \in Z\}$, respectively. We set $Stepsize = (Z_{TMax} - Z_{TMin})/40$. We solve the single objective sub-problems with a predefined optimality gap θ; hence find approximate nondominated solutions with a worst case quality guarantee. We report the results in Table 5.3. We also report results for $\theta = 5\%$ case with a fixed $Stepsize$ value of 1.

The results indicate that the solution times increase as n increases for fixed m although the number of solutions returned decreases or stays similar. It indicates that the single objective subproblems become more difficult when n increases. For fixed n, the solution times and the average number of solutions increase as we increase m. As expected, increasing the optimality gap parameter (θ) leads to a decrease in solution times. The number of calls to CPLEX also decreases as θ increases, this is because the algorithm starts with a solution that has larger imbalance values and hence returns solutions which lie at the center of the frontier.

We next perform experiments for 3-criteria problems where there is a single input and there are 2 different indicators. For these experiments we use I_2 (collective-bottleneck indicator) and I_3 (individual oriented-sum indicator) as the two additional criteria to the total output criterion. We express the two imbalance criteria in the form of constraints and set the corresponding *Stepsize* values as 0.05 and 0.5 for I_2 and I_3, respectively.

It is possible to observe the effect of the number of categories (n) to the solution times. The effect of the number of projects on solution times does not seem to be as predictable as the effect of the number of categories. In some settings we even observe smaller solution times as m gets larger for fixed n.

We also note that the correlation coefficient between the values of the indicators I_2 and I_3 is quite high: it is between 0.8 and 0.96 for all settings. This indicates that for most cases if a portfolio has a high I_2 value, it is likely to have a high I_3 value as well. As expected, highest correlation is observed for the settings with $n = 3$ categories. This is because, for such cases if the worst-off category has high deviation from the target it is likely that the sum of all deviations will be high as well. As the number of categories increases the effect of the worst-off value to the total deviation decreases, resulting in differences between sum-oriented and bottleneck-oriented indicators.

Finally, we consider the case where there are two inputs. In this setting the projects consume two inputs and return a single output. The output and input values are randomly generated integers between 10 and 100, as before. The resulting model is a 3-criteria model, where we have the total output criteria and two imbalance criteria corresponding to the distributions of the two inputs over project categories. We report the results of our experiments in Table 5.3 for $\theta = 5\%$, where we use I_3 as the imbalance indicator. The two imbalance criteria are incorporated in the form of constraints with the same *Stepsize* values of 0.5.

It is seen from Table 5.3 that the solution times and the number of solutions increase considerably when the number of categories increases. Moreover, as there are two different inputs, the correlation between the values of the imbalance indicator that correspond to distributions of these two inputs is expected to decrease compared to the previous 3-criteria instances. The correlation coefficients were between 0.46 and 0.78 for the all the settings.

Our computational results indicate that the heuristic version of the epsilon constraint approach with appropriate *Stepsize* and optimality gap parameters can be used for small to medium-size problems. We observe that the solution times tend to increase significantly as the number of categories increases. For large-size problems with more than 150 projects different heuristic algorithms can be employed to obtain solutions in reasonable time.

We have also attempted to obtain nondominated solutions for problems for which the imbalance criterion is defined over the output distribution to categories. We observe that these problems are harder to solve. Even for the smallest problems considered (m=50, n=3) the epsilon-constraint based heuristic with 5% optimality gap fails to return solutions in 1 hour for some instances. To obtain solutions to these problems in reasonable time heuristic algorithms can be explored. One such approach is described here and some preliminary results are provided.

We designed a tabu search (TS) heuristic that starts with the solution that maximizes the total output. Using this initial feasible solution, we try to find solutions with improved balance values by searching its neighborhood. Given a solution, we search its neighborhood by switching the status of the projects in a pairwise manner. That is, for each pair of projects one of which is in the portfolio and the other is not, we exclude the former and include the latter if such an interchange is feasible. We calculate the potential improvement in balance for each such move, and perform the move that leads to the maximum improvement. We terminate when the number of non-improving moves reaches to 250 or number of the iterations reaches to 1000. We set tabu tenure to 50, i.e., we do not undo a selected move for the next 50 iterations. We use aspiration criterion as the best solution, i.e., tabu status of the moves that improve the best solution is overridden. We keep the candidate solutions in a set called *incumbent set*.

The TS algorithm executes to improve the Z_I value. Meanwhile we keep track of the corresponding Z_T values and update the incumbent set whenever we find an eligible solution, i.e., a solution which is non-dominated by the incumbent set.

Our experiments showed that the TS has satisfactory performance in terms of solution quality and computational time. We now report computational results for TS. As it was not possible to obtain the exact nondominated set for the case where we seek balance in the output space we report the performance of the TS algorithm for the input-oriented

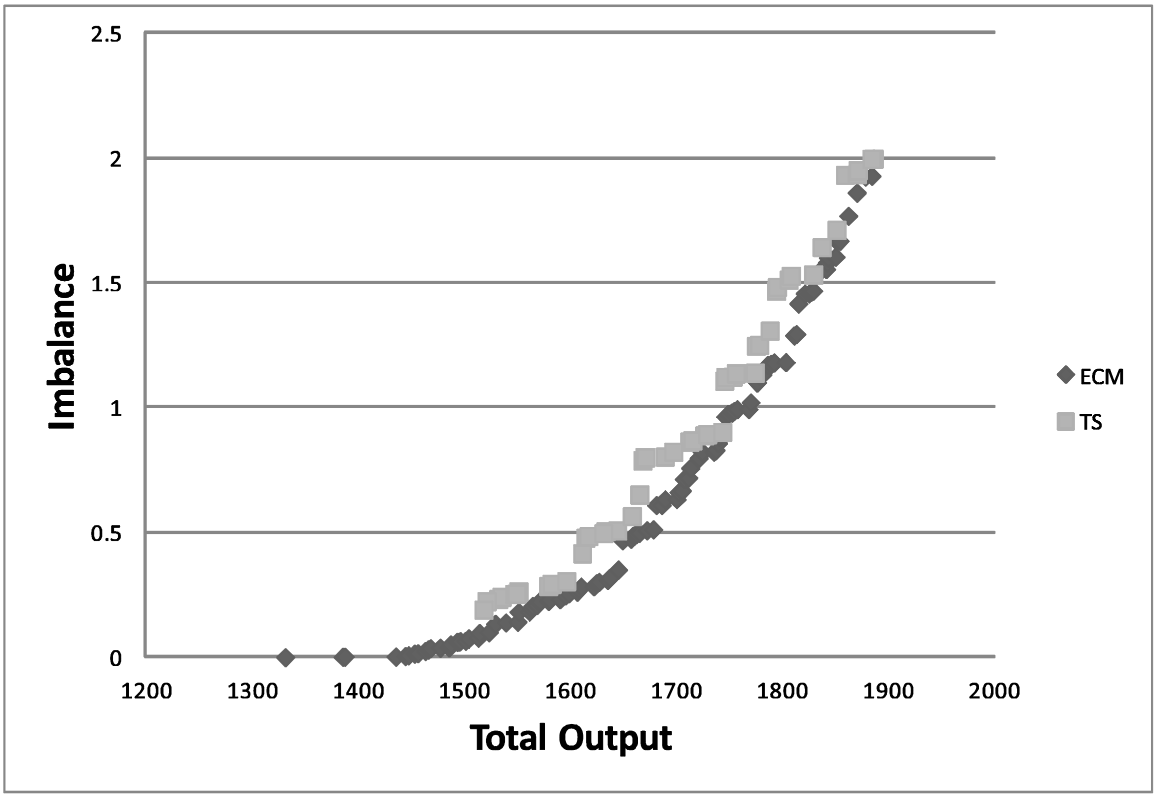

Figure 5.7: TS algorithm vs. ECM for a problem with m=50 n=3

case. Figure 5.7 shows the solutions obtained by the TS algorithm and the (exact) epsilon constraint method (ECM) in an example instance for m=50 n=3 case. As seen the TS approximates the nondominated set quite well.

We also compared TS with our (heuristic) epsilon constraint method (ECM) with 1% optimality gap (ECM(1%)) with variable *Stepsize* values (As reported in Table 5.3). In terms of solution time TS massively outperforms ECM as it takes less than 2 seconds for TS to return a set of candidate solutions even for the largest problem instances considered as opposed to 363 seconds for ECM. However, the TS method clearly does not give guarantees of optimality and so knowing how good the generated solutions are in general is problematic. To assess the quality of solutions returned by the algorithms in this particular case, we use three performance metrics, namely P, D1 and D2. We denote the solutions returned by the TS or the heuristic ECM as the ANS (approximate nondominated set). P is the percentage of exact non-dominated objective vectors returned by the TS (or heuristic ECM). D1 and D2 give information about the average and maximum distances between

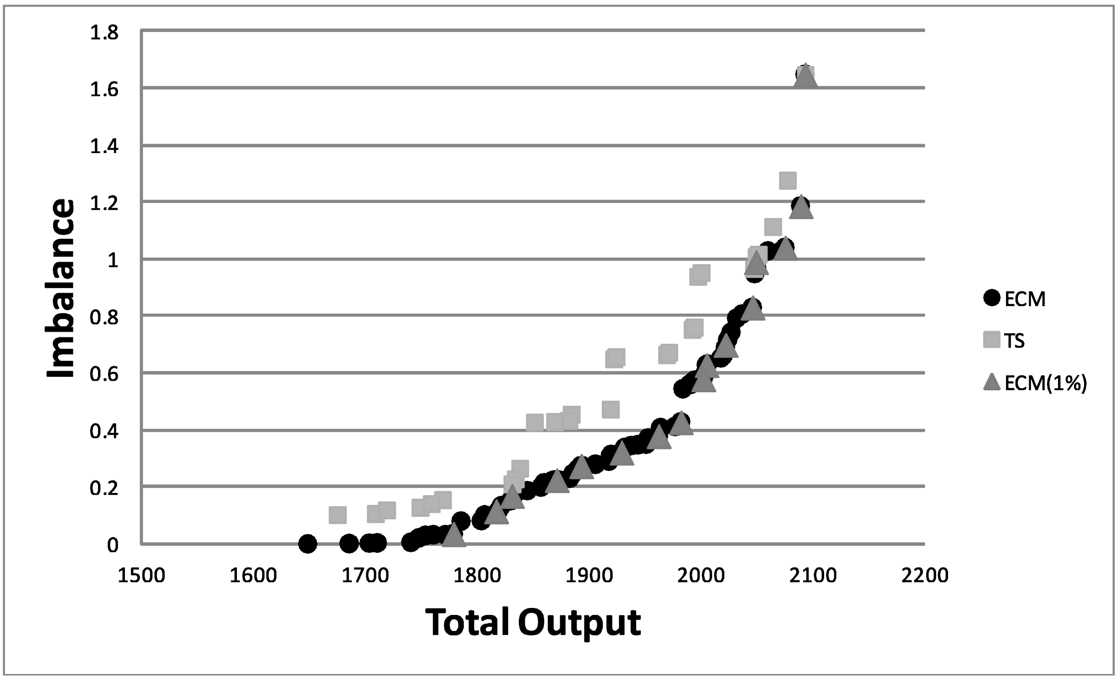

Figure 5.8: Comparison of ECM (Heuristic) and TS for an instance with average distance values

the points of the nondominated set and the points in set ANS, respectively (See [205] for the formulations of these metrics).

Table 5.4 shows the results. To give an idea about the scale of the distance metric we provide a graphical display of the solutions returned by the algorithms for an example instance which has the average distance values for both TS and ECM. For this instance TS has values of 0.07 and 0.13 and ECM(1%) has 0.01 and 0.03 for distance metrics D1 and D2, respectively.

As seen from the Table 5.4 and Figure 5.8, ECM outperforms the TS but the performance of TS algorithm is still satisfactory for these problems.

Table 5.4: Performance results for TS and ECM with 1% optimality gap

			P		D1		D2	
m	n	Algorithm	Avg	Min	Avg	Max	Avg	Max
50	3	TS	1.18	0	0.06	0.1	0.13	0.22
		ECM (1%)	1.69	0	0.01	0.02	0.04	0.05
50	5	TS	0.68	0	0.05	0.09	0.12	0.24
		ECM (1%)	13.83	0	0.01	0.02	0.05	0.13

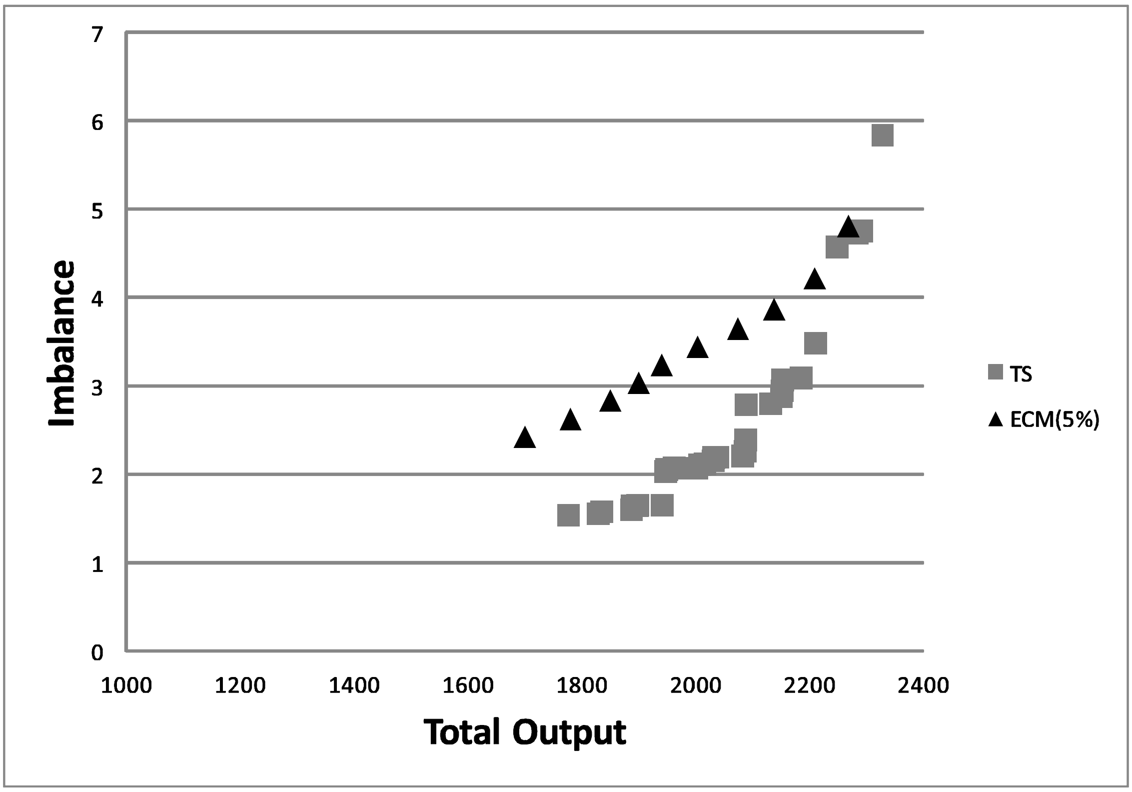

Figure 5.9: TS algorithm vs ECM with 5% optimality gap

For the output case we were unable to obtain the (approximate) nondominated set in reasonable time using the epsilon constraint approach with 5% optimality gap. Figure 5.9 shows the results of the ECM with 5% optimality gap and TS for an example instance where m=50 and n=3. We leave a detailed comparative study of different solution approaches for the output-oriented case to future research.

We observe that the TS algorithm returns a set of good solutions in negligible time for the input-oriented cases. For the output-oriented case the algorithm finds a set of solutions in negligible time. We have also done some explorations to extend the TS algorithm for multicriteria problems and observed that the solution times are negligible. However, further research should focus on generating a diverse set of solutions for multicriteria cases using algorithms that are computationally efficient. We hope this interesting and challenging question stimulates further research.

5.8 Conclusion

Allocation problems include a wide range of applications where inputs are allocated to entities so as to maximize the total output. Taking our motivation from various real life cases where a balanced (input/ output) distribution over categories is considered important as well as total output maximization we provide a framework to trade balance off against efficiency in such problems.

We define and categorize balance distribution based (im)balance indicators and show a way to incorporate these measures into the mathematical formulations of different allocation problems. We propose bi-criteria modelling by introducing balance as another criterion to the model alongside the total output criterion. We discuss an approach to obtain a subset of nondominated solutions. The solutions obtained are distributed over the entire nondominated set in a uniform way and range from the solution that has the maximum total output to the solution that has maximum balance.

We illustrate the approach by solving a real life project selection problem. Considering balance explicitly as another criterion and showing a subset of the efficient solutions to the DM has many advantages like bringing transparency to decisions and facilitating communication with the stakeholders. The generated graphs can help to initialize a structured discussion on balance. Observing how much one has to sacrifice to get closer to an ideally balanced distribution can provide justification for the decisions made for the final allocation.

We discuss the performance of the epsilon constraint approach by providing experimental results for larger bi-criteria and 3-criteria project selection problems. We are able to obtain a subset of (approximate) nondominated solutions that spread uniformly over the nondominated frontier, hence represent different regions of the frontier. We also suggest a TS approach for large-size problems and those with output-oriented imbalance criteria. We provide initial experimental results on the performance of the TS approach.

It is possible to use this modelling approach in other types of allocation problems where we allocate a homogeneous good to multiple entities. We note that the nature of the allocation, i.e. whether it is discrete or continuous, has an effect on the type of models developed. For example, for problems where the input allocation is continuous and balance is sought in the output space there is no obvious way to transform the decision model to a tractable mixed integer program when one is using the imbalance indicators discussed here: in these problems we cannot assume that the denominator is constant nor do we have binary variables and so cannot linearize using the idea we deployed in the discrete cases.

We have taken an initial step to bring in the perfect equality line concept to consider balance in resource (or output) distributions. There are possible further steps that can be taken. For example:

- Further research can be done on generalizing the proposed approach to a multi-criteria case where balance concerns are defined over multiple aspects and on developing ways to present the problem to the DM in a way that is easily communicated. Related algorithmic challenges can be addressed using appropriate methods such as metaheuristics.

- The balance line concept can also be extended by allowing a piecewise linear structure for the balance line. For example when the total amount available is very low the DM might tend to desire an equitable allocation, and as the total amount distributed increases, some other allocations may become more desirable than the equal allocation. The balance line approach can also be generalized to a *balance cone* approach where the extreme points and rays of the cone are generated based on the information given by the DM. Regarding any allocation within the cone as perfectly balanced one might assess the balance of alternative allocations and provide a subset of nondominated points to the DM.

- Axiomatic discussion of the difference imbalance indicators is another research area that we believe would be interesting. Presumably a key idea in axiomatizing balance would involve the observation that the points on the balance line are equally balanced and that as one moves towards the balance line one gets points which are better in terms of balance.

Acknowledgement 59 *We are thankful to the anonymous referees for their constructive comments which led to substantial improvement of the paper. We also thank Dr. Marco Laumanns for kindly sending us the code for the adaptive epsilon constraint method and Mr. Andy Jones for kindly providing us with the data for the real life example and many helpful discussions.*

5.A Proofs of Propositions 53 and 54

5.A.1 Proof of Proposition 53

We first prove the first part of the proposition and show that for $n = 2, 3$ we have $I_1(x) = 2I_2(x)$.

Let $n = 3$, and let the input/output distribution over which balance is sought be $a(x)$. $a(x) = (a(x)_1, a(x)_2, a(x)_3)$ and $a(x)_1 + a(x)_2 + a(x)_3 = a_T$. Suppose the balance distribution is $(\alpha_1, \alpha_2, \alpha_3)$. Let $d(x)_1$, $d(x)_2$ and $d(x)_3$ be the componentwise absolute deviations from the rescaled balance distribution. The following holds: $d(x)_1 + d(x)_2 + d(x)_3 = 2 * Max\{d(x)_1, d(x)_2, d(x)_3\}$. To see this, without loss of generality (w.l.o.g.) assume that $Max\{d(x)_1, d(x)_2, d(x)_3\} = d(x)_1$. Observe that the total negative componentwise deviation of $a(x)$ from $\overline{r}(x)$ should be equal to the total positive componentwise deviation. Hence we have $Max\{d(x)_1, d(x)_2, d(x)_3\} = d(x)_1 = d(x)_2 + d(x)_3$ and $d(x)_1 + d(x)_2 + d(x)_3 = 2 * Max\{d(x)_1, d(x)_2, d(x)_3\}$. Hence

$$I_1(x) = \frac{d(x)_1 + d(x)_2 + d(x)_3}{a_T} = \frac{2Max\{d(x)_1, d(x)_2, d(x)_3\}}{a_T} = 2I_2(x).$$

Note that it is easy to verify that $I_1(x) = 2I_2(x)$ for $n = 2$ in the same way.

We now prove the second part of the proposition: For $n = 2$, $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2) \iff I_3(x^1) \geq I_3(x^2) \iff I_4(x^1) \geq I_4(x^2)$. Note that $I_1(x^1) = 2I_2(x^1)$ (and $I_1(x^2) = 2I_2(x^2)$), hence $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2)$ for $n = 2$.

Let $a(x^1) = (a(x^1)_1, a(x^1)_2)$ and $a(x^2) = (a(x^2)_1, a(x^2)_2)$. Let $a(x^1)_1 + a(x^1)_2 = a_T^1$ and $a(x^2)_1 + a(x^2)_2 = a_T^2$. Suppose the balance allocation is (α_1, α_2). Let $\overline{r}(x^1)$ and $\overline{r}(x^2)$ be the corresponding (adjusted) balance distributions, i.e., $\overline{r}(x^1) = (a_T^1 * \alpha_1, a_T^1 * \alpha_2)$ and $\overline{r}(x^2) = (a_T^2 * \alpha_1, a_T^2 * \alpha_2)$. Note that $d(x^1)_1 = d(x^1)_2$ and $d(x^2)_1 = d(x^2)_2$.

1. We will first show that $I_1(x^1) \geq I_1(x^2) \iff I_3(x^1) \geq I_3(x^2)$.

   (a) $I_1(x^1) \geq I_1(x^2) \implies I_3(x^1) \geq I_3(x^2)$

   Suppose that $I_1(x^1) \geq I_1(x^2)$ while $I_3(x^1) < I_3(x^2)$.

   If $I_1(x^1) \geq I_1(x^2)$ then

$$\frac{2d(x^1)_1}{a_T^1} \geq \frac{2d(x^2)_1}{a_T^2} \tag{5.3}$$

   If $I_3(x^1) < I_3(x^2)$ then

$$\begin{aligned}
&\frac{d(x^1)_1}{a_T^1 * \alpha_1} + \frac{d(x^1)_1}{a_T^1 * \alpha_2} < \frac{d(x^2)_1}{a_T^2 * \alpha_1} + \frac{d(x^2)_1}{a_T^2 * \alpha_2} \implies \\
&\frac{d(x^1)_1}{a_T^1}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) < \frac{d(x^2)_1}{a_T^2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) \implies \\
&\frac{d(x^1)_1}{a_T^1} < \frac{d(x^2)_1}{a_T^2} \implies \frac{2d(x^1)_1}{a_T^1} < \frac{2d(x^2)_1}{a_T^2}
\end{aligned} \tag{5.4}$$

   From equations 5.3 and 5.4 we have a contradiction hence there is no x^1 and x^2 such that $I_1(x^1) \geq I_1(x^2)$ while $I_3(x^1) < I_3(x^2)$ for $n = 2$. It is easy to verify

   $I_3(x^1) \geq I_3(x^2) \implies I_1(x^1) \geq I_1(x^2)$ in the same way.

2. We will now show that $I_1(x^1) \geq I_1(x^2) \iff I_4(x^1) \geq I_4(x^2)$.

   $I_1(x^1) \geq I_1(x^2) \implies I_4(x^1) \geq I_4(x^2)$

   Suppose that $I_1(x^1) \geq I_1(x^2)$ while $I_4(x^1) < I_4(x^2)$. From previous result if $I_1(x^1) \geq I_1(x^2)$ equation 5.3 holds.

   If $I_4(x^1) < I_4(x^2)$ then

$$\begin{aligned}
Max\{\frac{d(x^1)_1}{a_T^1 * \alpha_1}, \frac{d(x^1)_2}{a_T^1 * \alpha_2}\} &< Max\{\frac{d(x^2)_1}{a_T^2 * \alpha_1}, \frac{d(x^2)_2}{a_T^2 * \alpha_2}\} \\
\frac{d(x^1)_1}{Min\{a_T^1 * \alpha_1, a_T^1 * \alpha_2\}} &< \frac{d(x^2)_1}{Min\{a_T^2 * \alpha_1, a_T^2 * \alpha_2\}}
\end{aligned}$$

Without loss of generality let $\alpha_1 < \alpha_2$. Then we have $\frac{d(x^1)_1}{a_T^1 * \alpha_1} < \frac{d(x^2)_1}{a_T^2 * \alpha_1} \implies \frac{d(x^1)_1}{a_T^1} < \frac{d(x^2)_1}{a_T^2}$. This is equation 5.4, hence the rest follows as in part 1 leading to a contradiction. Similarly, it is easy to show that $I_4(x^1) \geq I_4(x^2) \implies I_1(x^1) \geq I_1(x^2)$ also holds.

5.A.2 Proof of Remark 54

Consider the following counterexamples:

Example 60 *Consider two allocations x^1 and x^2 which have $a(x^1)$ and $a(x^2)$ as shown in the table below and suppose that the balance distribution is r. The pairwise comparisons of the two alternatives are different under $I_4(x)$. We have $I_1(x^1) < I_1(x^2)$; $I_3(x^1) < I_3(x^2)$ but $I_4(x^1) > I_4(x^2)$.*

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1)$ =*(16,16,13)*	*0.21*	*0.67*	*0.42*
$a(x^2)$ =*(18,20,20)*	*0.28*	*0.84*	*0.38*
r =*(36,20,24)*			

In the example case given below the pairwise comparisons of the two alternatives are different under $I_1(x)$. Note that for $n = 3$ we have $I_1(x^1) \geq I_1(x^2) \iff I_2(x^1) \geq I_2(x^2)$ so $I_2(x)$ is also not consistent with $I_3(x)$ and $I_4(x)$.

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1)$ = *(11,12,18)*	*0.17*	*0.49*	*0.24*
$a(x^2)$ = *(20,10,17)*	*0.16*	*0.51*	*0.28*
r =*(30,25,30)*			

In the example case given below the pairwise comparisons of the two alternatives are different under $I_3(x)$.

Allocation	$I_1(x)$	$I_3(x)$	$I_4(x)$
$a(x^1)$ =*(18,20,10)*	*0.25*	*0.80*	*0.42*
$a(x^2)$ = *(12,12,17)*	*0.26*	*0.78*	*0.47*
r =*(39,27,26)*			

Example 61 *To show that $I_1(x) = 2 * I_2(x)$ no longer holds when $n > 3$; consider the*

below example where $I_1(x^1) < I_1(x^2)$ *but* $I_2(x^1) > I_2(x^2)$.

Allocation	$I_1(x)$	$I_2(x)$
$a(x^1)$ =*(18,13,10,17)*	*0.25*	*0.13*
$a(x^2)$ =*(19,11,20,15)*	*0.28*	*0.11*
r =*(39,33,28,20)*		

5.B Models using other indicators

5.B.1 Using $I_2(x)$:

This model is very similar to Model 1, except for the constraints related to Z_I. We use decision variables I_j to denote componentwise misallocations, i.e., $I_j = d_j / \sum_{i \in I} c_i x_i$. We find upper and lower bounds for I_j. We use the same bounds for all I_j and denote them as I^{UB} and I^{LB} , respectively. The bounds are as follows (d^{UB} is as defined in Model 1):

$$
\begin{aligned}
I^{UB} &= \frac{d^{UB}}{Min_i\{c_i\}} \\
I^{LB} &= \frac{d^{LB}}{\sum_{i \in I} c_i x_i} = 0
\end{aligned}
$$

Z_I is the maximum componentwise deviation, i.e. $I_j \leq Z_I$ for all $j \in J$ and we minimize Z_I, hence $Z_I^{UB} = I^{UB}$. We have nonlinear terms in the equation defining I_js. We use the same techniques used in model 1 and obtain the following model.

Model 2

$Max\ \{Z_T, -Z_I\}$

Constraint sets 5.2b, 5.2c, 5.2d, 5.2e,5.2f, 5.2g

$I_j \leq Z_I \ \ \forall j \in J$

$d_j = \sum_{i \in I} c_i t_{ij} \ \forall j \in J$

$I^{LB} x_i \leq t_{ij} \leq I^{UB} x_i \ \forall i \in I,\ j \in J$

$I^{LB}(1 - x_i) \leq I_j - t_{ij} \leq I^{UB}(1 - x_i) \ \forall i \in I,\ j \in J$

$x_i \in \{0, 1\} \ \forall i \in I$

$y_j \in \{0, 1\} \ \forall j \in J$

$d_j \geq 0 \ \forall j,\ t_{ij} \geq 0 \ \forall i \in I,\ j \in J$

Model 2 has $mn + m + 2n + 2$ variables and $2mn + 6n + 2$ constraints excluding the set constraints.

5.B.2 Using $I_3(x)$:

This model uses $I_3(x)$ as the balance criterion. Recall that this indicator is the sum of the componentwise proportional deviations. We use decision variables I_j to denote the componentwise proportional deviations for the categories in the model. That is, $I_j = d_j/\alpha_j \sum_{i \in I} c_i x_i$. We use the following upper and lower bounds for I_j in the model, denoted as I_j^{UB} and I_j^{LB}, respectively (We set d^{UB} as before):

$$\begin{aligned} I_j^{UB} &= \left(\frac{\sum_{j \in J} r_j}{r_j}\right) \frac{d_j^{UB}}{Min_i\{c_i\}} = \left(\frac{\sum_{j \in J} r_j}{r_j}\right) \frac{d^{UB}}{Min_i\{c_i\}} \text{ for all } j \in J \\ I_j^{LB} &= \frac{d^{LB}}{\alpha_j \sum_{i \in I} c_i x_i} = 0 \text{ for all } j \in J. \end{aligned}$$

Using I_j^{UB} and I_j^{LB} we can set $Z_I^{UB} = \sum_{j \in J} I_j^{UB}$ and $Z_I^{LB} = 0$.

The resulting model is the following:

Model 3

$Max\ \{Z_T, -Z_I\}$

Constraint sets 5.2b, 5.2c, 5.2d, 5.2e,5.2f, 5.2g

$$\sum_{j\in J} I_j = Z_I$$

$$d_j = \sum_{i\in I} c_i \alpha_j t_{ij}\ \forall j \in J$$

$$I_j^{LB} x_i \leq t_{ij} \leq I_j^{UB} x_i\ \forall i \in I,\ j \in J$$

$$I_j^{LB}(1-x_i) \leq I_j - t_{ij} \leq I_j^{UB}(1-x_i)\ \forall i \in I,\ j \in J$$

$$x_i \in \{0,1\}\ \forall i \in I$$

$$y_j \in \{0,1\}\ \forall j \in J$$

$$d_j \geq 0\ \forall j, t_{ij} \geq 0\ \forall i \in I,\ j \in J$$

Model 3 has $mn + m + 2n + 2$ variables and $2mn + 5n + 3$ constraints excluding the set constraints.

5.B.3 Using $I_4(x)$:

This model uses $I_4(x)$ in the objective function. It is very similar to model 3 with a slight change in the constraint defining Z_I. We change it as follows:

$$I_j \leq Z_I\ \forall j \in J$$

Where I_j^{UB}, I_j^{LB} and d^{UB} are as in model 3 and $Z_I^{UB} = \underset{j}{Max}\{\frac{\sum_{j\in J} r_j}{r_j}\}d^{UB} = \underset{j}{Max}\{\frac{1}{\alpha_j}\}d^{UB}$. The resulting model has $mn + m + 2n + 2$ variables and $2mn + 6n + 2$ constraints excluding the set constraints.

Chapter 6

Conclusion

In this work we reflected upon handling equity concerns in operational research problems (see [127], [128], [206] for more discussion on equity and fairness).

We first provide a comprehensive review of the studies in the operational research (OR) literature that try to handle equity concerns in mathematical models. We categorize the equity related concerns into two main areas, namely equitability and balance.

We categorise the studies based on the approach used to address equity concerns and discuss pros and cons of these different approaches. Alongside the Rawlsian approach, which focuses on the worst off entity rather than the distribution vector and hence is rather crude, two approaches are used to handle equitability concerns. One approach uses inequality indices and the other uses equitable aggregations. Balance concerns are handled in similar ways: one approach uses imbalance indicators and another one uses a scaling approach that converts the problem into an equitability problem, for which equitability-handling methods can be used.

The equitable aggregation approach is based on a well-defined and commonly accepted set of axioms and can be used as a gold-standard for other approaches. Finding the equitably efficient set of solutions, however, may not be of much help to a decision maker who has to select the "best" solution or rank alternatives from best to the worst. In an effort to guide the decision maker through the set of candidate alternatives (the equitably efficient ones) we extend the current theory in interactive multi-criteria decision making literature. Specifically, we extend the theory of convex cones such that they can be used in a symmetric setting. After a discussion and review on the use of convex cones in non-symmetric environments, we introduce our theoretical results that extend this approach to symmetric environments. We then illustrate the use of our theoretical findings by designing an interactive ranking algorithm and solving example problem instances.

The equitable efficiency concept is based on an axiomatically justified notion of fairness; however, it may result in prohibitively high computational effort, especially in problems where the set of alternatives are implicitly defined by constraints. One other way to incorporate equity concerns is using inequality indices in the models. These indices return scalar values that show the degree of disparity in a distribution and have a value of zero at the perfect equality line, i.e. the line of distributions where everyone receives an equal amount. We generalize the perfect equality line concept and define the perfect balance line, which consists of points that have the same proportional allocation as a reference distribution the decision maker (DM) provided. This concept provides a way to address balance concerns, which occur in many resource allocation settings in real life. Not ignoring the widely encountered efficiency concerns, we discuss the use of bi-criteria modelling in resource allocation settings, the two criteria being efficiency and balance, respectively.

Some of the research directions that could be explored further are summarized below:

Algorithmic challenges:

There is a vast amount of potential real-life applications such as health care decision making, resource allocation, and supply chain design, for which mathematical modelling tools can provide solutions by considering the multiple concerns inherent in the problems. Most mathematical models considered in the recent OR literature are multi-criteria decision making models where one (or sometimes all) of the criteria is equity related. Multi-objective models are more difficult than their single objective counterparts ([207]). This renders it necessary to address the related algorithmic challenges and explore exact and/or heuristic methods to solve these multi-objective models.

As a specific example, in models incorporating equity concerns via inequality indices, as one moves away from simple inequality indices to more complicated but more realistic ones, the complexity of the resulting models may change significantly. For example, the indices that respect the strong version of the Pigou-Dalton Principle of transfers (PD), such as the Gini coefficient and the variance, often make the corresponding mathematical model harder to solve than some simpler measures that do not respect strong PD such as the range. As a result, there are relatively few studies in the literature which involve models that quantify equity using such measures. Hence, designing computationally feasible algorithms for more complicated models with indices which satisfy desired axioms on equitability and hence would be accepted by many inequity-averse DMs is a relevant and stimulating research topic.

Another example research question would be on handling computational challenges when one has a multi-criteria approach to equitability, i.e. considers a preference model with the symmetry axiom. We have so far discussed the problems where a discrete set of alternatives is explicitly given and proposed an interactive solution method for such problems. A natural extension of our work would be considering situations where the alternatives are implicitly defined by constraints. Such problems are relevant in many real-life applications. Hence, finding ways to deal with the computational challenges the symmetry axiom brings (due to the combinatorial number of permutations) and designing algorithms which, for example, return equitably efficient solutions in reasonable time is another relevant research topic. Moreover, further research can be performed in guiding the DM through the set of equitably efficient alternatives by using, for example, interactive approaches.

Another specific research question would be about resource allocation settings where the DMs have balance concerns over the output distribution. We have made initial attempts along this line by proposing solution approaches for discrete problems and observed that these models are significantly more difficult to solve than their counterparts, which consider balance concerns in the input distribution. Moreover, the output distribution may be generated by production functions based on the input allocation decision. In such settings inputs and outputs are connected by the "production functions", the forms of which will affect the complexity of the resulting mathematical models. For example, a nonlinear production function would be expected to lead to models, which are computationally challenging yet relevant in many applications and interesting to explore.

Decision support:

We see great potential for further research in improving the decision support process for real-life multi-criteria problems involving equity concerns. Researchers should reflect upon how the theoretical advances and analyses can be used within a real life decision support context.

One of the main issues is choosing the type of the decision support in terms of the timing of the interaction with the DM. As we have discussed in Chapter 3 there are three main multicriteria decision aiding (MCDA) methods: a priori, a posterior and interactive MCDA methods ([145]). All three approaches are meaningful and useful in their own way. For a decision support system to be successfully implemented the analyst should choose the appropriate type of support for the problem at hand. Hence, a relevant discussion in "inequity-averse decision making" can be held on the application contexts where each of these methods may make sense. Some example questions are the following: "Are there application contexts where more a priori approaches make sense as there is consensus about the aggregation mechanism or equity index to be used? Are there contexts (perhaps health care decision making) where making equity judgements is so sensitive that all that one can do is present the DM with the efficient set and ask her to choose one of the solutions in this set? Where do interactive approaches make sense? Such discussions would help the decision making practitioners in deciding the appropriate set (type) of support systems, which would eventually contribute to the success and impact of the relevant decision aiding applications and increase the practical value of the research outputs.

Some research questions regarding the interactive decision support can be considered in the future. Experimental studies on interactive approaches can be performed in order to answer questions like "How easy do people find it to express preferences over distributions?" or " What kind of questions result in the best performance of the interactive procedure used in terms of e.g. the cognitive effort or the time required from the DM?" "What kinds of inconsistencies arise?" . Alongside general discussions on these questions, specific interactive methods can be re-considered from these perspectives. For example, our work on convex cones can be re-considered in the light of such questions. Recall that, we propose an interactive method for symmetric cases that uses the convex cones approach, which is based on using preference information from the DM. In our method we used pairwise comparisons provided by the DM but other questioning modes are also possible. It would be interesting to conduct an experimental study on alternative types of interaction and analysis of inconsistencies that may arise within this context.

Designing a decision support system or choosing an appropriate one from the available systems would involve considering behavioural aspects as well. In inequity-averse decision making, questions like "What equity indices are closest to people's intuitive ideas of equity?" await further attention. Such studies exist in the Economics literature but more practice-oriented discussions can be held within the OR community, especially by behavioural scientists.

The appropriateness and computational handling of particular structural assumptions on the DM's preferences is also a fruitful area for further research. Within the aggregation functional approaches to equitability we discuss the equitable dominance relation, which is also the unanimity relation when DM's utility function is assumed to belong to any one of the following sets: additive, concave, quasi-concave or Schur-concave. When preference information from the DM is introduced, this unanimity no longer holds. In this thesis we have considered the case where the utility function is assumed to be quasi-concave and proposed a solution approach that incorporates the DM's preference information to obtain a most preferred solution or a subset of good solutions, or to refine the ranking of the set of solutions. An extension of this work that we have in our future research agenda involves considering all these four assumptions on the functional form of the utility function and proposing interactive solution approaches accordingly. We aim to discuss the advantages and limitations of each approach.

Robustness:

Robustness is crucial for the decision maker(s) to be comfortable with the proposed solution as a robust solution would make it easier to persuade the stakeholders to implement it (see [208] and [209] for a discussion of robustness concerns in operational research and decision aiding).

In general, robustness in multi-criteria decision making approaches is a relatively recent topic; hence to the best of our knowledge, there is almost no work on the multi-criteria decision making situations where equity is of concern. For example, in a decision making setting where the feasible alternatives are implicitly defined by constraints, the decision makers may want to obtain solutions which would always be among the "good" solutions within the possible parameter value intervals. Further research can be performed especially in MCDM settings with the symmetry property. In such settings some equitably efficient solutions may be less sensitive to changes in the considered parameters than the others. Hence, one can focus on designing solution approaches that specifically aim to find such robust solution sets.

Robustness concerns related to the fairness rules can also be explored further. For example, in settings that quantify equity using an inequality measure, further analysis could be performed to see the extent to which selecting the "wrong" inequality approach affects the solutions in different problem settings (see [42] for an example study that considers a single facility location problem). Such works would be of much help to the decision makers who find it difficult to select one inequality measure from a number of alternative measures. Having knowledge on the robustness of solutions with respect to different measures would guide the analyst on the amount of time to be devoted to discussion on the choice of the measure to be used. If the solutions do not change significantly when the index is changed, then less time could be devoted to such discussions. If solutions would change in a considerable way when different candidate measures are used, then more time would be required to deliberate over these candidate measures and to choose the "right" one.

Deepening our understanding of the concept and measurement of equity:

We have discussed various indices that can be used in the models to incorporate equity related concerns. One future line of work can focus on axiomatisation of these inequality and balance indices. An axiomatic analysis would make it possible to discuss limitations and advantages of different indices in a more structured way.

More research can be performed on how to "measure" or quantify equity in different settings such as the ones where ordinal measures rather than cardinal ones should be used. An example occurs in the ground delay problems, where there is a reference ordering in which entities are served or processed and one wants to deviate from this order as little as possible (see [108] for an example study). Imbalance measures can be defined and classified for such settings along with a discussion on the underlying axioms assumed for each measure.

Another extension of our work on imbalance indicators, which are based on the perfect balance line, can be to examine the cases with different balance line forms, such as piecewise linear ones. A piecewise linear balance line would reflect preferences of a decision maker, for whom the desired ratio allocations would change depending on the total amount of good. For example, when the amount of good is relatively scarce, then the concern may be allocating the good equitably (i.e. a perfectly balanced distribution allocates each entity the same amount) whereas when the total amount of good is plenty then a perfectly balanced allocation for the DM might involve different ratios than the equal ones. Such a piecewise linear extension of the perfect balance line concept would result in a more flexible decision support system in terms of capturing different ideas of balance.

Multidimensional equity also seems of interest and links to a topic in economics ([210], [211], [212], [213], [214], [215], [216], [217]). Multidimensional equity considerations are relevant in many practical applications and involve assessing inequity in bundles of different goods that different entities receive. Such cases are computationally and cognitively more challenging since the distribution (allocation) vectors seen in the single good case are now replaced with matrices in which each column (row) corresponds to the distribution of one of the multiple goods. Hence decision making in such settings is associated with comparing such allocation matrices. The definition of the anonymity property changes in such cases because the preference model is expected to have the anonymity property over the entities but not over the different goods. That is, symmetry holds over only one dimension of the matrix. Similarly, the PD is defined in a slightly different way: everything else being the same, an allocation matrix 1 should be considered more equal than another allocation matrix 2, when it is obtained by applying matrix 2 a series of PD transfers for one or more of the goods. Further research can be performed on generalizing the concepts and axioms used in the single dimensional settings to multidimensional ones, proposing operational means to compare two allocation matrices and designing solution approaches that can incorporate preference information in these settings.

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