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Essays on Intermediation in Trade Problems

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To Oya, the reason behind everything good that happens to me.

Declaration

I certify that the thesis I have presented for examination for the PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others.

I declare that my thesis consists of 46,871 words.

Abstract

This thesis studies the theory of intermediation in trade problems arising from the allocation of a single indivisible object.

Chapter I considers a general trade problem with a single seller and multiple buyers. I analyze a game, where multiple intermediaries compete with each other by designing contracts that determine the terms of trade between the bargaining parties. I show the existence and uniqueness of equilibrium outcomes. Repeating the analysis for the case of a single monopolist intermediary, I compare the equilibrium outcomes and show that allocative efficiency is strictly improved as a result of the competition among intermediaries.

Chapter II considers a bilateral trade problem with two-sided asymmetric information where the buyer's valuation may depend on the private information of both bargaining parties. I analyze the impact of intermediation by a profit-maximizing intermediary in a game, where the seller has the ability to trade directly with the buyer. I provide a necessary and sufficient condition for the equilibrium outcomes with the presence of an intermediary to be strictly more efficient than those that are attainable in its absence.

Lastly, in Chapter III, similar to the previous chapter, bilateral trade problems with informational externalities arising from interdependences are considered. I analyze a game, where the seller designs a contract at ex-ante stage before learning his private information. I characterize the optimal mechanisms and show that they attain second-best outcomes. The Pareto optimality of ex-ante contracting in the absence of an intermediary, in turn presents a natural limit to the benefits to be accrued from intermediation.

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Oya, I dedicate this thesis to you. Now, let's enjoy the rest of the ride!

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Preface

The aim of this thesis is to analyze the theory of intermediation in trade problems of a single indivisible object. To that end, I submit three papers, each of which analyzes a different problem related to intermediation in these single object bargaining scenarios.

In Chapter I, I consider a trade setup between a seller and multiple buyers over the transaction of a single indivisible object and analyze a game, where multiple intermediaries compete with each other to attract the seller. Agents have independent private valuations for this object and all players have quasilinear preferences. I show the existence and uniqueness of equilibrium outcomes. In any equilibrium, all intermediaries make zero expected profits, while the unique allocation rule and the expected payoffs agents receive are equal to those that are accrued under seller's revenue maximizing auction. Hence these unique outcomes can be implemented in an equilibrium in which all intermediaries announce the seller's optimal auction with discriminatory reserve prices. Characterizing the unique equilibrium outcomes when there is a single monopolist intermediary, I show that competition among intermediaries unambiguously improves allocative efficiency, as well as the agents' expected payoffs. However, I also show that, relative to the equilibrium under competition, a social planner may further increase total surplus.

In Chapter II, I consider a bilateral trade problem, where the seller and the buyer have quasilinear preferences. In this setup, however, I allow for interdependent valuations where the seller's private information becomes relevant for the buyer's valuation. I first characterize the equilibria of the informed seller's signaling game where the seller makes a take-it-or-leave-it offer. Then, I consider another game with an uninformed intermediary as the third player, who designs menus of prices to facilitate trade between the two parties. In this game, the seller decides whether to use the intermediary or to trade directly with the buyer on his own. The possibility of direct trading enables the seller to generate revenues from direct sales, which creates outside options depending on the seller's type. Consequently, the intermediary competes with the seller when designing the menu of prices. I provide a necessary and sufficient condition for the implemented trade probabilities in the game with an intermediary to be strictly more efficient than those in any equilibrium of the informed seller's game without an intermediary. Hence, this condition characterizes cases under which intermediation by a profit-maximizing third-party proves to be beneficial by improving welfare, as measured by expected gains from trade.

Lastly, Chapter III also considers a bilateral trade problem with quasi-linear preferences allowing for informational interdependent valuations. I analyze a game, where the seller designs a trading mechanism at ex-ante stage before learning his private information. I characterize the seller's ex-ante optimal mechanism and show the the unique equilibrium outcomes of this game. I also show that the seller's ex-ante optimal mechanism is Pareto optimal in the sense of ex-ante interim efficiency. Hence, ex-ante commitment leads seller's revenue maximizing behavior to result in second-best outcomes. In turn, the seller's ability to attain Pareto optimality without an intermediary can be interpreted as a natural limit to the benefits of intermediation.

CHAPTER I

Competing Intermediaries

I.1 Introduction

A central problem in mechanism design deals with bargaining situations between a seller and multiple buyers over the allocation of a single indivisible object in the presence of two-sided asymmetric information arising from independent private valuations (henceforth IPV). It has been shown in Myerson and Satterthwaite (1983) for the case of bilateral trade, and later in Williams (1999) for multiple buyer environments that there are no transfer rules which implement the efficient allocations in a budget balancing way. It is common for many scenarios that the bargaining is carried out via an intermediary. As shown in Myerson and Satterthwaite (1983), when there is a single monopolist intermediary who aims to maximize profits, the introduction of additional monopoly distortions exacerbates the efficiency losses. A natural next line of inquiry would be to consider the impact of competition among multiple intermediaries. In this paper, I analyze the competitive contract market among multiple profit-seeking intermediaries who compete over facilitating trade in these bargaining situations.

To achieve this goal, I consider a trade setting with a single seller possessing one unit of an indivisible good, n-many buyers who want to buy that good where n is at least one, and m-many intermediaries who compete over mediating the trade, where m is at least two. I assume that the agents have private valuations for the object that are independently drawn from the same interval.[1] These valuations represent the respective types of the agents, where for brevity I refer to

[1] I use the female pronoun for the seller, male pronouns for the buyers, and neuter pronouns for the intermediaries.

them as the cost for the seller and value for the buyers, respectively. All players have quasilinear preferences. The collection of $m + n + 1$-many players engage in the *Intermediation Game* played over three stages. In the first stage, the intermediaries simultaneously announce their mechanisms[2] to the seller. Following the announcements, in the second stage the seller moves by first choosing an intermediary and next by sending her private message to the chosen intermediary. Finally, in the last stage the buyers, observing only seller's chosen mechanism, send their private messages. The collection of messages sent to the seller's chosen mechanism determine the outcome.

I show two sets of results. Firstly, I characterize the unique equilibrium outcomes of the intermediation game whenever there are two or more intermediaries. Secondly, characterizing the unique equilibrium outcomes of the intermediation game when there is a single monopolist intermediary, I show that, compared to the case of monopoly, competition unambiguously improves welfare from an allocative efficiency point of view. However, I also show that relative to the equilibrium under competition, a social planner may further increase total surplus by designing a mechanism that satisfies incentive compatibility and individual rationality constraints of the agents.

The intermediation game with multiple intermediaries belong to the framework of common agency as the intermediaries compete over general mechanisms to attract the agents. As shown in Peters (2001), an immediate obstacle in the common agency literature is the failure of the standard solution techniques from the single principal mechanism design literature. In these games, it is with loss of generality to restrict attention to simpler and smaller mechanism spaces for the principals (i.e. the intermediaries in the context of this paper), because such restrictions may eliminate outcomes that are supported by an equilibrium in the original game, or conversely may create outcomes that were not supported by an equilibrium in the original game.[3] Hence, without any restrictions on the strategy spaces, I resort to the revelation principle, which provides a set of necessary conditions any equilibrium outcome has to satisfy. The conditions imply the familiar payoff equivalence result for the agents from the auction literature, which suggests that I can characterize the agents' equilibrium expected payoffs by only describing the equilibrium allocation rule and the equilibrium expected payoffs for the highest cost seller and lowest value buyers.[4] Nevertheless, a difficult problem

[2]Given the trade context, a mechanism is a collection of allocation and payment rules that are determined according to the messages sent by the seller and buyers.

[3]For a more detailed discussion, please refer to Martimort and Stole (2002).

[4]These types are referred to as the "worst" types of the respective agents.

still prevails where I need to identify the unique equilibrium allocation rule and expected payoffs for worst type agents and prune all others that satisfy the necessary conditions. Furthermore, for the intermediaries revelation principle only summarizes the sum of their nonnegative expected payoffs which is equal to the surplus net of the agents' ex-ante expected payoffs. Therefore, another complication may arise in characterizing the equilibrium payoffs to the intermediaries from a mere sum.

In light of the necessary conditions, I show the uniqueness of equilibrium outcomes in two steps. In the first step, I show that in any equilibrium the expected profits of the intermediaries and the expected payoffs of the lowest value buyers are zero. Furthermore, there exists a set of buyer specific reservation values such that the allocation rule has to award the object to the buyer with the highest virtual valuation[5] so long as it exceeds his reservation value. In the second step, I characterize the unique set of the reservation values, which pins down both the allocation rule and the expected payoff to the highest cost seller, and consequently the unique equilibrium outcomes.

In both of the steps, I prove the statements by contradiction where I show that, if the assumed equilibrium outcomes do not satisfy the claims, then there exists a profitable deviation mechanism for at least one intermediary. The intuition behind these deviation mechanisms is an undercutting argument. However, under the competing mechanism design context with two-sided asymmetric information, their construction is more complicated than small enough increments to the agents' payoffs. The problem is that a profitable deviation mechanism might require agents to play a particular equilibrium at the subgame following the deviation, which might be different than the strategy profile that supports the assumed equilibrium outcomes. Hence, the main challenge is to make sure the behavior of agents is precisely the way that deems the deviation mechanism profitable. A contribution of this paper at a technical level is to show that, one can construct strictly dominant strategy mechanisms for which the dominant strategy equilibrium yields the desired profitable deviation profits.

I show that for every cost, the unique equilibrium allocation rule implements same trade probabilities as those under the seller's revenue maximizing auction if her types were commonly known. There are several important implications of this equivalence. Firstly, when buyers do not observe seller's costs, the informed seller's revenue maximizing auction implements the same allocations because the

[5]Virtual valuation of a buyer's type can be interpreted as the marginal revenue he can generate, which is net of his information rents, if the good were to be allocated to him.

seller's costs are independently distributed of the buyers' valuations.[6] Secondly, the seller's revenue maximizing auction yields the same expected payoffs to the agents as those in the unique equilibrium of the intermediation game with multiple intermediaries, because the expected payoffs for the worst type agents are also equal to zero. These suggest that the seller's revenue maximizing auction is free from cross-subsidization across seller types as it awards maximal expected revenues generated, when her costs were observable. Thirdly, the unique equilibrium outcomes of the intermediation game can be attained by the intermediaries announcing the seller's optimal auctions. In other words, existence of equilibrium can be established by constructing one, where all intermediaries identically announce these auctions free from cross-subsidization. Finally, the equivalence also suggests that allowing intermediaries to have access to a larger space of mechanisms do not improve the outcomes from the equilibria of the informed seller's optimal auction. Furthermore, in the symmetric case where buyers all have the same distribution of types, the seller's optimal auction can be implemented by a classical auction; e.g. a second-price auction with reserve prices. Hence in a way, the result verifies the popularity of using auctions in real life applications, such as the allocation of a good in secondary markets, sale of a house in real estate markets, and sale of an antique or fine art in auction houses.

The intuition behind the uniqueness of equilibrium outcomes is in the spirit of Bertrand competition. The intermediaries compete with each other in order to attract the seller. However the existence of private information causes the intermediaries to contest over every type of the seller. This competition drives their profits down to zero and leads every seller type to be offered the maximum expected payoff she can receive subject to the necessary conditions. The remaining equilibrium outcomes have to equal to those accrued under the seller's revenue maximizing auction, so long as they satisfy the equilibrium conditions, which they do in the IPV framework. Furthermore, similar to the Bertrand competition, the unique equilibrium outcomes for the agents and the allocation rule remain the same for any number of intermediaries m larger than two.

There are two important aspects of the way I model competition that is crucial for the uniqueness result. Firstly, I assume that the buyers observe only the mechanism announced by the seller's chosen intermediary. This information structure eliminates a fundamental problem inherent in the common agency models, which

[6]This equivalence for the bilateral trade case is noted in Yilankaya (1999). More generally, Skreta (2011) establishes the irrelevance of private information in the IPV for the informed seller's revenue maximizing mechanism.

in turn clears the way for the result on zero intermediary profits. If the buyers also observed the whole array of mechanisms, then it would be possible to have equilibria where an intermediary makes positive expected profits, because it could bully away potential deviation from a competitor by awarding the seller with huge monetary transfers at the expense of huge expected losses. In this argument the array of mechanisms would be used to coordinate equilibrium play on the part of the agents. As the buyers observe only the seller's chosen mechanism, such schemes are not possible in this game.[7] Secondly, the timing assumption grants the choice of intermediary to the seller, which in turn brings the intermediary competition down to attracting the seller. Lack of cross-subsidization in the seller's revenue maximizing auction establishes the equivalence for the remaining unique equilibrium outcomes. I acknowledge the importance of this model of competition for the uniqueness result. At the same time I believe it is not unreasonable to assume that the seller chooses the mechanism while the buyers observe only seller's chosen mechanism. In particular, it is common that the seller has an advantageous position in the bargaining situation by choosing among alternatives and the buyers are uninformed about the whole set of mechanisms offered. When considering the real estate market, the owner of the property picks her preferred agent, while it is highly unlikely that the buyers are aware of the alternative mechanisms the seller faced from competitor estate agents. Similarly, in the case of fine arts or antiques auctions, the seller chooses the auction house and the buyers participate at the chosen location without knowing the alternative auction offers from competing auctioneers.

The second set of results relates to the welfare comparisons of equilibrium outcomes between different regimes. Considering the intermediation game with a single monopolist intermediary, I characterize the unique equilibrium outcomes. Similar to the competition case, the allocation rule also awards the object to the highest virtual valuation buyer, but subject to a different set of reservation values. In both regimes, each buyer's optimal reservation value in the equilibrium allocation rule is determined by equating his marginal revenue to the marginal cost. Although the marginal revenues are calculated the same way, the discrepancy arises from how the monopolist intermediary assesses the marginal costs. Under competition, marginal cost equals to only the seller's true cost of the good, while under monopoly the seller's information rents are also accounted for. As a result, for all costs the highest virtual valuation buyer exceeds the reservation

[7]There exists a subliterature that analyze folk theorem results for competing mechanism environments. Some examples include Yamashita (2010) and Peters and Troncoso-Valverde (2013).

value more often in the equilibrium with multiple intermediaries than under the monopolist intermediary. Furthermore, the additional trade scenarios award the object to a buyer, who values it more than the seller, which implies that competition unambiguously improves allocative efficiency for any possible vector of agent types. It is also shown that the efficiency improvements lead to increases in the ex-ante total surplus and the agents' expected payoff schedules. These results are in line with the intuition from received wisdom that competition improves efficiency.

Despite the increase in allocative efficiency, the equilibrium outcomes under competition are not ex-post efficient due to two reasons; the highest virtual valuation buyer might not be the buyer with the highest actual valuation, and in some cases the reservation values require the object to be retained by the seller even though the highest virtual valuation buyer has actual valuation that exceeds the true cost. Focusing on efficiency maximization, I show that a larger ex-ante expected surplus can be achieved relative to the equilibrium outcomes under multiple intermediaries. As a second-best benchmark, I characterize the constrained-efficient outcomes as those that are accrued under a benevolent social planner who maximizes ex-ante expected gains from trade subject to the necessary conditions implied by the revelation principle. The utilitarian welfare maximizing outcomes implement an allocation rule where the object is awarded to the buyer with highest α-weighted virtual valuation for some constant $\alpha \in (0, 1)$ subject to the social planner's optimal reservation values. Comparing the unique equilibrium outcomes from competition with those from the social planner, it is shown that a planner can increase the size of the expected surplus by improving total ex-ante expected payoffs of the buyers at the expense of seller's ex-ante expected payoffs. A more detailed analysis of allocative efficiency proves to be difficult in the general case of heterogeneous buyers. As the parameter α does not have an analytic solution, the differences in the allocation rules can not be compared.

Restricting attention to the case of homogeneous buyers with identical distributions, I show both equilibrium allocations award the object to the highest actual valuation buyer, albeit subject to different reservation values. More specifically, I show that the planner sets lower reservation values for low costs, while the reservation values for high costs are lower under the equilibrium of multiple intermediaries. Hence, the social planner improves total surplus by inducing more trade for low cost seller types, despite the reduction in the surplus arising from lowered trade for high costs. Comparing the expected payoffs of the agents does not yield clear cut predictions because of this aforementioned nonmonotonic relationship between the reservation values. Namely, the constrained-efficient outcomes implement more efficient trade for (sufficiently) low cost seller while the equilibrium outcomes under competition are more efficient for (sufficiently) high cost seller.[8] However, I show that compared to the case of competition, the ex-ante losses in seller's expected payoffs could be to the extreme that every seller type becomes weakly worse off under the social planner. I provide a necessary and sufficient condition for this Pareto dominance relation to hold. Interpreting the condition suggests that the seller types are better off under competition, whenever the average of the seller's interim expected trade probabilities across all their types are weakly greater than the corresponding average trade probability under the social planner. I also do the analogous analysis for the buyers and provide a necessary and sufficient condition for their interim expected payoffs under social planner to Pareto dominate the payoffs from the equilibrium of multiple intermediaries.

The rest of the paper is structured as follows. In the rest of this section, I will present the related literature. In Section I.2, I define the model, the equilibrium concept, and the revelation principle. Section I.3 establishes existence of equilibria and characterizes the unique equilibrium outcomes in terms of allocation and expected payoffs. In Section I.4, I derive the outcomes of the Intermediation Game under a single profit maximizing intermediary in the absence of competition. Section I.5 analyzes efficiency of the equilibria outcomes by comparing expected gains from trade and expected payoffs under competition and monopoly. I extend the comparative statics exercises by considering the constrained efficient mechanisms. Finally, Section I.6 concludes. All proofs are presented in Appendix A.

I.1.1 Related literature

There is a growing literature on competition among intermediaries. A large strand concentrates on general two-sided markets with multiple agents on both sides, where the intermediaries are interpreted as platforms that match these two sides; for example competing matchmakers in Caillaud and Jullien (2003) or credit cards in Rochet and Tirole (2003). Different than my paper, this literature concentrates on the network externalities that arise from matchings of the two sides as in Rochet and Tirole (2006) and restricts intermediaries' strategies from general mechanism spaces to specific tools such as prices in Armstrong (2006) or listing and closing fees in Matros and Zapechelnyuk (2012).

More recently, Condorelli et al. (2013) consider, similar to this paper, a single-unit auction setting with a single seller, multiple intermediaries and multiple buy-

[8] Similarly, social planner is more efficient for low valuation buyers while competition is for high valuation buyers.

ers. In their setup, each intermediary is linked to a subset of buyers, where the subsets are mutually exclusive across the intermediaries. They analyze a game, where the seller can trade the object one of two ways; either directly with a buyer, who was referred by an intermediary for some referral fee, or with an intermediary, who in turn resells the object to one of its exclusive buyers. The authors focus on the role of referrals, in particular impact on efficiency, in these markets with intermediaries. In their paper, the seller has all the bargaining power by designing the trading mechanism, whereas in my paper the intermediaries design the trading mechanisms. Furthermore, I consider a bargaining situation with two-sided asymmetric information, as the seller also possesses private information.

A few recent papers consider competition among intermediaries over mechanisms. In Feldman et al. (2010), inspired by the internet ad auctions, the authors consider multiple intermediaries competing for the purchase of a single good from an upstream seller, via a second price auction with reserve price, in order to sell to their respective "captive" buyers in the downstream auctions. The paper tries to identify the equilibrium downstream auctions for the intermediaries. This paper differs in the feature that the intermediaries do not have captive buyers and hence do not offer up and downstream auctions. Furthermore, the intermediaries have access to a larger strategy space than second-price auctions. Closer to my paper is Loertscher and Niedermayer (2008) and Loertscher and Niedermayer (2012), where the authors consider competition among intermediaries over announcements of auctions with a fee that is levied on the realized price. They consider a dynamic random matching model where a seller, a buyer and an intermediary are matched, which yields the intermediaries temporary monopoly powers over the traders until they are rematched. In my paper there is a single seller, who optimally chooses the intermediary as opposed to being randomly matched to an intermediary. Altogether, to the best of my knowledge, this is the first paper that attempts to analyze competition among intermediaries over general mechanisms in the context of trade.

Insofar as the intermediaries are competing designers, this paper also relates to literature on competing mechanism design. The existing papers, such as McAfee (1993), Peters and Severinov (1997), Burguet and Sákovics (1999), Pai (2009) and Virág (2010), all consider sellers as the designers. Furthermore, the papers restrict the sellers' strategy spaces to direct mechanism or auctions. In that regard, my paper differs in having the intermediaries as designers who have access to a larger set of mechanisms. An interesting result close to the findings presented here is found in McAfee (1993). In that paper, the author considers many sellers competing with each other over direct mechanisms in order to attract buyers. Concentrating on "large" market equilibria where the sellers' mechanism announcements do not alter the distribution of buyers at other sellers' mechanisms, he shows that auctions arise endogenously as trading institutions. I show that a similar result is retained in the model of competition considered here, where the intermediaries are the designers. The result is stronger in my paper, because it is maintained even though the intermediaries have access to a larger strategy space than the space of direct mechanisms.

The existence of multiple principals competing over mechanisms relates this paper to the competing mechanisms in common agency literature. There are several theoretical papers that offer methods to cope with the difficulties arising from common agency problem. The main methods offered in the literature to tackle common agency problems include universal type space approach from Epstein and Peters (1999) and the menu theorem approaches in Martimort and Stole (2002), Peters (2001) and Han (2006), where the last one develops menu theorems for bilateral contracting with multiple agents. More recently, Attar et al. (2011) consider multiple-principal and multiple-agent games of incomplete information in which each agent can at most participate with one principal. They show that in such games with a single agent, it is without loss of generality to restrict principals' strategies to direct mechanisms and the agent reporting his type truthfully. Unfortunately, as there are multiple agents in the setup considered in this paper, the results from Attar et al. (2011) can not be applied. Nevertheless, the particular model of competition I analyze skirts the greater difficulties related to the common agency framework and the revelation principle provides the means to characterizing the equilibrium outcomes.

The special case of a single buyer relates this paper to the literature on bilateral trade initiated by Myerson and Satterthwaite (1983) and Chatterjee and Samuelson (1983). The monopolist intermediary's optimal mechanism and the constrained-efficient mechanism have been identified in Myerson and Satterthwaite (1983). The double-auction analyzed in Chatterjee and Samuelson (1983) is a particular trading mechanism that implements the constrained-efficient outcomes.

The seller's revenue maximizing auctions are central to the analysis. In the context of bilateral trade, seller's optimal mechanism when her types are commonly known has been examined by Riley and Zeckhauser (1983). As summarized in Yilankaya (1999), in the IPV case the informed seller's optimal mechanism is equivalent to the ex-ante optimal mechanism of the seller, which was separately studied in Williams (1987). The equivalence between informed seller's ex-ante optimal mechanism and the optimal mechanism if her private information were commonly known holds generally for the IPV setup as shown in Skreta (2011).[9]

Finally, various techniques from the vast literature on optimal design are employed in the proofs. The optimal mechanisms are derived using approaches established in the seminal Myerson (1981) paper on optimal auction design. On a more technical level, in two of the contradiction proofs, I construct profitable deviation mechanisms that have strictly dominant strategies or satisfy strict ex-post incentive compatibility. Some results I find useful include the equivalence between Bayesian implementation and dominant strategy implementation as shown in Gershkov et al. (2013) and Manelli and Vincent (2010), and the characterization of ex-post incentive compatible mechanisms from Chung and Ely (2002). Accounting for entry decisions of the seller requires techniques from optimal mechanism in arbitrary (possibly nonconvex) type spaces to be applied. I appeal to Skreta (2006) where she establishes revenue equivalence results for optimal mechanisms defined on general type spaces. Lastly, my proof techniques are also related to the work in Jullien (2000) where the author considers the problem of optimal design under type dependent outside options. A similar scenario arises endogenously in the model considered here, as every intermediary faces the problem of attracting seller types that receive particular expected payoffs from the competitor intermediary's announced mechanisms.

I.2 Intermediation Game with Multiple Intermediaries

I.2.1 Primitives

Consider a trading problem where a seller, denoted s, owns a single indivisible object that a set of n buyers, each denoted by $j \in \{1, \dots, n\}$ want to buy. The agents have their own valuations for the object which are privately known. There is a set of intermediaries $I = \{1, \dots, m\}$ for $m \geq 2$ who try to intermediate the transaction between the agents, without knowing how much the object is worth

[9]Maskin and Tirole (1990) analyzes the general principal-agent relationship when the principal has private information. However, their analysis does not apply to the informed seller's optimal mechanism design in (bilateral) trade problems directly as several assumptions such as the finiteness of type spaces or the sorting assumption do not hold.

to either agent.[10]

All of the players in the game are risk neutral. They all have outside options with payoffs normalized to 0. The intermediaries care only about money. The agents have additively separable and linear preferences over both the good and money. I refer to the seller's valuation for the object, or her opportunity cost of selling it, as her cost and denote it by c, while the valuation of buyer j is denoted by v_j. I assume that c is distributed on interval $C = [0, 1]$ according to distribution F, while v_j's are independently distributed on the same interval, denoted by $V_j = [0, 1]$ according to distributions G_j. I also assume that the distribution functions F and G_j's are differentiable and are commonly known. In addition, their corresponding density functions f and g_j are assumed to be strictly positive everywhere, satisfying the usual monotone hazard rate properties; F/f is strictly increasing and $(1 - G_j)/g_j$ is strictly decreasing. The random variables c and v_j's are private information of the seller and the buyers, respectively, and hence will be referred to as the *types* of the agents.

I will find it convenient to define $\psi_j(v_j) = v_j - \frac{1-G_j(v_j)}{g_j(v_j)}$ as the *virtual valuation* of a buyer j. By monotone hazard rate property, $\psi_j(v_j)$ is strictly increasing in v_j for every j. Similarly define $\psi_s(c) = c + \frac{F(c)}{f(c)}$ to be the *virtual cost* of the seller. By the same token, $\psi_s(c)$ is also a strictly increasing function in c.

Finally, I use $\mathbf{V} = \times_{j=1}^n V_j = [0, 1]^n$ to denote the set of all possible vectors of valuations for all of the buyers, while $\mathbf{V}_{-j} = \times_{k \neq j} V_k = [0, 1]^{n-1}$ represents the set of all possible vectors of valuations for all of the buyers other than j. Similarly let $\mathbf{v} = (v_1, \ldots, v_n)$ denote a vector of valuations for all of the buyers, while $\mathbf{v}_{-j} = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n)$ denotes a vector of valuations for all of the buyers other than j. The joint distribution of $\mathbf{v}$ on $\mathbf{V}$ is denoted by $G = \times_{j=1}^n G_j$, while $G_{-j} = \times_{k \neq j} G_k$ denotes the joint distribution of $\mathbf{v}_{-j}$ on $\mathbf{V}_{-j}$. In that regard $g(\mathbf{v})$ and $g_{-j}(\mathbf{v}_{-j})$ denote the corresponding densities.

I.2.2 Timing of the Game

These players participate in the *intermediation game with multiple intermediaries* that is player over three stages where in the first stage the intermediaries announce their mechanisms simultaneously to the seller. After the mechanism announcements, at the second stage the seller moves, initially picking which intermediary mediates the bargaining with the buyers and later sending her private message to

[10]For the rest of the paper, I reserve subscripts for the agents while the superscripts for the intermediaries.

that chosen mechanism. Finally in the third and last stage, the buyers observe only the seller's chosen mechanism and send their private message to that previously chosen mechanism. The game ends with the chosen mechanism implementing its specified outcome.

At the first stage of the game, every intermediary $i \in I$ offers a mechanism, denoted γ^i. The vector of announced mechanisms is denoted by $\boldsymbol{\gamma} = \{\gamma^1, \ldots, \gamma^m\}$. I use Γ to refer to the space of mechanisms that each γ^i belongs to. Due to the symmetry of the intermediaries' preferences and information, it is without loss of generality to assume that all announced mechanisms come from the same space Γ. Hence $\Gamma^m = \times_{i=1}^m \Gamma$ represents the joint mechanism space to which the vector $\boldsymbol{\gamma}$ belongs.

Following the mechanism announcements, at the second stage the seller moves. She first takes her entry decision which is publicly observed. The seller chooses one of the announced mechanisms from $\boldsymbol{\gamma}$. Since the entry decision entails choosing an intermediary, a typical action is denoted by $i \in I$. Following an entry choice $i \in I$, the seller at subgame $s.i$ sends her private message to intermediary i, where a typical action is denoted by m_s^i and belongs to the set M_s^i, while the game is over for all the other intermediaries who receive outside option payoffs of 0. This concludes the second stage and the game proceeds to the third and final stage.

In the last stage, following seller's entry action of $i \in I$, the buyers observe only γ^i. Then each buyer j at subgame $j.i$ sends his private message to the chosen intermediary i. Again, a typical action is denoted by m_j^i and belongs to the set M_j^i for all buyers $j \in \{1, \ldots, n\}$. I denote the vector of buyers' messages by $\mathbf{m}_b^i = (m_1^i, \ldots, m_n^i)$. Finally, given a vector of messages $(m_s^i, \mathbf{m}_b^i)$, the chosen mechanism γ^i implements its outcome and the payoffs are realized.

I.2.3 Behavioral Strategies, Beliefs and Payoffs

I will define the behavioral strategies starting with the entry decision of the seller from the second stage. For all announced mechanisms $\boldsymbol{\gamma}$ and types $c \in C$, the seller picks an intermediary from the set of intermediaries I. A (pure) entry strategy of the seller is defined by the mapping $\eta : C \times \Gamma^n \to I$.

Following an entry decision of $i \in I$, the seller at subgame $s.i$ chooses her communication strategy which is a mapping $\mu_s^i : C \times \Gamma^m \to M_s^i$. I will use $\boldsymbol{\mu}_s = (\mu_s^1, \ldots, \mu_s^n)$ to denote the shorthand notation for the set of communication strategies of the seller at subgames $s.i$ for all $i \in I$.

Similarly at the third stage each buyer j at subgame $j.i$ chooses his communication strategy which is defined by a mapping $\mu_j^i : V_j \times \Gamma \rightarrow M_j^i$. Observe that for every buyer j, their communication strategy mapping accounts for the fact that they observe only the seller's chosen mechanism. Also note that for all agents, the messages specified by their strategies are sent only to the chosen intermediary i and no other intermediaries receive any messages. It will be convenient to define $\boldsymbol{\mu}_b^i = (\mu_1^i, \ldots, \mu_n^i)$ as the shorthand for the set of communication strategies of all buyers at subgame following entry to i. Similarly denote by $\boldsymbol{\mu}_{-j}^i = (\mu_1^i, \ldots, \mu_{j-1}^i, \mu_{j+1}^i, \ldots, \mu_n^i)$ as the collection of communication strategies of all buyers other than j following entry to i. Finally $\boldsymbol{\mu}_b = \{\boldsymbol{\mu}_b^i\}_{i=1}^m$ denotes the overall collection of communication strategies of all buyers at all subgames.

It will be assumed that for all agents and all intermediaries the message spaces M_s^i and M_j^i are rich enough to include a message that upon sending guarantees the outside option payoff of 0 to that agent. One way to interpret this message is "leaving the negotiation table".[11]

The behavioral strategies for the intermediaries are choices of their respective mechanisms. Given the bilateral trade context, a mechanism γ^i offered by intermediary $i \in I$ comprises the allocation rule $\mathbf{Q}^i = \{Q_1^i, \ldots, Q_n^i\}$ and payment rule $\boldsymbol{\tau}^i = \{\tau_s^i, \tau_1^i, \ldots, \tau_n^i\}$. Each term Q_j^i corresponds to the probability of buyer j receiving the good. The payment τ_s^i represents the transfer to the seller from intermediary i, while τ_j^i is the transfer from buyer j to intermediary i, respectively. It will be assumed that the transfers belong to the set $[-K, K]$ for a large enough constant K that satisfies $1 \ll K < \infty$.[12] Hence a mechanism γ^i is defined by the mapping $\gamma^i : M_s^i \times M_1^i \times \cdots \times M_n^i \rightarrow [0,1]^n \times [-K, K]^{n+1}$ where the inputs are the private messages sent by each agent to the intermediary and outputs are trade probabilities and transfers in the aforementioned order.

As is standard in the literature, it is assumed that intermediaries are not allowed to offer mechanisms which are contingent on other mechanisms and that the intermediaries can and will perfectly commit to the mechanisms they offer. A mechanism is said to satisfy *resource constraints* if for any vector of messages, the sum of trade probabilities are at most 1. Given there is a single indivisible object to be allocated, I restrict attention to mechanisms that satisfy resource constraints and denote the space of all such mechanisms by Γ, which is compact. Because it

[11]This assumption allows me to, without loss of generality, assume that the seller's entry choice is followed by participation of the buyers. In other words, any alternative model with "nonparticipation" among the set of entry choices for the seller or at the beginning of the third stage allowing the buyers first to choose whether to go to seller's chosen mechanism or to stay at home do not alter the results.

[12]Observe that K can be made arbitrarily large. This assumption makes the general mechanism space compact, however has no impact on the results.

is assumed that the good can not be destroyed, from resource constraints one can define $Q_s^i = \sum_{j=1}^n Q_j^i$ to be the probability of the good leaving the hands of the seller. Hence $1-Q_s^i = 1-\sum_{j=1}^n Q_j^i$ is the probability that the seller keeps the object. Hence a pure strategy of an intermediary is to announce a mechanism $\gamma^i \in \Gamma$. Denote by $\boldsymbol{\gamma}^{-i} = (\gamma^1, \ldots, \gamma^{i-1}, \gamma^{i+1}, \ldots, \gamma^m)$ the strategies of all intermediaries other than i.

Next, I will define the beliefs in the game. For all the intermediaries, their beliefs of seller's types c and buyers' types v_j are given by the prior distribution functions F and G_j's, respectively.

In order to define the beliefs of the agents, consider the information nodes $(c, \mathbf{v}, i, \boldsymbol{\gamma})$ which is composed of the vector of agents' types, seller's entry decision and vector of announced mechanisms, respectively. Seller at subgame $s.i$ forms beliefs denoted by $\beta_s(\mathbf{v}|c, i, \boldsymbol{\gamma})$ over buyers' types $\mathbf{v}$. It will be convenient to suppress seller's entry choice in the notation by using $\beta_s^i(\mathbf{v}|c, \boldsymbol{\gamma})$ and whenever there is no confusion about histories I will refer to seller's beliefs at subgame $s.i$ in shorthand by β_s^i. The collection of seller's beliefs over all subgames are denoted by $\boldsymbol{\beta}_s = (\beta_s^1, \ldots, \beta_s^n)$.

On the other hand, each buyer j at subgame $j.i$ forms beliefs over seller types c, types of the other buyers $\mathbf{v}_{-j}$ and the mechanisms $\boldsymbol{\gamma}^{-i}$ announced by the intermediaries other than i. Buyer j's beliefs will be denoted by $\beta_j(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, i, \gamma^i)$. Similar to the seller's beliefs, suppressing seller's entry choice, I will denote buyer j's beliefs at subgame $j.i$ by $\beta_j^i(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, \gamma^i)$ and refer to them by β_j^i in shorthand. The vector of all buyers' beliefs following entry to i are denoted by $\boldsymbol{\beta}_b^i = (\beta_1^i, \ldots, \beta_n^i)$ and the term $\boldsymbol{\beta}_b$ denotes the whole system of buyers' beliefs over all subgames.

Next I define the payoffs in the game. Consider a terminal node defined by a history of actions with array of mechanisms $\boldsymbol{\gamma}$, entry to intermediary i and messages sent m_s^i and $\mathbf{m}_b^i$. Then the realized payoffs are given by:

$$\begin{aligned}
U_s(\boldsymbol{\gamma}, i, m_s^i, \mathbf{m}_b^i|c) &= \tau_s^i(m_s^i, \mathbf{m}_b^i) - Q_s^i(m_s^i, \mathbf{m}_b^i)c \\
U_j(\boldsymbol{\gamma}, i, m_s^i, \mathbf{m}_b^i|v_j) &= Q_j^i(m_s^i, \mathbf{m}_b^i)v_j - \tau_j^i(m_s^i, \mathbf{m}_b^i) && \forall j \in \{1, \ldots, n\} \\
U^i(\boldsymbol{\gamma}, i, m_s^i, \mathbf{m}_b^i) &= \sum_{j=1}^n \tau_j^i(m_s^i, \mathbf{m}_b^i) - \tau_s^i(m_s^i, \mathbf{m}_b^i) \\
U^k(\boldsymbol{\gamma}, i, m_s^i, \mathbf{m}_b^i) &= 0 && \forall k \neq i
\end{aligned}$$

Finally, I define the expected payoffs at the relevant subgames where the corresponding players move. For the rest of the section, consider a profile of strategies $(\boldsymbol{\gamma}, \eta, \boldsymbol{\mu}_s, \boldsymbol{\mu}_b)$ and beliefs $(\boldsymbol{\beta}_s, \boldsymbol{\beta}_b)$. At the beginning of the third stage, for every buyer j and entry choice i, the expected payoff at subgame $j.i$ is denoted by $U_j\big(\boldsymbol{\gamma}, i, \mu_s^i, \boldsymbol{\mu}_b^i | v_j\big)$ which is equal to:

$$U_j\big(\boldsymbol{\gamma}, i, \mu_s^i, \boldsymbol{\mu}_b^i | v_j\big) = \int U_j\big(\boldsymbol{\gamma}, i, \mu_s^i(c, \boldsymbol{\gamma}), \mu_j^i(v_j, \gamma^i), \boldsymbol{\mu}_{-j}^i(\mathbf{v}_{-j}, \gamma^i) | v_j\big)$$
$$\beta_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i} | v_j, \gamma^i\big) d(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i})$$

For the seller at the second stage, there are two expected payoffs to be defined. Firstly, following a realized entry decision of i, seller at subgame $s.i$ receives an expected payoff denoted by $U_s\big(\boldsymbol{\gamma}, i, \mu_s^i, \boldsymbol{\mu}_b^i | c\big)$ which is equal to:

$$U_s\big(\boldsymbol{\gamma}, i, \mu_s^i, \boldsymbol{\mu}_b^i | c\big) = \int U_s\big(\boldsymbol{\gamma}, i, \mu_s^i(c, \boldsymbol{\gamma}), \boldsymbol{\mu}_b^i(\mathbf{v}, \gamma^i) | c\big) \beta_s^i(\mathbf{v} | c, \boldsymbol{\gamma}) d\mathbf{v}$$

Secondly, when deciding on the entry decision $\eta(c, \boldsymbol{\gamma})$, the seller receives an expected payoff denoted by $U_s\big(\boldsymbol{\gamma}, \eta, \boldsymbol{\mu}_s, \boldsymbol{\mu}_b | c\big)$ and is equal to:

$$U_s\big(\boldsymbol{\gamma}, \eta, \boldsymbol{\mu}_s, \boldsymbol{\mu}_b | c\big) = \sum_{i \in I} U_s\big(\boldsymbol{\gamma}, i, \mu_s^i, \boldsymbol{\mu}_b^i | c\big) \mathbb{I}\big(\eta(c, \boldsymbol{\gamma}) = i\big)$$

Lastly, in the first stage each intermediary $i \in I$ receives an expected profit denoted by $U^i\big(\boldsymbol{\gamma}, \eta, \boldsymbol{\mu}_s, \boldsymbol{\mu}_b\big)$ and is equal to:

$$U^i\big(\boldsymbol{\gamma}, \eta, \boldsymbol{\mu}_s, \boldsymbol{\mu}_b\big) = \int_c \int_{\mathbf{v}} U^i\big(\boldsymbol{\gamma}, i, \mu_s^i(c, \boldsymbol{\gamma}), \boldsymbol{\mu}_b^i(\mathbf{v}, \gamma^i)\big) dG(\mathbf{v}) \mathbb{I}\big(\eta(c, \boldsymbol{\gamma}) = i\big) dF(c)$$

I.2.4 Equilibrium

An assessment $\hat{\mathcal{E}}$ consisting of strategies $(\hat{\boldsymbol{\gamma}}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b)$ and beliefs $\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_s, \hat{\boldsymbol{\beta}}_b)$ forms a *sequential equilibrium*, or equilibrium for short, whenever the strategies satisfy sequential rationality with beliefs $\hat{\boldsymbol{\beta}}$ and the beliefs $\hat{\boldsymbol{\beta}}$ are consistent with the strategies. First consider the sequential rationality conditions of the strategies given the belief system $\hat{\boldsymbol{\beta}}$. The communication strategies $\hat{\mu}_s^i$ and $\hat{\mu}_j^i$ for all buyers at subgames $s.i$ and $j.i$ for any $i \in I$ have to satisfy:

$$U_s\big(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | c\big) \geq U_s\big(\boldsymbol{\gamma}, i, m_s^i, \hat{\boldsymbol{\mu}}_b^i | c\big)$$
$$U_j\big(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | v_j\big) \geq U_j\big(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, m_j^i, \hat{\boldsymbol{\mu}}_{-j}^i | v_j\big)$$

for every $c \in C$, $v_j \in V_j$, all mechanism announcements $\boldsymbol{\gamma}$, all entry strategies $i \in I$ and all messages $m_s^i \in M_s^i$ and $m_j^i \in M_j^i$, in the corresponding way. The entry strategy of the seller from the beginning of the second stage has to satisfy the following condition:

$$U_s\big(\boldsymbol{\gamma}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b|c\big) \geq U_s\big(\boldsymbol{\gamma}, i, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b|c\big)$$

for every $c \in C$, all strategies $\boldsymbol{\gamma}$, the optimal communication strategies $\hat{\boldsymbol{\mu}}_b$ and $\hat{\boldsymbol{\mu}}_s$, and any entry decision $i \in I$. At the first stage, by announcing $\hat{\gamma}^i$ each intermediary's expected profits have to satisfy:

$$U^i\big(\hat{\boldsymbol{\gamma}}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b\big) \geq U^i\big(\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b\big)$$

for every $\gamma_i \in \Gamma$.

Next consider the equilibrium beliefs of the agents. Consistency of $\hat{\boldsymbol{\beta}}$ requires that there exists a sequence of totally mixed strategies $\gamma^{i,h} \in \Delta\Gamma$ for each i and $\eta^h \in \Delta I$ that converge to the equilibrium strategies[13] and that the equilibrium beliefs satisfy $\hat{\boldsymbol{\beta}} = \lim_{h\to\infty} \boldsymbol{\beta}^h$ where the sequence of beliefs are generated from the sequence of strategies using Bayes' rule.

Consistency implies that the agents have correct beliefs at the information sets that are reached with positive probability on the equilibrium path. Then given an equilibrium vector of mechanisms $\hat{\boldsymbol{\gamma}}$, the beliefs of the seller at subgame $s.i$ that is reached with positive probability, i.e. $\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c) > 0$, for every type $c \in C$ are given by:

$$\hat{\beta}_s^i(\mathbf{v}|c, \hat{\boldsymbol{\gamma}}) = g(\mathbf{v})$$

Similarly, given an equilibrium mechanism $\hat{\gamma}^i$ that is chosen with positive probability on the equilibrium path, the beliefs for any type $v_j \in V_j$ of any buyer j at subgame $j.i$ are given by:

$$\hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i})|v_j, \hat{\gamma}^i\big) = \begin{cases} \frac{\mathbb{I}\left(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\right) f(c)}{\int_0^1 \mathbb{I}\left(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\right) dF(c)} g_{-j}(\mathbf{v}_{-j}) & \text{if } \boldsymbol{\gamma}^{-i} = \hat{\boldsymbol{\gamma}}^{-i} \\ 0 & \text{o/w} \end{cases}$$

If the equilibrium mechanism $\hat{\gamma}^i$ is not chosen by any type of the seller or an intermediary announces a deviation mechanism $\gamma^i \neq \hat{\gamma}^i$, the subgames are out-of-equilibrium path and hence are reached with zero probability. In either of those cases, the beliefs are equal to the limit of belief sequences generated by the cho-

[13]The sequences $\{\gamma^{1,h}, \ldots, \gamma^{m,h}, \eta^h\}_{h=1}^{\infty}$ satisfy $\lim_{h\to\infty} \gamma^{i,h} = \hat{\gamma}^i$ for every i and $\lim_{h\to\infty} \eta^h = \hat{\eta}$.

sen sequence of convergent strategy sequences. It turns out that the seller's equilibrium beliefs at any information node is equal to $\hat{\beta}_s^i(\mathbf{v}|c,\boldsymbol{\gamma}) = g(\mathbf{v})$, no matter whether the array of mechanisms are the equilibrium strategies or not. This result follows from the consistency of the beliefs and is proved in Appendix A.1.2.

Finally, it will be convenient to define shorthand notations for the equilibrium expected payoffs of the agents and the intermediaries. Letting $\hat{\mathcal{E}}$ be an equilibrium, then for any seller with type c, for any buyer j with type v_j and any intermediary i, denote by $\hat{U}_s(c)$, $\hat{U}_j(v_j)$ and $\hat{U}^i$ their expected payoffs on the equilibrium, respectively:

$$\begin{aligned}
\hat{U}_s(c) &= U_s\big(\hat{\boldsymbol{\gamma}}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b | c\big) \\
\hat{U}_j(v_j) &= \sum_{i \in I} U_j\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | v_j\big) \left[\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c)\right] \\
\hat{U}^i &= U^i\big(\hat{\boldsymbol{\gamma}}, \hat{\eta}, \hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\mu}}_b\big)
\end{aligned}$$

Similarly, I will denote by $\hat{\mathbf{Q}} = (\hat{Q}_1, \ldots, \hat{Q}_n)$ the allocation rule that is implemented on equilibrium. For any vector of types $(c, \mathbf{v})$, define for each j the probability of trade by:

$$\hat{Q}_j(c, \mathbf{v}) = \sum_{i \in I} \hat{Q}_j^i\big(\hat{\mu}_s^i(c, \hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v}, \hat{\gamma}^i)\big) \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big)$$

My aim is to provide a full characterization of the equilibrium outcomes in terms of the implemented allocation rule $\hat{\mathbf{Q}}$ and the expected payoffs $\hat{U}_s(c)$, $\hat{U}_j(v_j)$'s and $\hat{U}^i$'s.

I.2.5 Revelation Principle

In mechanism design problems with a single principal, the revelation principle pioneered in Myerson (1981) establishes that, it is without loss of generality to restrict attention to a simple class of mechanisms called direct mechanisms where the agents' message spaces are equal to their type spaces. As pointed out by Peters (2001) and Epstein and Peters (1999) among many others, existence of multiple principals poses a major challenge. In particular, in such environments it is *with* loss of generality to restrict attention to the principals announcing direct mechanisms.

Insofar as the multiple competing intermediaries are mechanism designers, this paper also falls into the latter category. Despite the difficulties with the revelation principle,[14] it turns out that in this setup I can use it to characterize the equilibrium outcomes in terms of the implemented trade probabilities and expected payoffs to the players. Before proceeding, I go over some terminology and definitions that are used in the rest of the paper.

Feasible Direct Revelation Mechanisms and Their Properties

The intermediation game considered in this paper is a Bayesian game of incomplete information with $m+n+1$ players. The aim of the subsection is to show that for any equilibrium of the intermediation game, there exists a *direct revelation mechanism* (DRM), an auxiliary Bayesian game where players report their types with a payoff equivalent truthful type-telling equilibrium. In the setup of this paper, as it is only the agents ($n+1$-many players) who have private information, it is also only them who report their types in the DRM and the m-many intermediaries do not take any actions.

In the context of the intermediation game, a DRM denoted by δ, is a collection of allocation rule $\bar{\mathbf{Q}} = \{\bar{Q}_1, \ldots, \bar{Q}_n\}$ and transfer rule $\bar{\boldsymbol{\tau}} = \{\bar{\tau}_s, \bar{\tau}_1, \ldots, \bar{\tau}_n, \bar{\tau}^1, \ldots, \bar{\tau}^m\}$ where the outcomes are determined by the agents reporting their types. Similar to an intermediary's mechanism, each term $\bar{Q}_j$ denotes the probability of buyer j receiving the good, and the sum of $\bar{Q}_j$'s corresponds to the probability of the good leaving the hands of the seller. Furthermore, $\bar{\tau}_s$ and $\bar{\tau}^i$'s are transfers made to the seller and intermediaries denoted by i, while $\bar{\tau}_j$'s are the transfers made from the corresponding buyer j. Hence a DRM is defined by the mapping $\delta : C \times \mathbf{V} \to [0,1]^n \times \mathbb{R}^{m+n+1}$.

Assuming agents report types truthfully, the expected trade probabilities, transfers and consequently payoffs for the agents are defined as follows:

$$\bar{q}_s(c) = \int_{\mathbf{V}} \bar{Q}_s(c,\mathbf{v})dG(\mathbf{v}) \qquad \bar{q}_j(v_j) = \int_0^1 \int_{\mathbf{V}_{-j}} \bar{Q}_j(c,\mathbf{v})dG_{-j}(\mathbf{v}_{-j})dF(c)$$

$$\bar{t}_s(c) = \int_{\mathbf{V}} \bar{\tau}_s(c,\mathbf{v})dG(\mathbf{v}) \qquad \bar{t}_j(v_j) = \int_0^1 \int_{\mathbf{V}_{-j}} \bar{\tau}_j(c,\mathbf{v})dG_{-j}(\mathbf{v}_{-j})dF(c)$$

$$\bar{U}_s(c) = \bar{t}_s(c) - \bar{q}_s(c)c \qquad \bar{U}_j(v_j) = \bar{q}_j(v_j)v_j - \bar{t}_j(v_j)$$

Similarly, the expected transfers and payoffs for the intermediaries are:

$$\bar{t}^i = \int_0^1 \int_{\mathbf{V}} \bar{\tau}^i(c,\mathbf{v})dG(\mathbf{v})dF(c) \qquad \bar{U}^i = \bar{t}^i$$

[14] See Martimort and Stole (2002) for a detailed discussion of the difficulties with the revelation principle in common agency framework.

A DRM δ is *feasible* if and only if it satisfies *individual rationality* (IR), *incentive compatibility* (IC), *resource constraints* (RES), and *budget-balancedness* (BB). A DRM satisfies IR if the expected payoffs players receive are weakly larger than their outside option payoffs, IC if it is an equilibrium for all agents to report their types truthfully, RES if for any vector of reported types the sum of trade probabilities are at most 1, BB if there are no net transfers in or out of the system. These constraints can be expressed as follows:

$$\text{IR}: \quad \bar{U}_s(c),\, \bar{U}_j(v_j),\, \bar{U}^i \geq 0 \qquad \forall c,\ \forall j \text{ and } v_j,\ \forall i$$

$$\text{IC}: \quad \bar{U}_s(c) \geq \bar{U}_s(c'|c), \text{ and } \bar{U}_j(v_j) \geq \bar{U}_j(v_j'|v_j) \qquad \forall c, c',\ \forall j \text{ and } v_j, v_j'$$

$$\text{RES}: \quad 0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1, \text{ and } \bar{Q}_s(c,\mathbf{v}) = \textstyle\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v}) \leq 1 \qquad \forall c, \mathbf{v}$$

$$\text{BB}: \quad \textstyle\sum_{j=1}^n \left[\int_0^1 \bar{t}_j(v_j)dG_j(v_j)\right] - \int_0^1 \bar{t}_s(c)dF(c) - \sum_{i=1}^m \bar{t}^i = 0$$

Consider the implications of feasibility for a DRM δ. Using standard arguments as in Myerson (1981) or Krishna (2009), IC for buyer j implies that the expected payoff function $\bar{U}_j(v_j)$ is a maximum of a family of affine functions. Thus $\bar{U}_j(v_j)$'s are absolutely continuous convex functions that are differentiable almost everywhere with derivative $\bar{q}_j(v_j)$ which is weakly increasing in v_j. By RES $\bar{q}_j(v_j)$'s are bounded and thus buyer j's expected payoffs are equal to $\bar{U}_j(v_j) = \bar{U}_j(0) + \int_0^{v_j} \bar{q}_j(y_j)dy_j$. Then IR constraint of buyer j simplifies to holding only at the "worst" type $v_j = 0$.

Similar arguments for the seller yield that $\bar{U}_s(c)$ is an absolutely continuous convex function that is differentiable almost everywhere with derivative $\bar{q}_s(c)$ which is weakly decreasing in c. Given $\bar{q}_s(c)$'s are bounded by RES, the seller's expected payoff for any type c can be written as $\bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx$. In turn the IR of the seller simplifies to hold only at $c = 1$.

Finally BB suggests that the sum of expected transfers to the intermediaries are equal to the expected revenue generated from the agents, which is the difference between the sum of expected transfers from the buyers and the seller. For a given feasible DRM δ, it will be convenient to denote this revenue from the agents by $R(\delta)$. The following remark summarizes the necessary conditions from above for feasibility of a DRM δ. The proofs are omitted as they are standard results from mechanism design:

Remark I.1.
If a DRM δ is feasible, then the following must be true:

$$\begin{array}{llll} \textit{i)} & \bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx, \quad \bar{U}_s(1) \geq 0, \quad \frac{d\bar{q}_s(c)}{dc} \leq 0 & & \forall c \\ \textit{ii)} & \bar{U}_j(v_j) = \bar{U}_j(0) + \int_0^{v_j} \bar{q}_j(y_j)dy_j, \quad U_j(0) \geq 0, \quad \frac{d\bar{q}_j(v_j)}{dv_j} \geq 0 & & \forall j \textit{ and } v_j \\ \textit{iii)} & \bar{U}^i = \bar{t}^i \geq 0, \qquad \sum_{i \in I} \bar{U}^i = \sum_{i=1}^{m} \bar{t}^i = R(\delta) & & \forall i \end{array}$$

where $R(\delta)$ is the expected revenue generated from the agents and is defined as:

$$\begin{aligned} R(\delta) &= \sum_{j=1}^{n} \left[\int_0^1 \bar{t}_j(v_j)dG_j(v_j) \right] - \int_0^1 \bar{t}_s(c)dF(c) \\ &= \int_0^1 \int_{\mathbf{v}} \left[\sum_{j=1}^{n} \bar{q}_j(c,\mathbf{v})\big[v_j - c\big] - \sum_{j=1}^{n} \bar{U}_j(v_j) - \bar{U}_s(c) \right] dG(\mathbf{v})dF(c) \end{aligned}$$

Observe that given a feasible DRM $\delta = (\bar{\mathbf{Q}}, \bar{\boldsymbol{\tau}})$, the agents' expected payoffs can equivalently be characterized by only considering the allocation rule $\bar{\mathbf{Q}}$ along with the endpoint expected payoffs $\bar{U}_s(1)$ and $\bar{U}_j(0)$'s for all j. Also observe that, the same information characterizes $R(\delta)$.

Payoff Equivalence to a Feasible DRM and Its Implications

Here's the main result of this subsection:

Lemma I.1.
Given any equilibrium $\hat{\mathcal{E}}$ of the intermediation game, there exists a feasible DRM δ that has a payoff equivalent truthful type-telling equilibrium.

The proof incorporates a standard indexation and composition argument. There are two important points to take away from Lemma I.1. Firstly, given any equilibrium $\hat{\mathcal{E}}$ of the intermediation game, the implemented allocation rule $\hat{\mathbf{Q}}$ is equivalent to the allocation rule implemented in the truthful type-telling equilibrium of some feasible DRM $\delta = (\bar{\mathbf{Q}}, \bar{\boldsymbol{\tau}})$.

Secondly, combining Remark I.1 with the equivalence between the agents' expected payoffs, the following equalities can be attained:

$$\hat{U}_s(c) = \bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx \geq 0 \qquad \forall c$$
$$\hat{U}_j(v_j) = \bar{U}_j(v_j) = \bar{U}_j(0) + \int_0^{v_j} \bar{q}_j(y_j)dy_j \geq 0 \qquad \forall j \text{ and } v_j$$

where $\bar{q}_s(c)$ and $\bar{q}_j(v_j)$'s are the weakly monotonic expected trade probabilities for the corresponding agents while $\bar{U}_s(1)$ and $\bar{U}_j(0)$'s are the expected payoffs of the agents for their corresponding "worst" types. Furthermore, by Remark I.1 the agents' equilibrium expected payoff schedules $\hat{U}_s(c)$ and $\hat{U}_j(v_j)$'s are absolutely continuous in their own types. Hence characterizing the allocation rule and the nonnegative expected payoffs at the corresponding endpoint types of the payoff equivalent feasible DRM's provides a full summary of equilibrium outcomes on the agents' side.

On the other side, however, the same knowledge of allocation rule and expected payoffs of agents at corresponding endpoints does not provide a full description of the intermediaries' expected payoffs on the equilibrium. Nevertheless, due to BB, the sum of the expected payoffs for the intermediaries is equal to $R(\delta)$:

$$\hat{U}^i = \bar{U}^i \geq 0 \qquad \forall i \in I \quad \Rightarrow \quad \sum_{i=1}^m \hat{U}^i = \sum_{i=1}^m \bar{U}^i = R(\delta)$$

In the next section, I will uniquely characterize the outcomes of the intermediation game using the set of feasible DRM's. The main challenges will be to pin down the allocation rule $\bar{\mathbf{Q}}$ and agents' expected payoffs at endpoints $\bar{U}_s(1)$ and $\bar{U}_j(0)$'s of a feasible DRM that are consistent with an equilibrium. It will also be another challenge to separate the intermediaries' equilibrium expected payoffs $\bar{U}^i$'s from an identified $R(\delta)$.

I.3 Equilibrium Outcomes with Multiple Intermediaries

In this section I present the main results of the paper including the existence and uniqueness of equilibrium outcomes in the intermediation game with multiple intermediaries. I begin by describing a feasible DRM that plays an important role in the rest of the analysis.

I.3.1 Seller-Optimal DRM

I start with the following definition:

Definition I.1.
A seller-optimal DRM, *denoted by $\delta^* = (\mathbf{Q}^*, \boldsymbol{\tau}^*)$, is a feasible DRM that maximizes seller's ex-ante expected payoff.*

Proposition I.1.
The seller-optimal DRM, denoted by δ^* is characterized by the following:

1. *There exists a unique allocation rule $\mathbf{Q}^* = (Q_1^*, \dots, Q_n^*)$ where for any vector of truthful type reports each term Q_j^* is equal to:*

$$Q_j^*(c, \mathbf{v}) = \begin{cases} 1 & \textit{if } \psi_j(v_j) > \max\left\{c, \max\limits_{k \neq j}\{\psi_k(v_k)\}\right\} \\ 0 & \textit{o/w} \end{cases} \tag{I.1}$$

2. *Agent's expected payoffs at corresponding endpoints satisfy $U_s^*(1) = U_j^*(0) = 0$ for all j.*

3. *The expected revenue generated from the agents satisfies $R(\delta^*) = 0$, which in turn implies $U^{i,*} = 0$ for all $i \in I$.*

There are several important properties associated with the seller-optimal DRM δ^*.

The unique allocation rule is dominant-strategy IC: For every j, given any reported vector of types the trade probability $Q_j^*(c, \mathbf{v})$ is weakly increasing in v_j and weakly decreasing in c. The latter property suggests that $Q_s^*(c, \mathbf{v})$ is also weakly decreasing in c as it is equal to the sum of Q_j^*'s across all j. Therefore the seller-optimal allocation rule $\mathbf{Q}^*$ is dominant-strategy implementable.[15]

The agents' expected payoffs are uniquely defined: For any feasible DRM δ, Remark I.1 establishes that the agents' expected payoffs are defined by the allocation rule and the constants $\bar{U}_s(1)$ and $\bar{U}_j(0)$'s. For δ^*, there is a unique allocation rule $\mathbf{Q}^*$ and the endpoint expected payoffs satisfy $U_s^*(1) = U_j^*(0) = 0$ for all j. Hence agents' expected payoff schedules are unique.

No cross-subsidization across seller types: Consider for the moment that there are no intermediaries and the seller is the designer in a situation where her

[15] Truthful type-telling is dominant strategy for the mechanism $\gamma^* = \{\mathbf{Q}^*, \boldsymbol{\tau}^*\}$ where the payment rules $\boldsymbol{\tau}^*$ are $\tau_s^*(c, \mathbf{v}) = Q_s^*(c, \mathbf{v})c + \int_c^1 Q_s^*(x, \mathbf{v})dx$, and $\tau_j^*(c, \mathbf{v}) = Q_j^*(c, \mathbf{v})v_j - \int_0^{v_j} Q_j^*(c, y, \mathbf{v}_{-j})dy$ for all j, and $\tau^{i,*}(c, \mathbf{v}) = 0$ for all i.

cost is commonly known by the buyers. Standard results from optimal mechanism design theory suggests that, the following auction $\gamma^S = \{\mathbf{Q}^S, \boldsymbol{\tau}^S\}$ is an optimal mechanism in the sense that it maximizes her expected revenues:[16]

$$Q_j^S(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j) > \max_{k\neq j}\{\psi_k(v_k)\} \text{ and } v_j > r_j^S(c) \\ 0 & \text{o/w} \end{cases} \qquad \forall j,\, c,\, \mathbf{v} \tag{I.2}$$

$$\tau_j^S(c,\mathbf{v}) = Q_j^S(c,\mathbf{v})v_j - \int_0^{v_j} Q_j^S(c,y,\mathbf{v}_{-j})dy \qquad \forall j,\, c,\, \mathbf{v} \tag{I.3}$$

where $r_j^S(c)$'s form a set of optimal discriminatory reserve prices defined by $r_j^S(c) = \psi_j^{-1}(c)$ for each j.[17] Note that the monotone hazard rate property implies $\psi_j(v_j)$ is strictly increasing in v_j for each j, and therefore ψ_j^{-1} is well defined.

Next, assume that the buyers do not observe the seller's cost. It turns out that in the IPV paradigm, the informed seller's revenue maximizing mechanism yields the same outcomes as in the truthful type-telling equilibrium of γ^S. In other words, γ^S is also an optimal mechanism for the informed seller. In particular, it implements the same allocation as $\mathbf{Q}^S$ shown above. This is because the seller's cost is distributed independently of the buyers' valuations. Hence her private information does not influence the maximal revenue she can generate.[18] For the rest of the paper, I will refer to γ^S described above, as the informed seller's optimal mechanism.

At a closer look, one can see that for every c, the two allocation rules $\mathbf{Q}^*$ and $\mathbf{Q}^S$ are the same; they both award the good to the buyer with the highest virtual valuation subject to the same discriminatory reservation values. Furthermore, under the seller's revenue maximizing auction, the expected payoffs for the buyers with lowest valuation $v_j = 0$ for all j and seller with highest cost $c = 1$ are equal to zero. To see why, observe that the reservation values are increasing in c and satisfy $r_j(c) \geq r_j(0) = \psi_j^{-1}(0) > 0$ for all c. Hence a buyer with lowest valuation never gets the object and receives zero expected payoffs. Similarly, seller with cost $c = 1$ sets the same reservation value $\psi_j^{-1}(1) = 1$ for all j, which in turn implies no probability of trade. Hence revenues are zero for highest cost seller yielding zero expected payoff.

[16] See for instance Myerson (1981), or Section 5.2 in Krishna (2009).

[17] Note that, there is slight abuse of notation as the optimal mechanism would not depend on the seller's private information as it is assumed that her cost is common knowledge. In order to make explicit connection in the rest of the analysis, the optimal mechanism under symmetric information is described this way.

[18] See Footnote 6.

These observations establish the equivalence in agents' payoffs and the allocation rules between the truthful equilibria of the seller-optimal DRM and the informed seller's revenue maximizing auction. An important implication of this equivalence is that δ^* is free from cross-subsidization across seller types. In other words, for every type c the seller's expected payoffs are exactly equal to the maximal expected revenue net of buyers' total information rents conditional on c. Since the seller's optimal auction when her types are commonly known does not display cross-subsidization of expected payoffs across seller types, the equivalence result implies that neither does the seller-optimal DRM.

The allocation rule is invariant to the seller's type distribution: One related point implied by the previous no cross-subsidization property is the invariance of the allocation rule to seller's type distribution $F(c)$. Because the revenue maximizing auction is optimal for every type of the seller, any change in the distribution does not alter the allocation rule. This is evident from the fact that the only point where the seller's private information appears in the allocation rule, i.e. the optimal reservation values, are free from distribution F.

Intermediaries make zero expected profits: By Remark I.1, it is established that in any feasible DRM δ, the sum of all intermediaries' payoffs is equal to $R(\delta)$, i.e. the revenue generated from the agents. For the seller-optimal DRM δ^*, this revenue $R(\delta^*)$ is equal to zero. Given the IR constraints imply nonnegativity of the intermediaries' payoffs, it must be the case that $U^{i,*} = 0$ for all i. This is not surprising, because any expected profits an intermediary makes can always be transferred to the seller as an increase in the additive constant $U_s(1)$ without violating any of the feasibility constraints. In light of the previous properties, this observation can be strengthened in the following sense.

Corollary I.1.
A feasible DRM $\bar{\delta}$ for which the truthful type-telling equilibrium yields $\bar{U}_s(c) \geq U_s^*(c)$ for all c has to satisfy $R(\bar{\delta}) = 0$, thus $\bar{U}^i = 0$ for all i and consequently $\bar{U}_s(c) = U_s^*(c)$ for all c.

From the corollary above, it follows that there is no feasible way of improving the seller's expected payoffs from those that accrued under δ^*. Hence the payoff schedule $U_s^*(c)$ is Pareto optimal and $\mathbf{Q}^*$ is a second best allocation. Furthermore, providing these optimal expected payoffs to the seller requires the intermediaries to make zero profits.

Seller-optimal DRM is invariant to the number of intermediaries: The derivation of δ^* did not rely on m, the number of intermediaries. Hence the characterization of Proposition I.1 is valid for any number of competing intermediaries.[19]

I.3.2 Uniqueness of Equilibrium Outcomes

In this subsection I prove the uniqueness of equilibrium outcomes in the intermediation game with multiple intermediaries. So far Lemma I.1 established the existence of a feasible DRM that has a payoff equivalent truthful type-telling equilibrium. It turns out that there is a unique feasible DRM that characterizes the outcomes in all equilibria of the intermediation game:

Theorem I.1.
Given any equilibrium $\hat{\mathcal{E}}$ of the intermediation game with multiple intermediaries, the unique outcomes are characterized by the truthful type-telling equilibrium of the seller-optimal DRM δ^* and hence are given by:

1. *The allocation rule satisfies $\hat{\mathbf{Q}} = \mathbf{Q}^*$.*
2. *The seller's expected payoffs satisfy $\hat{U}_s(1) = 0$, and $\hat{U}_s(c) = U_s^*(c)$ for all c.*
3. *For every buyer j, their expected payoffs satisfy $\hat{U}_j(0) = 0$, and $\hat{U}_j(v_j) = U_j^*(v_j)$.*
4. *Every intermediary i receives zero expected profits $\hat{U}^i = U^{i,*} = 0$.*

I prove the uniqueness of equilibrium outcomes in two steps. In the first step, I show that in any equilibrium $\hat{\mathcal{E}}$, it has to be the case that the intermediaries make zero expected profits, the buyers with lowest valuation receive zero expected payoffs and the equilibrium allocation rule awards the object to the buyer with the highest virtual valuation subject to some buyer specific reservation values. As a result, the first step establishes the unique equilibrium payoffs for the intermediaries. Furthermore, the full characterization of the remaining outcomes only requires a description of the reservation values and the highest cost seller's expected payoff. In the second step, I precisely show that the unique set of equilibrium reservation values are equal to those under the seller-optimal DRM, or equivalently the seller's revenue maximizing auction.

In both steps I prove the steps by showing the existence of a strictly profitable deviation for at least one intermediary if the payoff equivalent feasible DRM of the considered equilibrium violates the claimed properties. However, the main challenge is to circumvent the beliefs and consequently behavior of the agents at

[19]For that matter, characterization is also valid for the special cases of no intermediaries $m = 0$ and a single intermediary $m = 1$.

out-of equilibrium subgames. A technical contribution is the construction of the strictly dominant strategy deviation mechanisms.

The intuition for the uniqueness of equilibrium outcomes and their equivalence to the outcomes from the truthful type-telling equilibrium of the seller-optimal DRM is in the spirit of Bertrand competition. The intermediaries compete with each other in order to attract the seller. However the existence of private information results in the intermediaries contesting over every type of the seller. This competition drives their profits down to zero and leads every seller type to be offered the maximum expected payoffs they can receive subject to the feasibility constraints. In turn, this yields the equivalence of outcomes in any equilibrium to the unique DRM outcomes that achieve the maximal payoffs for the seller in a feasible manner.

The equivalence to the seller-optimal DRM result has other crucial implications. In the previous subsection, several important properties associated with δ^* were established. By virtue of equivalence, all those properties can be carried over to the equilibria of the intermediation game with multiple intermediaries. In particular, the outcomes are invariant to the number of competing intermediaries and the seller's underlying type distribution. Furthermore, because the allocation rule and the expected payoffs to the agents under the truthful type-telling equilibrium of δ^* are equivalent to those that are accrued under the truthful type-telling equilibrium of the seller's revenue maximizing auction, the equivalence carries over to the intermediation game. The remark below summarizes the last point:

Remark I.2.
In any equilibrium $\hat{\mathcal{E}}$ of the intermediation game, the expected payoffs to the agents and the implemented allocation rule are equivalent to those that are accrued under the truthful type-telling equilibrium of the informed seller's revenue maximizing auction γ^S.

Furthermore, in the symmetric case where all the buyers have the same type distribution, the optimal mechanism γ^S can be implemented as a second-price auction with reserve prices that are same for all buyers.

The importance of this equivalence result is that, availability of more complicated mechanisms do not improve the outcomes that are achieved by the informed seller's optimal auction. Furthermore, it provides guidance for what a candidate equilibrium. Namely, one where intermediaries announce mechanisms that imitate the informed seller's revenue maximizing auction γ^S. This is precisely the aim of the next subsection.

Lastly, Remark I.2 suggests that under the symmetric case, the equilibrium outcomes can be implemented by classical auctions, because the payoff equivalent mechanism γ^S is a second-price auction with reserve prices. Hence, the results of Theorem I.1 highlight the robustness of classical auctions under certain scenarios, which in turn might be interpreted as a verification for the popularity of their usage.

I.3.3 Existence of Equilibrium

The equilibrium outcome characterization relied on assuming the existence of an equilibrium. The results would be moot if there were no equilibria of the intermediation game with multiple intermediaries. As the next proposition establishes, that is not the case:

Proposition I.2.
An equilibrium of the intermediation game with multiple intermediaries exists.

In light of Theorem I.1 and Remark I.2, an equilibrium is constructed where all intermediaries announce the same mechanisms that mimic the informed seller's optimal mechanism γ^S as described before. Namely, the allocation rule is equivalent to the unique seller-optimal DRM allocation rule $\mathbf{Q}^*$ (or equivalently to $\mathbf{Q}^S$) and the transfer rule yields zero realized profits for all possible message reports. More specifically the transfer rule for the buyers are equivalent to $\tau_j^S(c, \mathbf{v})$ as decribed in (I.3) and the transfer rule for the seller simply equals the sum of all the buyer transfers, i.e. $\sum_j \tau_j^S(c, \mathbf{v})$. It is shown that these transfers make sure the agents' equilibrium expected payoffs are equivalent to those attained under the truthful type-telling equilibrium of the seller-optimal DRM.

These mechanisms implement the seller-optimal DRM by essentially replicating the informed seller's optimal mechanism with reserve prices. Namely, after buyers bid valuations and the seller submits cost, the submitted cost determines the buyer-specific reserve prices for every buyer. The winner then is the highest virtual valuation buyer subject to the discriminatory reserve prices who makes a payment of the smallest value that would still make him win.

I.4 Intermediation Game with a Single Intermediary

In this section the case of the monopolist intermediary is considered. The model is kept the same way as in the intermediation game with multiple intermediaries.

Hence this game is referred to as the *intermediation game with a single monopolist intermediary*. Namely, at the first stage monopolist intermediary announces a mechanism. At the second stage the seller moves by reporting her private message and at the last stage the buyers move by reporting their private messages. Note that as there is only one intermediary, the seller has a trivial entry strategy of participating in that single mechanism. Hence a single designer case is considered, where the agents play a Bayesian communication game.

It follows from standard results such as Myerson (1981) or Myerson and Satterthwaite (1983) that in this context due to the revelation principle, it is without loss of generality to restrict attention to equilibria where the intermediary announces a feasible DRM δ and the agents report their types truthfully.

Definition I.2.
A monopolist's optimal DRM, *denoted by δ^M is a feasible DRM that maximizes the monopolist intermediary's expected payoff.*

In the context of a single intermediary, δ^M satisfying BB implies the expected profit of the monopolist intermediary is equal to the revenue generated from the agents $R(\delta)$. Hence, in this case it is redundant to describe a set of transfers specific for the intermediary. Furthermore, by Remark I.1, all the outcomes are described by the allocation rule and the constant expected payoffs for the worst types of the agents. In light of that, the following proposition summarizes the monopolist's optimal DRM:

Proposition I.3.
The monopolist's optimal DRM δ^M is characterized by the following:

1. *There exists a unique allocation rule $\mathbf{Q}^M = (Q_1^M, \ldots, Q_n^M)$ where for any vector of truthful type reports each term Q_j^M is equal to:*

$$Q_j^M(c, \mathbf{v}) = \begin{cases} 1 & \textit{if } \psi_j(v_j) > \max\left\{\psi_s(c), \max_{k \neq j}\{\psi_k(v_k)\}\right\} \\ 0 & \textit{o/w} \end{cases} \tag{I.4}$$

2. *Agents' expected payoffs at corresponding endpoints satisfy $U_s^M(1) = U_j^M(0) = 0$ for all j.*

Remark I.3.
The unique equilibrium outcomes of the intermediation game with a single monopolist intermediary are summarized as follows:

1. *The implemented allocation rule is* $\mathbf{Q}^M$.
2. *The seller's expected payoffs satisfy* $U_s^M(c) = \int_c^1 q_s^M(x)dx$ *for all* c.
3. *For every buyer* j, *their expected payoffs satisfy* $U_j^M(v_j) = \int_0^{v_j} q_j^M(y_j)dy_j$ *for all* v_j.
4. *Intermediary receives strictly positive expected profits* $U^M = R(\delta^M)$.

Similar to the seller-optimal DRM δ^*, the monopolist's optimal DRM δ^M also has a unique allocation which is dominant-strategy IC and the agents' expected payoffs are uniquely defined. Furthermore, δ^M can also be implemented as an auction that awards the object to the highest virtual valuation buyer subject to reservation values.

The main difference between the allocation rules $\mathbf{Q}^*$ and $\mathbf{Q}^M$ is that in the monopolist's optimal DRM, the buyers are subject to a different set of reservation values, which are the unique solutions to $\psi_j\big(r_j^M(c)\big) = \min\{\psi_s(c), 1\}$ for all j and c. The intuition is simple. The monopolist intermediary tries to screen the agents in order to maximize its expected profits. The marginal revenue from allocation good to buyer j is given by $\psi_j(v_j)$ while the marginal cost is $\psi_s(c)$. Hence its expected profits achieve maximum when the object is awarded to highest marginal revenue buyer only if it is higher than the marginal cost.

These reservation values, however make the allocation rule responsive to the seller's type distribution. Hence unlike in the seller-optimal DRM, the allocation rule in the monopolist's optimal DRM is varying in seller's underlying distribution F. This follows from the previous point that the optimal screening requires assessment of the marginal costs which are changing in F.

I.5 Welfare Comparison

In this section, welfare of equilibrium outcomes are compared from an efficiency point of view. I start with an analysis of outcomes under the two regimes; competition and monopoly.

I.5.1 Competition versus Monopoly

In light of Theorem I.1 and Remark I.3, the unique equilibrium outcomes of the intermediation game with multiple intermediaries and single monopolist intermediary implement allocation rules $\mathbf{Q}^*$ and $\mathbf{Q}^M$, respectively. Both of these award the object to the highest virtual valuation buyer, but they differ in the buyer specific reservation values. Denoting these reservation values by $r_j^*(c)$ for the competition case and by $r_j^M(c)$ for the monopoly case, for each buyer j they are the unique solutions to:

$$\psi_j(r_j^*(c)) = c, \qquad \psi_j\big(r_j^M(c)\big) = \min\{\psi_s(c), 1\} \qquad \text{for all } c$$

Firstly observe that $\psi_s(c)$ is strictly increasing and satisfies $\psi_s(c) > c$ for all $c \in (0, 1]$. Secondly, $\psi_s(c^M) = 1$ at some interior value $c^M \in (0, 1)$. Then it can be seen that buyer j's reservation values under the two regimes equal each other only at the end points, and for all interior costs $r_j^*(c) < r_j^M(c)$. Furthermore, from the definition of ψ_j's it can be seen that $r_j^*(c) > c$ for all $c \in [0, 1)$. These imply that for any given type vector $(c, \mathbf{v})$, the trade probabilities satisfy $Q_j^*(c, \mathbf{v}) \geq Q_j^M(c, \mathbf{v})$ for all j. To be more precise, letting buyer k have the highest virtual valuation, the inequalities are strict if $r_k^*(c) < \psi_k(v_k) < r_k^M(c)$. As virtual valuation of a buyer is less than the actual valuation and reservation values are above seller's actual costs, such an increase in trade probability translates into an improvement in allocative efficiency.

The point is that in the equilibrium under competition, intermediaries choose mechanisms that award the object to buyers on the basis of virtual valuations, which represent the marginal revenues from trade. The equilibrium DRM of the monopolist also chooses the winning buyer using the same criteria. Where they differ, however, is their evaluation of the marginal costs. The profit maximizing monopolist evaluates marginal cost as the combination of the actual cost and costs incurred by the information rents. In the other case, however, competition leads an intermediary to transfer all of its generated profits to the seller. This results in the intermediaries internalizing the information rents and consequently setting lower reservation values in equilibrium. This argument works under the IPV case, because the lack of cross-subsidization brings along the incentive compatibility of these optimal reservation values.

Next consider the ex-ante expected surplus comparison. This is an important measure from an allocative efficiency point of view, because it equals the sum of ex-ante expected payoffs for all players. Given an equilibrium of the intermediation game and its payoff equivalent feasible DRM δ, the ex-ante expected total gains from trade is denoted by $W(\delta)$ and is defined as follows:

$$W(\delta) = \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c, \mathbf{v})\big[v_j - c\big]\, dG(\mathbf{v})\, dF(c)$$

When comparing the ex-ante total surplus generated at the equilibrium in the case of competition versus monopoly, it is seen that competition strictly increases the size of the pie. This follows from the fact that the previously stated inequalities highlighting the allocative efficiency increases hold strictly over a positive measure of types.

The monotonic relationship between the allocation rules also makes it possible to compare the agents' payoffs. Given two expected payoff schedules $U_s(c)$ and $\acute{U}_s(c)$, the schedule $U_s(c)$ *Pareto dominates* $\acute{U}_s(c)$ if it is the case that $U_s(c) \geq \acute{U}_s(c)$ for all c. The analogous definition holds for the buyers. Then, the following corollary summarizes the welfare comparison results:

Corollary I.2.
Comparing the equilibrium outcomes of the intermediation game under the case of multiple competing intermediaries versus the case of a single monopolist intermediary yields:

1. $Q_j^*(c, \mathbf{v}) \geq Q_j^M(c, \mathbf{v})$ *for all* $(c, \mathbf{v}) \in C \times \mathbf{V}$.
2. $W(\delta^*) > W(\delta^M)$.
3. $U_s^*(c)$ *Pareto dominates* $U_s^M(c)$.
4. $U_j^*(v_j)$ *Pareto dominates* $U_j^M(v_j)$ *for all* j.
5. $U^M > U^{i,*} = 0$.

All agents are better off under competition. The size of the pie is larger and allocative efficiency is improved. The monopolist intermediary receives positive expected profits while under competition each intermediary receives zero expected profits.

I.5.2 Competition versus Social Planner

Competition improves welfare in terms of allocative efficiency, however it does not implement first-best allocation which would require the object to be awarded to the highest valuation buyer whenever it is higher than the cost of the seller. There are two sources of ex-post inefficiencies; not allocating the good to a buyer who has higher valuation than the cost due to the reservation values and not allocating the good to the highest valuation buyer due to evaluating the "winning buyer" on the basis of virtual valuations.[20] It is well known that there is no feasible

[20]The second point is not an issue if the buyers have symmetric distributions, as the virtual valuations are derived using the same distributions.

mechanism that achieves ex-post efficiency in these trade situations.[21] As Krishna and Perry (1998) alternatively states, the VCG mechanisms in these scenarios run strict losses. Given the first-best outcomes are infeasible, the second-best outcomes are considered as a benchmark.

Maximizing Ex-ante Surplus

The second-best outcomes are characterized as those outcomes that are implemented by a benevolent social planner who aims to maximize ex-ante total surplus. Consider the intermediation game with a single seller, n-many buyers and m-many intermediaries where m can be any integer. Hence all cases are covered; monopolist intermediary and competing intermediaries.[22] By the revelation principle, it is without loss of generality to restrict attention to the social planner announcing a feasible DRM and the players reporting their types truthfully.[23]

Definition I.3.
A constrained-efficient DRM *is a feasible DRM that maximizes the ex-ante expected gains from trade.*

Before describing the constrained-efficient DRM, some useful notation will be introduced. For any parameter $\alpha \in [0,1]$, let the *α-weighted virtual valuations* be defined by:

$$\psi_s(c,\alpha) = c + \alpha\tfrac{F(c)}{f(c)} \qquad\qquad \psi_j(v_j,\alpha) = v_j - \alpha\tfrac{1-G_j(v_j)}{g_j(v_j)} \qquad \forall j$$

Given the monotone hazard rate properties, the following observations hold for all $c, v, \alpha \in [0,1]$:

$$\begin{aligned} \tfrac{\partial\psi_s(c,\alpha)}{\partial c} &= 1 + \alpha\tfrac{d}{dc}\tfrac{F(c)}{f(c)} > 0 & \qquad \tfrac{\partial c_s(c,\alpha)}{\partial\alpha} &= \tfrac{F(c)}{f(c)} > 0 \\ \tfrac{\partial\psi_j(v_j,\alpha)}{\partial v_j} &= 1 - \alpha\tfrac{d}{dv_j}\tfrac{1-G_j(v_j)}{g_j(v_j)} > 0 & \qquad \tfrac{\partial\psi_j(v_j;\alpha)}{\partial\alpha} &= -\tfrac{1-G_j(v_j)}{g_j(v_j)} < 0 \end{aligned}$$

Proposition I.4.
The constrained-efficient DRM, denoted by δ^E is characterized by the following:

[21] For example Myerson and Satterthwaite (1983) have shown the impossibility for the case of bilateral trade. Williams (1999) establishes the same results for more general trade situations with multiple buyers and sellers.

[22] Following results also hold when there are no intermediaries.

[23] As in the case of multiple intermediaries, it is only the agents who have private information and thus report types.

1. *There exists a unique allocation rule* $\mathbf{Q}^E = (Q_1^E, \ldots, Q_n^E)$ *where for any vector of truthful type reports, each term* Q_j^E *is equal to:*

$$Q_j^E(c, \mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j, \alpha) \geq \max\left\{\psi_s(c, \alpha), \max_{k \neq j}\{\psi_k(v_k, \alpha)\}\right\} \\ 0 & \text{o/w} \end{cases} \quad \forall j \tag{I.5}$$

where α *is a constant that satisfies* $\alpha \in (0, 1)$.

2. *Agents' expected payoffs at corresponding endpoints satisfy* $U_s^E(1) = U_j^E(0) = 0$ *for all* j.

3. *The expected payoffs to intermediaries satisfy* $U^{i,E} = 0$ *for all* i.

The proof closely follows Theorem 2 from Myerson and Satterthwaite (1983), where the authors derive the constrained-efficient DRM in the context of bilateral trade. The following corollary can be stated:

Corollary I.3.
Comparing the equilibrium outcomes of the intermediation game with multiple intermediaries to the social planner's second-best outcomes implies that:

1. $W(\delta^E) > W(\delta^*)$.

2. $U^{i,*} = U^{i,E} = 0$ *for all* i.

3. $\int_0^1 U_s^*(c)dF(c) > \int_0^1 U_s^E(c)dF(c)$.

4. $\sum_{j=1}^n \int_0^1 U_j^E(v_j)dG_j(v_j) > \sum_{j=1}^n \int_0^1 U_j^*(v_j)dG_j(v_j)$.

When comparing the equilibrium outcomes from social planner and the equilibrium of intermediation game with competing intermediaries, it can be seen that in both cases the intermediaries receive zero expected profits. Secondly, $W(\delta^E) > W(\delta^*)$ which follows from the fact that the social planner maximizes the ex-ante total surplus by implementing allocation rule $\mathbf{Q}^E$ that is different than $\mathbf{Q}^*$. In other words, the social planner increases the total expected gains from trade. By a similar argument, it can be deduced that the seller's ex-ante expected payoff is higher under competition than under a social planner. In turn this means that the buyers are collectively better off under the social planner, because the intermediaries make zero expected profits and the total gains from trade is strictly higher.

A comparison of allocative efficiency proves to be difficult for the general case. The main problem is that the constrained-efficient allocation rule $\mathbf{Q}^E$ is defined with parameter α, which depends on the underlying distributions and has no analytic solution. Nevertheless, an initial inspection suggests that in the constrained-efficient allocation rule $\mathbf{Q}^E$, the object is awarded to the highest α-weighted virtual valuation buyer subject to reservation values $r_j^E(c)$'s which are the unique solutions to $\psi_j\big(r_j^E(c), \alpha\big) = \psi_s(c, \alpha)$ for all c. Since the parameter satisfies $\alpha \in (0, 1)$, it follows that $r_j^E(0) < r_j^*(0)$. Furthermore, for every j there exists some interior cost $\bar{c}_j$ such that $r_j^E(\bar{c}_j) = 1 > r_j^*(\bar{c}_j)$. Hence the relationship between the sets of reservation values $r_j^*(c)$'s and $r_j^E(c)$'s are nonmonotonic. Another complication is that the "winning" buyer has the highest α-weighted virtual valuation under the social planner while it is the highest virtual valuation buyer[24] under the equilibrium of the intermediation game with competing intermediaries, and these winners may be two different buyers.

To illustrate the complications, consider a vector of types $(c, \mathbf{v})$. If c is low enough, it might be the case that constrained-efficient allocation transacts the good to a buyer improving allocative efficiency relative to the equilibrium allocation rule under competition. However, for higher c it could be the case that $\mathbf{Q}^*$ awards to a buyer who indeed has the highest true valuation while the $\mathbf{Q}^E$ awards it some other buyer, in turn harming allocative efficiency. In order to provide a more detailed discussion of the welfare comparisons, I will concentrate on the case of symmetric distributions for the buyers.

Homogeneous Buyers

In the rest of this section, assume that $G_j = G$ for all j. Then the virtual valuations for all buyers are the same, which shall be denoted by $\psi_b(v_j)$. Similarly, the α-weighted virtual valuation for some constant $\alpha \in (0, 1)$ is denoted by $\psi_b(v_j, \alpha)$.

Under symmetric buyers, the equilibrium allocation rules $\mathbf{Q}^*$ and $\mathbf{Q}^E$ have very simple interpretations. Both award the object to the highest value buyer subject to the reservation values $r^*(c)$ and $r^E(c)$ which are not buyer specific and are the unique solutions to:[25]

$$\psi_b\big(r^*(c)\big) = c, \qquad \psi_b\big(r^E(c), \alpha\big) = \min\{\psi_s(c, \alpha), 1\} \qquad \text{for all } c$$

[24]Equivalently $\alpha = 1$.

[25]The allocation rule $\mathbf{Q}^M$ from the monopolist's optimal DRM also awards to the highest valuation buyer subject to its reservation value $r^M(c)$ which solve $\psi\big(r^M(c)\big) = \min\{\psi_s(c), 1\}$ for every c.

Observe that both sets of reservation values are increasing functions. Furthermore, $r^E(0) < r^*(0)$ and there exists some interior cost $\bar{c}^E$ such that $r^E(\bar{c}^E) = 1 > r^*(\bar{c}^E)$. Therefore, the reservation values cross once at some interior value $\hat{c}$ at which point $r^E(\hat{c}) = r^*(\hat{c})$. Then given any vector of types $(c, \mathbf{v})$, the trade probabilities for all j satisfy $Q_j^*(c, \mathbf{v}) \leq Q_j^E(c, \mathbf{v})$ for any $c \leq \hat{c}$ and $Q_j^*(c, \mathbf{v}) \geq Q_j^E(c, \mathbf{v})$ for any $c \geq \hat{c}$.

It is important to note that the results from Corollary I.3 hold. Then any discrepancy in the allocation rules arise solely from the differences in the reservation values, and therefore interpretation of the surplus differences under symmetric buyers is straight forward. The social planner sets a lower reservation value for low cost seller types which increases the allocative efficiency and expected surpluses. On the other hand the reservation value for the high cost seller types are increased, which exerts the opposite effect on efficiency. Hence from an interim perspective, changes in the expected surplus are uncertain. However, when the ex-ante surplus is considered, it can be seen that these adjustments increase expected gains from trade. From an ex-ante perspective, the seller is made worse of in expectation. However the gains in buyers' ex-ante expected payoffs compensate those losses and yield the overall efficiency improvement. The following corollary elaborates on the differences in the agents' expected payoffs arising from the changes in the reservation values:

Corollary I.4.
Comparing the equilibrium outcomes of the intermediation game with multiple intermediaries with the social planner's second-best outcomes under symmetric distribution for the buyers yields the following:

1. *$U_s^*(c)$ Pareto dominates $U_s^E(c)$ iff average of seller's trade probabilities is higher under δ^*:*

$$U_s^*(c) \geq U_s^E(c)\ \forall c \qquad \Leftrightarrow \qquad \int_0^1 q_s^*(c)dc \geq \int_0^1 q_s^E(c)dc \tag{I.6}$$

2. *$U_j^E(v_j)$ Pareto dominates $U_j^*(v_j)$ iff average of buyer j's trade probabilities is higher under δ^E:*

$$U_j^E(v_j) \geq U_j^*(v_j)\ \forall v_j \qquad \Leftrightarrow \qquad \int_0^1 q_j^E(v_j)dv_j \geq \int_0^1 q_j^*(v_j)dv_j \tag{I.7}$$

Corollary I.4 highlights the changes in the expected payoffs of the agents. The losses in seller's expected payoffs under the social planner could be to the extreme that the seller's payoff schedule is Pareto dominated by the schedule from the equilibrium of the intermediation game with multiple intermediaries. A necessary and sufficient condition is provided, which relates the reservation value alterations by the social planner with the resulting changes in expected trade probabilities across the different types of seller. It has been shown that high cost types trade less often, while opposite is true for the low cost types. If on average the increments are less than the reductions in the trade probabilities,[26] then seller-optimal DRM yields a Pareto dominating payoff schedule for the seller. An analogous condition holds for each buyer, but from the other perspective as the social planner's constrained-efficient DRM improves the ex-ante expected payoff of the buyers.

I.6 Conclusion

A model of competition among intermediaries in a trade situation has been analyzed. I showed that the equilibrium exists and that all equilibria exhibit the same unique outcomes. Namely, the intermediaries make 0 expected profits and the agents' expected payoffs are equivalent to those that are accrued under the informed seller's revenue maximizing auction.

I also characterized the unique equilibrium outcomes of the intermediation game with a single monopolist intermediary. Comparing welfare from an allocative efficiency point of view, I showed that the outcomes under the case of competition are unambiguously better than those under the monopoly. In particular, the allocative efficiency is weakly increased for every possible type realization which leads to a higher ex-ante total surplus and Pareto improvements in the agents' expected payoff schedules.

Characterizing the second-best outcomes from the social planner's ex-ante surplus maximization problem, I compared them with the equilibrium outcomes from the intermediation game in the case of competition. The analysis highlighted that in the general case of heterogeneous distributions for the buyers, the planner's equilibrium achieves higher total surplus by implementing an allocation rule that differs both in the reservation values and the evaluation of the winning buyer. Restricting attention to the case of homogeneous buyers where the discrepancies in allocation are attributed solely to differences in the reservation values, I showed that the efficiency improvement is achieved by trading more often for low cost types at the expense of more restricted trade for high cost types.

There are several directions for future research. Firstly, I concentrated on in-

[26] Note that the condition looks at the arithmetic average of the expected trade probabilities, which is different than the ex-ante expected trade probability.

dependent private valuations. However, it would be interesting to consider correlated valuations and common values at the other extreme. Secondly, I assumed that all intermediaries are equally uninformed about the private information of the agents. It would be interesting to see if similar results are attained when one or some of the intermediaries have superior knowledge about agents' private information than others. Such a setup could be interpreted as experience on the market that comes with incumbency. Hence, one may also discuss robustness of these outcomes to entry of new, inexperienced intermediaries.

Finally the environment contained a single indivisible object. It would also be interesting to consider scenarios where the seller has more objects for sale. In the IPV framework, if the seller is required to "single-home",[27] then similar results may be expected. Allowing the seller to multi-home, on the other hand, could potentially alter the proof methods and hence requires further investigation.

[27] The seller has to choose a single intermediary to mediate the sale of all her goods.

CHAPTER II

Benefits of Intermediation under Asymmetric Information

II.1 Introduction

The aim of this paper is to analyze the role of intermediation in bilateral trade problems where the parties have the ability to bargain directly with each other bypassing the use of an intermediary. In particular, the emphasis will be on the impact of intermediation on allocative efficiency as measured by expected gains generated from trade. The question of focus can thus be stated as follows:

Can the presence of an intermediary lead to outcomes that are strictly more efficient than the outcomes that are attainable in its absence?

In order to answer this question, I consider a bargaining situation between a seller and a buyer over the allocation of a single indivisible good. In this bilateral trade problem, I assume that the private information of each bargaining party about their valuation for the object are drawn from binary type spaces. However, I allow for interdepence between the valuations. Namely, the buyer's valuations are assumed to be higher whenever the seller has a higher valuation for the object. Hence this situation may be interpreted as a simplified bargaining scenario with two-sided asymmetric information subject to a lemon problem à la Akerlof (1970).

In Section II.2, I analyze the informed seller's signaling game, denoted by Γ, where the two parties can trade at take-it-or-leave-it (henceforth TIOLI) price offers that are announced by the seller. I characterize the set of pure strategy sequential equilibria of this game in Propositions II.1,II.2 and II.3. I find that there are multiple equilibrium outcomes for any parameter configuration.[1] In anticipation of the subsequent analysis, I select and summarize the most efficient equilibria in Corollary II.2. It is important to note that, I show there are a range of parameter values, where there is inefficient underselling even in the most efficient equilibria. Lastly, in Corollary II.3 I characterize the equilibria that yield the highest expected payoffs to the seller.

Next, in Section II.3, I turn the attention to a variant of the previous game with an additional player; a third-party intermediary who only knows the prior probabilities of the agents' types, but not their realizations. In this game, denoted by Γ^i, the intermediary designs a menu of price pairs, where each pair describes the transfer to be made to the seller and the price charged to the buyer, respectively. The seller, observing the menu decides whether to trade via the intermediary or to choose the outside option of direct trade, where he can make a TIOLI offer to the buyer.[2]

Signaling elements prevail in the game Γ^i with intermediary, which in turn leads to multiplicities similar to the previous game Γ. However, the main aim in this section is to establish a result on the existence of desired equilibria as opposed to a full characterization. Namely, as previously mentioned, I aim to see whether there may exist an equilibrium of Γ^i that is strictly more efficient than the most efficient equilibrium of Γ. Furthermore, if there are more efficient equilibria, then I also want to describe the necessary and sufficient conditions for their existence. In this spirit, I restrict the rest of the analysis to the cases where there is inefficient underselling in the informed seller's signaling game Γ as characterized in Corollary II.2, and examine the existence of *efficient intermediated equilibria* in Γ^i, which satisfy strict allocative efficiency improvement property.

Let me digress and discuss an important feature of the game Γ^i. In this paper, analysis focuses on a bilateral trade problem, where the parties have the ability to bargain on their own in a direct manner without intermediation, albeit on seller's terms (seller makes TIOLI offers). This ability in turn enables the seller to guarantee himself minimum expected payoffs that depend on his type. The intermediary, on the other hand, tries to maximize its profits while having to respect the seller's endogenously determined and type-dependent outside options. Hence this situation may be interpreted as the intermediary competing with the seller.

The intermediary's competition with the seller is an important aspect of the

[1]This is a common feature in dynamic Bayesian games of incomplete information with signaling generally. For example consider the multiplicities of equilibrium outcomes in Spence (1973).

[2]For the rest of the paper, I use the male pronoun for the seller, the female pronoun for the buyer, and the neuter pronoun for the intermediary.

setup in this paper, which makes the analysis interesting from two angles. Firstly, the literature on bilateral trade has not paid any attention to analyzing the impact of competition on intermediation. To that extent, Chapter I of this thesis has considered the impact of competition that arises when there are multiple intermediaries who try to facilitate trade. Referring to that type of competition as being *external*, the current paper considers *internal competition* that is posed by one of the bargaining parties. Other than novelty, it is also the relevance of the setup that makes the analysis interesting. Considering situations where matching or search technologies do not have an impact on the bargaining situation, e.g. because the bargaining parties know the identities of each other, then it is reasonable to assume that they have the ability to negotiate with each other directly.

I first establish in Lemma II.6, that in any desired equilibrium, trade has to go through the intermediary as the seller types would choose the intermediary rather than engaging in direct trade outside option. I show this by arguing that, if at least one seller type chooses outside option to trade directly, then the strict efficiency improvement requires either the intermediary or the buyer to make strict losses, which contradicts the efficient intermediated equilibrium assumption. This observation suggests that the intermediary plays an active role in the efficiency improvements.

The main result of the paper is presented in Theorem II.1, where I provide a necessary and sufficient condition for the existence of efficient intermediated equilibria of Γ^i. The necessity part of the proof follows from nonnegativity of the sum of expected payoffs for the intermediary and the buyer in an equilibrium, while for the sufficiency part I pursue a constructive proof by showing the existence of an efficient intermediated equilibrium under the described condition.

There are several points to highlight with regards to the importance of the result in Theorem II.1. Firstly, due to the way I define the efficient intermediated equilibria, the efficiency improvements occur in ex-post sense. In other words, for all type pairs the implemented trade probabilities in the efficient intermediated equilibrium of Γ^i are weakly greater than their counterparts under the equilibria of the game Γ.

Secondly, the efficiency improvements arise in the presence of the intermediary, which in turn suggests that there are benefits to be gained from intermediation. The existing explanations about the importance and benefits of intermediation include quality certification, expertise provision, providing matching or other infrastructures that facilitate trade.[3] In this paper, the intermediary is the least

[3]See for example Spulber (1999) and Salanie (2011) for an overview on the roles of interme-

informed player in the game Γ^i and the agents have the ability to directly trade with each other. Therefore, the explanations related to expertise provision, certification, or acting as a platform to match the two sides do not apply. Alternatively, in this setup the intermediary overtakes the role of a coordination point for monetary transactions (cross-subsidization) and information flow (separation of seller types), which in turn improves the trade outcomes.

Lastly, the efficiency is strictly improved by a profit maximizing intermediary. The more interesting part in this observation is the fact that even though the intermediary is a strategic player who is acting in its self-interest, its presence could lead to these improved outcomes. Therefore, it is shown that ex-post efficiency improvements in the market may be attained without the need for the intervention of a benevolent social planner.

It is worth noting that timing plays an important role in the aforementioned results. To be more precise, in the informed seller's price announcement game without an intermediary, the buyer observes the TIOLI offers and evaluates her beliefs about the seller. Consequently, the seller faces ex-post constraints on his price offers. Along with the signaling difficulties arising from the lemon problem, this creates the inefficient underselling in equilibria. In the game Γ^i, on the other hand, the intermediary alleviates these efficiency losses by breaking its interim budget balance on one type, which is covered by the interim profits it makes from the other type; i.e. by cross-subsidizing.[4]

Finally, in Section II.4, I discuss several extensions to the analysis from the previous sections. Firstly, I analyze the impact of the out-of-equilibrium subgame outcomes of the direct trade outside option. Namely, I repeat the analysis for the case where in the direct trade outside option subgame the agents play the highest expected payoff yielding equilibria of Γ. I provide a revised necessary and sufficient condition for efficient intermediated equilibria to exist. The condition becomes more restrictive, as the outside option payoffs for the seller increase, making it less profitable for the intermediary to facilitate trade and consequently more difficult to achieve the efficiency improvements. Nevertheless, this shows the robustness of the previous result from Theorem II.1. Namely, that the efficiency improvements exist even in the most difficult case, when it is assumed that the seller receives the best possible expected payoffs in the direct trade outside option.

A second extension I consider is the special case of independent private values

diaries.

[4]Alternatively, if the seller had the ability to design and commit to prices ex-ante, then there could not be an equilibrium that is strictly more efficient in the presence of an intermediary. This is a key result covered in Chapter III of this thesis.

(henceforth IPV), or equivalently where the valuations for the objects do not exhibit any interdependence. I first revisit characterizing the equilibrium outcomes of Γ and show that the equilibrium outcomes are uniquely defined under the IPV case. However, the rest of the analysis leads to a negative conclusion. Namely I find that in the IPV case, the game Γ^i has no efficient intermediated equilibria.[5,6]

In the last Subsection II.4.3, I discuss another important point. Dropping the direct trade ability in the outside option subgame, I consider the intermediary's optimal mechanism design problem in Bayesian implementation, à la Myerson and Satterthwaite (1983). This is a standard screening problem that has been thoroughly addressed in optimal mechanism design literature. Applying the findings from the literature to the setup at hand, I characterize the intermediary's profit maximizing bargaining protocol. I show that there are parameter cases where this optimal mechanism implements strictly more efficient trade than the most efficient equilibrium of Γ, which is in line with the results from Jullien and Mariotti (2006). However, I point out that the condition from Theorem II.1 in this paper is not only different, but also easier to be satisfied. There are two main reasons for the differences between the two conditions. The intermediary's optimal mechanism is designed to be implemented in interim (Bayesian) sense for both agents, which is not the case for the intermediary's mechanism in the game Γ^i, i.e. menu of prices announced. The second reason relates to the absence of direct trade outside option and consequently the competition element in the optimal mechanism design problem. Although the competition creates additional constraints for the intermediary, it also curbs the monopolistic distortions. It turns out that the latter effect dominates, as the intermediary is more effective in efficiency improvements in the game Γ^i compared to the outcomes from the standard screening problem.

II.1.1 Related Literature

To the best of my knowledge, this is the first paper to model competition between an intermediary and a bargaining party in bilateral trade.

[5]This result is in line with those from Maskin and Tirole (1990), where the authors consider principal-agent relationships under private values. More specifically, in the case of quasi-linear preferences, Proposition 11 establishes that the three-stage mechanism selection game has unique equilibrium outcomes that are equivalent to the unconstrained Pareto optimum outcomes. Extending to the current model sheds light on the uniqueness of equilibria outcomes in Γ, and the inability of the intermediary to improve efficiency.

[6]In fact, the work from Maskin and Tirole (1990) does not directly apply for the seller's optimal mechanism problem in bilateral trade scenarios, because several assumptions are violated such as finiteness of types or sorting condition. Nevertheless, analogous results regarding Pareto optimality of the informed seller's revenue maximizing contract under quasilinear preferences and the IPV case have been shown in Yilankaya (1999).

The closest paper is Jullien and Mariotti (2006), where the authors consider an auction market with one seller and two buyers over the allocation of a single indivisible good that is subject to a lemon problem. Authors first analyze a game where the seller announces reserve prices and then holds a second price auction. They characterize the unique separating equilibrium outcomes of the informed seller's signaling game. Then, they consider an uninformed monopolist intermediary's optimal trading mechanism. Comparing the unique separating equilibrium of the signaling game to the equilibrium of the intermediary's screening game, authors show that the expected gains from trade may be larger under monopoly broker. Although the results about benefits from intermediation are similar, the setups are different. Namely, in this paper the intermediary faces competition from the seller due to the possibility of direct trade without the intermediary. The authors in Jullien and Mariotti (2006), however, consider the intermediary's optimal mechanism design problem in the standard Bayesian implementation sense. Therefore, the conditions under which the welfare comparisons favor the intermediary against the decentralized markets differ in the two papers.[7]

There are various strands in the literature that this work relates to. Firstly, the paper builds around the bilateral trade problem, about which a plethora of articles have been written. Starting with the seminal contributions of Myerson and Satterthwaite (1983), there are numerous papers written on efficient bilateral mechanisms, including, but not limited to Williams (1987), Krishna and Perry (1998), and Williams (1999). Similarly Myerson (1981) and Riley and Samuelson (1981) solve for the revenue maximizing optimal mechanisms.

Insofar as the TIOLI price announcement designs by the informed seller are concerned, this paper is related to the informed principal's mechanism design problem. Myerson (1983) introduces the concept of mechanism-selection by an informed principal and formulates the notion of *inscrutable mechanisms*. Another seminal paper is Maskin and Tirole (1990) where the authors consider a mechanism-selection game played over three stages (proposal, accept/reject, execute) in private value environments. Other notable contributions to informed principal's problem in private values environment include Skreta (2011) and Mylovanov and Tröger (2013). There have also been articles that concentrate on the informed principal in bilateral trade problems. Riley and Zeckhauser (1983) provides an ex-ante optimal mechanism's characterization, while Yilankaya (1999) and Tisljar (2003) consider the interim optimal mechanisms. A lot of these articles

[7]As mentioned above, a more detailed anaylsis of the comparisons between the results of this paper and Jullien and Mariotti (2006) can be found in Section II.4.3.

mentioned show the optimality of TIOLI price offers, such as Riley and Zeckhauser (1983) and Williams (1987).

The interdependence of valuations plays an important role in the analysis. To that extent, the literature has diverged on the coverage of informed principal's problem under common values. The seminal contribution is by Maskin and Tirole (1992), where the authors consider, similar to their companion paper Maskin and Tirole (1990), a mechanism-selection game player over three stages (proposal, accept/reject, execute), but this time in common values. It is shown that in these environments, the informed principal can guarantee himself what the authors name as the "RSW" allocation outcomes, where the mnemonic term "RSW" allocation refers to the zero-profit separating allocations that plays an important role in the competitive screening models from insurance markets analyzed in Rothschild and Stiglitz (1976) and Wilson (1977).

Finally, the seller's ability to engage in direct trade complicates the intermediary's price menu design problem, as the design has to respect these type-dependent outside options of the seller. To that extent, there exists a subliterature that concentrates on mechanism design problems with type-dependent outside options and endogenous participation decisions, starting with the analysis of countervailing incentives in Lewis and Sappington (1989). The papers Jehiel et al. (1996) and Jehiel et al. (1999) consider type-dependent negative externalities between buyers in trade problems. Jullien (2000) considers the optimal contract of the principal when agent's reservation utilities depend on her type. The author identifies the conditions for the contract to be bunching and separating, as well as the cases when it induces full participation of the agent. Lastly, Figueroa and Skreta (2009) and Figueroa and Skreta (2011) also consider revenue maximizing allocation mechanisms when buyer's outside options depend on their private information, and show the crucial dependence of the optimal allocations on the shape of the outside options. The main distinction in this setup with those aforementioned papers is that here, the outside options are determined by the equilibrium of a signaling game, and thus are endogenously chosen by the seller.

The remainder of the paper is structured as follows. In Section II.2, I describe the setup and analyze the signaling game in the absence of an intermediary. In Section II.3, I analyze the game with the presence of an intermediary. Lastly, in Section II.4, I discuss several extensions and provide a couple of numerical examples to illustrate the established results. All proofs are in the appendix.

II.2 Trading Game Without an Intermediary

II.2.1 Model

There are two risk-neutral agents, one seller and one buyer, denoted by s and b, respectively. The seller owns a single unit of an indivisible good which the buyer would like to buy.

The seller privately knows his valuation (henceforth cost for brevity) for this object. The seller's cost is referred as his type and is denoted it by t_s. It is assumed that t_s can take one of two values in $\{l, h\}$ where $0 \leq l < h$ and l is realized with probability $p \in (0,1)$. Similarly, the buyer privately knows her type t_b which can be low (L) or high (H) where $0 < L < H$ and the probability of L type is $q \in (0,1)$. On the other hand, the buyer's valuation for the object, denoted by $v(t_s, t_b)$, depends both on her and the seller's private information and is assumed to satisfy:

$$v(t_s, t_b) = \begin{cases} t_b & \text{if } t_s = l \\ t_b + \lambda_{t_b} & \text{if } t_s = h \end{cases}$$

for some parameters $\lambda_L, \lambda_H \geq 0$. Hence, valuation of each buyer type increases by a constant λ_{t_b} whenever the seller is of high type. It is possible to interpret this assumption as an extension of the standard lemons market with two-sided asymmetric information where the seller's cost represents the quality of the object and λ_{t_b} represents the premium of owning a higher quality object for the corresponding type of the buyer.[8]

The expected valuations conditional on buyer's type are given by $\mathbb{E}(v|t_b) = t_b + \lambda_{t_b}(1-p)$ for both $t_b \in \{L, H\}$. For the rest of the paper, it is assumed that the parameters satisfy the following:

Assumption II.1.
[A1] $l < v(l, L) < v(l, H)$ and $v(h, L) < h < v(h, H)$.
[A2] $\mathbb{E}(v|H) > h$.

Assumption II.1 has a couple of important implications. Firstly, due to **[A1]** there are uncertain gains from trade in this bilateral trade scenario, which implies that the seller has to screen the buyer. Secondly, the assumption allows me to without loss of generality normalize the parameters (apart from λ_{t_b}'s) so that low type of the seller equals $l = 0$. Finally, there also is a nontrivial signaling problem

[8] Note that $\lambda_L = \lambda_H = 0$ is the special case of independent private valuations (IPV).

for the seller as $h < \mathbb{E}(v|H)$. A high cost seller would like to signal his type to the high type buyer. Due to **[A2]**, however, there exists prices at which high cost seller would be willing to trade with the high type buyer, if buyer believes the likelihood of the seller being high type is sufficiently large.[9] This in turn gives more incentives for the low cost seller type to mimic the high cost type, further strengthening the signaling problem.

The trading game without an intermediary, denoted by Γ, can be described as follows. First, the seller puts the object for sale by announcing take-it-or-leave-it (TIOLI) price. Next, upon observing the price, buyer forms beliefs about the seller's type, and then decides whether to accept or reject to trade at that price.

II.2.2 Strategies, Beliefs and Payoffs

At the first stage of the game, the seller, after observing his type, announces a TIOLI offer denoted by m. Without loss of generality, attention is restricted to offers in the compact space $[0, H + \lambda_H]$. Hence a pure strategy for the seller is given by the mapping $m : \{l, h\} \to [0, H + \lambda_H]$. It will be convenient to use the shorthand notations m_l and m_h for the respective strategies of the two types of the seller.

In the second stage, the buyer, knowing her type, observes the price announcement m and decides whether to trade or not at that price, denoted by $d \in \{A, R\}$. Hence a pure strategy for the buyer is given by the mapping $d : \{L, H\} \times [0, H + \lambda_H] \to \{A, R\}$. Similarly, for any price announcement $m \in [0, H + \lambda_H]$, denote the strategies of the respective buyer types by $d_L(m)$ and $d_H(m)$.

Next I define the beliefs. Both types of the seller believe that the buyer can be of type L with probability q. On the other hand, the buyer forms beliefs $\pi(l|t_b, m)$ about the likelihood of facing a type l seller knowing her own type t_b and seller's announced price m. Abusing notation, I will denote these beliefs by $\pi_L(m)$ and $\pi_H(m)$.

Now, the payoffs can be defined. Given a price announcement m by the seller

[9] Note that beliefs close to the priors would be sufficiently high.

and the buyer's decision d, the payoffs to the agents are equal to:

$$U_{t_b}(m,d) = U(m,d|t_b) = \begin{cases} t_b + \lambda_{t_b}\big(1-\pi_{t_b}(m)\big) - m & \text{if } d = A \\ 0 & \text{if } d = R \end{cases} \quad \text{for } t_b \in \{L,H\}$$

$$U_{t_s}(m,d) = U(m,d|t_s) = \begin{cases} m - t_s & \text{if } d = A \\ 0 & \text{if } d = R \end{cases} \quad \text{for } t_s \in \{l,h\}$$

Define the expected probability of trade $Q(m) = q\mathbb{I}\big(d_L(m) = A\big) + (1-q)\mathbb{I}\big(d_H(m) = A\big)$. Then the expected payoff of the seller from announcing m at the first stage is equal to:

$$U_{t_s}(m) = U(m|t_s) = (m - t_s)Q(m) \qquad \text{for } t_s \in \{l,h\}$$

II.2.3 Characterization of Equilibria

In the game Γ, a pure strategy sequential equilibrium, or equilibrium for short, is the collection of strategies $\{(m_l^*, m_h^*), (d_L^*, d_H^*)\}$ and beliefs (π_L^*, π_H^*) where the strategies satisfy sequential rationality and the beliefs are consistent. More specifically, the strategies have to satisfy:

$$\begin{aligned} U_{t_b}(m, d_{t_b}^*) &\geq U_{t_b}(m,d) && \forall\, m, d \text{ and } t_b \\ U_{t_s}(m_{t_s}^*) &\geq U_{t_s}(m) && \forall\, m \text{ and } t_s \end{aligned}$$

Consistency of the beliefs require that the beliefs are defined according to the Bayes' rule at the prices chosen on equilibrium; $\pi_{t_b}^*(m) = \frac{p\mathbb{I}(m=m_l^*)}{\sum_{t_s}\mathbb{P}(t_s)\mathbb{I}(m=m_{t_s}^*)}$ for $m \in \{m_l^*, m_h^*\}$ and both buyer types. At out-of-equilibrium prices $m \in [0, H+\lambda_H] \setminus \{m_l^*, m_h^*\}$, the beliefs are equal to the limits of the belief sequences generated by a chosen sequence of convergent totally mixed strategy sequences.

Before the set of equilibria are characterized, start with a couple of useful preliminary results.

Lemma II.1.
In any equilibrium of the game Γ, the beliefs satisfy $\pi_L^*(m) = \pi_H^*(m) = \pi^*(m)$ for all m.

Lemma II.1 suggests that both types of the buyer hold the same beliefs on and off the equilibrium paths. This result arises from the fact that the types t_s and t_b are independently distributed, and hence buyer's private information has no impact in her consistent belief assignments. An immediate consequence of Lemma II.1 is the following corollary.

Corollary II.1.
Given an equilibrium of Γ, if at some price m the low type buyer accepts the offer, then so does the high type buyer.

Corollary II.1 simply follows from the sequential rationality of the buyer's optimal strategy. Finally, I provide the following lemma related to the out-of-equilibrium beliefs and behavior of the buyer which are crucial for the characterization of equilibria in this game.

Lemma II.2.
In any equilibrium, it is without loss of generality to assume that following any deviation price announcement $m' \neq m^*_{t_s}$, the buyer types believe that the seller is of low type; $\pi^*(m') = 1$. Furthermore, the optimal strategy $d^*_{t_b}(m')$ for each buyer type at out-of-equilibrium prices is to accept the offer whenever $m' \leq t_b$ and reject otherwise.

The game Γ is essentially a signaling game. Hence the equilibrium conditions are highly reliant on the out-of-equilibrium beliefs and strategies of the message receiver, i.e. the buyer. Lemma II.2 is very useful for the rest of the section, as it provides a clear structure for the out-of-equilibrium scenarios. Now I will characterize the equilibria of Γ. There may be two kinds of pure strategy sequential equilibria; pooling or separating. I will start by characterizing the set of pooling equilibria.

Pooling Equilibria

In a pooling equilibrium, both types of the seller announce the same price, hence $m^*_l = m^*_h = m^P$.[10] Consequently, the buyer's beliefs on the equilibrium price are not updated from the prior probabilities yielding $\pi^P(m^P) = p$. Below I characterize the set of pooling equilibria:

Proposition II.1.
There exists a pooling equilibrium where only the high type buyer trades at price m^P satisfying $\max\{\frac{L}{1-q}, h, H\} \leq m^P \leq \mathbb{E}(v|H)$, whenever

$$q \leq \frac{\mathbb{E}(v|H) - L}{\mathbb{E}(v|H)} = q^P \tag{II.1}$$

[10] I use the superscript P to refer to a pooling equilibrium.

Several remarks are in order. Firstly, only the high type buyer trading suggests that these equilibria are allocatively inefficient, as there is no trade between the low types of the agents, even though the low type buyer values the object more when the seller is low cost. Secondly, the threshold probability q^P below which there exists a pooling equilibrium is strictly between 0 and 1. The condition suggests that there is a pooling equilibrium whenever the probability of low type buyer is sufficiently small or the high type buyer is sufficiently high. The intuition arises from standard screening incentives of the low cost seller; that when the probability of high type buyer is large enough, l type seller prefers to trade exclusively with H type buyers. The difference here, however, is that by pooling with the high cost seller, low cost seller benefits from the buyer's valuation premium, which creates additional incentives for the low cost seller to pool with the high cost seller. As a result, the threshold q^P is larger than what it would be, if there were no premiums from consuming object sold by h-type.[11]

In a pooling equilibrium, each seller type t_s receives expected payoffs that are equal to $U_{t_s}(m^P) = (m^P - t_s)(1-q)$. It will be useful to define $U^P_{t_s}$ as the highest expected payoff to the seller types from a pooling equilibrium which equals $U^P_{t_s} = (\mathbb{E}(v|H) - t_s)(1-q)$.

Separating Equilibria

Next consider separating equilibria where the equilibrium prices are different, i.e. $m_l^* \neq m_h^*$, which suggests that the buyer correctly identifies the seller types, i.e. $\pi^*(m_l^*) = 1$ and $\pi^*(m_h^*) = 0$. Before characterizing these equilibria, the following preliminary result is established.

Lemma II.3.
In any separating equilibrium, the low type seller trades strictly more often than the high type seller. Consequently the low type seller announces a price that is strictly less than the high type seller.

The result follows from the sequential rationality constraints for the seller types under separation. An important implication of Lemma II.3 is that there are three possible cases that a buyer can play in a separating equilibrium; both buyer types accept the low price but only high type buyer accepts the high price, both buyer types only accept the low price, or only the high type buyer accepts the low price and low type rejects both prices. Observe that in the latter two case, the high

[11] If $\lambda_H = 0$ so $\mathbb{E}(v|H) = H$, then the threshold for pooling would be $\frac{H-L}{H}$. Similarly, if the buyer knew the seller's type, then seller would screen out low type buyers whenever $q < \frac{H-L}{H}$.

type seller does not trade in equilibrium. The first case will be considered separately from the latter two cases, and based on allocative efficiency properties these cases will be referred to as the *efficient* and *inefficient* separating equilibria, respectively.[12]

Efficient Separating Equilibria: In these equilibria, the low type seller trades with both buyer types and the high type seller trades only with the high type buyer. Under assumption **[A1]**, these trades achieve first best outcomes in terms of allocative efficiency. The following result characterizes S^e, the efficient separating equilibria:

Proposition II.2.
There exists an efficient separating equilibrium where the low cost seller trades with both types of buyer at price $m_l^{S^e} = L$*, while the high cost seller trades only with the high type buyer at price* $m_h^{S^e}$ *satisfying* $\max\{h, H\} \leq m_h^{S^e} \leq \min\{\frac{L}{1-q}, H + \lambda_H\}$*, whenever*

$$q \geq \max\left\{\frac{h-L}{h}, \frac{H-L}{H}\right\} = q^{S^e} \tag{II.2}$$

Observe that if $q^{S^e} \leq q \leq \frac{H+\lambda_H-L}{H+\lambda_H}$, then the maximum price for the high type seller is capped at $\frac{L}{1-q}$ which is less than $H + \lambda_H$. This is due to the incentive compatibility constraints of the seller for the separation to occur, because any higher price for the high cost seller would violate the sequential rationality of low cost seller's price announcement.

Inefficient Separating Equilibria: Lemma II.3 established three possible cases for separating equilibria, one of which are the efficient separating equilibria S^e described above. The other two cases fall under the category of inefficient separating equilibria S^i, because in both of those remaining cases the high type seller does not trade in equilibrium. These cases arise when the high type seller's price makes the high type buyer (who is the only eligible buyer type to generate gains from trade with the high type seller) indifferent between trading or not and she breaks her indifference in favor of no trade. In order for such equilibria to exist however, it has to be the case that the high type seller should not be able to trade for any beliefs of the buyer. However, when $h > H$ it is profitable for the high type seller to trade with the high type buyer, even if she thinks that the seller is of low type. Therefore these equilibria exist if and only if $h \geq H$. The following result characterizes S^i:

[12]It will be convenient to use superscripts S^e and S^i for the efficient and inefficient separating equilibria, respectively.

Proposition II.3.
The inefficient separating equilibrium exist if and only if $h \geq H$. In these equilibria, the high cost seller announces $m_h^{S^i} = H + \lambda_H$ and does not trade as both buyer types reject the offer. The equilibrium price announcement of the low cost seller and the decisions of the buyer types, on the other hand, are summarized below:

$$(m_l^{S^i}, d_L^{S^i}(m_l^{S^i}), d_H^{S^i}(m_l^{S^i})) = \begin{cases} (L, A, A) & \textit{if } q \geq q^{S^i} = \frac{H-L}{H} \\ (H, R, A) & \textit{if } q \leq q^{S^i} \end{cases} \tag{II.3}$$

Observe that whenever $h \geq H$, then there always exists an inefficient separating equilibrium for all probabilities q. On the other hand, the magnitude of q impacts low cost seller's price announcement and consequently his equilibrium trade probabilities. Namely, if q is low enough then the low cost seller would rather trade only with the high type buyer.

Summarizing Equilibria: As established in the previous three propositions, there are multiple equilibria of the game Γ for any given parameter configuration. The following two figures summarize the characterization results.

Figure II.1: When $H > h$

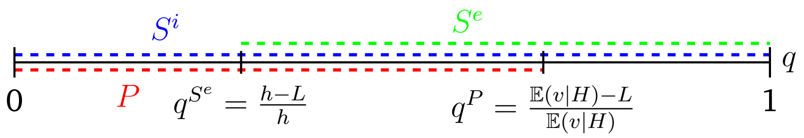

Figure II.2: When $H \leq h$

Figures II.1 and II.2 summarize the pure strategy equilibria of Γ under different parameter configurations. One can observe that no matter what the relative values h and H are, there always exists a pure strategy equilibrium in this game. Another observation is that there are ranges of parameters where multiple equilibria exist.

The focus in the subsequent sections will be on efficiency properties of equilibria, where efficiency is measured in terms of expected gains from trade generated in equilibrium. In that regard, it will be useful to establish the following corollary, which characterizes the most efficient equilibria of the informed seller's signaling game Γ:

Corollary II.2.
The most efficient equilibria of Γ are:

$$i)\ H > h \Rightarrow \begin{cases} S^e & \text{if } q \geq q^{S^e} = \frac{H-L}{H} \\ P & \text{o/w} \end{cases}$$

$$ii)\ H \leq h \Rightarrow \begin{cases} S^e & \text{if } q \geq q^{S^e} = \frac{h-L}{h} \\ S^i \text{ (with } m_l^{S^i} = L) & \text{if } \frac{h-L}{h} > q \geq \min\left\{\frac{h-L}{h}, \frac{(1-p)(H+\lambda_H-h)}{(1-p)(H+\lambda_H-h)+pL}\right\} \\ P & \text{o/w} \end{cases}$$

Observe that when $H \leq h$, there might be an intermediate range of values of q for which the inefficient equilibrium with low cost seller trading with both buyer types at price $m_l^{S^i} = L$ is more efficient than the pooling equilibrium. This range exists whenever:

$$p > p^{S^i} = \frac{H + \lambda_H - h}{H + \lambda_H - L} \tag{II.4}$$

The main point to take away from Corollary II.2 is that whenever $q < q^{S^e}$, even the most efficient equilibrium suffers from inefficient underselling. The frictions arising from asymmetric information leads the seller to leave informational rents to the buyer. This in turn creates distortions in equilibrium allocations, resulting in the inefficient equilibria P or S^i. The following figures summarize the configurations for the most efficient equilibria described in Corollary II.2:

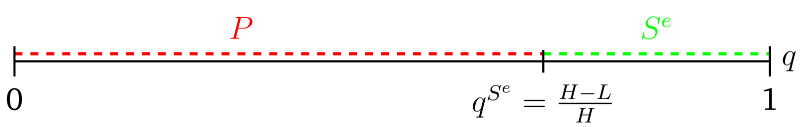

Figure II.3: When $H > h$

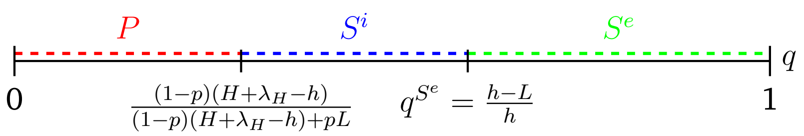

Figure II.4: When $H \leq h$

Note that for the case where $H \leq h$, the existence of the range over which equilibrium S^i is the most efficient depends on the parameters; namely on p satisfying

$p > p^{S^i}$, i.e. the inequality from (II.4). In Figure II.4, the depicted case assumes the inequality is satisfied and thus the range exists. Otherwise, the range over which S^i is the most efficient would disappear and the graph visually would be similar to the one in Figure II.3, except for the threshold of q equaling $q^{S^e} = \frac{h-L}{h}$.

Finally, the following corollary characterizes the equilibria that generates the highest expected payoffs to the seller types:

Corollary II.3.
The equilibria of Γ that yield the highest expected payoffs to the respective seller types, along with the expected payoffs are:

$$i)\ q \leq q^P \quad \Rightarrow \quad P \quad \textit{w/ prices} \quad m^P = \mathbb{E}(v|H)$$

$$\textit{payoffs} \quad \begin{cases} U_l^P &= \mathbb{E}(v|H)(1-q) \\ U_h^P &= [\mathbb{E}(v|H) - h](1-q) \end{cases}$$

$$ii)\ q \geq q^P \quad \Rightarrow \quad S^e \quad \textit{w/ prices} \quad \begin{cases} m_l^{S^e} &= L \\ m_h^{S^e} &= \min\left\{\frac{L}{1-q}, H + \lambda_H\right\} \end{cases}$$

$$\textit{payoffs} \quad \begin{cases} U_l^{S^e} &= L \\ U_h^{S^e} &= [\min\{\frac{L}{1-q}, (H+\lambda_H)\} - h](1-q) \end{cases}$$

The following Figure II.5 summarizes the configurations described in Corollary II.3 for the equilibria that yield the highest expected payoffs to the seller types:[13]

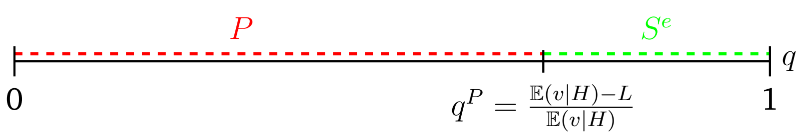

Figure II.5: For all values of H and h satisfying Assumption II.1

Note that, when $q \geq q^P$ and the best payoffs are attained under S^e, the high type seller's expected payoff depends on the magnitude of q. This observation is related to the comment made following Proposition II.2. Namely that if $q^P \leq q \leq \frac{H+\lambda_H-L}{H+\lambda_H}$, then $U_h^{S^e} = L - h(1-q)$ because $m_h^{S^e} = \frac{L}{1-q} \leq H + \lambda_H$, whereas if $q > \frac{H+\lambda_H-L}{H+\lambda_H}$, then $U^{S^e} = [H + \lambda_H - h](1-q)$.

[13] Note that the characterization of equilibria outcomes in Corollary II.3 satisfy the definition of IE* (interim efficient) allocation from Maskin and Tirole (1992) where buyer's beliefs are equal to the priors p.

II.3 Trading Game with an Intermediary

II.3.1 Model

This section introduces an uninformed risk-neutral intermediary, denoted by i, who tries to maximize expected profits while trying to facilitate trade between the seller and the buyer. The primitives related to the bargaining situation between the seller and the buyer are maintained as in the previous section, however the players (s, b and i) are assumed to engage in the trading game with an intermediary, denoted by Γ^i, played over three stages.

In the first stage, the intermediary i announces a menu of price pairs, denoted by $\mathbf{M}^i = \{(m_s^i(j), m_b^i(j)\}_{j=1}^J$, where a price pair $\mathbf{m}^i(j) = (m_s^i(j), m_b^i(j))$ represents the prices to be transferred to and from the seller and buyer, respectively if the object is to be traded.

In the second stage the seller moves, where after observing the menu of price pairs, he first decides his entry strategy, denoted by e. The entry actions can either be to use the intermediary, i.e. $e = i$, or to trade with the buyer directly as his outside option, i.e. $e = o$. In both subgames following his entry decision, the seller updates his beliefs regarding the buyer's types. If he decides to use the intermediary, then he chooses one of the price pairs from the announced menu $\mathbf{M}^i$, denoted by action $c^i \in \{1, \ldots, J\}$. If, on the other hand, he decides to choose the outside option of trading directly, then he announces a TIOLI price, denoted by action $m^o \in [0, H + \lambda_H]$.

In the third and final stage, the buyer moves. If in the previous stage, the seller's entry decision was in favor of the intermediary, then she observes the menu of prices and the seller's choice of price pair, updates her beliefs regarding the seller types and decides whether to accept or reject trading according to the proposed price pair, where her decision is denoted by $d^i \in \{A, R\}$. Otherwise, in the direct trade outside option subgame, the buyer observes the announced TIOLI price, updates her beliefs and decides whether to accept or reject trading at the proposed offer, where analogously her decision is denoted by $d^o \in \{A, R\}$.

II.3.2 Strategies, Beliefs and Payoffs

Consider the strategies of the players, starting with the intermediary. Firstly, observe that as there are only two types of the seller, it is without loss of generality to restrict attention to the intermediary announcing two price pairs, or that

$J = 2$. Secondly, as in the previous section, restrict attention to the intermediary announcing prices coming from the compact space $[0, H+\lambda_H]$. Lastly, it is assumed that the intermediary does not announce menus that would generate strictly negative expected payoffs whenever chosen.[14] More specifically, denoting a menu of price pairs by $\mathbf{M}^i = \{(m_s^i(1), m_b^i(1)), (m_s^i(2), m_b^i(2))\}$, consider the space of menus $\mathcal{M}^i = \{\mathbf{M}^i \in [0, H + \lambda_H]^4 | \max\{m_b^i(1) - m_s^i(1), m_b^i(2) - m_s^i(2)\} \geq 0\}$. Observe that this space of menus excludes those menus of price pairs where $m_b^i(j) < m_s^i(j)$ for both pairs, i.e. the intermediary makes strict losses on both price pairs. Thus a pure strategy for the intermediary is to choose a menu of price pairs $\mathbf{M}^i \in \mathcal{M}^i$.

At the beginning of second stage, after observing $\mathbf{M}^i$, each type of the seller chooses his entry strategy, which is defined by the mapping $e : \{l, h\} \times \mathcal{M}^i \to \{i, o\}$. These strategies are denoted by $e_l(\mathbf{M}^i)$ and $e_h(\mathbf{M}^i)$ for the respective types. Following the entry decision, in the subgame $e = i$, the seller chooses one of the price pairs. This strategy is defined by the mapping $c^i : \{l, h\} \times \mathcal{M}^i \to \{1, 2\}$. Similarly, define the shorthand notations $c_l^i(\mathbf{M}^i)$ and $c_h^i(\mathbf{M}^i)$ for the respective price pair choices of the seller types. Finally, in the other subgame $e = o$, the seller makes a TIOLI offer where the strategy is defined by $m^o : \{l, h\} \times \mathcal{M}^i \to [0, H+\lambda_H]$. Again, $m_l^o(\mathbf{M}^i)$ and $m_h^o(\mathbf{M}^i)$ are used as shorthand notations.

The buyer in the third stage chooses to accept or reject the standing offers in the respective subgames. If $e = i$, then her strategy is defined by the mapping $d^i : \{L, H\} \times \mathcal{M}^i \times \{1, 2\} \to \{A, R\}$. These strategies are denoted by $d_L^i(\mathbf{M}^i, j)$ and $d_H^i(\mathbf{M}^i, j)$ for $j \in \{1, 2\}$ for the respective buyer types. In the subgame $e = o$, the buyer's strategy is defined by the mapping $d^o : \{L, H\} \times [0, H + \lambda_H] \to \{A, R\}$, along with the shorthand notations $d_L^o(m^o)$ and $d_H^o(m^o)$, respectively.

At the beginning of the game, the intermediary's beliefs regarding the seller and buyer types are equal to the priors. Similarly, at the beginning of the second stage, the seller's beliefs regarding buyer's types when making the entry decision are equal to the priors. In each subgame $e \in \{i, o\}$, both seller types form beliefs $\pi(L|t_s, e, \mathbf{M}^i)$ for any given price pairs $\mathbf{M}^i$. Abusing notation, these beliefs are denoted by $\pi_{t_s}^e(\mathbf{M}^i)$ for each combination of entry decision e and seller type t_s.

Similarly, the buyer forms beliefs regarding the seller's types in each of the subgames. In the subgame where the intermediary is chosen, the beliefs are denoted by $\pi(l|t_b, i, \mathbf{M}^i, c^i)$. Abusing notation, these beliefs are denoted by $\pi_{t_b}^i(\mathbf{M}^i, c^i)$ for each type of the buyer. In the other subgame, the buyer's beliefs are $\pi(l|t_b, o, m^o)$, which again is shortened to $\pi_{t_b}^o(m^o)$.

Now, the payoffs can be defined. I start with the payoffs from the subgame

[14]Note that such menus are weakly dominated.

following entry decision $e = i$. Given an announcement of price pair menu $\mathbf{M}^i$ by the intermediary, seller's entry decision $e = i$ and price pair choice $c^i \in \{1, 2\}$, along with the buyer's decision $d^i \in \{A, R\}$, the payoffs to the seller, buyer and the intermediary are equal to, respectively:

$$U^i_{t_b}(\mathbf{M}^i, c^i, d^i) = U_{t_b}(\mathbf{M}^i, i, c^i, d^i) = \begin{cases} t_b + \big(1 - \pi^i_{t_b}(\mathbf{M}^i, c^i)\big)\lambda_{t_b} - m^i_b(c^i) & \text{if } d^i = A \\ 0 & \text{o/w} \end{cases}$$

$$U^i_{t_s}(\mathbf{M}^i, c^i, d^i) = U_{t_s}(\mathbf{M}^i, i, c^i, d^i) = \begin{cases} m^i_s(c^i) - t_s & \text{if } d^i = A \\ 0 & \text{o/w} \end{cases}$$

$$V^i(\mathbf{M}^i, c^i, d^i) = V(\mathbf{M}^i, i, c^i, d^i) = \begin{cases} m^i_b(c^i) - m^i_s(c^i) & \text{if } d^i = A \\ 0 & \text{o/w} \end{cases}$$

Let the expected trade probability be denoted by $Q^i(m^i_b(c^i)) = \pi^i_{t_s}(\mathbf{M}^i)\mathbb{I}\big(d^i_L(\mathbf{M}^i, c^i) = A\big) + \big(1 - \pi^i_{t_s}(\mathbf{M}^i)\big)\mathbb{I}\big(d^i_H(\mathbf{M}^i, c^i) = A\big)$. Then the expected payoff to the seller from choosing price pair $c^i \in \{1, 2\}$ is equal to:

$$U^i_{t_s}(\mathbf{M}^i, c^i) = U_{t_s}(\mathbf{M}^i, i, c^i) = \big[m^i_s(c^i) - t_s\big]Q^i(m^i_b(c^i))$$

In the other subgame following entry decision $e = o$, for any given actions $\mathbf{M}^i$, m^o and d^o, the payoffs to the three players are equal to:

$$U^o_{t_b}(\mathbf{M}^i, m^o, d^o) = U_{t_b}(\mathbf{M}^i, o, m^o, d^o) = \begin{cases} t_b + \big(1 - \pi^o_{t_b}(m^o)\big)\lambda_{t_b} - m^o & \text{if } d^o = A \\ 0 & \text{o/w} \end{cases}$$

$$U^o_{t_s}(\mathbf{M}^i, m^o, d^o) = U_{t_s}(\mathbf{M}^i, o, m^o, d^o) = \begin{cases} m^o - t_s & \text{if } d^o = A \\ 0 & \text{o/w} \end{cases}$$

$$V^o(\mathbf{M}^i, m^o, d^o) = V(\mathbf{M}^i, o, m^o, d^o) = 0$$

Again, defining the expected trade probability $Q^o(m^o) = \pi^o_{t_s}(\mathbf{M}^i)\mathbb{I}\big(d^o_L(m^o) = A\big) + \big(1 - \pi^o_{t_s}(\mathbf{M}^i)\big)\mathbb{I}\big(d^o_H(m^o) = A\big)$, the expected payoff to the seller from choosing price m^o on the direct trade outside option is equal to:

$$U^o_{t_s}(\mathbf{M}^i, m^o) = U_{t_s}(\mathbf{M}^i, o, m^o) = \big[m^o - t_s\big]Q^o(m^o)$$

At the beginning of the second stage, the seller's expected payoff from making entry decision $e \in \{i, o\}$ is given by:

$$U_{t_s}(\mathbf{M}^i, e) = U^i_{t_s}(\mathbf{M}^i, c^i)\mathbb{I}(e = i) + U^o_{t_s}(\mathbf{M}^i, m^o)\mathbb{I}(e = o)$$

Finally, the expected payoff to the intermediary from announcing menu $\mathbf{M}^i$ at the beginning of the first stage is equal to:

$$V(\mathbf{M}^i) = \sum_{t_s} \mathbb{P}(t_s)\mathbb{I}\big(e_{t_s}(\mathbf{M}^i = i)\big)\left[\mathbb{I}\big(c^i_{t_s}(\mathbf{M}^i) = j\big)\Big[\sum_{t_b} \mathbb{P}(t_b)V^i(\mathbf{M}^i, j, d^i_{t_b}(\mathbf{M}^i, j))\Big]\right]$$

II.3.3 Equilibria

In the game Γ^i with an intermediary, a pure strategy sequential equilibrium (equilibrium) is a collection of strategies $\{\hat{\mathbf{M}}^i, (\hat{e}_l, \hat{e}_h, \hat{c}^i_l, \hat{c}^i_h, \hat{m}^o_l, \hat{m}^o_h), (\hat{d}^i_L, \hat{d}^i_H, \hat{d}^o_L, \hat{d}^o_H)\}$ and beliefs $\{(\hat{\pi}^i_l, \hat{\pi}^i_h, \hat{\pi}^o_l, \hat{\pi}^o_h), (\hat{\pi}^i_L, \hat{\pi}^i_H, \hat{\pi}^o_L, \hat{\pi}^o_H)\}$, where the strategies satisfy sequential rationality and the beliefs are consistent. More specifically, strategies for the buyer satisfy:

$$\begin{aligned} U^i_{t_b}(\mathbf{M}^i, c^i, \hat{d}^i_{t_b}(\mathbf{M}^i, c^i)) &\geq U^i_{t_b}(\mathbf{M}^i, c^i, d^i) && \forall \mathbf{M}^i, c^i, d^i, \text{ and } t_b \\ U^o_{t_b}(\mathbf{M}^i, m^o, \hat{d}^o_{t_b}(m^o)) &\geq U^o_{t_b}(\mathbf{M}^i, m^o, d^o) && \forall \mathbf{M}^i, m^o, d^o, \text{ and } t_b \end{aligned}$$

Given these optimal strategies, the seller in the second stage has to choose strategies that satisfy:

$$\begin{aligned} U^i_{t_s}(\mathbf{M}^i, \hat{c}^i_{t_s}(\mathbf{M}^i)) &\geq U^i_{t_s}(\mathbf{M}^i, c^i) && \forall \mathbf{M}^i, c^i, \text{ and } t_s \\ U^o_{t_s}(\mathbf{M}^i, \hat{m}^o_{t_s}(\mathbf{M}^i)) &\geq U^o_{t_s}(\mathbf{M}^i, m^o) && \forall \mathbf{M}^i, m^o, \text{ and } t_s \\ U_{t_s}(\mathbf{M}^i, \hat{e}_{t_s}(\mathbf{M}^i)) &\geq U_{t_s}(\mathbf{M}^i, e) && \forall \mathbf{M}^i, e, \text{ and } t_s \end{aligned}$$

Finally, the intermediary announces a menu of price pairs that satisfy:

$$V(\hat{\mathbf{M}}^i) \geq V(\mathbf{M}^i) \qquad \forall \mathbf{M}^i$$

The beliefs, on the other hand, are defined by using Bayes' rule at information sets that are reached with positive probability on the equilibrium. Otherwise, they are determined by taking the limit of a sequence of beliefs that are generated from a chosen sequence of totally mixed strategies converging to the equilibrium strategies.

It will be convenient to define the following shorthand notations for the following equilibrium objects. Given any equilibrium, for each seller type t_s, let $\hat{m}_s[t_s]$ and $\hat{m}_b[t_s]$ be the prices chosen on the equilibrium path which are equal to:

$$\hat{m}_s[t_s] = \begin{cases} \hat{m}^o_{t_s}(\hat{\mathbf{M}}^i) & \text{if } \hat{e}_{t_s}(\hat{\mathbf{M}}^i) = i \\ \hat{m}^i_s(\hat{c}^i_{t_s}(\hat{\mathbf{M}}^i)) & \text{if } \hat{e}_{t_s}(\hat{\mathbf{M}}^i) = o \end{cases}$$
$$\hat{m}_b[t_s] = \begin{cases} \hat{m}^o_{t_s}(\hat{\mathbf{M}}^i) & \text{if } \hat{e}_{t_s}(\hat{\mathbf{M}}^i) = i \\ \hat{m}^i_b(\hat{c}^i_{t_s}(\hat{\mathbf{M}}^i)) & \text{if } \hat{e}_{t_s}(\hat{\mathbf{M}}^i) = o \end{cases}$$

Then the implemented trade probabilities can be denoted by $\hat{Q}[t_s, t_b]$ for any type pair (t_s, t_b) on the equilibrium path, which are defined as:

$$\hat{Q}[t_s, t_b] = \mathbb{I}\big(\hat{d}_{t_b}(\hat{m}_b[t_s]) = A\big)$$

Extending further, for each seller type t_s define the expected equilibrium trade probabilities $\hat{Q}(\hat{m}_b[t_s]) = q\hat{Q}[t_s, L] + (1-q)\hat{Q}[t_s, H]$.

Finally, define the following shorthand notations for the expected payoffs to the seller types. The expected payoff from subgame $e = o$ is denoted by $\hat{U}^o_{t_s} = U^o_{t_s}(\hat{\mathbf{M}}^i, \hat{m}^o_{t_s}(\hat{\mathbf{M}}^i))$, while from subgame $e = i$ is denoted by $\hat{U}^i_{t_s} = U^i_{t_s}(\hat{\mathbf{M}}^i, \hat{c}^i_{t_s}(\hat{\mathbf{M}}^i))$. The expected payoff of the seller type in the equilibrium is denoted by $\hat{U}_{t_s} = U_{t_s}(\hat{\mathbf{M}}^i, \hat{e}_{t_s}(\hat{\mathbf{M}}^i))$. Observe that $\hat{U}_{t_s} = (\hat{m}_s[t_s] - t_s)\hat{Q}(\hat{m}_b[t_s])$.

I first provide some preliminary observations about the equilibria of the game Γ^i, which will be useful in the rest of the analysis. First result relates to the beliefs in any equilibrium.

Lemma II.4.
In any equilibrium of Γ^i*;*

1. *The seller's beliefs satisfy* $\hat{\pi}^e_{t_s}(\mathbf{M}^i) = q$ *for all* e *and* t_s.

2. *The buyer's beliefs, on the other hand, are the same across her two types in each subgame; i.e.* $\hat{\pi}^i_{t_b}(\mathbf{M}^i, c^i) = \hat{\pi}^i(\mathbf{M}^i, c^i)$ *for all* $\mathbf{M}^i$ *and* c^i *in subgame* $e = i$*, and* $\hat{\pi}^o_{t_b}(m^o) = \hat{\pi}^o(m^o)$ *for all* m^o *in subgame* $e = o$.

The result follows from the fact that the types of the seller and buyer are independently distributed. The next result relates to the seller's expected payoffs in any equilibrium.

Lemma II.5.
In any equilibrium of Γ^i*;*

1. *The expected payoff $\hat{U}^o_{t_s}$ from direct trade outside option subgame (i.e. $e = o$) satisfies the following:*

$$\hat{U}^o_{t_s} = U^o_{t_s}(\hat{\mathbf{M}}^i, \hat{m}^o_{t_s}(\hat{\mathbf{M}}^i)) = \max_{m^o}\{U^o_{t_s}(\mathbf{M}^i, m^o)\} \qquad \forall\, \mathbf{M}^i$$

   Furthermore, the expected payoffs from the outside option subgame necessarily satisfy:

$$\hat{U}^o_l \geq \max\{L, H(1-q)\}$$
$$\hat{U}^o_h \geq \max\{(H-h)(1-q), 0\}$$

2. *Similarly, the equilibrium expected payoffs for each seller type satisfy the following:*

$$\hat{U}_l \geq \max\{\hat{U}^o_l, \hat{U}_h + h\hat{Q}(\hat{m}_b[h])\}$$
$$\hat{U}_h \geq \max\{\hat{U}^o_h, \hat{U}_l - h\hat{Q}(\hat{m}_b[l])\}$$

Both parts of Lemma II.5 follow from the optimality of seller's equilibrium strategies. An important implication of the first part of Lemma II.5 is that, each seller type t_s receives the same expected payoff $\hat{U}^o_{t_s}$ from the subgame $e = o$ no matter what menu $\mathbf{M}^i$ the intermediary offers. The lower bounds on the expected payoffs for the direct trade outside option subgame are attained by evaluating the expected payoffs under the worst beliefs of the buyer where she puts full probability to the seller being of low type, i.e. $\pi^o(m^o) = 1$. The second part suggests that each seller type should be receiving at least the maximum of their outside option they receive from directly trading and their payoff from mimicking the other type on the overall equilibrium paths. These observations will play an important role in the subsequent analysis of the next subsection.

II.3.4 Efficient Intermediated Equilibria

In this subsection, the main result of the paper is provided. The main interest rests in answering the following questions:

1. *Are there any equilibria of Γ^i with an intermediary that is strictly more efficient than the most efficient equilibrium of the game Γ without an intermediary?*

2. *If so, under what conditions are these more efficient equilibria attained?*

In order to answer the above questions, I concentrate on those parameter values where the most efficient equilibrium of Γ can be improved upon. Proposition II.2 established that whenever $q \geq q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\}$, there exists an efficient separating equilibrium. Since in an efficient separating equilibrium the first best allocation rule is implemented, an allocative efficiency improvement is not possible. Hence I focus on the cases where $q < q^{S^e}$.

Assumption II.2.
[A3] *For the rest of the paper, assume that* $q < q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\}$.

Corollary II.2 showed that, in the most efficient equilibria of Γ under Assumption II.2, either only the low type seller trades with both buyer types (inefficient separating equilibrium S^i with $m_l^{S^i} = L$) or only the high type buyer trades with both seller types (pooling equilibrium P). Hence, if an equilibrium of Γ^i is strictly more efficient than the most efficient equilibria of Γ when $q < q^{S^e}$, then it has to implement trade in all seller-buyer type realizations except for the case where the seller has type h and buyer has type L. In light of these observations, the following provides a definition of the equilibria to be analyzed:

Definition II.1.
An ***efficient intermediated equilibrium*** *is an equilibrium of* Γ^i *in which the implemented trades on the equilibrium satisfy* $\hat{Q}[l, L] = \hat{Q}[l, H] = \hat{Q}[h, H] = 1$ *while* $\hat{Q}[h, L] = 0$.

In light of this definition, the main question can be rephrased as follows:

Given Assumption II.2 is satisfied, does there exist an efficient intermediated equilibrium of Γ^i*? If so, what are the necessary and sufficient conditions for the existence of such equilibria?*

Before answering these questions, the following lemma is provided:

Lemma II.6.
In an efficient intermediated equilibrium, if it exists, both seller types have to necessarily choose the intermediary; i.e. $\hat{e}_l(\hat{\mathbf{M}}^i) = \hat{e}_h(\hat{\mathbf{M}}^i) = i$.

Lemma II.6 suggests that any allocative efficiency improvement relative to the most efficient equilibrium of Γ has to arise due to the presence of the intermediary. Furthermore, the intermediary plays an active role in coordinating trades of the two seller types, as both types choose the intermediary in any such equilibrium. This result, in a way, legitimizes the name choice for the specific equilibrium of analysis.

Next, a necessary condition for the existence of efficient intermediated equilibria is provided.

Proposition II.4.
An efficient intermediated equilibrium exists only if the following necessary condition holds:

$$pL + (1-p)(1-q)(H + \lambda_H - h) \geq p\hat{U}_l + (1-p)\hat{U}_h \tag{II.5}$$

Observe that the left-hand side term is equal to the ex-ante expected gains from trade, while the right-hand side equals to the ex-ante expected payoff to the seller. The difference of the two sides simply yields the sum of the ex-ante expected payoffs to the intermediary and the buyer. Since each of those payoffs have to be nonnegative, Proposition II.4 follows immediately.

Now the main result of the paper can be stated.

Theorem II.1.
An efficient intermediated equilibrium of Γ^i *exists if and only if:*

$$p\frac{L}{1-q} + (1-p)(H + \lambda_H) \geq \max\{H, h\} \tag{II.6}$$

The necessity part of the proof for Theorem II.1 follows from Proposition II.4. For the sufficiency part, I prove the existence of an equilibrium whenever condition (II.6) is satisfied.

The intuition behind why the intermediary is able to achieve these efficiency improvements lies in cross-subsidization. Namely, whenever condition (II.6) is satisfied, the intermediary makes large enough gains from the trade between the high type seller and the high type buyer to sufficiently compensate the losses it makes on the low type seller's trades. The losses arise due to the ability of the seller to trade directly with the buyer in his outside option, and this ability, in turn, creates competition against the intermediary which is referred to as the "internal competition".

To understand the last point better, remember that the seller types can always guarantee themselves expected payoffs of $\hat{U}^o_{t_s}$ which have lower bounds as described in Lemma II.5. These lower bounds on the equilibrium expected payoffs have to be respected by the intermediary. Combining the impact of this "internal competition" along with the incentive compatibility constraints, it can be seen that in any equilibrium $\hat{U}_l \geq \max\{H(1-q), h(1-q)\}$. On the other hand, for the desired trade probabilities to be exercised, the low type seller's equilibrium price $\hat{m}_b[l]$ for the buyer should be at most L. By Assumption II.2, $L < \max\{H(1-q), h(1-q)\}$

which means that the intermediary has to make expected losses over facilitating the trade of the low type seller.

Theorem II.1 suggests that the presence of an intermediary leads to a strict allocative efficiency improvement in the trade outcomes. There are a few points to be made with regards to the importance of the result.

Firstly, the efficiency is improved in ex-post sense. In other words, for all type pairs, the implemented trade probabilities are weakly higher than their counterparts under the equilibria of game Γ. Furthermore, where the trade probability is strictly higher, the buyer values the object more than the seller, hence the efficiency improvement.[15]

Secondly, the efficiency improvements are carried out by the intermediary, who is the least informed player in the game Γ^i. Similarly, the agents having the ability to trade directly implies that the intermediary is not providing matchmaking technology. Thus, the analysis provides a different motivation for the benefits of intermediation, other than the explanations related to expertise provision and certification, or acting as a platform to match the two sides. Alternatively, it is shown that, in this setup the intermediary overtakes the role of a coordination point for monetary transactions (cross-subsidization) and information flow (separation of seller types), which in turn improves the trade outcomes.

Lastly, the efficiency is strictly improved by a profit maximizing intermediary. The more interesting part in this observation is the fact that, even though the intermediary is a strategic player who is acting in its self-interest, its presence could lead to these improved outcomes. In other words, findings highlight the possibility of ex-post efficiency improvements in the market without the need for the intervention of a benevolent social planner.

II.4 Discussion

In this section, I elaborate on the results from the previous section by discussing some alternative cases. At the end of the section, I also provide a couple of numeric examples to illustrate the points covered in the analysis.

[15] Given the binary type setup, the efficient intermediated equilibria actually implement the first best allocation that achieves ex-post efficiency.

II.4.1 Seller's Optimal Prices In the Direct Trade Outside Option

In the previous section, Lemma II.6 established that an equilibrium of Γ^i that is strictly more efficient than the most efficient equilibrium of Γ has to have trades going through the intermediary. Therefore, in any efficient intermediated equilibria, direct trading at the outside option subgame remains out-of-equilibrium. The main result in Theorem II.1 established existence of efficient intermediated equilibria if and only if Condition (II.6) is satisfied, however it did not impose any restrictions on the outcomes of the out-of-equilibrium outside option subgame. Apart from Lemma II.5, which defines the lower bounds for the expected payoffs of each seller type from the outside option subgame equilibria, there are no restrictions on how large these expected payoffs can be.[16]

It was shown in Corollary II.3, that the highest expected payoff yielding equilibria of Γ are the efficient separating equilibrium when $q \geq q^P = \frac{\mathbb{E}(v|H)-L}{\mathbb{E}(v|H)}$ and the pooling equilibrium with price $m^P = \mathbb{E}(v|H)$ when $q \leq q^P$. Since $q^P > q^{S^e}$, under Assumption II.2, the pooling equilibrium is the highest payoff yielding equilibrium of the game Γ.

Now consider the game Γ^i, where the seller and the buyer play the pooling equilibrium with the optimal pooling prices in the direct trade outside option subgame, and repeat the analysis for the existence of efficient intermediated equilibrium. The seller's optimal outside option prices and the associated out-of-equilibrium payoffs provide a good robustness check for the previous existence result from Theorem II.1. Furthermore, it can be argued whether there exists a strictly more efficient equilibrium with the presence of an intermediary not only relative to the most efficient equilibria of Γ, but also relative to the seller's highest expected payoff yielding equilibria of Γ. The following proposition provides the necessary and sufficient condition:

Proposition II.5.

An efficient intermediated equilibrium, where in the direct trade outside option subgame both seller types trade the object with the high type buyer at the pooling equi-

[16]As a matter of fact, in the constructive proof for the sufficiency part of Theorem II.1, these lower bounds are used as the expected payoffs for the direct trade outside option subgame equilibria. This was required for providing the necessary and sufficient condition for the existence of the efficient intermediated equilibria. As the analysis in this subsection suggests, however, higher expected payoffs from the outside option subgame provide tougher restraints, which in turn constrain the parameter scenarios for existence of efficient intermediated equilibria.

librium price of $\hat{m}_l^o = \hat{m}_h^o = \mathbb{E}(v|H)$, exists if and only if:

$$h > \frac{L}{1-q} \geq H \tag{II.7}$$

There are several remarks to be made. Firstly, condition (II.7) requires $h > H$. To see why, remember that Assumption II.2 requires $q < q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\}$, or equivalently put $\frac{L}{1-q} < \max\{H, h\}$, which is equal to H whenever $H \geq h$. Since $H > \frac{L}{1-q} \geq H$ is inconsistent, the efficient intermediated equilibria exists only in the cases where $h > H$. The range of q values for existence are given by $\frac{H-L}{H} \leq q < \frac{h-L}{h}$, where the right-most term is equal to q^{S^e} in this parameter case. Rearranging the terms yields condition (II.7) shown above.

Secondly, compared to the condition (II.6) in Theorem II.1, it can be seen that condition (II.7) is more restrictive, i.e. as the former condition is implied by the latter. To see why, assume that $h > \frac{L}{1-q} \geq H$, in which case it holds that:

$$p\frac{L}{1-q} + (1-p)(H + \lambda_H) \geq pH + (1-p)(H + \lambda_H) = \mathbb{E}(v|H) > h = \max\{H, h\}$$

where the last inequality follows from **[A2]** in the initial Assumption II.1. This is not surprising, because as the direct trade outside option subgame equilibrium prices increase, they make the outside option payoffs of the seller types stronger. In turn, it becomes more difficult to maintain an equilibrium where both seller types choose the intermediary, because it becomes less profitable for the intermediary to facilitate the trade. Hence, higher outside option equilibrium prices can be interpreted as the intermediary facing a more intensified internal competition posed by the seller.

Thirdly, observe that in any efficient intermediated equilibrium characterized by Proposition II.5, both the seller and the buyer's payoffs are Pareto dominant compared to their counterparts from the direct trade outside option subgame equilibrium, or equivalently the expected payoffs from the pooling equilibrium of Γ with price $m^P = \mathbb{E}(v|H)$. The Pareto improvement for the seller is immediate from the sequential rationality of entry choice in the game Γ^i. To see why the buyer types are also weakly better off, observe that in the pooling equilibrium of Γ with the highest possible price, the buyer's expected payoffs are equal to $U_L^P = U_H^P = 0$. Low type buyer receives 0, because she does not trade while the high type buyer pays precisely her expected value of the object. In the efficient intermediated equilibrium, on the other hand, the implemented trade probabilities require that the prices the buyer pays in equilibrium satisfy $\hat{m}_b[l] \leq L$ for low type seller's equilibrium menu option and $\hat{m}_b[h] \leq H + \lambda_H$ for the high type seller menu option. These suggest that in an efficient intermediated equilibrium, the buyer types receive expected payoffs that are at least:

$$\hat{U}_L = p(L - \hat{m}_b[l]) \geq 0 \qquad \hat{U}_H = p(H - \hat{m}_b[l]) + (1-p)(H + \lambda_H - \hat{m}_b[h]) > 0$$

The Pareto improvement property suggests the robustness of the efficient intermediated equilibria in these cases. In particular, the strict allocative efficiency improvement does not come at the expense of making the buyer side worse off. On the contrary the additional gains from trade accrued are allocated between the seller, the buyer and the intermediary in a way that every type of both the seller and the buyer are made weakly better off.

Lastly, Proposition II.5 highlights that the result of Theorem II.1 is not an artifact of constructing the equilibrium with the lowest possible out-of-equilibrium outside option subgame payoffs. On the contrary, it is shown that indeed there are cases where, it is possible to have strict improvement of allocative efficiency with the presence of an intermediary, even relative to the highest expected payoff yielding equilibria of the game Γ. This in turn represents a strong robustness check in favor of the benefits of intermediation.

II.4.2 Independent Private Values

So far it was assumed that the valuations of the seller and the buyer display interdependence. Namely for both types of the buyer, her valuation for the object increases by some constant λ_{t_b}, whenever the seller is of high type. A natural case to consider is the extreme situation where $\lambda_L = \lambda_H = 0$. This is precisely the case of independent private values (IPV), as the buyer's valuations satisfy $v(l, L) = v(h, L) = L$ and $v(l, H) = v(h, H) = H$. Note that, Assumption II.1 in this case boils down to having the four parameters satisfying the following ordering $l = 0 < L < h < H$.

The analysis in Section II.2 remains valid, however the conditions need to adjusted to reflect the case $\lambda_L = \lambda_H = 0$. In light of the adjustments, the equilibria of Γ under the IPV case can be characterized as follows:

Lemma II.7.
In the IPV case, the equilibria of Γ *are:*

$$\begin{array}{llll} i)\ P & \Rightarrow & m^P = H & \textit{whenever} \quad q \leq \frac{H-L}{H} \\ ii)\ S^e & \Rightarrow & m_l^{S^e} = L,\ m_h^{S^e} = H & \textit{whenever} \quad q \geq \frac{H-L}{H} \end{array}$$

Observe that the equilibrium characterization of Γ is a lot simpler under the IPV case. Firstly, there are no multiplicities at any given parameters. Secondly, the inefficient separating equilibria no longer exist. Furthermore, there is only pooling or efficient separating equilibrium depending on the magnitude of q.

The emphasis is again on the existence of efficient intermediated equilibria. Hence, restrict attention to the cases where Assumption II.2 is satisfied, i.e. $q < \frac{H-L}{H}$. Note that, by Lemma II.7, the unique equilibria of Γ in those cases are the pooling equilibria P with $m^P = H$. It turns out that the following impossibility result can be stated:

Proposition II.6.
In the IPV case, there are no efficient intermediated equilibria of Γ^i.

Proposition II.6 is a strong negative result. The intuition of the proof relies in the following argument. In the IPV case, there is no signaling aspect to the equilibria of the direct trade outside option subgame. This is because, buyer's beliefs have no impact on the optimal decision to accept or reject different TIOLI offers. Hence both seller types have a unique optimal price choice, which uniquely pins down the outside option payoffs. However, I show in the proof of Proposition II.6 that any strict efficiency improvement from these unique outside option subgame outcomes requires an intermediary to run strict losses. Hence it is not possible to have an efficient intermediated equilibrium in the IPV case.

An important implication of this impossibility result is that the benefits of intermediation arise only when there are interdependencies across the valuations of the buyer and the seller. This result is in line with Proposition 11 from Maskin and Tirole (1990).[17]

II.4.3 Intermediary's Optimal Mechanism Design

A relevant benchmark to consider for comparison is the optimal mechanism design problem the intermediary faces á la Myerson and Satterthwaite (1983). Namely, the buyer and the seller play a Bayesian game that is designed according to the

[17]See Footnotes 5 and 6.

intermediary's bargaining protocol. The primitives related to the bilateral trade scenario, such as valuations and type spaces, are kept the same way as before. However, the game in consideration differs in two important aspects. Firstly, it is assumed that trade has to go through the intermediary for there is no possibility of direct trade as their outside option. This means that there is no entry choice to be made. Secondly, it is also assumed that the buyer and the seller move simultaneously in the trading mechanism rather than sequentially. This subsection closely follows Section 5 of Myerson and Satterthwaite (1983). By Revelation Principle,[18] it is without loss of generality to restrict attention to the truthful type-telling equilibria of feasible direct revelation mechanisms (DRM) that satisfy incentive compatibility (IC) and individual rationality (IR).

A DRM, denoted by $\{Q, m_s, m_b\}$, consists of an allocation rule $Q : \{l, h\} \times \{L, H\} \to [0, 1]$, which determines the probability of the object changing hands from seller to buyer, and payments rules $m_s : \{l, h\} \times \{L, H\} \to \mathbb{R}$ and $m_b : \{l, h\} \times \{L, H\} \to \mathbb{R}$, where $m_s(t_s, t_b)$ describes the payment made to the seller by the intermediary and $m_b(t_s, t_b)$ describes the payment made by the buyer to the intermediary. Given truthful type reporting, the expected trade probabilities of each seller and buyer types are defined by $Q_s(t_s) = qQ(t_s, L) + (1 - q)Q(t_s, H)$ and $Q_b(t_b) = pQ(l, t_b) + (1 - p)Q(h, t_b)$, respectively. The expected payments $m_s(t_s)$ and $m_b(t_b)$ for each types of the respective agents are defined analogously. Then, given the buyer tells her type truthfully, the expected payoff to the seller of type t_s from reporting some type t'_s is equal to $U_s(t'_s|t_s) = m_s(t'_s) - t_s Q_s(t'_s)$. Similarly, given the seller tells his type truthfully, the expected payoff to the buyer of type t_b from reporting type t'_b is equal to $U_b(t'_b|t_b) = [pt_b Q(l, t'_b) + (1-p)(t_b + \lambda_{t_b})Q(h, t'_b)] - m_b(t'_b)$. Finally, a DRM is called *feasible* if and only if it satisfies the following IC and IR constraints:

$U_s(t_s) \geq 0$	IR for seller
$U_s(t_s) \geq U_s(t'_s\|t_s)$	IC for seller
$U_b(t_b) \geq 0$	IR for buyer
$U_b(t_b) \geq U_b(t'_b\|t_b)$	IC for buyer

Intermediary's expected profits in a truth-telling equilibrium are simply given by $V^i = [qm_b(L) + (1-q)m_b(H)] - [pm_s(l) + (1-p)m_s(h)]$. Thus, the intermediary solves the maximization problem with the objective function V^i subject to the IC and IR constraints from above. Observe that there are a total of 8 inequalities arising

[18] See for instance Myerson (1979), Myerson (1981).

from these constraints. One can show that at the optimal mechanism four of the constraints will bind; the IC constraints for l type seller and H type buyer while the IR constraints for h type seller and L type buyer. The proofs are omitted, as they follow from standard arguments. Plugging in these binding constraints into the objective function, the pointwise maximizer yields the optimal mechanism, denoted by $\{\tilde{Q}, \tilde{m}_s, \tilde{m}_b\}$. Below is the characterization of the optimal mechanism:

Remark II.1.
The optimal mechanism designed by the intermediary implements the allocation function:

$$\tilde{Q}(l,L) = \begin{cases} 1 & \textit{if } q \geq \frac{H-L}{H} \\ 0 & \textit{o/w} \end{cases} \qquad \tilde{Q}(l,H) = 1$$

$$\tilde{Q}(h,L) = 0 \qquad \tilde{Q}(h,H) = \begin{cases} 1 & \textit{if } p \leq \frac{H+\lambda_H-h}{H+\lambda_H} \\ 0 & \textit{o/w} \end{cases}$$

The optimal expected transfers can be summarized as follows:

$$\tilde{m}_s(l) = (1-q)h\tilde{Q}(h,H) \qquad \tilde{m}_s(h) = (1-q)h\tilde{Q}(h,H)$$
$$\tilde{m}_b(L) = pL\tilde{Q}(l,L) \qquad \tilde{m}_b(H) = p[H - L\tilde{Q}(l,L)] + (1-p)(H+\lambda_H)\tilde{Q}(h,H)$$

Consequently, the expected payoffs are:

$$\tilde{U}_s(l) = \begin{cases} 0 & \textit{if } p > \frac{H+\lambda_H-h}{H+\lambda_H} \\ h(1-q) & \textit{o/w} \end{cases} \qquad \tilde{U}_s(h) = 0$$

$$\tilde{U}_b(L) = 0 \qquad \tilde{U}_b(H) = \begin{cases} p(H-L) & \textit{if } q \geq \frac{H-L}{H} \\ 0 & \textit{o/w} \end{cases}$$

Comparing the most efficient equilibria of the informed seller's signaling game Γ under Assumption II.2 with the equilibrium outcomes of the intermediary's optimal mechanism characterized above, it can be seen that the intermediary attains strictly more efficient outcomes whenever $q \geq \frac{H-L}{H}$ and $p \leq \frac{H+\lambda_H-h}{H+\lambda_H}$. Observe that the inequalities can be satisfied only when $H \leq h$. In the case where $H > h$, the conditions yield an empty interval for probability q, i.e. $\frac{H-L}{H} = q^{S^e} > q \geq \frac{H-L}{H}$.

In the case where $H \leq h$, the strict improvement in the expected gains from trade is in parallel with the points made in Section 7 in Jullien and Mariotti (2006). The authors show a sufficient condition under which the monopolist intermediary's equilibrium allocation may generate strictly higher surplus than those accrued under the signaling equilibrium. The specific conditions are not comparable as the authors consider a setup with continuum of types for both the bargaining agents, unlike the binary types considered here. However, the analogy between the results prevail.

It is important to note that, the cases where the intermediary improves outcomes in the game Γ^i are more abundant. This point can be illustrated by comparing the condition (II.6) from Theorem II.1 with the conditions mentioned above. The comparison is obvious when $H > h$, because the conditions are not satisfied. In other words, the intermediary can not improve outcomes in the optimal design problem, whereas the game Γ^i has efficient intermediated equilibria. If $H \leq h$, on the other hand, rearranging (II.6) yields that the probability parameter p needs to satisfy $p \leq \frac{H+\lambda_H-h}{H+\lambda_H-\frac{L}{1-q}}$ for efficient intermediated equilibrium to exist in Γ^i. A simple comparison yields that the upper threshold for p is strictly lower in the case of intermediary's optimal design problem solution, which makes the condition more difficult to be satisfied.

To that extent, it can be seen that although a centralized structure under a monopolist intermediary may generate strictly higher surplus than the signaling outcomes, the efficiency improvements are even more frequent when the intermediary has to provide the seller at least expected payoffs he could generate from the direct trade outside option subgame. This is not surprising, because the competition between the intermediary and the seller's ability to trade directly mitigates the intermediary's distortions arising from market power and leads to more frequent efficiency improvements.

II.4.4 Examples

Example 1

Consider the bilateral trade scenario where the seller type's are $l = 0$ and $h = 7$, while the buyer's types are $L = 3$ and $H = 5$. Also assume that the constants representing the premium in buyer's valuation whenever the seller is high type are $\lambda_L = 3$ and $\lambda_H = 5$, or equivalently the buyer's valuation is doubled whenever the seller is of high type. The following matrix summarizes the valuations for the two bargaining parties:

$t_s \setminus t_b$	L	H
$l = 0$	3	5
$h = 7$	6	10

$$\Rightarrow \quad p = \mathbb{P}(t_s = l) = \tfrac{1}{2} = \mathbb{P}(t_b = L) = q$$

Finally, assume that the prior probabilities for the low types are $p = q = \frac{1}{2}$ for both the seller and the buyer.

Observe that, $\max\{H, h\} = \max\{5, 7\} = 7 = h$. Evaluating condition (II.4) yields that $p = \frac{1}{2} > p^{S^i} = \frac{H+\lambda_H - h}{H+\lambda_H - L} = \frac{3}{7}$. Thus by Corollary II.2 the most efficient equilibria of Γ is S^i, where $m_l^{S^i} = 3$ with low type seller trading with both buyer types while $m_h^{S^i} = 10$ with both buyer types rejecting the offers.[19] Clearly, this equilibrium suffers from inefficient underselling as there is no trade between the high types of the agents, despite the gains from trade to be made. Equivalently, it can be seen that the configuration above satisfies Assumption II.2 as $q = \frac{1}{2} < q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\} = \max\{\frac{2}{5}, \frac{4}{7}\} = \frac{4}{7}$.

It is easy to show that condition (II.6) from Theorem II.1 is satisfied:

$$\frac{1}{2}\frac{3}{1/2} + \frac{1}{2}10 = \frac{16}{2} = 8 > \max\{5, 7\} = 7$$

Therefore, there exists an efficient intermediated equilibrium of the Γ^i under the parameter configuration described above. As an example consider the following equilibrium path. The intermediary announces $\hat{M}^i = \{(3.5, 3), (7, 10)\}$, both seller types choose the intermediary followed by the low type choosing menu 1 and the high type choosing menu 2. The high type buyer accepts both menus while the low type buyer accepts only when menu 1 is chosen. The expected payoffs for the high type seller and the buyers are the same as under S^i in Γ, while the low type seller receives $\hat{U}_l = 3.5$, which is strictly higher than the expected payoff of 3 under S^i of Γ. Lastly, the intermediary makes an expected profit of $\hat{V} = \frac{1}{2}(3 - 3.5) + \frac{1}{2}\frac{1}{2}(10 - 7) = \frac{1}{2}$, which is strictly positive.

Evaluating condition (II.7) from Proposition II.5 suggests that $h = 7 > \frac{L}{1-q} = 6 > H = 5$. Hence, there also exists an efficient intermediated equilibrium even when in the out-of-equilibrium direct trade outside option subgame, the agents play the pooling equilibrium that yields the highest expected payoffs for the seller. More specifically, in the outside option subgame, both seller types announce $\hat{m}^o = \mathbb{E}(v|H) = \frac{15}{2}$ and they are accepted only by the high type buyer.[20] In turn, the

[19]Note that the high type buyer rejecting is optimal by indifference.

[20]Note that these strategies represent the seller's optimal equilibrium in the game Γ as charac-

equilibrium in the direct trade subgame yields the seller types expected payoffs of $\hat{U}_l^o = \frac{15}{4}$ and $\hat{U}_h^o = \frac{1}{4}$. In this case, an example efficient intermediated equilibrium has menu $\hat{M}^i = \{(\frac{15}{4}, 3), (\frac{15}{2}, 10)\}$ announced in equilibrium. Similar to the previous case, the agents in equilibrium play the same way. Namely that both seller types choose the intermediary followed by the low type choosing menu 1 and the high type choosing menu 2. The high type buyer accepts both menus, while the low type buyer accepts only when menu 1 is chosen. The expected payoffs for all types of the agents remain the same as those accrued from the direct trade outside option subgame (or equivalently the corresponding P of the game Γ). The intermediary makes an expected profit of $\hat{V} = \frac{1}{2}(3 - \frac{15}{4}) + \frac{1}{2}\frac{1}{2}(10 - \frac{15}{2}) = \frac{1}{4} > 0$.

Finally, note that by Remark II.1 from Subsection II.4.3, the intermediary's optimal mechanism implements trade probabilities that satisfy $\tilde{Q}(l, L) = \tilde{Q}(l, H) = 1$ while $\tilde{Q}(h, L) = \tilde{Q}(h, H) = 0$. To see why, observe that $p = \frac{1}{2} > \frac{H+\lambda_H-h}{H+\lambda_H} = \frac{3}{10}$. This is clearly less efficient than S^i, i.e. the most efficient equilibrium of Γ. Therefore, it follows that the intermediary generates a strict efficiency improvement in the game Γ^i, which are not attained in the absence of the internal competition from the seller.

Example 2

Consider the bilateral trade scenario where the seller type's are $l = 0$ and $h = 4$, while the buyer's types are $L = 2$ and $H = 5$. Also assume that the constants representing the premium in buyer's valuation whenever the seller is high type are $\lambda_L = 1$ and $\lambda_H = 2$. The following matrix summarizes the valuations for the two bargaining parties:

$t_s \setminus t_b$	L	H
$l = 0$	2	5
$h = 4$	3	7

$$\Rightarrow \quad p = \mathbb{P}(t_s = l) = \tfrac{1}{2} = \mathbb{P}(t_b = L) = q$$

Finally, similar to the previous Example 1, assume that the prior probabilities for the low types are $p = q = \frac{1}{2}$ for both the seller and the buyer.

This time, it holds that $\max\{H, h\} = \max\{5, 4\} = 5 = H$. Evaluating condition (II.2) yields $q = \frac{1}{2} < q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\} = \max\{\frac{2}{4}, \frac{3}{5}\} = \frac{3}{5}$. Thus by Corollary II.2, the most efficient equilibria of the game Γ is P, where $m^P = 6$ with only the high type buyer trading. Clearly, this equilibrium suffers from inefficient underselling as there is no trade between the low types of the agents, despite the

terized in Corollary II.3.

gains from trade to be made. Note that, again the configuration above satisfies Assumption II.2, as it was shown above that $q < q^{S^e}$.

By Lemma II.5, in any equilibrium of Γ^i the outside option subgame equilibrium payoffs satisfy $\hat{U}_l^o \geq \max\{2, 5(1/2)\} = 5/2$ and $\hat{U}_h^o \geq \max\{(5-4)(1/2), 0\} = 1/2$. This is because, in the direct trade subgame, both seller types can announce a TIOLI offer of $m^o = 5$ which will be accepted by the high type buyer, no matter what beliefs she has.

Again, it is easy to show that condition (II.6) from Theorem II.1 is satisfied:

$$\tfrac{1}{2}\tfrac{2}{1/2} + \tfrac{1}{2}7 = \tfrac{11}{2} > \max\{5, 4\} = 5$$

Therefore, there exists an efficient intermediated equilibrium of the Γ^i under the parameter configuration described above. As an example consider the following equilibrium. Let both seller types announce $\hat{m}_l^o = \hat{m}_h^o = H = 5$ in the out-of-equilibrium direct trade outside option subgame, which is only accepted by the high type buyer. Then on the equilibrium path, the intermediary announces $\hat{M}^i = \{(2.5, 2), (5, 7)\}$, both seller types choose the intermediary followed by the low type choosing menu 1 and the high type choosing menu 2. The high type buyer accepts both menus while the low type buyer accepts only when menu 1 is chosen. The expected payoffs for the agents are the same as in a pooling equilibrium with $m^P = 5$ of Γ. Lastly, the intermediary makes an expected profit of $\hat{V} = \frac{1}{2}(2 - 2.5) + \frac{1}{2}\frac{1}{2}(7-5) = \frac{1}{4} > 0$.

Condition (II.7) from Proposition II.5 is violated here, as the parameters satisfy $h = 4 < H = 5$. Therefore, there can not exist an efficient intermediated equilibrium where the outside option subgame equilibrium has both seller types trading with the high type buyer at prices $\hat{m}_l^o = \hat{m}_h^o = \mathbb{E}(v|H) = 6$. To see why, observe that the direct trade outside option subgame equilibrium payoffs at those TIOLI offers are $\hat{U}_l^o = 3$ and $\hat{U}_h^o = 1$. In an efficient intermediated equilibrium, the maximum amounts the intermediary can charge the buyer are $\hat{m}_b[l] \leq 2$ for low type seller's menu choice and $\hat{m}_b[h] \leq 7$ for high type seller's menu choice. Similarly, in light of the outside option payoffs, the minimum amounts the seller types must be paid are $\hat{m}_s[l] \geq 3$ and $\hat{m}_s[h] \geq 6$. Thus, in an efficient intermediated equilibrium, the expected profits of the intermediary satisfy $\hat{V} \leq \frac{1}{2}(2-3) + \frac{1}{2}\frac{1}{2}(7-6) = -\frac{1}{4} < 0$, which violates equilibrium optimality for the intermediary.

The intermediary's optimal mechanism can not yield strictly more efficient trade outcomes. From Remark II.1, it follows that $\tilde{Q}(l, L) = 0$ as $q = \frac{1}{2} < \frac{H-L}{H} = \frac{3}{5}$.

Lastly, consider the IPV case where $\lambda_L = \lambda_H = 0$. Then the revised valuation matrix is shown below:

$t_s \setminus t_b$	L	H
$l = 0$	2	5
$h = 4$	2	5

$$\Rightarrow \quad p = \mathbb{P}(t_s = l) = \tfrac{1}{2} = \mathbb{P}(t_b = L) = q$$

Under the parameter configuration, Lemma II.7 suggests that the unique equilibrium of Γ is P where $m^P = 5$ and the expected payoffs are $U_l^P = \frac{5}{2}$ and $U_h^P = \frac{1}{2}$. Due to Proposition II.6, there are no efficient intermediated equilibria of the above game Γ^i in the IPV case, because the intermediary would run a strict loss. To see why, observe that the maximum prices charged to the buyer in equilibrium would be given by $\hat{m}_b[l] \leq 2$ and $\hat{m}_b[h] \leq 5$, while the minimum prices paid to the seller would satisfy $\hat{m}_s[l] \leq 2.5$ and $\hat{m}_s[h] \geq 5$. Therefore the expected profits of the intermediary satisfy $\hat{V} \leq \frac{1}{2}(2-2.5)+\frac{1}{2}\frac{1}{2}(5-5) = -\frac{1}{4} < 0$, which violates equilibrium condition for the intermediary.

CHAPTER III

Ex-Ante Contracting and Limits of Intermediation

III.1 Introduction

In this paper, I consider a bargaining situation between a seller and a buyer over the allocation of a single indivisible object. I restrict attention to the case of quasi-linear preferences for the bargaining parties, allowing for informational interdependences in valuations. Namely, I allow the buyer's valuation for the object to depend both on her and the seller's private information, which in turn creates informational externalities.

The aim of the paper is to analyze the equilibrium outcomes of a game where the seller designs the trading mechanism at ex-ante stage before learning his private information, and then the seller and the buyer execute the trading mechanism at interim stage after both have learned their respective private information.

There are two important features to the game being analyzed; (i) seller has all the bargaining power, (ii) seller has full commitment. The former assumption implies that the seller, when designing the contract, will try to maximize his ex-ante expected payoff. The second feature implies that, the seller can credibly commit to the mechanism he designs. In other words, he does not change the rules to the bargaining protocol as defined by the contract, after he learns his type.

I first characterize the seller's ex-ante optimal mechanism that maximizes his ex-ante expected payoff. Due to the revelation principle, it is without loss of generality to restrict attention to the seller designing direct mechanisms and the agents reporting their types truthfully in equilibrium. I show that there exists a unique allocation rule as well as payoffs that are accrued in the equilibrium of this ex-ante contracting game.

Next, I discuss some properties of this optimal mechanism. I highlight how the allocation rule differs, when the valuations have no informational interdependence versus when there is interdependence. I also establish existence of the seller's ex-ante optimal mechanism by construction. Finally, I discuss efficiency of the ex-ante optimal mechanism for the seller. Using ex-ante incentive efficiency[1] from Holmström and Myerson (1983) as the notion of Pareto efficiency, I show that the seller's ex-ante optimal mechanism is also ex-ante incentive efficient.

Altogether the findings suggest that, the seller's ex-ante payoff maximizing contract achieves Pareto optimal outcomes. Furthermore, as it is the seller who designs the optimal contract, the implementation does not require the presence of a third-party intermediary. Hence one way to interpret these results is that, they describe a natural limit to the benefits of intermediation.

III.1.1 Related Literature

This paper relates to the vast literature on mechanism design in (bilateral) trade problems. The pioneering works in Myerson (1979) and Myerson (1981) initiated the literature on optimal mechanism design that maximizes revenues for the designer. Another focus area has been efficient mechanism design, where for example Krishna and Perry (1998) considers the efficient VCG mechanisms in general Bayesian environments. Mechanism design in trade problems for the case of valuations with informational interdependences have also been anaylzed. For these environments, Mezzetti (2004) considers efficiency aspects while the companion paper Mezzetti (2007) considers surplus maximization for the seller. Note that in the latter paper, the seller does not possess any private information. More specifically in the context of bilateral trade, Riley and Zeckhauser (1983) and Samuelson (1984) analyze seller's revenue maximization problem while Myerson and Satterthwaite (1983) consider efficient mechanisms, all under the independent private values realm. In this paper, the emphasis is on the ex-ante expected payoff maximizing mechanism for the seller, when there are informational interdependences in buyer's valuation.

The benchmark notion of Pareto-efficiency for the mechanisms in Bayesian games have been defined in the seminal work of Holmström and Myerson (1983).

[1] A mechanism is *incentive efficient*, if it attains second-best outcomes subject to the familiar feasibility constraints of incentive compatibility and individual rationality. In a Bayesian setting, however, there exists three stages to evaluate the Pareto relationships; ex-ante, interim and ex-post. The differences reflect the knowledge of the players' private information at the time of efficiency evaluation.

There have also been various papers that aim to characterize the constrained-efficient, or equivalently the incentive efficient, mechanisms in the bilateral trade context. Samuelson (1984), Wilson (1985) and Williams (1987) analyze ex-ante incentive efficiency in the case of independent private values, while Gresik (1991) characterizes the ex-ante incentive efficient mechanisms in the general valuations with informational externalities. For the interim incentive efficient mechanisms, on the other hand, Gresik (1996) and Ledyard and Palfrey (2007) consider the case of independent private values while Kucuksenel (2012) addresses the interdependent valuations cases.

The rest of the paper is organized as follows. In Section III.2 I present the model and the ex-ante contracting game. In Section III.3, the main result on the characterization of the ex-ante optimal mechanism for the seller is presented. Its properties are also discussed in that section. All proofs are in the appendix.

III.2 Model

III.2.1 Setup

There are two risk-neutral agents, one seller and one buyer, denoted by s and b, respectively. The seller owns a single unit of an indivisible good which the buyer would like to buy. Each agent $i \in \{s, b\}$ privately observes a signal $\theta_i \in \Theta_i = [0, 1]$ where θ_s and θ_b are drawn independently from the cumulative distribution functions $F(\theta_s)$ and $G(\theta_b)$, respectively. It is assumed that these distribution functions are differentiable and are commonly known. In addition, their corresponding density functions, $f(\theta_s)$ and $g(\theta_b)$, are assumed to be strictly positive everywhere, satisfying the usual monotone hazard rate properties; F/f is strictly increasing in θ_s for the seller and $(1 - G)/g$ is strictly decreasing in θ_b for the buyer. The random variables $\boldsymbol{\theta} = (\theta_s, \theta_b)$ are private information of the seller and the buyer, respectively, and hence will be referred to as the *types* of the respective agents.

The valuation of the seller for the object $c(\boldsymbol{\theta})$ is referred to as his *cost* for brevity and satisfies $c(\boldsymbol{\theta}) = \theta_s$. The valuation of the buyer for the object $v(\boldsymbol{\theta})$, on the other hand, is referred to as her *value* and satisfies $v(\boldsymbol{\theta}) = \alpha\theta_s + (1 - \alpha)\theta_b$ for some constant $\alpha \in [0, 1)$ that is commonly known.[2] Observe that the seller's type equals his cost, while for the buyer the seller's private information appears in her valuation whenever $\alpha \in (0, 1)$, i.e. the interdependence of valuations. Note that when $\alpha = 0$, the valuations satisfy the special case of independent private values

[2] I use the male pronoun for the seller, female pronoun for the buyer.

(henceforth IPV).

III.2.2 Timing of the Trading Game

The seller and the buyer play a Bayesian bargaining game or a trading mechanism to determine (i) who gets the object and (ii) how much should the buyer pay the seller. The aim is to analyze the outcomes of this bargaining situation, when the seller designs the optimal trading mechanism that maximizes his expected payoff at ex-ante stage, i.e. before he learns his private information. Thus the timing of the trading game can be summarized as follows:

1. The seller designs a trading mechanism before learning his type.
2. The seller and the buyer observe their private information.
3. They play the trading mechanism.

The seller designing the contract ex-ante before learning his valuation implies that there is no signaling, and consequently there is no informed seller's problem. Due to the Revelation Principle,[3] all equilibrium feasible allocations and the associated outcomes of an arbitrary trading mechanism can be characterized, without loss of generality, by restricting attention to the truth-telling equilibria of direct revelation mechanisms that satisfy incentive compatibility (henceforth IC) and individual rationality (henceforth IR) constraints. For completeness, a proof of the revelation principle in this setup is provided in Appendix C.2.

III.2.3 Feasible Direct Revelation Mechanisms

A *direct revelation mechanism* (henceforth DRM), denoted by $\gamma = \{Q, \tau\}$, consists of an allocation rule $Q : \Theta_s \times \Theta_b \to [0, 1]$ and a payment rule $\tau : \Theta_s \times \Theta_b \to \mathbb{R}$. Given any pair of reported types $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_s, \tilde{\theta}_b)$, the allocation rule $Q(\tilde{\boldsymbol{\theta}})$ specifies the probability of the object changing hands from the seller to the buyer, while the payment rule $\tau(\tilde{\boldsymbol{\theta}})$ specifies the monetary transfer from buyer to seller.

Given a DRM γ and any pair of type realization $\boldsymbol{\theta}$, the payoffs to the agents under truthful type reporting are equal to:

$$u_s(\boldsymbol{\theta}) = \tau(\boldsymbol{\theta}) - \theta_s Q(\boldsymbol{\theta})$$
$$u_b(\boldsymbol{\theta}) = [\alpha\theta_s + (1-\alpha)\theta_b]Q(\boldsymbol{\theta}) - \tau(\boldsymbol{\theta})$$

[3] See for example Myerson (1979), Myerson (1981).

The following expressions define the (interim) expected trade probabilities and transfers for an agent i with type θ_i from reporting type $\tilde{\theta}_i$ whenever the other agent reports their type truthfully:

$$q_s(\tilde{\theta}_s|\theta_s) = \int_0^1 Q(\tilde{\theta}_s, \theta_b) dG(\theta_b), \qquad q_b(\tilde{\theta}_b|\theta_b) = \int_0^1 Q(\theta_s, \tilde{\theta}_b) dF(\theta_s)$$
$$t_s(\tilde{\theta}_s|\theta_s) = \int_0^1 \tau(\tilde{\theta}_s, \theta_b) dG(\theta_b), \qquad t_b(\tilde{\theta}_b|\theta_b) = \int_0^1 \tau(\theta_s, \tilde{\theta}_b) dF(\theta_s)$$

Using the expressions above, the (interim) expected utility for an agent i with type θ_i from reporting type $\tilde{\theta}_i$, whenever the other agent reports their type truthfully, can be described as follows:

$$U_s(\tilde{\theta}_s|\theta_s) = t_s(\tilde{\theta}_s|\theta_s) - \theta_s q_s(\tilde{\theta}_s|\theta_s)$$
$$U_b(\tilde{\theta}_b|\theta_b) = \int_0^1 [\alpha\theta_s + (1-\alpha)\theta_b] Q(\theta_s, \tilde{\theta}_b) dF(\theta_s) - t_b(\tilde{\theta}_b|\theta_b)$$

A DRM satisfies IC if and only if honest type reporting defines a Bayesian-Nash equilibrium. This means that for each agent $i \in \{s, b\}$, the expected payoffs satisfy $U_i(\theta_i) \equiv U_i(\theta_i|\theta_i) \geq U_i(\tilde{\theta}_i|\theta_i)$, where $\tilde{\theta}_i$ is an arbitrary type report of agent i. Similarly, a DRM satisfies IR if and only if the expected payoffs under truthful type reporting are weakly greater than the respective outside options of the agents, which are normalized to 0. Therefore, IR is satisfied iff $U_i(\theta_i) \geq 0$ for all possible types of each agent. Finally, a DRM is *feasible* if and only if it satisfies IC and IR constraints.

III.2.4 Preliminaries

Before presenting the main results, some preliminary results that are well-known from the literature on mechanism design are presented. The following remark summarizes the necessary conditions implied by a feasible DRM.

Remark III.1.
If A DRM $\gamma = \{Q, \tau\}$ is feasible, then the following must be true:

- *The allocation function satisfies* monotonicity *where,*

$$\frac{dq_s(\theta_s)}{d\theta_s} \leq 0 \quad \forall\theta_s \qquad \textit{and} \qquad \frac{dq_b(\theta_b)}{d\theta_b} \geq 0 \quad \forall\theta_b$$

- *The expected payoffs satisfy:*

$$U_s(\theta_s) = U_s(1) + \int_{\theta_s}^{1} q_s(x)dx \qquad \forall\theta_s \quad \textit{where} \quad U_s(1) \geq 0$$

$$U_b(\theta_b) = U_b(0) + (1-\alpha)\int_{0}^{\theta_b} q_b(y)dy \qquad \forall\theta_b \quad \textit{where} \quad U_b(0) \geq 0$$

The proofs are omitted as they follow from standard results.[4] Note that the expressions for the buyer's expected payoffs are attained by exploiting the binding downward IC constraints. This yields the expression for the expected payoffs (up to the constant $U_b(0)$) for each type θ_b, which equals her expected information rents. Although seller's private information influences her valuation $v(\theta_s, \theta_b)$, the independence of the distributions between the private information leaves θ_b as the only source of her information rents, which in turn explains the fraction $(1-\alpha)$ appearing in the expression for $U_b(\theta_b)$.

Observe that given a feasible DRM $\gamma = \{Q, \tau\}$, the outcomes from the truthful equilibrium can be characterized by only considering the allocation rule $Q(\theta_s, \theta_b)$, along with the constant expected payoffs for the respective "worst" types, i.e. $U_s(1)$ and $U_b(0)$.

It will be convenient to define the following functions.

$$\psi_b(\theta_b;\mu) = \theta_b - \mu\frac{1-G(\theta_b)}{g(\theta_b)} \qquad \text{and} \qquad \psi_s(\theta_s;\mu) = \theta_s + \mu\frac{F(\theta_s)}{(1-\alpha)f(\theta_s)}$$

where $\mu \in [0,1]$ is a constant. When $\mu = 1$, the functions $\psi_b(\theta_b;1) \equiv \psi_b(\theta_b)$ and $\psi_s(\theta_s;1) \equiv \psi_s(\theta_s)$ are the familiar *virtual valuations*. Thus, the general functions $\psi_b(\theta_b;\mu)$ and $\psi_s(\theta_s;\mu)$ with parameter μ are referred to as the *μ-weighted virtual valuations*. Observe that the monotone hazard rate property implies that both the μ-weighted virtual valuations are strictly increasing over their entire domains, i.e. for all $\theta_s, \theta_b \in [0,1]$, as well as for any weight $\mu \in [0,1]$.

Lastly, as the virtual valuation function for the buyer $\psi_b(\theta_b;1) = \psi_b(\theta_b)$ is strictly increasing in θ_b, its inverse function can also be defined. Let $\psi_b^{-1} : [-\frac{1}{g(0)}, 1] \to [0,1]$ be the inverse function where $\psi_b^{-1}(y) = \{x|\psi_b(x) = y\}$. Observe that $\psi_b^{-1}(y)$ is also strictly increasing over its entire domain.

[4]Derivations in the context of bilateral trade may be found in Myerson and Satterthwaite (1983) for the case of IPV and Gresik (1991) for the case of interdependent valuations.

III.3 Ex-ante Optimal Mechanism for the Seller

The main aim is to characterize the equilibrium outcomes of the seller's ex-ante contracting game. Due to the revelation principle, this amounts to characterizing the outcomes of the truthful equilibria of feasible DRM's that maximize the seller's ex-ante expected payoff, which are referred to as the *ex-ante optimal mechanisms for the seller*. The following result provides the characterization:

Proposition III.1.
The seller's ex-ante optimal mechanism is characterized as follows:

- *There exists a unique allocation rule Q^* where for any pair of truthful type reports $\boldsymbol{\theta} = (\theta_s, \theta_b)$, the trade probabilities equal:*

$$Q^*(\theta_s, \theta_b) = \begin{cases} 1 & \text{if } \theta_b \geq \kappa^*(\theta_s) = \min\{\psi_b^{-1}(\psi_s(\theta_s; \mu)), 1\} \\ 0 & \text{o/w} \end{cases} \tag{III.1}$$

  where $\mu = 0$ for $\alpha = 0$ and $\mu \in (0,1)$ for $\alpha \in (0,1)$.

- *The expected payoffs to the agents at "worst" types satisfy $U_s^*(1) = U_b^*(0) = 0$.*

Then $\gamma^* = \{Q^*, \tau^*\}$ is an ex-ante optimal mechanism for the seller where;

$$\begin{aligned} \tau^*(\theta_s, \theta_b) &= Q^*(\theta_s, \theta_b)[\alpha\theta_s + (1-\alpha)\theta_b] - (1-\alpha)\int_0^{\theta_b} Q^*(\theta_s, y)dy \\ &+ \int_{\theta_s}^1 \int_0^1 Q^*(x, \theta_b)dG(\theta_b)dx - (1-\alpha)\int_0^1 Q^*(\theta_s, \theta_b)[\psi_b(\theta_b) - \theta_s]dG(\theta_b) \end{aligned} \tag{III.2}$$

A direct proof is provided by solving the optimization program of ex-ante expected payoff maximization subject to the feasibility constraints (IC and IR) of the DRM. There are a few points to highlight.

Firstly, the uniqueness of the allocation rule along with the expected payoffs for the "worst" types imply that the equilibrium outcomes of seller's ex-ante optimal mechanism are unique. Namely, the interim expected transfers and trade probabilities are uniquely defined. This in turn also suggests that the seller and the buyer receive a unique interim expected payoff schedule.

Secondly, observe that the optimal allocation rule is different for when $\alpha = 0$, i.e. the IPV case, versus when $\alpha \in (0,1)$, i.e. the informational interdependent valuations case. Although in both cases, the optimal allocation rule is deterministic, in the sense that there is no randomization or $Q^*(\theta_s, \theta_b) \in \{0,1\}$ for all (θ_s, θ_b), the cutoff values for the buyer's private information differ. This arises because, the parameters μ differ.[5]

In the former case of IPV, observe that $\psi_s(\theta_s; 0) = \theta_s$. This, in turn implies that $\psi_b^{-1}(\psi_s(\theta_s; 0)) = \psi_b^{-1}(\theta_s)$. Because $\psi_b^{-1}(\theta_s) \leq 1$ for all $\theta_s \leq 1$,[6] it holds that $\kappa^*(\theta_s) = \psi_b^{-1}(\theta_s)$. Therefore, the optimal allocation rule implemented in the case of IPV is equivalent to those allocations that would be implemented if the private information of the seller were commonly known, which is a result that has been previously highlighted in the literature.[7]

In the other case of informational interdependence, the value of the parameter μ is strictly between 0 and 1. Unfortunately, there is no analytic solution for the exact value of the Lagrangian multipler λ, and consequently the parameter μ. However, it can be seen that in these cases $\psi_s(\theta_s; \mu) = 1$ at some intermediate value $\bar{\theta}_s^{\mu} \in (0, 1)$. Consequently, this implies $\psi_b^{-1}(\psi_s(\bar{\theta}_s^{\mu}; \mu)) = 1$. Hence, the cutoff function satisfies $\kappa^*(\theta_s) = \begin{cases} \psi_b^{-1}(\psi_s(\theta_s; \mu)) & \text{if } \theta_s \leq \bar{\theta}_s^{\mu} \\ 1 & \text{o/w} \end{cases}$.

Last point relates to the implementation of these uniquely characterized outcomes. In the second part of Proposition III.1, an example of ex-ante optimal mechanism for the seller is provided. The important implication is that, the outcomes of seller's ex-ante optimal mechanism can be attained without the intermediation of a third party. In other words, there is no need to have an intermediary to coordinate the monetary transfers, because the construction of the transfer rule defined in (III.2) guarantees that the payments between the seller and the buyer balance each other for every possible type pair realization.

III.3.1 Pareto Optimality of Seller's Ex-ante Optimal Mechanism

An important inquiry relates to the efficiency properties of the seller's ex-ante optimal mechanism. In these environments, it has been shown that there is no feasible mechanism that implements the ex-post efficient allocation rule[8] due to the informational frictions arising from asymmetric information.[9] In that regard,

[5]This parameter appears from the optimization problem and is equal to $\mu = \frac{\lambda}{1+\lambda}$ where λ is a Lagrange-multiplier for the seller's IR constraint, i.e. $U_s(1) \geq 0$.

[6]In fact $\psi_b^{-1}(\theta_s) = 1$ only when $\theta_s = 1$ and otherwise is strictly less than 1 for any $\theta_s < 1$.

[7]See for instance Yilankaya (1999).

[8]Ex-post efficient allocation rule is given by $Q^e(\theta_s, \theta_b) = \begin{cases} 1 & \text{if } v(\boldsymbol{\theta}) \geq c(\boldsymbol{\theta}) \;\Leftrightarrow\; \theta_b \geq \theta_s \\ 0 & \text{o/w} \end{cases}$, where the simplified condition follows from plugging in the specific valuation functions.

[9]See for instance Myerson and Satterthwaite (1983) for IPV and Gresik (1991) for the case of general interdependent valuations.

the aim of this subsection is to see whether the ex-ante optimal mechanism for the seller yield second-best outcomes that achieve Pareto efficiency.

In the context of a Bayesian game, the benchmark notion of Pareto efficiency is *incentive efficiency* as defined in Holmström and Myerson (1983). According to the definition of the authors, a feasible DRM satisfies incentive efficiency if and only if there is no other feasible DRM that makes some type of an agent better off without making the other types (for all agents) worse off. The authors also highlight that, there are three stages to evaluate such a dominance relationship; ex-ante, interim and ex-post. The interest of this paper lies in the first notion, that is ex-ante incentive efficiency, which requires that there be no other feasible DRM that ex-ante Pareto dominates the payoffs for the agents.

Let $W(\gamma)$ be the social welfare function where:

$$W(\gamma) = w \int_0^1 U_s(\theta_s)dF(\theta_s) + (1-w)\int_0^1 U_b(\theta_b)dG(\theta_b) \tag{III.3}$$

where $w \in [0,1]$ is a welfare weight parameter. Then a feasible DRM γ^e is *ex-ante incentive efficient* if and only if it maximizes $W(\gamma)$ among the class of feasible DRM's, i.e. subject to the IC and IR (feasibility) constraints. Observe that ex-ante incentive efficiency is the strongest notion as it implies interim incentive efficiency, which in turn implies ex-post incentive efficiency.

In light of this definition, the following result holds.

Proposition III.2.
Ex-ante optimal mechanism for the seller is ex-ante incentive efficient.

The result simply follows from the equality between the optimization programs for the seller's ex-ante optimal mechanism problem and the ex-ante incentive efficient mechanisms. For the latter optimization program, the welfare weights w may vary between 0 and 1. In the extreme case when $w = 1$, the welfare function is equivalent to the seller's ex-ante expected utility. Since both programs are solving the same objective functions subject to the feasibility constraints, they yield the same optimization programs and consequently, the same solutions.

This implies that the seller's ex-ante optimal mechanism achieves second-best outcomes. Thus, it would not be possible to Pareto improve the outcomes. Hence, ex-ante commitment enables the bargaining parties to reach (constrained) efficient outcomes. In turn, these results highlight a case where the intermediation can not yield Pareto improvements. In other words, the analysis identifies a natural limit to the benefits of intermediation.

Finally, in light of the similarities in the setup, it is worth comparing the result with the findings from Chapter II. Previously, it was shown in Proposition II.6 that, when the buyer's valuations satisfied the IPV framework, then the presence of an intermediary would never lead to a strict improvement in allocative efficiency. In other words, the informed seller's optimal TIOLI offers attain second-best outcomes, which can not be Pareto improved. In the IPV framework, Yilankaya (1999) establishes the equivalence between the informed seller's interim-optimal mechanism and his ex-ante optimal mechanism. Since Proposition III.2 establishes the Pareto optimality of the ex-ante optimal mechanism for the seller, it also validates the aforementioned result for the special case of IPV from the previous chapter.

In the case of interdependent valuations, Theorem II.1 establishes that an intermediary can lead to Pareto improvements in allocative efficiency if and only if condition (II.6) is satisfied. At the core of the intuition is the following: When the informed seller tries to maximize its revenues by making TIOLI offers, the presence of lemon problem arising from informational externalities (i.e. the valuation interdepence) leads to inefficient underselling. An intermediary may alleviate these inefficiencies by means of cross-subsidization, even if the intermediary aims to maximize its expected profits. Here, on the other hand, it is shown that, when the seller has the ability to commit and design the terms of trade ex-ante, the intermediary can no longer provide efficiency improvements. This is simply because the parties can reach second-best outcomes by the seller designing a Pareto optimal mechanism equipped with a payment rule such as the one described in (III.2) that is more sophisticated than simple TIOLI offers. This, in turn, suggests that the ability to ex-ante contract is a strong remedy to Pareto inefficiencies and an impediment to intermediation providing any added value.

APPENDIX A

Appendix to Chapter I

A.1 Useful Results

A.1.1 Mapping General Messages to Type Reports

In several proofs of the intermediation game with multiple intermediaries, initial steps are taken by considering direct mechanisms, where the message spaces for the agents are restricted to their respective type spaces. However, such mechanisms need to be represented in a way that they accommodate general message spaces. In order to address this issue, I define the following mappings $\phi_s : M_s^i \to C$ and $\phi_j : M_j^i \to V_j$ where latter is defined for all buyers j:

$$\phi_s(m_s^i) = \begin{cases} m_s^i & \text{if } m_s^i = c' \in [0,1] \\ 1 & \text{o/w} \end{cases} \qquad \phi_j(m_j^i) = \begin{cases} m_j^i & \text{if } m_j^i = v_j' \in [0,1] \\ 0 & \text{o/w} \end{cases} \tag{A.1}$$

For all intermediaries i, these mappings are defined identically where each takes the corresponding agent's message and maps it into their respective type space. Given any vector of messages $(m_s^i, \mathbf{m}_b^i)$, denote the vector of message mappings by $\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i) = \big(\phi_s(m_s^i), \phi_1(m_1^i), \ldots, \phi_n(m_n^i)\big)$. Then, taking composition of a general mechanism with the mappings yields a direct mechanism. Hence, in the proofs general mechanisms may be defined retrospectively from a described direct mechanism.

A.1.2 Seller's Equilibrium Beliefs

In the intermediation game with multiple intermediaries, I look for sequential equilibria. Below is an observation that is used in several proofs.

Lemma A.1.
Given an equilibrium $\hat{\mathcal{E}}$ of the intermediation game, seller's beliefs $\hat{\boldsymbol{\beta}}_s$ satisfy:

$$\hat{\beta}_s(\mathbf{v}|c,i,\boldsymbol{\gamma}) = g(\mathbf{v}) \qquad \forall c \in C,\ \forall i \in I,\ \textit{and}\ \forall \boldsymbol{\gamma} \in \Gamma^m$$

Proof of Lemma A.1. For an equilibrium $\hat{\mathcal{E}}$, I will derive the seller's consistent beliefs. Given equilibrium strategies $\hat{\eta}$ and $\hat{\boldsymbol{\gamma}}$, consider a sequence of totally mixed strategies $\{\eta^h, \boldsymbol{\gamma}^h\}_{h=1}^{\infty}$ where $\eta^h \in \Delta I$ and $\gamma^{i,h} \in \Delta\Gamma$ for each i, and $\lim_{h\to\infty}(\gamma^{1,h},\ldots,\gamma^{m,h},\eta^h) = (\hat{\gamma}^1,\ldots,\hat{\gamma}^m,\hat{\eta})$. Let $\boldsymbol{\gamma}^h = (\gamma^{1,h},\ldots,\gamma^{m,h})$ denote the shorthand for the sequences of totally mixed strategies for all intermediaries. Also let $\beta_s^h(\mathbf{v}|c,i,\boldsymbol{\gamma})$ denote the seller's beliefs at information set $(c,i,\boldsymbol{\gamma})$ when the totally mixed strategies $(\eta^h,\boldsymbol{\gamma}^h)$ are played. Lastly, let $\mathbb{P}(\mathbf{v},c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)$ denote the probability of the information node $(\mathbf{v},c,i,\boldsymbol{\gamma})$ being reached under the totally mixed strategies $(\eta^h,\boldsymbol{\gamma}^h)$. Then the consistent beliefs in any equilibrium are equal to:

$$\begin{aligned}
\hat{\beta}_s(\mathbf{v}|c,i,\boldsymbol{\gamma}) &= \lim_{h\to\infty} \beta_s^h(\mathbf{v}|c,i,\boldsymbol{\gamma}) \\
&= \lim_{h\to\infty} \frac{\mathbb{P}(\mathbf{v},c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)}{\int_{\mathbf{v}} \mathbb{P}(\mathbf{v},c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)d\mathbf{v}} \\
&= \lim_{h\to\infty} \frac{g(\mathbf{v})\mathbb{P}(c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)}{\int_{\mathbf{v}} g(\mathbf{v})\mathbb{P}(c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)d\mathbf{v}} \\
&= \lim_{h\to\infty} \frac{g(\mathbf{v})}{\int_{\mathbf{v}} g(\mathbf{v})d\mathbf{v}} \frac{\mathbb{P}(c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)}{\mathbb{P}(c,i,\boldsymbol{\gamma}|\eta^h,\boldsymbol{\gamma}^h)} = g(\mathbf{v})
\end{aligned}$$

Observe that from second to third line, the joint density function $g(\mathbf{v})$ is taken out of the information node probabilities. This is due to the independence of the type distributions.

□

A.2 Proofs

Proof of Lemma I.1. Let $\hat{\mathcal{E}}$ be an equilibrium of the intermediation game with multiple intermediaries. Consider the DRM $\delta = (\bar{\mathbf{Q}},\bar{\boldsymbol{\tau}})$ which, for a given vector of type reports $(c',\mathbf{v}')$, implements the outcomes constructed as follows:

$$\begin{aligned}
\bar{Q}_j(c',\mathbf{v}') &= \sum_{i\in I} \hat{Q}_j^i\big(\hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v}',\hat{\gamma}^i)\big)\mathbb{I}\big(\hat{\eta}(c',\hat{\boldsymbol{\gamma}}) = i\big) \\
\bar{\tau}_j(c',\mathbf{v}') &= \sum_{i\in I} \hat{\tau}_j^i\big(\hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v}',\hat{\gamma}^i)\big)\mathbb{I}\big(\hat{\eta}(c',\hat{\boldsymbol{\gamma}}) = i\big) \\
\bar{\tau}_s(c',\mathbf{v}') &= \sum_{i\in I} \hat{\tau}_s^i\big(\hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v}',\hat{\gamma}^i)\big)\mathbb{I}\big(\hat{\eta}(c',\hat{\boldsymbol{\gamma}}) = i\big) \\
\bar{\tau}^i(c',\mathbf{v}') &= \left[\sum_{j=1}^{n} \hat{\tau}_j^i\big(\hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v}',\hat{\gamma}^i)\big) - \hat{\tau}_s^i\big(\hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v}',\hat{\gamma}^i)\big)\right]\mathbb{I}\big(\hat{\eta}(c',\hat{\boldsymbol{\gamma}}) = i\big)
\end{aligned}$$

Assuming the agents report their types truthfully, the expected trade probabilities and transfers for the players can be defined as follows:

$$
\begin{aligned}
\bar{q}_j(v_j) &= \int_0^1 \int_{\mathbf{v}_{-j}} \bar{Q}_j(c, v_j, \mathbf{v}_{-j}) dG_{-j}(\mathbf{v}_{-j}) dF(c) \\
\bar{q}_s(c) &= \int_{\mathbf{v}} \left[\sum_{j=1}^{n} \bar{Q}_j(c, \mathbf{v}) \right] dG(\mathbf{v}) \\
\bar{t}_j(v_j) &= \int_0^1 \int_{\mathbf{v}_{-j}} \bar{\tau}_j(c, v_j, \mathbf{v}_{-j}) dG_{-j}(\mathbf{v}_{-j}) dF(c) \\
\bar{t}_s(c) &= \int_{\mathbf{v}} \bar{\tau}_s(c, \mathbf{v}) dG(\mathbf{v}) \\
\bar{t}^i &= \int_0^1 \int_{\mathbf{v}} \bar{\tau}^i(c, \mathbf{v}) dG(\mathbf{v}) dF(c)
\end{aligned}
$$

I want to show that δ as described above is a feasible DRM and its truthful type-telling equilibrium is payoff equivalent to $\hat{\mathcal{E}}$.

Given $\hat{\mathcal{E}}$ of the original game, it will be convenient to abuse notation to define the expected payoff to the seller with type c at subgame $s.i$ who deviates and announces $m_s^i = \hat{\mu}_s^i(c', \hat{\boldsymbol{\gamma}})$ as follows:

$$
\hat{U}_s^i(c'|c) = U_s\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c', \hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i | c\big) = U_s\big(\hat{\boldsymbol{\gamma}}, i, m_s^i, \hat{\boldsymbol{\mu}}_b^i | c\big)
$$

Similarly, define the corresponding shorthand notation for the expected payoff to buyer j with type v_j at subgame $j.i$ who deviates and announces $m_j^i = \hat{\mu}_j^i(v_j', \hat{\gamma}_i)$:

$$
\hat{U}_j^i(v_j'|v_j) = U_j\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i, \hat{\mu}_j^i(v_j', \hat{\gamma}_i), \hat{\boldsymbol{\mu}}_{-j}^i | v_j\big) = U_j\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i, m_j^i, \hat{\boldsymbol{\mu}}_{-j}^i | v_j\big)
$$

Observe that $\hat{U}_s^i(c)$ and $\hat{U}_j^i(v_j)$'s are the expected payoffs prescribed by the equilibrium strategies for the corresponding agents' types from subgames $s.i$ and $j.i$'s, respectively. Furthermore, the agents' expected payoffs from equilibrium can be written as $\hat{U}_s(c) = \sum_{i \in I} \hat{U}_s^i(c) \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big)$ and $\hat{U}_j(v_j) = \sum_{i \in I} \hat{U}_j^i(v_j) \big[\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c)\big]$.

Going back to the DRM, assume that the buyers report types truthfully. Then by changing order of integration and summation, the expected payoff for a seller of type c from reporting c' can be expressed as follows:

$$\begin{aligned}\bar{U}_s(c'|c) &= \bar{t}_s(c') - \bar{q}_s(c')c \\ &= \int_{\mathbf{v}} \left[\sum_{i \in I} U_s\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c', \hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v}, \hat{\gamma}_i)\big) \mathbb{I}\big(\hat{\eta}(c', \hat{\boldsymbol{\gamma}}) = i\big) \right] dG(\mathbf{v}) \\ &= \sum_{i \in I} \left[\int_{\mathbf{v}} U_s\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c', \hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v}, \hat{\gamma}_i)\big) dG(\mathbf{v}) \right] \mathbb{I}\big(\hat{\eta}(c', \hat{\boldsymbol{\gamma}}) = i\big) \\ &= \sum_{i \in I} \hat{U}_s^i(c'|c) \mathbb{I}\big(\hat{\eta}(c', \hat{\boldsymbol{\gamma}}) = i\big)\end{aligned}$$

Before considering the corresponding expected payoffs for the buyers, remember that if an equilibrium mechanism $\hat{\gamma}^i$ is reached with positive probability, i.e. $\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c) > 0$, then each buyer j has equilibrium beliefs $\hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i})|v_j, \hat{\gamma}^i\big)$ given by:

$$\hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, \hat{\gamma}^i\big) = \begin{cases} \frac{\mathbb{I}\left(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\right) f(c)}{\int_0^1 \mathbb{I}\left(\hat{\eta}(x, \hat{\boldsymbol{\gamma}}) = i\right) dF(x)} g_{-j}(\mathbf{v}_{-j}) & \text{if } \boldsymbol{\gamma}_{-i} = \hat{\boldsymbol{\gamma}}_{-i} \\ 0 & \text{o/w} \end{cases}$$

Then for any subgame that is reached with positive probability, the following rearranging holds:

$$\begin{aligned}\hat{U}_j^i(v_j'|v_j) &= \int U_j\big(\hat{\gamma}^i, \boldsymbol{\gamma}^{-i}, i, \hat{\mu}^{s.i}(c, \hat{\gamma}_i, \boldsymbol{\gamma}_{-i}), \hat{\mu}_j^i(v', \hat{\gamma}^i), \hat{\boldsymbol{\mu}}_{-j}^i(\mathbf{v}_{-j}, \hat{\gamma}^i)\big) \\ &\qquad \hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i})|v_j, \hat{\gamma}^i\big) d(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}) \\ &= \int_0^1 \int_{\mathbf{v}_{-j}} U_j\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c, \hat{\boldsymbol{\gamma}}), \hat{\mu}_j^i(v', \hat{\gamma}^i), \hat{\boldsymbol{\mu}}_{-j}^i(\mathbf{v}_{-j}, \hat{\gamma}^i)\big) dG_{-j}(\mathbf{v}_{-j}) \frac{\mathbb{I}\left(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\right) dF(c)}{\int_0^1 \mathbb{I}\left(\hat{\eta}(x, \hat{\boldsymbol{\gamma}}) = i\right) dF(x)}\end{aligned}$$

Otherwise $\hat{\gamma}^i$ is not reached with positive probability, or $\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c) = 0$. In that case the following equalities hold:

$$0 = \hat{U}_j^i(v_j'|v_j) \Big[\int_0^1 \mathbb{I}\big(\hat{\eta}(c, \hat{\boldsymbol{\gamma}}) = i\big) dF(c) \Big]$$

Then this observation can be used to expand the expected payoffs to buyers in the DRM. Assume that buyer j with type v_j reports v_j', while the seller and all other buyers report types truthfully. Then, by changing order of integration and summation, his expected payoff can be written as follows:

$$\begin{aligned}\bar{U}_j(v_j'|v_j) &= \bar{q}_j(v_j')v_j - \bar{t}_j(v_j')\\ &= \int_0^1 \int_{\mathbf{v}_{-j}} \sum_{i\in I} U_j\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}}), \hat{\mu}_j^i(v',\hat{\gamma}^i), \hat{\boldsymbol{\mu}}_{-j}^i(\mathbf{v}_{-j},\hat{\gamma}^i)\big)\\ &\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\big)dG_{-j}(\mathbf{v}_{-j})dF(c)\\ &= \sum_{i\in I} \hat{U}_j^i(v_j'|v_j)\left[\int_0^1 \mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\big)dF(c)\right]\end{aligned}$$

From previous equalities, it is immediate to see that the agents' equilibrium payoffs from $\hat{\mathcal{E}}$ are equivalent to those from the DRM under truthful type reporting; i.e. $\bar{U}_s(c) = \hat{U}_s(c)$ for all c and $\bar{U}_j(v_j) = \hat{U}_j(v_j)$ for all j and v_j. In order to verify that the DRM under truthful type reporting yields expected payoffs to the intermediaries that are same as under the original equilibrium, consider the following:

$$\begin{aligned}\bar{U}^i &= \bar{t}^i\\ &= \int_0^1 \int_{\mathbf{v}} \left[\sum_{j=1}^n \hat{\tau}_j^i\big(\hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\hat{\gamma}^i)\big) - \hat{\tau}_s^i\big(\hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\hat{\gamma}^i)\big)\right] \mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\big)dG(\mathbf{v})dF(c)\\ &= \int_0^1 \int_{\mathbf{v}} U^i\big(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\hat{\gamma}^i)\big)dG(\mathbf{v})\mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}})=i\big)dF(c)\\ &= \hat{U}^i\end{aligned}$$

This establishes the payoff equivalence of δ under truthful type reporting. Next, I need to show that δ is a feasible DRM, hence satisfies IR, IC, RES and BB. It is immediate to see that IR is satisfied since the equilibrium payoffs $\hat{U}_s(c)$, $\hat{U}_j(v_j)$ and $\hat{U}^i$'s are all weakly greater than the outside option payoff of 0. Allocation rule $\bar{\mathbf{Q}}$ satisfies RES, because it is constructed as a convex combination of the allocation rules $\hat{\mathbf{Q}}^i$ where each satisfies RES. BB is also satisfied as it can simply be shown by changing the order of summations across intermediaries and buyers.

Only IC remains to be shown. Starting with the seller, observe that the equilibrium of the original game implies that seller's communication strategy $\hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}})$ is optimal at every subgame $s.i$ for every type c. In other words, it holds that:

$$\hat{U}_s^i(c) = U_s(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i(c,\hat{\boldsymbol{\gamma}}), \hat{\boldsymbol{\mu}}_b^i|c) \geq U_s(\hat{\boldsymbol{\gamma}}, i, m_s^i, \hat{\boldsymbol{\mu}}_b^i|c) \qquad \forall m_s^i \in M_s^i$$

Hence $\hat{U}_s^i(c) \geq \hat{U}_s^i(c'|c)$ where deviation message considered is $m_s^i = \hat{\mu}_s^i(c',\hat{\boldsymbol{\gamma}})$. Furthermore, combining with the optimality of the seller's entry choice yields:

$$\begin{aligned}\bar{U}_s(c) = \hat{U}_s(c) &= \max_{i\in I}\left\{\hat{U}^i_s(c)\right\} \\ &= \sum_{i\in I}\hat{U}^i_s(c)\mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}}) = i\big) \\ &\geq \sum_{i\in I}\hat{U}^i_s(c)\mathbb{I}\big(\hat{\eta}_i(c',\hat{\boldsymbol{\gamma}}) = i\big) \geq \sum_{i\in I}\hat{U}^i_s(c'|c)\mathbb{I}\big(\hat{\eta}(c',\hat{\boldsymbol{\gamma}}) = i\big) = \bar{U}_s(c'|c)\end{aligned}$$

Next consider the IC for the buyers. Observe that, due to the decomposition of $\bar{U}_j(v'_j|v_j)$, optimality of truthful type-reporting should be argued only for expected payoffs $\hat{U}^i_j(v'_j|v_j)$ from subgames that are reached with positive probability. In that regard, the equilibrium of the original game implies that buyer's communication strategy $\hat{\mu}^i_j(v_j,\hat{\gamma}^i)$ is optimal for every type v_j at such subgames. In other words, it holds that:

$$\hat{U}^i_j(v_j) = U^i_j(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}^i_s, \hat{\mu}^i_j(v_j,\hat{\gamma}^i), \hat{\boldsymbol{\mu}}^i_{-j}|v_j) \geq U^i_j(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}^i_s, m^i_j, \hat{\boldsymbol{\mu}}^i_{-j}|v_j) \qquad \forall m^i_j \in M^i_j$$

Hence $\hat{U}^i_j(v_j) \geq \hat{U}^i_j(v'_j|v_j)$ where deviation message considered is $m^i_j = \hat{\mu}^i_j(v'_j,\hat{\gamma}^i)$. Then it follows that:

$$\begin{aligned}\bar{U}_j(v_j) = \hat{U}_j(v_j) &= \sum_{i\in I}\hat{U}^i_j(v_j)\left[\int_0^1 \mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}}) = i\big)dF(c)\right] \\ &\geq \sum_{i\in I}\hat{U}^i_j(v'_j|v_j)\left[\int_0^1 \mathbb{I}\big(\hat{\eta}(c,\hat{\boldsymbol{\gamma}}) = i\big)dF(c)\right] = \bar{U}_j(v'_j|v_j)\end{aligned}$$

□

Proof of Proposition I.1. The seller-optimal DRM solves the following program $P1$:

$$P1 \rightarrow \qquad \max_{\{\bar{\mathbf{Q}},\bar{\boldsymbol{\tau}}\}}\left\{\int_0^1 \bar{U}_s(c)dF(c)\right\}$$

subject to

$$\begin{aligned}&\text{IR}: && \bar{U}_s(c),\ \bar{U}_j(v_j),\ \bar{U}^i \geq 0 && \forall c,\ \forall j \text{ and } v_j,\ \forall i \\ &\text{IC}: && \bar{U}_s(c) \geq \bar{U}_s(c'|c),\ \text{and } \bar{U}_j(v_j) \geq \bar{U}_j(v'_j|v_j) && \forall c, c',\ \forall j \text{ and } v_j, v'_j \\ &\text{RES}: && 0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1,\ \text{and } \bar{Q}_s(c,\mathbf{v}) = \textstyle\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v}) \leq 1 && \forall c, \mathbf{v} \\ &\text{BB}: && \textstyle\sum_{j=1}^n\left[\int_0^1 \bar{t}_j(v_j)dG_j(v_j)\right] - \int_0^1 \bar{t}_s(c)dF(c) - \sum_{i=1}^m \bar{t}^i = 0 &&\end{aligned}$$

From Remark I.1, a feasible DRM δ has to satisfy the following necessary conditions:

$$\bar{U}_s(1) \geq 0, \qquad \bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx, \qquad \frac{d\bar{q}_s(c)}{dc} \leq 0 \qquad \forall c$$
$$\bar{U}_j(0) \geq 0, \qquad \bar{U}_j(v_j) = \bar{U}_j(0) + \int_0^{v_j} \bar{q}_j(y_j)dy_j, \qquad \frac{d\bar{q}_j(v_j)}{dv_j} \geq 0 \qquad \forall j \text{ and } v_j$$

$$\bar{U}_s(1) = \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c, \mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v})dF(c) - \sum_{j=1}^n \bar{U}_j(0) - \sum_{i \in I} \bar{t}^i$$

Plugging in the expected payoff for $\bar{U}_s(c)$ the objective function becomes:

$$\int_0^1 \bar{U}_s(c)dF(c) = \int_0^1 \left[\bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx\right] dF(c)$$
$$= \bar{U}_s(1) + \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c, \mathbf{v})dG(\mathbf{v})F(c)dc$$

Firstly, observe that $\bar{U}_j(0) = 0$ optimally for every j, as otherwise one can decrease $\bar{U}_j(0)$ and increase $\bar{U}_s(1)$ by the same amount and improve the objective function's value. Similarly, $\bar{t}^i = 0$ optimally for every i, as otherwise $\bar{U}_s(1)$ could be increased without violating the conditions, thereby improving the value of the objective function. Note that, these are in line with Proposition I.1 where $U_j^*(0) = 0$ and $U^{i,*} = 0$.

Simplifications suggest that $\bar{U}_s(1) = \int_0^1 \int_{\mathbf{v}} \left[\sum_{j=1}^n \bar{Q}_j(c, \mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right]\right] dG(\mathbf{v})dF(c)$. Plugging into $P1$ yields the following simplified problem, $P1'$:

$$P1' \rightarrow \qquad \max_{\bar{\mathbf{Q}}} \left\{ \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c, \mathbf{v}) \left[\psi_j(v_j) - c\right] dG(\mathbf{v})dF(c) \right\}$$

$$\text{subject to} \qquad \frac{d\bar{q}_s(c)}{dc} \leq 0 \qquad \forall c$$
$$\frac{d\bar{q}_j(v_j)}{dv_j} \geq 0 \qquad \forall j \text{ and } v_j$$
$$0 \leq \bar{Q}_j(c, \mathbf{v}) \leq 1, \text{ and } \bar{Q}_s(c, \mathbf{v}) = \sum_{j=1}^n \bar{Q}_j(c, \mathbf{v}) \leq 1 \qquad \forall c, \mathbf{v}$$
$$\bar{U}_s(1) = \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c, \mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v})dF(c) \geq 0$$

Ignoring the monotonicity and seller's simplified IR constraints momentarily, the linearity of the objective function in the choice variable along with the RES constraints suggest that the pointwise maximizer is a simple step function given by:

$$Q_j^*(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j) \geq \max\left\{c, \max_{k\neq j}\{\psi_k(v_k)\}\right\} \\ 0 & \text{o/w} \end{cases} \qquad \forall j \in \{1,\dots,n\}$$

This is precisely the same allocation function as (I.1) from Proposition I.1. Next, I check that the pointwise maximizer $\mathbf{Q}^*$ satisfies the conditions of the simplified problem. RES is satisfied, because for any given vector of type reports $(c,\mathbf{v})$, at most one $Q_j^*(c,\mathbf{v})$ equals 1. Also observe that for any reported vector of types the trade probability $Q_j^*(c,\mathbf{v})$ is weakly increasing in v_j and weakly decreasing in c for every j. Each trade probability being weakly decreasing in c suggests that $Q_s^*(c,\mathbf{v})$ is also weakly decreasing in c, as it is equal to the sum of Q_j^*'s across all j. Hence the monotonicity constraints are also satisfied. Finally, consider the seller's IR constraint. I want to show that $U_s^*(1) = 0$ as stated in the proposition. By rearranging terms, the following is attained:

$$\begin{aligned} U_s^*(1) &= \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n Q_j^*(c,\mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v}) dF(c) \\ &= \int_0^1 \left[\int_{\mathbf{v}} \sum_{j=1}^n \left[Q_j^*(c,\mathbf{v}) \left[\psi_j(v_j) - c\right] - \int_c^1 Q_j^*(x,\mathbf{v}) dx \right] dG(\mathbf{v}) \right] dF(c) \end{aligned}$$

Noting that $\psi_j(v_j) < 1$ for all $v_j < 1$, consider the subtracted integral term from above:

$$\int_c^1 Q_j^*(x,\mathbf{v}) dx = \begin{cases} 0 & \text{if } Q_j^*(c,\mathbf{v}) = 0 \\ \psi_j(v_j) - c & \text{if } Q_j^*(c,\mathbf{v}) = 1 \end{cases}$$

Hence plugging in above observation yields $U_s^*(1) = 0$. Finally, note that for δ^* the revenue generated from agents $R(\delta^*)$ is also equal to $U_s^*(1)$. This again verifies the earlier point that $U^{i,*} = t^{i,*} = 0$ for all i. □

Proof of Corollary I.1. Consider the following program $P2$:

$$P2 \rightarrow \qquad \max_{\{\bar{\mathbf{Q}},\bar{\boldsymbol{\tau}}\}} \left\{ \sum_{i\in I} \bar{U}^i + \int_0^1 \bar{U}_s(c) dF(c) \right\}$$

subject to

$$\begin{aligned} &\text{IR}: && \bar{U}_s(c),\ \bar{U}_j(v_j),\ \bar{U}^i \geq 0 && \forall c,\ \forall j \text{ and } v_j,\ \forall i \\ &\text{IC}: && \bar{U}_s(c) \geq \bar{U}_s(c'|c), \text{ and } \bar{U}_j(v_j) \geq \bar{U}_j(v_j'|v_j) && \forall c, c',\ \forall j \text{ and } v_j, v_j' \\ &\text{RES}: && 0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1, \text{ and } \bar{Q}_s(c,\mathbf{v}) = \textstyle\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v}) \leq 1 && \forall c, \mathbf{v} \\ &\text{BB}: && \textstyle\sum_{j=1}^n \left[\int_0^1 \bar{t}_j(v_j) dG_j(v_j)\right] - \int_0^1 \bar{t}_s(c) dF(c) - \sum_{i=1}^m \bar{U}^i = 0 \end{aligned}$$

Plugging in the expected payoff for $\bar{U}_s(c)$ and $\bar{U}^i$'s implied by the necessary conditions of a feasible DRM from Remark I.1, the objective function becomes:

$$\sum_{i\in I}\bar{U}^i + \int_0^1 \bar{U}_s(c)dF(c) = \int_0^1\int_{\mathbf{v}}\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})[\psi_j(v_j)-c]dG(\mathbf{v})dF(c) - \sum_{j=1}^n \bar{U}_j(0)$$

Observe that $\bar{U}_j(0)=0$ optimally for every j, as otherwise one can decrease $\bar{U}_j(0)$ without violating IR of buyer j and increase the value of the objective function. The simplifications yield:

$$P2' \rightarrow \qquad \max_{\bar{\mathbf{Q}}}\left\{\int_0^1\int_{\mathbf{v}}\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\left[\psi_j(v_j)-c\right]dG(\mathbf{v})dF(c)\right\}$$

subject to

$$\frac{d\bar{q}_s(c)}{dc}\le 0 \qquad \forall c$$

$$\frac{d\bar{q}_j(v_j)}{dv_j}\ge 0 \qquad \forall j \text{ and } v_j$$

$$0\le \bar{Q}_j(c,\mathbf{v})\le 1, \text{ and } \bar{Q}_s(c,\mathbf{v})=\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\le 1 \qquad \forall c,\mathbf{v}$$

$$\bar{U}^i\ge 0 \qquad \forall i$$

$$\bar{U}_s(1)=\int_0^1\int_{\mathbf{v}}\sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\left[\psi_j(v_j)-\psi_s(c)\right]dG(\mathbf{v})dF(c)-\sum_{i\in I}\bar{U}^i\ge 0$$

Observe that $P2'$ is the same problem as in $P1'$ with slightly different IR constraints. Remember that the solution of $P1'$ satisfies $\bar{U}^i=\bar{U}_s(1)=0$. Hence here the same solution, namely δ^*, solves $P2'$ and consequently $P2$.

Assume that some feasible DRM $\bar{\delta}$ has a truthful type-telling equilibrium where $\bar{U}_s(c)\ge U_s^*(c)$ for all c. Then δ^* solving $P2$ suggests that for all feasible DRM $\bar{\delta}$, the following holds:

$$\sum_{i\in I}U^{i,*}+\int_0^1 U_s^*(c)dF(c)=\int_0^1 U_s^*(c)dF(c)\ge\sum_{i\in I}\bar{U}^i+\int_0^1\bar{U}_s(c)dF(c)$$

Since $\bar{U}_s(c)\ge U_s^*(c)$ for all c, it follows that $\sum_{i\in I}\bar{U}^i\le 0$. Feasibility of $\bar{\delta}$ then suggests $\bar{U}^i=0$ for all i. Finally, plugging in $\bar{U}^i=0$ to the above inequality yields $\int_0^1 U_s^*(c)dF(c)\ge\int_0^1\bar{U}_s(c)dF(c)$. Hence, it follows that the condition $\bar{U}_s(c)\ge U_s^*(c)$ for all c is satisfied only if the inequalities hold with equalities. □

Proof of Theorem I.1. I prove the uniqueness of equilibrium outcomes in two steps.

Step 1: I show that at any equilibrium $\hat{\mathcal{E}}$:

1. $\hat{U}^i=0$ for all i.

2. $\hat{U}_j(0)=0$ for all j.

3. $\hat{\mathbf{Q}}$ is such that for every j there exists reservation values $r_j(c)$ that are weakly increasing in c and the probability of trade $\hat{Q}_j$ is given by:

$$\hat{Q}_j(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j) > \max_{k\neq j} \psi_k(v_k) \text{ and } v_j > r_j(c) \\ 0 & \text{o/w} \end{cases}$$

Proof. Let $\hat{\mathcal{E}}$ be an equilibrium of the intermediation game and $\bar{\delta}$ be the feasible DRM that has a payoff equivalent truthful type-telling equilibrium. Firstly, remember that the sum of intermediaries' expected payoffs is equal to the revenue generated from the agents due to the BB:

$$R(\bar{\delta}) = \sum_{i\in I} \bar{U}^i = \int_0^1 \left[\int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\big[\psi_j(v_j) - c\big] dG(\mathbf{v}) - \bar{U}_s(c) \right] dF(c) - \sum_{j=1}^n \bar{U}_j(0)$$

Secondly at the truthful type-telling equilibrium of $\bar{\delta}$ for all c, it holds that $\bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx$ where $\bar{U}_s(1) \geq 0$. Denote by $\kappa(c)$ the unique solution that satisfies $\prod_{j=1}^n G_j(\psi_j^{-1}(\kappa(c))) = 1 - \bar{q}_s(c)$ for every c. Observe that the right hand side is the probability of the seller keeping the object, while the left hand side can be interpreted as the cumulative probability of each buyer having valuation less than $\psi_j^{-1}(\kappa(c))$.

By monotone hazard property $\psi_j^{-1}(x)$ is strictly increasing in x. Similarly, $G_j(v_j)$ is strictly increasing as the distribution is assumed to have positive density over whole support. Since $\bar{q}_s(c)$ is weakly decreasing in c, it implies that $\kappa(c)$ is weakly increasing in c. Now define the following allocation rule $\tilde{\mathbf{Q}}$ where each trade probability is defined as:

$$\tilde{Q}_j(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j) > \max_{k\neq j} \psi_k(v_k) \text{ and } v_j > \psi_j^{-1}(\kappa(c)) \\ 0 & \text{o/w} \end{cases}$$

Observe that by construction of $\kappa(c)$'s, it holds that $\tilde{q}_s(c) = \bar{q}_s(c)$ for all c. Hence, if $\tilde{U}_s(1) = \bar{U}_s(1)$, then $\tilde{U}_s(c) = \bar{U}_s(c)$ for all c, as well. Furthermore, consider a feasible DRM $\tilde{\delta}$ with allocation rule $\tilde{\mathbf{Q}}$ as described above. Let $\tilde{U}_s(1) = \bar{U}_s(1)$ and $\tilde{U}_j(0) = 0$ for all j. Then the revenue generated from agents in $\tilde{\delta}$ is given by:

$$R(\tilde{\delta}) = \int_0^1 \left[\int_{\mathbf{v}} \sum_{j=1}^n \tilde{Q}_j(c,\mathbf{v})\big[\psi_j(v_j) - c\big] dG(\mathbf{v}) - \tilde{U}_s(c) \right] dF(c)$$

Clearly $R(\tilde{\delta}) \geq R(\bar{\delta})$, because $\tilde{q}_s(c) = \bar{q}_s(c)$ and $\tilde{U}_s(c) = \bar{U}_s(c)$ for all c, while $\tilde{Q}_j(c,\mathbf{v})$ puts full probability to highest virtual valuation buyer.

Now assume that $\hat{\mathcal{E}}$ violates at least one of the three conditions listed above. Then, for the payoff equivalent feasible DRM $\bar{\delta}$ it means that either $\bar{U}^i > 0$ for some i, or $\bar{U}_j(0) > 0$

for some j, or $\bar{\mathbf{Q}}$ is different than $\tilde{\mathbf{Q}}$ over some positive measure of agent types (or any combination of those three). In any of those cases, it holds that $R(\tilde{\delta}) > 0$.

I will show that in this situation, there exists at least one intermediary who has a strictly profitable deviation mechanism. In particular, since there are at least two intermediaries, there is certainly one intermediary whose equilibrium payoff satisfies $\hat{U}^i \leq R(\tilde{\delta})/2$. Hence it is enough to show that the expected profits from deviation are strictly more than $R(\tilde{\delta})/2$.

In order to describe the deviation mechanism, first consider the direct mechanism $\acute{\gamma}$ with allocation rule $\acute{\mathbf{Q}}$ and transfer rule $\acute{\boldsymbol{\tau}}$, where for any vector of type reports $(c', \mathbf{v}')$:

$$\acute{Q}_j(c', \mathbf{v}') = \begin{cases} (1-2\epsilon) + \frac{\epsilon}{n}\big(v'_j + (1-c')\big) & \text{if } \psi_j(v'_j) \geq \max\Big\{\kappa(c'), \max_{k \neq j}\{\psi_k(v'_k)\}\Big\} \\ \frac{\epsilon}{n}\big(v'_j + (1-c')\big) & \text{o/w} \end{cases} \qquad \forall j$$

$$\acute{\tau}_j(c', \mathbf{v}') = \acute{Q}_j(c', \mathbf{v}')v'_j - \int_0^{v'_j} \acute{Q}_j(c', y_j, \mathbf{v}_{-j})dy_j \qquad \forall j$$

$$\acute{\tau}_s(c', \mathbf{v}') = \acute{Q}_s(c', \mathbf{v}')c' + \int_{c'}^1 \acute{Q}_s(x, \mathbf{v}')dx + \acute{U}_s(1)$$

where $\epsilon \in (0, \frac{1}{2})$ is a constant, $\acute{U}_s(1)$ is a nonnegative constant (expected payoff for worst type of the seller), and $\kappa(c)$ is the weakly increasing function described earlier from $\bar{q}_s(c)$'s. Observe firstly that by construction $\acute{Q}_j(c', \mathbf{v}') \in [0, 1]$ and $Q_s(c', \mathbf{v}') = \sum_{j=1}^n Q_j(c', \mathbf{v}') \leq 1$ for every possible vector $(c', \mathbf{v}') \in C \times \mathbf{V}$. These suggest that $\acute{\mathbf{Q}}$ satisfies RES. Secondly given any vector of type reports, for all j the trade probability $\acute{Q}_j(c', \mathbf{v}')$ is strictly monotonic (decreasing) in c' and strictly monotonic (increasing) in v'_j.

Now I want to construct the deviation mechanism for intermediary i using $\acute{\gamma}$. I adjust $\acute{\gamma}$ into a general mechanism by using the mappings $\boldsymbol{\phi} = (\phi_s, \phi_1, \ldots, \phi_n)$ defined in (A.1) from Appendix A.1.1. Given any vector of messages $(m_s^i, \mathbf{m}_b^i)$, denote the vector of message mappings by $\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i) = \big(\phi_s(m_s^i), \phi_1(m_1^i), \ldots, \phi_n(m_n^i)\big)$. Define the deviation mechanism $\check{\gamma}^i = (\check{\mathbf{Q}}^i, \check{\boldsymbol{\tau}}^i)$ by taking the composition of the direct mechanism $\acute{\gamma}$ and message mapping $\boldsymbol{\phi}$. In particular for any vector of messages $(m_s^i, \mathbf{m}_b^i)$, the mechanism $\check{\gamma}^i$ has outcome functions defined as follows:

$$\check{Q}_j^i(m_s^i, \mathbf{m}_b^i) = \acute{Q}_j\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{Q}_j(c', \mathbf{v}') \qquad \forall j$$

$$\check{\tau}_j^i(m_s^i, \mathbf{m}_b^i) = \acute{\tau}_j\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{\tau}_j(c', \mathbf{v}') \qquad \forall j$$

$$\check{\tau}_s^i(m_s^i, \mathbf{m}_b^i) = \acute{\tau}_s\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{\tau}_s(c', \mathbf{v}')$$

where $\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i) = (c', \mathbf{v}') \in C \times \mathbf{V}$. In the remainder of the proof, I will show that for a correct choice of ϵ, deviation to announcing mechanism $\check{\gamma}^i$ instead of $\hat{\gamma}^i$ yields a strictly higher expected profits for intermediary i. To be more specific, deviation profits will be strictly more than $R(\tilde{\delta})/2$ and hence contradict the initial assumption that $\hat{\mathcal{E}}$ were an equilibrium of the intermediation game.

For the moment assume that some intermediary i unilaterally deviates and announces $\check{\gamma}^i$ while the rest continue announcing the assumed equilibrium mechanisms $\hat{\boldsymbol{\gamma}}^{-i}$. I claim that it is strictly dominant for both agents to report their true types if they are at the subgame following entry to intermediary i. In other words, $\hat{\mu}_s^i(c, \check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = c$ for all c and $\hat{\mu}_j^i(v_j, \check{\gamma}^i) = v_j$ for all j and v_j.

Let me start by looking at buyer j's optimal communication strategy. In order to argue that it is strictly dominant for every buyer to report his type v_j truthfully, the following needs to hold for every v_j:

$$\check{U}_j(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, v_j, \mathbf{m}_{-j}^i | v_j) > \check{U}_j(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, m_j^i, \mathbf{m}_{-j}^i | v_j) \qquad \forall m_s^i, m_j^i, \mathbf{m}_{-j}^i$$

Namely, the ex-post realized payoff of buyer j with type v_j must be strictly better than any other payoff he gets by sending any other message no matter what message the seller and other buyers send. Denote the message mapping by $(c', \mathbf{v}') = \boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)$. If the vector of messages sent is $(m_s^i, \mathbf{m}_b^i)$, then the ex-post realized payoff for buyer j with type v_j :

$$\begin{aligned}
&\check{U}_j(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, \mathbf{m}_b^i | v_j) \\
&\quad = \check{Q}_j^i(m_s^i, \mathbf{m}_b^i) v_j - \check{\tau}_j^i(m_s^i, \mathbf{m}_b^i) \\
&\quad = \acute{Q}_j(c', \mathbf{v}')(v_j - v_j') + \int_0^{v_j'} \acute{Q}_j(c', y_j, \mathbf{v}_{-j}) dy_j \\
&\quad = \begin{cases} (1-2\epsilon)\big(v_j - \kappa(c')\big) + \frac{\epsilon}{n}(1-c')v_j \\ \qquad + \frac{\epsilon}{n} v_j' \big(v_j - \frac{v_j'}{2}\big) & \text{if } \psi_j(v_j') > \max\Big\{\kappa(c'), \max\limits_{k \neq j}\{\psi_k(v_k')\}\Big\} \\ \frac{\epsilon}{n}(1-c')v_j + \frac{\epsilon}{n} v_j' \big(v_j - \frac{v_j'}{2}\big) & \text{o/w} \end{cases}
\end{aligned}$$

In order to find the payoff maximizing message, I look at the first order condition which yields $v_j' = v_j$. Since the second derivative is equal to $-\epsilon/n < 0$, the candidate optimum is the unique maximizer. Hence for all buyers it is strictly dominant to report their type v_j truthfully. Equivalently put $\hat{\mu}_j^i(v_j, \check{\gamma}^i) = v_j$ or every v_j type buyer reports his type truthfully no matter what his (out-of-equilibrium) beliefs $\hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i})|v_j, \check{\gamma}^i\big)$ are. Note that, this is the case, even though the deviation mechanism $\check{\gamma}^i$ is not a direct mechanism.

Next looking at the seller's ex-post payoffs, strict dominance of truthful type reporting for every c requires the following inequality to hold:

$$\check{U}_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, c, \mathbf{m}_b^i | c) > \check{U}_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, \mathbf{m}_b^i | c) \qquad \forall m_s^i, \mathbf{m}_b^i$$

Again denoting by $(c', \mathbf{v}') = \boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)$, for any vector of sent messages $(m_s^i, \mathbf{m}_b^i)$, the ex-post realized payoff for a c type seller is equal to:

$$\begin{aligned}
&\check{U}_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, \mathbf{m}_b^i|c) \\
&\quad = \check{\tau}_s^i(m_s^i, \mathbf{m}_b^i) - \check{Q}_s^i(m_s^i, \mathbf{m}_b^i)c' \\
&\quad = \acute{U}_s(1) + \acute{Q}_s(c', \mathbf{v}')(c' - c) + \int_{c'}^{1} \acute{Q}_s(x, \mathbf{v}')dx \\
&\quad = \begin{cases} \acute{U}_s(1) + (1-2\epsilon)\big(\kappa^{-1}(\psi_k(v_k')) - c\big) & \\ \qquad + \frac{\epsilon}{n}\big[\sum_{j=1}^{n} v_j'\big](1-c) + \epsilon(1-c')\big(\frac{1+c'}{2} - c\big) & \text{if } \psi_k(v_k') > \kappa(c') \\ \acute{U}_s(1) + \frac{\epsilon}{n}\big[\sum_{j=1}^{n} v_j'\big](1-c) + \epsilon(1-c')\big(\frac{1+c'}{2} - c\big) & \text{o/w} \end{cases}
\end{aligned}$$

where $\psi_k(v_k')$ is the highest virtual valuation among all buyers and the inverse of weakly increasing $\kappa(c)$ is defined as $\kappa^{-1}(\psi_k(v_k')) = \inf\{x : \kappa(x) = \psi_k(v_k')\}$. Similar to before, in order to find the payoff maximizing message, I look at the first order condition which yields $c' = c$. Since the second derivative is equal to $-\epsilon < 0$, the candidate optimum is the unique maximizer. Hence for all seller types, it is strictly dominant to report c truthfully, or $\hat{\mu}_s^i(c, \check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = c$ no matter what the buyers report.

I have shown that whenever i announces the deviation mechanism $\check{\gamma}^i$, it is strictly dominant for all agents to report their types truthfully in the subgame where i is chosen by the seller. Furthermore, since their realized payoffs are nonnegative for all possible reports, so are their expected payoffs from the subgame.

Next, consider the seller's entry decision upon observing the mechanism profile $\{\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}\}$. Firstly observe that for every c, the expected payoff $U_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c)$ for any intermediary $k \neq i$ is at most $\hat{U}_s(c)$. To see why, note that as all the other intermediaries continue announcing $\hat{\gamma}^k$, for any buyer j his communication strategy remains to be $\hat{\mu}_j^k(v_j, \hat{\gamma}^k)$ at any subgame $j.k$ for $k \neq i$. Hence the expected payoff $U_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c)$ can not be more than $U_s(\hat{\boldsymbol{\gamma}}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c)$ for any c, as otherwise it would violate the optimality of seller's assumed equilibrium communication strategy $\hat{\mu}_s^k(c, \hat{\boldsymbol{\gamma}})$. Similarly, the optimality of seller's equilibrium entry strategy implies that $\hat{U}_s(c) = U_s(\hat{\boldsymbol{\gamma}}, \hat{\eta}, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c) \geq U_s(\hat{\boldsymbol{\gamma}}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c)$. It is worth stressing that the information structure in the model eliminates the common agency problem and in turn these important inequalities hold. Then, it follows that for all c and $k \neq i$:

$$\bar{U}_s(c) = \hat{U}_s(c) \geq U_s(\hat{\boldsymbol{\gamma}}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c) \geq U_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k|c)$$

Abusing notation, denote by $\check{U}_s^i(c)$ the expected payoff from choosing $i$following the deviation mechanism $\check{\gamma}^i$ announcement. Hence, $\check{U}_s^i(c) = U_s(\check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i|c)$. Given

$\hat{\mu}^i_j(v_j, \check{\gamma}^i) = v_j$ for all j and v_j, the expected payoff $\check{U}^i_s(c)$ is equal to:

$$\begin{aligned}
\check{U}^i_s(c) &= \int_{\mathbf{v}} \left[\check{\tau}^i_s(c,\mathbf{v}) - \check{Q}_s(c,\mathbf{v})c\right] dG(\mathbf{v}) \\
&= \int_c^1 \int_{\mathbf{v}} \acute{Q}_s(x,\mathbf{v}) dG(\mathbf{v}) dx + \acute{U}_s(1) \\
&= \int_c^1 \int_{\mathbf{v}} \left[(1-2\epsilon)\tilde{Q}_s(x,\mathbf{v}) + \sum_{j=1}^n \frac{\epsilon}{n}\left[v_j + (1-x)\right]\right] dG(\mathbf{v}) dx + \acute{U}_s(1) \\
&= (1-2\epsilon)\int_c^1 \bar{q}_s(x)dx + \epsilon\left[(1-c)\frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} + \frac{(1-c)^2}{2}\right] + \acute{U}_s(1)
\end{aligned}$$

where $\mathbb{E}(v_j) = \int_0^1 v_j dG_j(v_j)$. Observe that from second to third line, I exploited the construction of $\acute{Q}$ and substituted in $\tilde{Q}$. This manipulation is possible, because $\tilde{Q}$ equals to 1 precisely in the cases where the term $(1-2\epsilon)$ appears in $\acute{Q}$. In that regard, the first term of the last line simply follows from the equality of $\tilde{q}_s(c) = \bar{q}_s(c)$. The middle term in the last line is a simplified version of the middle term (multiplied by $\frac{\epsilon}{n}$) from the third line. Letting $\acute{U}_s(1) = \bar{U}_s(1) + 2\epsilon$ yields that for every $c \in [0,1]$:

$$\begin{aligned}
\check{U}^i_s(c) &= (1-2\epsilon)\int_c^1 \bar{q}_s(x)dx + \epsilon\left[(1-c)\tfrac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} + \tfrac{(1-c)^2}{2}\right] + \acute{U}_s(1) \\
&= \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx + \epsilon\left[2 - 2\int_c^1 \bar{q}_s(x)dx + (1-c)\tfrac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} + \tfrac{(1-c)^2}{2}\right] \\
&> \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx = \bar{U}_s(c) = \tilde{U}_s(c) = \hat{U}_s(c)
\end{aligned}$$

where the strict inequality follows from the fact that both ϵ and the large summation inside square brackets are strictly positive for all c. The equality from the last line follows from the payoff equivalence of $\bar{\delta}$. Given the strict inequality, it can be seen that if i deviates and offers $\check{\gamma}^i$, then all the seller types would choose i.

Finally, evaluate the expected profits generated by $\check{\gamma}^i$. Given $\hat{\eta}(c, \check{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i$ for all c and the agents report types truthfully, the expected profits of intermediary i are given by:

$$\begin{aligned}
\check{U}^i &= \int_0^1 \int_{\mathbf{v}} \left[\sum_{j=1}^n \check{\tau}^i_j(c,\mathbf{v}) - \check{\tau}^i_s(c,\mathbf{v})\right] dG(\mathbf{v}) dF(c) \\
&= \int_0^1 \left[\int_{\mathbf{v}} \sum_{j=1}^n \acute{Q}^i_j(c,\mathbf{v})\left[\psi_j(v_j) - c\right] dG(\mathbf{v}) - \check{U}^i_s(c)\right] dF(c) \\
&= (1-2\epsilon)\int_0^1 \left[\int_{\mathbf{v}} \sum_{j=1}^n \tilde{Q}_j(c,\mathbf{v})\left[\psi_j(v_j) - c\right] dG(\mathbf{v}) - \tilde{U}_s(c)\right] dF(c) - \epsilon\Delta \\
&= (1-2\epsilon)R(\tilde{\delta}) - \epsilon\Delta
\end{aligned}$$

where the term Δ is the following constant:

$$\Delta = \int_0^1 \left[\left[2 + 2\bar{U}^s(1) + \frac{(1-c)\sum_{j=1}^n \mathbb{E}(v_j)}{n} + \frac{(1-c)^2}{2} \right] - \sum_{j=1}^n \int_{\mathbf{v}} \left[\frac{v_j+1-c}{n} \right] \left[\psi_j(v_j) - c \right] dG(\mathbf{v}) \right] dF(c) < 8$$

I want to show that $\check{U}^i$ is strictly more than $\frac{R(\tilde{\delta})}{2}$. Evaluating the inequality:

$$\begin{aligned} \check{U}^i &\geq (1-2\epsilon)R(\tilde{\delta}) - \epsilon\Delta > (1-2\epsilon)R(\tilde{\delta}) - 8\epsilon > \frac{R(\tilde{\delta})}{2} \\ &\Rightarrow \epsilon < \frac{R(\tilde{\delta})}{4R(\tilde{\delta})+16} \end{aligned}$$

Hence, if $\epsilon = \frac{R(\tilde{\delta})}{8R(\tilde{\delta})+32} > 0$, then there exists some intermediary i such that deviating to $\check{\gamma}^i$ is strictly profitable, which in turn contradicts the equilibrium assumption. □

This concludes Step 1. Hence it has been shown that in any equilibrium, the intermediaries make zero expected profits. Furthermore, worts type of buyers receive zero expected payoffs and there exists a reservation value $r_j(c) = \psi_j^{-1}(\kappa(c))$ such that the allocation rule awards the object to the highest virtual valuation whenever his valuation is above the corresponding reservation value. For a full equilibrium outcome characterization, all that needs to be done is to define these reservation values and highest cost seller's expected payoff.

Step 2: I show that at any equilibrium $\hat{\mathcal{E}}$, the unique set of reservation values used in the allocation rule are given by $r_j(c) = \psi_j^{-1}(c)$. This implies that $\hat{U}_s(1) = 0$.

Hence in any equilibrium $\hat{\mathcal{E}}$, the unique payoff equivalent feasible DRM is the seller-optimal δ^*.

Proof. In order to show that for any equilibrium the payoff equivalent DRM is equal to δ^*, I pursue a proof by contradiction. Hence, assume that there exists an equilibrium $\hat{\mathcal{E}}$ where the payoff equivalent direct mechanism $\bar{\delta}$ satisfies the results from Step 1, yet the reservation value schedules $\bar{r}_j(c)$ are different than $r_j^*(c) = \psi_j^{-1}(c)$. Then the expected trade probability of the seller under $\bar{\delta}$ equals $\bar{q}_s(c) = 1 - \prod_{j=1}^n G_j(\bar{r}_j(c))$. Since G_j's are strictly increasing, a different reservation value schedule yields a different expected trade probability schedule and consequently expected payoff schedule; $\bar{U}_s(c)$ is different than $U_s^*(c)$. Note that it can not be the case that $\bar{U}_s(c) \geq U_s^*(c)$ for all c, as otherwise by Corollary I.1 it would have to be the case that $\bar{U}_s(c) = U_s^*(c)$ for all c. In light of this, since $\bar{\delta}$ is not the seller-optimal DRM, then there exists some positive measure of seller types for which $\bar{U}_s(c) < U_s^*(c)$.

Because both $\bar{U}_s(c)$ and $U_s^*(c)$ for all c are absolutely continuous and bounded payoff schedules, the following maximum can be defined. Let $\acute{c} = \text{argmax}_c\{U_s^*(c) - \bar{U}_s(c)\}$ and $\Delta = U_s^*(\acute{c}) - \bar{U}_s(\acute{c})$. Observe that $\Delta > 0$ and $\acute{c} < 1$. Then define two types $0 \le \underline{c} < \bar{c} < 1$ where $U_s^*(c) - \bar{U}_s(c) \ge \frac{\Delta}{2}$ for all $c \in [\underline{c}, \bar{c}]$. The existence of the interval follows from the continuity of the two payoff schedules. Clearly $\acute{c} \in [\underline{c}, \bar{c}]$. Also note that, by feasibility (more specifically IR) of $\bar{\delta}$, the expected payoff for worst type seller satisfies $\bar{U}_s(1) \ge 0$, which in turn implies that $\bar{c} < 1$ since $U_s^*(1) - \bar{U}_s(1) = -\bar{U}_s(1) \le 0 < \frac{\Delta}{2}$.

Now for every j, define the reservation values $\acute{r}_j(c) = \begin{cases} r^*(c+\alpha) & \text{if } c \le 1-\alpha \\ 1 & \text{o/w} \end{cases}$ where α is a small positive constant and $r_j^*(c) = \psi_j^{-1}(c)$ for every c represents the optimal reservation value schedules from δ^*. Noting that $r_j^*(c)$ is strictly increasing in c and $0 < r_j^*(0) < r_j^*(1) = 1$, here by construction the adjusted reservation value schedule $\acute{r}_j(c)$ is also strictly increasing over $c \in [0, 1-\alpha)$ and is flat at $\acute{r}_j(c) = 1$ for all $c \in [1-\alpha, 1]$.

In order to describe the deviation mechanism, first consider the direct mechanism $\acute{\gamma}$ with allocation rule $\acute{\mathbf{Q}}$ and transfer rule $\acute{\boldsymbol{\tau}}$, where for any vector of type reports $(c', \mathbf{v}')$:

$$\acute{Q}_j(c', \mathbf{v}') = \begin{cases} (1-\epsilon) + \frac{\epsilon}{n} v_j' & \text{if } \psi_j(v_j') > \max\limits_{k \neq j} \psi_k(v_k') \text{ and } v_j' > \acute{r}_j(c) \\ \frac{\epsilon}{n} v_j' & \text{o/w} \end{cases}$$

$$\acute{\tau}_j(c', \mathbf{v}') = \acute{Q}_j(c', \mathbf{v}') v_j' - \int_0^{v_j'} \acute{Q}_j(c', y_j, \mathbf{v}_{-j}) dy_j$$

$$\acute{\tau}_s(c', \mathbf{v}') = \acute{Q}_s(c', \mathbf{v}') c' + \int_{c'}^1 \acute{Q}_s(x, \mathbf{v}') dx$$

where ϵ is a small positive constant and $\acute{r}_j(c)$ is the adjusted reservation value defined above. Firstly by construction $\acute{Q}_j(c', \mathbf{v}') \in [0,1]$ and $Q_s(c', \mathbf{v}') = \sum_{j=1}^n Q_j(c', \mathbf{v}') \le 1$ for every possible vector $(c', \mathbf{v}') \in C \times \mathbf{V}$. These suggest that $\acute{\mathbf{Q}}$ satisfies RES. Secondly given any vector of type reports, for all j the trade probability $\acute{Q}_j(c', \mathbf{v}')$ is monotonic (decreasing) in c' and strictly monotonic (increasing) in v_j'.

Now I want to construct the deviation mechanism for intermediary i using $\acute{\gamma}$. However, I need to accommodate the general message space. Again taking advantage of the agents' message mappings $\boldsymbol{\phi}$ the same way as in Step 1, define the deviation mechanism $\breve{\gamma}^i = (\breve{\mathbf{Q}}^i, \breve{\boldsymbol{\tau}}^i)$ by taking the composition of the direct mechanism $\acute{\gamma}$ and message mapping $\boldsymbol{\phi}$. In particular for any vector of messages $(m_s^i, \mathbf{m}_b^i)$, the mechanism $\breve{\gamma}^i$ has outcome functions defined as follows:

$$\breve{Q}_j^i(m_s^i, \mathbf{m}_b^i) = \acute{Q}_j\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{Q}_j(c', \mathbf{v}') \qquad \forall j$$

$$\breve{\tau}_j^i(m_s^i, \mathbf{m}_b^i) = \acute{\tau}_j\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{\tau}_j(c', \mathbf{v}') \qquad \forall j$$

$$\breve{\tau}_s^i(m_s^i, \mathbf{m}_b^i) = \acute{\tau}_s\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \acute{\tau}_s(c', \mathbf{v}')$$

where $\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i) = (c', \mathbf{v}') \in C \times \mathbf{V}$. In the remainder of the proof, I will show that for particular choices of ϵ and α, deviation to announcing mechanism $\breve{\gamma}^i$ instead of $\hat{\gamma}^i$ yields strictly positive expected profits for intermediary i. This yields a contradiction with the initial assumption that $\hat{\mathcal{E}}$ were an equilibrium of the intermediation game.

For the moment assume that some intermediary i unilaterally deviates and announces $\breve{\gamma}^i$ while the rest of the intermediaries continue announcing the assumed equilibrium mechanisms $\hat{\boldsymbol{\gamma}}^{-i}$. I will first examine the agents' optimal communication strategies at the subgame following entry to intermediary i when the profile of mechanisms is $\{\breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}\}$.

Start by looking at buyer j's optimal communication strategy. I claim that for all j at subgame $j.i$ upon observing $\breve{\gamma}^i$, it is strictly dominant for the buyer to report his true type. In other words, $\hat{\mu}_j^i(v_j, \breve{\gamma}^i) = v_j$ for all v_j. To verify this claim I need to show that for every v_j the following (strict) inequality holds:

$$\breve{U}_j(\breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, v_j, \mathbf{m}_{-j}^i | v_j) > \breve{U}_j(\breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, m_j^i, \mathbf{m}_{-j}^i | v_j) \qquad \forall m_s^i, m_j^i, \mathbf{m}_{-j}^i$$

Namely, the ex-post realized payoff of a v_j type for buyer j must be strictly better than any other payoff he gets by sending any other message no matter what message the seller and other buyers send. Again denoting by $(c', \mathbf{v}') = \boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)$, the ex-post realized payoff for a v_j type of buyer j whenever the vector of messages sent is equal to $(m_s^i, \mathbf{m}_b^i)$:

$$\begin{aligned}
&\breve{U}_j(\breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, \mathbf{m}_b^i | v_j) \\
&\quad = \breve{Q}_j^i(m_s^i, \mathbf{m}_b^i) v_j - \breve{\tau}_j^i(m_s^i, \mathbf{m}_b^i) \\
&\quad = \acute{Q}_j(c', \mathbf{v}')(v_j - v_j') + \int_0^{v_j'} \acute{Q}_j(c', y_j, \mathbf{v}_{-j}) dy_j \\
&\quad = \begin{cases} (1-\epsilon)\big(v_j - \kappa(c')\big) + \frac{\epsilon}{n} v_j'\big(v_j - \frac{v_j'}{2}\big) & \text{if } \psi_j(v_j') > \max\Big\{\psi_j(\acute{r}_j(c')), \max\limits_{k \neq j}\{\psi_k(v_k')\}\Big\} \\ \frac{\epsilon}{n} v_j'\big(v_j - \frac{v_j'}{2}\big) & \text{o/w} \end{cases}
\end{aligned}$$

In order to find the payoff maximizing message, look at the first order condition which yields $v_j' = v_j$. Since the second derivative is equal to $-\epsilon/n < 0$, the candidate optimum is the unique maximizer. Hence for all buyers it is strictly dominant to report their type v_j truthfully. Equivalently put $\hat{\mu}_j^i(v_j, \breve{\gamma}^i) = v_j$ or every v_j type buyer reports his type truthfully no matter what his (out-of-equilibrium) beliefs $\hat{\beta}_j^i\big(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}) | v_j, \breve{\gamma}^i\big)$ are.

Next consider the seller's optimal communication strategy. The seller understands that the buyers report their types truthfully by virtue of strict dominance. Then letting

$c' = \phi^s(m_s^i)$, the expected trade probabilities $\breve{q}_s^i(m_s^i)$ for every c equals:

$$\begin{aligned}
\breve{q}_s^i(m_s^i) = \acute{q}_s(c') &= \int_0^1 \acute{Q}_s(c', \mathbf{v}) dG(\mathbf{v}) \\
&= (1-\epsilon)\left[1 - \int_0^{\acute{r}_1(c')} \dots \int_0^{\acute{r}_n(c')} dG_1(v_1)\dots dG_n(v_n)\right] + \frac{\epsilon}{n}\sum_{j=1}^n \int_0^1 v_j dG_j(v_j) \\
&= (1-\epsilon)\Big[1 - \prod_{j=1}^n G_j\big(\acute{r}_j(c')\big)\Big] + \epsilon \frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n}
\end{aligned}$$

where $\mathbb{E}(v_j) = \int_0^1 v_j dG_j(v_j) \in (0,1)$. Remember that for every j, the adjusted reservation value schedule $\acute{r}_j(c)$ is strictly increasing for all $c \in [0, 1-\alpha)$ and is flat over the remaining types $[1-\alpha, 1]$. Then it follows that $\acute{q}_s(c')$ is strictly decreasing over the range $c' \in [1-\alpha)$, while it is flat over the remaining range. Abusing notation, denote by $\breve{U}_s^i(m_s^i|c)$ the expected payoff for the seller with cost c from reporting m_s^i:

$$\begin{aligned}
\breve{U}_s^i(m_s^i|c) &= U_s(\breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, i, m_s^i, \hat{\boldsymbol{\mu}}_b^i|c) \\
&= \int_{\mathbf{v}} \big[\breve{\tau}_s^i(m_s^i, \mathbf{v}) - \breve{Q}_s^i(m_s^i, \mathbf{v})c\big] dG(\mathbf{v}) \\
&= \int_{\mathbf{v}} \left[\acute{Q}_s(c', \mathbf{v})(c'-c) + \int_{c'}^1 \acute{Q}_s(x, \mathbf{v})dx\right] dG(\mathbf{v}) \\
&= \acute{q}_s(c')(c'-c) + \int_{c'}^1 \acute{q}_s(x)dx
\end{aligned}$$

The first order condition from the expected payoff maximization problem a c type seller faces yields that for $c < 1-\alpha$ reporting the unique maximizer is to report type truthfully. To see why, observe that the second order condition at the only candidate optimum is equal to $\frac{d\acute{q}_s(c')}{dc'}|_{c'=c}$ which is strictly negative when $c \in [0, 1-\alpha)$. For any $c \in [1-\alpha, 1]$, on the other hand, it is optimal to report any message $m_s^i \in M_s^i \setminus [0, 1-\alpha)$ which yields a constant payoff of $\epsilon(1-c)\frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n}$. In light of this, evaluated at the subgame equilibrium communication strategies, the expected payoff to a c type seller is equal to:

$$\breve{U}_s^i(c) = \begin{cases} \epsilon(1-c)\frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} + (1-\epsilon)U_s^*(c+\alpha) & \text{if } c \leq 1-\alpha \\ \epsilon(1-c)\frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} & \text{o/w} \end{cases}$$

where the equality follows from exploiting $\acute{r}_j(c) = r_j^*(c+\alpha)$ for $c \leq 1-\alpha$ and applying integration by substitution as shown below:

$$\begin{aligned}
\int_c^{1-\alpha} \Big[1 - \prod_{j=1}^n G_j\big(\acute{r}_j(x)\big)\Big] dx &= \int_c^{1-\alpha} \big[1 - \prod_{j=1}^n G_j\big(r_j^*(x+\alpha)\big)\big] dx \\
&= \int_c^{1-\alpha} q_s^*(x+\alpha)dx = \int_{c+\alpha}^1 q_s^*(y)dy = U_s^*(c+\alpha)
\end{aligned}$$

For the moment assume that some c type seller where $c < 1-\alpha$ participates in $\breve{\gamma}^i$. Define the expected profit i makes conditional on (attracting) such a c type, denoted by abusing notation as $\breve{U}^i(c|0 \le c < 1-\alpha)$:

$$\begin{aligned}
&\breve{U}^i(c|0 \le c < 1-\alpha)\\
&\qquad = \int_{\mathbf{v}} \Big[\sum_{j=1}^{n} \breve{\tau}_j^i\big(\hat{\mu}_s^i(c,\breve{\gamma}^i,\hat{\boldsymbol{\gamma}}^{-i}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\breve{\gamma}^i)\big) - \breve{\tau}_s^i\big(\hat{\mu}_s^i(c,\breve{\gamma}^i,\hat{\boldsymbol{\gamma}}^{-i}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\breve{\gamma}^i)\big)\Big] dG(v)\\
&\qquad = \int_{\mathbf{v}} \sum_{j=1}^{n} \acute{Q}_j(c,\mathbf{v})\Big[\psi_j(v_j) - c\Big] dG(\mathbf{v}) - \breve{U}_s^i(c)\\
&\qquad = \int_{\mathbf{v}} \sum_{j=1}^{n} \Big[(1-\epsilon)Q_j^*(c+\alpha,\mathbf{v}) + \tfrac{\epsilon}{n}v_j\Big]\Big[\psi_j(v_j) - c\Big] dG(\mathbf{v}) - \breve{U}_s^i(c)\\
&\qquad = (1-\epsilon)\left[\int_{\mathbf{v}} \sum_{j=1}^{n} Q_j^*(c+\alpha,\mathbf{v})\big[\psi_j(v_j) - c\big] dG(\mathbf{v}) - U_s^*(c+\alpha)\right]\\
&\qquad\qquad + \tfrac{\epsilon}{n}\sum_{j=1}^{n}\int_{\mathbf{v}} v_j\big[\psi_j(v_j) - 1\big] dG(\mathbf{v})\\
&\qquad = \underbrace{(1-\epsilon)\alpha q_s^*(c+\alpha)}_{\text{Strictly positive}} - \epsilon\underbrace{\int_{\mathbf{v}} \frac{\sum_{j=1}^{n} v_j\big[1-\psi_j(v_j)\big]}{n} dG(\mathbf{v})}_{\le 1}
\end{aligned}$$

where going from penultimate to the last line, I simplified the first component by substituting in the equality $U_s^*(c) = \int_{\mathbf{v}}\sum_{j=1}^{n} Q_j^*(c,\mathbf{v})[\psi_j(v_j) - c]dG(\mathbf{v})$, which holds for all c as it is a property of δ^* as shown in the Proof of Proposition I.1. It is important to note that for $c < 1-\alpha$, it holds that $r^*(c+\alpha) < 1$ and thus $q_s^*(c+\alpha) > 0$ guaranteeing that the first term in the last line is strictly positive.

Alternatively, consider some seller with type $c \in [1-\alpha, 1]$ participating in $\breve{\gamma}^i$. Then for any equilibrium message $\breve{m}_s^i = \hat{\mu}_s^i(c,\breve{\gamma}^i,\hat{\boldsymbol{\gamma}}^{-i}) \in M_s^i \setminus [0, 1-\alpha)$ she sends, the allocation outcomes equal $\breve{Q}_s^i(\breve{m}_s^i, v) = \frac{\epsilon\sum_{j=1}^{n} v_j}{n}$ with probability 1. Hence the expected profit i makes conditional on (attracting) such a c type seller is equal to:

$$\begin{aligned}
&\breve{U}^i(c|1-\alpha \le c \le 1)\\
&\qquad = \int_{\mathbf{v}} \Big[\sum_{j=1}^{n} \breve{\tau}_j^i\big(\hat{\mu}_s^i(c,\breve{\gamma}^i,\hat{\boldsymbol{\gamma}}^{-i}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\breve{\gamma}^i)\big) - \breve{\tau}_s^i\big(\hat{\mu}_s^i(c,\breve{\gamma}^i,\hat{\boldsymbol{\gamma}}^{-i}),\hat{\boldsymbol{\mu}}_b^i(\mathbf{v},\breve{\gamma}^i)\big)\Big] dG(\mathbf{v})\\
&\qquad = \int_{\mathbf{v}} \sum_{j=1}^{n} \acute{Q}_j(c,\mathbf{v})\Big[\psi_j(v_j) - c\Big] dG(\mathbf{v}) - \breve{U}_s^i(c)\\
&\qquad = -\tfrac{\epsilon}{n}\sum_{j=1}^{n}\int_{\mathbf{v}} v_j\big[1-\psi_j(v_j)\big] dG(\mathbf{v}) \ge -\epsilon
\end{aligned}$$

Inequalities above imply that, the potential losses $\breve{\gamma}^i$ makes from attracting any seller type are bounded below by $-\epsilon$. Furthermore, depending on α and ϵ's values, the mechanism might make expected profits by attracting seller with types $c < 1 - \alpha$.

Now consider the optimal entry decisions of the seller types. If the following inequalities hold, then $\hat{\eta}(c, \breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i$ for all $c \in [\underline{c}, \bar{c}]$:

$$\breve{U}_s^i(c) = (1-\epsilon)U_s^*(c+\alpha)+\epsilon(1-c)\frac{\sum_{j=1}^n \mathbb{E}(v_j)}{n} > (1-\epsilon)U_s^*(c+\alpha) \geq U_s^*(c)-\tfrac{\Delta}{2} \geq \bar{U}_s(c) \quad \text{(A.2)}$$

First (strict) inequality follows from $\epsilon > 0$, $\mathbb{E}(v_j) > 0$ for all j and $c \leq \bar{c} < 1$. The last inequality follows from the definition of the interval $[\underline{c}, \bar{c}]$. It only remains to describe the conditions under which the middle (weak) inequality is satisfied. Note that it is immediate to see that middle inequality requires α to be small enough, in particular, $\alpha < 1 - \bar{c}$.

It is established that $U_s^*(c)$ is decreasing in c. Hence $(1 - \epsilon)U_s^*(c + \alpha) < U_s^*(c + \alpha) < U_s^*(c)$ for $c \leq \bar{c} < 1 - \alpha$. Now consider the first derivative with respect to c of the following difference:

$$\frac{d}{dc}\big[(1 - \epsilon)U_s^*(c + \alpha) - U_s^*(c)\big] = q_s^*(c) - (1 - \epsilon)q_s^*(c + \alpha) > 0 \qquad \forall c \in [\underline{c}, \bar{c}]$$

Hence the expected payoff difference between $(1 - \epsilon)U_s^*(c + \alpha)$ and $U_s^*(c)$ gets smaller (closer to 0) as c gets larger. In other words, over the interval $[\underline{c}, \bar{c}]$, the gap is largest in magnitude at $\underline{c}$. Then this means that the inequality on the left-hand side below guarantees the desired (weak) inequality from above to be satisfied:

$$-\tfrac{\Delta}{2} \leq (1 - \epsilon)U_s^*(\underline{c} + \alpha) - U_s^*(\underline{c}) \quad \Rightarrow \quad -\tfrac{\Delta}{2} \leq (1 - \epsilon)U_s^*(c + \alpha) - U_s^*(c) \qquad \forall c \in [\underline{c}, \bar{c}]$$

Hence the following inequality needs to be satisfied:

$$\tfrac{\Delta}{2} - [U_s^*(\underline{c}) - U_s^*(\underline{c} + \alpha)] \geq \epsilon U_s^*(\underline{c} + \alpha) \quad \text{(A.3)}$$

In order to have a well defined inequality, the left-hand side in (A.3) must be strictly positive. Then observe the following:

$$U_s^*(\underline{c}) - U_s^*(\underline{c} + \alpha) = \int_{\underline{c}}^{\underline{c}+\alpha} q_s^*(c)dc \leq \alpha q_s^*(\underline{c})$$

Hence if the rightmost term above is at most $\frac{\Delta}{4}$ (any value strictly between $(0, \frac{\Delta}{2})$ suffices), then the left-hand side becomes strictly positive. Define α^* as the following:

$$\alpha^* = \min\left\{\frac{1 - \bar{c}}{2}, \frac{\Delta}{4q_s^*(\underline{c})}\right\}$$

Observe that by construction α^* is strictly positive. Furthermore $\bar{c} < 1 - \alpha^*$. Now from inequality (A.3), it follows that if ϵ satisfies the following:

$$0 < \epsilon \leq \frac{\frac{\Delta}{2} - \big[U_s^*(\underline{c}) - U_s^*(\underline{c} + \alpha^*)\big]}{U_s^*(\underline{c} + \alpha^*)}$$

then inequalities in (A.2) are satisfied. Hence, for all types $c \in [\underline{c}, \bar{c}]$, it will be strictly optimal to participate in $\breve{\gamma}_i$. Note that the entry strategy for $c \in [0,1] \setminus [\underline{c}, \bar{c}]$ were not specified. Yet as shown below, attracting $c \in [\underline{c}, \bar{c}]$ is enough to show the strict profitability of the deviation mechanism $\breve{\gamma}^i$.

Consider the total expected profits $\breve{U}^i$ intermediary i makes from deviation, using the previously declared notation $\breve{U}^i(c|0 \leq c \leq 1 - \alpha^*)$ and $\breve{U}^i(c|1 - \alpha^* \leq c \leq 1)$ to denote the expected profits conditional on attracting those corresponding types:

$$\begin{aligned}
\breve{U}^i = \int_0^{1-\alpha^*} & \breve{U}^i(c|c \leq 1 - \alpha^*)\mathbb{I}\big(\hat{\eta}(c, \breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dF(c) \\
& \qquad + \int_{1-\alpha^*}^{1} \breve{U}^i(c|1 - \alpha^* \leq c)\mathbb{I}\big(\hat{\eta}(c, \breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dF(c) \\
& \geq (1 - \epsilon)\alpha^* \int_0^{1-\alpha^*} q_s^*(c + \alpha^*)\mathbb{I}\big(\hat{\eta}(c, \breve{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dF(c) - \epsilon \\
& \geq (1 - \epsilon)\alpha^* \big[F(\bar{c}) - F(\underline{c})\big] q_s^*(\bar{c} + \alpha^*) - \epsilon
\end{aligned}$$

The last line from above can be rearranged so that it provides an upper bound for ϵ. In particular define ϵ as follows:

$$\epsilon^* = \min \left\{ \frac{\frac{\Delta}{2} - \big[U_s^*(\underline{c}) - U_s^*(\underline{c} + \alpha^*)\big]}{2U_s^*(\underline{c} + \alpha^*)}, \frac{\alpha^* \big[F(\bar{c}) - F(\underline{c})\big] q_s^*(\bar{c} + \alpha^*)}{2\Big[1 + \alpha^* \big[F(\bar{c}) - F(\underline{c})\big] q_s^*(\bar{c} + \alpha^*)\Big]} \right\}$$

By construction $\epsilon^* > 0$. Also, the value of ϵ^* guarantees that inequality (A.3) is satisfied. Furthermore, it is also true that given the values of α^* and ϵ^*, the expected profit from deviation satisfies $\breve{U}^i > 0$ which means it is profitable for i to deviate contradicting the initial assumption of equilibrium. □

This concludes Step 2. Hence I have shown that the unique payoff equivalent feasible DRM is δ^*, which establishes the uniqueness of equilibrium outcomes in the intermediation game with multiple intermediaries. □

Proof of Proposition I.2. Consider the following direct mechanism $\tilde{\gamma}^i = (\tilde{\mathbf{Q}}^i, \tilde{\boldsymbol{\tau}}^i)$ for intermediary i where the agents' messages are restricted to their type spaces:

$$\begin{aligned}
\tilde{Q}_j^i(c', \mathbf{v}') &= Q_j^*(c', \mathbf{v}') \\
\tilde{\tau}_j^i(c', \mathbf{v}') &= Q_j^*(c', \mathbf{v}')v_j' - \int_0^{v_j'} Q_j^*(c', y_j, \mathbf{v}'_{-j})dy_j \\
\tilde{\tau}_s^i(c', \mathbf{v}') &= \sum_{j=1}^n \tilde{\tau}_j^i(c', \mathbf{v}')
\end{aligned}$$

Define the general mechanism $\hat{\gamma}^i = (\hat{\mathbf{Q}}^i, \hat{\boldsymbol{\tau}}^i)$ by taking the composition of the direct mechanism $\tilde{\gamma}$ with the message mappings $\boldsymbol{\phi}$ as defined in A.1. In particular for any vector of messages $(m_s^i, \mathbf{m}_b^i)$, the mechanism $\hat{\gamma}^i$ has outcome functions defined as follows:

$$\begin{aligned}
\hat{Q}_j^i(m_s^i, \mathbf{m}_b^i) &= \tilde{Q}_j^i\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \tilde{Q}_j^i(c', \mathbf{v}') \qquad \forall j \\
\hat{\tau}_j^i(m_s^i, \mathbf{m}_b^i) &= \tilde{\tau}_j^i\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \tilde{\tau}_j^i(c', \mathbf{v}') \qquad \forall j \\
\hat{\tau}_s^i(m_s^i, \mathbf{m}_b^i) &= \tilde{\tau}_s^i\big(\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i)\big) = \tilde{\tau}_s^i(c', \mathbf{v}')
\end{aligned}$$

where $\boldsymbol{\phi}(m_s^i, \mathbf{m}_b^i) = (c', \mathbf{v}') \in C \times \mathbf{V}$. Then the following assessment is an equilibrium. Every intermediary i announces $\hat{\gamma}^i$ as defined above. The remaining strategies and beliefs are defined as follows:

$$\hat{\eta}(c, \boldsymbol{\gamma}) = \begin{cases} 1 & \text{if } \boldsymbol{\gamma} = \hat{\boldsymbol{\gamma}} \\ i^* & \text{o/w} \end{cases} \qquad \forall c$$

$$\hat{\mu}_s^i(c, \boldsymbol{\gamma}) = \begin{cases} c & \text{if } \gamma^i = \hat{\gamma}^i \\ \hat{m}_s^i & \text{o/w} \end{cases} \qquad \forall c, i$$

$$\hat{\mu}_j^i(v_j, \gamma^i) = \begin{cases} v_j & \text{if } \gamma^i = \hat{\gamma}^i \\ \hat{m}_j^i & \text{o/w} \end{cases} \qquad \forall j,\ v_j,\ i$$

$$\hat{\beta}_s^i(\mathbf{v}|c, \boldsymbol{\gamma}) = g(\mathbf{v}) \qquad \forall c,\ i,\ \boldsymbol{\gamma}$$

$$\hat{\beta}_j^i(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, \gamma^i) = \begin{cases} \frac{\mathbb{I}\left(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i\right)f(c)}{\int_0^1 \mathbb{I}\left(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i\right)dF(c)} g_{-j}(\mathbf{v}_{-j}) & \text{if } \boldsymbol{\gamma}^{-i} = \hat{\boldsymbol{\gamma}}^{-i} \text{ and } \mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) > 0 \\ f(c)g_{-j}(\mathbf{v}_{-j}) & \text{if } \boldsymbol{\gamma}^{-i} = \hat{\boldsymbol{\gamma}}^{-i} \text{ and } \mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) = 0 \\ 0 & \text{if } \boldsymbol{\gamma}^{-i} \neq \hat{\boldsymbol{\gamma}}^{-i} \end{cases}$$

$$\forall j,\ v_j,\ \gamma^i,\ i$$

$$
\begin{aligned}
\text{where} \qquad i^* &= \min_i \left\{ i | i \in \underset{k \in I}{\text{argmax}} \{ U_s(\boldsymbol{\gamma}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k | c) \} \right\} && \forall c, \, \boldsymbol{\gamma} \\
\hat{m}_j^i &\in \underset{m_j^i \in M_j^i}{\text{argmax}} \left\{ U_j(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, m_j^i, \hat{\mu}_{-j}^i | v_j) \right\} && \forall j, \, v_j, \, i, \, \gamma^i \neq \hat{\gamma}^i \\
\hat{m}_s^i &\in \underset{m_s^i \in M_s^i}{\text{argmax}} \left\{ U_s(\boldsymbol{\gamma}, i, m_s^i, \hat{\boldsymbol{\mu}}_b^i | c) \right\} && \forall c, \, i, \, \gamma_i \neq \hat{\gamma}_i \\
\mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) &= \int_0^1 \mathbb{I}\big(\hat{\eta}(c, \gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dF(c)
\end{aligned}
$$

I first verify the sequential rationality of the strategies for the agents given the described beliefs. By construction of $\hat{\gamma}^i$'s, it is weakly dominant for buyers to report types truthfully. As $\mathbf{Q}^*$ is an incentive compatible allocation, it is also optimal for the seller to report her type truthfully whenever the buyers report truthfully. These verify the optimality of agents' communication strategies when $\gamma^i = \hat{\gamma}^i$. For the other cases, the strategy profiles prescribe $\hat{m}_s^i$ and $\hat{m}_j^i$'s which satisfy rationality by definition.

Next consider the optimality of seller's entry strategy $\hat{\eta}(c, \boldsymbol{\gamma})$. Start with the intermediaries announcing the equilibrium profile of mechanisms $\hat{\boldsymbol{\gamma}}$. Given the truthful type reporting at all subgames following entry to intermediary i, the expected payoff to the seller satisfies $U_s(\hat{\boldsymbol{\gamma}}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | c) = U_s^*(c)$ for all c and i. Hence every type of the seller is indifferent between the intermediaries, and it is optimal for all types to choose intermediary $i = 1$. Alternatively, when an out-of-equilibrium mechanism profile $\boldsymbol{\gamma} \neq \hat{\boldsymbol{\gamma}}$ is considered, then equilibrium entry strategy prescribes choosing $\hat{\eta}(c, \boldsymbol{\gamma}) = i^*$. By construction of i^*, the following is satisfied:

$$
U_s(\boldsymbol{\gamma}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b | c) = U_s(\boldsymbol{\gamma}, i^*, \hat{\mu}_s^{i^*}, \hat{\boldsymbol{\mu}}_b^{i^*} | c) = \max_{i \in I} \left\{ U_s(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | c) \right\} \geq U_s(\boldsymbol{\gamma}, i, \hat{\mu}_s^i, \hat{\boldsymbol{\mu}}_b^i | c) \qquad \forall i
$$

This establishes the optimality of entry strategies for all announced mechanism profiles, as the inequality holds for all i, c, and $\boldsymbol{\gamma}$. Lastly, I will show that for every intermediary i, it is optimal to announce $\hat{\gamma}^i$. In other words, for all i and every $\gamma^i \in \Gamma$, the equilibrium expected profits satisfy the following inequality:

$$
0 = \hat{U}^i = U^i(\hat{\gamma}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b) \geq U^i(\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b)
$$

I prove that above inequality is satisfied by contradiction. Assume that there exists some intermediary i, who unilaterally deviates and announces some $\acute{\gamma}^i \neq \hat{\gamma}^i$, which yields expected profits $U^i(\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b) > 0$. As $\acute{\gamma}^i$ yields strictly positive profits, it has to be the case that $\int_0^1 \mathbb{I}(\hat{\eta}(c, \acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}) = i) dF(c) > 0$ or that some positive measure of seller types choose i. On the other hand, despite the deviation by intermediary i, agents continue to play the truthful type reporting equilibrium at all other subgames where some intermediary k other than i are chosen. Hence the expected payoffs seller types receive from those subgames are $U_s(\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, k, \hat{\mu}_s^k, \hat{\boldsymbol{\mu}}_b^k | c) = U_s^*(c)$ for all c and $k \neq i$. Then optimality of seller's entry requires that any c type seller who chooses i's deviation mechanism $\acute{\gamma}^i$ receives at least $U_s^*(c)$ and all other types who choose $k \neq i$ receive $U_s^*(c)$.

Consider the subgame following the deviation mechanism profile $\acute{\boldsymbol{\gamma}} = (\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i})$. The assumed strategies make up an equilibrium for this out-of-equilibrium subgame. Namely, the entry strategies $\hat{\eta}(c, \acute{\boldsymbol{\gamma}})$ and the communication strategies $\hat{\boldsymbol{\mu}}_s(c, \acute{\boldsymbol{\gamma}})$ and $\hat{\boldsymbol{\mu}}_j(v_j, \acute{\boldsymbol{\gamma}})$ for all c and v_j, along with the beliefs $\hat{\boldsymbol{\beta}}_s$ and $\hat{\boldsymbol{\beta}}_s$ make up an equilibrium. It is possible to invoke the revelation principle argument for the equilibrium outcomes of this subgame. In other words, by using the same indexation and composition arguments as in the equilibrium of the overall game, it is possible to construct a feasible DRM $\bar{\delta}$ with a truthful type-telling equilibrium that is payoff equivalent with the subgame equilibrium. As a result, the payoffs in the truthful type-telling equilibrium of $\bar{\delta}$ satisfy:

$$\begin{aligned}
\bar{U}^i &= U^i(\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b) \qquad \forall i \\
\bar{U}_s(c) &= U_s(\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b | c) \qquad \forall c \\
\bar{U}_j(v_j) &= U_j(\acute{\gamma}^i, \hat{\boldsymbol{\gamma}}^{-i}, \hat{\eta}, \hat{\mu}_s, \hat{\boldsymbol{\mu}}_b | v_j) \qquad \forall j,\, v_j
\end{aligned}$$

The profitable deviation assumption implies that $\bar{U}^i > 0$. It was established in the previous points that $\bar{U}_s(c) \geq U_s^*(c)$ for all c. However by Corollary I.1 it needs to be the case that $\bar{U}^i = 0$ which yields a contradiction. Hence no intermediary has a profitable deviation.

Lastly, consistency of the beliefs must be verified. Lemma A.1 from Appendix A.1.2 establishes that the beliefs $\hat{\boldsymbol{\beta}}_s$ defined above are the consistent beliefs for any equilibrium. Below establishes consistency of the described beliefs for the buyers. Given the equilibrium strategies $\hat{\eta}$ and $\hat{\boldsymbol{\gamma}}$ consider a sequence of totally mixed strategies $\{\eta^h, \boldsymbol{\gamma}^h\}_{h=1}^{\infty}$ where $\eta^h \in \Delta I$ and $\gamma^{i,h} \in \Delta\Gamma$ for each i, and $\lim_{h\to\infty}(\gamma^{1,h}, \ldots, \gamma^{m,h}, \eta^h) = (\hat{\gamma}^1, \ldots, \hat{\gamma}^m, \hat{\eta})$. Let $\boldsymbol{\gamma}^h = (\gamma^{1,h}, \ldots, \gamma^{m,h})$ denote the shorthand for the sequences of totally mixed strategies for all intermediaries. Similarly, for any i let $\boldsymbol{\gamma}^{-i,h} = (\gamma^{1,h}, \ldots, \gamma^{i-1,h}, \gamma^{i+1,h}, \ldots, \gamma^{m,h})$ be the shorthand notation for the sequences of totally mixed strategies of all intermediaries other than i. Also let $\beta_j^h(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i} | v_j, i, \gamma^i)$ denote buyer j's beliefs at information set (v_j, i, γ^i) when the totally mixed strategies $(\eta^h, \boldsymbol{\gamma}^h)$ are played. Lastly, let $\mathbb{P}(\mathbf{v}, c, i, \boldsymbol{\gamma} | \eta^h, \boldsymbol{\gamma}^h)$ denote the probability of the information node $(\mathbf{v}, c, i, \boldsymbol{\gamma})$ being reached under the totally mixed strategies $(\eta^h, \boldsymbol{\gamma}^h)$. Then the consistent beliefs are given by:

$$\begin{aligned}
\hat{\beta}^i_j(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, \gamma^i) &= \hat{\beta}_j(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, i, \gamma^i) \\
&= \lim_{h\to\infty} \beta^h_j(c, \mathbf{v}_{-j}, \boldsymbol{\gamma}^{-i}|v_j, i, \gamma^i) \\
&= \lim_{h\to\infty} \frac{\mathbb{P}(\mathbf{v}, c, i, \boldsymbol{\gamma}|\eta^h, \boldsymbol{\gamma}^h)}{\int_{\mathbf{v}_{-j}} \int_{\boldsymbol{\gamma}^{-i}} \int_c \mathbb{P}(\mathbf{v}, c, i, \boldsymbol{\gamma}|\eta^h, \boldsymbol{\gamma}^h) dc d\boldsymbol{\gamma}^{-i} d\mathbf{v}_{-j}} \\
&= g_{-j}(\mathbf{v}_{-j}) \lim_{h\to\infty} \frac{\mathbb{P}(c, i, \boldsymbol{\gamma}^{-i}|\eta^h, \boldsymbol{\gamma}^h)}{\int_{\boldsymbol{\gamma}^{-i}} \int_c \mathbb{P}(c, i, \boldsymbol{\gamma}^{-i}|\eta^h, \boldsymbol{\gamma}^h) dc d\boldsymbol{\gamma}^{-i}}
\end{aligned}$$

Note that, from penultimate to the last line, the joint probability distribution of all other buyers are taken out due to the independence of type distributions. Also, the probability of $\gamma^{i,h} = \gamma^i$ drops out, because no matter what probability the realized mechanism γ^i is played with in the totally mixed strategy $\gamma^{i,h}$, buyer j observes γ^i and cancels out of the limit as it appears both in the numerator and the denominator.

Consider the fraction that for which the limit will be taken:

$$\begin{aligned}
&\frac{\mathbb{P}(c, i, \boldsymbol{\gamma}^{-i}|\eta^h, \boldsymbol{\gamma}^h)}{\int_{\boldsymbol{\gamma}^{-i}} \int_c \mathbb{P}(c, i, \boldsymbol{\gamma}^{-i}|\eta^h, \boldsymbol{\gamma}^h) dc d\boldsymbol{\gamma}^{-i}} \\
&= \frac{f(c)\mathbb{P}\big(\eta^h(c, \gamma^i, \boldsymbol{\gamma}^{-i}) = i\big)\mathbb{P}\big(\boldsymbol{\gamma}^{-i,h} = \boldsymbol{\gamma}^{-i}\big)}{\begin{array}{l}\mathbb{P}\big(\boldsymbol{\gamma}^{-i,h} = \hat{\boldsymbol{\gamma}}^{-i}\big) \int_c f(c)\mathbb{P}\big(\eta^h(c, \gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dc \\ \quad + \int_{\boldsymbol{\gamma}^{-i} \neq \hat{\boldsymbol{\gamma}}^{-i}} \int_c f(c)\mathbb{P}\big(\eta^h(c, \gamma^i, \boldsymbol{\gamma}^{-i}) = i\big)\mathbb{P}\big(\boldsymbol{\gamma}^{-i,h} = \boldsymbol{\gamma}^{-i}\big) dc d\boldsymbol{\gamma}^{-i}\end{array}}
\end{aligned} \tag{A.4}$$

Now consider the totally mixed strategies to define the following probabilities:

$$\mathbb{P}\big(\eta^h(c, \gamma^i, \boldsymbol{\gamma}^{-i}) = i\big) = \begin{cases} 1 - \epsilon^h & \text{if } \hat{\eta}(c, \gamma^i, \boldsymbol{\gamma}^{-i}) = i \\ \frac{\epsilon^h}{m-1} & \text{o/w} \end{cases}$$

$$\mathbb{P}(\gamma^{i,h} = \gamma^i) = \begin{cases} 1 - \epsilon^{2h} & \text{if } \gamma^i = \hat{\gamma}^i \\ \frac{\epsilon^{2h}}{[2K]^{n+1}} & \text{if } \gamma^i \neq \hat{\gamma}^i \end{cases}$$

Observe that, both totally mixed strategies uniformly randomize over the out-of-equilibrium strategies within their respective strategy spaces. On the other hand, it is also important to observe that $\gamma^{i,h}$ converges to $\hat{\gamma}^i$ infinitely faster than $\eta^h(c, \gamma^i, \boldsymbol{\gamma}^{-i})$ converges to $\hat{\eta}$.

Finally denote by $\mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) = \int_0^1 f(c)\mathbb{I}\big(\hat{\eta}(c, \gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) = i\big) dc$, the overall expected probability of the intermediary i being chosen in the out-of-equilibrium subgame where only i has deviated.

A closer look at expression in (A.4) suggests that for any mechanism γ^i, if $\mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i}) > 0$, then in the denominator as h gets larger the first term approaches $\mathbb{P}(i|\gamma^i, \hat{\boldsymbol{\gamma}}^{-i})$ (which is strictly positive) while the second term approaches 0. Thus, limits can be taken. Evaluating the limits in that case yields:

$$\lim_{h\to\infty}\frac{\mathbb{P}(c,i,\boldsymbol{\gamma}^{-i}|\eta^h,\boldsymbol{\gamma}^h)}{\int_{\boldsymbol{\gamma}^{-i}}\int_c \mathbb{P}(c,i,\boldsymbol{\gamma}^{-i}|\eta^h,\boldsymbol{\gamma}^h)dcd\boldsymbol{\gamma}^{-i}} = \begin{cases} 0 & \text{if } \boldsymbol{\gamma}^{-i}\neq\hat{\boldsymbol{\gamma}}^{-i} \\ \frac{f(c)\mathbb{I}(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i)}{\int_0^1 f(c)\mathbb{I}\left(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i\right)dc} & \text{if } \boldsymbol{\gamma}^{-i}=\hat{\boldsymbol{\gamma}}^{-i}\end{cases}$$

In the other case where $\mathbb{P}(i|\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=0$, observe that the first term in the denominator equals $[1-\epsilon^{2h}]^{m-1}\frac{\epsilon^h}{m-1}$. Then, carrying the $\epsilon^h/(m-1)$ term to the numerator, the following can be observed:

$$\lim_{h\to\infty}\frac{\mathbb{P}\big(\eta^h(c,\gamma^i,\boldsymbol{\gamma}^{-i})=i\big)\mathbb{P}\big(\boldsymbol{\gamma}^{-i,h}=\boldsymbol{\gamma}^{-i}\big)}{\epsilon^h/(m-1)} = \begin{cases}\lim\limits_{h\to\infty}[1-\epsilon^{2h}]^{m-1}=1 & \text{if } \boldsymbol{\gamma}^{-i}=\hat{\boldsymbol{\gamma}}^{-i} \\ 0 & \text{if } \boldsymbol{\gamma}^{-i}\neq\hat{\boldsymbol{\gamma}}^{-i}\end{cases}$$

The second case follows, because $\mathbb{P}\big(\boldsymbol{\gamma}^{-i,h}=\boldsymbol{\gamma}^{-i}\big)$ has at least one multiplicative term equaling $\frac{\epsilon^{2h}}{[2K]^{n+1}}$. This means that, even after the simplifying cancellations, ϵ^h remains, which in turn brings the limit to 0. Then, in the case where $\mathbb{P}(i|\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=0$, the limits of the fraction are equal to:

$$\lim_{h\to\infty}\frac{\mathbb{P}(c,i,\boldsymbol{\gamma}^{-i}|\eta^h,\boldsymbol{\gamma}^h)}{\int_{\boldsymbol{\gamma}^{-i}}\int_c \mathbb{P}(c,i,\boldsymbol{\gamma}^{-i}|\eta^h,\boldsymbol{\gamma}^h)dcd\boldsymbol{\gamma}^{-i}} = \begin{cases} 0 & \text{if } \boldsymbol{\gamma}^{-i}\neq\hat{\boldsymbol{\gamma}}^{-i} \\ \frac{f(c)\mathbb{I}(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i)}{\int_0^1 f(c)\mathbb{I}\left(\hat{\eta}(c,\gamma^i,\hat{\boldsymbol{\gamma}}^{-i})=i\right)dc} & \text{if } \boldsymbol{\gamma}^{-i}=\hat{\boldsymbol{\gamma}}^{-i}\end{cases}$$

Combining these limits under the different cases with $g_{-j}(\mathbf{v}_{-j})$ yields the consistent beliefs, which are equivalent to the description of the equilibrium assessment. □

Proof of Proposition I.3. Under a single monopolist intermediary, a feasible DRM δ satisfying BB implies that intermediary's expected payoff is equal to $R(\delta)$. Thus it is unnecessary to define transfers to the intermediary. Then the monopolist intermediary's problem can be described as follows:

$$\begin{array}{rcl} P3 \rightarrow & \max\limits_{\{\bar{\mathbf{Q}},\bar{\tau}_s,\bar{\boldsymbol{\tau}}_b\}} \{\bar{U}\} & \\ \text{IR}: & \bar{U}_s(c),\, \bar{U}_j(v_j), \bar{U}\geq 0 & \forall c,\ \forall j \text{ and } v_j \\ \text{IC}: & \bar{U}_s(c)\geq\bar{U}_s(c'|c), \text{ and } \bar{U}_j(v_j)\geq\bar{U}_j(v_j'|v_j) & \forall c,c',\ \forall j \text{ and } v_j,v_j' \\ \text{RES}: & 0\leq\bar{Q}_j(c,\mathbf{v})\leq 1, \text{ and } \bar{Q}_s(c,\mathbf{v})=\sum_{j=1}^n\bar{Q}_j(c,\mathbf{v})\leq 1 & \forall c,\mathbf{v} \\ \text{BB}: & \bar{U}=\sum_{j=1}^n\left[\int_0^1\bar{t}_j(v_j)dG_j(v_j)\right]-\int_0^1\bar{t}_s(c)dF(c) & \end{array}$$

Using the necessary conditions of a feasible DRM from Remark I.1, the objective function becomes:

$$\bar{U}=\int_0^1\int_{\mathbf{v}}\sum_{j=1}^n\bar{Q}_j(c,\mathbf{v})\big[\psi_j(v_j)-\psi_s(c)\big]dG(\mathbf{v})dF(c)-\sum_{j=1}^n\bar{U}_j(0)-\bar{U}_s(1)$$

Observe that at the solution, $U_j^M(0) = U_s^M(1) = 0$ optimally, as otherwise decreasing either would strictly improve the objective function's value without violating the constraints. Note that these are in line with Proposition I.3. Then, plugging these into $P3$ yields the following simplified problem $P3'$:

$$P3' \rightarrow \qquad \max_{\bar{\mathbf{Q}}} \left\{ \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v}) dF(c) \right\}$$

$$\text{subject to} \qquad \frac{d\bar{q}_s(c)}{dc} \leq 0 \qquad \forall c$$

$$\frac{d\bar{q}_j(v_j)}{dv_j} \geq 0 \qquad \forall j \text{ and } v_j$$

$$0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1, \text{ and } \bar{Q}_s(c,\mathbf{v}) = \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \leq 1 \qquad \forall c, \mathbf{v}$$

$$\int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v}) dF(c) \geq 0$$

Ignoring the monotonicity and intermediary's IR constraints momentarily, the linearity of the objective function in the choice variable suggests that in light of RES constraints the pointwise maximizer is a simple step function given by:

$$Q_j^M(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j) \geq \max\left\{\psi_s(c), \max_{k \neq j}\{\psi_k(v_k)\}\right\} \\ 0 & \text{o/w} \end{cases} \qquad \forall j \in \{1,\ldots,n\}$$

This is precisely the same allocation function as in (I.4). Lastly, I check that the pointwise maximizer $\mathbf{Q}^M$ satisfies the simplified conditions. RES is satisfied because for any given vector of type reports $(c,\mathbf{v})$, at most one $Q_j^M(c,\mathbf{v})$ equals 1 and so is $Q_s^M(c,\mathbf{v})$. Also observe that $\psi_s(c)$ is strictly increasing in c and $\psi_j(v_j)$'s are strictly increasing in v_j's. Thus for any reported vector of types, for each j the trade probabilities $Q_j^M(c,\mathbf{v})$ are weakly increasing in v_j and weakly decreasing in c. Each trade probability being weakly decreasing in c suggests that $Q_s^M(c,\mathbf{v})$ is also weakly decreasing in c as it is equal to the sum of Q_j^M's across all j. Hence the monotonicity constraints are also satisfied. Finally, consider the intermediary's IR constraint. For all $c < 1$ there exists a positive measure of buyer types with $\psi_j(v_j) > \psi_s(c)$. As the allocation rule $\mathbf{Q}^M$ implements trade only when the highest virtual valuation is above the virtual cost, the expected profits are strictly positive, satisfying intermediary's IR. □

Proof of Corollary I.2. $Q_j^*(c,\mathbf{v}) \geq Q_j^M(c,\mathbf{v})$ for all $(c,\mathbf{v})$, because $r_j^*(c) \leq r_j^M(c)$ for all j and c. However $c \leq r_j^*(c) \leq r_j^M(c)$ for all c. This implies that the total surplus under δ^* is higher as it implements more efficient trade. Observe that the inequality from the trade probabilities imply the Pareto dominance relationship for the agents. Consider buyer j's expected payoffs:

$$\begin{aligned} U_j^*(v_j) &= \int_0^{v_j}\int_0^1\int_{\mathbf{v}_{-j}} Q_j^*(c, y_j, \mathbf{v}_{-j})dG_{-j}(v_{-j})dF(c)dy_j \\ &\geq \int_0^{v_j}\int_0^1\int_{\mathbf{v}_{-j}} Q_j^M(c, y_j, \mathbf{v}_{-j})dG_{-j}(v_{-j})dF(c)dy_j = U_j^M(v_j) \end{aligned}$$

Similarly $U_s^*(c) \geq U_s^M(c)$ for all c. Finally, expected profit to the monopolist intermediary is strictly positive as shown in Proposition I.3. □

Proof of Proposition I.4. I closely follow the Proof of Theorem 2 from Myerson and Satterthwaite (1983). The constrained efficient DRM solves the following program:

$$P4 \rightarrow \qquad \max_{\{\bar{\mathbf{Q}},\bar{\tau}\}} \left\{ \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\big(v_j - c\big)dG(\mathbf{v})dF(c) \right\}$$

subject to

$$\begin{array}{lll} \text{IR}: & \bar{U}_s(c),\ \bar{U}_j(v_j), \bar{U} \geq 0 & \forall c,\ \forall j \text{ and } v_j \\ \text{IC}: & \bar{U}_s(c) \geq \bar{U}_s(c'|c),\ \text{and } \bar{U}_j(v_j) \geq \bar{U}_j(v_j'|v_j) & \forall c, c',\ \forall j \text{ and } v_j, v_j' \\ \text{RES}: & 0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1,\ \text{and } \bar{Q}_s(c,\mathbf{v}) = \sum_{j=1}^n \bar{Q}_j(c,\mathbf{v}) \leq 1 & \forall c, \mathbf{v} \\ \text{BB}: & \bar{U} = \sum_{j=1}^n \left[\int_0^1 \bar{t}_j(v_j)dG_j(v_j)\right] - \int_0^1 \bar{t}_s(c)dF(c) & \end{array}$$

From Remark I.1, a feasible DRM δ has to satisfy the following necessary conditions:

$$\begin{array}{llll} \bar{U}_s(1) \geq 0, & \bar{U}_s(c) = \bar{U}_s(1) + \int_c^1 \bar{q}_s(x)dx, & \dfrac{d\bar{q}_s(c)}{dc} \leq 0 & \forall c \\ \bar{U}_j(0) \geq 0, & \bar{U}_j(v_j) = \bar{U}_j(0) + \int_0^{v_j} \bar{q}_j(y_j)dy_j, & \dfrac{d\bar{q}_j(v_j)}{dv_j} \geq 0 & \forall j \text{ and } v_j \end{array}$$

$$0 = \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n \bar{Q}_j(c,\mathbf{v})\left[\psi_j(v_j) - \psi_s(c)\right] dG(\mathbf{v})dF(c) - \sum_{j=1}^n \bar{U}_j(0) - \bar{U}_s(1) - \sum_{i\in I} \bar{U}^i$$

Firstly, observe that $U^{i,E} = U_s^E(1) = U_j^E(0) = 0$ optimally for all i and j. To see why, assume that $U_j^E(0) > 0$ for some j. Then the objective function's value can be increased by decreasing the worst type buyer's constant payoff and simultaneously increasing the trade probability for $(c, \mathbf{v})$ where $\max_j\{v_j\} > c$ just enough to maintain the BB. The existence of such $(c, \mathbf{v})$ pairs are guaranteed due to the fact that ex-post efficiency is not achieved. Similar arguments hold for $U^{i,E}$ and $U_s^E(1)$. Plugging into $P4$ yields the simplified problem

$P4'$:

$$P4' \rightarrow \qquad \max_{\bar{\mathbf{Q}}} \left\{ \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \big[v_j - c \big] dG(\mathbf{v}) dF(c) \right\}$$

$$\text{subject to} \qquad \frac{d\bar{q}_s(c)}{dc} \leq 0 \qquad \forall c$$

$$\frac{d\bar{q}_j(v_j)}{dv_j} \geq 0 \qquad \forall j \text{ and } v_j$$

$$0 \leq \bar{Q}_j(c,\mathbf{v}) \leq 1, \text{ and } \bar{Q}_s(c,\mathbf{v}) = \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \leq 1 \qquad \forall c, \mathbf{v}$$

$$\int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \left[\psi_j(v_j) - \psi_s(c) \right] dG(\mathbf{v}) dF(c) = 0$$

Observe that the unconstrained pointwise maximizer of the objective function in $P4'$ is the ex-post efficient allocation. However, the ex-post efficient allocation does not solve $P4'$ as it leads the BB to be violated, i.e. the left-hand side of the last constraint is strictly negative as opposed to 0.[1] Ignoring the monotonicity constraints for the moment and denoting the Lagrange multiplier for the BB constraint by λ, the Lagrangian for the above problem can be written as the following:

$$\begin{aligned} L(\bar{\mathbf{Q}};\lambda) &= \int_0^1 \int_{\mathbf{v}} \sum_{j=1} \bar{Q}_j(c,\mathbf{v}) \Big[(v-c) + \lambda \big[\psi_j(v_j) - \psi_s(c) \big] \Big] dG(\mathbf{v}) dF(c) \\ &= (1+\lambda) \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^{n} \bar{Q}_j(c,\mathbf{v}) \Big[\psi_j\big(v, \tfrac{\lambda}{1+\lambda}\big) - \psi_s\big(c, \tfrac{\lambda}{1+\lambda}\big) \Big] dG(\mathbf{v}) dF(c) \end{aligned}$$

The linearity of the Lagrangian function in the choice variable suggests that in light of RES constraints the pointwise maximizer is a simple step function given by:

$$Q_j^E(c,\mathbf{v}) = \begin{cases} 1 & \text{if } \psi_j(v_j,\alpha) \geq \max\Big\{ \psi_s(c,\alpha), \max_{k \neq j} \{ \psi_k(v_k,\alpha) \} \Big\} \\ 0 & \text{o/w} \end{cases} \qquad \forall j \in \{1,\ldots,n\}$$

where $\alpha = \frac{\lambda}{1+\lambda} \in (0,1)$. This is precisely the same allocation function as (I.5) from Proposition I.4. Lastly, I check that the pointwise maximizer $\mathbf{Q}^E$ satisfies the conditions. RES is satisfied because for any given vector of type reports $(c,\mathbf{v})$, at most one $Q_j^E(c,\mathbf{v})$ equals 1. Also observe that for any reported vector of types the trade probability $Q_j^E(c,\mathbf{v})$ is weakly increasing in v_j and weakly decreasing in c for every j. Each trade probability being weakly decreasing in c suggests that $Q_s^E(c,\mathbf{v})$ is also weakly decreasing in c as it is equal to the sum of Q_j^E's across all j. Hence the monotonicity constraints are also satisfied.

[1] See for example Krishna and Perry (1998) for a proof of the VCG mechanism running an expected loss, and hence violating BB.

Finally, the BB constraint is satisfied as the Lagrange multiplier satisfies $\lambda > 0$. $\square$

Proof of Corollary I.3. The ex-ante expected surplus under social planner is higher as δ^E solves the maximization of the surplus subject to feasibility constraints. Similarly, δ^* solves the maximization of seller's ex-ante expected payoffs and hence they are higher. As the intermediaries receive zero expected profits, the following result attains:

$$\int_0^1 U_s^*(c)dF(c) > \int_0^1 U_s^E(c)dF(c)$$

$$\int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n Q_j^*(c,\mathbf{v})\big[v_j - c\big]dG(\mathbf{v})dF(c) < \int_0^1 \int_{\mathbf{v}} \sum_{j=1}^n Q_j^E(c,\mathbf{v})\big[v_j - c\big]dG(\mathbf{v})dF(c)$$

$$\Rightarrow \qquad \sum_{j=1}^n \int_0^1 U_j^*(v_j)dG_j(v_j) < \sum_{j=1}^n \int_0^1 U_j^E(v_j)dG_j(v_j)$$

The last inequality follows from the BB, where sum ex-ante expected of all buyers is equal to the surplus minus the seller's ex-ante expected payoff. $\square$

Proof of Corollary I.4. Consider the seller's interim expected payoff schedules $U_s^*(c)$ and $U_s^E(c)$ for all c from equilibrium outcomes of the intermediation game with competing intermediaries and social planner's surplus maximization, respectively. Firstly it has been established that both schedules are absolutely continuous and convex decreasing functions. Secondly $U_s^*(c)$ is strictly decreasing throughout the whole domain while $U_s^E(c)$ is strictly decreasing for all $c \in [0, \bar{c}^E)$. Thirdly, for costs slightly below $c = 1$, $U_s^*(c) > U_s^E(c)$. Finally, for costs slightly above $c = 0$, the derivatives satisfy $\frac{d}{dc}U_s^E(c) = -\Big[1 - \big[G\big(r^E(c)\big)\big]^n\Big] < -\Big[1 - \big[G\big(r^*(c)\big)\big]^n\Big] = \frac{d}{dc}U_s^*(c)$ as $r^*(c) > r^E(c)$, which implies that $U_s^E(c)$ decreases at a faster rate than $U_s^*(c)$.

The aim is to evaluate whether $U_s^*(c) \geq U_s^E(c)$ for all c. Pareto relationship is violated if and only if there exists an interior type $c' \in (0,1)$ such that $U_s^*(c') = U_s^E(c')$ and for all $c < c'$, the payoff schedules satisfy $U_s^E(c) > U_s^*(c)$. All of the previous points combined imply that the Pareto relationship inequalities for all types hold if and only if there is no interior type c' such that $U_s^*(c') = U_s^E(c')$. Given the convexity and continuity, the continuum of inequalities hold if and only if the inequality at the end point $c = 0$, i.e. $U_s^*(0) \geq U_s^E(0)$, is satisfied. Plugging in for the interim expected payoffs from Remark I.1 yields the desired condition as stated in (I.6).

As the reservation values are increasing functions, it is possible define their inverses, which are denoted by $h^*(v_j)$ and $h^E(v_j)$ and are the unique solutions to:

$$h^*(v_j) = \max\{\psi_b(v_j), 0\}, \qquad \psi_s(h^E(v_j), \alpha) = \max\{\psi_b(v_j, \alpha), 0\} \qquad \text{for all } v_j$$

Hence the functions $h^*(v_j)$ and $h^E(v_j)$ represent the maximum cost type for which the reservation value equals the valuation v_j. Then pursuing an analogous argument regarding the convexity and continuity of buyer j's interim expected payoffs yields that $U_j^E(v_j) \geq U_j^*(v_j)$ for all v_j if and only if the inequality at the endpoint $v_j = 1$ is satisfied. Again, plugging in for the interim expected payoffs from Remark I.1, the condition stated in (I.7) is attained. $\square$

APPENDIX B

Appendix to Chapter II

B.1 Proofs

Proof of Lemma II.1. Given an equilibrium, the beliefs on the equilibrium path satisfying $\pi^*_L(m^*_{t_s}) = \pi^*_H(m^*_{t_s})$ for both t_s follows from the Bayes' rule. To see why the beliefs are the same for any $m \in [0, H + \lambda_H] \setminus \{m^*_l, m^*_h\}$, let $m^k_{t_s}$ consider a sequence of totally mixed strategies that converge to $m^*_{t_s}$. Then, consistency requires the out-of-equilibrium beliefs to satisfy $\pi^*_{t_b}(m) = \lim_{k\to\infty} \pi^k_{t_b}(m)$ where $\pi^k_{t_b}(m)$ is are equal to;

$$\begin{aligned}\pi^k_{t_b}(m) &= \frac{\mathbb{P}\big(m^k_l = m\big)q\mathbb{P}(t_b)}{\mathbb{P}\big(m^k_l = m\big)q\mathbb{P}(t_b) + \mathbb{P}\big(m^k_h = m\big)(1-q)\mathbb{P}(t_b)} \\ &= \frac{\mathbb{P}\big(m^k_l = m\big)q}{\mathbb{P}\big(m^k_l = m\big)q + \mathbb{P}\big(m^k_h = m\big)(1-q)}\end{aligned}$$

Hence for both types of the buyer, the out-of-equilibrium beliefs $\pi^*_{t_b}(m)$ are equal to the limits of the same sequence of beliefs. □

Proof of Corollary II.1. Given an equilibrium, consider some price announcement $m \in [0, H + \lambda_H]$ where $d^*_L(m) = A$. By sequential rationality, low type buyer accepts only when $m \leq L + \big(1 - \pi^*(m)\big)\lambda_L$. Observe that $L + \big(1 - \pi^*(m)\big)\lambda_L < H + \big(1 - \pi^*(m)\big)\lambda_H$ for any belief $\pi^*(m)$. Since the high type buyer holds the same beliefs, then it follows that $d^*_H(m) = A$, as well. □

Proof of Lemma II.2. Consider any equilibrium $\{(m^*_l, m^*_h), (d^*_L, d^*_H), \pi^*\}$. Then, I will show that the assessment $\{(m_l^*, m_h^*), (\tilde{d}_L, \tilde{d}_H), \tilde{\pi}\}$, where

$$\tilde{\pi}(m) = \begin{cases} \pi^*(m) & \text{if } m \in \{m_l^*, m_h^*\} \\ 1 & \text{o/w} \end{cases} \qquad \tilde{d}_{t_b}(m) = \begin{cases} d_{t_b}^*(m_{t_s}^*) & \text{if } m \in \{m_l^*, m_h^*\} \\ A & \text{if } m \le t_b \\ R & \text{if } m > t_b \end{cases}$$

is a payoff equivalent equilibrium. Firstly, the payoff equivalence follows from construction as the equilibrium-path beliefs and strategies for both buyer types are the same as in the original equilibrium. Hence, it only remains to be checked that the new profile is indeed a sequential equilibrium.

The optimality of $d_{t_b}^*$ along with the equality between the equilibrium path beliefs, i.e. $\tilde{\pi}(m_{t_s}^*) = \pi^*(m_{t_s}^*)$, suggests that $\tilde{d}_{t_b}(m_{t_s}^*)$ is also optimal. Following any price announcement $m' \neq m_{t_s}^*$, sequential rationality requires buyer's strategies to satisfy:

$$\tilde{d}_{t_b}(m') = \begin{cases} A & \text{if } m' < t_b + \big(1 - \tilde{\pi}(m')\big)\lambda_{t_b} \\ R & \text{if } m' > t_b + \big(1 - \tilde{\pi}(m')\big)\lambda_{t_b} \\ \{A, R\} & \text{if } m' = t_b + \big(1 - \tilde{\pi}(m')\big)\lambda_{t_b} \end{cases}$$

Since the out-of-equilibrium beliefs are $\tilde{\pi}(m') = 1$, then $\tilde{d}_{t_b}(m)$ is indeed optimal for both types.

Next, consider the strategies for the seller types. The sequential rationality requires that for each seller type, the equilibrium expected payoffs satisfy $U_{t_s}(m_{t_s}^*) \ge U_{t_s}(m') = (m' - t_s)\tilde{Q}(m')$ for all $m' \neq m_{t_s}^*$ where $\tilde{Q}(m')$ denotes the expected trade probabilities for deviation prices in the conjectured equilibrium. Given the original profile is an equilibrium, it is true that $U_{t_s}(m_{t_s}^*) \ge (m' - t_s)Q^*(m')$ for all $m' \neq m_{t_s}^*$ where $Q^*(m')$ is the corresponding expected trade probability for deviation prices. Thus, if it holds that $Q^*(m') \ge \tilde{Q}(m')$ for any m', then sequential rationality would be satisfied. If $\pi^*(m') < \tilde{\pi}(m') = 1$, then whenever $\tilde{d}_{t_b}(m') = A$, it is also the case that $d_{t_b}^*(m') = A$ or that $\tilde{Q}(m') \le Q^*(m')$ as desired. If $\pi^*(m') = \tilde{\pi}(m') = 1$, then for any $m' > t_b$ it holds that $d_{t_b}^*(m') = \tilde{d}_{t_b}(m') = R$, while for any $m' < t_b$ it holds that $d_{t_b}^*(m') = \tilde{d}_{t_b}(m') = A$. Altogether they imply that $\tilde{Q}(m') = Q^*(m')$ as required.

The only case left to consider is if $m' = t_b$ with $\pi^*(t_b) = \tilde{\pi}(t_b) = 1$ and the original equilibrium has type t_b buyer breaking her tie in favor of rejection, $d_{t_b}^*(t_b) = R$ whereas $\tilde{d}_{t_b}(t_b) = A$. It needs to be shown that sequential rationality condition is still satisfied, i.e. $U_{t_s}(m_{t_s}^*) \ge U_{t_s}(m') = (m' - t_s)\tilde{Q}(m') = (t_b - t_s)\tilde{Q}(t_b)$. To pursue a contradiction, assume $U_{t_s}(m_{t_s}^*) < (t_b - t_s)\tilde{Q}(t_b)$. Then for $m'' < t_b$, it holds that $\tilde{Q}(m'') \le Q^*(m'')$, since $d_{t_b}^*(m'') = A$ as it was shown above. However by continuity, for m'' close enough to t_b, it holds that $U_{t_s}(m_{t_s}^*) < (m'' - t_s)\tilde{Q}(m'') \le (m'' - t_s)Q^*(m'')$ which contradicts the original equilibrium condition for $m_{t_s}^*$. Thus, it has to be the case that the original equilibrium strategies $m^*_{t_s}$ for both t_s also satisfy sequential rationality under the conjectured equilibrium.

Finally to see why the beliefs $\tilde{\pi}(m)$ are consistent, consider the following convergent sequence of mixed strategies $\{m^k_{t_s}\}^{\infty}_{k=0}$, where for each seller type t_s, prices $m \neq m^*_{t_s}$ are announced uniformly with probability $\frac{\epsilon^k_{t_s}}{H+\lambda_H}$ for some $\epsilon_{t_s} \in (0,1)$ and $m = m^*_{t_s}$ is announced with probability $1 - \epsilon^k_{t_s}$. By construction $\lim_{k\to\infty} m^k_{t_s} = m^*_{t_s}$. Then, there exists a sequence of beliefs $\{\pi^k(m)\}$ generated by (m^k_l, m^k_h) where:

$$\pi^k(m) = \begin{cases} \frac{\epsilon^k_l p}{\epsilon^k_l p + \mathbb{P}(m^k_h = m)(H+\lambda_H)(1-p)} & \text{if } m \neq m^*_l \\ \frac{(1-\epsilon^k_l)p}{(1-\epsilon^k_l)p + \mathbb{P}(m^k_h = m)(1-p)} & \text{if } m = m^*_l \end{cases}$$

Letting $\epsilon_l = \epsilon$ while $\epsilon_h = \epsilon^2$ for $\epsilon \in (0,1)$, it is straight forward to see that the limits satisy

$$\tilde{\pi}(m) = \lim_{k\to\infty} \pi^k(m) = \begin{cases} 1 & \text{if } m \neq \{m^*_l, m^*_h\} \\ \pi^*(m^*_{t_s}) & \text{if } m = m^*_{t_s} \end{cases}. \qquad \square$$

Proof of Proposition II.1. In a pooling equilibrium, the equilibrium price m^P induces the buyer types to have beliefs $\pi^P(m^P) = p$, which yields the prior expected value $\mathbb{E}(v|t_b)$ of each buyer type as their maximum willingness to pay for the object, respectively. At the same time, m^P has to be larger or equal to h, as otherwise it would not be optimal for the high cost seller to announce m^P. However, as $\mathbb{E}(v|L) < h$, there is no pooling equilibrium in which low type buyer trades, i.e. $d^P_L(m^P) = R$. On the other hand, there is trade with the high type buyer or $d^P_H(m^P) = A$, because otherwise both seller types would receive 0 expected payoffs and the low type seller could always deviate to $m' = L/2 > 0$ and trade with both buyer types for any belief. Sequential rationality for the high type buyer to trade implies that $m^P \leq \mathbb{E}(v|H)$.

In light of Lemma II.2, given any deviation price $m' \neq m^P$, the buyer believes $\pi^P(m') = 1$ and plays the optimal strategy $d^P_{t_b}(m') = \begin{cases} A & \text{if } m' \leq t_b \\ R & \text{o/w} \end{cases}$. Then, the sequential rationality constraints for the seller types are given by:

$$U_{t_s}(m^P) \geq U_{t_s}(m') = (m' - t_s)Q(m') = \begin{cases} (m' - t_s) & \text{if } 0 \leq m' \leq L \\ (m' - t_s)(1-q) & \text{if } L < m' \leq H \\ 0 & \text{if } H < m' \leq H + \lambda_H \end{cases}$$

These inequalities suggest that $m^P \geq \max\{\frac{L}{1-q}, h, H\}$. Along with the condition $m^P \leq \mathbb{E}(v|H)$, the range of pooling equilibrium prices emerges, which is nonempty if and only if $q \leq \frac{\mathbb{E}(v|H) - L}{\mathbb{E}(v|H)} = q^P$ as provided in condition (II.1). $\square$

Proof of Lemma II.3. Consider a separating equilibrium where $m^*_l \neq m^*_h$. Then, sequential rationality for the seller types require that:

$$\left.\begin{aligned} U_l(m_l^*) = m_l^* Q(m_l^*) \geq U_l(m_h^*) = m_h^* Q(m_h^*) \\ U_h(m_h^*) = (m_h^* - h)Q(m_h^*) \geq U_h(m_l^*) = (m_l^* - h)Q(m_l^*) \end{aligned}\right\} \quad Q(m_l^*) \geq Q(m_h^*)$$

If $Q(m_l^*) = Q(m_h^*)$, then the sequential rationality constraints imply that $m_l^* = m_h^*$, contradicting the separating equilibrium assumption. Hence, it has to be the case that $Q(m_l^*) > Q(m_h^*)$ in any separating equilibrium.

To see why the second part of the lemma, i.e. $m_l^* < m_h^*$, holds assume that $m_l^* \geq m_h^*$. Then, as the beliefs satisfy $\pi^*(m_l^*) = 1$ and $\pi^*(m_h^*) = 0$, it would follow that $Q(m_h^*) \geq Q(m_l^*)$ contradicting the first part. □

Proof of Proposition II.2. In an efficient separating equilibrium, it holds that $d_L^{S^e}(m_l^{S^e}) = d_H^{S^e}(m_l^{S^e}) = A$, which suggests $m^{S^e} \leq L$ by sequential rationality of buyer's decision under the separating equilibrium beliefs $\pi^{S^e}(m_l^{S^e}) = 1$. On the other hand, it is also true that $d_L^{S^e}(m_h^{S^e}) = R$, while $d_H^{S^e}(m_h^{S^e}) = A$, which requires that $L+\lambda_L \leq m_h^{S^e} \leq H+\lambda_H$. Observe that high type seller would be willing to sell the object for at least h. As $h > L + \lambda_L$, the high cost seller's equilibrium price range can be revised as $h \leq m_h^{S^e} \leq H + \lambda_H$. In light of these observations, the equilibrium expected payoffs for the seller types are $U_l(m_l^{S^e}) = m_l^{S^e}$ and $U_h(m_h^{S^e}) = (m_h^{S^e} - h)(1 - q)$.

In light of Lemma II.2, the out-of-equilibrium beliefs at any deviation price $m' \neq m_{t_s}^{S^e}$ for either seller type satisfies $\pi^{S^e}(m') = 1$ and buyer plays $d_{t_b}^{S^e}(m') = \begin{cases} A & \text{if } m' \leq t_b \\ R & \text{o/w} \end{cases}$.
Then the sequential rationality constraints for the seller types are given by:

$$U_{t_s}(m_{t_s}^{S^e}) \geq U_{t_s}(m') = (m' - t_s)Q(m') = \begin{cases} (m' - t_s) & \text{if } 0 \leq m' \leq L \\ (m' - t_s)(1 - q) & \text{if } L < m' \leq H \\ 0 & \text{if } H < m' \leq H + \lambda_H \end{cases}$$

For the low type seller, the inequalities require that $m_l^{S^e} \geq L$. Combined the with earlier constraint, the low type seller's equilibrium price is sandwiched at $m_l^{S^e} = L$. In turn, plugging in the equilibrium price for the low seller type yields the inequality that $L \geq H(1 - q)$. For the high type seller, the inequalities require that $m_h^{S^e} \geq H$.

Lastly, the equilibrium prices also have to satisfy sequential rationality against each other, i.e. incentive compatibility constraints. Evaluated at $m_l^{S^e} = L$, constraints require $m_h^{S^e} \leq \frac{L}{1-q}$. Combining all these constraints suggests that, in an efficient separating equilibrium the high type seller announces a price satisfying $\max\{h, H\} \leq m_h^{S^e} \leq \min\{\frac{L}{1-q}, H + \lambda_H\}$.

The valuation $H+\lambda_H$ is assumed to be larger than h by assumption **[A1]**. Similarly, the valuation is larger than H as $\lambda_H \geq 0$. Hence, the range of prices are nonempty whenever

$\frac{L}{1-q} \geq \max\{h, H\}$. Rearranging the terms yields the condition provided in (II.2). $\square$

Proof of Proposition II.3. Inefficient separating equilibria are characterized as those, where only the low cost seller trades while the high cost seller does not trade. In other words, there exists prices $m_l^{S^i} < m_h^{S^i}$ such that $d_{t_b}^{S^i}(m_h^{S^i}) = R$ for both t_b, while $d_{t_b}^{S^i}(m_l^{S^i}) = A$ for either H type or for both L and H type buyers.

First, I prove that such equilibria exist if and only $h \geq H$. To see the necessity, prove the contrapositive, i.e. if $H > h$ then there is no S^i. In order to pursue a contradiction, assume there exists an inefficient separating equilibrium under the condition. As the high cost seller does not trade, his equilibrium expected payoff is $U_h(m_h^{S^i}) = 0$. However, if he deviates and announces $m' = \frac{H+h}{2} \in (h, H)$, then the high type buyer accepts m' for any belief she has, i.e. $d_H^{S^i}(m') = A$. Because $m' > h$, the deviation yields an expected payoff of $U_h(m') = (m' - h)(1 - q) > 0$, yielding the contradiction.

For the sufficiency part, consider the following equilibrium. By Lemma II.2, the equilibrium beliefs satisfy $\pi^{S^i}(m_h^{S^i}) = 0$ and $\pi^{S^i}(m) = 1$ for all $m \neq m_h^{S^i}$. Sequential rationality of the buyer's decision requires that $m_h^{S^i} = H + \lambda_H$, as otherwise the high type buyer would optimally play $d_H^{S^i}(m_h^{S^i}) = A$. For the low cost seller, the optimal price announcements are easy to evaluate, as he only needs to decide whether to engage in trade with only the high type buyer, or both types of buyer. Noting that $d_L^{S^i}(m) = \begin{cases} A & \text{if } m \leq L \\ R & \text{o/w} \end{cases}$, announcing $m \leq L$ yields the low cost seller $U_l(m) = (m - l) = m$, while announcing $m \in (L, H]$ yields $U_l(m) = (m - l)(1 - q) = m(1 - q)$. A simple comparison suggests that announcing $m_l^{S^i} = L$ is optimal whenever $L \geq H(1 - q)$, or equivalently stated whenever $q \geq \frac{H-L}{H} \equiv q^{S^i}$ as in (II.3). $\square$

Proof of Corollary II.2. It is clear that whenever S^e exists, it is the most efficient equilibrium as it achieves first best efficiency. Hence characterization in the case where $H > h$ follows immediately. In the other case where $H \leq h$, for the range where there is no efficient separating equilibrium, i.e. $q \leq q^{S^e} = \max\{\frac{h-L}{h}, \frac{H-L}{H}\} = \frac{h-L}{h}$, efficiency of the pooling equilibrium and the inefficient separating equilibrium need to be compared. If $q \leq \frac{H-L}{H}$, then in the inefficient separating equilibrium S^i, there is trade only between low cost seller and high type buyer. This is strictly less efficient than the pooling equilibrium outcomes where high type buyer trades with both seller types, leading to an expected surplus improvement of $(1 - p)(1 - q)(H + \lambda_H - h) > 0$.

Assuming that $\frac{H-L}{H} \leq q \leq \frac{h-L}{h}$, there are two equilibria; P which generates expected surplus of $(1-p)(1-q)(H+\lambda_H-h)+p(1-q)H$ versus S^i where $m_l^{S^i} = L$, which generates expected surplus of $pqL + p(1 - q)H$. A comparison yields that the pooling equilibrium is more efficient than inefficient separating equilibrium whenever:

$$q \leq \frac{(1 - p)(H + \lambda_H - h)}{(1 - p)(H + \lambda_H - h) + pL}$$

It is not for certain whether the threshold above is less than $\frac{h-L}{h}$. Whenever threshold is below $\frac{h-L}{h}$, then S^i is more efficient for q in the intermediate range and below the threshold the pooling equilibrium is more efficient. Otherwise, there is no range for which S^i is the most efficient and thus P is the most efficient for all $q \leq \frac{h-L}{h}$. This is summarized via $\min$ operator in the statement of Corollary II.2. $\square$

Proof of Corollary II.3. It has been established in Proposition II.1 that there are a continuum of pooling equilibria P varying in the equilibrium price m^P whenever they exist; i.e. $q \leq q^P$. Because the aim is to characterize equilibria with the highest expected payoff for the seller, consider the pooling equilibrium with the highest price or $m^P = \mathbb{E}(v|H)$ with the payoffs $U^P_{t_s} = [\mathbb{E}(v|H) - t_s](1-q)$. On the other hand, the characterization of the efficient separating equilibria in Proposition II.2 suggests that the expected payoff for the low type seller is $U^{S^e}_l = L$ while for the high type is $U^{S^e}_h = [\min\{\frac{L}{1-q}, H + \lambda_H\} - h](1-q)$. Similarly, Proposition II.3 characterizes that in the inefficient separating equilibria, the seller's expected payoffs are $U^{S^i}_l = \max\{L, H(1-q)\}$ and $U^{S_i}_h = 0$. A simple comparison yields that the payoffs for both seller types are greatest under P whenever $q \leq q^P$, and otherwise under S^e. Note that when comparing the payoffs from P with S^e in the cases where P exists, the efficient separating equilibrium payoff for the high type equals $U^{S^e}_h = L - h(1-q)$, which is less than U^P_h. This, in turn allows a clear ranking between payoffs for both seller types. Namely the expected payoff to the high type seller in S^e equals $(H + \lambda_H - h)(1-q)$, which is larger than U^P_h only when $q \geq \frac{H+\lambda_H-L}{H+\lambda_H} > q^P$ or equivalently only when P does not exist. $\square$

Proof of Lemma II.4. Given any equilibrium, consider the type t_s seller's beliefs at subgames $e \in \{i, o\}$:

$$\hat{\pi}^e_{t_s}(\mathbf{M}^i) = \frac{\mathbb{P}(t_b = L)\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^i)\mathbb{P}(e_{t_s}(\mathbf{M}^i) = e)}{\mathbb{P}(t_b = L)\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^i)\mathbb{P}(e_{t_s}(\mathbf{M}^i) = e) + \mathbb{P}(t_b = H)\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^i)\mathbb{P}(e_{t_s}(\mathbf{M}^i) = e)}$$

On the equilibrium path, where information set $\hat{\mathbf{M}}^i$ and $\hat{e}_{t_s}(\hat{\mathbf{M}}^i)$ is reached with positive probability, it is immediate to see that the beliefs for both seller types are equal to q. For any out-of-equilibrium information set, on the other hand, the beliefs are defined as the limits of the sequence of beliefs generated by a sequence of totally mixed strategies converging to the equilibrium strategies. Then, for any such sequence of totally mixed strategies $\{\mathbf{M}^{i,k}\}_{k=1}^{\infty}$ and $\{e^k_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$, Bayes' rule yields the sequence of beliefs with the following elements:

$$\begin{aligned}\pi^{e,k}_{t_s}(\mathbf{M}^i) &= \pi^k(L|t_s, e, \mathbf{M}^i) \\ &= \frac{q\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e^k_{t_s}(\mathbf{M}^i) = e)}{\begin{array}{l} q\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e^k_{t_s}(\mathbf{M}^i) = e) \\ \quad + (1-q)\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e^k_{t_s}(\mathbf{M}^i) = e)\end{array}} = q\end{aligned}$$

Hence, $\hat{\pi}^e_{t_s}(\mathbf{M}^i) = \lim_{k\to\infty} \pi^{e,k}_{t_s}(\mathbf{M}^i) = q$ for both types t_s, all $\mathbf{M}^i$ and e.

Next consider the buyer's beliefs at subgame $e = i$ where menu $\mathbf{M}^i$ is announced and choice c^i is made:

$$\hat{\pi}^i_{t_b}(\mathbf{M}^i, c^i) = \frac{\mathbb{P}(t_s = l)\mathbb{P}(e_l(\mathbf{M}^i) = i)\mathbb{P}(c^i_l(\mathbf{M}^i) = c^i)\mathbb{P}(t_b)\mathbb{P}(\mathbf{M}^i)}{\Big[\mathbb{P}(t_s = l)\mathbb{P}(e_l(\mathbf{M}^i) = i)\mathbb{P}(c^i_l(\mathbf{M}^i) = c^i) + \mathbb{P}(t_s = h)\mathbb{P}(e_h(\mathbf{M}^i) = i)\mathbb{P}(c^i_h(\mathbf{M}^i) = c^i)\Big]\mathbb{P}(t_b)\mathbb{P}(\mathbf{M}^i)}$$

By grouping the same terms in the denominator, it can be seen that the probability of buyer's type gets eliminated out of the beliefs. This follows from the fact that the types are independently distributed. Interestingly, the probability of the intermediary's menu choice also gets eliminated, although the specific choice of menu still appears in the beliefs through seller's entry and price pair decisions. It will be convenient to define the consistent beliefs for future reference. Consider sequence of strategies $\{\mathbf{M}^{i,k}\}_{k=1}^{\infty}$, $\{e^k_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ and $\{c^{i,k}_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ converging to their respective equilibrium strategies. Then the beliefs generated are:

$$\begin{aligned}\hat{\pi}^i_{t_b}(\mathbf{M}^i, c^i) &= \lim_{k\to\infty} \pi^{i,k}_{t_b}(\mathbf{M}^i, c^i) \\ &= \lim_{k\to\infty} \frac{p\mathbb{P}(e^k_l(\mathbf{M}^i) = i)\mathbb{P}(c^{i,k}_l(\mathbf{M}^i) = c^i)}{p\mathbb{P}(e^k_l(\mathbf{M}^i) = i)\mathbb{P}(c^{i,k}_l(\mathbf{M}^i) = c^i) + (1-p)\mathbb{P}(e^k_h(\mathbf{M}^i) = i)\mathbb{P}(c^{i,k}_h(\mathbf{M}^i) = c^i)} \\ &= \hat{\pi}^i(\mathbf{M}^i, c^i)\end{aligned} \tag{B.1}$$

Finally consider buyer's beliefs at subgame $e = o$, where she observes price announcement m^o from the seller. Just as in the other subgame, due to the independence of the types, probability of buyer's type can be factored out in the denominator and hence is eliminated out of her beliefs. Again, a description of the consistent beliefs are provided for future reference. Consider sequence of strategies $\{\mathbf{M}^{i,k}\}_{k=1}^{\infty}$, $\{e^k_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ and $\{m^{o,k}_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ converging to their respective equilibrium strategies. Then the beliefs generated are:

$$\begin{aligned}\hat{\pi}^o_{t_b}(m^o) &= \lim_{k\to\infty} \pi^{o,k}_{t_b}(m^o) \\ &= \lim_{k\to\infty} \frac{p\int_{\mathbf{M}^i} \mathbb{P}(e^k_l(\mathbf{M}^i) = o)\mathbb{P}(m^{o,k}_l(\mathbf{M}^i) = m^o)\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)d\mathbf{M}^i}{\sum_{t_s\in\{l,h\}}\Big[\mathbb{P}(t_s)\int_{\mathbf{M}^i} \mathbb{P}(e^k_{t_s}(\mathbf{M}^i) = o)\mathbb{P}(m^{o,k}_{t_s}(\mathbf{M}^i) = m^o) \mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)d\mathbf{M}^i\Big]} \\ &= \hat{\pi}^o(m^o)\end{aligned} \tag{B.2}$$

□

Proof of Lemma II.5. Consider an equilibrium. Then in the subgame following $e = o$, both of the buyer's types have the same beliefs $\hat{\pi}^o(m^o)$ which are constant for any menu announcement $\mathbf{M}^i$. Hence, each buyer type's optimal decision strategy $\hat{d}^o_{t_b}(m^o)$ is the same across all menus $\mathbf{M}^i$. This implies, due to sequential rationality, that each seller type announces $\hat{m}^o_{t_s}(\mathbf{M}^i)$ which maximizes $U^o_{t_s}(\mathbf{M}^i, m^o)$ over all $m^o \in [0, H+\lambda_H]$. Then, for two different menu announcements $\mathbf{M}^i$ and $\acute{\mathbf{M}}^i$, it has to be the case that $U^o_{t_s}(\mathbf{M}^i, \hat{m}^o_{t_s}(\mathbf{M}^i)) = U^o_{t_s}(\acute{\mathbf{M}}^i, \hat{m}^o_{t_s}(\acute{\mathbf{M}}^i))$, as otherwise it would be optimal to announce the TIOLI offer that yields the strictly higher expected payoff in the subgame $e = o$ following either menu announcements. This equality also extends to the case where the intermediary announces the equilibrium path menu $\hat{\mathbf{M}}^i$, which is the first statement in the first part of Lemma II.5.

In order to understand the lower bounds on $\hat{U}^o_{t_s}$, assume that the beliefs of the buyer in subgame $e = o$ are equal to $\pi^o(m^o) = 1$ for all m^o. These are the worst possible beliefs for the seller in terms of his prospects of securing high expected payoffs in the subgame, simply because it lowers the maximum TIOLI offer that the buyer types would be willing to accept. Under these beliefs, it is optimal for the buyer of type t_b to accept any offer $m^o \leq t_b$.[1] Then, each seller type could always guarantee securing a minimum payoff of $\max\{(L - t_s), (H - t_s)(1 - q), 0\}$, where 0 is received by announcing an offer of $m^o > H$, which the buyer optimally rejects. This implies that, for the low type seller $\hat{U}^o_l \geq \max\{L, H(1 - q)\}$ due to the two terms in the max operator being strictly greater than 0. For the high type, however, $\hat{U}^o_h \geq \max\{(H - h)(1 - q), 0\}$ where the first term drops out since $L < h$.

The second part also follows from sequential rationality of seller's equilibrium strategies. In any equilibrium, $\hat{U}_{t_s}$ has to equal $\hat{U}^o_{t_s}$ if $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = o$. If, on the other hand, equilibrium entry strategy is to choose $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = i$, then $\hat{U}_{t_s} = \hat{U}^i_{t_s}$, which has to be larger or equal to $\hat{U}^o_{t_s}$ due to the optimality of the entry decision. Similarly, each seller type might mimic the other type's equilibrium strategies. Sequential rationality requires that the expected payoffs resulting from this mimicking behavior has to be less than or equal to the type's actual equilibrium expected payoff. For each type of the seller, this condition amounts to requiring:

$$\hat{U}_l = \big[\hat{m}_s[l] - l\big]\hat{Q}\big(\hat{m}_b[l]\big) \geq \big[\hat{m}_s[h] - l\big]\hat{Q}\big(\hat{m}_b[h]\big) = \hat{U}_h + (h - l)\hat{Q}\big(\hat{m}_b[h]\big)$$
$$\hat{U}_h = \big[\hat{m}_s[h] - h\big]\hat{Q}\big(\hat{m}_b[h]\big) \geq \big[\hat{m}_s[l] - h\big]\hat{Q}\big(\hat{m}_b[l]\big) = \hat{U}_l - (h - l)\hat{Q}\big(\hat{m}_b[l]\big)$$

Plugging in the normalization of $l = 0$ and combining the two conditions using max operator, the two inequalities in the second part of Lemma II.5 are attained. □

Proof of Lemma II.6. Firstly observe that in an efficient intermediated equilibrium, the trade probabilities can be implemented only when the two types of the seller choose dif-

[1] Note that due to similar reasons as in the proof of Lemma II.2, it is optimal for the buyer to accept an offer of $m^o = t_b$ even though she is indifferent between A and R. If the buyer were to reject the offer, then the seller types could get arbitrarily close to the expected payoff from when the buyer accepts by approaching $m^o = t_b$ from below.

ferent prices. Since $\hat{m}_b[l] \neq \hat{m}_b[h]$, there is full separation on the equilibrium path, i.e. abusing notation the buyer's equilibrium beliefs are $\hat{\pi}(\hat{m}_b[l]) = 1$ and $\hat{\pi}(\hat{m}_b[h]) = 0$. Furthermore, the implemented trade probabilities require that $\hat{m}_b[h] \in [L+\lambda_L, H+\lambda_H]$ while $\hat{m}_b[l] \in [0, L]$.

In order show that in an efficient intermediated equilibrium both seller types have to choose the intermediary as the entry decision, pursue a contradiction by assuming the entry decision of at least one seller type is for the direct trade outside option, i.e. $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = o$ for at least one t_s. Then it has to be the case that $\hat{m}_s[t_s] \leq \hat{m}_b[t_s]$. To see why, observe that if for type t_s entry decision is $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = o$, then the inequality binds as both terms equal $\hat{m}^o_{t_s}(\hat{\mathbf{M}}^i)$. Otherwise, the intermediary is chosen only by that seller type and sequential rationality of the intermediary's strategy implies the inequality.

Furthermore, optimality of equilibrium prices also require the high type seller's price to satisfy $\hat{m}_s[h] \geq h$, as otherwise he would not be trading with the H type buyer. Combining these observations yields $\hat{m}_s[l] \leq \hat{m}_b[l] \leq L < h \leq \hat{m}_s[h] \leq \hat{m}_b[h]$.

From the inequalities above, it can be seen that $\hat{U}_l = \hat{m}_s[l]\hat{Q}(\hat{m}_b[l]) = \hat{m}_s[l] \leq L$. On the other hand, Lemma II.5 suggests that the low type seller's expected equilibrium payoff $\hat{U}_l$ has to satisfy:

$$\begin{aligned}\hat{U}_l &\geq \max\{\hat{U}^o_l, \hat{U}_h + h\hat{Q}(\hat{m}_b[h])\}\\ &\geq \max\{L, H(1-q), \hat{U}_h + h(1-q)\}\\ &\geq \max\{H(1-q), h(1-q)\} > L\end{aligned}$$

where the second to last inequality follows from the fact that $\hat{U}_h \geq 0$ and the last inequality follows from Assumption II.2; i.e. $q < q^{S^e} \;\Leftrightarrow\; L < \max\{H(1-q), h(1-q)\}$. Combining above inequalities yields a contradiction, because $L \geq \hat{U}_l > L$. In turn this implies that, if there exists an efficient intermediated equilibrium, then both seller types have to choose the intermediary. □

Proof of Proposition II.4. In an efficient intermediated equilibrium, the total expected gains from trade is equal to $pL + (1-p)(1-q)(H + \lambda_H - h)$, while each seller type receives expected payoff of $\hat{U}_{t_s}$. The difference is the sum of the intermediary's expected payoff $\hat{V} = V(\hat{\mathbf{M}}^i)$ and buyer's ex-ante expected payoff. Sequential rationality requires these payoffs to be nonnegative, which in turn yields the inequality in (II.5). □

Proof of Theorem II.1. For the necessity part, consider the inequalities from Lemma II.5:

$$\begin{aligned}\hat{U}_l &\geq \hat{U}_h + h(1-q) \geq \hat{U}^o_h + h(1-q) \geq \max\{H, h\}(1-q)\\ \hat{U}_h &\geq \hat{U}^o_h \geq \max\{(H-h)(1-q), 0\} = \max\{H, h\}(1-q) - h(1-q)\end{aligned}$$

Plugging in these to the condition (II.5) from Proposition II.4 and rearranging the terms yields the condition (II.6):

$$\begin{aligned}
pL + (1-p)(1-q)(H+\lambda_H - h) &\geq p\hat{U}_l + (1-p)\hat{U}_h \\
&\geq \max\{H,h\}(1-q) - (1-p)(1-q)h \\
\Rightarrow\ p\frac{L}{1-q} + (1-p)(H+\lambda_H) &\geq \max\{H,h\}
\end{aligned}$$

To show the sufficiency, I will show that there exists an efficient intermediated equilibrium whenever condition (II.6) holds. Before describing the equilibrium, I define the following notation, which will be used in the rest of the proof. Consider the subgame $e = i$, where the intermediary announces a menu $\mathbf{M}^i$, the seller type t_s makes a choice $c^i \in \{1, 2\}$ and buyer's beliefs are $\pi^i(\mathbf{M}^i, c^i) \in [0, 1]$. When the seller chooses between the two options of the menu, he assumes that each buyer type plays her corresponding equilibrium decision strategy $\hat{d}^i_{t_b}(\mathbf{M}^i, c^i)$, which depends on her beliefs π^i. Given a price pair choice c^i, denote the expected trade probability by $Q^i(m^i_b(c^i)|\pi^i)$ for any possible belief. Clearly, the equilibrium strategies are evaluated using the consistent equilibrium beliefs, i.e. $\pi^i = \hat{\pi}^i$. Similarly, denote by $U^i_{t_s}(\mathbf{M}^i, c^i|\pi^i)$ type t_s seller's expected payoff from making choice c^i when the buyer types play their optimal strategies under some beliefs π^i. Again, the equilibrium choice of the seller types $\hat{c}^i_{t_s}(\mathbf{M}^i)$ are evaluated when the buyer has her equilibrium beliefs $\pi^i = \hat{\pi}^i$. It is also important to note that both $Q^i(m^i_b(c^i)|\pi^i)$ and $U^i_{t_s}(\mathbf{M}^i, c^i|\pi^i)$ are weakly decreasing in π^i. To see why, observe that each buyer type's updated value for the object is equal to $t_b + (1 - \pi^i(\mathbf{M}^i, c^i))\lambda_{t_b}$, which is decreasing in π^i. Hence, as π^i increases, it might be the case that the updated value of some buyer type falls below $m^i_b(c^i)$ leading that type to reject the offer, which consequently decreases the expected trade probability and the payoff to the seller types. In the rest of the proof, the terms $U^i_{t_s}(\mathbf{M}^i, c^i)$ and $Q(m^i_b(c^i))$ omitting π^i refer to the cases, where the respective terms are evaluated at the equilibrium beliefs $\pi^i = \hat{\pi}^i$.

Consider the following assessment:

$$\hat{\mathbf{M}}^i = \{(\max\{H,h\}(1-q), L), (\max\{H,h\}, H+\lambda_H)\}$$

$$\hat{e}_{t_s}(\mathbf{M}^i) = \begin{cases} i & \text{if } \mathbf{M}^i = \hat{\mathbf{M}}^i \text{ OR } U^i_{t_s}(\mathbf{M}^i, \hat{c}^i_{t_s}(\mathbf{M}^i)) \geq \hat{U}^o_{t_s} \text{ and } m^i_b(\hat{c}^i_{t_s}(\mathbf{M}^i) \leq H \\ o & \text{o/w} \end{cases}$$

$$\hat{c}^i_{t_s}(\mathbf{M}^i) = \begin{cases} 1 & \text{if } t_s = l \text{ and } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ 2 & \text{if } t_s = h \text{ and } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ 1 & \text{if } U^i_{t_s}(\mathbf{M}^i, 1) > U^i_{t_s}(\mathbf{M}^i, 2) \\ 2 & \text{if } U^i_{t_s}(\mathbf{M}^i, 2) > U^i_{t_s}(\mathbf{M}^i, 1) \\ \{1,2\} & \text{o/w} \end{cases}$$

$$\hat{m}^o_l(\mathbf{M}^i) = \begin{cases} L & \text{if } \frac{H-L}{H} \leq q < q^{S^e} \quad (\text{requires } h > H) \\ H & \text{if } q < \frac{H-L}{H} \end{cases}$$

$$\hat{m}^o_h(\mathbf{M}^i) = \max\{H,h\}$$

$$\hat{d}^i_{t_b}(\mathbf{M}^i, c^i) = \begin{cases} A & \text{if } m^i_b(c^i) \leq t_b + (1-\hat{\pi}^i(\mathbf{M}^i, c^i))\lambda_{t_b} \\ R & \text{o/w} \end{cases}$$

$$\hat{d}^o_{t_b}(m^o) = \begin{cases} A & \text{if } m^o \leq t_b + (1-\hat{\pi}^o(m^o))\lambda_{t_b} \\ R & \text{o/w} \end{cases}$$

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} 0 & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}^i_h(\mathbf{M}^i) = c^i \neq \hat{c}^i_l(\mathbf{M}^i) \\ 0 & \text{if } \hat{e}_h(\mathbf{M}^i) = i,\ \hat{e}_l(\mathbf{M}^i) = o,\ \hat{c}^i_h(\mathbf{M}^i) = c^i, \text{ and } \hat{c}^i_l(\mathbf{M}^i) \in \{1,2\} \\ p & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}^i_l(\mathbf{M}^i) = \hat{c}^i_h(\mathbf{M}^i) = c^i \\ 1 & \text{o/w} \end{cases}$$

$$\hat{\pi}^o(m^o) = 1$$

I will show that above strategies and beliefs form an efficient intermediated equilibrium. First consider the equilibrium path play and verify that it satisfies the definition of efficient intermediated equilibrium. On the equilibrium path, intermediary announces menu $\hat{\mathbf{M}}^i$, both seller types choose i, low type seller chooses option 1 while high type chooses option 2.

The buyer's beliefs on this path are equal to $\hat{\pi}^i(\hat{\mathbf{M}}^i, 1) = 1$ and $\hat{\pi}^i(\hat{\mathbf{M}}^i, 2) = 0$. Then her optimal decision strategies satisfy $\hat{d}^i_L(\hat{\mathbf{M}}^i, 1) = \hat{d}^i_H(\hat{\mathbf{M}}^i, 1) = A$ as $\hat{m}^i_b(1) = L \leq t_b + (1 - \hat{\pi}^i(\hat{\mathbf{M}}^i, 1))\lambda_{t_b} = t_b$, and $\hat{d}^i_H(\hat{\mathbf{M}}^i, 2) = A$ while $\hat{d}^i_L(\hat{\mathbf{M}}^i, 2) = R$, because $\hat{m}^i_b(2) = H + \lambda_H = H + (1 - \hat{\pi}^i(\hat{\mathbf{M}}^i, 2))\lambda_H > L + (1 - \hat{\pi}^i(\hat{\mathbf{M}}^i, 2))\lambda_L$.

The seller's choices $\hat{c}^i_{t_s}(\hat{\mathbf{M}}^i)$ are optimal as $U^i_l(\hat{\mathbf{M}}^i, 1) = \max\{H,h\}(1-q) \geq U^i_l(\hat{\mathbf{M}}^i, 2) = \max\{H,h\}(1-q)$ and $U^i_h(\hat{\mathbf{M}}^i, 2) = [\max\{H,h\} - h](1-q) > U^i_h(\hat{\mathbf{M}}^i, 1) = \max\{H,h\}(1-$

$q) - h$.

On the out-of-equilibrium subgame $e = o$, buyer's beliefs are $\hat{\pi}^o(m^o) = 1$ for all TIOLI offers. Hence, buyer's optimal strategy is to accept any offer less than or equal to her updated belief, i.e. $m^o \leq t_b$. For low type seller, announcing $\hat{m}_l^o$ yields him an expected payoff of $\hat{U}_l^o = \max\{L, H(1-q)\}$. On the other hand, for the h type seller if $h > H$, then there is no TIOLI offer that yields him a strictly positive expected payoff. Thus $\hat{U}_h^o = \max\{(H-h)(1-q), 0\} = [\max\{H, h\} - h](1-q)$.

In light of these, it follows that seller's entry strategies $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = i$ are optimal, as each type receives $\hat{U}_{t_s}^i \geq \hat{U}_{t_s}^o$. Finally, the intermediary receives an expected payoff of $V(\hat{\mathbf{M}}^i) = pL + (1-p)(1-q)(H + \lambda_H) - \max\{H, h\}(1-q)$, which is nonnegative given the condition (II.6) is satisfied.

Next, I show the sequential rationality of the strategies starting with the buyer and work backwards. The buyer's optimal strategy should be to accept, only when the price she pays is less than or equal to her updated value for the good, which is precisely what the strategies formulate.

The seller in subgame $e = i$ should choose the option among the two price pairs that gives him the higher expected payoff. Clearly, when he is indifferent, he could choose either of the two options. In the subgame $e = o$, however, the seller's optimal TIOLI offer strategies $\hat{m}_{t_s}^o(\mathbf{M}^i)$ depend on his type. Given the buyer's beliefs are equal to $\hat{\pi}^o(m^o) = 1$ for all TIOLI offers, it follows that $\hat{Q}^o(m^o) = \begin{cases} 1 & \text{if } m^o \leq L \\ 1-q & \text{if } m^o \in (L, H] \\ 0 & \text{o/w} \end{cases}$. Then the low type seller would choose between trading only with the high type buyer or both types. Hence, he announces H when q is sufficiently small or L otherwise. Note that the optimal TIOLI satisfies $\hat{m}_l^o = L$ if q is above $\frac{H-L}{H}$ which can happen only when $h > H$, due to Assumption II.2. The high type seller, on the other hand, would never want to trade with both types of the buyer, as that would cap the maximum TIOLI offer at L which is strictly less than h. Hence, he chooses to trade with H type buyer at price H only when $H \geq h$. Otherwise, he gets an expected payoff of 0 by announcing any price $m^o > H$, and clearly $\hat{m}_h^o = h$ is one such optimal announcement.

The seller, when choosing his entry decision, compares the expected payoffs from the two subgames evaluated using the continuation equilibrium strategies $\hat{c}_{t_s}^i$, $\hat{d}_{t_b}^i$, $\hat{m}_{t_s}^o$ and $\hat{d}_{t_b}^o$. Apart from the equilibrium menu announcement $\hat{\mathbf{M}}^i$, both types choose entry to intermediary subgame whenever the expected payoff from intermediary exceeds the direct trade outside option subgame expected payoff and the buyer's price of the subsequent menu choice satisfies $m_b^i(\hat{c}_{t_s}^i(\mathbf{M}^i)) \leq H$. Observe that, if the latter condition is violated, i.e. $m_b^i(j) > H$ where $j = \hat{c}_{t_s}^i(\mathbf{M}^i)$, then the buyer's equilibrium beliefs are equal to $\hat{\pi}^i(\mathbf{M}^i, j) = 1$. In turn, both buyer types would choose $\hat{d}_{t_b}^i(\mathbf{M}^i, j) = R$, as $m_b^i(j) > H \geq t_b$. Thus $U_{t_s}^i(\mathbf{M}^i, j) = 0 \leq \hat{U}_{t_s}^o$ for both types, and $\hat{e}_{t_s}(\mathbf{M}^i) = o$ is indeed optimal.

Lastly, consider the optimality of intermediary's menu announcement strategy. In order to check that announcing $\hat{\mathbf{M}}^i$ satisfies sequential rationality, I discuss what happens following any deviation menu announcement $\mathbf{M}^i \in \mathcal{M}^i \setminus \{\hat{\mathbf{M}}^i\}$. In particular, I want to verify that for any deviation menu announcement, the aforementioned sequentially rational strategies form subgame equilibria, yet the deviation yields the intermediary an expected payoff that is less than the equilibrium payoff $V(\hat{\mathbf{M}}^i)$. I will consider three cases of a deviation menu $\mathbf{M}^i$; first where $m_b^i(j) > H$ for both $j \in \{1,2\}$, second where (WLOG) $m_b^i(1) \leq H$ while $m_b^i(2) > H$ and lastly where $m_b^i(j) \leq H$ for both $j \in \{1,2\}$.

Case 1) Starting with the first case, if both $m_b^i(j) > H$, then optimal entry strategies specify $\hat{e}_{t_s}(\mathbf{M}^i) = o$ for both seller types. Then for any price pair choices for both seller types, the buyer's beliefs are $\hat{\pi}^i(\mathbf{M}^i, c^i) = 1$ for both c^i, which suggests that $\hat{d}^i_{t_b}(\mathbf{M}^i, c^i) = R$ for both buyer types and choices c^i. The expected payoff for both seller types from either choice is equal to $U^i_{t_s}(\mathbf{M}^i, c^i) = 0$, hence it is optimal to have $\hat{c}^i(\mathbf{M}^i) \in \{1,2\}$. Since $\hat{U}^o_{t_s}(\mathbf{M}^i) \geq 0$, it holds that $\hat{e}_{t_s}(\mathbf{M}^i) = o$ is indeed sequentially rational.

To recap, whenever $\mathbf{M}^i$ satisfies $m_b^i(j) > H$ for both j, then in subgame equilibrium the intermediary does not attract either seller type. Hence the expected payoff is equal to $V(\mathbf{M}^i) = 0$. As $V(\hat{\mathbf{M}}^i) \geq 0$, there is no profitable deviation from any such menu.

Case 2) Consider the second case where $m_b^i(1) \leq H < m_b^i(2)$. If buyer's belief satisfies $\hat{\pi}^i(\mathbf{M}^i, 2) = 1$, then $\hat{d}^i_{t_b}(\mathbf{M}^i, 2) = R$ for both buyer types, which consequently yields $U^i_{t_s}(\mathbf{M}^i, 2) = 0$ for both seller types. Because $m_b^i(1) \leq H$, for any belief $\pi^i = \pi^i(\mathbf{M}^i, 1) \in [0,1]$, the high type buyer plays $\hat{d}^i_H(\mathbf{M}^i, 1) = A$ as $m_b^i(1) \leq H \leq H + (1-\pi^i)\lambda_H$, which implies that $Q^i(m_b^i(1)|\pi^i) \geq 1-q$. Thus, it holds that $U^i_l(\mathbf{M}^i, 1|\pi^i) \geq U^i_l(\mathbf{M}^i, 2) = 0$, or that it is optimal for low type seller to play $\hat{c}^i(\mathbf{M}^i) = 1$. For the high type seller, on the other hand, $U^i_h(\mathbf{M}^i, 1|\pi^i) = (m_s^i(1) - h)Q^i(m_b^i(1)|\pi^i) \geq (m_s^i(1) - h)(1-q)$ for any π^i. If $m_s^i(1) \geq h$, then the expected payoff from choosing 1 is greater or equal to 0 yielding $\hat{c}^i_h(\mathbf{M}^i) = \begin{cases} 1 & \text{if } m_s^i(1) \geq h \\ 2 & \text{o/w} \end{cases}$.

Next, consider the entry strategies of the seller types. First consider the high type seller's expected payoff $U^i_h(\mathbf{M}^i, 1|p)$ from choosing price pair $c^i = 1$, when buyer's beliefs equal $\pi^i(\mathbf{M}^i, 1) = p$. If $U^i_h(\mathbf{M}^i, 1|p) \geq \hat{U}^o_h = [\max\{H, h\} - h](1-q)$, then indeed the high type seller enters, i.e. $\hat{e}_h(\mathbf{M}^i) = i$. In turn, this implies that $U^i_l(\mathbf{M}^i, 1|p) = U^i_h(\mathbf{M}^i, 1|p) + hQ^i(m_b^i(1)|p) \geq [\max\{H, h\} - h](1-q) + h(1-q) = \max\{H, h\}(1-q) \geq \hat{U}^o_l = \max\{L, H(1-q)\}$, where last inequality is due to Assumption II.2. The previous inequalities altogether suggest that low type seller also enters, i.e. $\hat{e}_l(\mathbf{M}^i) = i$. Observe that, in this case, given both seller types would enter and choose price pair 1, indeed the equilibrium beliefs would equal $\hat{\pi}^i(\mathbf{M}^i, 1) = p$ and $\hat{\pi}^i(\mathbf{M}^i, 2) = 1$.

If, on the other hand, $U^i_h(\mathbf{M}^i, 1|p) < \hat{U}^o_h$, then the optimal entry would be $\hat{e}_h(\mathbf{M}^i) = o$, in which case $\hat{\pi}^i(\mathbf{M}^i, c^i) = 1$ for both c^i. Note that, $U^i_h(\mathbf{M}^i, 1|1) \leq U^i_h(\mathbf{M}^i, 1|p)$ which maintains the optimality of $e = o$ choice for h type seller. Then the entry strategy of the low type seller equals $\hat{e}_l(\mathbf{M}^i) = i$ whenever $U_l^i(\mathbf{M}^i, 1|1) \geq \hat{U}_l^o$ and equals o otherwise.

To recap, whenever $\mathbf{M}^i$ satisfies (WLOG) $m_b^i(1) \leq H < m_b^i(2)$, then in subgame equilibrium either both seller types enter choosing option 1, or only the low type seller chooses option 1 under the intermediary while the high type seller opts for the direct trade outside option. In the former case, it needs to be the case that $m_s^i(1) \geq \max\{H, h\}$ for the high type to come. Consequently, the expected profits to the intermediary are equal to $V(\mathbf{M}^i) = [m_b^i(1) - m_s^i(1)]Q^i(m_b^i(1)|p) \leq [H - \max\{H, h\}]Q^i(m_b^i(1)|p) \leq 0$. In the latter case, low type seller choosing intermediary requires $U_s^i(\mathbf{M}^i, 1|1) = m_s^i(1)Q^i(m_b^i(1)|1) \geq \hat{U}_l^o = \max\{L, H(1-q)\}$. Then the expected profits for the intermediary from deviation are equal to $V(\mathbf{M}^i) = p[m_b^i(1) - m_s^i(1)]Q^i(m_b^i(1)|1) \leq p\big[\max\{H(1-q), L\} - m_s^i(1)Q^i(m_b^i(1)|1)\big] \leq 0$. Therefore, deviation to such a menu is not profitable.

Case 3) Finally consider the last case, where $m_b^i(j) \leq H$ for both j. Assume, WLOG, that $U_l^i(\mathbf{M}^i, 1|1) \geq U_l^i(\mathbf{M}^i, 2|1)$.

Subcase 3.1) If $\max_j\{U_h^i(\mathbf{M}^i, j|0)\} < \hat{U}_h^o$, then it would be optimal for high type seller to choose $\hat{e}_h(\mathbf{M}^i) = o$, where the beliefs equal $\hat{\pi}^i(\mathbf{M}^i, j) = 1$ for both $j \in \{1, 2\}$. To see why, observe that $\max_j\{U_h^i(\mathbf{M}^i, j|1)\} \leq \max_j\{U_h^i(\mathbf{M}^i, j|0)\} < \hat{U}_h^o$. Then equilibrium requires high type seller to play $\hat{c}_h^i(\mathbf{M}^i) = \begin{cases} 1 & \text{if } U_h^i(\mathbf{M}^i, 1|1) \geq U_h^i(\mathbf{M}^i, 2|1) \\ 2 & \text{o/w} \end{cases}$ as his optimal choice strategy, where WLOG his indifference is broken in favor of price pair 1, while low type seller playing $\hat{c}_l^i(\mathbf{M}^i) = 1$ and entry strategy $\hat{e}_l(\mathbf{M}^i) = \begin{cases} i & \text{if } U_l^i(\mathbf{M}^i, 1|1) \geq \hat{U}_l^o \\ o & \text{o/w} \end{cases}$.

Subcase 3.2) If $\max_j\{U_h^i(\mathbf{M}^i, j|0)\} \geq \hat{U}_h^o$, on the other hand, specifics of the subgame equilibrium following deviation depend on whether high type seller prefers price pair 1 or 2 under beliefs $\pi^i = p$.

3.2.1) In the first case where $U_h^i(\mathbf{M}^i, 1|p) \geq U_h^i(\mathbf{M}^i, 2|p)$, it would be optimal for both seller types to enter and choose price pair 1 with beliefs equal to $\hat{\pi}^i(\mathbf{M}^i, 1) = p$ and $\hat{\pi}^i(\mathbf{M}^i, 2) = 1$. To see why the low type seller would also enter, observe that $U_l^i(\mathbf{M}^i, 1|p) = U_h^i(\mathbf{M}^i, 1|p) + hQ^i(m_b^i(1)|p) \geq U_h^i(\mathbf{M}^i, 1|p) + h(1-q) \geq \hat{U}_h^o + h(1-q) \geq \max\{H, h\}(1-q) \geq \hat{U}_l^o$. Furthermore, $\hat{c}_l^i(\mathbf{M}^i) = 1$ is also optimal here since $U_l^i(\mathbf{M}^i, 1|p) \geq U_l^i(\mathbf{M}^i, 1|1) \geq U_l^i(\mathbf{M}^i, 2|1)$.

3.2.2) In the other case where $U_h^i(\mathbf{M}^i, 1|p) < U_h^i(\mathbf{M}^i, 2|p)$, it further depends on low type seller's optimal price pair choice for different beliefs.

A: If $U_l^i(\mathbf{M}^i, 2|p) \geq U_l^i(\mathbf{M}^i, 1|1) \geq U_l^i(\mathbf{M}^i, 2|1)$, then both seller types enter, i.e. $\hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i$, and both choose price pair 2, i.e. $\hat{c}_l^i(\mathbf{M}^i) = \hat{c}_h^i(\mathbf{M}^i) = 2$, which in turn generates beliefs $\hat{\pi}^i(\mathbf{M}^i, 1) = 1$ and $\hat{\pi}^i(\mathbf{M}^i, 2) = p$.

B: Otherwise, $U_l^i(\mathbf{M}^i, 1|1) > U_l^i(\mathbf{M}^i, 2|p) \geq U_l^i(\mathbf{M}^i, 2|1)$ in which case final two scenarios have to be considered.

B.i: For the first subcase where $U_l^i(\mathbf{M}^i, 1|1) \geq U_l^i(\mathbf{M}^i, 2|0) \geq U_l^i(\mathbf{M}^i, 2|p) \geq U_l^i(\mathbf{M}^i, 2|1)$,[2]

[2]Note that one of the first two inequalities is strict.

there is an equilibrium such that each seller type chooses a different price pair, which leads the buyer to infer the types, i.e. $\hat{\pi}^i(\mathbf{M}^i, 1) = 1$ and $\hat{\pi}^i(\mathbf{M}^i, 2) = 0$. The optimal strategies are $\hat{e}_{t_s}(\mathbf{M}^i) = i$ for both seller types, while choosing $\hat{c}_l^i(\mathbf{M}^i) = 1$ and $\hat{c}_h^i(\mathbf{M}^i) = 2$.

B.ii: The other subcase with $U_l^i(\mathbf{M}^i, 2|0) > U_l^i(\mathbf{M}^i, 1|1) > U_l^i(\mathbf{M}^i, 2|p) \geq U_l^i(\mathbf{M}^i, 2|1)$ is a situation where both seller types would like to enter, the high type seller would like to choose price pair 2, however low type seller would like to choose price pair 2 only for sufficiently high beliefs and otherwise would like to strictly choose price pair 1. I will show that, it is not possible to have this kind of a problematic case.

Firstly, note that $U_{t_s}^i(\mathbf{M}^i, j|\pi^i) = [m_s^i(j) - t_s]Q^i(m_b^i(j)|\pi^i)$ and $Q^i(m_b^i(j)|\pi^i) \geq (1-q)$ for both $j \in \{1, 2\}$ and all π^i, where the latter inequality follows from $m_b^i(j) \leq H$.

Then in light of the inequality $U_l^i(\mathbf{M}^i, 2|0) > U_l^i(\mathbf{M}^i, 2|p)$, it has to be the case that $Q^i(m_b^i(2)|0) = 1 > Q^i(m_b^i(2)|p) = 1-q$ or that $m_b^i(2) \in (L+(1-p)\lambda_L, L+\lambda_L]$. Plugging in $Q^i(m_b^i(2)|p) = (1-q)$ yields $U_h^i(\mathbf{M}^i, 2|p) = [m_s^i(2)-h](1-q) \geq \hat{U}_h^o = [\max\{H, h\}-h](1-q)$ or that $m_s^i(2) \geq \max\{H, h\}$. Combining these inequalities yields $m_s^i(2) \geq \max\{H, h\} > L+\lambda_H \geq m_b^i(2)$. This implies that for $\mathbf{M}^i \in \mathcal{M}^i$, it needs to be the case that $m_s^i(1) \leq m_b^i(1)$.

On the other hand, one of the conditions in the subcase is that $U_h^i(\mathbf{M}^i, 2|p) = [m_s^i(2) - h](1-q) > U_h^i(\mathbf{M}^i, 1|p) = [m_s^i(1) - h]Q^i(m_b^i(1)|p) \geq [m_s^i(1) - h](1-q)$, which suggests that $m_s^i(1) < m_s^i(2)$. This inequality, in turn implies that in order for $U_l^i(\mathbf{M}^i, 1|1) = m_s^i(1)Q^i(m_b^i(1)|1) > U_l^i(\mathbf{M}^i, 2|p) = m_s^i(2)(1-q)$ to hold, $m_b^i(1) \leq L$ so that $Q^i(m_b^i(1)|1) = 1$. Plugging these observations into low type seller's inequalities yields $U_l^i(\mathbf{M}^i, 1|1) = m_s^i(1) > U_l^i(\mathbf{M}^i, 2|p) = m_s^i(2)(1-q) \geq \max\{H, h\}(1-q)$. Combining these inequalities over the price pair for choice $c^i = 1$, it can be deduced that $m_b^i(1) \leq L < \max\{H, h\}(1-q) \leq m_s^i(1)$, where strict inequality follows from Assumption II.2. Hence, all the conditions combined require $m_s^i(j) > m_b^i(j)$ for both price pairs, which contradicts with $\mathbf{M}^i \in \mathcal{M}^i$.

To recap, whenever $\mathbf{M}^i$ satisfies $m_b^i(j) \leq H$ for both j, then in subgame equilibrium it could be that neither seller type enters, or both seller types enter and then choose the same option or different options, or it could be that only the low type seller enters chooses one of the two options.[3] In the cases where neither seller type enters, the expected profits are 0, as in **Case 1**. In the other cases, where only the low type enters or both seller types enter choosing the same option, the expected profits from deviation are similar to **Case 2** that was previously covered; i.e. $\mathbf{M}^i$ satisfies $m_b^i(1) \leq H < m_b^i(2)$ and in equilibrium either only low type seller enters choosing option 1, or both seller types enter choosing option 1. Since in those cases deviation is not profitable, neither will it be in this case. The only case to check is, if both types enter and choose different options; (WLOG) low type chooses 1 and high type chooses 2. In this case, optimality of the entry decision to intermediary for each type requires that $m_s^i(1)Q^i(m_b^i(1)|1) \geq \max\{L, H(1-q)\}$ for the low type seller and $m_s^i(2) \geq \max\{H, h\}$ for the high type seller. Then the expected profits

[3]In other words, it can never be that only high type seller enters, while the low type chooses the direct trade outside option.

are:

$$
\begin{aligned}
V(\mathbf{M}^i) &= p[m_b^i(1) - m_s^i(1)]Q^i(m_b^i(1)|1) + (1-p)[m_b^i(2) - m_s^i(2)]Q^i(m_b^i(2)|0) \\
&\leq p\big[\max\{H, L(1-q)\} - m_s^i(1)Q^i(m_s^i(1)|1)\big] + (1-p)[H - \max\{H,h\}]Q^i(m_b^i(2)|0) \\
&\leq p\big[\max\{H, L(1-q)\} - \max\{L, H(1-q)\}\big] + (1-p)[H - \max\{H,h\}]Q^i(m_b^i(2)|0) \\
&\leq 0
\end{aligned}
$$

This concludes the sequential rationality checks for the strategies. Lastly, I need to verify that the beliefs $\hat{\pi}^i(\mathbf{M}^i, c^i)$ and $\hat{\pi}^o(m^o)$ at all information sets (both on and off equilibrium path) are defined consistently.

I start by showing the consistency of beliefs $\hat{\pi}^o(m^o) = 1$. Given convergent sequences of totally mixed strategies $\{\mathbf{M}^{i,k}\}_{k=1}^{\infty}$, $\{e_{t_s}^k(\mathbf{M}^i)\}_{k=1}^{\infty}$ and $\{m_{t_s}^{o,k}(\mathbf{M}^i)\}_{k=1}^{\infty}$ converging to their respective equilibrium strategy counterparts, the consistent beliefs must equal to the limit of the beliefs generated by these strategy sequences. Consider the limit of the beliefs as defined in (B.2) in the Proof of Lemma II.4:

$$
\begin{aligned}
\hat{\pi}^o(m^o) &= \lim_{k\to\infty} \frac{p\int_{\mathbf{M}^i} \mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e_l^k(\mathbf{M}^i) = o)\mathbb{P}(m_l^{o,k}(\mathbf{M}^i) = m^o)d\mathbf{M}^i}{\sum_{t_s\in\{l,h\}}[\mathbb{P}(t_s)\int_{\mathbf{M}^i} \mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e_{t_s}^k(\mathbf{M}^i) = o)\mathbb{P}(m_{t_s}^{o,k}(\mathbf{M}^i) = m^o)d\mathbf{M}^i]} \\
&= \lim_{k\to\infty} \frac{p\mathbb{P}(\mathbf{M}^{i,k} = \hat{\mathbf{M}}^i)\mathbb{P}(e_l^k(\hat{\mathbf{M}}^i) = o)\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_l}{\sum_{t_s\in\{l,h\}}\Big[\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^{i,k} = \hat{\mathbf{M}}^i)\mathbb{P}(e_{t_s}^k(\hat{\mathbf{M}}^i) = o)\mathbb{P}(m_{t_s}^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_{t_s}\Big]}
\end{aligned}
\tag{B.3}
$$

where $\Delta_{t_s} = \mathbb{P}(t_s)\int_{\mathbf{M}^i\neq\hat{\mathbf{M}}^i} \mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e_{t_s}^k(\mathbf{M}^i) = o)\mathbb{P}(m_{t_s}^{o,k}(\mathbf{M}^i) = m^o)d\mathbf{M}^i$ for each seller type. Note that $\Delta_{t_s} \in (0,1)$ for both t_s as the strategies are totally mixed along the sequences.

Now let the convergent strategy sequences satisfy the following:

$$
\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i) = \begin{cases} 1-\epsilon^{4k} & \text{if } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ \frac{2\epsilon^{4k}}{[H+\lambda_H]^4} & \text{o/w} \end{cases}
$$

$$
\mathbb{P}(e_l^k(\hat{\mathbf{M}}^i) = e) = \begin{cases} 1-\epsilon^{k} & \text{if } e = i \\ \epsilon^{k} & \text{if } e = o \end{cases}
$$

$$
\mathbb{P}(e_h^k(\hat{\mathbf{M}}^i) = e) = \begin{cases} 1-\epsilon^{3k} & \text{if } e = i \\ \epsilon^{3k} & \text{if } e = o \end{cases}
$$

$$
\mathbb{P}(m_{t_s}^{o,k}(\mathbf{M}^i) = m^o) = \begin{cases} 1-\epsilon^{k} & \text{if } m^o = \hat{m}_{t_s}^o \\ \frac{\epsilon^{k}}{[H+\lambda_H]} & \text{o/w} \end{cases} \qquad t_s \in \{l,h\} \text{ and } \forall \mathbf{M}^i
$$

where $\epsilon \in (0,1)$. Note that the space of menus $\mathcal{M}^i$ has Lebesgue measure $|\mathcal{M}^i| = \frac{[H+\lambda_H]^4}{2}$. To see why, observe that for any $\mathbf{M}^i \in \mathcal{M}^i$, the prices satisfy $m_s^i(j) \leq m_b^i(j)$ for at least one

$j \in \{1,2\}$. WLOG letting $m_s^i(2) \leq m_b^i(2)$ yields the stated measure. Then the convergent sequence $\{\mathbf{M}^{i,k}\}_k$ simply uniformly randomizes across the non-equilibrium menus.

Observe that, the sequences for some other strategies (e.g. $\{e_{t_s}^k(\mathbf{M}^i)\}_k$ for $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$) were omitted. This is because, the limit to the belief sequences remain the same, regardless of what convergent sequences are chosen for the omitted strategies. Plugging in the above probabilities for the sequence of strategies, the limit of belief sequences becomes:

$$\begin{aligned}\hat{\pi}^o(m^o) &= \lim_{k\to\infty} \frac{p(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_l}{\begin{aligned}&p(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)\\ &\quad + (1-p)(1-\epsilon^{4k})\epsilon^{3k}\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_l + \Delta_h\end{aligned}}\\ &= \lim_{k\to\infty} \frac{p + \Delta_l'}{p + (1-p)\epsilon^{2k}\frac{\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i)=m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i)=m^o)} + \Delta_l' + \Delta_h'}\end{aligned}$$

where from first to second equality, both numerator and denominator were divided by the same expression $(1-\epsilon^{4k})\epsilon^k\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)$. Hence, the constant terms are equal to $\Delta_{t_s}' = \frac{2\epsilon^{3k}\mathbb{P}(t_s)}{(1-\epsilon^{4k})[H+\lambda_H]^4}\int_{\mathbf{M}^i\neq\hat{\mathbf{M}}^i}\mathbb{P}(e_{t_s}^k(\mathbf{M}^i) = o)\frac{\mathbb{P}(m_{t_s}^{o,k}(\mathbf{M}^i)=m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i)=m^o)}d\mathbf{M}^i$ for each seller type. It is important to note that for both seller types $\Delta_{t_s}' \to 0$ as $k \to \infty$, for any sequence of totally mixed strategies $\{e_{t_s}^k(\mathbf{M}^i)\}_k$ for $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$.

A closer inspection of the middle term in the denominator simplifies the limits, because there are three possible values for the ratios between h and l types of $\mathbb{P}(m_{t_s}^{o,k}(\hat{\mathbf{M}}^i) = m^o)$:

$$\epsilon^{2k}\frac{\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i) = m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)} = \begin{cases}\epsilon^{2k} & \text{if } m^o = \hat{m}_l^o = \hat{m}_h^o\\ \epsilon^{2k} & \text{if } m^o \in [0, H+\lambda_H]\setminus\{\hat{m}_l^o, \hat{m}_h^o\}\\ \epsilon^k(1-\epsilon^k)[H+\lambda_H] & \text{if } m^o = \hat{m}_l^o \neq \hat{m}_h^o\\ \frac{\epsilon^{3k}}{(1-\epsilon^k)[H+\lambda_H]} & \text{if } m^o = \hat{m}_h^o \neq \hat{m}_l^o\end{cases}$$

In any case after cancellations, ϵ^k term remains in the expression $(1-p)\epsilon^{2k}\frac{\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i)=m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i)=m^o)}$, which in turn converges to 0 as k goes to infinity. Along with Δ_{t_s}''s converging to 0 as k becomes larger, the limit satisfies $\hat{\pi}^o(m^o) = \frac{p}{p} = 1$ as desired.

Finally, consider the beliefs $\hat{\pi}^i(\mathbf{M}^i, c^i)$. Similar to before, sequences of totally mixed strategies $\{e_{t_s}^k(\mathbf{M}^i)\}_{k=1}^\infty$ and $\{c_{t_s}^{i,k}(\mathbf{M}^i)\}_{k=1}^\infty$ converging to their equilibrium counterparts. The consistent beliefs must equal to the limit of the beliefs generated by these strategy sequences. Consider the limit of the beliefs as defined in (B.1) in the Proof of Lemma II.4:

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \lim_{k\to\infty}\frac{p\mathbb{P}(e_l^k(\mathbf{M}^i) = i)\mathbb{P}(c_l^{i,k}(\mathbf{M}^i) = c^i)}{p\mathbb{P}(e_l^k(\mathbf{M}^i) = i)\mathbb{P}(c_l^{i,k}(\mathbf{M}^i) = c^i) + (1-p)\mathbb{P}(e_h^k(\mathbf{M}^i) = i)\mathbb{P}(c_h^{i,k}(\mathbf{M}^i) = c^i)}$$

Observe that the probability $\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)$ is eliminated out of the beliefs along the sequence, because they are part of the observed information by the buyer in the subgame $e = i$. This feature has the following implication. Consider an out-of-equilibrium information set $(\mathbf{M}^i, c^i)$ with a deviation menu $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$. If it is the case that at least one seller type enters in the equilibrium strategy and chooses the option c^i, then Bayes' rule can be applied to define the sequence of beliefs. Hence, the consistent beliefs can be summarized as follows:

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} 0 & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}_h^i(\mathbf{M}^i) = c^i \neq \hat{c}_l^i(\mathbf{M}^i) \\ 0 & \text{if } \hat{e}_h(\mathbf{M}^i) = i,\ \hat{e}_l(\mathbf{M}^i) = o,\ \hat{c}_h^i(\mathbf{M}^i) = c^i, \text{ and } \hat{c}_l^i(\mathbf{M}^i) \in \{1,2\} \\ p & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}_l^i(\mathbf{M}^i) = \hat{c}_h^i(\mathbf{M}^i) = c^i \\ 1 & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}_l^i(\mathbf{M}^i) = c^i \neq \hat{c}_h^i(\mathbf{M}^i) \\ 1 & \text{if } \hat{e}_l(\mathbf{M}^i) = i,\ \hat{e}_h(\mathbf{M}^i) = o,\ \hat{c}_l^i(\mathbf{M}^i) = c^i \text{ and } \hat{c}_h^i(\mathbf{M}^i) \in \{1,2\} \end{cases}$$

Then in all the remaining cases, the equilibrium beliefs are equal to $\hat{\pi}^i(\mathbf{M}^i, c^i) = 1$ and all that remains to be checked are the consistency of these beliefs. First consider the following sequences of totally mixed strategies:

$$\mathbb{P}(e_l^k(\mathbf{M}^i) = i) = \begin{cases} 1 - \epsilon^k & \text{if } \hat{e}_l(\mathbf{M}^i) = i \\ \epsilon^k & \text{o/w} \end{cases} \qquad \mathbb{P}(e_h^k(\mathbf{M}^i) = i) = \begin{cases} 1 - \epsilon^{3k} & \text{if } \hat{e}_h(\mathbf{M}^i) = i \\ \epsilon^{3k} & \text{o/w} \end{cases}$$

$$\mathbb{P}(c_l^{i,k}(\mathbf{M}^i) = c^i) = \begin{cases} 1 - \epsilon^k & \text{if } \hat{c}_l^i(\mathbf{M}^i) = c^i \\ \epsilon^k & \text{o/w} \end{cases} \qquad \mathbb{P}(c_h^{i,k}(\mathbf{M}^i) = c^i) = \begin{cases} 1 - \epsilon^{3k} & \text{if } \hat{c}_h^i(\mathbf{M}^i) = c^i \\ \epsilon^{3k} & \text{o/w} \end{cases}$$

where $\epsilon \in (0, 1)$. Now the beliefs in the remaining cases are as follows. Given an out-of-equilibrium information set $(\mathbf{M}^i, c^i)$, if:

1. $\hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i$ and $\hat{c}_l^i(\mathbf{M}^i) = \hat{c}_h^i(\mathbf{M}^i) \neq c^i$;

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \lim_{k \to \infty} \frac{p(1-\epsilon^k)\epsilon^k}{p(1-\epsilon^k)\epsilon^k + (1-p)(1-\epsilon^{3k})\epsilon^{3k}} = 1$$

2. $\hat{e}_l(\mathbf{M}^i) = o$, $\hat{e}_h(\mathbf{M}^i) = i$, $\hat{c}_l^i(\mathbf{M}^i) \in \{1, 2\}$ and $\hat{c}_h^i(\mathbf{M}^i) \neq c^i$;

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} \lim_{k \to \infty} \frac{p\epsilon^k(1-\epsilon^k)}{p\epsilon^k(1-\epsilon^k) + (1-p)(1-\epsilon^{3k})\epsilon^{3k}} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) = c^i \\ \lim_{k \to \infty} \frac{p\epsilon^k\epsilon^k}{p\epsilon^k\epsilon^k + (1-p)(1-\epsilon^{3k})\epsilon^{3k}} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) \neq c^i \end{cases}$$

3. $\hat{e}_l(\mathbf{M}^i) = i$, $\hat{e}_h(\mathbf{M}^i) = o$, $\hat{c}_l^i(\mathbf{M}^i) \neq c^i$ and $\hat{c}_h^i(\mathbf{M}^i) \in \{1, 2\}$;

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} \lim_{k \to \infty} \frac{p(1-\epsilon^k)\epsilon^k}{p(1-\epsilon^k)\epsilon^k + (1-p)\epsilon^{3k}(1-\epsilon^{3k})} = 1 & \text{if } \hat{c}_h^i(\mathbf{M}^i) = c^i \\ \lim_{k \to \infty} \frac{p(1-\epsilon^k)\epsilon^k}{p(1-\epsilon^k)\epsilon^k + (1-p)\epsilon^{3k}\epsilon^{3k}} = 1 & \text{if } \hat{c}_h^i(\mathbf{M}^i) \neq c^i \end{cases}$$

4. $\hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = o$, $\hat{c}_l^i(\mathbf{M}^i) \in \{1,2\}$ and $\hat{c}_h^i(\mathbf{M}^i) \in \{1,2\}$;

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} \lim\limits_{k\to\infty} \frac{p\epsilon^k(1-\epsilon^k)}{p\epsilon^k(1-\epsilon^k)+(1-p)\epsilon^{3k}(1-\epsilon^{3k})} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) = \hat{c}_h^i(\mathbf{M}^i) = c^i \\ \lim\limits_{k\to\infty} \frac{p\epsilon^k(1-\epsilon^k)}{p\epsilon^k(1-\epsilon^k)+(1-p)\epsilon^{3k}\epsilon^{3k}} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) = c^i \neq \hat{c}_h^i(\mathbf{M}^i) \\ \lim\limits_{k\to\infty} \frac{p\epsilon^k\epsilon^k}{p\epsilon^k\epsilon^k+(1-p)\epsilon^{3k}(1-\epsilon^{3k})} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) \neq \hat{c}_h^i(\mathbf{M}^i) = c^i \\ \lim\limits_{k\to\infty} \frac{p\epsilon^k\epsilon^k}{p\epsilon^k\epsilon^k+(1-p)\epsilon^{3k}\epsilon^{3k}} = 1 & \text{if } \hat{c}_l^i(\mathbf{M}^i) = \hat{c}_h^i(\mathbf{M}^i) \neq c^i \end{cases}$$

This concludes the consistency check of the beliefs and the proof altogether. □

Proof of Proposition II.5. For the necessity part, consider the inequalities from Lemma II.5 again, noting that $\hat{U}^o_{t_s} = [\mathbb{E}(v|H) - t_s](1-q)$;

$$\begin{aligned} \hat{U}_l &\geq \hat{U}_h + h(1-q) \geq \hat{U}^o_h + h(1-q) = \hat{U}^o_l = \mathbb{E}(v|H)(1-q) \\ \hat{U}_h &\geq \hat{U}^o_h = [\mathbb{E}(v|H) - h](1-q) \end{aligned}$$

Plugging in these to condition (II.5) from Proposition II.4 and rearranging terms yields the following condition:

$$\begin{aligned} pL + (1-p)(1-q)(H + \lambda_H - h) &\geq p\hat{U}_l + (1-p)\hat{U}_h \geq \mathbb{E}(v|H)(1-q) - (1-p)(1-q)h \\ \Rightarrow\ p\tfrac{L}{1-q} + (1-p)(H+\lambda_H) &\geq \mathbb{E}(v|H) = pH + (1-p)(H+\lambda_H) \\ \tfrac{L}{1-q} &\geq H \end{aligned}$$

The other part, i.e. $h > \frac{L}{1-q}$, is implied by Assumption II.2, because both $H \leq \frac{L}{1-q}$ and $\frac{L}{1-q} < \max\{H, h\}$ can be simultaneously satisfied only when $h > \frac{L}{1-q} \geq H$, which is precisely the condition (II.7).

For the sufficiency part, it is enough to show the existence of an equilibrium whenever

(II.7) is satisfied. Consider the following assessment:

$$\hat{\mathbf{M}}^i = \{(\mathbb{E}(v|H)(1-q), L), (\mathbb{E}(v|H), H + \lambda_H)\}$$

$$\hat{e}_{t_s}(\mathbf{M}^i) = \begin{cases} i & \text{if } \mathbf{M}^i = \hat{\mathbf{M}}^i \text{ OR } U^i_{t_s}(\mathbf{M}^i, \hat{c}^i_{t_s}(\mathbf{M}^i)) \geq \hat{U}^o_{t_s} \text{ and } m^i_b(\hat{c}^i_{t_s}(\mathbf{M}^i) \leq H \\ o & \text{o/w} \end{cases}$$

$$\hat{c}^i_{t_s}(\mathbf{M}^i) = \begin{cases} 1 & \text{if } t_s = l \text{ and } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ 2 & \text{if } t_s = h \text{ and } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ 1 & \text{if } U^i_{t_s}(\mathbf{M}^i, 1) > U^i_{t_s}(\mathbf{M}^i, 2) \\ 2 & \text{if } U^i_{t_s}(\mathbf{M}^i, 2) > U^i_{t_s}(\mathbf{M}^i, 1) \\ \{1, 2\} & \text{o/w} \end{cases}$$

$$\hat{m}^o_{t_s}(\mathbf{M}^i) = \mathbb{E}(v|H)$$

$$\hat{d}^i_{t_b}(\mathbf{M}^i, c^i) = \begin{cases} A & \text{if } m^i_b(c^i) \leq t_b + (1 - \hat{\pi}^i(\mathbf{M}^i, c^i))\lambda_{t_b} \\ R & \text{o/w} \end{cases}$$

$$\hat{d}^o_{t_b}(m^o) = \begin{cases} A & \text{if } m^o \leq t_b + (1 - \hat{\pi}^o(m^o))\lambda_{t_b} \\ R & \text{o/w} \end{cases}$$

$$\hat{\pi}^i(\mathbf{M}^i, c^i) = \begin{cases} 0 & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}^i_h(\mathbf{M}^i) = c^i \neq \hat{c}^i_l(\mathbf{M}^i) \\ 0 & \text{if } \hat{e}_h(\mathbf{M}^i) = i,\ \hat{e}_l(\mathbf{M}^i) = o,\ \hat{c}^i_h(\mathbf{M}^i) = c^i, \text{ and } \hat{c}^i_l(\mathbf{M}^i) \in \{1, 2\} \\ p & \text{if } \hat{e}_l(\mathbf{M}^i) = \hat{e}_h(\mathbf{M}^i) = i,\ \hat{c}^i_l(\mathbf{M}^i) = \hat{c}^i_h(\mathbf{M}^i) = c^i \\ 1 & \text{o/w} \end{cases}$$

$$\hat{\pi}^o(m^o) = \begin{cases} p & \text{if } m^o = \hat{m}^o_l = \hat{m}^o_h \\ 1 & \text{o/w} \end{cases}$$

Observe that following the equilibrium menu announcement $\hat{\mathbf{M}}^i$, both seller types choose the intermediary and separate contracts; low type chooses 1 and high type chooses 2. Thus, by Bayes' rule, the beliefs are $\hat{\pi}^i(\hat{\mathbf{M}}^i, 1) = 1$ and $\hat{\pi}^i(\hat{\mathbf{M}}^i, 2) = 0$, which suggests that the desired trade probabilities are implemented. More precisely, $\hat{d}^i_L(\hat{\mathbf{M}}^i, 1) = \hat{d}^i_H(\hat{\mathbf{M}}^i, 1) = A$ as $\hat{m}^i_b(1) = L \leq t_b$ for both types, while $\hat{d}^i_L(\hat{\mathbf{M}}^i, 2) = R$ and $\hat{d}^i_H(\hat{\mathbf{M}}^i, 2) = A$ as $L + \lambda_L < \hat{m}^i_b(2) = H + \lambda_H \leq H + \lambda_H$.

Next, in order to verify the optimality of menu option choices observe that for low type $U^i_l(\hat{\mathbf{M}}^i, 1) = \mathbb{E}(v|H)(1-q) \geq U^i_l(\hat{\mathbf{M}}^i, 2) = \mathbb{E}(v|H)(1-q)$ and for high type $U^i_h(\hat{\mathbf{M}}^i, 2) = [\mathbb{E}(v|H) - h](1-q) > U^i_h(\hat{\mathbf{M}}^i, 1) = \mathbb{E}(v|H)(1-q) - h$.

To verify optimality of the entry choice, first consider the expected payoffs from subgame $e = o$. In the subgame, given buyer's equilibrium beliefs, if the seller announces $\hat{m}^o$, then the buyer types would optimally decide $\hat{d}^o_L(\hat{m}^o) = R$ and $\hat{d}^o_H(\hat{m}^o) = A$ as $H + (1-p)\lambda_H = \mathbb{E}(v|H) > L + (1-p)\lambda_L$. By deviating to any other price $m^o \neq \hat{m}^o$, the probability of trade, i.e. $Q^o(m^o)$, decreases as $\hat{\pi}^o(m^o) = 1$. Furthermore, in the case where $m^o > \hat{m}^o$, it holds that $Q^o(m^o) = 0$. Altogether, indeed both seller types would announce $\hat{m}^o$ in the direct trade outside option subgame, which yields expected payoffs $\hat{U}^o_{t_s} = U^o_{t_s}(\hat{m}^o) = [\mathbb{E}(v|H) - t_s](1-q)$. Note that for both types $\hat{m}^o = \mathbb{E}(v|H)$ is optimal for all possible menu announcements $\mathbf{M}^i \in \mathcal{M}^i$, which obviously includes the equilibrium menu $\hat{\mathbf{M}}^i$. In light of these, if follows that both seller types are indifferent between intermediary and the outside option for entry choice. Hence $\hat{e}_{t_s}(\hat{\mathbf{M}}^i) = i$ for both types are indeed optimal.

The rest of the proof for sequential rationality is analogous to the corresponding section from the sufficiency proof for Theorem II.1. Namely, for any deviation menu announcement $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$, there exists a continuation equilibrium in the aforementioned strategies and a deviation mechanism is not profitable as it yields $V(\mathbf{M}^i) = 0 \leq V(\hat{\mathbf{M}}^i)$. Thus, all the described strategies satisfy sequentially rationality.

In terms of verifying the consistency of the beliefs, the same sequences of convergent strategies from the previous proof work for the beliefs $\hat{\pi}^i(\mathbf{M}^i, c^i)$ and hence are omitted. Hence, only the consistency of the out-of-equilibrium beliefs from subgame $e = o$ need to be verified. First, consider the following limit of the beliefs as simplified in (B.3) from the proof of Theorem II.1, where the beliefs are generated from the convergent sequences of totally mixed strategies $\{\mathbf{M}^{i,k}\}_{k=1}^{\infty}$, $\{e^k_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ and $\{m^{o,k}_{t_s}(\mathbf{M}^i)\}_{k=1}^{\infty}$ converging to their respective equilibrium strategy counterparts:

$$\hat{\pi}^o(m^o) = \lim_{k\to\infty} \frac{p\mathbb{P}(\mathbf{M}^{i,k} = \hat{\mathbf{M}}^i)\mathbb{P}(e^k_l(\hat{\mathbf{M}}^i) = o)\mathbb{P}(m^{o,k}_l(\hat{\mathbf{M}}^i) = m^o) + \Delta_l}{\sum_{t_s\in\{l,h\}}[\mathbb{P}(t_s)\mathbb{P}(\mathbf{M}^{i,k} = \hat{\mathbf{M}}^i)\mathbb{P}(e^k_{t_s}(\hat{\mathbf{M}}^i) = o)\mathbb{P}(m^{o,k}_{t_s}(\hat{\mathbf{M}}^i) = m^o) + \Delta_{t_s}]}$$

where $\Delta_{t_s} = \mathbb{P}(t_s)\int_{\mathbf{M}^i\neq\hat{\mathbf{M}}^i} \mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i)\mathbb{P}(e^k_{t_s}(\mathbf{M}^i) = o)\mathbb{P}(m^{o,k}_{t_s}(\mathbf{M}^i) = m^o)d\mathbf{M}^i$ for each seller type. Note that $\Delta_{t_s} \in (0,1)$ for both t_s as the strategies are totally mixed along the sequences.

Now let the convergent strategy sequences satisfy the following:

$$\mathbb{P}(\mathbf{M}^{i,k} = \mathbf{M}^i) = \begin{cases} 1-\epsilon^{4k} & \text{if } \mathbf{M}^i = \hat{\mathbf{M}}^i \\ \frac{2\epsilon^{4k}}{[H+\lambda_H]^4} & \text{o/w} \end{cases}$$

$$\mathbb{P}(e^k_{t_s}(\hat{\mathbf{M}}^i) = e) = \begin{cases} 1-\epsilon^k & \text{if } e = i \\ \epsilon^k & \text{if } e = o \end{cases} \qquad t_s \in \{l,h\}$$

$$\mathbb{P}(m^{o,k}_l(\hat{\mathbf{M}}^i) = m^o) = \begin{cases} 1-\epsilon^k & \text{if } m^o = \hat{m}^o \\ \frac{\epsilon^k}{[H+\lambda_H]} & \text{o/w} \end{cases}$$

$$\mathbb{P}(m^{o,k}_h(\hat{\mathbf{M}}^i) = m^o) = \begin{cases} 1-\epsilon^{2k} & \text{if } m^o = \hat{m}^o \\ \frac{\epsilon^{2k}}{[H+\lambda_H]} & \text{o/w} \end{cases}$$

where $\epsilon \in (0,1)$. Note that the space of menus $\mathcal{M}^i$ has Lebesgue measure $|\mathcal{M}^i| = \frac{[H+\lambda_H]^4}{2}$. To see why, observe that for any $\mathbf{M}^i \in \mathcal{M}^i$, the prices satisfy $m_s^i(j) \leq m_b^i(j)$ for at least one $j \in \{1,2\}$. WLOG letting $m_s^i(2) \leq m_b^i(2)$, yields the stated measure. Then the convergent sequence $\{\mathbf{M}^{i,k}\}_k$ simply uniformly randomizes across the non-equilibrium menus.

Observe that, the sequences for some other strategies (e.g. $\{e_{t_s}^k(\mathbf{M}^i)\}_k$ for $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$) were omitted. This is because, the limit to the belief sequences remain the same, regardless of what convergent sequences are chosen for the omitted strategies. Plugging in the above probabilities for the sequence of strategies, the limit of belief sequences becomes:

$$\begin{aligned}\hat{\pi}^o(m^o) &= \lim_{k\to\infty} \frac{p(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_l}{\begin{array}{l} p(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o) \\ \qquad + (1-p)(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i) = m^o) + \Delta_l + \Delta_h \end{array}} \\ &= \lim_{k\to\infty} \frac{p + \Delta_l'}{p + (1-p)\frac{\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i) = m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)} + \Delta_l' + \Delta_h'}\end{aligned}$$

where from first to second equality, both numerator and denominator were divided by the same expression $(1-\epsilon^{4k})\epsilon^k \mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)$. Hence the constant terms are equal to $\Delta_{t_s}' = \frac{2\epsilon^{3k}\mathbb{P}(t_s)}{(1-\epsilon^{4k})[H+\lambda_H]^4} \int_{\mathbf{M}^i \neq \hat{\mathbf{M}}^i} \mathbb{P}(e_{t_s}^k(\mathbf{M}^i) = o) \frac{\mathbb{P}(m_{t_s}^{o,k}(\mathbf{M}^i) = m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)} d\mathbf{M}^i$ for each seller type. It is important to note that, for both seller types $\Delta_{t_s}' \to 0$ as $k \to \infty$, no matter what probabilities are assigned to $\{e_{t_s}^k(\mathbf{M}^i)\}_k$ and $\{m_{t_s}^{o,k}(\mathbf{M}^i)\}_k$ for $\mathbf{M}^i \neq \hat{\mathbf{M}}^i$.

A closer inspection of the middle term in the denominator simplifies the limits, because there are two possible values for the ratios between h and l types of $\mathbb{P}(m_{t_s}^{o,k}(\hat{\mathbf{M}}^i) = m^o)$:

$$\frac{\mathbb{P}(m_h^{o,k}(\hat{\mathbf{M}}^i) = m^o)}{\mathbb{P}(m_l^{o,k}(\hat{\mathbf{M}}^i) = m^o)} = \begin{cases} \frac{1-\epsilon^{2k}}{1-\epsilon^k} & \text{if } m^o = \hat{m}^o \\ \epsilon^k & \text{if } m^o \neq \hat{m}^o \end{cases}$$

Plugging in the above ratios shows that the beliefs at subgame $e = o$ satisfy consistency:

$$\hat{\pi}^o(m^o) = \begin{cases} \lim_{k\to\infty} \frac{p+\Delta_l'}{p+(1-p)+\Delta_l'+\Delta_h'} = p & \text{if } m^o = \hat{m}^o \\ \lim_{k\to\infty} \frac{p+\Delta_l'}{p+(1-p)\epsilon^k+\Delta_l'+\Delta_h'} = 1 & \text{if } m^o \neq \hat{m}^o \end{cases}$$

□

Proof of Lemma II.7. The thresholds q^P and q^{S^e} are both equal to $q^P = q^{S^e} = \frac{H-L}{H}$ as $\mathbb{E}(v|H) = H$ and $H > h$. Furthermore, whenever condition (II.1) in Proposition II.1 is satisfied, the pooling equilibrium equilibrium exists where the price satisfies $\max\{\frac{L}{1-q}, h, H\} = H \leq m^P \leq \mathbb{E}(v|H) = H$. Thus a pooling equilibrium can exist only at price $m^P = H$ whenever $q \leq q^P$. Similarly, whenever condition (II.2) in Proposition II.2 is satisfied, the efficient separating equilibrium exists with prices $m_l^{S^e} = L$ and $\max\{h, H\} = H \leq m_h^{S^e} =$

$H \leq \min\{\frac{L}{1-q}, H\} = H$, where the last equality is due to $q \geq q^{S^e}$. Thus, in an efficient separating equilibrium, unique price for high type seller is $\hat{m}_h^{S^e} = H$. Finally, there are no longer inefficient separating equilibria as $h < H$, which violates the necessary condition for their existence provided in Proposition II.3. □

Proof of Proposition II.6. Firstly, observe that in the IPV case, under Assumption II.2, the seller's expected payoff from subgame $e = o$ are equal to $\hat{U}^o_{t_s} = [H - t_s](1-q)$. To see why, remember that in the IPV case the parameters satisfy $l = 0 < L < h < H$. Furthermore, in the subgame $e = o$, the buyer accepts a TIOLI offer $m^o \leq H$ for any beliefs she might have. Then, when $q < q^{S^e} = \max\{\frac{H-L}{H}, \frac{h-L}{h}\} = \frac{H-L}{H}$, it is optimal for both seller types to announce $\hat{m}^o_{t_s} = H$ no matter what beliefs $\hat{pi}^o(m^o)$ the buyer has.

By Lemma II.5, it holds that $\hat{U}_{t_s} \geq \hat{U}^o_{t_s} = [H - t_s](1 - q)$. On the other hand, Proposition II.4 suggests that an efficient intermediated equilibrium of Γ^i exists, only when condition (II.5) holds. Plugging in the inequality into condition (II.5) yields:

$$
\begin{aligned}
pL + (1-p)(1-q)(H-h) \geq p\hat{U}_l + (1-p)\hat{U}_h &\geq pH(1-q) + (1-p)[H-h](1-q) \\
p\frac{L}{1-q} + (1-p)[H-h] &\geq H - h(1-p) \\
\frac{L}{1-q} &\geq H
\end{aligned}
$$

However Assumption II.2 implies that the necessary condition can not be satisfied, because $H > \frac{L}{1-q}$. Hence there can not exist an efficient intermediated equilibrium in the IPV case. □

APPENDIX C

Appendix to Chapter III

C.1 Proofs

Proof of Proposition III.1. The seller's ex-ante optimal mechanism is characterized by the solution to the following optimization program:

$$P1 \rightarrow \qquad \max_{\{Q,\tau\}} \left\{ \int_0^1 U_s(\theta_s) dF(\theta_s) \right\}$$

subject to

$$\begin{aligned}
\text{IR}_s: &\quad U_s(\theta_s) \geq 0 &&\forall \theta_s \\
\text{IR}_b: &\quad U_b(\theta_b) \geq 0 &&\forall \theta_b \\
\text{IC}_s: &\quad U_s(\theta_s) \geq U_s(\tilde{\theta}_s|\theta_s) &&\forall \tilde{\theta}_s, \theta_s \\
\text{IC}_b: &\quad U_b(\theta_b) \geq U_b(\tilde{\theta}_b|\theta_b) &&\forall \tilde{\theta}_b, \theta_b
\end{aligned}$$

where the constraints follow from the feasibility of the mechanism.

The ex-ante expected payoff for the seller is equal to the ex-ante expected gains from trade minus the ex-ante expected payoff of the buyer, as the object is not destroyed. Plugging in the expected payoffs from Remark III.1 and rearranging yields:

$$\begin{aligned}
\int_0^1 U_s(\theta_s) dF(\theta_s) &= U_s(1) + \int_0^1 \int_{\theta_s}^1 q_s(x) dx dF(\theta_s) \\
&= \int_0^1 \int_0^1 Q(\theta_s, \theta_b)[v(\theta_s, \theta_b) - c(\theta_s, \theta_b)] dF(\theta_s) dG(\theta_b) - \int_0^1 U_b(\theta_b) dG(\theta_b) \\
&= (1-\alpha) \int_0^1 \int_0^1 Q(\theta_s, \theta_b)[\psi_b(\theta_b) - \theta_s] dF(\theta_s) dG(\theta_b) - U_b(0) \qquad \text{(C.1)}
\end{aligned}$$

Observe that, the objective function in maximization problem $P1$ is equivalently the last equation above. From the necessary conditions required by feasibility of γ, the constants $U_s(1)$ and $U_b(0)$ have to be nonnegative. It is clear from the equivalent objective function that, at the optimal mechanism, it has to be $U_b^*(0) = 0$, as otherwise decreasing $U_b(0)$ would increase the value of the solution. Rearranging above equalities also provides a description of $U_s(1)$ only in terms of the allocation rule. In light of these observations, the maximization problem can be restated as follows:

$$P1' \rightarrow \qquad \max_{Q(\theta_s,\theta_b)} \left\{ (1-\alpha)\int_0^1\int_0^1 Q(\theta_s,\theta_b)[\psi_b(\theta_b)-\theta_s]dF(\theta_s)dG(\theta_b) \right\}$$

subject to
$$\frac{dq_s(\theta_s)}{d\theta_s} \leq 0 \qquad \forall\theta_s$$
$$\frac{dq_b(\theta_b)}{d\theta_b} \geq 0 \qquad \forall\theta_b$$
$$U_s(1) = (1-\alpha)\int_0^1\int_0^1 Q(\theta_s,\theta_b)[\psi_b(\theta_b)-\psi_s(\theta_s)]dG(\theta_b)dF(\theta_s) \geq 0$$

where the first two inequalities are the necessary monotonicity of the allocation rule required by the IC constraints, while last inequality amounts to the seller's simplified IR constraint.

Ignoring the monotonicity and the seller's IR constraints momentarilty, consider the pointwise maximizer of the linear objective function which is equal to:

$$\hat{Q}(\theta_s,\theta_b) = \begin{cases} 1 & \text{if } \psi_b(\theta_b) \geq \theta_s \;\Leftrightarrow\; \theta_b \geq \hat{\kappa}(\theta_s) = \psi_b^{-1}(\theta_s) \\ 0 & \text{o/w} \end{cases}$$

If the neglected constraints are also satisfied, then this allocation rule defines the solution. It is easy to see that the monotonicity constraints are satisfied, as $\hat{\kappa}(\theta_s) = \psi_b^{-1}(\theta_s)$ is strictly increasing in θ_s. To see why, observe that $q_s(\theta_s) = 1 - G(\psi_b^{-1}(\theta_s))$ which is strictly decreasing in θ_s and $q_b(\theta_b) = F(\psi_b(\theta_b))$ which is strictly increasing in θ_b. Hence only the seller's IR constraint needs to be checked. Evaluate the expected payoff $U_s(1)$ when the allocation rule from above is plugged in:

$$
\begin{aligned}
U_s(1) &= (1-\alpha)\int_0^1\int_0^1 \hat{Q}(\theta_s,\theta_b)[\psi_b(\theta_b)-\psi_s(\theta_s)]dG(\theta_b)dF(\theta_s) \\
&= (1-\alpha)\int_0^1\int_{\hat{\kappa}(\theta_s)}^1 [\psi_b(\theta_b)-\psi_s(\theta_s)]dG(\theta_b)dF(\theta_s) \\
&= (1-\alpha)\int_0^1 [\hat{\kappa}(\theta_s)-\psi_s(\theta_s)][1-G(\hat{\kappa}(\theta_s))]dF(\theta_s) \\
&= \int_0^1 \underbrace{\Big[(1-\alpha)[\hat{\kappa}(\theta_s)-\theta_s)][1-G(\hat{\kappa}(\theta_s))] - \int_{\theta_s}^1 [1-G(\hat{\kappa}(x))]dx\Big]}_{I(\theta_s)} dF(\theta_s)
\end{aligned}
$$

where going from penultimate to the last line, the following substitutions are made:

$$
\begin{aligned}
(1-\alpha)\int_0^1 \tfrac{F(\theta_s)}{(1-\alpha)f(\theta_s)}[1-G(\kappa(\theta_s))]dF(\theta_s) &= \int_0^1 F(\theta_s)[1-G(\kappa(\theta_s))]d\theta_s \\
&= \int_0^1\int_{\theta_s}^1 [1-G(\kappa(x))]dxdF(\theta_s)
\end{aligned}
$$

Observe that using integration by parts, the last line can be rewritten as follows:

$$
\begin{aligned}
U_s(1) = \int_0^1 I(\theta_s)dF(\theta_s) = \big[I(\theta_s)F(\theta_s)\big]\Big|_{\theta_s=0}^1 &- \int_0^1 I'(\theta_s)F(\theta_s)d\theta_s \\
&= 0 - \int_0^1 I'(\theta_s)F(\theta_s)d\theta_s \quad \text{(C.2)}
\end{aligned}
$$

where $I'(\theta_s) = \frac{dI(\theta_s)}{d\theta_s}$. To see why the term in brackets equals 0, observe that $I(1) = F(0) = 0$ where the former is due to $\hat{\kappa}(1) = \psi_b^{-1}(1) = 1$. Therefore, it follows that $I(1) = (1-\alpha)[1-1][1-1] - \int_1^1 [1-G(\hat{\kappa}(x))]dx = 0$. Next, evaluate the sign of $I'(\theta_s)$:

$$
\begin{aligned}
\frac{dI(\theta_s)}{d\theta_s} &= \frac{d}{d\theta_s}\Big[(1-\alpha)[\hat{\kappa}(\theta_s)-\theta_s)][1-G(\hat{\kappa}(\theta_s))] - \int_{\theta_s}^1 [1-G(\hat{\kappa}(x))]dx\Big] \\
&= (1-\alpha)\Big[\big[\hat{\kappa}'(\theta_s)-1\big]\big[1-G(\hat{\kappa}(\theta_s))\big] - \big[\hat{\kappa}(\theta_s)-\theta_s\big]g(\hat{\kappa}(\theta_s))\hat{\kappa}'(\theta_s)\Big] \\
&\qquad + \big[1-G(\hat{\kappa}(\theta_s))\big] \\
&= -(1-\alpha)g(\hat{\kappa}(\theta_s)\hat{\kappa}'(\theta_s)\underbrace{\Big[\hat{\kappa}(\theta_s) - \frac{1-G(\hat{\kappa}(\theta_s))}{g(\hat{\kappa}(\theta_s))} - \theta_s\Big]}_{=\psi_b(\hat{\kappa}(\theta_s)-\theta_s=0} + \alpha\big[1-G(\hat{\kappa}(\theta_s))\big] \\
&= \alpha\big[1-G(\hat{\kappa}(\theta_s))\big]
\end{aligned}
$$

where $\hat{\kappa}'(\theta_s)$ denotes the derivative with respect to θ_s. The term inside the brackets equals $\psi_b(\hat{\kappa}(\theta_s)) - \theta_s$, which in turn equals 0. To see why, note that $\hat{\kappa}(\theta_s) = \psi_b^{-1}(\theta_s)$, thus

$\psi_b(\hat{\kappa}(\theta_s)) = \psi_b(\psi_b^{-1}(\theta_s)) = \theta_s$. Plugging the derivative of $I(\theta_s)$ into (C.2) yields:

$$U_s(1) = -\int_0^1 I'(\theta_s)F(\theta_s)d\theta_s = -\alpha\int_0^1 \big[1 - G(\hat{\kappa}(\theta_s))\big]F(\theta_s)d\theta_s$$

It is easy to see that, whenever $\alpha = 0$, the expected payoff constant satisfies $U_s(1) = 0$. Therefore the allocation $\hat{Q}(\theta_s, \theta_b)$ along with the cutoff function $\hat{\kappa}(\theta_s)$ solves the relaxed program $P1'$ and consequently the original program $P1$. Observe that the cutoff function $\hat{\kappa}(\theta_s)$ is equal to $\kappa^*(\theta_s) = \min\{\psi_b^{-1}(\psi_s(\theta_s;\mu)), 1\} = \psi_b^{-1}(\theta_s)$ for $\mu = 0$, satisfying the description in Proposition III.1. Along with $U_s^*(1) = 0$, the characterization for the IPV case is attained, i.e. $\alpha = 0$.

When $\alpha \in (0, 1)$ and there is interdependence, the allocation rule $\hat{Q}$ along with cutoff function $\hat{\kappa}$ do not satisfy the IR constraint for the seller as $U_s(1) < 0$. This means that in any solution to $P1'$ in the case of interdependence, the seller's IR constraint optimally binds. Letting λ denote the Lagrange multiplier for the seller's IR constraint, consider the Lagrangian for the relaxed program after rearranging as follows:

$$\begin{aligned} L(Q;\lambda) &= (1-\alpha)(1+\lambda)\int_0^1\int_0^1 Q(\theta_s,\theta_b)\Big[\psi_b(\theta_b) - \theta_s - \frac{\lambda}{1+\lambda}\frac{F(\theta_s)}{(1-\alpha)f(\theta_s)}\Big]dG(\theta_b)dF(\theta_s) \\ &= (1-\alpha)(1+\lambda)\int_0^1\int_0^1 Q(\theta_s,\theta_b)\Big[\psi_b(\theta_b) - \psi_s(\theta_s;\mu)\Big]dG(\theta_b)dF(\theta_s) \end{aligned}$$

where $\mu = \frac{\lambda}{1+\lambda}$. Observe that as the constraint optimally binds whenever $\alpha \in (0, 1)$, the multiplier λ will be strictly positive. In turn, this implies that $\mu \in (0, 1)$. Now consider the pointwise maximizer for the Lagrangian:

$$Q^*(\theta_s,\theta_b) = \begin{cases} 1 & \text{if } \psi_b(\theta_b) \geq \psi_s(\theta_s;\mu) \\ 0 & \text{o/w} \end{cases}$$

However, observe that when $\mu > 0$, for θ_s large enough, it holds that $\psi_s(\theta_s;\mu) > 1$. Therefore, employing the $\min$ operator yields the following accurate description of the cutoff function; $\kappa^*(\theta_s) = \min\{\psi_b^{-1}(\psi_s(\theta_s;\mu)), 1\}$. Note that, it is exactly the cutoff function described in (III.1). Furthermore, $U_s^*(1) = 0$ by the binding IR constraint. Hence, the characterization as described in Proposition III.1 is attained.

Finally, consider the following transfer rule from (III.2) as shown below:

$$\begin{aligned} \tau^*(\theta_s,\theta_b) = {} & Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b] - (1-\alpha)\int_0^{\theta_b} Q^*(\theta_s, y)dy \\ & + \int_{\theta_s}^1\int_0^1 Q^*(x,\theta_b)dG(\theta_b)dx - (1-\alpha)\int_0^1 Q^*(\theta_s,\theta_b)[\psi_b(\theta_b) - \theta_s]dG(\theta_b) \end{aligned}$$

The aim is to show that $\gamma^* = \{Q^*, \tau^*\}$ is an ex-ante optimal mechanism for the seller. This amounts to showing that γ^* in truthful type-telling equilibrium implements the outcomes described previously. Clearly, the allocation function is precisely the optimal allocation rule that was characterized. Hence, it only remains to be checked that the transfer rule yields the expected payoffs under the truthful type-telling equilibrium strategies.

Starting with the seller, consider the expected transfer $t_s^*(\theta_s)$:

$$\begin{aligned}
t_s^*(\theta_s) &= \int_0^1 \tau^*(\theta_s,\theta_b)dG(\theta_b) \\
&= \int_0^1 Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b]dG(\theta_b) - (1-\alpha)\int_0^1\int_0^{\theta_b} Q^*(\theta_s,y)dydG(\theta_b) \\
&\quad + \int_0^1\int_{\theta_s}^1\int_0^1 Q^*(x,\theta_b)dG(\theta_b)dxdG(\theta_b) \\
&\quad - (1-\alpha)\int_0^1\int_0^1 Q^*(\theta_s,\theta_b)[\psi_b(\theta_b)-\theta_s]dG(\theta_b)dG(\theta_b) \\
&= \theta_s\int_0^1 Q^*(\theta_s,\theta_b)dG(\theta_b) + \int_{\theta_s}^1\int_0^1 Q^*(x,\theta_b)dG(\theta_b)dx
\end{aligned}$$

These expected transfers yield $U_s^*(\theta_s) = t_s^*(\theta_s) - q_s^*(\theta_s)\theta_s = \int_{\theta_s}^1\int_0^1 Q^*(\theta_s,\theta_b)dG(\theta_b)dx$. It is easy to see that $U_s^*(1)=0$ and $U_s^*(\theta_s)$ is indeed the characterized payoff schedule for the seller.

Lastly, consider the buyer's expected transfers:

$$\begin{aligned}
t_b^*(\theta_b) &= \int_0^1 \tau^*(\theta_s,\theta_b)dF(\theta_s) \\
&= \int_0^1 Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b]dF(\theta_s) - (1-\alpha)\int_0^1\int_0^{\theta_b} Q^*(\theta_s,y)dydF(\theta_s) \\
&\quad + \int_0^1\int_{\theta_s}^1\int_0^1 Q^*(x,\theta_b)dG(\theta_b)dxdF(\theta_s) \\
&\quad - (1-\alpha)\int_0^1\int_0^1 Q^*(\theta_s,\theta_b)[\psi_b(\theta_b)-\theta_s]dG(\theta_b)dF(\theta_s) \\
&= \int_0^1 Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b]dF(\theta_s) - (1-\alpha)\int_0^1\int_0^{\theta_b} Q^*(\theta_s,y)dydF(\theta_s) \\
&\quad + \underbrace{\int_0^1\int_{\theta_s}^1 q_s^*(x)dxF(\theta_s) - (1-\alpha)\int_0^1\int_0^1 Q^*(\theta_s,\theta_b)[\psi_b(\theta_b)-\theta_s]dG(\theta_b)dF(\theta_s)}_{=0} \\
&= \int_0^1 Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b]dF(\theta_s) - (1-\alpha)\int_0^{\theta_b} q_b^*(y)dy
\end{aligned}$$

where the whole expression with the underbrace equals 0, as was shown in (C.1). Then evaluating the expected payoffs yields $U_b^*(\theta_b) = \int_0^1 Q^*(\theta_s,\theta_b)[\alpha\theta_s + (1-\alpha)\theta_b]dF(\theta_s) - t_b^*(\theta_b) = (1-\alpha)\int_0^{\theta_b} q_b^*(y)dy$. Again, it is easy to see that $U_b^*(0)=0$ and $U_b^*(\theta_b)$ is indeed the characterized payoff schedule for the buyer. □

Proof of Proposition III.2. A mechanism γ is ex-ante incentive efficient mechanism if it solves maximization of $W(\gamma)$ subject to IC and IR constraints. Observe that when $w = 1$, the objective function equals $W(\gamma) = \int_0^1 U_s(\theta_s)dF(\theta_s)$. Thus the problem is equivalent to the optimization program $P1$ presented in the proof of Proposition III.1. Hence seller's ex-ante optimal mechanism γ^* is also ex-ante incentive efficient. □

C.2 Revelation Principle

Consider the seller's ex-ante contracting game. At the first stage, the seller announces a mechanism, before either party learns their private information. Then both the seller and the buyer learn their private information, followed by them sending their respective messages and the mechanism implementing its specified outcome.

First, I define the strategies of the players starting from the last stage. The seller, after he learns his private information, sends a message where a typical action is denoted by m_s and belongs to the set M_s. Similarly, the buyer sends her message where a typical action is denoted by m_b and it belongs to the set M_b. The communication strategy of a player i is defined by the mapping $\mu_i : \Theta_i \to M_i$ for each $i \in \{s, b\}$.

In the first stage, the strategy of the seller is to announce a contract, or equivalently a mechanism. In the context of bilateral trade, a mechanism $\gamma = \{Q, \tau\}$ consists of an allocation rule Q and a payment rule τ, where for any given pair of messages (m_s, m_b) submitted by the players, the allocation rule determines the probability of the object changing hands from the seller to the buyer, while the payment (or equivalently transfer) rule determines the monetary transfer from the buyer to the seller. Hence a mechanism can be defined by the mapping $\gamma : M_s \times M_b \to [0, 1] \times \mathbb{R}$.

It is assumed that the message spaces M_s and M_b are rich enough to include a particular message such that upon sending it, that agent is guaranteed to his/her outside option payoff of 0.[1]

Finally, a direct mechanism $\bar{\gamma} = \{\bar{Q}, \bar{\tau}\}$ is simply a mechanism where the message spaces of the seller and the buyer are equal to their type spaces. Hence a direct mechanism is defined by the mapping $\bar{\gamma} : \Theta_s \times \Theta_b \to [0, 1] \times \mathbb{R}$. A direct mechanism satisfies IC if truthful type-telling forms a Bayesian Nash equilibrium. Similarly, a direct mechanism satisfies IR if the truthful type-telling equilibrium payoffs are weakly greater than the outside option payoff, which is normalized to 0. A direct mechanism is feasible if and only if it satisfies IC and IR.

In this game, the equilibrium notion is a perfect Bayesian Nash equilibrium (henceforth PBNE) denoted by $(\hat{\gamma}, \hat{\mu}_s, \hat{\mu}_b)$, where the communication strategies of the seller and the buyer $(\hat{\mu}_s, \hat{\mu}_b)$ form a Bayesian Nash equilibrium of the mechanism $\hat{\gamma}$ and the seller

[1] It is also possible to allow the buyer to choose whether she wants to participate in the mechanism or she wants to "stay at home" before sending her message. As the results do not change, this formulation is preferred for ease of notation.

announces $\hat{\gamma}$ that maximizes his ex-ante expected payoff. The beliefs of the players in the second stage are equal to the priors, as both players participate in the mechanism and there is no signaling.

The aim of this appendix is to show the following.

Lemma C.1.
Given any PBNE $(\hat{\gamma}, \hat{\mu}_s, \hat{\mu}_b)$, there exists a feasible direct mechanism $\bar{\gamma} = \{\bar{Q}, \bar{\tau}\}$ that has a payoff equivalent truthful type-telling equilibrium.

Proof. Let $(\hat{\gamma}, \hat{\mu}_s, \hat{\mu}_b)$ be an equilibrium. The equilibrium expected payoffs are given as follows:

$$U_s(\hat{\mu}_s|\theta_s) = \int_{\theta_b} \big[\hat{\tau}(\hat{\mu}_s(\theta_s), \hat{\mu}_b(\theta_b)) - \hat{Q}(\hat{\mu}_s(\theta_s), \hat{\mu}_b(\theta_b))\theta_s\big] dG(\theta_b)$$
$$U_b(\hat{\mu}_b|\theta_b) = \int_{\theta_s} \big[\hat{Q}(\hat{\mu}_s(\theta_s), \hat{\mu}_b(\theta_b))[\alpha\theta_s + (1-\alpha)\theta_b] - \hat{\tau}(\hat{\mu}_s(\theta_s), \hat{\mu}_b(\theta_b))\big] dF(\theta_s)$$

The optimality of the communication strategies imply that expected payoffs satisfy:

$$U_s(\hat{\mu}_s|\theta_s) \geq U_s(m_s|\theta_s) \qquad \forall\theta_s \text{ and } m_s \in M_s$$
$$U_b(\hat{\mu}_b|\theta_b) \geq U_b(m_b|\theta_b) \qquad \forall\theta_b \text{ and } m_b \in M_b$$

where the terms $U_i(m_i|\theta_i)$ denote the expected payoffs from deviating to any different message m_i for each player $i \in \{s, b\}$.

Now consider the direct mechanism $\bar{\gamma} = \{\bar{Q}, \bar{\tau}\}$, where for any given type pair announcement $(\tilde{\theta}_s, \tilde{\theta}_b)$, the allocation and transfer functions are defined as follows:

$$\bar{Q}(\tilde{\theta}_s, \tilde{\theta}_b) = \hat{Q}(\hat{\mu}_s(\tilde{\theta}_s), \hat{\mu}_b(\tilde{\theta}_b))$$
$$\bar{\tau}(\tilde{\theta}_s, \tilde{\theta}_b) = \hat{\tau}(\hat{\mu}_s(\tilde{\theta}_s), \hat{\mu}_b(\tilde{\theta}_b))$$

Observe that under truthful type-reporting, the expected payoff schedules are equivalent; i.e. $\bar{U}_s(\theta_s|\theta_s) = U_s(\hat{\mu}_s|\theta_s)$ and $\bar{U}_b(\theta_b|\theta_b) = U_b(\hat{\mu}_b|\theta_b)$ for all respective types. Furthermore, truthful type-reporting is indeed an equilibrium of $\bar{\gamma}$. To see why, observe for the seller that:

$$U_s(\hat{\mu}_s|\theta_s) = \bar{U}_s(\theta_s|\theta_s) \geq \bar{U}_s(\tilde{\theta}_s|\theta_s) = U_s(\tilde{m}_s|\theta_s)$$

where $\tilde{m}_s = \hat{\mu}_s(\tilde{\theta}_s)$. Analogous arguments apply to the buyer. Lastly, observe that the expected payoffs in the truthful type-telling equilibrium of $\bar{\gamma}$ also satisfies IR. To see why, note that the payoffs from the PBNE of the original game are weakly greater than 0 for the seller and the buyer. Then, due to the equivalence of the expected payoffs, the truth-telling equilibrium payoffs of the direct mechanism are also nonnegative. $\square$

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